Vector Fields

USTC, 2024 Spring

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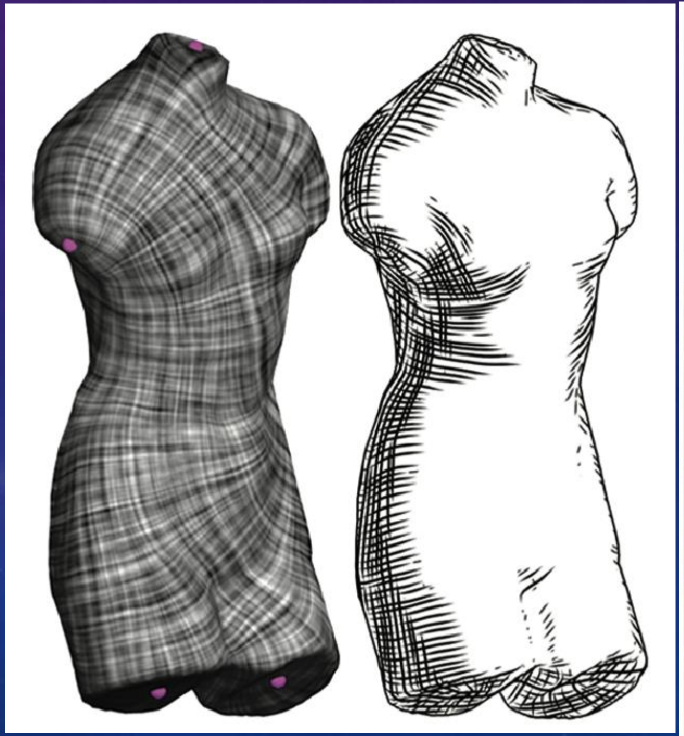
https://qingfang1208.github.io/

