(a)

Since we know that Φ has a solution, we can simply let y to be such a solution. Then based on y, we can construct z such that $y_i + z_i = 1$ for all $i = 1, \cdots, n$ (z is not necessary a solution of Φ). Then we can prove that such a construction of y and z will satisfy the inequality system: For the first inequality, since all y_i and z_i have only two values to take, either true (1) or false (0). Hence, $y_i, z_i \geq 0$ will always be satisfied; For the second inequality, since we just constructed z using this equation, it is definitely always satisfied; For the third inequality, y is a solution of Φ. We know that $P_j = \{i \text{ such that } x_i \in C_j\}$, and $N_j = \{i \text{ such that } \bar{x}_i \in C_j\}$. z is the "opposite" of y as we have constructed, so all $z_i (i \in N_j)$ are the same variables as $y_i (i \in P_j)$ with a negative sign ($\bar{x}$). For example, assuming a clause C_j is $(y_1 \vee \bar{y}_2 \vee y_3)$, then $y_i (i \in P_j) = \{y_1, y_3\}$. According to our construction, the corresponding z_j should be $(\bar{y}_1 \vee y_2 \vee \bar{y}_3)$, so $z_i (i \in N_j) = \{\bar{y}_1, \bar{y}_3\}$. Therefore the left side of the third inequality will be pairs of $(x_i, \bar{x}_i)$. Since there will always be a "True" (1) in $(x_i, \bar{x}_i)$, the left side will always be greater or equal to 1, which satisfies our third inequality. Hence, we showed that id Φ has a solution, then this inequality system is also satisfiable.

(b)

We consider the case that the probability that C is not satisfied (then just use 1 minus unsatisfied probability we will get satisfied probability): There is only one case that C is not satisfied – both variables are false, the probability of which is $z_1 z_2$. According to the construction, since x_i and $\bar{x}_i$ are opposite, then the probability that $\bar{x}_i$ being false is exactly the same as x_i being true. Hence, $\sum_{i \in P_j} y_i + \sum_{i \in N_j} z_i = \sum_{i \in C_j} y_i$, which is $y_1 + y_2$ in this case. Given the third inequality, we then have $y_1 + y_2 \geq 1$. Therefore we have $(1 - z_1) + (1 - z_2) \geq 1$, which means $z_1 + z_2 \leq 1$. From CauchySchwarz inequality we know $z_1 z_2 \leq \left(\frac{z_1 + z_2}{2}\right)^2$, so $z_1 z_2 \leq \left(\frac{1}{2}\right)^2 = \frac{1}{4}$. Hence, the probability that C is not satisfied is $\frac{1}{4}$. Then the probability that C is satisfied is $1 - \frac{1}{4} = \frac{3}{4}$.

(c)

From the last question we have already known that for each clause with 2 terms, the probability that such clause is satisfied is 0.75. Now we consider the clause with only 1 term. This is even simpler: since we have proved that the third inequality is equivalent to $\sum_{i \in C_j} y_i \geq 1$, then we have $y_1 \geq 1$, which means $y_1 = 1$ because a probability cannot be greater than 1. Therefore for each clause with 1 term, the probability that such clause is satisfied is 1. Hence, the expected number of clauses satisfied by the above method in this case is within $[\mathbf{0.75m}, \mathbf{m}]$ (we calculated the expectations based on Binomial Distribution rule), where the lower bound is reached when all clauses have 2 terms and the upper bound is reached when all clauses have only 1 term. Hence, we conclude that the expected number of clauses satisfied by the above method is at least $0.75m$.

(d)

Supposing the number of satisfied clauses is $(m - X)$, then the number of unsatisfied clauses is X. If the $(m-X)$ is at least $\frac{1}{2}$ of the clauses $(m - X \geq \frac{1}{2}m)$, then $X \leq \frac{1}{2}m$. So, if we would like to prove that the probability of $m - X \geq \frac{1}{2}m$ is at least $\frac{1}{2}$, we can prove that the probability of $X \leq \frac{1}{2}m$ is at least $\frac{1}{2}$. Then we take the opposite side, and **we only need to prove that the probability of $\mathbf{X} \geq \frac{1}{2}\mathbf{m}$ is at most $\frac{1}{2}$.** According to the MARKOV inequality:

$$P(X \geq \frac{1}{2}m) \leq \frac{E(X)}{\frac{1}{2}m} = \frac{2E(X)}{m}$$

From the last question we know that $E(m - X) \geq 0.75m$, so we have $E(X) \leq 0.25m$. Then we get the following:

$$P(X \geq \frac{1}{2}m) \leq \frac{2E(X)}{m} \leq \frac{2 \times 0.25m}{m} = \frac{1}{2}$$

Hence we proved that the probability of $X \geq \frac{1}{2}m$ is at most $\frac{1}{2}$, which means that the probability that at least $\frac{1}{2}$ of the clauses are satisfied, is at least $\frac{1}{2}$.