

AOPS AMC12 class Note 2

functions

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1 Class 1: Quadratics, Vieta, and Factorization

Today, we will look at quadratic functions, Vieta's formula and factorizations

1.1 Quadratic Equation

A quadratic equation is an equation of the form

$$ax^2 + bx + c = 0 \quad (1)$$

where a , b and c are constants and $a \neq 0$

1.2 Roots of Quadratic Equation

The roots of Quadratic Equation can be determined by the Quadratic Formula, or more formally, Sridharacharya Formula

Assume r_1 and r_2 are both roots of the Quadratic Equation $ax^2 + bx + c = 0$

$$r_1 = \frac{-(b) + \sqrt{b^2 - 4ac}}{2a}, r_2 = \frac{-(b) - \sqrt{b^2 - 4ac}}{2a} \quad (2)$$

r_1 and r_2 may be imaginary numbers if $\sqrt{b^2 - 4ac} < 0$

1.3 Vieta's Formula

Assume we have a polynomial formula with degree of n

$$P(x) = a_n x^n + a_{n-1} x^{n-1} + \cdots + a_1 x + a_0 \quad (3)$$

and r_1, r_2, \dots, r_n are the roots of the polynomial
we can get:

$$r_1 + r_2 + \cdots + r_{n-1} + r_n = \frac{a_{n-1}}{a_n} \quad (4)$$

$$(r_1 r_2 + r_1 r_3 + \cdots + r_1 r_{n-1} + r_1 r_n) + (r_2 r_3 + r_2 r_4 + \cdots + r_2 r_{n-1} + r_2 r_n) + \cdots + r_{n-1} r_n = \frac{a_{n-2}}{a_n} \quad (5)$$

$$r_1 r_2 \cdots r_{n-1} r_n = (-1)^n \frac{a_0}{a_n} \quad (6)$$

1.4 Factorizations

Some key factorizations that you should be familiar with are difference of square, difference of cubes, and sum of cubes

one good example is the "Simon's favorite factoring trick"

for example:

Factor the equation, $mn - 2m - 4n + 8 = 8$

we can get:

$$(m - 4)(n - 2) = 8$$

2 Class 2: Solving equations and systems

Summary:

When dealing with several variables, look for ways of eliminating them to get what you want. And make sure you read the problem carefully so that you know what you want, and you do not end up doing more work than you have to.

3 Class 3: Sequences and Series

3.1 Arithemtic Sequences and Series

An arithmetic sequence is a sequence of the form $a, a + d, a + 2d$, and so on (for example, 7,11,15,19 is an arithmetic sequence). In other words, we begin with a first term a , and repeatedly add a common difference d to obtain the terms that follow.

The sum of the arithmetic series with n terms is:

$$a + (a + d) + \cdots + [a + (n - 1)d] = \frac{2a + (n - 1)d}{2} * n \quad (7)$$

3.2 Geometric Sequences and Series

A geometric sequence is a sequence of the form a, ar, ar^2 , and so on. In other words, we have a first term a , and repeatedly multiply by a common ratio r to obtain the terms that follow

The sum of the geometric series with n term is:

$$a + ar + \cdots + ar^{n-1} = \frac{a(r^n - 1)}{r - 1} = \frac{a(1 - r^n)}{1 - r} \quad (8)$$

The numerator can be viewed as the difference of two terms, a and ar^n .
Notice in particular that these are not the first term and last term.

