Introduction to Problem Solving and Proof

Introduction To Math Proofs

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Chapter 1

Proof

1.1 What is a Proof

Example. Let's proof that $\sqrt{2}$ is irrational

To start, we need to know what is the definition of irrational number

Definition: An irrational real number cannot be expressed in the form $\frac{m}{n}$, where n and m are integers

Proof. Assume, $\sqrt{2}$ is rational

$$\sqrt{2} = \frac{m}{n} \quad \text{(m and n are integers and } \frac{m}{n} \text{ is a reduced fraction)} \tag{1.1}$$

We can square both side using the principle of Algebra

$$2 = \frac{m^2}{n^2} \tag{1.2}$$

$$2n^2 = m^2 \tag{1.3}$$

$$\because \quad \text{Definition: A real number } N \text{ is even if it can be written as } N = 2k\text{, where } k \text{ is an integer} \tag{1.4}$$

$$\therefore m^2 \quad \textit{is even} \tag{1.5}$$

$$\therefore m \quad \textit{is even} \tag{1.6}$$

According to the definition, assume $m = 2k$, where $k \in \mathbb{I}$

Sub $m = 2k$ into equation (3)

$$2n^2 = (2k)^2 \tag{1.7}$$

$$n^2 = 2k^2 \tag{1.8}$$

$$n \text{ is even} \tag{1.9}$$

$$\because \text{According to (9) and (6), } n \text{ and } m \text{ are both even} \tag{1.10}$$

$$\therefore m \text{ and } n \text{ have a common factor of 2} \tag{1.11}$$

$$\therefore \frac{m}{n} \text{ is not an reduced fraction} \tag{1.12}$$

$$\text{This violate our original assumpution in (1)} \tag{1.13}$$

Therefore, $\sqrt{2}$ is not rational, $\sqrt{2}$ must be irrational □

1.2 Logical Rules

1.2.1 Modus Ponens

Definition: If p is ture and p implies q, then q is ture
Logical notation: $p, p \rightarrow q \therefore q$
Example: $p =$ "It is raining", $q =$ "The ground is wet"
Given: "it is raining" and "It is raining implies the ground is wet"
Conclusion: "The ground is wet"

1.2.2 Modus Tollens

Definition: If not q is ture and p implies q, then not p is true
Logical notation: $-q, p \rightarrow q \therefore -q$
Example: $p =$ "It is raining", $q =$ "The ground is wet"
Given: "it is raining" and "It is raining implies the ground is wet"
Conclusion: "It is not raining"

1.2.3 Hypothetical Syllogism

Definition: If p implies q and q implies r, then p implies r.
Logical notation: $(p \rightarrow q), (q \rightarrow r) \therefore (p \rightarrow r)$
Example: $p =$ "It is raining", $q =$ "The ground is wet", $r =$ "People use umbrellas"
Given: "It is raining implies the ground is wet" and "The ground is wet implies people use umbrellas"
Conclusion: "It is not raining implies people use umbrellas"

1.2.4 Disjunctive Syllogism

Definition: If not p is true and p or q is true, then q is true.
Logical notation: $-p, (p \vee q) \therefore q$
Example: $p =$ "It is raining", $q =$ "I will stay indoors"
Given: "It is not raining" and "It is raining or I will stay indoors"
Conclusion: "I will stay indoors"

1.2.5 Additon

Definition: If p is true, then p or q is true
Logical notation: $p \therefore (p \wedge q)$
Example: $p =$ "It is raining", $q =$ "I will go for a run"
Given: "It is raining"
Conclusion: "It is raining or I will go for a run"

1.2.6 Simplification

Definition: If p and q are true, then p is true
Logical notation: $(p \wedge q) \therefore p$

Example: $p =$ "It is raining", $q =$ "The ground is wet"
Given: "It is raining and The ground is wet"
Conclusion: "It is raining"

1.2.7 Conjunction

Definition: If p is true and q is true, then p and q are true.
Logical notation: $p, q \therefore (p \wedge q)$
Example: $p =$ "It is raining", $q =$ "The ground is wet"
Given: "It is raining", "The ground is wet"
Conclusion: "It is raining and The ground is wet"