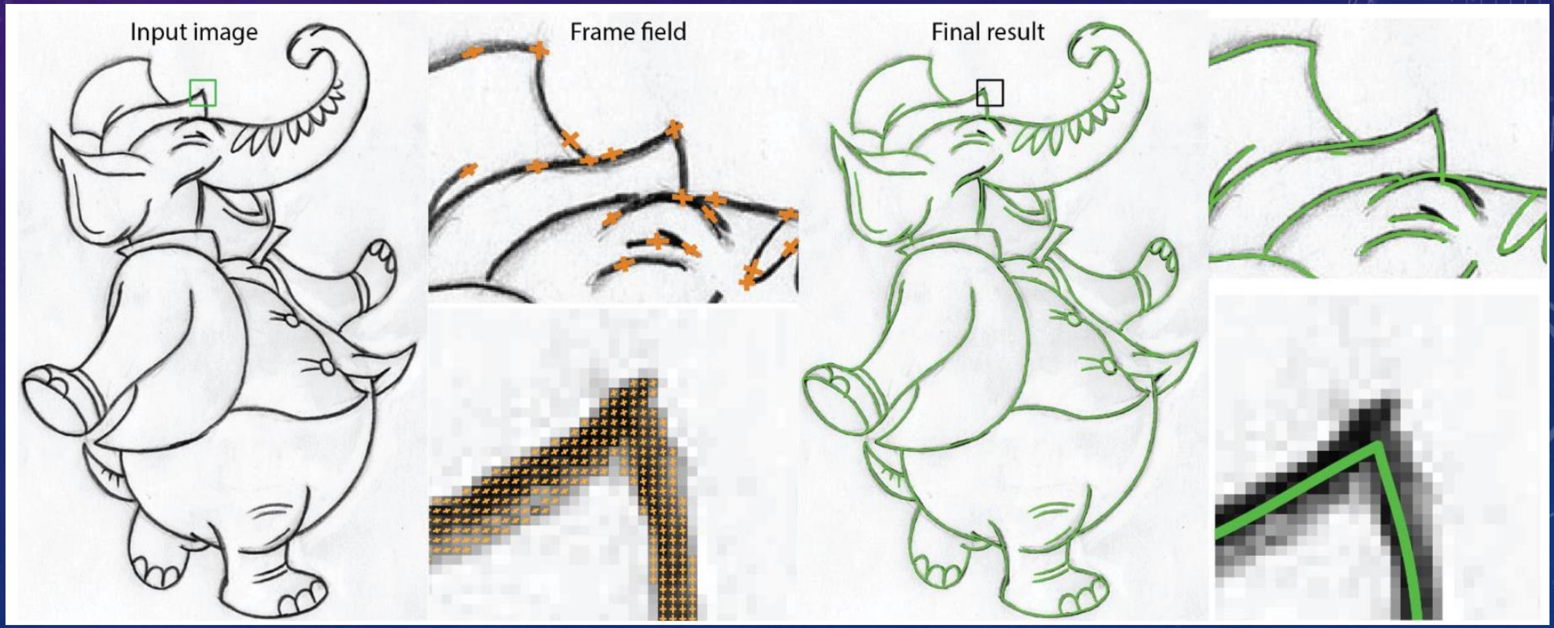
Background

- Vector field in design
  - Interpolation



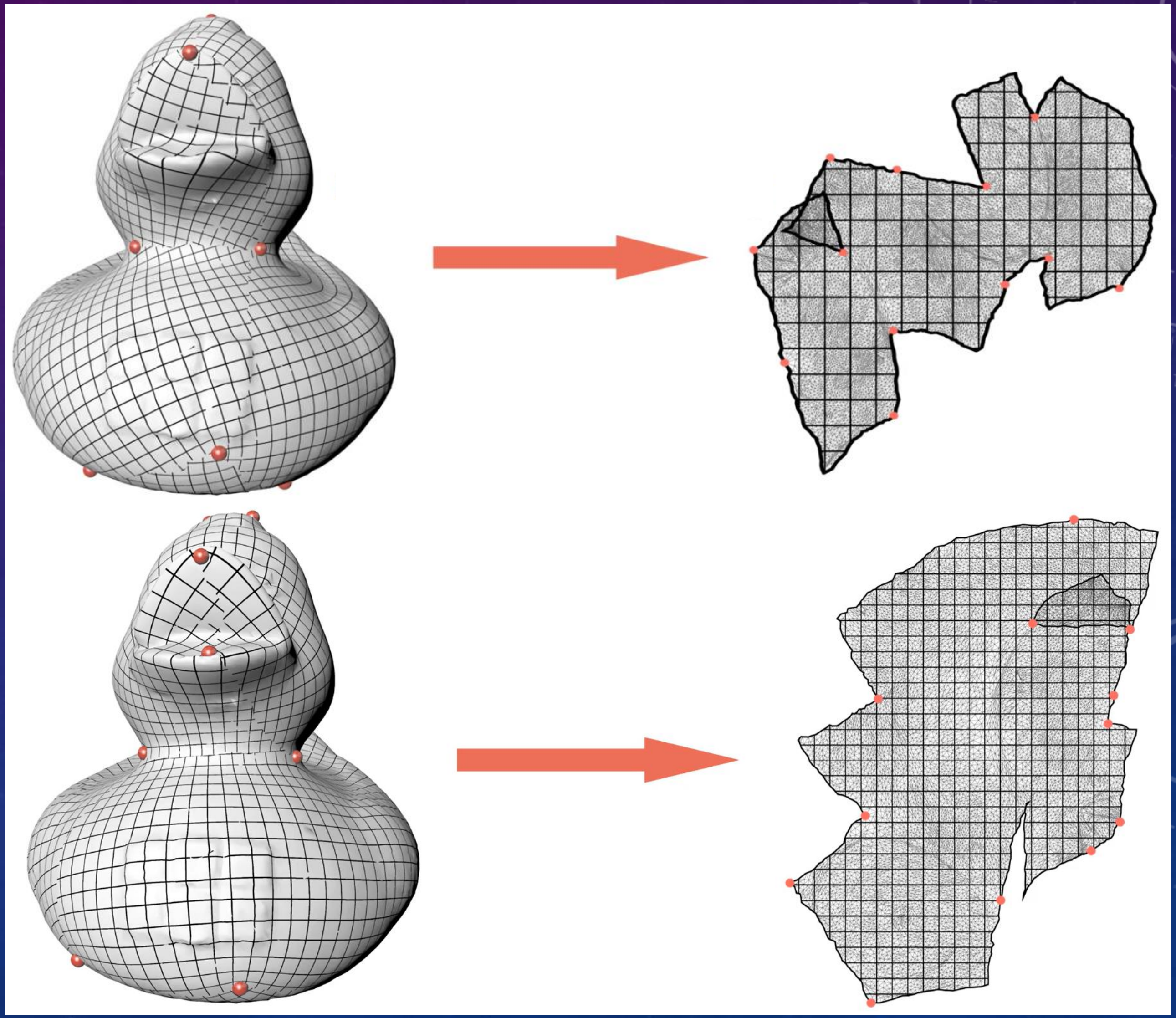

Background

- Vector field in design
  - Interpolation
  - Vectorization



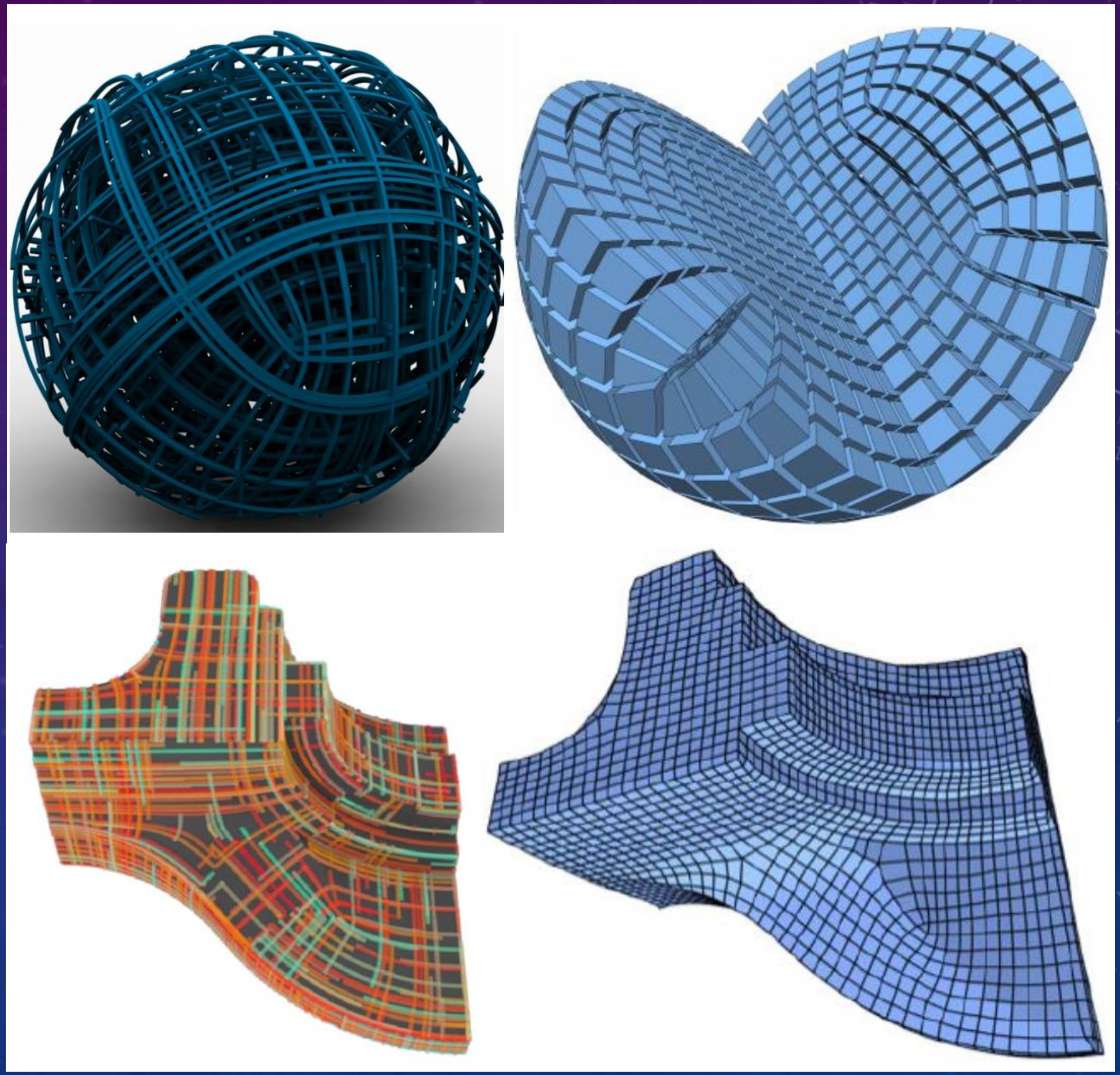
Background

- Vector field in design
  - Interpolation
  - Vectorization
  - Texture mapping



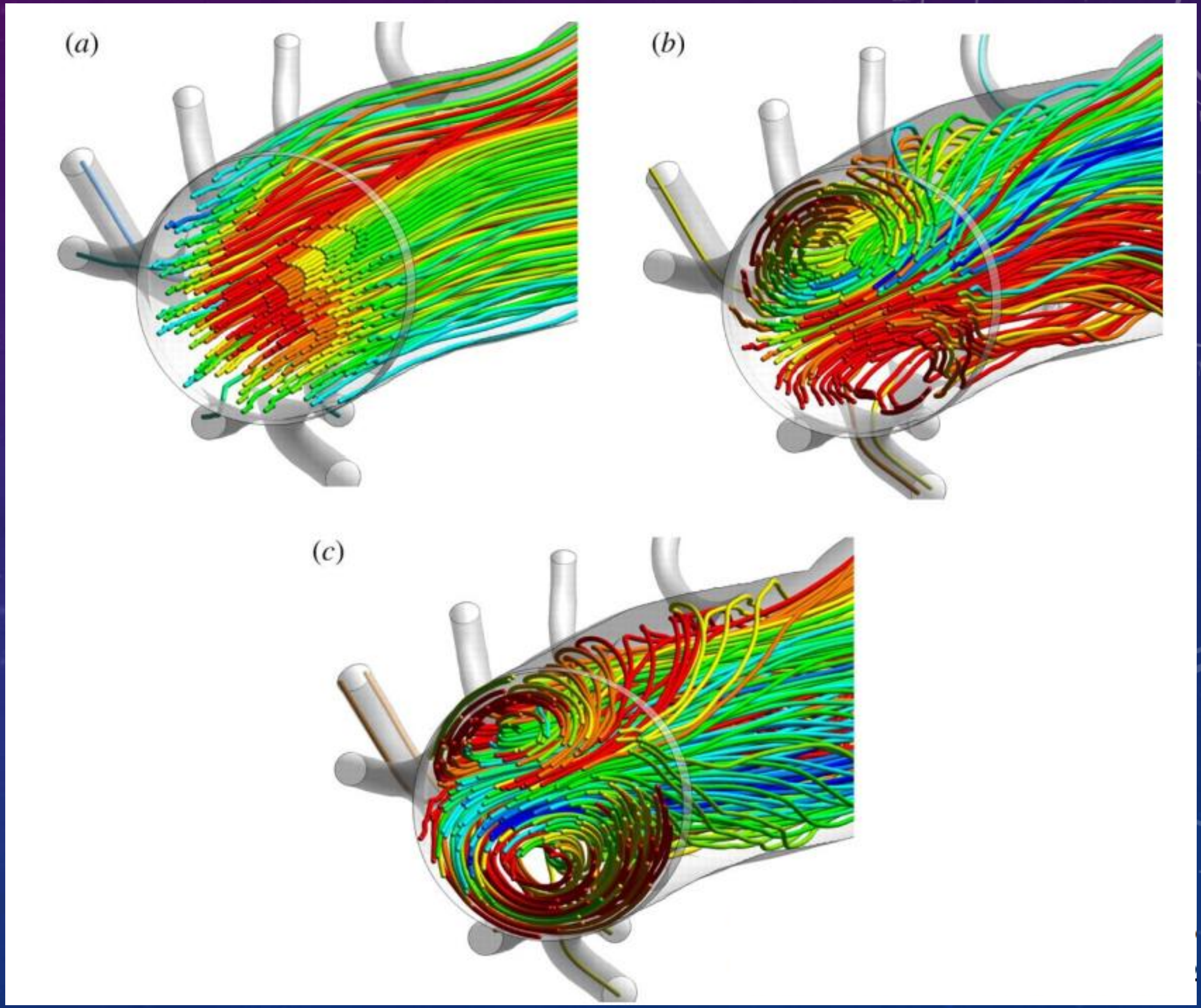
Background

- Vector field in design
- Vector field in simulation
  - Quad and hex meshing



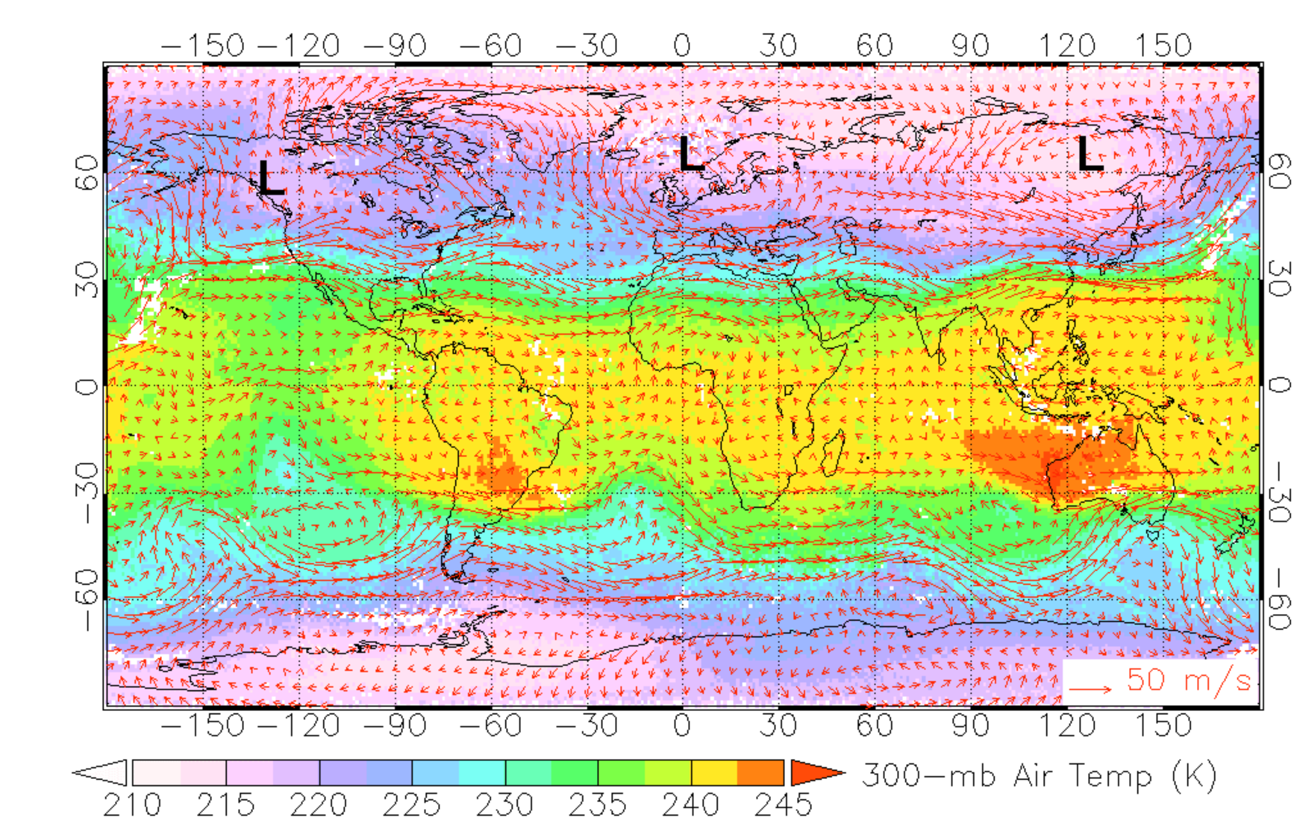
Background

- Vector field in design
- Vector field in simulation
  - Quad and hex meshing
  - Biological science



Background

- Vector field in design
- Vector field in simulation
  - Quad and hex meshing
  - Biological science
  - Weather forecast



Background

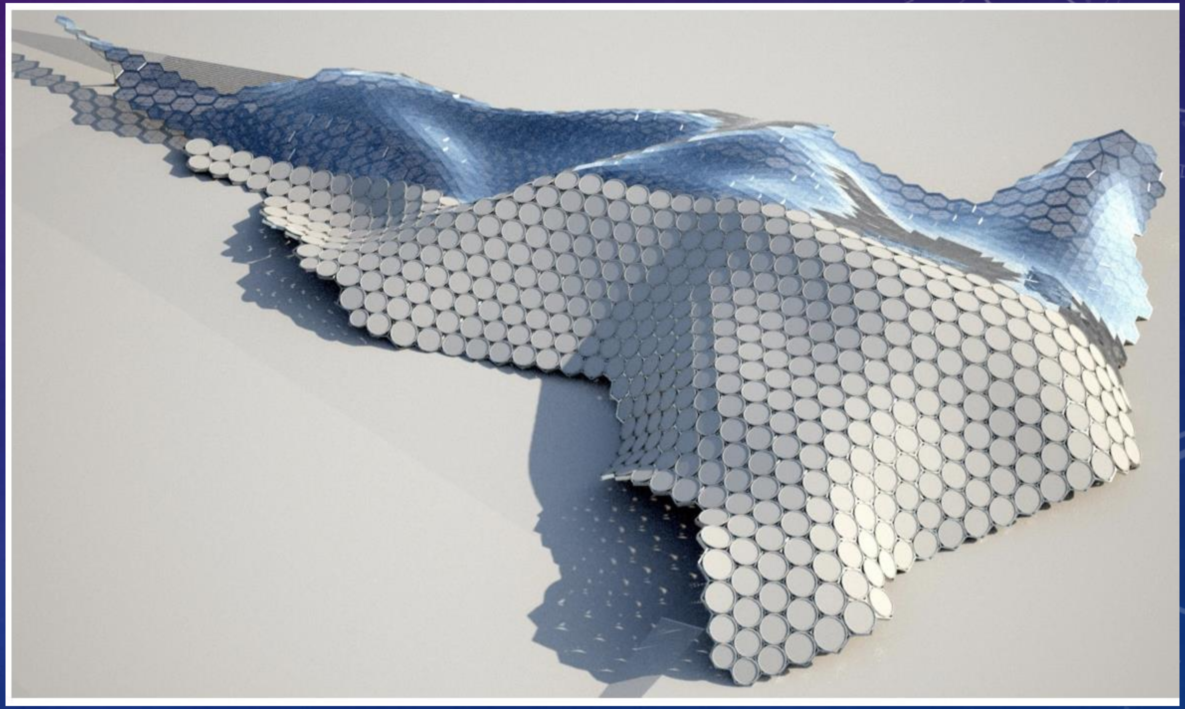
- Vector field in design
- Vector field in simulation
  - Quad and hex meshing
  - Biological science
  - Weather forecast
  - Engineering

Background

- Vector field in design
- Vector field in simulation
- Vector field in manufacturing
    - Pastry design

Background

- Vector field in design
- Vector field in simulation
- Vector field in manufacturing
  - Pastry design
  - Architecture



Background

- Vector field in design
- Vector field in simulation
- Vector field in manufacturing
    - Pastry design
    - Architecture
    - Weaving

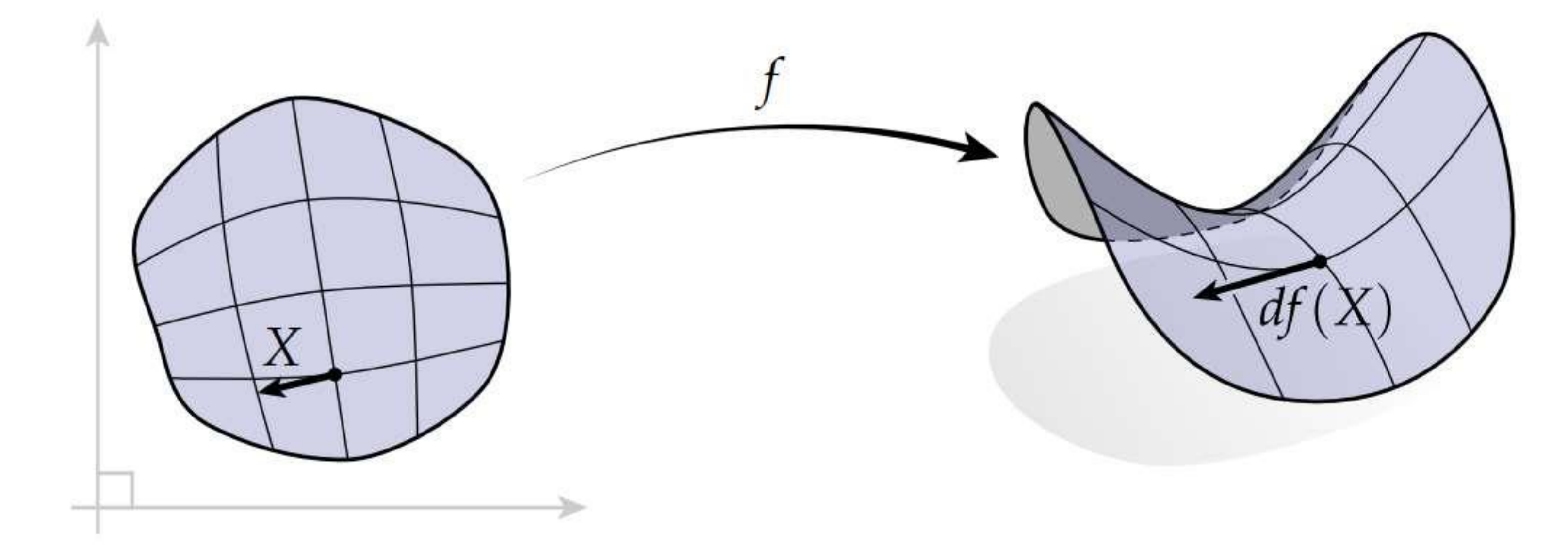
Smooth theory

- Differential of vector field
- Parallel transport
- Singularities
- Direction fields

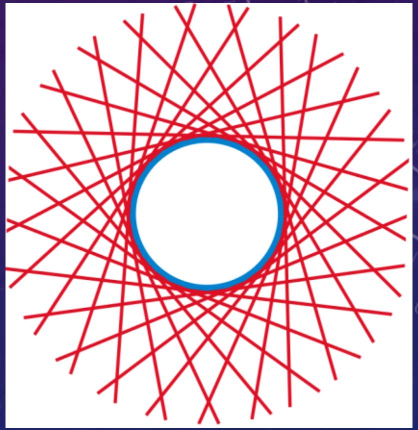
Differential of a surface

- $df: X \in \Omega \rightarrow T_P f \in \mathbb{R}^3$ push forward X

$$df\,(X) = \lim_{t\to 0} \frac{f(p+tX) - f(p)}{t} \Longrightarrow \begin{aligned} df(X) &= df(c_1 e_1 + c_2 e_2) \\ &= c_1 df(e_1) + c_2 df\,(e_2), c_1, c_2 \in \mathbb{R} \end{aligned}$$



