Vector line integral

Idea: Fundamental Theorem of Calculus

integrate the Vector field $\vec{F}$ over the curve C, where $\vec{F}(x, y) = (f_1(x, y), f_2(x, y))$

Both $\vec{F}$ and C are 1D object living in 2D space. Their dimensions are the same. Works in 3D as well.

Why do this?

- We can talk about the projection of F onto C
- It's an analogy of total work done in moving particle

Definition

C is given by $\vec{r}(t)$, then $\vec{r}'(t)$ is tangent to curve C, then $\vec{F} \cdot \vec{r}'(t)$ is the Tangential component of $\vec{F}$ along curve C.

$$\int_C \vec{F} \cdot \mathrm{d}\vec{r} = \int_C \vec{F} \cdot \vec{r}' \mathrm{d}t = \int_C \vec{F}(\vec{r}(t)) \cdot \vec{r}' \mathrm{d}t$$

Also, unit tangent $\vec{T} = \frac{\vec{r}'(t)}{|\vec{r}'(t)|}$, then $\mathrm{d}\vec{r} = \vec{r}'(t)\mathrm{d}t = \vec{T}\left|\vec{r}'(t)\right|\mathrm{d}t$.

So $\int_C \vec{F} \cdot \mathrm{d}\vec{r} = \int_C \vec{F} \cdot \vec{T}\left|\vec{r}'(t)\right|\mathrm{d}t$

Note that $\vec{F} \cdot \vec{T}$ is a number, so this is a Scalar line Integral, where $\left|\vec{r}'(t)\right|$ is the Magic factor in Mapping > From 1D to 2Dand Mapping > From 1D to 3D.

So $\left|\vec{r}'(t)\right|\mathrm{d}t = \mathrm{d}C$. TheF

$$\int_C \vec{F} \cdot \mathrm{d}\vec{r} = \int_C \vec{F} \cdot \vec{T}\left|\vec{r}'(t)\right|\mathrm{d}t = \int_C \vec{F} \cdot \vec{T}\mathrm{d}C$$

Calculate in "pieces"

Let $\vec{F} = (P(x, y), Q(x, y))$, and we have $\vec{r}'(t) = (x'(t), y'(t))$

$$\int_C \vec{F} \cdot \mathrm{d}\vec{r} = \int_C \vec{F} \cdot \vec{r}'\mathrm{d}t = \int_C P(x, y)x'(t)\mathrm{d}t + \int_C Q(x, y)y'(t)\mathrm{d}t$$
$$= \int_C P(x, y)\mathrm{d}x + \int_C Q(x, y)\mathrm{d}y$$

Intuitively, that's from $\mathrm{d}\vec{r} = (\mathrm{d}x, \mathrm{d}y)$

Integration with respect to x in Scalar line Integral > Integrate in pieces

Rotation

You can also view closed Vector line integral as a representation of rotation of $\vec{F}$ over the boundary, which makes sense, because if the Orientation of curve C is positive, then more Tangential component occurs when $\vec{F}$ is in the same direction as $\mathrm{d}\vec{r}$, namely counterclockwise.

(see: Fundamental Theorem of Calculus > Surface integral)