Physics 115B Notes

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I. LECTURE 1

We first attempt to address the mysterious stuff such as the uncertainty principle in quantum mechanics:

$$\Delta x \Delta p \geq \frac{\hbar}{2} \tag{1}$$

$$\Delta t \Delta E \geq \frac{\hbar}{2} \tag{2}$$

In Classical mechanics, we have Noether's theorem which states that for every symmetry of the Lagrangian, there is a conserved quantity. We have some favorites: time translation gives energy conservation; space translation gives momentum conservation:

$$x \to p \tag{3}$$

$$t \to E \tag{4}$$

So we know these values are connected, by why? We start with momentum, which is a key quantity in physics. We find that it is actually really hard to define momentum. In first year, we define momentum of the Newtonian definition which calls momentum the "quantity of motion":

$$F = ma = \frac{dp}{dt}, \quad p = \int F dt = \int m \frac{dv}{dt} dt = mv \tag{5}$$

In physics 1c, we see that momentum is defined:

$$p = \gamma mv = \frac{mv}{\sqrt{1 - \left(\frac{v}{c}\right)^2}} \tag{6}$$

In our second year physics classes. we see that momentum can be defined using the de brolie definition:

$$o = \frac{h}{\lambda} \quad \text{or} \quad p = \hbar k \quad \text{with} k \equiv \frac{2\pi}{\lambda} \tag{7}$$

Then, in physics 105, we see that the generalized momentum is defined:

$$p_j = \frac{\partial \mathcal{L}}{\partial \dot{q}} \tag{8}$$

where $\mathcal{L} \equiv T - U$ (sometimes) and $\mathcal{L} = \sum_j p_j \dot{q}_j - \mathcal{H}$ from the Legendre transform and $\mathcal{H}(q, p)$ is the Hamiltonian and $\mathcal{L}(q, \dot{q})$ is the Lagrangian.

Going into our 3rd year, we learned that:

$$p = \frac{\hbar}{i}\nabla; \quad p_x = \frac{\hbar}{i}\frac{d}{dx} \tag{9}$$

In this class we are going to find that p in the generator of translations. We will also learn that:

$$\Pi = p - qA \tag{10}$$

where Π is the mechanical or kinetic observable, and p is the canonical or the conjugate momentum. They are both linear momentum (nothing to do with angular momentum). From this list, we hgave at least 9 possible definitions for p! We can rule out a few of them(mv or $\frac{\hbar}{i}\nabla$), because we want definitions to be always true. We also note a bootstrapping problem. For exmaple, in the formula $F = ma$, the definitions of F and m is quite circular which is another we're trying to avoid.

Spo here's the upshot: we shouldn't be discussing momentum at all, if we have to:

$$\vec{p} \equiv h\vec{K}, \quad \text{with} \quad |K| = \frac{1}{\lambda} \tag{11}$$

where $\vec{K}$ is the wave-vector or the spatial frequency, is a fundamental quantity, not $\vec{p}$.

What special relativity teaches us is that space and time are not separate, they are combined to create spacetime. In a change of coordinates in space (rotation), we have:

$$\begin{pmatrix} x' \\ y \end{pmatrix} = \begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} \tag{12}$$

Rotation in space corresponds to a boost in spacetime, we have an analagous rotation matrix:

$$\begin{pmatrix} ct' \\ x' \end{pmatrix} \begin{pmatrix} \cosh\theta & \sinh\theta \\ \sinh\theta & \cosh\theta \end{pmatrix} \begin{pmatrix} ct \\ x \end{pmatrix} \tag{13}$$

where

$$\cosh\theta = \gamma, \quad \gamma = \frac{1}{\sqrt{1-\beta^2}} \tag{14}$$

$$\sinh\theta = \beta\gamma, \quad \beta = \frac{v}{c} \tag{15}$$

We know that rotations preserves distances (regardless of angle θ), that is:

$$x'^2 + y'^2 = x^2 + y^2 \tag{16}$$

Meanwhile boost preserve the interval:

$$(ct')^2 - (x'^2 + y'^2 + z'^2) = (ct)^2 - (x^2 + y^2 + z^2) \tag{17}$$

this is reflected from the fact that the determinant of both matrices are 1. There is also a nice geometry intuition which is the spacetime is in fact hyperbolic.

One ugly thing is that we need to put ct so that the unit works out. In some sense, C is a historical legacy that hides the geometry. It also means that we are using two sets of units in the same setting. For more details, see Parables of the surveys in Taylor and Wheeler.

That was special relativity, now let's go back to quantum mechanics: the wave equation says:

$$\frac{1}{V^2}\frac{\partial^2 u}{\partial t^2} = \nabla^2 u = \frac{\partial^2 u}{\partial x^2} \tag{18}$$

solution looks like $u(ft - kx) = u(\phi)$. We can check it quite easily:

$$\frac{\partial u}{\partial t} = \frac{\partial u}{\partial \phi}\frac{\partial \phi}{\partial t} = \dot{u}f \tag{19}$$

$$\frac{\partial^2 u}{\partial t^2} = \ddot{u}f^2 \tag{20}$$

$$\frac{\partial u}{\partial x} = \frac{\partial u}{\partial \phi}\frac{\partial \phi}{\partial x} = \dot{u}(-k) \tag{21}$$

$$\frac{\partial^2 u}{\partial x^2} = \ddot{x}k^2 \tag{22}$$

This gives:

$$\frac{1}{V^2}\cdot f^2\ddot{u} = k^2\ddot{u} \tag{23}$$

This, this is a solution if $\left(\frac{f}{k}\right)^2 = v^2$. We recognize that V is the phase velocity. We can show that by picking some constant $\phi_{\text{const}} = ft - kx$:

$$kx = ft - \phi_c \tag{24}$$

$$x = \frac{f}{k}t - \frac{\phi_c}{k} \tag{25}$$

$$\frac{dx}{dt} = \frac{f}{k} = v_{\text{phase}} \tag{26}$$

Here, ϕ is a scalar number. Now, we want to connect the wave equation to relativity. Things we can count don't transform under a Lorentz boost. This discussion of scalar leads us to vectors, namely four-vectors.

Everyone agree that:

$$(ct)^2 - x^2 = S^2 = \begin{pmatrix} ct & x \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}\begin{pmatrix} ct \\ x \end{pmatrix} = \Gamma^\mu\Gamma_\mu \tag{27}$$

We can write that:

$$ft - Kx = \phi = \begin{pmatrix} \frac{f}{c} & x \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}\begin{pmatrix} ct \\ x \end{pmatrix} = K^\mu\Gamma^\mu \tag{28}$$

where K^μ is the four-wave vector and Γ_μ is the four-position vector. The final product (dot/scalar product) ($K^\mu \Gamma^\mu$) is invariant under rotations (even hyperbolic rotations = boost). We can form another dot product:

$$K^\mu K_\mu = \begin{pmatrix} \frac{f}{c} & K \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \begin{pmatrix} \frac{f}{c} \\ K \end{pmatrix} = \left(\frac{f}{c}\right)^2 - K^2 = K_c^2 \tag{29}$$

and this value (Compton wave vector) also must be invariant. We calculate $x(hc)^2$:

$$(hf)^2 - (h\vec{k}c)^2 = (hK_c c)^2 \tag{30}$$

We define:

$$E \equiv hf \tag{31}$$

$$\vec{p} \equiv h\vec{k} \tag{32}$$

$$m \equiv \frac{hK_c}{c} \tag{33}$$

We derived the Einstein Energy-Momentum principle:

$$E^2 - \vec{p}^2 c^2 = m^2 c^4 \tag{34}$$

II. LECTURE 2

We first review the material from last class. We made 2 assumptions: 1, the geometric of spacetime is hyperbolic; 2, we have waves. We found 2 4-vectors:

$$r = (ct, \vec{r}), \quad \text{position in real space(time)} \tag{35}$$

$$k = (\frac{f}{c}.\vec{k}), \quad \text{position in reciprocal space(time)} \tag{36}$$

We can think of the second as the frequency space (or momentum space), with the first being the temporal frequency and the second as spacial frequency. From these four-vectors, we get 3 variants:

$$rr = \text{"interval" } S^2 \tag{37}$$

$$Kr = \text{phase } \phi \tag{38}$$

$$KK = \text{Compton wave-vector } K_c^2 \tag{39}$$

We define all our classical quantities based on these wave parameters. For instance:

$$\text{Energy}: \quad E \equiv hf \tag{40}$$

$$\text{Momentum}: \quad \vec{p} \equiv h\vec{K} \quad (k = 1/\lambda) \tag{41}$$

$$\text{Mass}: \quad m \equiv \frac{hK_c}{c} \tag{42}$$

$$\text{action}: \quad S \equiv h\phi \tag{43}$$

In fact, we don't need any of these "new" old-fashioned concept to do physics. (we do need them to communicate with more old-fashioned physicists.)

Let's demonstrate that we can consider classical mechanics as a limiting case of wave mechanics (QM).

Let's consider a a probability function $|\Psi(x)|^2$ and in the reciprocal space $|\bar{\Psi}(k)|^2$. We know that its spread are inverse of each other: $\sigma_x \sim \frac{1}{\sigma_k}$, or the uncertainty principle: $\sigma_x \sigma_k \geq \frac{1}{4\pi}$.

