OSSD - Calculus and Vector

MCV4U

Qinghao Hu

February 20, 2026

Unit 1: Rate of change. In the first unit, we will discuss the concept of Rate of Change, Continuity and Limits. Also, the the introduction to the Derivatives will be discussed.

Contents

1 Slope

1.1 Measuring Rates of Change on a Table or a Graph

Definition 1.1 (slope/Rate of Change)**.** the ratio of the vertical distance to the horizontal distance between two points.

In Grade 9, we all studied the slope formula:

$$m = \frac{y_2 - y_1}{x_2 - x_1}$$

m represents the slope because linear function is written as:

$$f(x) = mx + b$$

1.1.1 Intervial of the function

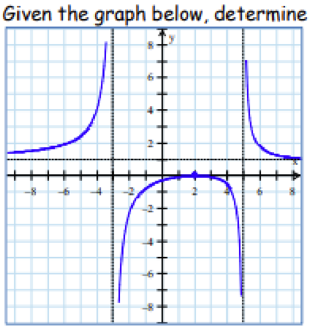
When you are asked to give the intervial of a function, one must response in one of these forms:

$$x \in (-\infty, a) \cap (b, \infty)$$

Remark. We look the function from $-\infty$ to ∞

Example 1.2. Here is a question by Mr.Hoken

8. Given the graph below, determine the regions on the function where:



a) the y values are increasing

$x \in (-\infty, -3)$ or $-3 < x < 2$, $x \in R$

b) the rate of change (r.o.c.) is increasing

$x < -3 \cup x > 5$, $x \in R$

c) the y values are negative and thE slopes are positive

$-3 < x < 2$, $x \in R$

d) the slopes are negative and the r.o.c. is decreasing

$x \in (2, 5)$, $x \in R$

The answer for a) should be $x \in (-\infty, -3)$
The answer for b) should be $x \in (-\infty, -3) \cup (5, \infty)$

1.1.2 Determine approximation for the slope of the tangent

To start off, we need to two points which have equal difference of x to the point that tangent line cross over.

Let's assume we have points $1, 2,$ and 3. Point 2 touches the tangent line. Solve for the slope between point 1 and point 2:

$$m_1 = \frac{y_2 - y_1}{x_2 - x_1}$$

Solve for the slope between point 2 and point 3:

$$m_2 = \frac{y_3 - y_2}{x_3 - x_2}$$

The slope at point 2 that touches tangent line should be:

$$m = \frac{m_2 + m_1}{2}$$

1.1.3 Trends and increasing/decreasing

Here are the summary Mr. Hoken made for this class

- An increasing trend in y values corresponds to positive slopes of tangents in that intervial
- A decreasing trend in y values corresponds to negative slopes of tangents in that interval
- "Rate of Change" in y values refers to your slope values. We use standard expressions to describe the trend in Slope values
  * "The rate of change is increasing" means that the slope values are climbing.
    ex. (0.5, 3, 4, 5, 6, 20, $\cdots$) or (-10, -9, $\cdots$, 1, 2)
  * "The rate of change is decreasing" means that the slope values are dropping.
    ex. (25, 11, 9, 5, $\cdots$, -3, -7, -20)

1.2 Measuring Rates of Change Using a Defined Relation

1.2.1 Difference Quotient

Definition 1.3 (Difference Quotient)**.** a measure of the average rate of change of the function over an interval. In other words, it is a simplified expression for the slope of a secant. It can be expressed as

$$m = \frac{f(x+h) - f(x)}{h}$$

where $h \neq 0$

Let's proof the difference Quotient

Proof. We have the formula of the slope:

$$m = \frac{y_2 - y_1}{x_2 - x_1}$$

Assume we have two points: $(x, f(x))$ and $(x+h, f(x+h))$, sub these two points into the formula of slope.

$$m = \frac{f(x+h) - f(x)}{(x+h) - x}$$
$$m = \frac{f(x+h) - f(x)}{h}$$

□

Let's look at an example.

Example 1.4. Determine a general expression for the slope given the function $f(x) = -x^2 + 4x + 5$

$$m = \frac{f(x+h) - f(x)}{h}$$
$$m = \frac{-(x+h)^2 + 4(x+h) + 5 - (-x^2 + 4x + 5)}{h}$$
$$m = \frac{-x^2 - 2xh - h^2 + 4x + 4h + 5 + x^2 - 4x - 5}{h}$$
$$m = \frac{-2xh - h^2 + 4h}{h}$$
$$m = -2x + 4 - h, h \neq 0$$

1.2.2 Approximating the slope of the tangent

Recall:

- The slope of a secant line will give the average rate of change of the 'y' values over a givern interval of 'x'.
- The slope of a tangent line will give the instantaneous rate of change of the 'y' value at a givern value of 'x'.

For difference quotient

$$\frac{f(x+h)-f(x)}{h}$$

if h is extremely small, we can approximate the slope of the tangent line.

In later sections, we will discuss about principle of limits. At that time, you will have a better understanding about today's topic.