For $|r| < 1$, the sum of the infinite geometric series is:

$$a + ar + ar^2 + \cdots = \frac{a}{1 - r}$$

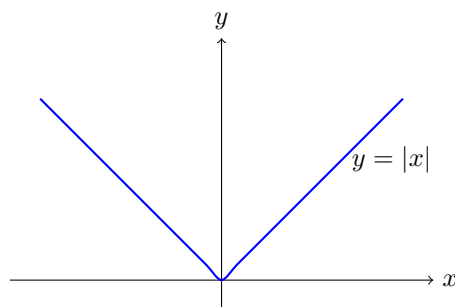
4 Lesson 4: Functions and Polynomials

Today we will look at the properties of certain functions, such as the "floor" function and logarithm, as well as polynomials, which form a very special class of functions

4.1 ABSOLUTE VALUE

The absolute value signs make equation difficult to work with. How might we deal with those pesky bars ($|a|$)

Example. if $x < 0$, then what happens to the equation $|x| + x + y = 10$, if $x > 0$, then what happens to the equation $|x| + x + y = 10$



4.2 Floor Function

In case you have not seen it before, $\lfloor x \rfloor$ is the greatest integer less than or equal to x , also called the floor of x . In other words, $\lfloor x \rfloor$ is x rounded down to the nearest integer.

$$\begin{aligned} & \text{(In general} \\ & \lfloor x + n \rfloor = \lfloor x \rfloor + n \\ & \text{for any integer } n) \end{aligned}$$

4.3 Logarithms

Logarithms identity:

$$\begin{aligned} \log_b x + \log_b y &= \log_b xy \text{ (Product law)} \\ \log_b x - \log_b y &= \log_b \frac{x}{y} \text{ (Quotient Law)} \\ \log_b x^n &= n \log_b x \text{ (Power law)} \\ \log_b x &= \frac{\log_a x}{\log_a b} \text{ (Power Law)} \\ \log_{b^n} x^n &= \log_b x \end{aligned}$$

4.4 POLYNOMIALS

Let $F(x)$ and $G(x)$ be polynomials. If we divide $G(x)$ into $F(x)$, then we will obtain a quotient $Q(x)$ and a remainder $R(x)$, where the degree of $R(x)$ is less than the degree of $G(x)$. The quotient $Q(x)$ and $R(x)$ are unique.

Also, if a polynomial has all real coefficients, then its nonreal roots must come in conjugate pairs. (*Complex Conjugate Root Theorem*)

if:

$$c_n x_n^{e_n} + c_{n-1} x_{n-1}^{e_{n-1}} + \cdots + c_0 x_0^{e_0} = 0$$

and

for i from $\{0$ to $n\}$:

$$i \in \mathbb{R}$$

We can get:

if $a + bi$ is a root for the polynomial, $a - bi$ will also be the root of the polynomial

5 Tools for Algebra

Today, we will look at what we will describe as the tools of algebra. This tools include working with radicals (square roots and cube roots), and finding the extrema (minima maxima) of functions

5.1 RADICALS

Example. 1. What does $\frac{2\sqrt{6}}{\sqrt{2}+\sqrt{3}+\sqrt{5}}$ equal to

To solve this problem, we need to rationalize this term. Why not we multiple both numerator and denominator by $(\sqrt{2} + \sqrt{3} + \sqrt{5})$

$$\frac{2\sqrt{6}}{\sqrt{2} + \sqrt{3} + \sqrt{5}} = \frac{2\sqrt{6} * (\sqrt{2} + \sqrt{3} - \sqrt{5})}{(\sqrt{2} + \sqrt{3} + \sqrt{5}) * (\sqrt{2} + \sqrt{3} - \sqrt{5})}$$

We can expand the denominator using difference of squares:

$$(\sqrt{2} + \sqrt{3} + \sqrt{5}) * (\sqrt{2} + \sqrt{3} - \sqrt{5}) = (\sqrt{2} + \sqrt{3})^2 - (\sqrt{5})^2$$

We expand, we will get:

$$(\sqrt{2} + \sqrt{3})^2 - (\sqrt{5})^2 = 2 + 2\sqrt{6} + 3 - 5 = 2\sqrt{6}$$

Finally:

$$\frac{2\sqrt{6} * (\sqrt{2} + \sqrt{3} - \sqrt{5})}{2\sqrt{6}} = (\sqrt{2} + \sqrt{3} - \sqrt{5})$$

Example. 2. Which of the following is closest to $\sqrt{65} - \sqrt{63}$?