Classically, the "probability function" in position and momentum space would just be a delta-function (since they are well-localized). Well, it seems like this violate the uncertainty principle.

The resolution of this seemingly paradox is that we do in fact has $\sigma_x \sim \frac{1}{\sigma_k}$. Classically, we have $\bar{x} >> \sigma_x$ and $\bar{k} >> \sigma_k$. Or:

$$\frac{\sigma_x}{\bar{x}} \to 0 \quad \text{as} N(\text{number of particles}) \to \text{large} \tag{44}$$

$$\frac{\sigma_k}{\bar{k}} \to 0 \quad \text{as} N(\text{number of particles}) \to \text{large} \tag{45}$$

There's no contradiction here. In one dim: $\bar{x} \sim N$ and $\sigma_x \sim \sqrt{N}$; $\bar{k} \sim N$ and $\sigma_k \sim \sqrt{N}$. Thus, both distributions look like delta-function in the classical limit: "object" can be well-localized in real and reciprocal space.

Classical "objects"(waves) follow the path where the phase(action) is stationary. With this wave interpretation, we can justify the classical postulate.

Consider we have two paths from (x_1, y_1) to (x_2, y_2) separate by δ_y. We enforce the phase in stationary:

$$-\phi = \vec{k} \cdot \vec{r} - f \tag{46}$$

$$= \int \vec{k} \cdot d\vec{r} - f dt \tag{47}$$

This way we allow for a potential, i.e. $\vec{k}$ and f might change over the path. We require ϕ ton be a min or max (extremum) along the path: $\delta\phi = 0$.(Marion and Thorton "modified Hamilton's principle"). We get:

$$-\phi = \int (k\dot{r} - f)dt \quad \text{where} \dot{r} \equiv \frac{dr}{dt} \tag{48}$$

Here the integrand in the Lagrangian: $\mathcal{L} = h\dot{\phi}$. We have:

$$\delta\phi = 0 = \int (\delta k\dot{r} + k\delta\dot{r} - \delta f)dt \tag{49}$$

then

$$\int k\delta\dot{r}dt = \int k\delta dr = \int kd(\delta r) = k\delta r\Big|_1^2 - \int \delta r dk \tag{50}$$

The surface term disappear:

$$0 = \int (\delta k \dot{r} - \delta r \dot{k} - \frac{\partial f}{\partial r}\delta r - \frac{\partial f}{\partial k}\delta k)dt \tag{51}$$

$$= \int \left[\delta k \left(\dot{r} - \frac{\partial f}{\partial k}\right) - \delta r \left(\dot{k} + \frac{\partial f}{\partial r}\right)\right] dt \tag{52}$$

For this to be true, both terms nee to be 0, this means that:

$$\dot{r} = \frac{\partial f}{\partial k} \tag{53}$$

$$\dot{k} = -\frac{\partial f}{\partial r} \tag{54}$$

This is our equation of motion. The first is also our group velocity $v_{\text{group}} = \frac{\partial \mathcal{H}}{\partial p}$ (compared to $v_{\text{phase}} = \frac{f}{k}$). We time the second equation by h:

$$\dot{p} = \frac{d\vec{p}}{dt} = -\frac{\partial \mathcal{H}}{\partial \vec{r}} = -\frac{\partial}{\partial \vec{r}}(T + U) = -\frac{\partial}{\partial \vec{r}}U = -\nabla U \equiv \vec{F} \tag{55}$$

This gives us $\vec{F} = \frac{d\vec{p}}{dt}$ which is Newton's 2nd law. Using this analysis, we can write:

$$\frac{df}{dt} = \frac{\partial f}{\partial r}\frac{dr}{dt} + \frac{\partial f}{\partial k}\frac{dk}{dt} + \frac{\partial f}{\partial t} \tag{56}$$

$$= \frac{\partial f}{\partial r}\frac{\partial f}{\partial k} - \frac{\partial f}{\partial k}\frac{\partial f}{\partial r} + \frac{\partial f}{\partial t} = \frac{\partial f}{\partial t} \tag{57}$$

This tells us that if Hamiltonian is time independent, then energy is conserved. The equation $\dot{k} = -\frac{\partial f}{\partial r}$ is Noether's theorem. If f(i.e. $\mathcal{H}$) is independent of some coordinate q. then the "momentum conjugate to q is conserved:

$$\frac{\partial f}{\partial x} = 0 \to k_x, p_x \text{ is conserved} \tag{58}$$

$$\frac{\partial f}{\partial \theta} = 0 \to L \text{ is conserved} \tag{59}$$

We have recovered all of classical physics from our discussion of wave. We don't really need $\vec{p}, E, S$.... We can rewrite the energy momentum relation as follows:

$$f^2 - \vec{k}^2 = m^2 \tag{60}$$

For QM, we are much better off without them. The classical quantum mechanical uncertainty principle says:

$$\Delta x \Delta p \geq \frac{\hbar}{2} \tag{61}$$

$$\Delta t \Delta E \geq \frac{\hbar}{2} \tag{62}$$

where now we can write:

$$\Delta x \Delta k \geq \frac{1}{4\pi} \quad (k = \lambda) \tag{63}$$

$$\Delta t \Delta f \geq \frac{1}{4\pi} \tag{64}$$

III. LECTURE 3

Now we see that we don't need $E, p, S....$ Now for a few examples. The first example was the uncertainty principle which we did in the first lecture. The second example is the Schrodinger equation and momentum in quantum mechanics. The schrodinger equation:

$$-\hbar\frac{\partial\Psi}{\partial t} = \mathcal{H}\Psi \tag{65}$$

$$-i\hbar\frac{\partial\Psi}{\partial x} = p\Psi \tag{66}$$

where $\omega = 2\pi f = 2\pi/T$ amd $k = 2\pi/\lambda = 1\Theta$ depending on our conversion. Unit of ω is radians/s ($1/s$); the unit of f is cycles/s ($1/s$). In our standard notation, this could be a bit confusing. We can also write:

$$i\frac{\partial\Psi}{\partial t} = \hat{\omega}\Psi \tag{67}$$

$$-i\frac{\partial\Psi}{\partial x} = \hat{k}\Psi \quad \text{where} \quad k = \frac{2\pi}{\lambda} \tag{68}$$

We want to answer the question: How does the wavefunction change in time? In traditional units, the answer would be according to the Hamiltonian (momentum), or according to its frequency (spatial frequency) in the modern units.

Now the 3rd example. The plane wave in traditional units:

$$\Psi = \Psi_0 e^{\phi(\vec{p}\cdot r - \vec{E}t)/\hbar} \tag{69}$$

where in our new units (more concise and more physical):

$$\Psi = \Psi_0 e^{i(\vec{k}\vec{r} - \omega t)} \tag{70}$$

In Griffiths, the Fourier transform is defined:

$$\Psi(x) = \frac{1}{\sqrt{2\pi\hbar}}\int_{-\infty}^{\infty} e^{ipx/\hbar}\bar{\Psi}(p)dp \tag{71}$$

and

$$\Psi(x) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} e^{-ikx}\bar{\Psi}(k)dk \tag{72}$$

where $k = \frac{2\pi}{\lambda}$ or we can write

$$\Psi(x) = \int_{-\infty}^{\infty} e^{2\pi ik\cdot x}\bar{\Psi}(k)dk \tag{73}$$

where $k = \frac{1}{\lambda}$.

In some sense, the Fourier transform in Dirac notation:

$$\langle p|x\rangle = \frac{1}{\sqrt{2\pi\hbar}} e^{-ipx/\hbar} \tag{74}$$

and in new units:

$$\langle k|x\rangle = e^{-2\pi i k\cdot x} \tag{75}$$

there are many more examples (9n chapter 6, time evolution, translations, boost, rotation operators...) Before we can use E, p we must convert it out of traditional units. Using 2 line summary, we have two defined constants: c relates to x and t; while h relates to x, t to m. While the numerical meaning of c, it does indicates that we have a limiting velocity. Also, we don't have a physical distinction between E, f or $\vec{p}, \vec{k}$.

We now move to QM and talk about the postulates of QM (ch3).

1. pure states are described by vectors(rays) $|\Psi\rangle$ in Hilbert space.

2. every observable is described by an operator A acting in this space

3. Measurements of the observable only return eigenvalues of A.

4. Principle of spectral decomposition: $A|u_n\rangle = a_n|u_n\rangle$ (abbrv to $A_0|n\rangle = n|n\rangle$) and $p(n) = |\langle n|\Psi\rangle|^2$. In the discretw, non-degenerate case: $|\Psi\rangle = \sum c_n|n\rangle$ and $P(n) = |c_n|^2$. In the Dirac notation,

$$|\Psi\rangle = \mathbb{1}|\Psi\rangle = (\sum_n |n\rangle\langle n|)|\Psi\rangle = \sum_n |n\rangle\langle n|\Psi\rangle = \sum_n |n\rangle c_n \tag{76}$$

. This is called closure or "completeness". There are some "obvious" extensions, if we have some degeneracy:

$$P(n) = \sum_{i=1}^{g_n} |\langle n_i|\Psi\rangle|^2 \tag{77}$$

where g_n is the degree of degeneracy and the $|n_i\rangle$ are the g_n orthonormal vectors spanning the subspace with eigenvalue n.