Tangent vector field



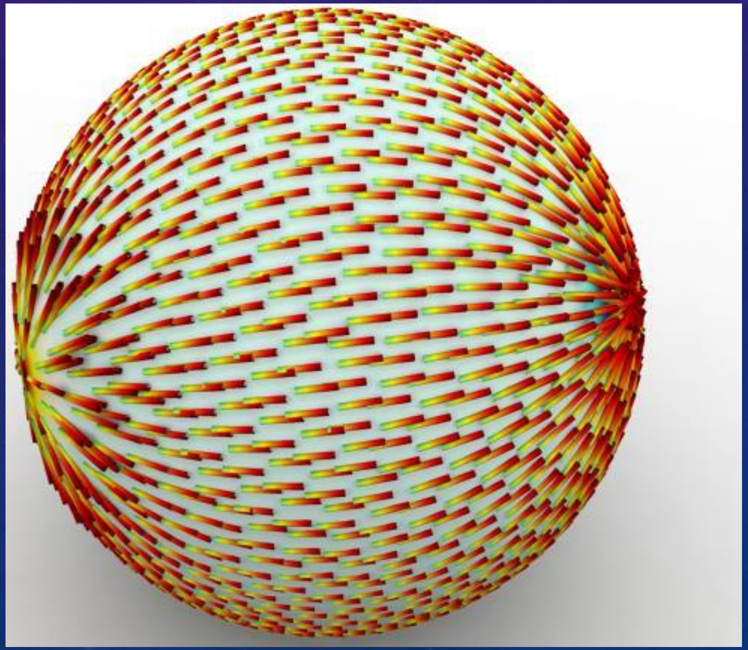
- Scalar field f :  $\to \mathbb{R}$

- Tangent bundle : $TM = \{(p, v) : v \in T_pM\}$

- Vector field $g: M \to TM$ with $g(p) = (p, v), v \in T_pM$

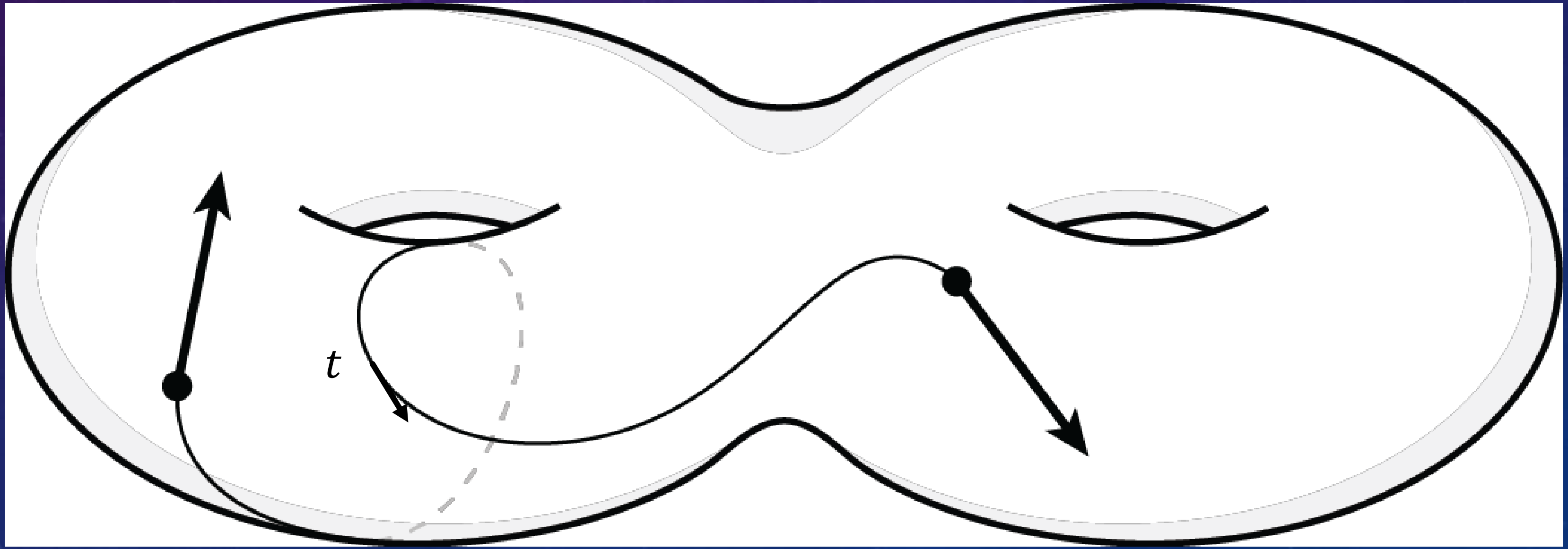
Gradient vector field

- For each $p \in M$, there exists a unique vector $\nabla f(p) \in T_pM$ so that $df_p(v) = v^T \nabla f(p)$ for all $v \in T_pM$

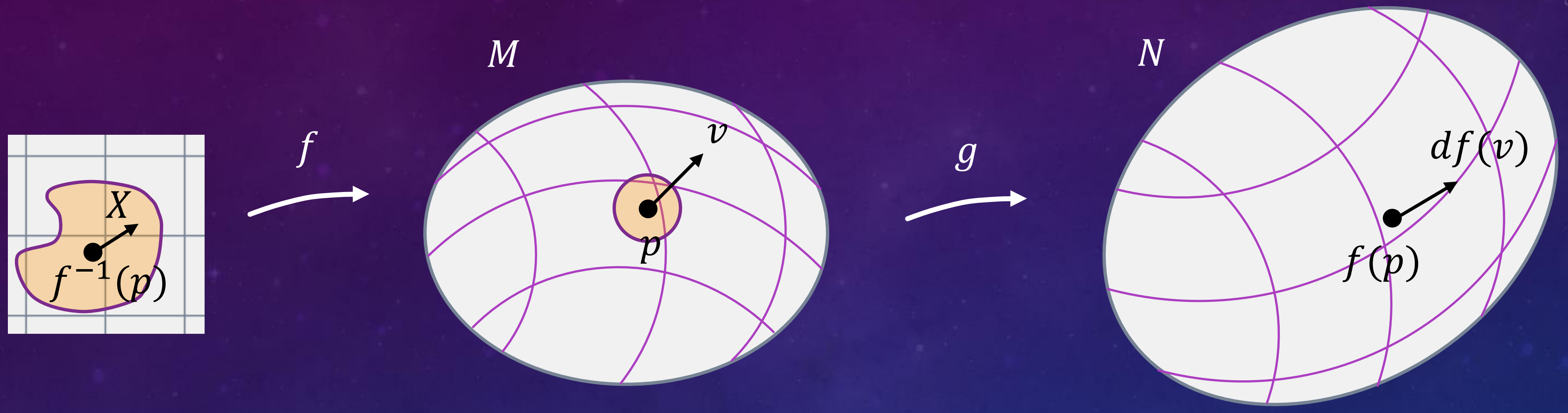


Differential of a vector field

- Key - How to identify different tangent spaces?



Differential of a map



Independent of f

$$df\,(X) = \lim_{t\to 0}\frac{f(f^{-1}(p)+tX)-p}{t} \quad \Rightarrow \quad dg(v) = \lim_{t\to 0}\frac{g\circ f(f^{-1}(p)+t(df)^{-1}v)-g(p)}{t} \in TN$$

$$\begin{cases} df : X \in \mathbb{R}^2_{f^{-1}(p)} \to v \in T_pM \\ (df)^{-1} : v \in T_pM \to X \in \mathbb{R}^2_{f^{-1}(p)} \end{cases}$$

Linear : $dg(v) = dg(c_1e_1 + c_2e_2)$

$= c_1dg(e_1) + c_2dg\,(e_2) \quad e_1, e_2 \in T_pM$

Vector field flows : diffeomorphism

- $f_t : M \to M$ is a diffeomorphism with inverse f_{-t}

Property $f_{t+s} = f_t(f_s(x))$

- $\frac{d}{dt} f_t = V \circ f_t$

Vector field flows : diffeomorphism

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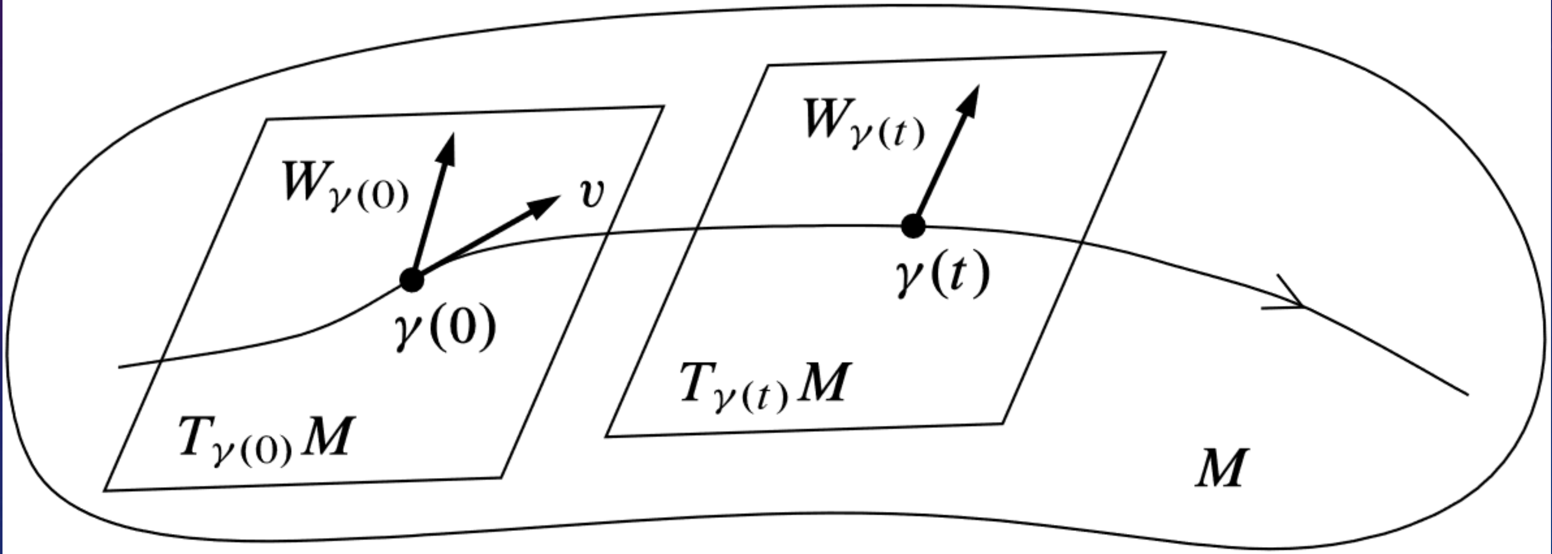
- Example : killing vector fields

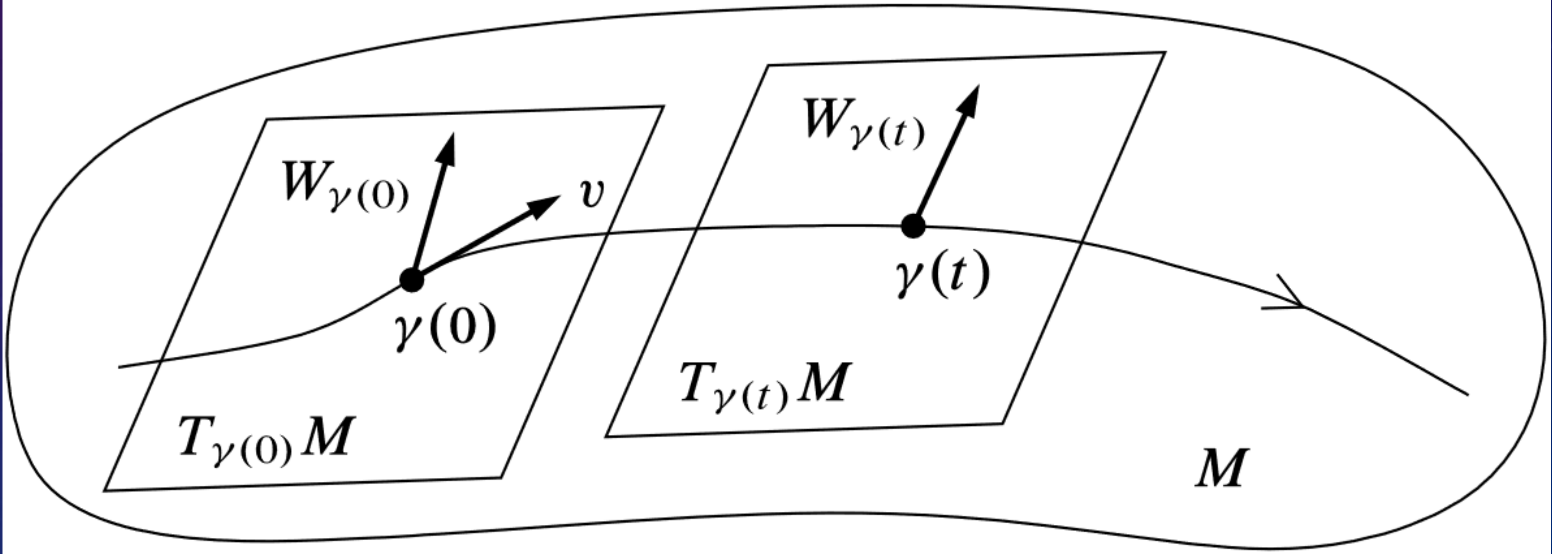
Differential of a vector field flow

- $df_t(p) : T_pM \to T_{f_t(p)}M$



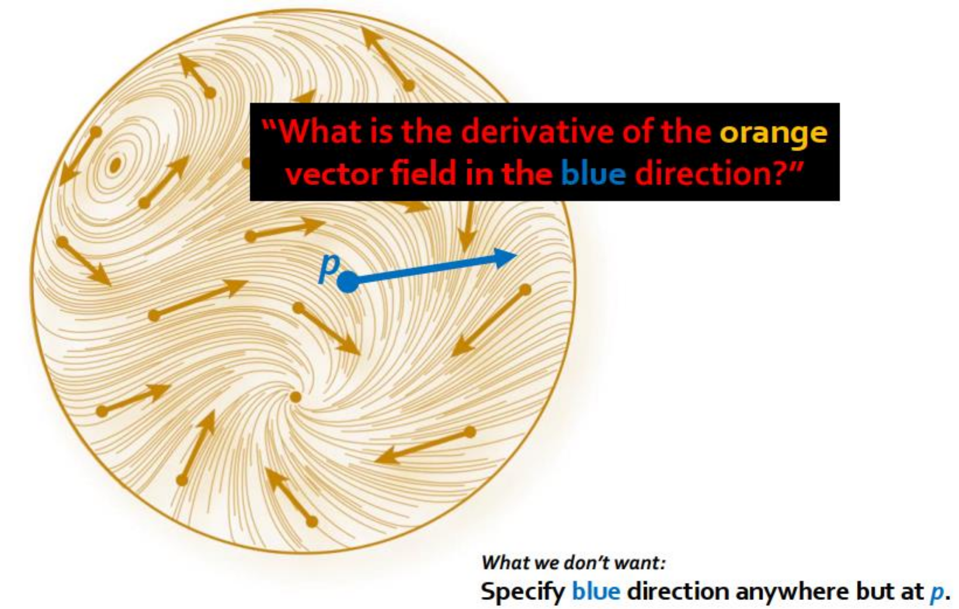
Lie derivate

- $(\mathcal{L}_V W)_p = \lim\limits_{t \to 0} \frac{1}{t} \left[(df_{-t})_{f_t(p)} \left(W_{f_t(p)} \right) - W_p \right]$



What we want

- The Lie derivative requires a local diffeomorphism
- How to define directional derivatives? – extra geometric structure