We can use a less common technique that is called "Rationalize the numerator"

$$\frac{\sqrt{65} - \sqrt{63}}{1}$$

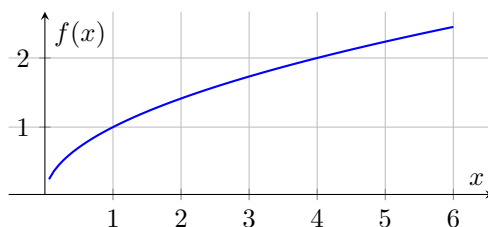
Rationalize the numerator, we get:

$$\frac{(\sqrt{65} - \sqrt{63}) * (\sqrt{65} + \sqrt{63})}{(\sqrt{65} + \sqrt{63})} = \frac{2}{\sqrt{65} + \sqrt{63}}$$

$\sqrt{65} + \sqrt{63}$ is quite close to $8 + 8$. So our fraction is really close to $\frac{2}{8+8} = 0.125$. It's really close to the (A) and (B) options.

What can we do to find out whether $\sqrt{65} + \sqrt{63}$ is a little more or a little less than $8 + 8$

We can graph the function $f(x) = \sqrt{x}$:



The root function $y = \sqrt{x}$ is getting less and less steep as x increases.

So $\sqrt{65} - \sqrt{63}$ is less than 16

\therefore the number should be greater than 0.125

Choose (B)

Example. If

$$N = \frac{\sqrt{\sqrt{5}+2} + \sqrt{\sqrt{5}-2}}{\sqrt{\sqrt{5}+1}} - \sqrt{3-2\sqrt{2}}$$

the N equals

(A) 1 (B) $2\sqrt{2}-1$ (C) $\frac{\sqrt{5}}{2}$ (D) $\sqrt{\frac{5}{2}}$ (E) none of these

We can try to rationalize the first section:

$$\begin{aligned} &= \frac{\sqrt{\sqrt{5}+2} + \sqrt{\sqrt{5}-2}}{\sqrt{\sqrt{5}+1}} * \frac{\sqrt{\sqrt{5}-1}}{\sqrt{\sqrt{5}-1}} \\ &= \frac{\sqrt{3+\sqrt{5}} - \sqrt{7-3\sqrt{5}}}{2} \end{aligned}$$

We can go a step forward using radicals

set $a, b, \{a, b \in \mathbb{Q}\}$

$$\begin{aligned} \sqrt{a} + \sqrt{b} &= \sqrt{3 + \sqrt{5}} \\ a + b + 2\sqrt{ab} &= 3 + \sqrt{5} \end{aligned}$$

We can list a linear equation system:

$$\begin{cases} a + b = 3 \\ 4ab = 5 \end{cases}$$

$$a = \frac{1}{2}, b = \frac{5}{2}$$

So,

$$\sqrt{3 + \sqrt{5}} = \sqrt{\frac{1}{2}} + \sqrt{\frac{5}{2}}$$

Apply the rule to all the terms, we can get:

.....

$$\sqrt{2} - (\sqrt{2} - 1) = 1$$

So choose (A)

Example. 3. What is the smallest integer larger than $(\sqrt{3} + \sqrt{2})^6$?

(A) 972 (B) 971 (C) 970 (D) 969 (E) 968

$$(\sqrt{3} + \sqrt{2})^6 = (5 + 2\sqrt{6})^3 = 485 + 198\sqrt{6}$$

It's really hard to say which is it greater than 970

But there is an interesting fact, the conjugate of the number, $\sqrt{3} - \sqrt{2} = 485 - 198\sqrt{6}$ So,

$$(\sqrt{3} + \sqrt{2})^6 + (\sqrt{3} - \sqrt{2})^6 = (485 + 198\sqrt{6}) + (485 - 198\sqrt{6}) = 970$$

$(\sqrt{3} - \sqrt{2})^6$ is a really small number (less than 1), so $(\sqrt{3} - \sqrt{2})^6$ must be between 969 and 970
Select (C)

5.2 MINIMA AND MAXIMA

When given you multiple function and solve for Minima and maxima, just graph it.
Think about the elements of Minima and Maxima.