If our basis are continuous:

$$dP(\alpha) = |\langle\alpha|\Psi\rangle|^2 d\alpha \tag{78}$$

is the probability of finding a result in $[\alpha, \alpha + d\alpha]$, where $|\alpha\rangle$ is the eigenvector corresponding to the eigenvalue α

5. If measuring A on $|\Psi\rangle$ gives n, the state of the system immediately after the measurement is the normalized projection: $\frac{P_n|\Psi\rangle}{\sqrt{\langle\Psi|P-n|\psi\rangle}}$. For the discrete, non-deg case, our projection operator is $p_n = |n\rangle\langle n|$ and $\frac{|n\rangle\langle n|\Psi\rangle}{\sqrt{\langle\Psi|n\rangle\langle n|\Psi\rangle}} = |n\rangle$

6. The time evolution of $|\Psi\rangle$ is governed by the Schrodinger equation:

$$i\hbar\frac{d}{dt}|\Psi(t)\rangle = \mathcal{H}(t)|\Psi(t)\rangle \tag{79}$$

or:

$$i\frac{d}{dt}|\Psi(t)\rangle = \hat{\omega}(t)|\Psi(t)\rangle \tag{80}$$

To summarize:

1. States $|\Psi\rangle$ are in Hilber space
2. Observables are Hermitian operators
3. Measurements give eigenvalues
4. spectral decomposition
5. measurements project (wavefunction collapse)
6. time evolution (Schrodinger equation)

Previously, we wrote $|\Psi\rangle = \sum_n c_n|n\rangle$. We can consider the result of a measurement:

$$\langle n|\Psi\rangle = \langle n|\sum_m c_m|m\rangle \tag{81}$$

$$= \sum_m c_m\langle n|m\rangle \tag{82}$$

$$= \sum_m c_m\delta_{nm} = c_n \tag{83}$$

or:

$$\langle n|\Psi\rangle = \langle n|\mathbb{1}|\Psi\rangle \tag{84}$$

$$= \langle n|\sum_m |m\rangle\langle m|\Psi\rangle \tag{85}$$

A better way to understand the Dirac notation is to see the analogy to regular vectors in linear algebra. All

Dirac Notation	regular vectors
Ket $\|\Psi\rangle$	Column $\vec{a}$
bra $\langle\Psi\|$	Row $\vec{b}$
Inner product $\langle\psi\|\phi\rangle$	$\vec{b}\cdot\vec{a}$
Operator (time evolution $e^{-i\mathcal{H}t/\hbar}$)	rotation R
Matrix element $\langle\phi\|e^{-i\mathcal{H}t/\hbar}\|\psi\rangle$ (time evolve $\|\Psi\rangle$ and then project on $\|\Psi\rangle$)	$\vec{b}R\vec{a}$ (rotate $\vec{a}$ and then project on $\vec{b}$)

these statements are independent of basis. We can pick a specific coordinate system:

$$\begin{pmatrix} bx & b_y & b_z \end{pmatrix} \begin{pmatrix} \cos\theta & \sin\theta & 0 \\ -\sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} a_x \\ a_y \\ a_z \end{pmatrix} \tag{86}$$

Then we have:

$$\vec{a} = (\vec{a}\cdot\hat{x})\hat{x} + (\vec{a}\cdot\hat{y})\hat{y} + (\vec{z}\cdot\hat{z})\hat{z} \tag{87}$$

In Dirac Notation:

$$\mathbb{1}|\Psi\rangle = \sum |n\rangle\langle n|\Psi\rangle \tag{88}$$

IV. LECTURE 4

Completeness means that our basis vectors span the space:

$$\mathbb{1} = \sum |n\rangle\langle n| \tag{89}$$

We define:

$$\hat{x} = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} \quad \hat{y} = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} \quad \hat{z} = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} \quad \rightarrow \quad \text{with regular vectors} \tag{90}$$

We have:

$$\mathbb{1} = \hat{x}\hat{x} + \hat{y}\hat{y} + \hat{z}\hat{z} \tag{91}$$

$$= \begin{pmatrix}1\\0\\0\end{pmatrix}\begin{pmatrix}1 & 0 & 0\end{pmatrix} + \begin{pmatrix}0\\1\\0\end{pmatrix}\begin{pmatrix}0 & 1 & 0\end{pmatrix} + \begin{pmatrix}0\\0\\1\end{pmatrix}\begin{pmatrix}0 & 0 & 1\end{pmatrix} \tag{92}$$

$$= \begin{pmatrix}1 & 0 & 0\\0 & 0 & 0\\0 & 0 & 0\end{pmatrix} + \begin{pmatrix}0 & 0 & 0\\0 & 1 & 0\\0 & 0 & 0\end{pmatrix} + \begin{pmatrix}0 & 0 & 0\\0 & 0 & 0\\0 & 0 & 1\end{pmatrix} = \begin{pmatrix}0 & 0 & 0\\0 & 0 & 0\\0 & 0 & 1\end{pmatrix} \tag{93}$$

$$= \mathbb{1} \tag{94}$$

$$\vec{a} = \mathbb{1}\vec{a} = \mathbb{1}\mathbb{1}\vec{a} \tag{95}$$

$$= \sum_{j=x,y,z} \hat{j}(\hat{j}\cdot\vec{a}) = \sum_{i,j} \hat{i}(\hat{i}\hat{j})(\hat{j}\vec{a}) \tag{96}$$

and

$$\sum_{i,j} \hat{i}\cdot\hat{j} = \begin{pmatrix}\hat{x}\hat{x}' & \hat{x}\hat{y}' & \hat{x}\hat{z}'\\ \hat{y}\hat{x}' & \hat{y}\hat{y}' & \hat{y}\hat{z}'\\ \hat{z}\hat{x}' & \hat{z}\hat{y}' & \hat{z}\hat{z}'\end{pmatrix} \tag{97}$$

we have:

$$\begin{pmatrix}a_x\\a_y\\a_z\end{pmatrix} = \begin{pmatrix}\hat{x}\hat{x}' & \hat{x}\hat{y}' & \hat{x}\hat{z}'\\ \hat{y}\hat{x}' & \hat{y}\hat{y}' & \hat{y}\hat{z}'\\ \hat{z}\hat{x}' & \hat{z}\hat{y}' & \hat{z}\hat{z}'\end{pmatrix}\begin{pmatrix}a'_x\\a'_y\\a'_z\end{pmatrix} = \begin{pmatrix}\cos\theta & \sin\theta & 0\\ -\sin\theta & \cos\theta & 0\\ 0 & 0 & 1\end{pmatrix} \tag{98}$$

Quantum mechanics problems are often just a change of basis. It might be easy to write $V(x)$. but easier to solve it in $\psi(\bar{k})$. One example is the infinite square well: Suppose:

$$\psi(x) = \begin{cases} A & 0 < x < \frac{L}{2} \\ 0 & \text{else} \end{cases}$$

We normalize in the Dirac notation:

$$\langle\psi|\psi\rangle = 1 = \langle\psi|\mathbb{1}|\psi\rangle = \langle\psi|\int dx|x\rangle\langle x||\psi\rangle \tag{99}$$

$$= \int dx\langle\psi|x\rangle\langle x|\psi\rangle = \int dx\psi^*(x)\psi(x) \tag{100}$$

$$= \int_0^{L/2} dx A^* A = A^2\frac{L}{2} \to A = \sqrt{\frac{2}{L}} \tag{101}$$

Let's say we want this state $|\psi\rangle$ in the $|k\rangle$ representation:

$$\bar{\psi(k)} = \langle k|\psi\rangle = \langle k|\mathbb{1}|\psi\rangle \tag{102}$$

$$= \int dx \langle k|x\rangle\langle x|\psi\rangle = \int_0^{L/2} e^{-2\pi ikx}\sqrt{\frac{2}{L}}\psi(x)dx \quad \text{where we pick} \quad k = \frac{1}{\lambda} \tag{103}$$

$$= \sqrt{\frac{2}{L}}\frac{1}{-2\pi ik}e^{-2\pi ikx}|_0^{L/2} \tag{104}$$

$$= \sqrt{\frac{2}{L}}\frac{1}{2\pi ik}(-e^{-2\pi ik\frac{L}{2}}+1) = \sqrt{\frac{2}{L}}\frac{e^{-2\pi ik\frac{L}{4}}}{2\pi ik}(e^{2\pi ik\frac{L}{4}} - e^{-2\pi ik\frac{L}{4}}) \tag{105}$$

$$= \sqrt{\frac{2}{L}}\frac{e^{-2\pi ik\frac{L}{4}}}{2\pi ikL/4}\frac{L}{4}\sin\left(\frac{2\pi kL}{4}\right) \tag{106}$$

$$= \sqrt{\frac{2}{L}}e^{-2\pi ik\frac{L}{4}}\,\text{sinc}\left(\frac{2\pi kL}{4}\right) \quad \text{where} \quad \text{sinc}(x) = \frac{\sin x}{4} \tag{107}$$

