# Forward Euler Simulation of the 1D Heat Equation

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## 1 System Description

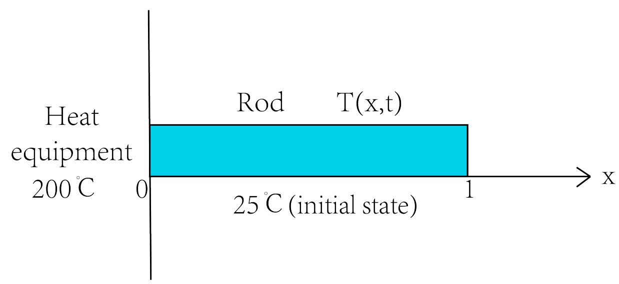


Figure 1: The heated rod system, with the temperature at any point given by $u(x,t)$

Our system is comprised of a rod fixed to a heated wall kept at constant temperature. The length of the rod is arbitrarily taken as 1m. The initial temperature of the rod is in equilibrium with the ambient temperature at 25°C and the temperature of the wall is 200°C. We represent temperature at a point on the rod at some time with the function $u(x,t)$. Note that the rod is thin with respect to its length, thus is modeled as a 1D system, that is, temperature is solely a function of rod position and time. The goal of this simulation is to show the variations in temperature of various points on the rod over time.

## 2 Initial State and Dynamics with time/space steps

We take our initial state as:

- $T(0,t) = 200$°C
- $T(x,t) = 25$°C, $x \neq 0$

The partial differential equation of the 1D heat propagation:

$$\frac{\partial u}{\partial t} = \alpha \frac{\partial^2 u}{\partial x^2} \tag{1}$$

We explicitly state that $u(\dot{})$ is a function of $x$ and $t$

$$\frac{\partial}{\partial t} u(x,t) = \alpha \frac{\partial^2}{\partial x^2} u(x,t) \tag{2}$$

We use a finite difference approximation to get compute the derivatives in space and time

$$\frac{u_i^{N+1} - u_i^N}{\Delta t} = \alpha \frac{u_{i+1}^N - 2u_i^N + u_{i-1}^N}{\Delta x^2} \tag{3}$$

Rearranging we reach the final form we need for our Forward Euler approximation. Note that the quantity $\alpha\frac{\Delta t}{\Delta x^2}$ is Fourier's number, with $\alpha$ being the thermal diffusivity of a material. In the following simulations, we arbitrarily choose Aluminium with $\alpha = 9.7e-5$.

$$u_i^{N+1} = u_i^N + \alpha\frac{\Delta t}{\Delta x^2}\left[u_{i+1}^N - 2u_i^N + u_{i-1}^N\right] \tag{4}$$

# 3 Matlab Code

```
% Values arbitrarily chosen. It's a useful exercise to vary these and look at the results
T_final = 300;
N_t = 2000;
X_final = 1;
N_x = 100;

% Calculate time and x steps based on sampling size and # of samples
T = linspace(0, T_final, N_t+1);
X = linspace(0, X_final, N_x+1);
dt = T(2) - T(1);  % Calculate delta t
dx = X(2) - X(1);  % Calculate delta x

alpha = 9.7e-5; % Thermal diffusivity of Aluminium in m^2/s
Fo = (alpha*dt)/(dx^2); % Fouriers number = diffusive transport rate/storage rate

% Define your initial condition here. This could be some function IC(x),
%  however for simplicity's sake we take a rod with a uniform temperature
%  and in contact with a hot plate at one end
wall_temp = 200;
init_temp = 25;

% Initialize the N state and the N-1 state
u_old = zeros(1, N_x+1);
u_old(:) = init_temp;
u_old(1) = wall_temp;
u_cur = u_old;
u_plot = u_old;

for t = 1:N_t
    for i = 2:N_x
      % Forward Euler solution to heat equation
        u_cur(i) = Fo*(u_old(i+1) - 2*u_old(i) + u_old(i-1)) + u_old(i);
    end
    u_cur(1) = wall_temp; % Set the left boundary to be our high of 200
    u_old(:) = u_cur; % We move to the next time step, reset N-1 state
    u_plot = [u_plot; u_cur];
end

[X_plot, T_plot] = meshgrid(X,T); % Create 2D meshgrid to create surface plot

surf(X_plot,T_plot,u_plot,'EdgeColor','none') % Create surface plot
c = colorbar; % Attach colour bar and create scale
c.Label.String = 'Temperature [C]';

xlim([0 1]) % Add axis limits
xlabel('Distance along rod [m]'); % Add descriptive axis labels
ylabel('Time [s]');
zlabel('Temperature [C]')
```

Listing 1: Forward Euler and Plotting in MATLAB

# 4 Simulation Figures

We can create plots of the rods temperature as a function of time and position.