1.3 Mathematical Sets

(Collection of Objects)
$\{\}$
{All Triangles}
$\{3, 6, 11, 117\}$, {Real Numbers}

{What's in the set — The condition to be in the set}
Example: $\{x$ — x is even and $x > 0\}$

1.3.1 Important Sets

$\mathbb{N}$ - The Natural Numbers(Counting Numbers): $\{1, 2, 3, \cdots\}$
$\mathbb{W}$ - The Whole Numbers: $\{0, 1, 2, 3, \cdots\}$
$\mathbb{Z}$ - The Integers: $\{\cdots, -3, -2, -1, 0, 1, 2, 3, \cdots\}$
$\mathbb{Q}$ - The Rational Numbers: $\{\frac{p}{q} | p, q \in \mathbb{Z}, q \neq 0\}$
$\mathbb{I}$ - The Irrational Numbers: $\{x | x \notin \mathbb{Q}, x \in \mathbb{R}\}$
$\mathbb{R}$ - The Real Numbers: $\{x\} = \{x | x \in \mathbb{Q}$ or $x \in \mathbb{I}\}$
$\mathbb{C}$ - The Complex (Imaginary) Numbers: $\{a + bi | a, b \in \mathbb{R}, i = \sqrt{-1}\}$
ø - Empty Set: $\{\}$
Sets do not care about orders, or duplicates.

1.3.2 Relationship of sets

Set A and B are equal ($A = B$).
$A \subseteq B$, $B \subseteq A$.
A is a subset of B if every element of A is also an element of B:
$(a \in A \rightarrow a \in B)$.
The Power Set of A, $P(A)$, is the set of all possible subsets of A.
The Complement, A^C or A', of a set A is the set of all elements in the universal set that are NOT elements of A.
The Union of two sets, $A \cup B$, is the set containing all the elements from either A or B.
Theorem: if $A \subseteq B, B \subseteq C$, then $A \subseteq C$

1.4 Quantifiers

1.4.1 Universal Quantify

"For all"
$\forall$
Example: $\forall y \in \mathbb{R}, y^2 \geqslant 0$

1.4.2 Essential Quantify

"There Exists"
$\exists$
Example: $\exists x$ such that $x + 3 = 5$

1.4.3 Negations

$\neg$
This is a dog
This is not a dog
Examples:
$\neg(A \text{ or } B) = \neg A \text{ and } \neg B$
$\neg(A \text{ and } B) = \neg A \text{ or } \neg B$
$\neg(A \Rightarrow B) = A \text{ and } \neg B$
$\neg(\forall x,\ y) = \exists x \text{ such that } \neg y$

1.5 Proofs

1.5.1 Direct Proofs

Assumption
$\Rightarrow$ something
$\Rightarrow \cdots$
$\Rightarrow$ conclusion

Example. Prove: The sum of two Even integers equals an even integer

Proof.

$$\text{Let } x, y \in \mathbb{Z} \quad \text{Assume } x \text{ and } y \text{ are both even.} \tag{1.14}$$
$$\Rightarrow x = 2a, y = 2b, (a, b \in \mathbb{Z}) \tag{1.15}$$
$$\Rightarrow x + y = 2a + 2b \tag{1.16}$$
$$\Rightarrow x + y = 2(a + b) \tag{1.17}$$
$$\text{Given that each Even number N can be present in this form } N = 2k \tag{1.18}$$
$$\Rightarrow \therefore \text{ The sum of two Even integers equals an even integer} \tag{1.19}$$