We can see that the normalization is the inverse of our previous normalization. This is familiar from single slit diffraction (narrow in real space $\rightarrow$ wide reciprocal space). The second term (exponential) is a phase factor. (in Chapter 6, we will see that it is a translation operator that translates by $\frac{L}{4}$)

Now we have two bases $\psi(x), \bar{\psi(k)}$. We shall move to a 3rd basis (we are using the short hand $|\psi_n\rangle = |n\rangle$):

$$c_n = \langle n|\psi\rangle = \langle n|\mathbb{1}|\psi\rangle = \langle n|\int dx|x\rangle\langle x||\psi\rangle \tag{108}$$

$$= \int dx\langle n|x\rangle\langle x|\psi\rangle \tag{109}$$

$$= \int dx\varphi_n^*(x)\psi(x) \quad \text{where} \quad \varphi_n(x) = \frac{2}{L}\sin\left(\frac{n\pi x}{L}\right),\ n = 1, 2, ... \tag{110}$$

$$= \int_0^{L/2} dx\sqrt{\frac{2}{L}}\sin\left(\frac{n\pi x}{L}\right) = \frac{2}{L}\frac{-L}{n\pi}\cos\left(\frac{n\pi x}{L}\right)|_0^{L/2} = \frac{2}{n\pi}\left(1-\cos\frac{n\pi}{2}\right) \tag{111}$$

So we have:

$$c_1 = \frac{2}{\pi} \quad c_2 = \frac{2}{\pi}(1-\cos\pi) = \frac{2}{\pi} \quad c_3 = \frac{2}{3\pi} \quad c_4 = 0, \quad \text{and} \quad \sum_n^\infty |c_n|^2 = 1 \tag{112}$$

We see that the completeness relation holds.

We can describe our state in 3 different ways:

$$|\psi\rangle = \mathbb{1}|\psi\rangle = \int dx|x\rangle\langle x||\psi\rangle = \int_0^{L/2} dx|x\rangle\sqrt{\frac{2}{L}} \tag{113}$$

$$= \int dk|k\rangle\langle k|\psi\rangle = \int_{-\infty}^{\infty} dk|k\rangle\sqrt{\frac{L}{2}}e^{-\frac{2\pi ikL}{4}}\,\text{sinc}\left(\frac{2\pi kL}{4}\right) \tag{114}$$

$$= \sum|\varphi_n\rangle\langle\varphi_n|\psi\rangle = \frac{2}{\pi}|1\rangle + \frac{2}{\pi}|2\rangle + \frac{2}{3\pi}|3\rangle + ... \tag{115}$$

with

$$\langle x|\varphi_n\rangle = \sqrt{\frac{2}{L}}\sin\left(\frac{n\pi x}{L}\right) \quad \langle x|\psi\rangle = \frac{2}{\pi}\sqrt{\frac{2}{L}}\left[\frac{\sin\pi x}{L} + \frac{\sin 2\pi x}{L} + \frac{\sin 3\pi x}{L} + ...\right] \tag{116}$$

To summarize:

$$\mathbb{1} = \int dx|x\rangle\langle x| = \int dk|k\rangle\langle k| = \int dp|p\rangle\langle p| = \sum|\varphi_n\rangle\langle\varphi_n| \tag{117}$$

and these are the energy eigenstates for a bound state.

$$|\psi\rangle = \int dx|x\rangle\langle x|\psi\rangle \tag{118}$$

$$= \int dk|k\rangle\langle k|\psi\rangle \tag{119}$$

$$= \int dp|p\rangle\langle p|\psi\rangle \tag{120}$$

$$= \sum_n |\varphi_n\rangle\langle\varphi_n|\psi\rangle \tag{121}$$

and:

$$c_n = \langle\varphi_n|\psi\rangle \tag{122}$$

$$= \langle\varphi|\mathbb{1}|\psi\rangle \tag{123}$$

$$= \int dx\langle\varphi_n|x\rangle\langle x|\psi\rangle \tag{124}$$

$$= \int dx\varphi_n^*(x)\psi(x)... \tag{125}$$

This should be familiar from chapter 2 of Griffiths Couple identities:

$$|\psi\rangle = \mathbb{1}\mathbb{1}|\psi\rangle \tag{126}$$

$$\mathbb{1}|\psi\rangle = \int dx|x\rangle\langle x|\psi\rangle \tag{127}$$

$$\mathbb{1}\mathbb{1}|\psi\rangle = \int dx'|x'\rangle\langle x'|\int dx|x\rangle\langle x|\psi\rangle \tag{128}$$

$$= \int dx'dx|x'\rangle\langle x'|x\rangle\langle x|\psi\rangle \tag{129}$$

$$\langle x'|x\rangle = \delta(k-k') \tag{130}$$

$$\langle x'|x\rangle = \delta(k-k') \tag{131}$$

$$\langle n'|n\rangle = \langle\phi_{n'}|\psi_n\rangle = \delta_{n'n} \tag{132}$$

For infinite square well:

$$\langle n|x\rangle = \left(\sqrt{\frac{2}{L}}\sin\left(\frac{n\pi x}{2}\right)\right)^* \tag{133}$$

$$\langle x|n\rangle = \sqrt{\frac{2}{L}}\sin\left(\frac{n\pi x}{2}\right) \tag{134}$$

Other interesting cases

$$\langle\psi|\psi\rangle = \int dx\langle\psi|x\rangle\langle x|\psi\rangle \quad \text{and} \quad \langle p|x\rangle = \frac{1}{\sqrt{2\pi\hbar}}e^{-2\pi ipx/\hbar} \tag{135}$$

We attempt to write $|\psi\rangle = |x\rangle$:

$$|x\rangle = \mathbb{1}|x\rangle = \int dx'|x'\rangle\langle x'|x\rangle \tag{136}$$

$$= \int dx'|x'\rangle\delta(x-x') \tag{137}$$

$$\psi(x) = \langle x'|x\rangle = \delta(x-x') \tag{138}$$

V. LECTURE 5

: On Monday, we were talking about some useful identities, such as:

$$\langle x'|x\rangle = \delta(x-x') \tag{139}$$

$$\langle k'|k\rangle = \delta(k-k') \tag{140}$$

One thing we didn't get to is that:

$$|\psi\rangle = \mathbb{1}\mathbb{1}|\psi\rangle = \mathbb{1}|\psi\rangle \tag{141}$$

$$= \int dp|p\rangle\langle p|\int dx|x\rangle\langle x|\psi\rangle \tag{142}$$

$$= \int\int dpdx|p\rangle\langle p|x\rangle\psi(x) \tag{143}$$

$$= \cap dp|p\rangle\langle p|\psi\rangle = \int dp|p\rangle\bar{\psi}(p) \tag{144}$$

If we compare this with the definition of the Fourier transform, it basically tells us:

$$\langle p|x\rangle = \frac{1}{\sqrt{2\pi\hbar}}e^{-\frac{ipx}{\hbar}} \tag{145}$$

$$\langle k|x\rangle = \frac{1}{\sqrt{2\pi}}e^{-ikx} \quad \text{where} \quad k = \frac{2\pi}{\lambda} \tag{146}$$

$$\langle k|x\rangle = e^{-i2\pi kx} \quad \text{where} \quad k = \frac{1}{\lambda} \tag{147}$$

This is just elegant notation, nothing deep mathematically here. Now, we look at other Dirac notations that are essential. We need to know how to form the adjoint, or the Hermitian conjugate.

1. $c \to c^*$
2. $|\varphi\rangle \to \langle\varphi|$
3. $\langle\psi| \to \ker\psi$
4. $A \to A^\dagger \quad (A^\dagger = (A^T)^* = (A^*)^T)$
5. Reverse the other of the factors

$$(\lambda\langle u|A|v\rangle|\omega\rangle|\psi\rangle)^\dagger = |\psi\rangle\langle\psi|\langle v|A^\dagger|u\rangle\lambda^* \tag{148}$$

$$= \lambda^*\langle v|A^\dagger|u\rangle|\psi\rangle\langle\omega| \tag{149}$$

Say we have:

$$A|\psi\rangle = \lambda|\psi\rangle \tag{150}$$

where we call A the operator, the $|\psi\rangle$ is the eigenket, λ is the eigenvalue (maybe complex). Now say $A = A^\dagger$ (meaning that A is Hermitian). Then, the expectation value of A is given:

$$A = \langle\psi|A|\psi\rangle = \langle\psi|\lambda|\psi\rangle = \lambda\langle\psi|\psi\rangle = \lambda \tag{151}$$

Then:

$$(\langle\psi|A|\psi\rangle)^\dagger = \langle\psi|A^\dagger|\psi\rangle = \langle\psi|A|\psi\rangle = \lambda = \lambda^* \tag{152}$$

so we have $(\lambda)^\dagger = \lambda^*$, for $A = A^\dagger$. Thus, the eigenvalues λ must be real. So, we require that measurements return real results $\to$ operators corresponding to observables must be Hermitian. We then follows that:

$$A|\psi\rangle = \lambda|\psi\rangle \tag{153}$$

$$\langle\psi|A^\dagger = |\psi\rangle A = |\psi\rangle\lambda^* = \langle\psi|\lambda \tag{154}$$

$$|\psi\rangle A = |\psi\rangle\lambda \tag{155}$$

$$\langle\psi|A|\psi\rangle = \langle\psi|\lambda|\psi\rangle = \lambda\langle\psi|\psi\rangle \tag{156}$$