What we want

- The Lie derivative requires a local diffeomorphism
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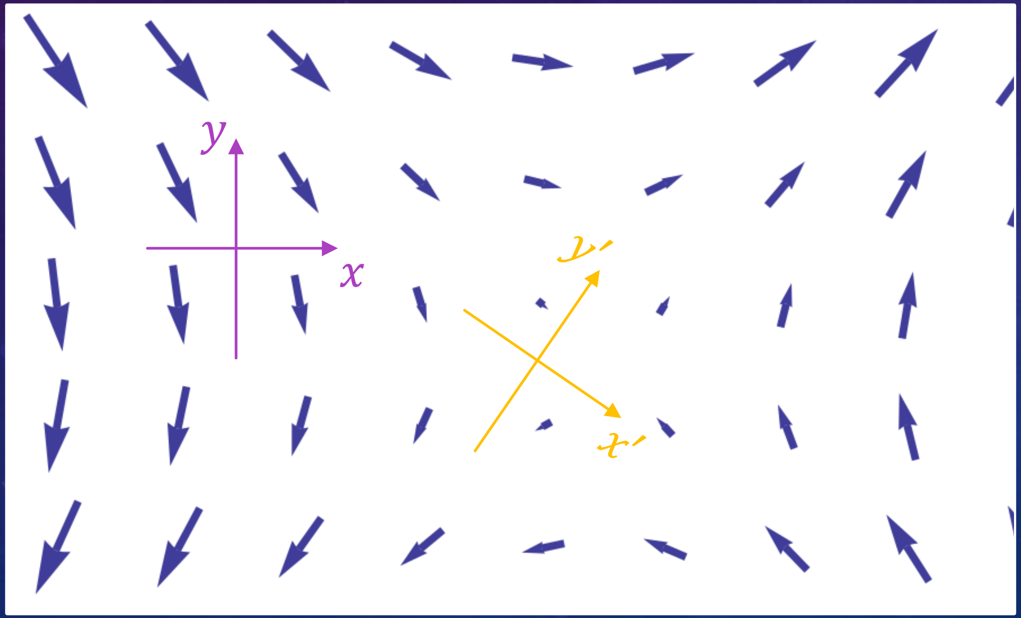
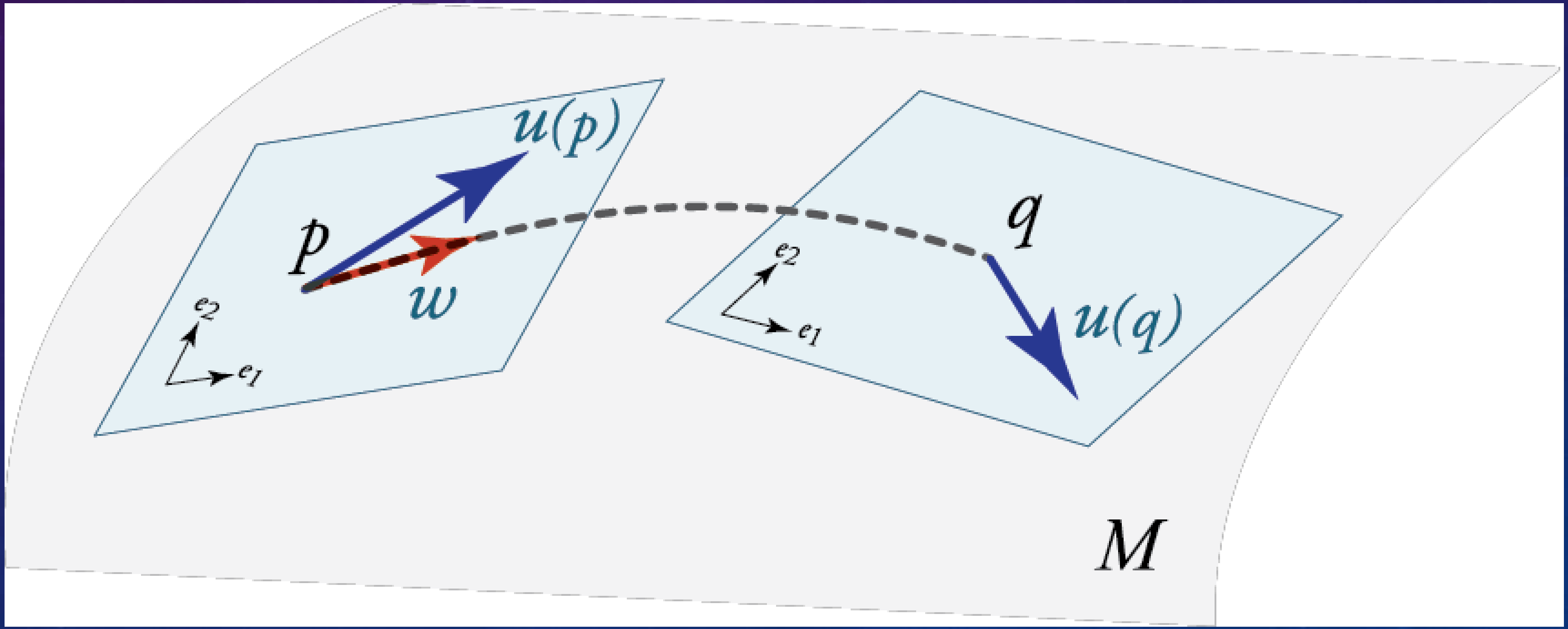


$$\nabla u = \begin{pmatrix} \frac{\partial u_x}{\partial x} & \frac{\partial u_y}{\partial x} \\ \frac{\partial u_x}{\partial y} & \frac{\partial u_y}{\partial y} \end{pmatrix}$$

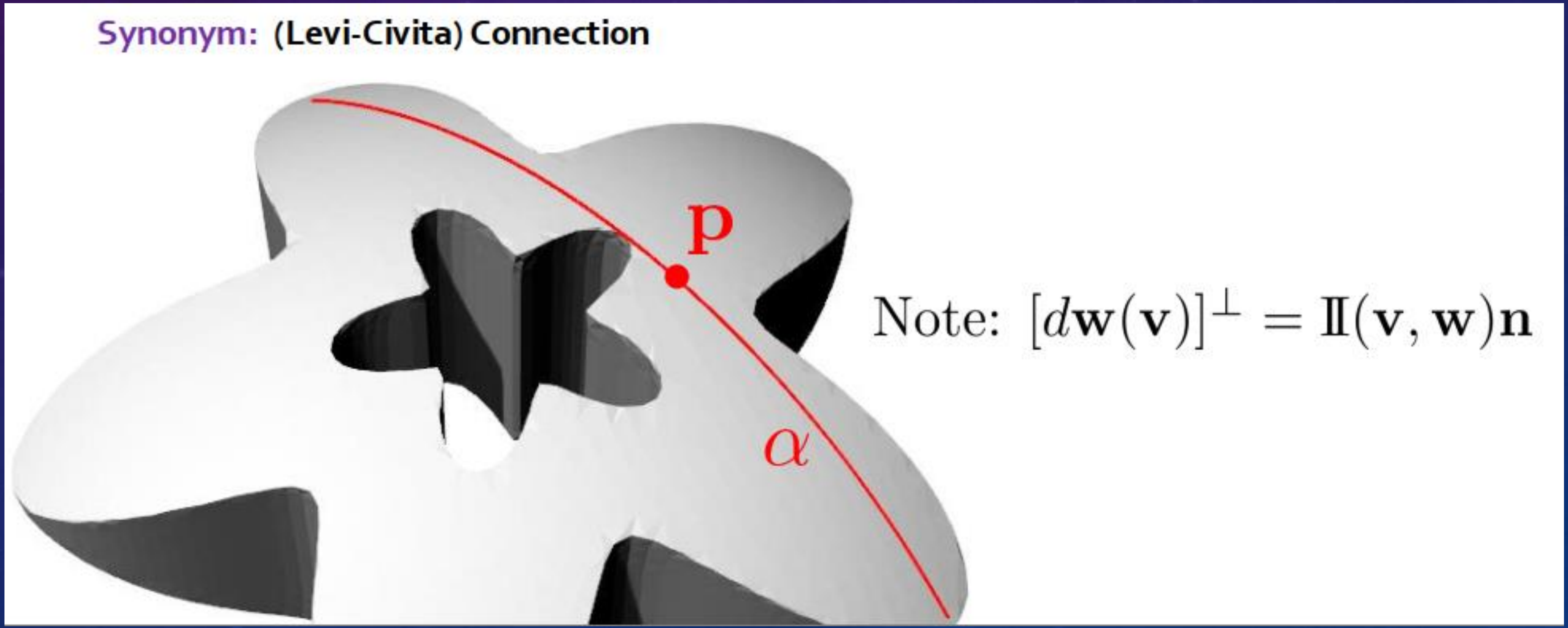
Parallel transport

- Geodesic path



Covariant derivate

- $\nabla_v W = [dW(v)]^{\parallel} = \mathrm{proj}_{T_pM}(W \circ \alpha)'(0)$, α integral curve of v through p



Properties

Properties of the Covariant Derivative

As defined, $\nabla_V Y$ depends only on V_p and Y to first order along c.

Also, we have the Five Properties:

1. C^∞-linearity in the V-slot:
   $\nabla_{V_1+fV_2} Y = \nabla_{V_1} Y + f\,\nabla_{V_2} Y$ where $f : S \to \mathbb{R}$
2. $\mathbb{R}$-linearity in the Y-slot:
   $\nabla_V(Y_1 + aY_2) = \nabla_V Y_1 + a\nabla_V Y_2$ where $a \in \mathbb{R}$
3. Product rule in the Y-slot:
   $\nabla_V(f\,Y) = f \cdot \nabla_V Y + (\nabla_V f) \cdot Y$ where $f : S \to \mathbb{R}$
4. The metric compatibility property:
   $\nabla_V\langle Y, Z\rangle = \langle \nabla_V Y, Z\rangle + \langle Y, \nabla_V Z\rangle$
5. The "torsion-free" property:
   $\nabla_{V_1} V_2 - \nabla_{V_2} V_1 = [V_1, V_2]$

The Lie bracket

$$[V_1, V_2](f) := D_{V_1} D_{V_2}(f) - D_{V_2} D_{V_1}(f)$$

Defines a vector field, which is tangent to S if V_1, V_2 are!

Geodesic equation

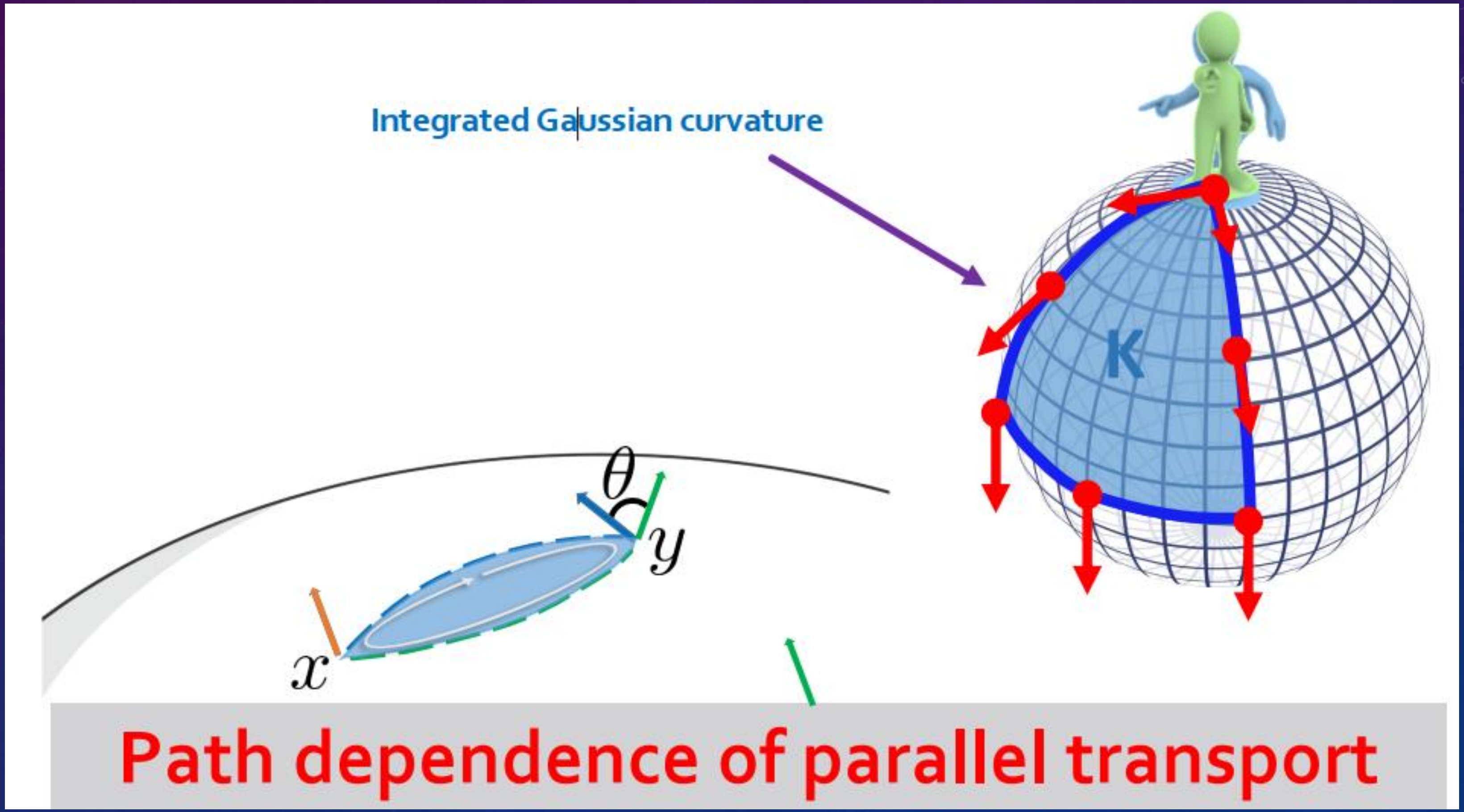
$$\mathrm{Proj}_{T_{\gamma(s)}M}[\gamma''(s)] = 0$$