2 Continuity and Limits

2.1 The Concept of Limit

Definition 2.1 (Limits)**.** Assume we have function $f(x)$, x, and

$$\lim_{x\to a} f(x) = L$$

Using the $\delta - \epsilon$ language:

$$\forall\epsilon > 0, \exists\delta > 0 \text{ such that } 0 < |a - x| < \delta \implies |f(x) - L| < \epsilon$$

Remember, $f(a)$ does not have to be defined.

For limts, we are solving for the expected value instead of defined value. When the following conditions are met, the limits exists

–
$$\lim_{x\to a^-} f(x) \text{ must exist}$$

–
$$\lim_{x\to a^+} f(x) \text{ must exist}$$

–
$$\lim_{x\to a^-} f(x) = \lim_{x\to a^+} f(x)$$

Definition 2.2 (Continuity)**.** a function which there are no "breaks" or "gaps" in the function.

These conditions must be met

(i)
$$\lim_{x\to a} f(x) \text{ exist}$$

(ii)
$$f(a) \text{ must be defined}$$

(iii)
$$\lim_{x\to a} = f(a)$$

If a function does not meet any of the conditions, it is a discontinue function.

2.2 Continuity and Limits

Definition 2.3 (Limits Law)**.** Assume we have function $f(x)$ and $g(x)$, which

$$\lim_{x\to a} f(x) \text{ and } \lim_{x\to a} g(x) \text{ both exist}$$

We also have constant c We get these properties:

(i)
$$\lim_{x\to a} c = c$$

(ii)
$$\lim_{x\to a} x = a$$

(iii)
$$\lim_{x\to a} [f(x) + g(x)] = \lim_{x\to a} f(x) + \lim_{x\to a} g(x)$$

(iv)
$$\lim_{x\to a} [f(x) - g(x)] = \lim_{x\to a} f(x) - \lim_{x\to a} g(x)$$

(v)
$$\lim_{x\to a} [f(x) * g(x)] = \lim_{x\to a} f(x) * \lim_{x\to a} g(x)$$

(vi)
$$\lim_{x\to a} \frac{f(x)}{g(x)} = \frac{\lim_{x\to a} f(x)}{\lim_{x\to a} g(x)}$$

(vii)
$$\lim_{x\to a} [f(x)]^n = (\lim_{x\to a} f(x))^n$$

(viii)
$$\lim_{x\to a} \sqrt[n]{f(x)} = \sqrt[n]{(\lim_{x\to a} f(x))}$$

Somes cases you should be careful

Example 2.4. Evaluate and simplify

$$\lim_{x\to a} \sqrt{x}$$

Recall: if a limit exist

(i)
$$\lim_{x\to a^-} f(x) \text{ exists}$$

(ii)
$$\lim_{x\to a^+} f(x) \text{ exists}$$

(iii)
$$\lim_{x\to a^-} f(x) = \lim_{x\to a^+} f(x)$$

In this case,

$$\lim_{x \to 0^-} \sqrt{x} \text{ does not exist}$$

because

$$D : x \in \mathcal{R} | x \geq 0$$

Example 2.5. Evaluate and simplify:

$$\lim_{x \to 1} x^2 - 5x$$

Remark. Please be carefull with the bracket, don't mix with

$$\lim_{x \to 1} (x^2 - 5x)$$

The solution should be:

$$\begin{aligned} &= (\lim_{x \to 1} x)^2 - 5x \\ &= (1)^2 - 5x \\ &= 1 - 5x \end{aligned}$$

2.3 Discontinuity and Limits

2.3.1 Jump Discontinuity

Example 2.6. Evaluate the limit of $f(x)$ at $x = 7$

$$f(x) = \begin{cases} \sqrt{x-3}, x \geq 7 \\ x-4, x < 7 \end{cases}$$

If we approach from the left side

$$\lim_{x \to 7^-} (x-4) = 3$$

If we approach from the right side

$$\lim_{x \to 7^+} (\sqrt{x-3}) = 2$$

$$\because \lim_{x \to 7^-} f(x) \neq \lim_{x \to 7^+} f(x)$$
$$\therefore \lim_{x \to 7} f(x) \text{ does not exist.}$$

This is called a jump discontinuity.

2.3.2 Infinite Discontinuity

Example 2.7. Evaluate the limit of $f(x)$ at $x = 2$

$$f(x) = \frac{x+3}{x-2}$$

If we approach from the left side

$$\lim_{x \to 2^-} (\frac{x+3}{x-2}) = \infty$$

If we approach from the right side

$$\lim_{x \to 2^+} (\frac{x+3}{x-2}) = -\infty$$

$$\because \lim_{x \to 2^-} f(x) \neq \lim_{x \to 2^+} f(x)$$
$$\therefore \lim_{x \to 2} f(x) \text{ does not exist.}$$

2.3.3 Removable Discontinuity

Example 2.8. Evaluate this limit.