This is true for any ket $|\varphi\rangle$, no necessarily an eigenket. These are groundwork for an awesome theorem: Twp eigenvectors of a Hermitian operator corresponding to different eigenvalues are orthogonal. Say we have:

$$A|\psi\rangle = \lambda|\psi\rangle \tag{157}$$

$$A|\varphi\rangle = \mu|\varphi\rangle \tag{158}$$

This gives:

$$\langle\psi|A|\varphi\rangle = \mu\langle\psi|\varphi\rangle \tag{159}$$

$$\langle\psi|A|\varphi\rangle = 1\langle\psi|\varphi\rangle \tag{160}$$

$$0 = (\mu - \lambda)\langle\psi|\mathrm{var}\phi\rangle \tag{161}$$

So if $\mu \neq \lambda$, then we have $\langle\psi|\varphi\rangle = 0$. ($|\psi\rangle$ and $|\varphi\rangle$ are orthogonal). Thus, we can see that that the momentum eigenstates with different momenta are orthogonal. The inifinite square well Hamiltonian eigenstates with different energies are orthogonal. (same with the Harmonic oscillator)

Now we start Chapter 4 of Griffiths. (QM in 3D). We start with our Schrodinger equation:

$$i\hbar = \frac{\partial|\psi\rangle}{\partial t} = \mathcal{H}|\psi\rangle \tag{162}$$

or better:

$$-\frac{\partial}{\partial t}|\psi\rangle = \hat{\omega}|\psi\rangle \tag{163}$$

Here we have

$$p_i \longrightarrow \frac{\hbar}{i}\frac{\partial}{\partial x} \quad \text{or} \quad p \longrightarrow \frac{\hbar}{i}\nabla \tag{164}$$

For now, let's say that ψ is scalar. Recall where $\mathcal{H}$ comes from. We set $c = 1$:

$$E^2 - p^2 = m^2 \tag{165}$$

and

$$E = \sqrt{m^2 + p^2} = m\sqrt{1 + \left(\frac{p}{m}^2\right)} = m\left(1 + \frac{1}{2}\left(\frac{p}{m}\right)^2 + ...\right) \tag{166}$$

We ignore the first term which is the constant and we add a potential:

$$\mathcal{H} = \frac{\vec{p}^2}{2m} + V(\vec{r}) \tag{167}$$

We know that:

$$\vec{p}^2 = \vec{p} \cdot \vec{p} = -\hbar^2 \nabla^2 \tag{168}$$

where the Laplacian is:

$$\nabla^2 \equiv \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2} \tag{169}$$

So:

$$i\hbar \frac{\partial \psi}{\partial t} = \mathcal{H}\psi = \left(-\frac{\hbar^2}{2m}\nabla^2 + V(\vec{r})\right)|\psi\rangle \tag{170}$$

Thus, closure in 3D is different:

$$1D: \quad \mathbb{1} = \int dx |x\rangle\langle x| \tag{171}$$

$$3D: \quad \mathbb{1} = \int d^3r |\vec{r}\rangle\langle\vec{r}| \tag{172}$$

Normalization is:

$$1 = \langle\psi|\psi\rangle = \langle\psi|\mathbb{1}|\psi\rangle \tag{173}$$

$$= \langle\psi| \int d^3\vec{r} |\vec{r}\rangle\langle\vec{r}|\psi\rangle \tag{174}$$

$$= \int d^3\vec{r} \langle\psi|\vec{r}\rangle\langle\vec{r}|\psi\rangle \tag{175}$$

$$= \int d^3\vec{r} \psi^*(\vec{r})\psi(\vec{r}) = 1 \tag{176}$$

Sometimes we can use separation of variables and that is the case here:

$$\Psi(x, y, z, t) = A(x)B(y)C(z)D(t) \tag{177}$$

Then we plug this guess into the Shrodinger equation:

$$i\hbar ABC\dot{D} = -\frac{\hbar^2}{2m}(\ddot{A}BCD + A\ddot{B}CD + AB\ddot{C}D) + V(\vec{r})ABCD \tag{178}$$

Then, divided by Ψ:

$$i\hbar \frac{\dot{D}}{D} = -\frac{\hbar}{2m}\left(\frac{\ddot{A}}{A} + \frac{\ddot{B}}{B} + \frac{\ddot{C}}{C}\right) = E \text{ (constant)} \tag{179}$$

Say we have an infinite square well in 3D:

$$V(\vec{r}) \begin{cases} 0 & \text{for } x, y, z \text{ in} [0, L] \\ \infty & \text{else} \end{cases}$$

Starting from the Shrodinger equation:

$$i\hbar\frac{\dot{D}}{D} = -\frac{\hbar^2}{2m}\left(\frac{\ddot{A}}{A}+\frac{\ddot{B}}{B}+\frac{\ddot{C}}{C}\right)+V(\vec{r}) \tag{180}$$

we have:

$$D = e^{-i\omega t} \tag{181}$$

with $E \equiv \hbar\omega$. Then:

$$-\frac{\hbar^2}{2m}(k_x^2+k_y^2+k_z^2) \equiv E \tag{182}$$

$$A = \sqrt{\frac{2}{L_x}}\sin\left(\frac{n_x\pi x}{2x}\right) \quad \text{with} \quad n_x = 1,2,3,...\text{and} \quad K_x = \frac{n_x\pi}{L_x} \tag{183}$$

$$B = \sqrt{\frac{2}{L_y}}\sin\left(\frac{n_y\pi y}{2y}\right) \quad \text{with} \quad n_y = 1,2,3,...\text{and} \quad K_y = \frac{n_y\pi}{L_y} \tag{184}$$

$$C = \sqrt{\frac{2}{L_z}}\sin\left(\frac{n_z\pi z}{2z}\right) \quad \text{with} \quad n_z = 1,2,3,...\text{and} \quad K_z = \frac{n_z\pi}{L_z} \tag{185}$$

For our next important case:

$$V(\vec{r}) = V(|\vec{r}|) = V(r) \tag{186}$$

For example, the Hydrogen atom. The Laplacian in the spherical coordinate:

$$\nabla^2 = \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial}{\partial r}\right)+\frac{1}{r^2}\left[\frac{1}{\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial}{\partial\theta}\right)+\frac{1}{\sin^2\theta}\frac{\partial^2}{\partial\phi^2}\right] \tag{187}$$

We again assume separability:

$$\Psi(x,y,z,t) = R(r)Y(\theta,\phi)D(t) \quad \text{where} \quad D = e^{-i\omega t} \tag{188}$$

The time-independent Shrodinger equation:

$$\mathcal{H}\psi = E\psi \quad \text{where} \quad \psi = RY \tag{189}$$

VI. LECTURE 6

We are looking at the special case that $V(\vec{r}) = V(r)$. We also assumed separability as usual. In this case, $\Psi = \psi(r)\phi(t)$ and $\phi(t) = e^{-i\omega t}$ as usual. This leaves us the time-independent Schrodinger equation:

$$\mathcal{H}\psi = E\psi \quad \text{where } E \text{ is the separation constant} \tag{190}$$

Our Hamiltonian:

$$\mathcal{H} = \frac{p^2}{2m} + V(r) \tag{191}$$

and the Schrodinger equation:

$$\frac{p^2}{2m}\psi = (E - V(r))\psi \tag{192}$$

We use separation of variables again:

$$\psi = R(r)Y(\theta,\phi) \quad \text{using} \quad \nabla^2 = \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial}{\partial r}\right) + \frac{1}{r^2}\left[\frac{1}{\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial}{\partial\theta}\right) + \frac{1}{\sin^2\theta}\frac{\partial^2}{\partial\phi^2}\right] \tag{193}$$

Putting this all together, we have:

$$-\frac{\hbar^2}{2m}\left(\frac{1}{R}\frac{1}{r^2}\frac{\partial}{\partial r}r^2\frac{\partial R}{\partial r} + \frac{1}{Y}\frac{1}{r^2}\left[\frac{1}{\sin\theta}\frac{\partial Y}{\partial\theta} + \frac{1}{\sin^2\theta}\frac{\partial^2 Y}{\partial\phi^2}\right]\right) = E - V(r) \tag{194}$$

Then, we times $\left(-\frac{2m}{\hbar^2}r^2\right)$ on both sides:

$$\frac{1}{R}\frac{\partial}{\partial r}r^2\frac{\partial R}{\partial r} + \frac{1}{Y}\left[\frac{1}{\sin\theta}\frac{\partial}{\partial\theta}\frac{\partial Y}{\partial\theta} + \frac{1}{\sin^2\theta}\frac{\partial^2 Y}{\partial\phi^2}\right] = \left(-\frac{2m}{\hbar^2}r^2\right)(E - V(r)) \tag{195}$$

$$\frac{1}{R}\frac{\partial}{\partial r}r^2\frac{\partial R}{\partial r} + \left(+\frac{2m}{\hbar^2}r^2\right)(E - V(r)) = -\frac{1}{Y}\left[\frac{1}{\sin\theta}\frac{\partial}{\partial\theta}\frac{\partial Y}{\partial\theta} + \frac{1}{\sin^2\theta}\frac{\partial^2 Y}{\partial\phi^2}\right] \tag{196}$$

On the left side, we have a function of r only, and on the right, we have a function of (θ,ϕ) only. We set this quantity to be constant and call it $l(l+1)$ in anticipation of future results.