- No steering wheel
- The only acceleration is out of the surface

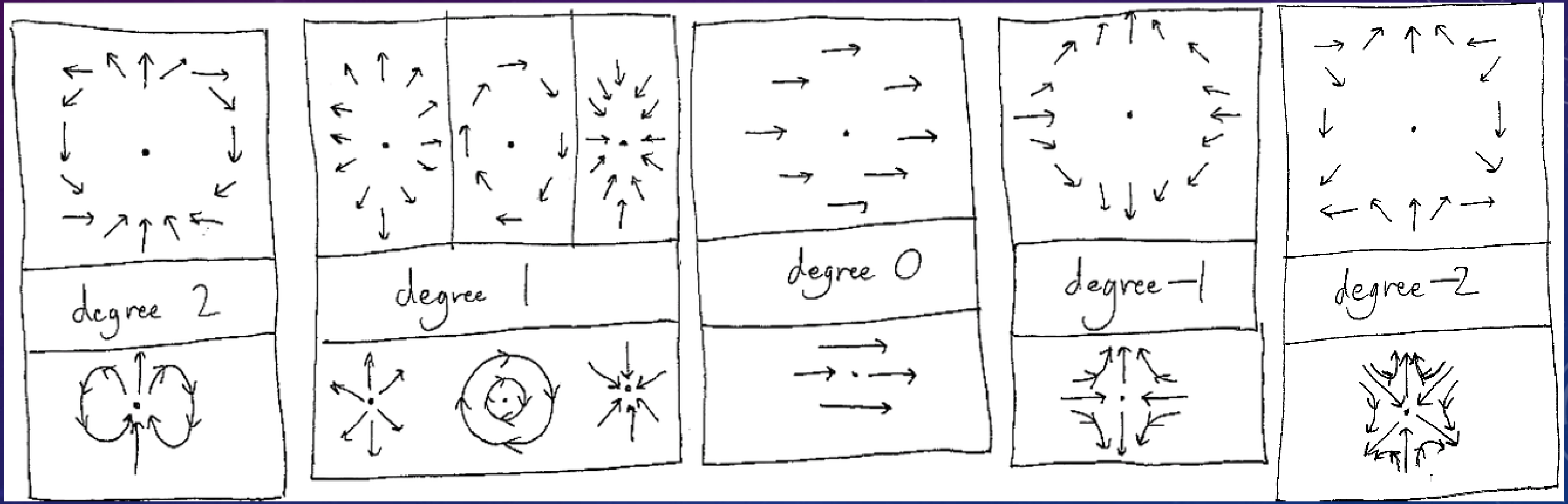
$$\nabla_{\gamma'(t)}[\gamma'(t)] = 0$$

- No steering wheel
- No stepping on the accelerator

Parallel transport



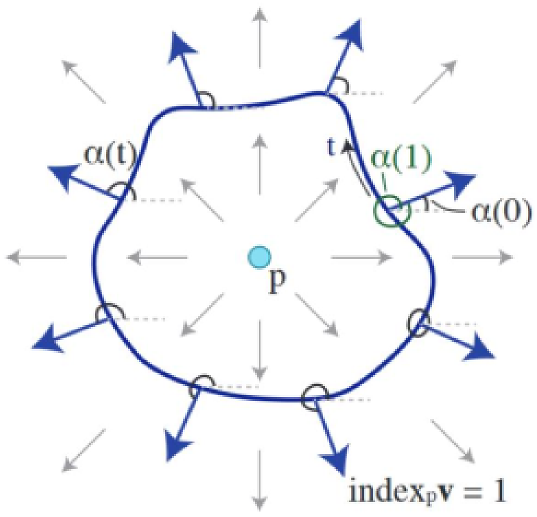
Singularities



Poincare-Hopf theorem

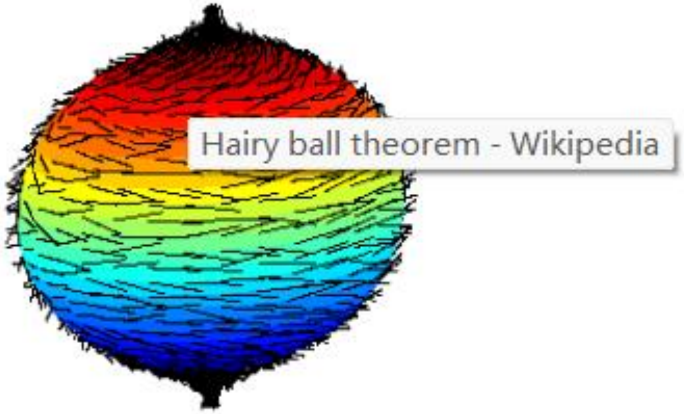
$$\sum_i \mathrm{index}_{x_i}(v) = \chi(M)$$

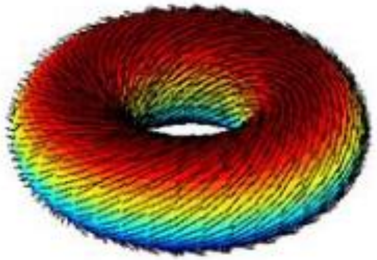
where vector field v has isolated singularities $\{x_i\}$.

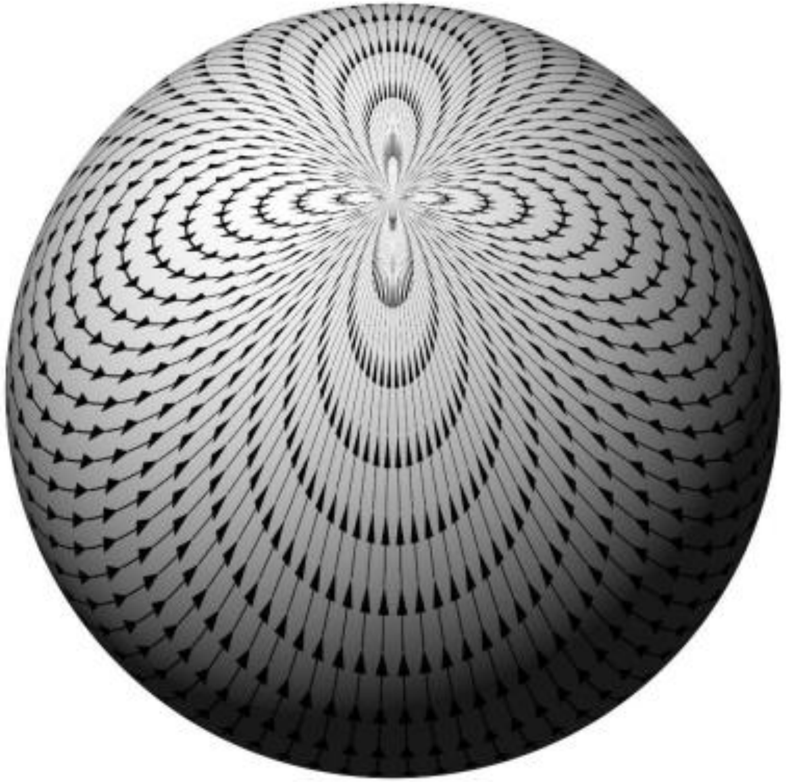


$$v(c(t)) = \|v(c(t))\| \begin{pmatrix} \cos\alpha(t) \\ \sin\alpha(t) \end{pmatrix}$$

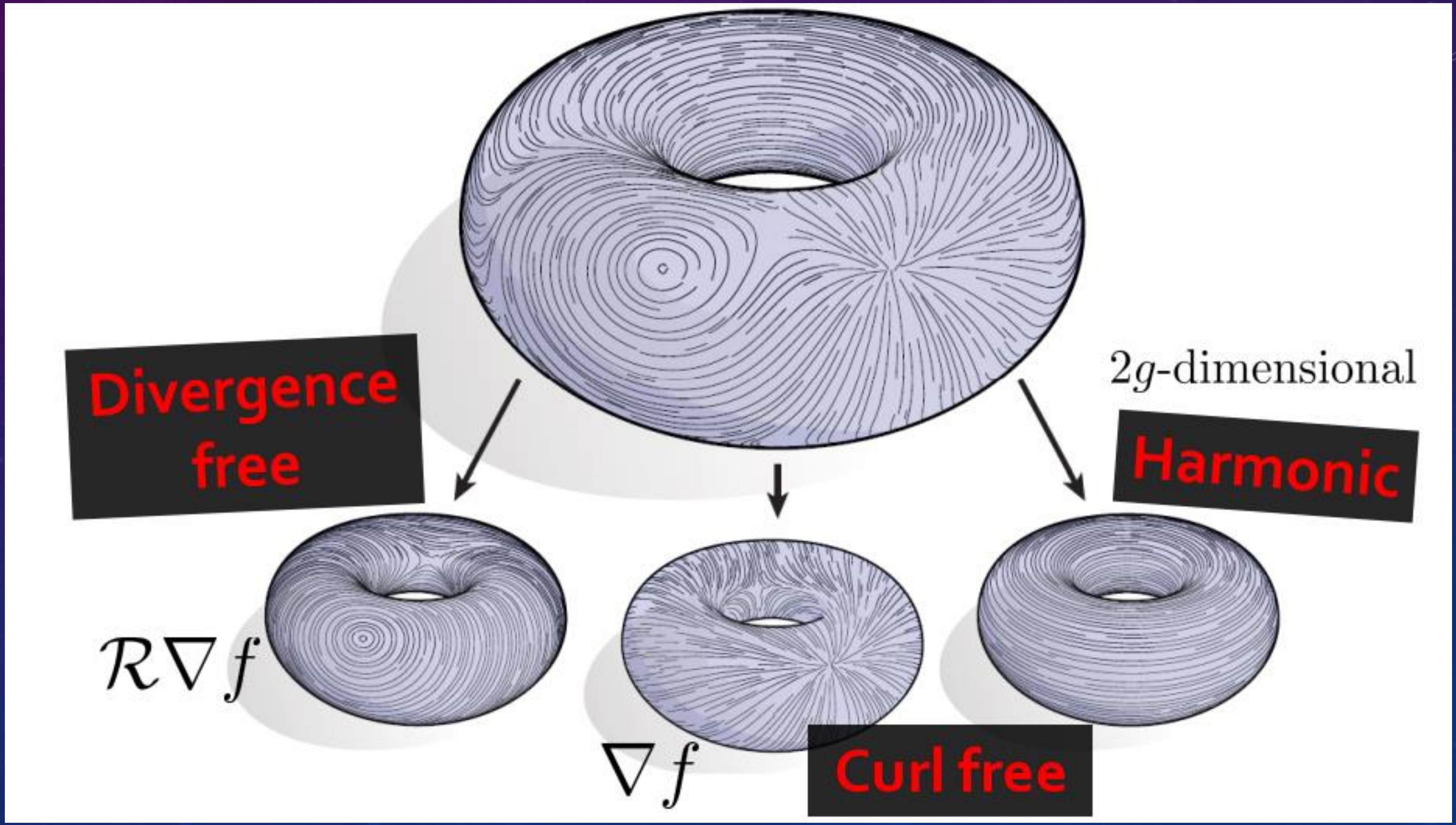
Hairy ball theorem







Helmholtz-Hodge decomposition



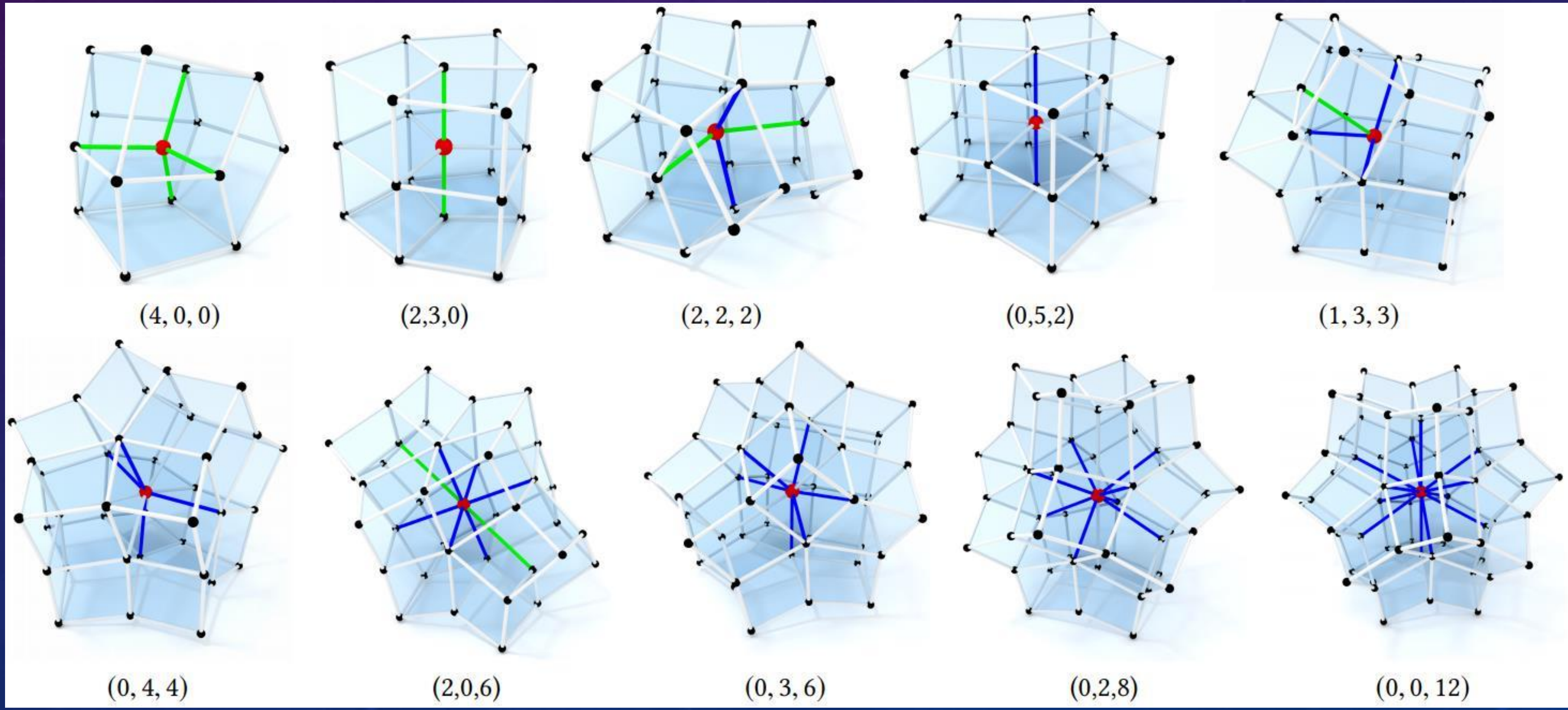
2D directional fields

- Directional Field Synthesis, Design, and Processing [Vaxman et al., EG STAR 2016]

	1-vector field	One vector, classical “vector field”
	2-direction field	Two directions with π symmetry, “line field”, “2-RoSy field”
	1^3-vector field	Three independent vectors, “3-polyvector field”
	4-vector field	Four vectors with $\pi/2$ symmetry, “non-unit cross field”
	4-direction field	Four directions with $\pi/2$ symmetry, “unit cross field”, “4-RoSy field”
	2^2-vector field	Two pairs of vectors with π symmetry each, “frame field”
	2^2-direction field	Two pairs of directions with π symmetry each, “non-ortho. cross field”
	6-direction field	Six directions with $\pi/3$ symmetry, “6-RoSy”
	2^3-vector field	Three pairs of vectors with π symmetry each

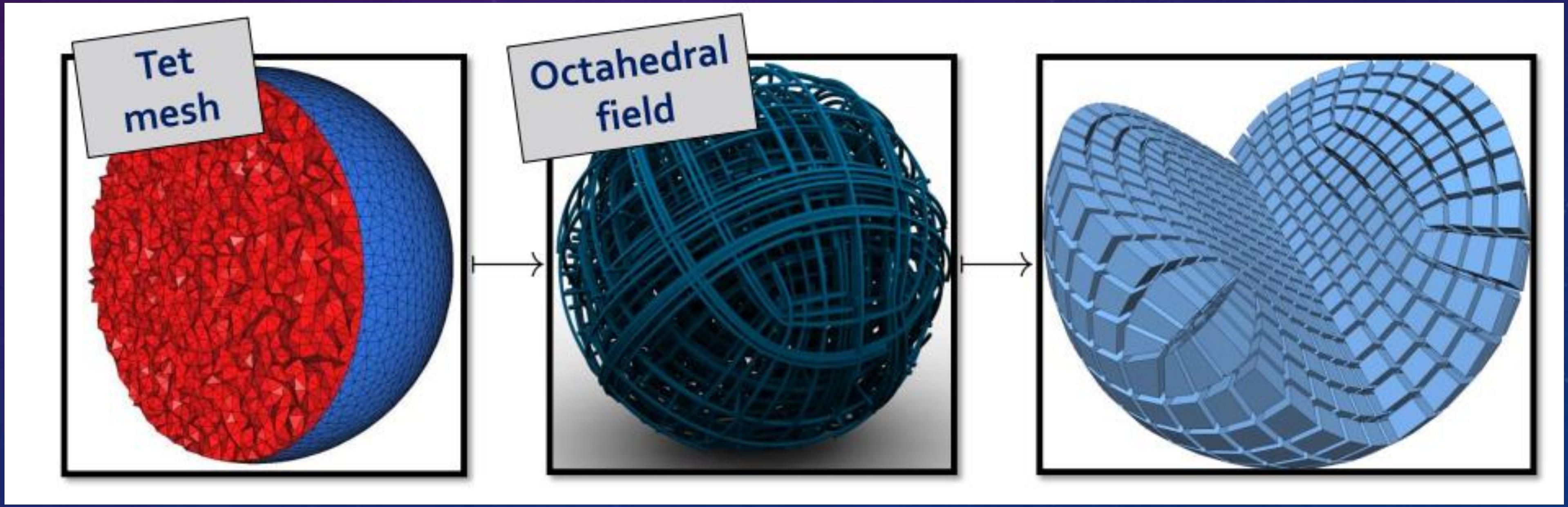
3D frame fields

- What singular structures are possible?



