(a)

There will be only 4 scenarios that the index of the median of the 3 points is in the middle range of $n/4 \leq i \leq 3n/4$: All 3 points within $[n/4, 3n/4]$; 2 points within $[n/4, 3n/4]$ and 1 point less than $n/4$; 2 points within $[n/4, 3n/4]$ and 1 point greater than $3n/4$; 1 point within $[n/4, 3n/4]$, 1 point less than $n/4$, and 1 point greater than $3n/4$.

The probabilities of each Scenario are stated as follow:

$$\mathrm{Pr}_1 = \left(\frac{1}{2}\right)^3 = \frac{1}{8} \tag{1}$$

$$\mathrm{Pr}_2 = C_3^2 \times \left(\frac{1}{2}\right)^2 \times \frac{1}{4} = \frac{3}{16} \tag{2}$$

$$\mathrm{Pr}_3 = C_3^2 \times \left(\frac{1}{2}\right)^2 \times \frac{1}{4} = \frac{3}{16} \tag{3}$$

$$\mathrm{Pr}_4 = A_3^3 \times \frac{1}{2} \times \frac{1}{4} \times \frac{1}{4} = \frac{3}{16} \tag{4}$$

Hence, adding them up, we get the probability:

$$\mathrm{Pr} = \frac{1}{8} + \frac{3}{16} + \frac{3}{16} + \frac{3}{16} = \mathbf{\frac{11}{16}} \tag{5}$$

(b)

The algorithm is stated as follow:

Algorithm 1 Algorithm for problem 1

Guarded select (S, k)
If $|S| < 4$
    Sort & find kth
Else $i = 0$
While ($n/4 \leq i \leq 3n/4$ not true)
    Select $x \in S$ as stated in the problem
    Find $S-, S+$; $i = |S-| + 1$
Endwhile
If $i = k$: **Return** x
Elif $i > k$: **Guarded select** $(S-, k)$
Else: **Guarded select** $(S+, k - i)$

From lecture, we proved that if the probability of success in one interation is p, then the expected number of iterations is $1/p$. Hence in this case, it takes $1/(11/16) = 16/11$ tries to find an element in the desired range. We need 3 comparisons when finding the median, and using such a median to compare to other $(n-3)$ numbers will cost $(n-3)$ comparisons. Hence the total number of comparison for a round will be $3+n-3=n$, and the expected number of comparisons for an element will be $16n/11$. Hence, according to what we have discussed during the class, the recurrence will be:

$$T(n) \leq \frac{16}{11}n + T\left(\frac{3n}{4}\right) \tag{6}$$

The expected number of comparisons is at most:

$$\frac{16n}{11} \times \frac{1}{1-\frac{3}{4}} = \mathbf{\frac{64}{11}n \approx 5.818n} \tag{7}$$

From Equation (7) we find that the expected number of comparisons made by this method is upper-bounded by cn for a $c < 6$. $(c = 5.818)$