We are going to solve the angular equation first. We use the separation of variable again:

$$Y(\theta,\phi) = \Theta(\theta)\Psi(\phi) \tag{197}$$

$$= A(\theta)B(\phi) \tag{198}$$

Then,

$$\frac{1}{Y}\left[\frac{1}{\sin\theta}\frac{\partial}{\partial\theta}\frac{\partial Y}{\partial\theta} + \frac{1}{\sin^2\theta}\frac{\partial^2 Y}{\partial\phi^2}\right] = -l(l+1)Y \;\rightarrow\; Y = AB \tag{199}$$

$$\frac{1}{A}\frac{1}{\sin\theta}\frac{\partial}{\partial\theta}\sin\dot{A} + \frac{1}{\sin^2\theta}\frac{\ddot{B}}{B} = -l(l+1) \tag{200}$$

We take the last equation and times $\sin^2\theta$:

$$\frac{1}{A}\sin\theta\frac{\partial}{\partial\theta}\sin\theta\dot{A} + l(l+1)\sin^2\theta = -\frac{\ddot{B}}{B} = \text{constant} = m^2 \tag{201}$$

So the left hand side, we have a function of θ only, and the right hadn side a function of ϕ only. We set this constant to be constant and call it m^2 (not mass!!). So we have:

$$-\ddot{B} = Bm^2 \tag{202}$$

the solution is then:

$$B = Ae^{im\phi} + B'e^{-im\phi} \tag{203}$$

$$= A\sin m\phi + B'\cos m\phi \tag{204}$$

This reduce to $B = e^{im\phi}$ (Griffiths says m can be $+$ or $-$).

$$\frac{1}{A}\frac{1}{\sin\theta}\frac{\partial}{\partial\theta}\sin\dot{A} + l(l+1)\frac{1}{\sin^2\theta} = m^2 \tag{205}$$

$$\frac{1}{\sin\theta}\frac{\partial}{\partial\theta}\sin\dot{A} + (l(l+1)\frac{1}{\sin^2\theta} - m^2)A = 0 \tag{206}$$

The solutions are:

$$A(\theta) = A'P_l^m(\cos\theta) \tag{207}$$

where:

$$P_l^m(x) \equiv (-1)^m(1-x^2)^{m/2}\left(\frac{\partial}{\partial x}\right)^m P_l(x) \tag{208}$$

This is called the associated Legendre function. Here:

$$P_l(x) \equiv \frac{1}{2^l l!}\left(\frac{\partial}{\partial x}\right)^l (x^2-1)^l \tag{209}$$

is the Legendre polynomial. Now, we can write our total solution:

$$Y(\theta,\phi) = A(\theta)B(\phi) \tag{210}$$

and the so called shperical harmonics are:

$$Y_l^m(\theta,\phi) = \sqrt{\frac{2l+1}{4\pi}\frac{(l-m)!}{(l+m)!}}e^{im\phi}P_l^m(\cos\theta) \tag{211}$$

The Y_l^m are normalized and orthogonal, or orthonormal. That is, if we integrate it over a sphere, it gives 0 or 1:

$$\int_0^{2\pi}\int_0^{\pi} d\phi\sin\theta d\theta\,[Y_l^m(\cos\theta)]^*\,Y_{l'}^{m'}(\cos\theta) = \delta_{ll'}\delta_{mm'} \tag{212}$$

This comes up in QM and also in the cosmic microwave background. That solves the angular part of the problem (the radial part $V(r)$ has not come in at all).

Now we look at the radial part:

$$\frac{\partial}{\partial r}r^2\frac{\partial R}{\partial r} + R\left(\frac{2m}{\hbar^2}r^2\right)(E - V(r)) = l(l+1)R \tag{213}$$

We apply change of variables $u = rR$, $R = \frac{u}{r}$, and $\frac{\partial R}{\partial r} = \frac{1}{r}\frac{\partial u}{\partial r} - \frac{u}{r^2} = \frac{1}{r^2}\left(r\frac{\partial u}{\partial r} - u\right)$. Then:

$$\frac{\partial}{\partial r} r^2 \frac{\partial R}{\partial r} = \frac{\partial}{\partial r}\left(r\frac{\partial u}{\partial r} - u\right) \tag{214}$$

$$= \frac{\partial u}{\partial r} + r\frac{\partial^2 u}{\partial r^2} - \frac{\partial u}{\partial r} \tag{215}$$

$$= r\frac{\partial^2 u}{\partial r^2} \tag{216}$$

Then:

$$r\frac{\partial^2 u}{\partial r^2} + \frac{2m}{\hbar} ru(E - V(r)) = l(l+1)\frac{u}{r} \tag{217}$$

$$-\frac{\hbar^2}{2m}\frac{d^2 u}{dr^2} + \left[V(r) + \frac{\hbar^2}{2m}\frac{l(l+1)}{r^2}\right] u = Eu \tag{218}$$

This last equation is called the radial equation, it looks just like our time independent Schrodinger equation $\mathcal{H}\psi = E\psi$ for $\mathcal{H} = \frac{p^2}{2m} + V(x)$. Here:

$$V_{\text{eff}} = V(r) + \frac{\hbar^2}{2m}\frac{l(l+1)}{r^2} \tag{219}$$

The second term is our new centrifugal term. This term being positive means it's repulsive. Still, we haven't said anything about $V(r)$. Some of the possibilities include

1. Infinite square well. For $l = 0$, it's just the square well. For $l \neq 0$, we have special function.
2. Finite Square Well
3. Hydrogen Atom

VII. LECTURE 7

From the previous lecture, we arrived at the radial equation:

$$-\frac{\hbar^2}{2m}\frac{d^2 u}{dr^2} + \left[V(r) + \frac{\hbar^2}{2m}\frac{l(l+1)}{r^2}\right] u = Eu \tag{220}$$

The Hydrogen atom has potential:

$$v(r) = -\frac{e^2}{4\pi\epsilon_0}\frac{1}{r} \tag{221}$$

We know that potential is an energy which we can think as a frequency (i.e. inverse time or distance). The $\frac{1}{r}$ part has unit 1/distance. Thus, the middle part $\frac{e^2}{4\pi\epsilon_0}$ should be dimensionless. Here, the $1/4\pi$ suggests spherical symmetry. (This appears here because the MKS system is rationalized). In the MKS system, the capacitance takes form:

$$C = \frac{A\epsilon\epsilon}{d} \tag{222}$$

There is no 2π because this is in cartesian. Here, we also have:

$$\epsilon \to 8.85 \times 10^{-12} F/s \tag{223}$$

This is completely a historical legacy. In fact, it would be nice if capacitance is to be measured in meters. Back to the original equation, the e^2 is where the physics is. We have:

$$\frac{e^2}{4\pi\epsilon_0 \hbar c} \equiv \alpha = \frac{1}{137} << 1 \tag{224}$$

where α is called the fine constant which is completely independent of the units. For the Hydrogen atom, we attribute the potential to the proton and assume that it is stationary. We plug it into the radial equation and solve it:

$$-\frac{\hbar^2}{2m}\frac{d^2u}{dr^2} + \left[-\frac{e^2}{4\pi\epsilon_0}\frac{1}{r} + \frac{\hbar^2}{2m}\frac{l(l+1)}{r^2}\right] u = Eu \tag{225}$$

We call this term in the middle:

$$\left[-\frac{e^2}{4\pi\epsilon_0}\frac{1}{r} + \frac{\hbar^2}{2m}\frac{l(l+1)}{r^2}\right] \tag{226}$$

the effective potential V_{eff}. We note that there is a gap called the centrifugal barrier(why planets don't fall into the sun). Now, we define:

$$\kappa \equiv = \frac{\sqrt{-2mE}}{\hbar} \tag{227}$$

where $\hbar\kappa = \sqrt{-2mE}$ is a momentum, in fact, it is the Bohr momentum (semi-classical momentum). We see that:

$$\frac{\hbar^2\kappa^2}{2m} = -E \tag{228}$$

Then:

$$\frac{E}{\kappa^2}\frac{d^2u}{dr^2} + \left[\frac{-e^2}{4\pi\epsilon_0 r} - \frac{E}{\kappa^2}\frac{l(l+1)}{r^2}\right]\kappa = Eu \tag{229}$$

$$\frac{1}{\kappa^2}\frac{d^2u}{dr^2} + \left[\frac{-e^2\kappa}{4\pi\epsilon_0 r E\kappa} - \frac{1}{\kappa^2}\frac{l(l+1)}{r^2}\right]\kappa = u \tag{230}$$

