Conservative

idea: The existence of Derivative to be integrated in Fundamental Theorem of Calculus

A Vector field $\vec{F}$ is conservative if there exists a Scalar function f such that $\nabla f = \vec{F}$ (see: Gradient)

$\vec{F}$ is conservative $\leftrightarrow$ $\vec{F}$ is Path-independent

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$\int_C \vec{F} \cdot \mathrm{d}\vec{r} = \int_C \nabla f \cdot \mathrm{d}\vec{r} = f[\vec{r}(t=b)] - f[\vec{r}(t=b)]$ by Fundamental Theorem of Calculus > Vector Line Integral in Higher Dimensions

$\vec{F}$ is conservative $\leftarrow$ $\vec{F}$ is Path-independent

Letting $\vec{F}(x,y) = (P(x,y), Q(x,y))$
Define $f(x,y) = \int_{(a,b)}^{(x,y)} \vec{F} \cdot \mathrm{d}\vec{r}$
Then $f(x,y) = \int_{(a,b)}^{(a,y)} \vec{F} \cdot \mathrm{d}\vec{r} + \int_{(a,y)}^{(x,y)} \vec{F} \cdot \mathrm{d}\vec{r}$, because of path-independent
Then $f(x,y) = \int_b^y Q(a,y)\mathrm{d}y + \int_a^x P(x,y)\mathrm{d}x$, because $\mathrm{d}\vec{r} = (0, \mathrm{d}y)$ for the first part, and $\mathrm{d}\vec{r} = (\mathrm{d}x, 0)$ for the second part (see: Vector line integral > Calculate in pieces)
So $f_x = P$, $f_y = Q$ by Fundamental Theorem of Calculus > 1D

$\vec{F}$ is conservative $\leftrightarrow$ $P_y = Q_x$

Letting $\vec{F}(x,y) = (P(x,y), Q(x,y))$

$\vec{F}$ is conservative $\rightarrow$ $P_y = Q_x$

Because if $\vec{F}$ is conservative, then there exists $f(x,y)$ such that $\nabla f = (f_x, f_y) = \vec{F} = (P, Q)$

So, $P_y = f_{xy}$ and $Q_x = f_{yx}$, and they are the same by Clairaut's Theorem

$\vec{F}$ is conservative $\leftarrow$ $P_y = Q_x$

if that region is simply connected (no holes) ...