

1、求极限. (32 分)

$$(1) \lim_{n \rightarrow \infty} \left(\frac{1}{\sqrt{n^2+2}} + \frac{1}{\sqrt{n^2+3}} + \cdots + \frac{1}{\sqrt{n^2+n+1}} \right) \quad (2) \lim_{n \rightarrow \infty} \left(\frac{n}{n+1} \right)^{n+1}$$

$$(3) \lim_{x \rightarrow 0} \left(\frac{\tan x - \sin x}{x^3} \right) \quad (4) \lim_{x \rightarrow 0} \left(\frac{1}{\ln(1+x)} - \frac{1}{x} \right)$$

2、求导数. (20 分)

$$(1) \text{ 设 } y = \ln \tan \frac{x}{3} + e^{\sqrt{x}} \sin x^2, \text{ 求 } y'.$$

$$(2) \text{ 设函数 } y = y(x) \text{ 由方程 } e^y - xy = e \text{ 所确定, 求 } y'(0).$$

$$3、\text{ 已知 } f(x) = \begin{cases} x^2 \cos \frac{1}{x^2} & x \neq 0 \\ a & x = 0 \end{cases} \text{ 在 } x=0 \text{ 处连续, 求 } a \text{ 的值, 并讨论此时 } f(x) \text{ 在}$$

$x=0$ 处是否可导, 若可导, 则求出 $f'(0)$; 若不可导, 说明理由. (16 分)

$$4、\text{ 设 } \lim_{x \rightarrow +\infty} f'(x) = 3, \text{ 求 } \lim_{x \rightarrow +\infty} [f(x+5) - f(x)]. \quad (16 \text{ 分})$$

5、设某同学在操场跑步时速度函数为 $S(t) = 2t^3 - 9t^2 + 12t$, 时间 $t \in [0, 3]$. 试判断该同学在这段时间内有几次加速过程和几次减速过程? 并给出具体时间段以及加速度为零的时刻. (16 分)

1. 求导.

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1. (1) $\lim_{n \rightarrow \infty} \left(\frac{1}{\sqrt{n^2+2}} + \frac{1}{\sqrt{n^2+3}} + \dots + \frac{1}{\sqrt{n^2+n+1}} \right)$

$$\frac{n}{\sqrt{n^2+n+1}} \leq \text{原式} \leq \frac{n}{\sqrt{n^2+2}}$$

$$\lim_{n \rightarrow \infty} \frac{n}{\sqrt{n^2+n+1}} = \lim_{n \rightarrow \infty} \sqrt{\frac{n^2}{n^2+n+1}} = \lim_{n \rightarrow \infty} \sqrt{\frac{1}{1+\frac{1}{n}+\frac{1}{n^2}}} = 1$$

$$\lim_{n \rightarrow \infty} \frac{n}{\sqrt{n^2+2}} = \lim_{n \rightarrow \infty} \sqrt{\frac{n^2}{n^2+2}} = \lim_{n \rightarrow \infty} \sqrt{\frac{1}{1+\frac{2}{n^2}}} = 1$$

$$\therefore \lim_{n \rightarrow \infty} \left(\frac{1}{\sqrt{n^2+2}} + \frac{1}{\sqrt{n^2+3}} + \dots + \frac{1}{\sqrt{n^2+n+1}} \right) = 1$$

(3) $\lim_{x \rightarrow 0} \left(\frac{\tan x - \sin x}{x^3} \right)$

$$= \lim_{x \rightarrow 0} \left[\frac{\sin x (\frac{1}{\cos x} - 1)}{x^3} \right]$$

$$= \lim_{x \rightarrow 0} \left[\frac{x (-\frac{1}{2}x^2)}{x^3} \right]$$

$$= -\frac{1}{2}$$

(4) $\lim_{x \rightarrow 0} \left(\frac{1}{\ln(1+x)} - \frac{1}{x} \right)$

$$= \lim_{x \rightarrow 0} \left[\frac{x - \ln(1+x)}{x \ln(1+x)} \right] \text{ 是 } \frac{0}{0} \text{ 型}$$

$$\therefore = \frac{1 - \frac{1}{1+x}}{\ln(1+x) + \frac{x}{1+x}}$$

$$\therefore = \frac{\frac{x}{1+x}}{\frac{x}{1+x} + \frac{x}{1+x^2}} = \lim_{x \rightarrow 0} \frac{1}{1 + \frac{1}{1+x}} = \frac{1}{2}$$

2. 求导

(1) $y = \ln \tan \frac{x}{3} + e^{\sqrt{x}} \sin x^2$

$$y' = \frac{1}{3} \times \frac{1}{\tan \frac{x}{3}} \times \frac{1}{3} + \frac{1}{2\sqrt{x}} e^{\sqrt{x}} \sin x^2 + 2x \cdot \cos x^2 \cdot e^{\sqrt{x}}$$

$$= \frac{1}{3 \tan \frac{x}{3}} + e^{\sqrt{x}} \left(\frac{\sin x^2}{2\sqrt{x}} + 2x \cdot \cos x^2 \right)$$

(2) $y = y$

$$e^y - xy = e$$

$$y' \cdot e^y - y - y' \cdot x = 0$$

$$y' = \frac{y}{e^y - x}$$

$$\text{当 } x=0 \text{ 时 } y=1$$

$$\therefore y'(0) = \frac{1}{e^1 - 0} = \frac{1}{e}$$

$$3. f(x) = \begin{cases} x^2 \ln \frac{1}{x} & x \neq 0 \\ a & x = 0 \end{cases} \quad \therefore \lim_{x \rightarrow 0^+} f(x) = \lim_{x \rightarrow 0^-} f(x) \text{ 存在}$$

$$\cancel{f(x)_{x=0}} = f(x)_0 \quad \therefore 0 = a$$

$$\lim_{x \rightarrow 0^+} x^2 \ln \frac{1}{x} = \lim_{x \rightarrow 0^+} x \cdot \ln \frac{1}{x} \quad \text{且连续, } \therefore f(x) \text{ 在 } x=0 \text{ 处可导}$$

$$f'(0) = 0$$

$$4. \lim_{x \rightarrow \infty} f'(x) = 3$$

$$\cancel{f(x) \text{ 在 } x \rightarrow \infty \text{ 处连续可导, } \therefore}$$

$$f(x+5) - f(x) = dy = f'(x) \cdot dx = 5f'(x)$$

$$\therefore \lim_{x \rightarrow \infty} [f(x+5) - f(x)] = \lim_{x \rightarrow \infty} 5f'(x) = 15 \quad -14$$

$$5. s(t) = 4t^3 - 9t^2 + 12t \quad t \in [0, 3]$$

$$s'(t) = 12t^2 - 18t + 12 = 0$$

$$t = 1 \text{ 或 } t = 2 \quad t \in [0, 3]$$

$$s(0) = 0 \quad s(3) = 9$$

- 共有 2 次加速, 1 次减速

0 到 1, [1, 2] 加速

[2, 3] 减速

$t=1$ 与 $t=2$ 时 加速量为 0