# Fundamental Theorem of Calculus

Integral of Derivative over a region = function over a boundary

Sum up the total weight or amount of a function described in input space.

Given a boundary, you can have anything you want as the region.

You can't integrate a Vector field on a boundary nor on an interior, but you can **derive *Scalar function* from that** Vector field which then can be integrated.

# 1D

$f : R \to R$

$$\int_a^b \frac{\mathrm{d}f}{\mathrm{d}x}\mathrm{d}x = f(b) - f(a)$$

Integral of Derivative over a 1D region (or line) living in 1D space = difference of function at two 0D boundary (or endpoint)

# Scalar line Integral in Higher Dimensions

$f : R^n \to R$

$$\int_C f(\vec{r})\mathrm{d}C = \int_{t_{start}}^{t_{end}} f(\vec{r}(t))\mathrm{d}C = \int_{t_{start}}^{t_{end}} f(\vec{r}(t))|\vec{r}_t|\mathrm{d}t$$

Integral of Derivative over a 1D Open region (or line) living in nD space = difference of function at two 0D boundary (or endpoint) (see: Fundamental Theorem of Calculus > 1D)

# Vector line integral in Higher Dimensions

$f : R^n \to R$, $\nabla f : R^n \to R^n$ (see: Gradient)

$$\int_C \nabla f \cdot \mathrm{d}\vec{r} = f[\vec{r}(t = b)] - f[\vec{r}(t = a)]$$

Integral of Derivative over a 1D Open region (or line) living in nD space = difference of function at two 0D boundary (or endpoint)

## Proof

Assign Vector field $\vec{F}(x, y, z) = \nabla f$

Chain rule

Fundamental Theorem of Calculus > 1D

# Surface integral Lying in 2D

$\vec{F}(x,y) = (P,Q) : R^2 \to R^2$

At any point on the boundary, we can decompose the Vector field $\vec{F}$ into a Tangential component and a Normal component

$$\vec{F} = [\vec{F} \cdot \vec{T}]\vec{\tau} + [\vec{F} \cdot \vec{n}]\vec{n}$$

where $\vec{\tau}$ and $\vec{n}$ are unit vectors in tangential and normal direction, respectively.

## Collecting Tangential component along Curve

Green's Theorem:

$$\oint_C \vec{F} \cdot \vec{T} \mathrm{d}C = \oint_C \vec{F} \cdot \mathrm{d}\vec{r} = \oint_C P\mathrm{d}x + Q\mathrm{d}y = \iint_A \left[\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right] \mathrm{d}x\mathrm{d}y$$

(see: Vector line integral > Definition)

Integral of "*Curl*" over a Closed 2D region (or surface) living in 2D space = (**all the *Tangential component* of) *function*** $\vec{F}$ over a 1D boundary (or Closed curve)

Suppose $\vec{F} = (P(x,y), Q(x,y), 0)$ , then

$$\oint_C \vec{F} \cdot \vec{T} \mathrm{d}s = \oint_C \vec{F} \cdot \mathrm{d}\vec{r} = \iint_A (\nabla \times \vec{F}) \cdot \vec{k} \mathrm{d}A$$

(see: Green's Theorem > Curl Representation)
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Also, Sum up rotations inside = rotation of the outside (boundary) (which is a Vector line integral)

(see: Curl for rotation inside, Vector line integral > Rotation for rotation outside)

> "Curl" inside = all Tangential component outside

## Collecting Normal component along Curve

$\vec{F} \cdot \vec{n} \mathrm{d}s$ is the Normal component

Unit tangent vector is $\vec{T}(t) = (\frac{x'(t)}{|\vec{r}'(t)|}, \frac{y'(t)}{|\vec{r}'(t)|})$

Since $\vec{T} \cdot \vec{n} = 0$, unit normal vector is $\vec{n}(t) = (\frac{y'(t)}{|\vec{r}'(t)|}, -\frac{x'(t)}{|\vec{r}'(t)|})$ (see: Orientation > Normal of Curve in 2D)

$$\int_C \vec{F} \cdot \vec{n} \mathrm{d}s = \int_a^b (\vec{F} \cdot \vec{n}(t)) \left|\vec{r}'\right| \mathrm{d}t = \int_a^b \left[P\frac{y'(t)}{|\vec{r}'(t)|} - Q\frac{x'(t)}{|\vec{r}'(t)|}\right] |\vec{r}'(t)| \mathrm{d}t$$

$$= \int_a^b [Py'(t) - Qx'(t)] \mathrm{d}t = \int_C -Q\mathrm{d}x + P\mathrm{d}y$$

From Green's Theorem, we have $\oint_C -Q\mathrm{d}x + P\mathrm{d}y = \iint_A \left[\frac{\partial P}{\partial x} + \frac{\partial Q}{\partial y}\right] \mathrm{d}x\mathrm{d}y$, then

$$\oint_C \vec{F} \cdot \vec{n} \mathrm{d}s = \iint_A \left[ \frac{\partial P}{\partial x} + \frac{\partial Q}{\partial y} \right] \mathrm{d}x \mathrm{d}y = \iint_A \nabla \cdot \vec{F} \mathrm{d}x \mathrm{d}y$$

Integral of *Divergence* (which are 2 derivatives) over a Closed 2D region (or surface) living in 2D space = **all the *Normal component* *of function*** $\vec{F}$ over a 1D boundary (or Closed curve)

Also, Sum up expansion inside = expansion of the outside (boundary)

> Divergence inside = all Normal component outside

# Surface integral Lying in 3D

$\vec{F} : R^3 \to R^3$

## Collecting Tangential component

(see: Flux)

Stoke's Theorem

$$\oint_C \vec{F} \cdot \vec{T} \mathrm{d}s = \int_C \vec{F} \cdot \mathrm{d}\vec{r} = \iint_S (\nabla \times \vec{F}) \cdot \vec{n} \mathrm{d}S = \iint_S (\nabla \times \vec{F}) \cdot \mathrm{d}\vec{S}$$

Integral of the *Flux of the Curl of function* $\vec{F}$ over an Open 2D region (surface) living in 3D space = all the *Tangential component of function* $\vec{F}$ over a 1D boundary (or Closed curve)

This interpretation applies to Green's Theorem > Stoke's Theorem Representation: (see Fundamental Theorem of Calculus > Collecting Tangential component along Curve)

$$\oint_C \vec{F} \cdot \vec{T} \mathrm{d}s = \oint_C \vec{F} \cdot \mathrm{d}\vec{r} = \iint_A (\nabla \times \vec{F}) \cdot \vec{k} \mathrm{d}A$$

> Flux of Curl inside = all Tangential component outside

## Collecting Normal component

Divergence Theorem

$$\iint_S \vec{F} \cdot \vec{n} ds = \iiint_E (\nabla \cdot \vec{F}) dE$$

Integral of *Divergence* over a Closed 3D region (or volume) living in 3D space = all the *Normal component of function* $\vec{F}$ over a Closed 2D boundary (or Closed surface)

This interpretation applies to Fundamental Theorem of Calculus > Collecting Normal component along Curve

> Divergence inside = all Normal component outside