We define $\rho = \kappa r$:

$$\frac{d^2u}{d\rho^2} + \left[-1 - \frac{e^2\kappa}{4\pi\epsilon_0\rho} - \frac{1}{\kappa^2}\frac{l(l+1)}{r^2}\right] u = 0 \tag{231}$$

We define:

$$\rho_0 = -\frac{e^2\kappa}{4\pi\epsilon E} = \frac{e^2\kappa 2m}{4\pi\epsilon\hbar^2\kappa^2} = \frac{2\alpha mc}{\hbar\kappa} \tag{232}$$

Then,

$$\frac{d^2u}{d\rho^2} = \left[1 - \frac{\rho_0}{\rho} + \frac{l(l+1)}{\rho^2}\right] u \tag{233}$$

Now we review the Bohr Model: First, we write down the expression for Energy:

$$E = \frac{1}{2}mv^2 = \frac{e^2}{4\pi\epsilon r} \tag{234}$$

We then assume circular orbit:

$$\frac{mv^2}{r} = \frac{e^2}{4\pi\epsilon_0 r^2} \tag{235}$$

Then, we say that the angular momentum is quantized in the unit of $\hbar$:

$$mvr = n\hbar \tag{236}$$

We solve the three equations:

$$pr = n\hbar \tag{237}$$

$$\hbar kr = n\hbar \tag{238}$$

$$\frac{2\pi}{\lambda}r = n \tag{239}$$

so we have:

$$2\pi r = n\lambda \tag{240}$$

Another important theorem is the virial theorem, which states that:

$$U \propto r^n \rightarrow \langle T \rangle = \frac{n}{2}\langle u \rangle \tag{241}$$

In out case where $n = -1$, we haveL

$$\langle T \rangle = -\frac{1}{2}\langle u \rangle \tag{242}$$

Then,

$$E = \langle T \rangle + \langle u \rangle = -\frac{1}{2}\langle u \rangle + \langle u \rangle = \langle u \rangle / 2 \tag{243}$$

also:

$$E = \langle T \rangle - 2\langle T \rangle = -\langle T \rangle \tag{244}$$

We write down the results in Regan's favorite units:

$$E = -\frac{1}{2}\frac{\alpha^2 mc^2}{n^2} \tag{245}$$

$$r = \frac{n^2}{\alpha}\frac{\hbar c}{mc^2} \tag{246}$$

$$\frac{v}{c} = \frac{\alpha}{n} \tag{247}$$

where,

$$\hbar c = 197\text{eV} \cdot \text{nm} \tag{248}$$

$$mc^2 = 511 \times 10^3\text{eV} \tag{249}$$

$$\frac{e^2}{4\pi\epsilon_0} = 1.44\text{eV} \cdot \text{nm} \tag{250}$$

. So, we know that:

$$E, \frac{1}{r} \propto \frac{1}{n^2} \tag{251}$$

$$v, p \propto \frac{1}{n} \tag{252}$$

$$\lambda, L = rp \propto n \tag{253}$$

Note that:

$$\frac{v}{c} = \frac{\alpha}{n} \tag{254}$$

$$vn = c\alpha \tag{255}$$

$$mvn = mc\alpha \tag{256}$$

Earlier, we have:

$$\rho_0 = \frac{2\alpha mc}{\hbar\kappa} = \frac{2mvn}{\hbar\kappa} = \frac{2pn}{p} = 2n \tag{257}$$

In some sense, ρ_0 is a dimensionless number where $\rho = \kappa r$. Previously, we had arrived at:

$$\frac{d^2u}{d\rho^2} = \left[1 - \frac{\rho_0}{\rho} + \frac{l(l+1)}{\rho^2}\right] u \tag{258}$$

Now, we look at the asymptotic behavior where $\rho \to \infty$. We have:

$$\frac{d^2u}{dp^2} = u \tag{259}$$

The solution is given $u = Ae^{-\rho} + Be^{\rho}$ and the second term dissappear since it blow up as ρ foes to ∞. Thus, we have $u = Ae^{-\rho}$ for large ρ. Then, for $\rho \to 0$, we have $\frac{d^2u}{d\rho^2} = \frac{l(l+1)}{\rho^2}u$. The solution is given:

$$u = c\rho^{l+1} + D^{-l} \tag{260}$$

$$\dot{u} = c(l+1)\rho^{l} - Dl\rho^{-(l+1)} \tag{261}$$

$$\ddot{u} = cl(l+1)\rho^{l-1} + Dl(l+1)\rho^{-(l+2)} = \frac{l(l+1)u}{\rho^2} \tag{262}$$

So, we have $u = C\rho^{l+1}$ for small ρ. Now, we factor out the asympotic behavior:

$$u(\rho) = \rho^{l+1}e^{-\rho}V(\rho) \tag{263}$$

Then,

$$\dot{u} \equiv \frac{\partial u}{\partial \rho} = (l+1)\rho^{l}e^{-\rho}v + \rho^{l+1}(-1)e^{-\rho}v + \rho^{l+1}e^{-\rho}\dot{v} \tag{264}$$

$$= \rho^{l}e^{-\rho}[(l+1-\rho)v + \rho\dot{v}] \tag{265}$$

$$\ddot{u} = l\rho^{l-1}e^{-\rho}[\,] + \rho^{l}(-1)e^{-\rho}[\,] + \rho^{l}e^{-\rho}[-v + (l+1-\rho)\dot{v} + \dot{v} + \rho\ddot{v}] \tag{266}$$

$$= \rho^{l}e^{-\rho}\left(\left[\frac{l(l+1-\rho)v}{\rho} + l\dot{v}\right] - ((l+1-\rho)v + \rho\dot{v}) - v + (l+2-\rho)\dot{v} + \rho\ddot{v}\right) \tag{267}$$

$$= \rho^{l}e^{-\rho}\left(v\left(-2-2l+\rho+\frac{l(l+1)}{\rho}\right) + \dot{v}\,(2l+2-2\rho) + \rho\ddot{v}\right) \tag{268}$$

$$= \left[1 - \frac{\rho_0}{\rho} + \frac{l(l+1)}{\rho^2}\right]\rho^{l+1}e^{-\rho}v(\rho) \tag{269}$$

VIII. LECTURE 8

So, we have:

$$v(-2l-2+\rho+\frac{l(l+1)}{\rho}) + 2\dot{v}(l+1-\rho) + \rho\ddot{v} = \left[\rho - \rho_0 + \frac{l(l+1)}{\rho}\right]v \tag{270}$$

$$p\ddot{v} + 2(l+1-\rho)\dot{v} + (\rho_0 - 2(l+1))v = 0 \tag{271}$$

Then, we assume a power series for solution:

$$V(\rho) = \sum_{j=0}^{\infty} c_j\rho^j \tag{272}$$

Taking the derivative:

$$\dot{v} = \sum_{j=0} jc_j\rho^{j-1} = \sum_{k=-1}^{\infty}(k+1)c_{k+1}\rho^k = \sum_{k=0}^{\infty}(k+1)c_{k+1}\rho^k \tag{273}$$

Second derivative:

$$\ddot{v} = \sum_{k=0} k(k+1)c_{k+1}\rho^{k-1} \tag{274}$$

Then:

$$\sum_{k=0} \Big[k(k+1)c_{k+1}\rho^k + 2(l+1-\rho)(k+1)c_{k+1}\rho^k + (\rho_0 - 2(l+1))c_k\rho^k\Big] = 0 \tag{275}$$

This expression must hold for every power of ρ individually: We use:

$$-2(k+1)c_{k+1}\rho^{k+1} = -2kc_k\rho^k \tag{276}$$

to simplify:

$$k(k+1)c_{k+1} + 2(l+1-\rho)(k+1)ck+1 + (\rho_0 - 2(l+1))c_k = 0 \tag{277}$$

$$k(k+1)c_{k+1} + 2(l+1)(k+1)ck+1 + (\rho_0 - 2(l+1))c_k - 2kc_k = 0 \tag{278}$$

Then:

$$c_{k+1}(k(k+1) + 2(l+1)(k+1)) = c_k(2k - \rho_0 + 2(l+1)) \tag{279}$$

$$c_{k+1} = \frac{2(k+l+1) - \rho_0}{(k+1)(k+2(l+1))}c_k \tag{280}$$

This last equation is our recursion relation which can be used to determine all the terms in this power series. We notice that for large K:

$$c_{k+1} = \frac{2k}{k(k+1)}c_k = \frac{2c_k}{k+1} \tag{281}$$

$$c_k = \frac{2^k c_0}{k!} \tag{282}$$

If this is true for all k,

$$v(\rho) = \sum_{k=0}^{\infty} c_k\rho^k = \sum \frac{2^k}{k!}\rho^k c_0 = \sum \frac{(2\rho)^k}{k!}c_0 = e^{2\rho}c_0 \tag{283}$$

since we know that $e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!}$. This simplifies:

$$u(\rho = e^{-\rho}\rho^{l+1}v(\rho) = e^{-\rho}\rho^{l+1}e^{2\rho}c_0 \propto e^{\rho} \tag{284}$$