5.3 AM-GM inequality

The AM-GM inequality states that the arithmetic mean of any set of nonnegative real numbers is greater than or equal to the geometric mean of those numbers.

In other words, if $x_1, x_2, \dots, x_n \geq 0$, then:

$$\frac{x_1 + x_2 + \dots + x_n}{n} \geq \sqrt[n]{x_1 * x_2 * x_3 \dots x_n}$$

Equality occurs if and only if $x_1 = x_2 = x_3 = \dots = x_n$. That is, the arithmetic mean is actually strictly greater than the geometric mean unless all the numbers are the same.

5.4 THE TRIANGLE INEQUALITY

Assume a, b and c are sides of a triangle

$$a + b > c \tag{9}$$

$$a + c > b \tag{10}$$

$$b + c > a \tag{11}$$

6 Triangles

6.1 Right Triangles

In many geometry problems, building right triangles is an important step, because right triangles give us a way of computing distances via the *Pythagorean theorem*. It is also important to recognize special triangle, such as the 30-60-90 and 45-45-90 right triangles

Example. In rectangle $ABCD$, angle C is trisected by \overline{CF} and \overline{CE} , where E is on \overline{AB} , $BE = 6$, and $AF = 2$. Which of the following is closest to the area of the rectangle $ABCD$? unitsize(0.3 cm);

6.2 Similar Triangle

When you see a question, try to find similar triangles first. They may help you find information about unknown variables.

6.3 Law of Sines and Cosines

Law of Sines

In triangle ABC , let the sides opposite angle A , B and C have lengths a , b and c respectively. Then

$$\frac{a}{\sin A} = \frac{b}{\sin B} = \frac{c}{\sin C}$$

6.4 Law of Cosines

In triangle ABC , let the sides opposite angles A , B and C have lengths a , b and c respectively. Then

$$c^2 = a^2 + b^2 - 2ab \cos C$$

6.5 Heron's Formula

Assume a , b and c are side lengths of a triangle. Assume s are half of the perimeter of the triangle

$$area = \sqrt{s * (s - a) * (s - b) * (s - c)}$$

7 Circle

7.1 Example Question

Always think about using dot lines to split the picture to get datas that are calculatable

7.2 Example Question 2

When we have three circles a , b and c , and they are tangent to lines or other circles and we have the relationship:

$$\frac{1}{\sqrt{a}} + \frac{1}{\sqrt{b}} = \frac{1}{\sqrt{c}}$$

7.3 Power of a Point

Circles have very important properties with respect to angles. They also have important properties with respect to lengths, most notably the **Power of a Point**.

This is what the Power of a Point theorem says: Suppose that P is a point not on a circle ω . Suppose that one line through P intersects circle ω at points A and B , and that another line through P intersects the circle at points C and D . Then the following holds:

$$PA \cdot PB = PC \cdot PD$$

This product is called the **power of point P** with respect to the circle.

7.4 The inradius of a quadrilateral

r is radius of the incircle, K and s are the area and semi-perimeter of the quadrilateral

$$r = \frac{K}{s}$$

8 Class 8: Polygons and Three-Dimensional Geometry

8.1 Polygons

For a regular polygon with m sides, each interior angle measure is equal to $\frac{180(m-2)}{m}$ (in degree), which we can also write as $180^\circ - \frac{360^\circ}{m}$

9 Class 9: Number Theory

- Number Theory can be boardly described as the study of the properties of integers, such as divisibility
- In number theory, we often restrict our attention to positive integers, but sometimes we must also extend our attention to negative integers as well. Remember to look for presence or absence of the word "positive" in the problem statement

For the positive integer x , its factor from is

$$x = a_1^{e_1} * a_2^{e_2} * \cdots * a_n^{e_n}$$

The number of dividend is defined by:

$$\text{Number of dividend} = (e_1 + 1) * (e_2 + 1) * \cdots * (e_n + 1)$$