$\square$

1.5.2 Contrapositive proof

$$p \to q \equiv \neg q \to \neg p$$

Example. Theorem: if x is an irrational number, then $\frac{1}{x}$ is also an irrational number.
we want to show: $\neg(\frac{1}{x}$ is not irrational) $\Rightarrow \neg(x$ is irrational)

$$x \in \mathbb{R} \setminus \mathbb{Q} \Rightarrow x \neq \frac{p}{q}, (p, q \in \mathbb{Z}) \tag{1.20}$$
$$\text{Assume } \frac{1}{x} \text{ is Not Irrational} \tag{1.21}$$
$$\Rightarrow \frac{1}{x} \text{ is rational, } (\frac{1}{x} \in \mathbb{Q}) \tag{1.22}$$
$$\Rightarrow \frac{1}{x} = \frac{m}{n}, (m, n \in \mathbb{Z}) \tag{1.23}$$
$$\Rightarrow x = \frac{n}{m} \tag{1.24}$$
$$\Rightarrow \text{x is rational} \tag{1.25}$$

1.5.3 Two Way Proof

Also known as (Two way Proof)

Example. Theorem: A whole number is divisible by 9 iff the sum of its digits is divisible by 9.

Proof.

$$x \in \mathbb{W} = \{0, 1, 2, 3, \cdots\} \tag{1.26}$$
$$x \text{ has digits } , a_n, a_{n-1}, \cdots, a_2, a_1, a_0 \tag{1.27}$$
$$x = 10^n a_n + 10^{n-1} a_{n-1} + \cdots + 100a_2 + 10a_1 + a_0 \tag{1.28}$$
$$\text{Assume } x \text{ is divisible by 9.} \tag{1.29}$$
$$10^n a_n + 10^{n-1} a_{n-1} + \cdots + 100a_2 + 10a_1 + a_0 (\text{divisible by 9}) \tag{1.30}$$
$$-(999 \ldots 99a_n + \cdots + 99a_2 + 9a_1)(\text{is divisible by 9}) \tag{1.31}$$
$$= a_n + a_{n-1} + \cdots + a_2 + a_1 + a_0 \text{ (is divisible by 9)} \tag{1.32}$$
$$\tag{1.33}$$
$$\text{Assume } a_n + a_{n-1} + a_{n-2} + \cdots + a_2 + a_1 + a_0 \text{ is divisible by 9} \tag{1.34}$$
$$x = a_n + a_{n-1} + a_{n-2} + \cdots + a_2 + a_1 + a_0 \tag{1.35}$$
$$+((999 \ldots 99)a_n + \cdots + 99a_2 + 9a_1) \text{ (is divisible by 9)} \tag{1.36}$$
$$\Rightarrow x \text{ is divisible by 9} \tag{1.37}$$

□

1.5.4 Proof by Contradiction

Assume the opposite of what we want to prove, then show a contradiction.

Example. Theorem: $\sqrt{3}$ is irrational.

Proof.

$$\text{Assume } \sqrt{3} \in \mathbb{Q} \tag{1.38}$$
$$\Rightarrow \sqrt{3} = \frac{m}{n} \text{ is a reduced fraction,} (n, m \in \mathbb{Z} \text{ and } n \neq 0) \tag{1.39}$$
$$\Rightarrow 3 = \frac{m^2}{n^2} \tag{1.40}$$
$$\Rightarrow 3 * n^2 = m^2 \tag{1.41}$$
$$\Rightarrow m^2 \text{ is a multiple of 3 and } m \text{ is a multiple of 3} \tag{1.42}$$
$$\Rightarrow m = 3k, (k \in \mathbb{Z}) \tag{1.43}$$
$$\Rightarrow 3n^2 = (3k)^2 \tag{1.44}$$
$$\Rightarrow 3n^2 = 9k^2 \tag{1.45}$$
$$\Rightarrow 3n^2 = 3k^2 \tag{1.46}$$
$$\Rightarrow n^2 \text{ is a multiple of 3 and } n \text{ is a multiple of 3} \tag{1.47}$$
$$\Rightarrow \frac{m}{n} \text{ is NOT in lowest terms} \tag{1.48}$$

□

Chapter 2

Problem Solving

2.1 Problem Solving Techniques

2.1.1 Intro

Example. Problem 6
Connie has a number of gold bars, all of different masses. She gives the 24 lightest bars, which represent 45of the total mass, to Brennan. She gives the 13 heaviest bars, which represent 26rest of the bars to Blair. How many bars did Blair receive.

2.1.2 Trial and error

This method not always works in the question. You should avoid this question for big number problem