3D frame fields

- Field-guided meshing



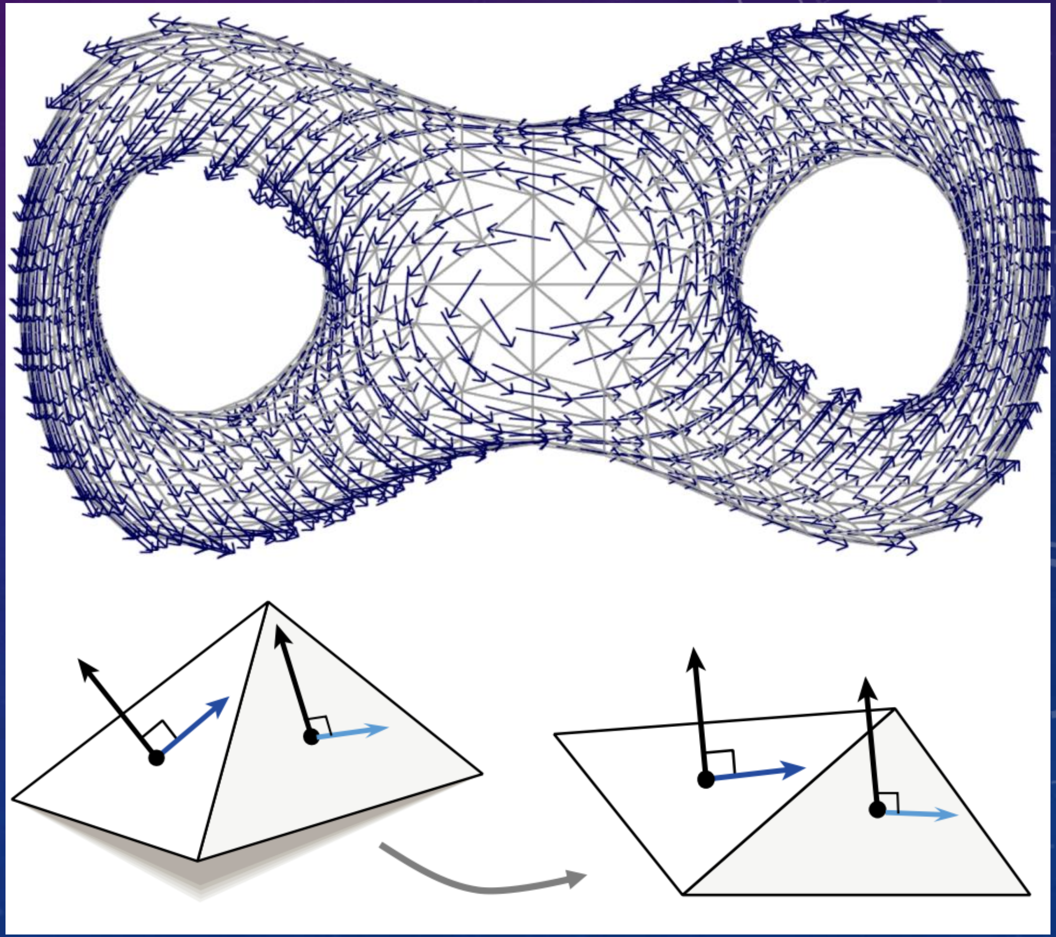
Algebraic Representations for Volumetric Frame Fields [Palmer et al. 2020]

Discretization

- Triangle-based vector fields
- Edge-based vector fields
- Vertex-based vector fields

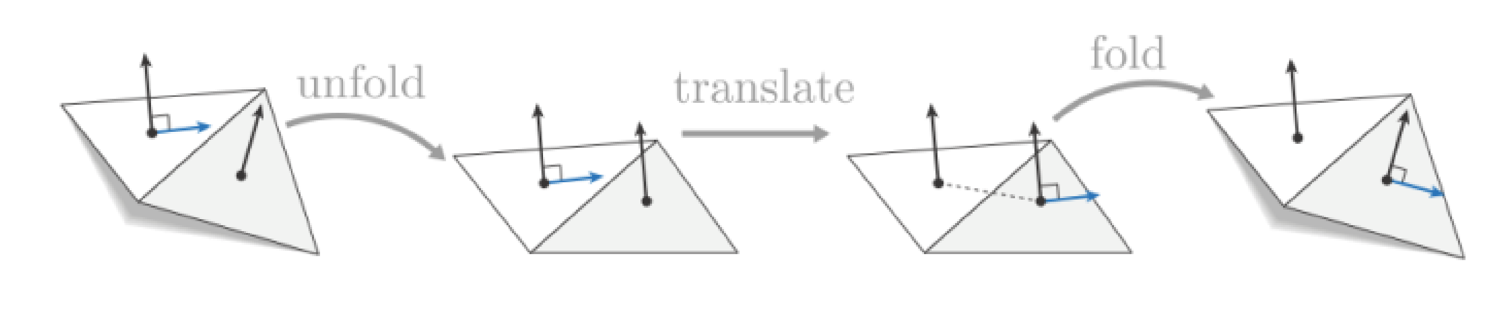
Triangle-based

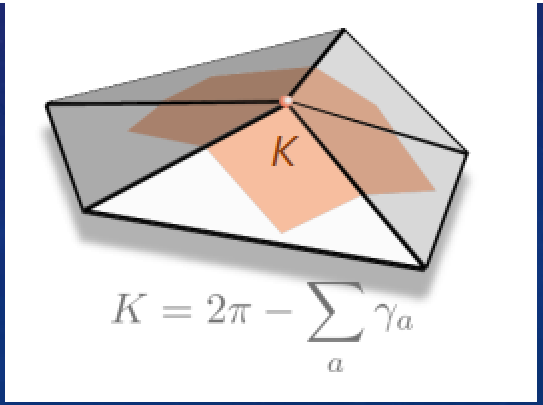
- Triangle as its own tangent plane
- One vector per triangle
  - Piecewise constant
  - Discontinuous at edges/vertices
- Easy to unfold/hinge



Discrete Levi-Civita connection

- Simple notion of parallel transport
- Transport around vertex
  - Excess angle
  - Gaussian curvature

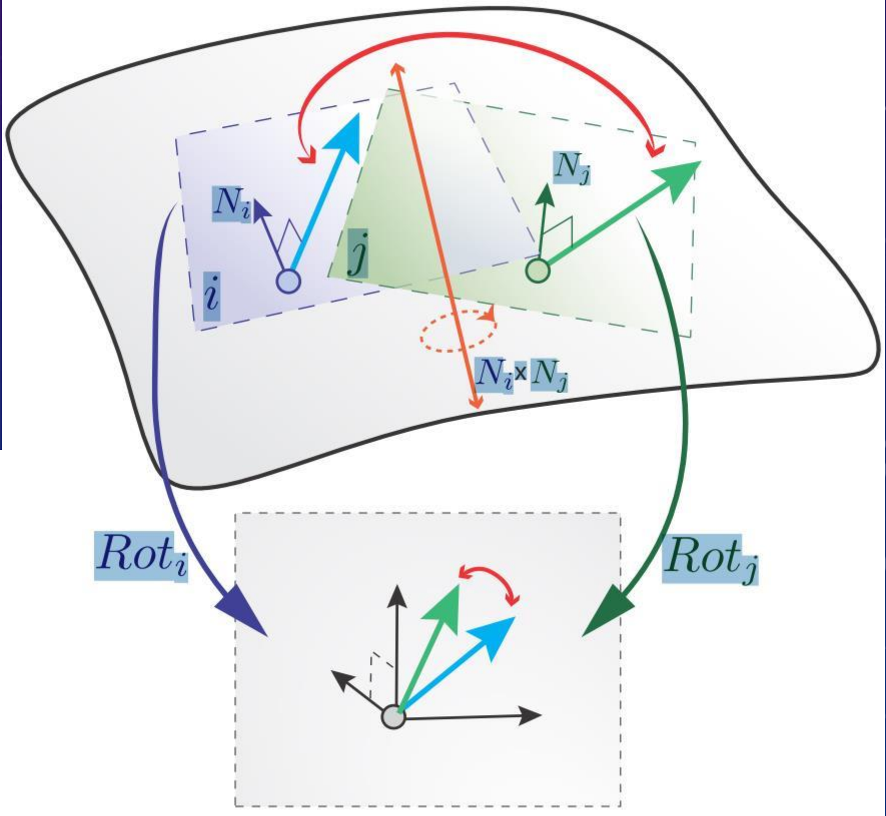


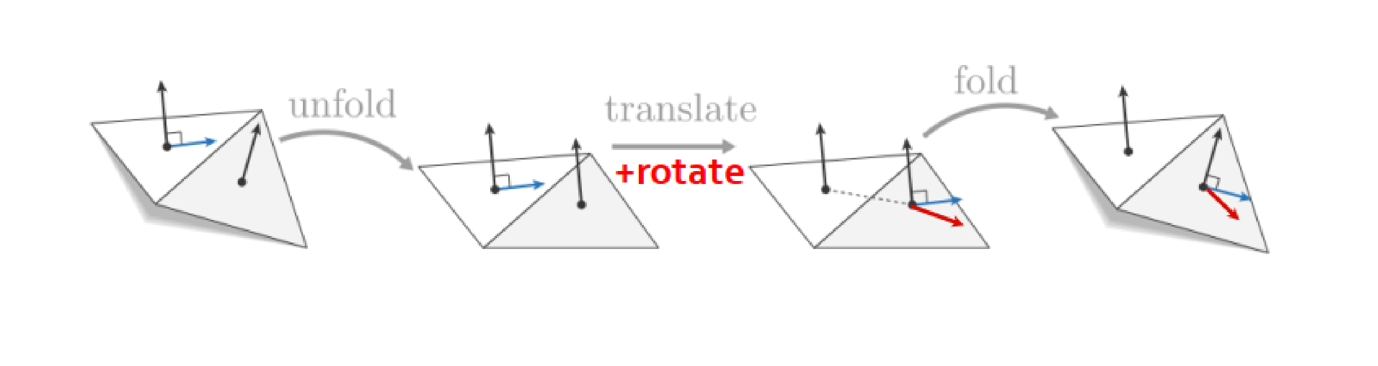


Representation in angle

- Vector field

$$\delta_{ij} + 2\pi k_{ij}, \delta_{ij} \in [0,2\pi), k_{ij} \in \mathbb{Z}$$





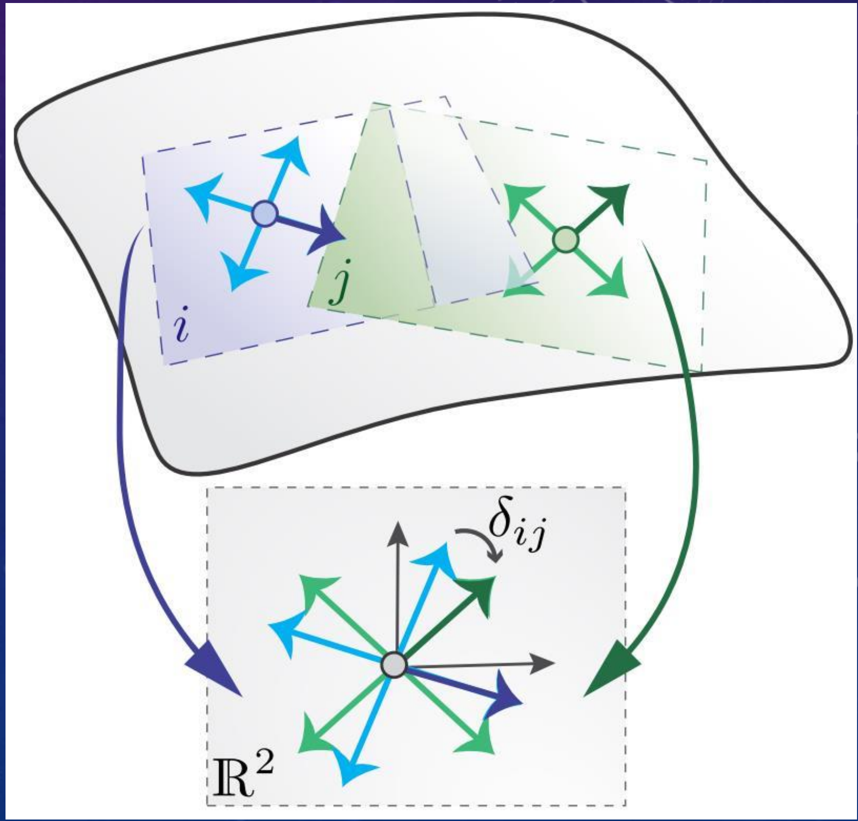
Representation in angle

- Vector field

$$\delta_{ij} + 2\pi k_{ij}, \delta_{ij} \in [0,2\pi), k_{ij} \in \mathbb{Z}$$

- N-directional field

$$\delta_{ij} + \frac{2\pi}{N} k_{ij}, \delta_{ij} \in \left[0, \frac{2\pi}{N}\right), k_{ij} \in \mathbb{Z}$$