$$\lim_{x \to 5} \frac{2x^2 - 7x - 15}{x-5}$$

$$= \lim_{x \to 5} \frac{(2x+3)(x-5)}{(x-5)} \tag{1}$$
$$= \lim_{x \to 5} 2x + 3 \tag{2}$$
$$= 13 \tag{3}$$

However, this function is not defined at $x = 5$
So there is a Removable discontinuity

Example 2.9. Evaluate the following limit

$$\lim_{x \to 4} \frac{1}{(x+4)^2}$$

If you approach this limit from both side, you will figure it out that this limit equals to ∞ This is not a "limit" because simply $\infty \notin \mathbb{R}$

Definition 2.10 (Indeterminate form)**.** is an expression in which generate values such as $\frac{0}{0}$ or $\frac{\infty}{\infty}$. Certain limits cannot be evaluated by direct substitution since we may end up with one of these indeterminate form.

Remark. See the teacher's notes for more details!!!

3 Derivatives

3.1 Introduction to Derivatives

4 Reference Sheet

4.1 Assume Knowledge

4.1.1 Log Laws

$$\log_b A + \log_b B = \log_b(A \times B)$$
$$\log_b(\tfrac{M}{N}) = \log_b M - \log_b N$$
$$log_b M^k = k \log_b M$$
$$log_b b^k = k$$
$$b^{\log_b k} = k$$

4.1.2 Trig Laws

$$\tan\theta = \frac{\sin\theta}{\cos\theta}$$
$$\cot\theta = \frac{\cos\theta}{\sin\theta}$$
$$\csc\theta = \frac{1}{\sin\theta}$$
$$\sec\theta = \frac{1}{\cos\theta}$$
$$\sin^2\theta + \cos^2\theta = 1$$
$$1 + \cot^2\theta = \csc^2\theta$$
$$\tan^2\theta + 1 = \sec^2\theta$$
$$\sin(A \pm B) = \sin A \cos B \pm \sin B \cos A$$
$$\cos(A \pm B) = \cos A \cos B \mp \sin A \sin B$$
$$\tan(\alpha \pm \beta) = \frac{\tan\alpha \pm \tan\beta}{1 \mp \tan\alpha \tan\beta}$$

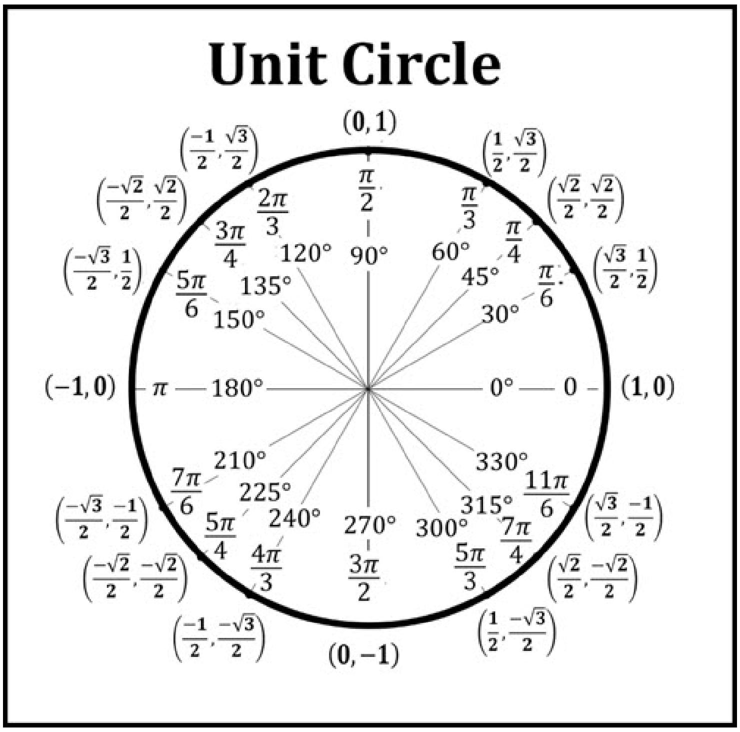
sin Law:

$$\frac{\sin A}{a} = \frac{\sin B}{b} = \frac{\sin C}{c}$$

Cosine Law:

$$a^2 = b^2 + c^2 - 2bc\cos A$$

Unit Circle:


Unit Circle
(0,1)
π/2
90°
(1/2, √3/2)
π/3
60°
(√2/2, √2/2)
π/4
45°
(√3/2, 1/2)
π/6
30°
(1,0)
0
0°
330°
11π/6
(√3/2, −1/2)
315°
7π/4
(√2/2, −√2/2)
300°
5π/3
(1/2, −√3/2)
(0,−1)
3π/2
270°
240°
4π/3
(−1/2, −√3/2)
225°
5π/4
(−√2/2, −√2/2)
210°
7π/6
(−√3/2, −1/2)
(−1,0)
π
180°
150°
5π/6
(−√3/2, 1/2)
135°
3π/4
(−√2/2, √2/2)
120°
2π/3
(−1/2, √3/2)