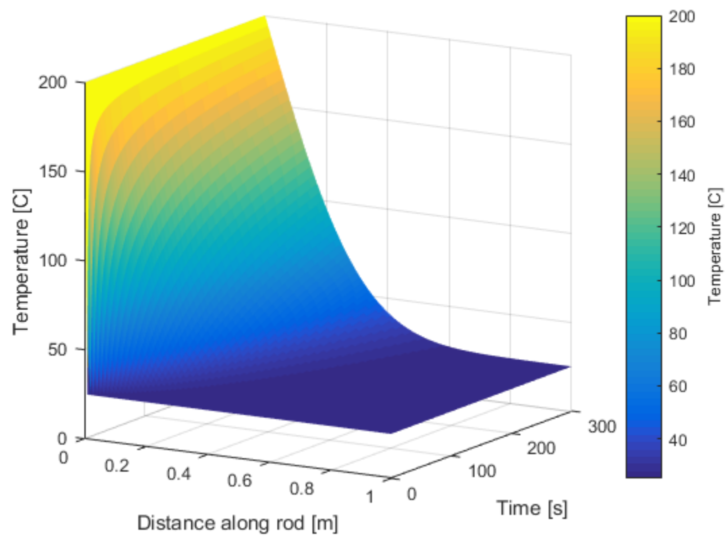


Figure 2: Temperature variation for the whole rod over 300 seconds

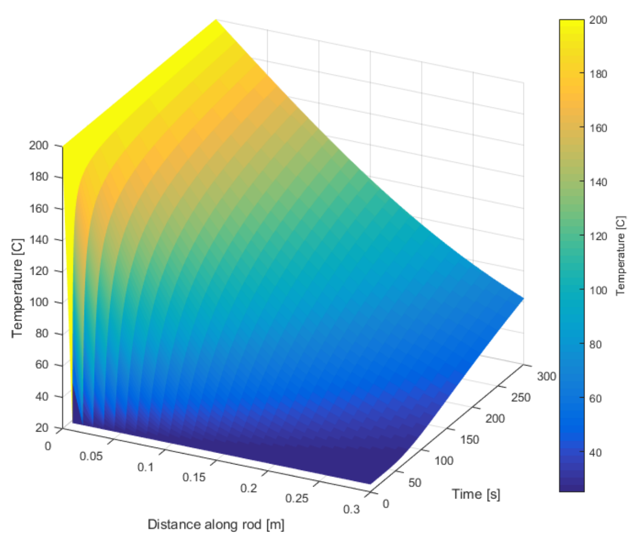


Figure 3: Temperature variation for the rod segment with x varies from 0 to 0.3m over 300 seconds

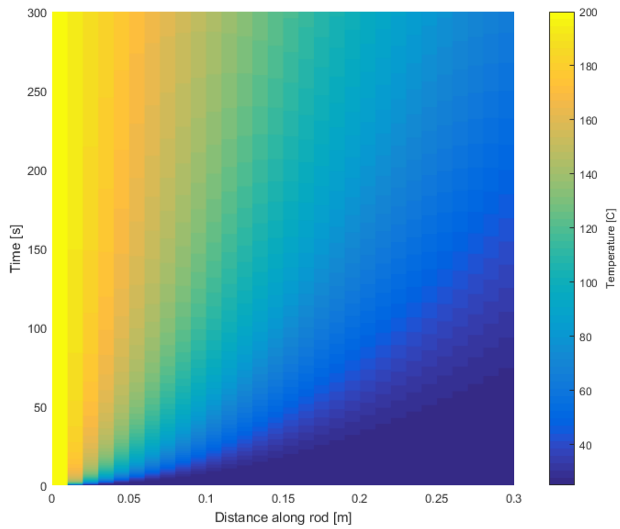


Figure 4: Top view of the temperature variation for the whole rod over 300 seconds

# 5 Python Code

```python
import numpy as np

# Values arbitrarily chosen. It's a useful exercise to vary these and look at the results
T_final = 300
N_t = 1000
X_final = 1
N_x = 100

# Calculate time and x steps based on sampling size and # of samples
_,dt = np.linspace(0,T_final,N_t+1,0,1)
_,dx = np.linspace(0,X_final,N_x+1,0,1)

alpha = 9.7e-5          # Thermal diffusivity of Aluminium in m^2/s
Fo = (alpha*dt)/(dx*dx) # Fourier number = diffusive transport rate/storage rate

u_cur = u_old = np.zeros(N_x+1) # Initialize the N state and the N-1 state

# Define your initial condition here. This could be some function IC(x),
#  however for simplicity's sake we take a rod with a uniform temperature
#  and in contact with a hot plate at one end

u_old[:] = 25  # Set the whole rod to be a constant temperature
u_old[0] = 200 # Set a high temperature at the boundary wall

for t in range(0,N_t):
    for i in range(1,N_x):
        # Forward Euler solution to 1D Heat Eq. PDE
        u_cur[i] = Fo*(u_old[i+1] - 2*u_old[i] + u_old[i-1]) + u_old[i]

    u_cur[0] = 200   # Set the left boundary to be our high of 200
    u_old[:] = u_cur # We move to the next time step, reset N-1 state
```

Listing 2: Forward Euler in Python