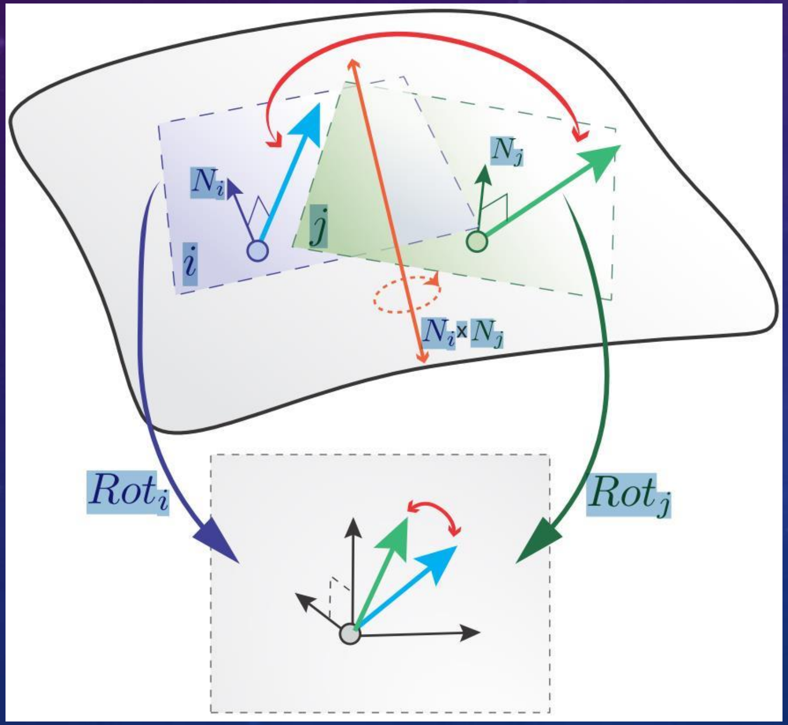


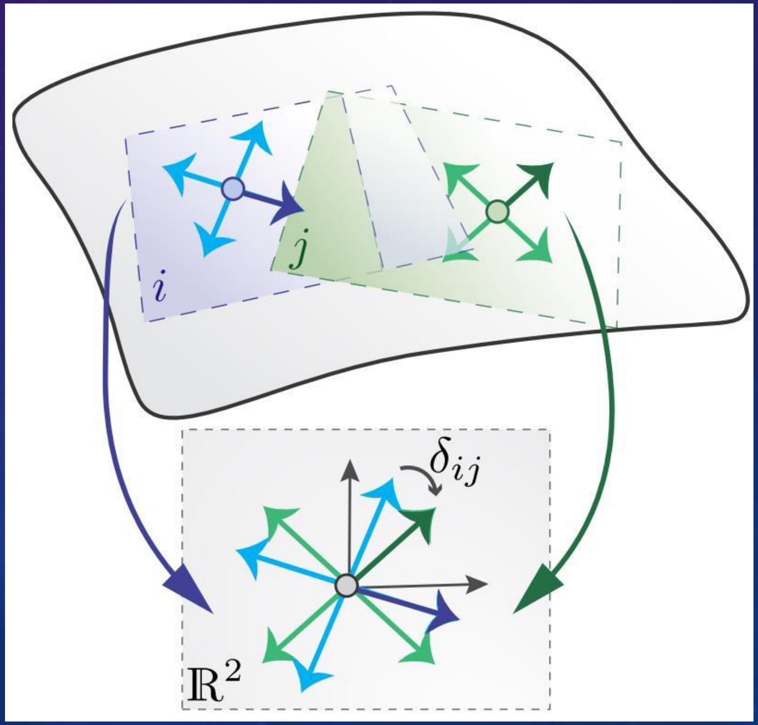
Pros and cons

- Advantage
  - Directions, as well as possible period jumps, are represented explicitly.
  - A linear expression of the rotation angle.
- Disadvantage
  - The use of integer variables, which leads to discrete optimization problems.

Representation in complex

- Complex $z = e^{i\theta}$ and complex polynomials $p(z) = (z - u_1) \dots (z - u_N)$





Representation in tensor

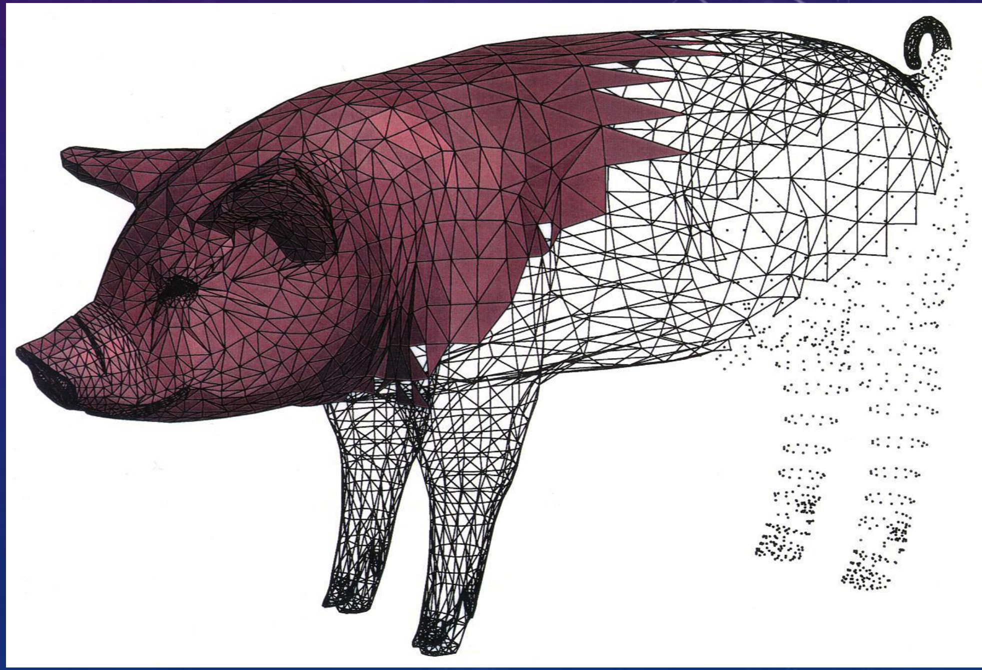
- Real-valued 2×2 matrices in local coordinates $T = \begin{pmatrix} T_{11} & T_{12} \\ T_{21} & T_{22} \end{pmatrix}$
- Symmetric Tensors
- An eigen-decomposition $T = U \Lambda U^T$
- $\Lambda = \mathrm{diag}(\lambda_1, \lambda_2)$ two real eigenvalues
- $U = (u_1 \quad u_2)$, two (orthogonal) eigenvectors with $\|u_i\| = 1$
- Since eigenvectors are only determined up to sign, a rank-2 tensor field can in fact be interpreted as two orthogonal 2-direction fields $\pm u_i$

Edge-based

- Discrete Exterior Calculus (DEC)

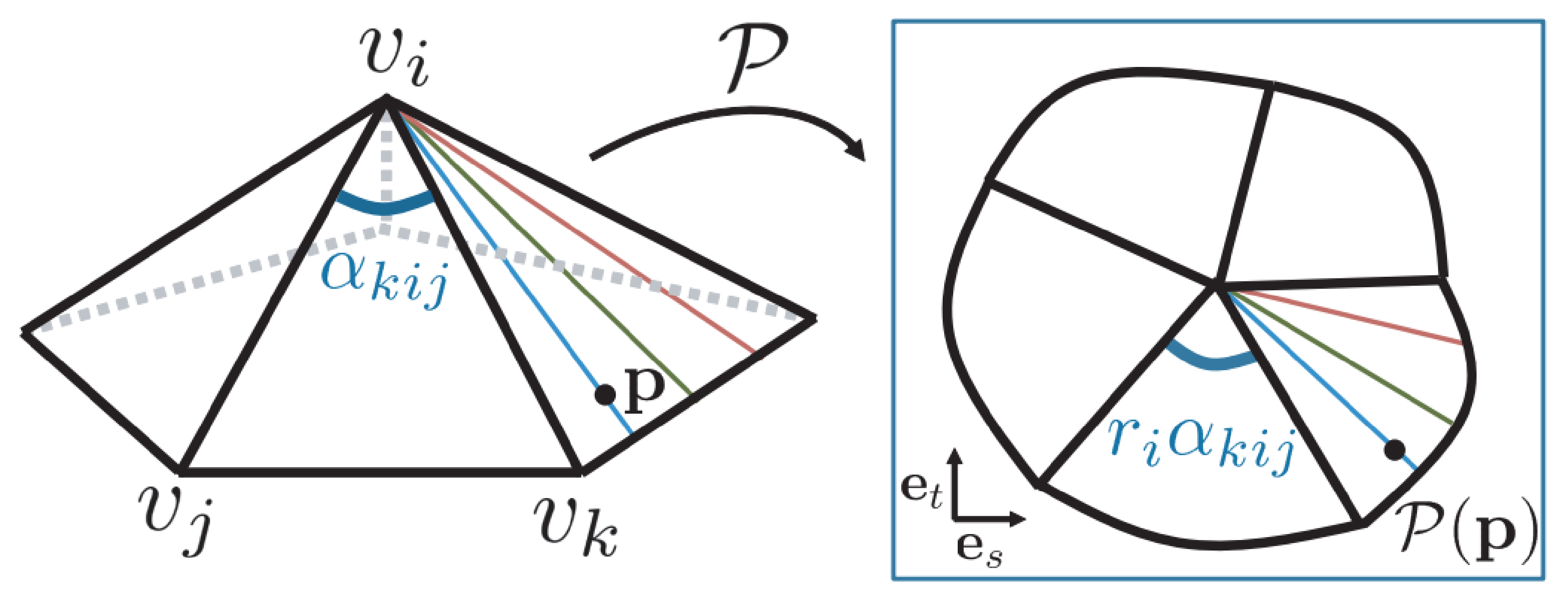
Vector Field Processing on Triangle Meshes

SIGGRAPH Asia Course 2015



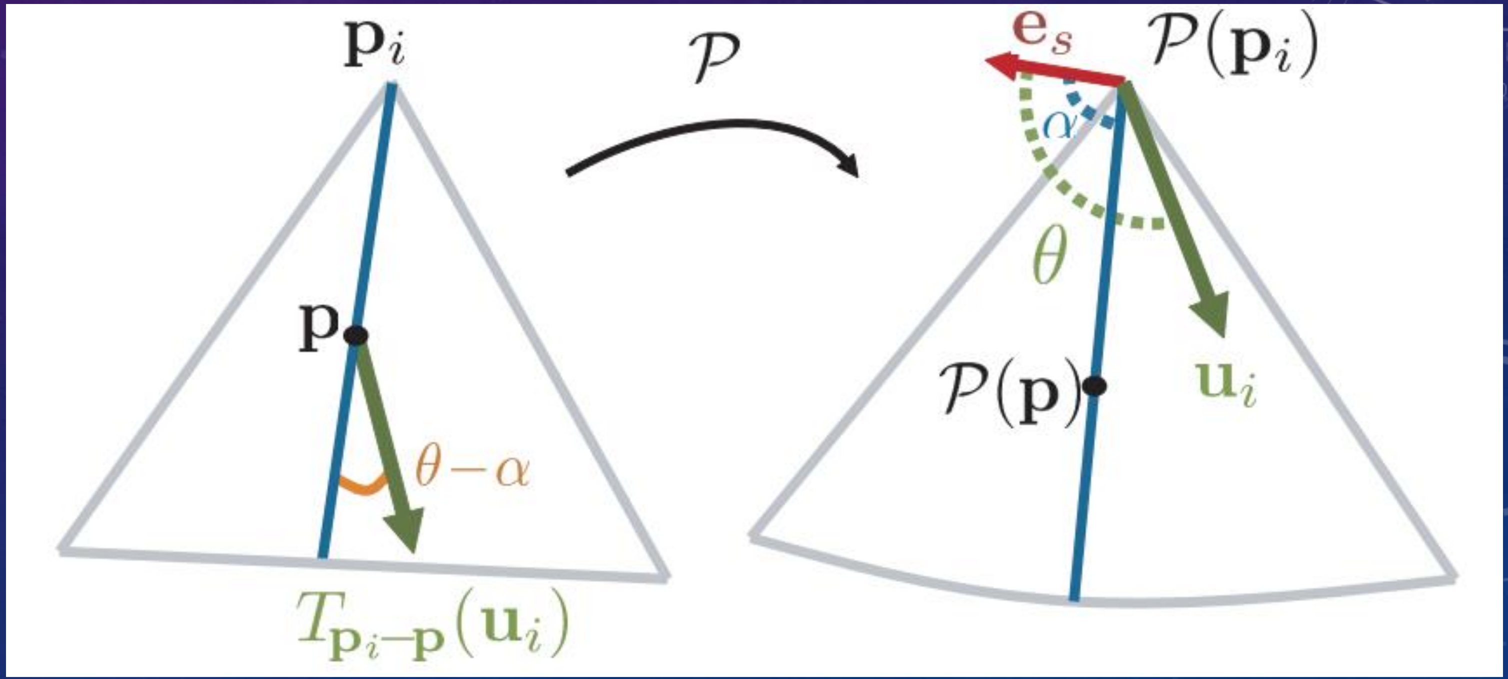
Vertex-based

- Geodesic polar map



Vertex-based

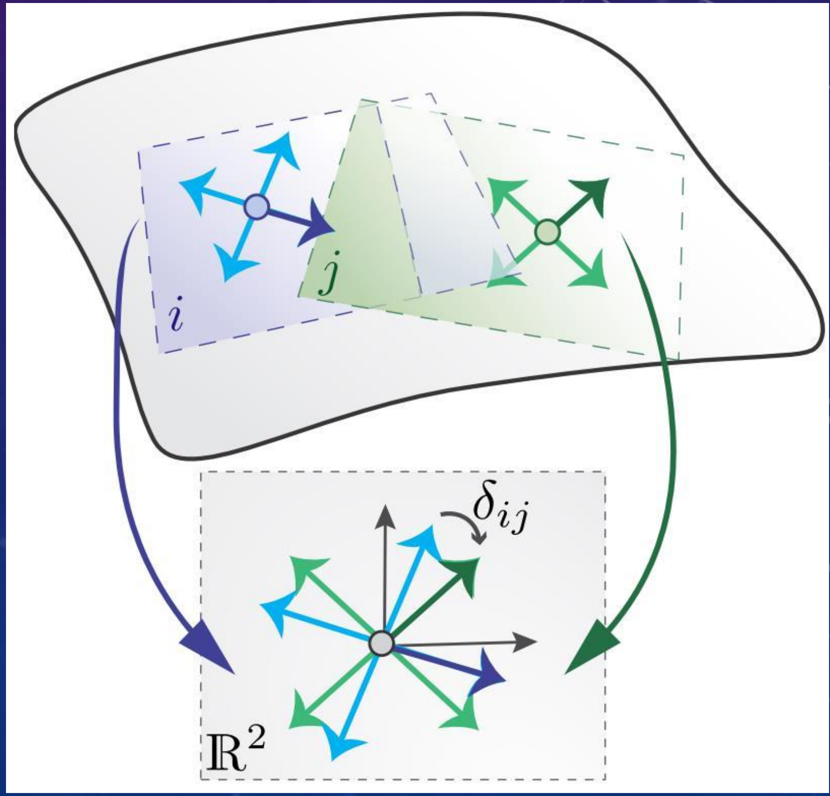
- Geodesic polar map
- Parallel transport



