# (a)

Suppose the number of trucks used is $G$ if we use this Truck Loading algorithm, and the number of trucks used is $OPT$ in the optimal solution. We would like to prove $G \leq 2OPT$.

If $G$ is a even number (let's say $G = 2m$), we divide the used trucks into consecutive groups of 2, so there will be $m$ groups in total. In each group, the total weight of the items have to be strictly greater than $K$ (otherwise the second truck in that group will not be used since the first truck can load all items in the group by itself). So adding up all groups, we have $W > mK$. Now we consider the optimal number of trucks $OPT$: Ideally there should be no room left for all trucks (i.e., items in all trucks have reached weight $K$). However, in real cases, this cannot be guaranteed. Hence the number of trucks in this ideal scenario should be less than or equal to $OPT$, so we have $\frac{W}{K} \leq OPT$. From $W > mK$, we can get $\frac{W}{K} > m = \frac{G}{2}$. Therefore $\frac{G}{2} < \frac{W}{K} \leq OPT$, which means $G < 2OPT$.

If $G$ is an odd number (let's say $G = 2m + 1$, which is only slightly different from the even case), we still divide the trucks into consecutive groups of 2, but now we have $m + 1$ groups where the last one only got 1 truck. Some as above, the total weight of items in the first $2m$ groups should be strictly greater than $K$. So we still get $W > mK$. Considering the ideal scenario as above, we still get $\frac{W}{K} > m = \frac{G-1}{2}$, which means $G < 2OPT + 1$. However, since we know that $G$ and $OPT$ are both integers, we can safely conclude that $G \leq 2OPT$.

Hence we have proved $G \leq 2OPT$, which means that Truck Loading algorithm is 2-optimal.

# (b)

To prove that this 2-optimal bound is tight, we constructed the following strategy: we set the total number of weights (which is $n$) to be $2OPT - 1$. This makes sense since $n$ is always valid for all possible $OPT$ (from 1 to infinity). We divide all items into group of 2, so we have $(OPT - 1)$ number of groups plus a single item. For each group, we set the total weight of the 2 items to be $K$, where the first one $w_{i1}$ is strictly greater than $0.5K$. Hence we have $w_{i1} + w_{i2} = K$ for all $i$ in $1,\ 2,\ \cdots,\ (OPT - 1)$. We also set all $w_{i1}$ to be distinct, namely, for each $i$, $w_{i1}$ is different. For the last single item, we set the weight to $K$. Hence, the optimal solution is simple: we just put the groups of two together, which is $OPT - 1$, and finally we put the last one into a truck. Hence the optimal solution is $OPT - 1 + 1 = OPT$.

Now we would like to arrange the order like this: we first sort all groups $(w_{i1}, w_{i2})$ by $w_{i1}$ in ascending order, so we can get pairs such that $w_{i1} < w_{j1}$ if $i < j$. **We construct the item order like this:**

$$w_{1,1}, w_{2,1}, w_{1,2}, w_{3,1}, w_{2,2}, w_{4,1}, w_{3,2}, \cdots, w_{(OPT-1),1}, w_{(OPT-2),2}, w_K, w_{(OPT-1),2}$$

where $w_K$ is the single one with weight $K$.

We show it in a graph in Fig. 1.

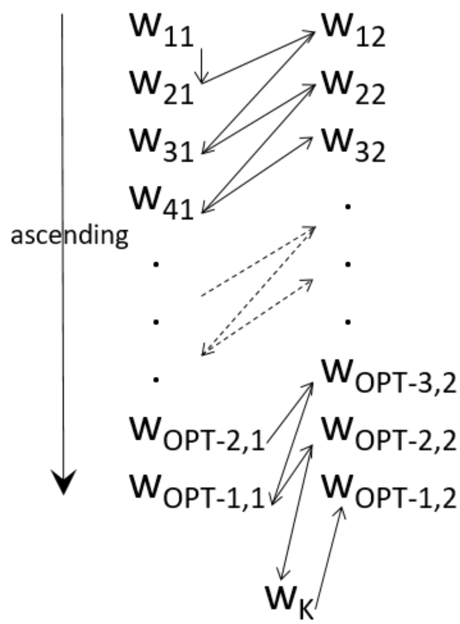


Figure 1: Construction strategy

To make this more understandable, we provide an example with $n = 9$ and $K = 1$ as shown in Fig. 2.

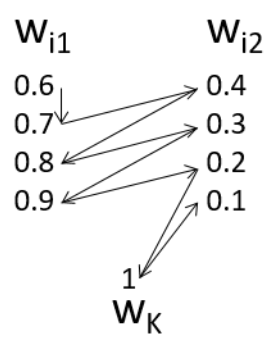


Figure 2: An example with $K = 1$

So in this case the order will be 0.7, 0.4, 0.8, 0.3, 0.9, 0.2, 1, 0.1, which we know that if use this algorithm we get the result of $2OPT - 1 \rightarrow 9$ ($OPT$ is 5 in this scenario).

We can prove that this works for all cases as we constructed (all OPT): In the order we constructed, for each consecutive pair of 2, they are in different "original pairs" $(w_{i1}, w_{i2})$. The ascending order with distinct value for each $w$ can make sure that such pair will always exceed

$K$. For the rest of 2 items, which is the first item in $w_i1$ and the last item in $w_i2$, we simply add the $W_K$ between them so there will be no 2 items loading into the same truck during the entire loading process. Therefore we ensured that the solution using TRUCK LOADING algorithm uses at least $2OPT - 1$ trucks.

Hence, we successfully constructed a sequence of objects for any value of $OPT$ that will ensure that while the optimal algorithm uses $OPT$ trucks, the TRUCK LOADING algorithm uses at least $2OPT - 1$ trucks, which means this 2-optimal bound is tight!