Vector field design

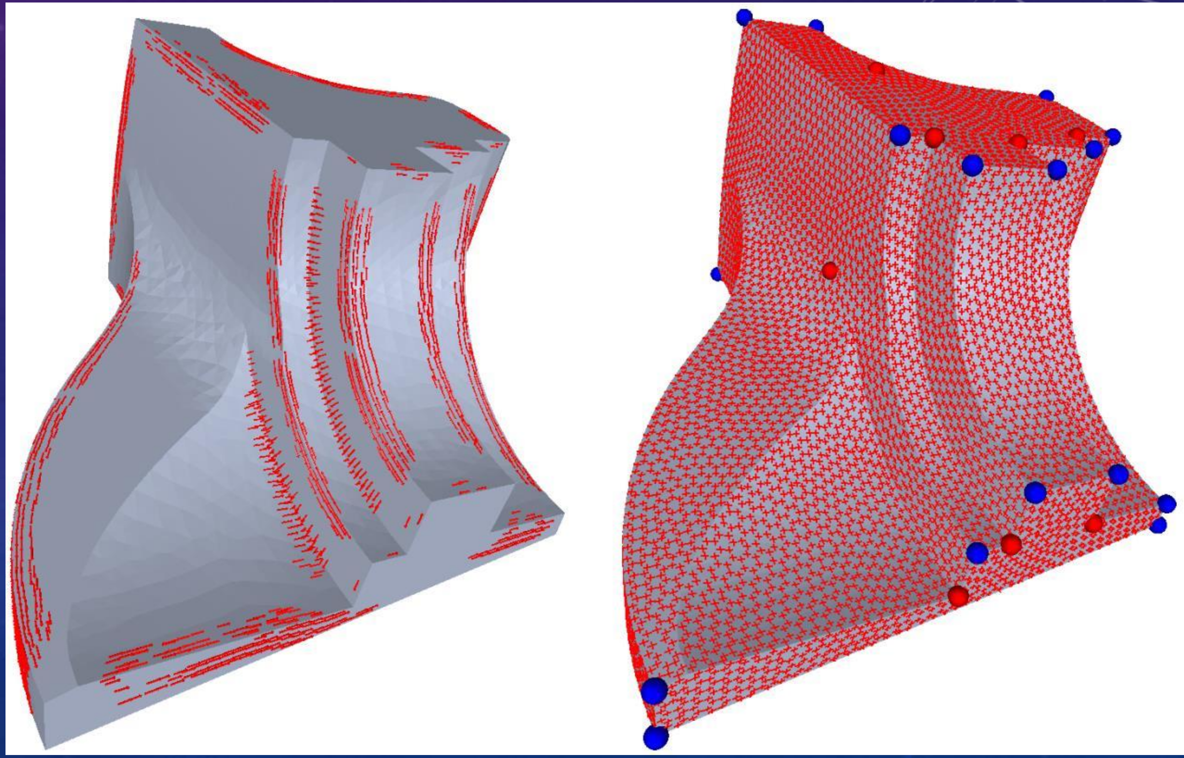
- Smooth vector fields

Angle-based : $E_{fair} = \sum_{ij}\left(\delta_{ij} + \frac{2\pi}{N}k_{ij}\right)^2$



Vector field design

- Smooth vector fields
- Alignment
  - Principal curvature
  - Strokes given by an artist
  - Boundary curves
  - Feature lines



Vector field design

- Smooth vector fields
- Alignment
- Specified requirements
  - Symmetry

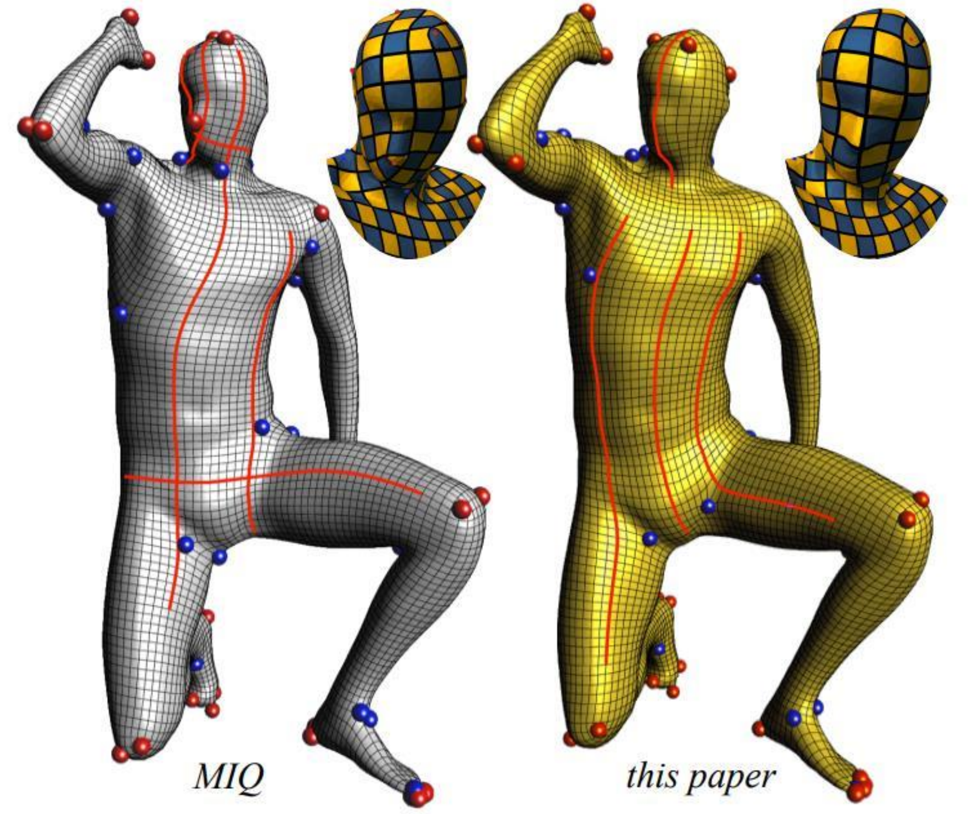


Figure 1: *Field-aligned parametrization of the knelt human model using the symmetry field construction method developed in this paper, and using the MIQ technique of Bommes* et. al.*[2009]. Red/blue bullets represent field singularities with positive/negative index. Red lines trace flows of the cross field.*

Vector field design

- Smooth vector fields
- Alignment
- Specified requirements
  - Symmetry
  - Correspondence

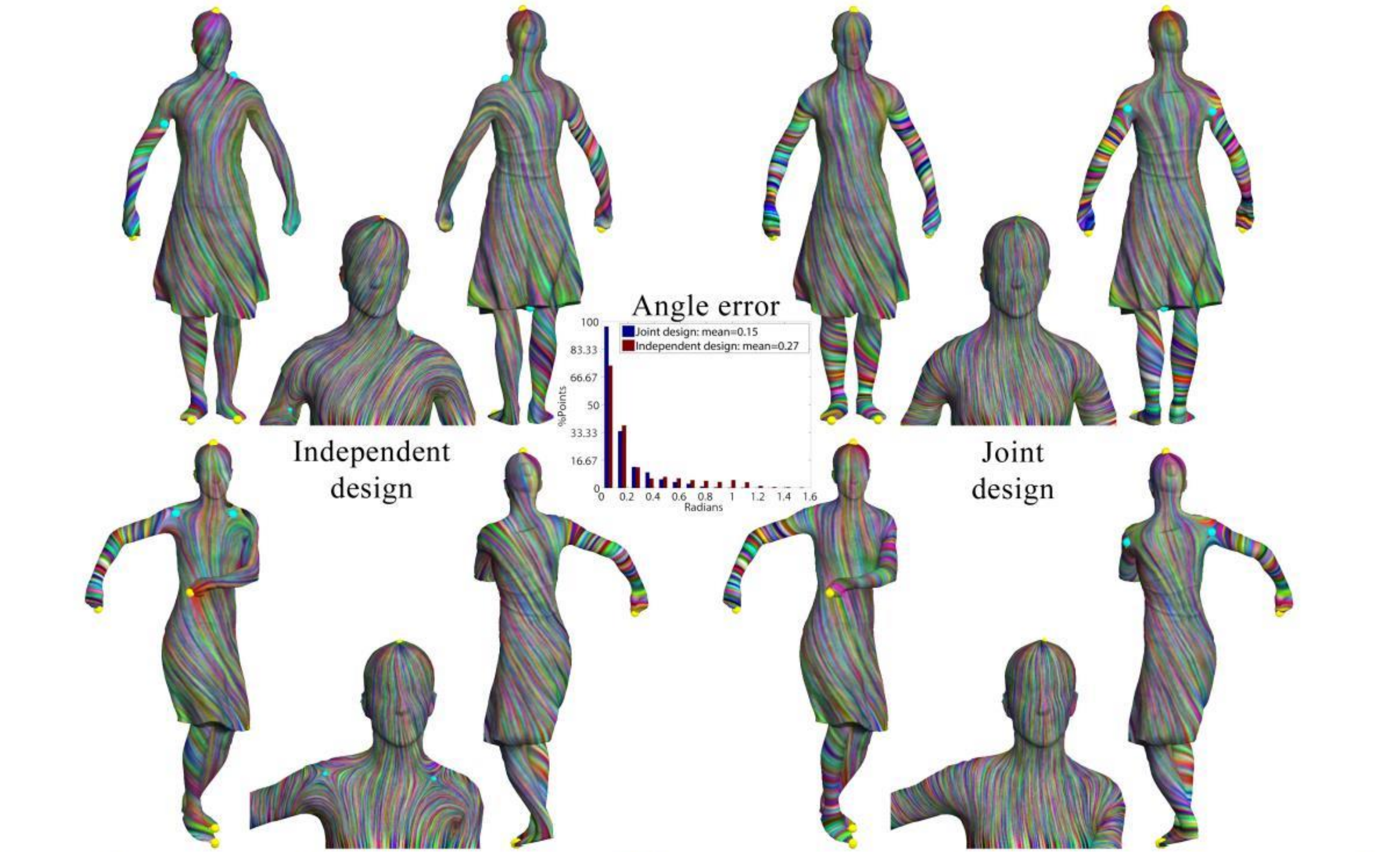
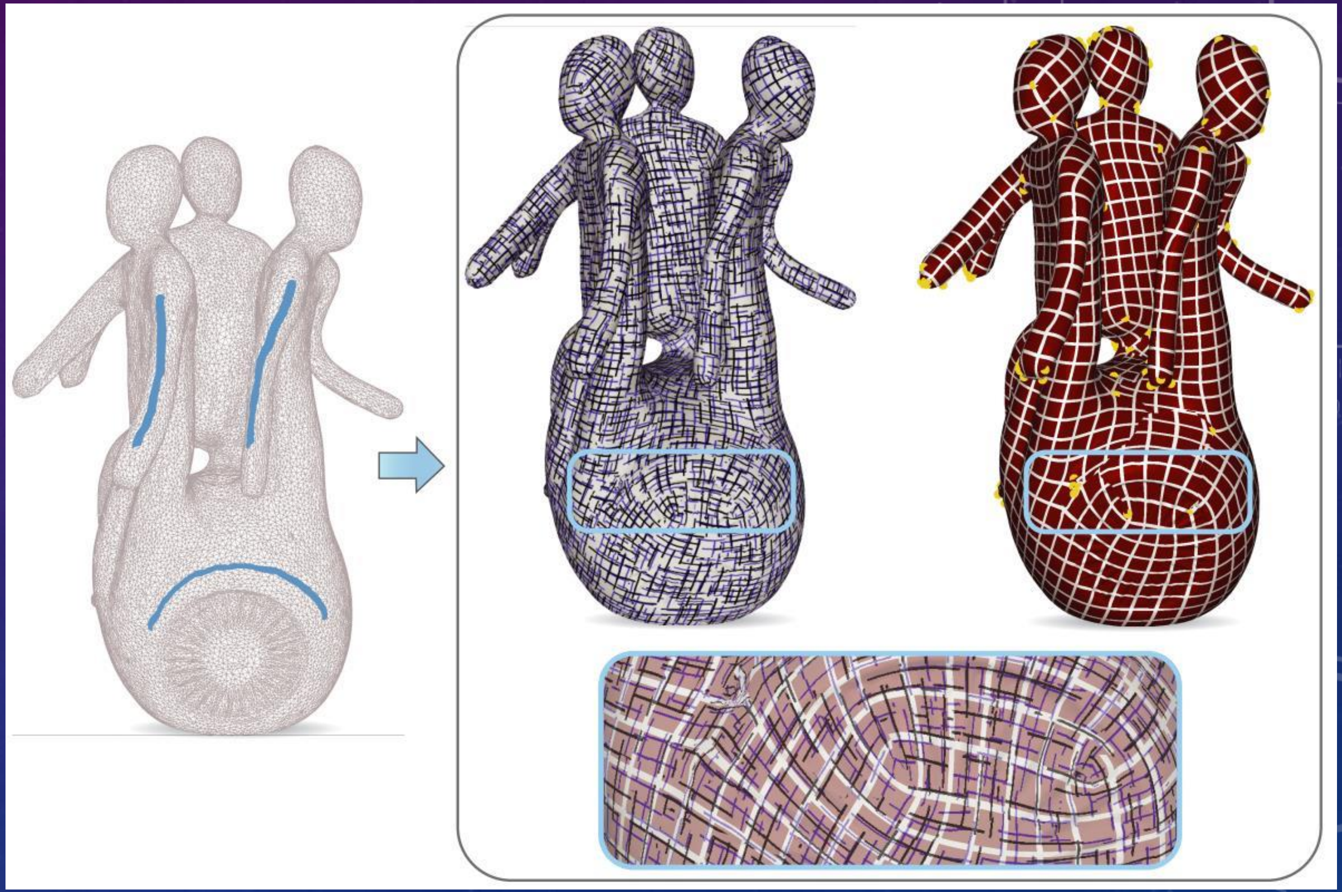


Figure 12: *(left) Independent design on two shapes which are in correspondence does not yield a consistent vector field, even if compatible constraints are used. (right) Solving jointly using our framework yields consistent vector fields (note the corresponding locations of the singularities on the back of the shape). See the text for details.*

Vector field design

- Smooth vector fields
- Alignment
- Specified requirements
  - Symmetry
  - Correspondence
  - Integrable for parameterization



Vector field design

- Smooth vector fields
- Alignment
- Specified requirements
  - Symmetry
  - Correspondence
  - Integrable for parameterization
  - Killing energy …