However, this blows up at $\rho \to \infty$, and this is unacceptable. We put a restraint on the power series: k can not $\to \infty$. This recursion series must terminate. So, for $k = \text{some}N$, we must find $c_n = 0$ (but $c_{n-1} \neq 0$) and $c_{n+1} = 0$. Our recursion relation gives:

$$c_{k+1} = \frac{2(k+l+1) - \rho_0}{(k+1)(k+2(l+1))}c_k \tag{285}$$

Then:

$$c_N = 2(N+l) - \rho_0 = 0 \tag{286}$$

This tells us:

$$2(N+l) = \rho_0 = 2n \quad \text{(from the Bohr model)} \tag{287}$$

We know that $N + l = n$. So, in other words, for this series to terminate, we need $N, l, n \in$ integers. We note that:

$$\rho_0 = 2n = \frac{-e^2\kappa}{4\pi\epsilon E} = \frac{-e^2}{4\pi\epsilon}\frac{\sqrt{-2mE}}{\hbar} = \frac{-e^2}{4\pi\epsilon_0\hbar}\sqrt{\frac{-2m}{E}} \tag{288}$$

then,

$$4n^2 = \left(\frac{e^2}{4\pi\epsilon_0\hbar}\right)^2 \cdot \frac{-2m}{E} \tag{289}$$

$$E = -\left(\frac{e^2}{4\pi\epsilon_0\hbar c}\right)^2 \frac{mc^2}{2n^2} = -\frac{1}{2}\alpha^2\frac{mc^2}{n^2} \quad \text{(Bohar Formula)} \tag{290}$$

So, we have redefined Bohr's semiclassical formula using the full Schrodinger equation. Bohr assumed the quantization of angular moment. Where, we derived it by following from the angular periodicity of Ψ and its asymptotic behavior.

To recap, we first apply the separation of variables:

$$\Psi = YR\phi \tag{291}$$

We get three separation constants: $l, (l+1)$ and m. We have the quantized n from the radial term and lastly E from the time dependent term. We write the radial term again:

$$\left[\frac{\hbar}{2m}\left(-\frac{d^2}{dr^2} + \frac{l(l+1)}{r^2}\right) + v(r)\right]u = Eu \quad \text{where} \quad u \equiv rR \tag{292}$$

We note that the terms in the parenthesis is $\frac{p^2}{2m}$ combining the parallel and perpendicular terms:

$$\frac{p^2}{2m} = \frac{\vec{p}_\parallel^2}{2m} + \frac{\vec{p}_\perp^2}{2m} \tag{293}$$

$$= \frac{\vec{p}_\parallel^2}{2m} + \frac{L^2}{2mr^2} \quad \text{where} \quad \vec{L} \equiv \vec{r}\times\vec{p} = |r||p_\perp| \tag{294}$$

IX. LECTURE 9

We're talking about the Hydrogen atom now. We have the time dependent Schrodinger equation:

$$i\hbar\frac{\partial\Psi}{\partial t} = \mathcal{H}\Psi \tag{295}$$

Then, we move to the Time-independent Schrodinger equation:

$$\Psi = R_{nl}(r)Y_l^m(\theta,\phi)\Phi(t) \tag{296}$$

and we have:

$$\mathcal{H}\psi = E\psi \quad \text{where} \quad \psi = Ry \tag{297}$$

Now, taking the results from last week: $\rho_0 = 2n$ and our recursion relation:

$$c_{k+1} = \frac{2(k+l+1-n)}{(k+1)(k+2l+2)}c_k \tag{298}$$

If we know that our c_0 is set by normalization condition, then we can calculate alll the c_k for $k \neq 0$. We defined:

$$u = rR = \rho^{l+1}e^{-\rho}v(p) \quad v(\rho) \equiv \sum_{k=0}^{\infty} c_k\rho^k \tag{299}$$

and

$$R = \frac{1}{r}\rho^{l+1}e^{-\rho}\sum_{k=0}^{\infty} c_k\rho^k \tag{300}$$

$v(\rho)$ can also be defined in terms of special functions:

$$v(\rho = L_{n-l-1}^{2l+1}(2\rho) \tag{301}$$

where the "associated Lagrerre polynomials" is defined:

$$L_q^p \equiv (-1)^p \left(\frac{d}{dx}\right)^p L_{p+q}(x) \tag{302}$$

they are written interms of the Laguerre polynomials:

$$L_q(x) = \frac{e^x}{q!}\left(\frac{d}{dx}\right)^q (e^{-x}x^q) \tag{303}$$

This is not to be confused with the associated Legendre functions and the Legendre polynomials seen in the solutions to the angular equation (Y_l^m). This wraps up the solution of the radial equation. Please see full solution $\psi_{nlm}(r,\theta,\phi)$ in equation 4.89 in the book.

Note that the ψ_{nlm} are orthogonal:

$$\int_0^\infty \int_0^\pi \int_0^{2\pi} \psi_{nlm}(r,\theta,\phi)\psi_{n'l'm'}(r,\theta,\phi)r^2 dr \sin\theta d\theta d\phi = \delta_{nn'}\delta_{ll'}\delta_{mm'} \tag{304}$$

Problem illustrating 3 good things to know: Find $\langle p^2 \rangle$ for a hydrogen atom in the $2s$ state. ($n = 2, l = 0, m = 0$). For different values of l, we have(Sharp(0), Principal(1), Diffuse(2), Fundamental(3)). One good way to remember this is using phrase: Sober Physicists Don't Find Giraffes Hiding In Kitchens.

Because $l = 0$, there is not angular dependent, so:

$$\psi_{200}(r) = R_{20}Y_0^0 = \frac{1}{\sqrt{2a^3}}\left(1 - \frac{r}{2a}\right)e^{-\frac{r}{2a}}\frac{1}{\sqrt{4\pi}} \tag{305}$$

Now, we find $\langle p^2 \rangle$:

$$\langle p^2 \rangle = \langle \psi_{200}|p^2|\psi_{200}\rangle \tag{306}$$

$$= -\frac{\hbar^2}{4\pi}\int\int_0^\infty \frac{1}{2a^3}\left(1 - \frac{r}{2a}\right)e^{-r/2a}\nabla^2\left[\left(1 - \frac{r}{2a}\right)e^{-r/2a}\right]r^2 dr d\Omega \tag{307}$$

Note that the r^2 is not differentiated by the ∇^2. We have:

$$\nabla^2 = \frac{1}{r^2}\frac{\partial}{\partial r}r^2\frac{\partial}{\partial r} \tag{308}$$

After 10 lines of algebra, ... note that:

$$\int_0^\infty x^n e^{-x} dx = n! \tag{309}$$

We find that:

$$\langle p^2 \rangle = \frac{\hbar^2}{4a^2} \tag{310}$$

While there is another way to do this, using the virial theorem:

$$\langle T \rangle = \langle \frac{p^2}{2m} \rangle = \frac{\langle p^2 \rangle}{2m} = -E = -\left(-\frac{\alpha^2 mc^2}{2n^2}\right) \tag{311}$$

then,

$$\langle p^2 \rangle = \frac{\alpha^2 m^2 c^2}{n^2} = \left(\frac{\alpha mc}{\hbar}\right)^2 \frac{\hbar^2}{n^2} = \frac{\hbar^2}{a^2 n^2} \tag{312}$$

where

$$a \equiv \frac{\hbar}{\alpha mc} \tag{313}$$

is the Bohr radius. Thus, for $n = 2$, we have:

$$\langle p^2 \rangle = \frac{\hbar^2}{4a^2} \tag{314}$$

But this expression works for all n in general. (note that previously, we noted the fine structure constant $\alpha = \frac{e^2}{4\pi\epsilon_0 \hbar c} \approx \frac{1}{137}$). Now, we move on to a new topic, that is angular momentum.

We've seen that the $Y_l^m(\theta, \phi)$ are more general than $v(r) = \frac{-e^2}{4\pi\epsilon_0 r}$ and this is good for any $v(t)$. Griffiths and all the other old school authors write:

$$\vec{L} = \vec{r} \times \vec{p} = \begin{pmatrix} \hat{x} & \hat{y} & \hat{z} \\ x & y & z \\ p_x & p_y & p_z \end{pmatrix} = (yp_z - zp_y)\hat{x} + (zp_x - xp_z)\hat{y} + (xp_y - yp_x)\hat{z} \tag{315}$$

While, we would prefer to have:

$$\vec{L} = \vec{r} \times \vec{k} \tag{316}$$

which is dimensionless. (Before $\vec{L}$ and S have the same dimension just like $\vec{\tau}$ and E). Note that the components of $\vec{L}$ do not commute with each other:

$$[L_x, L_y] = [yp_z - zp_y, zp_x - xp_z] \tag{317}$$

$$= [yp_z, zp_x - xp_z] - [zp_y, zp_x - xp_z] \tag{318}$$

$$= y[p_z, z]p_x + x[z, p_z]p_y \tag{319}$$

$$= yp_x(-i\hbar) + x(i\hbar)p_y \tag{320}$$

$$= i\hbar L_z \tag{321}$$

We have this result and it follows cyclic permutation. This is totally different from $[x, y] = 0$ and $[p_x, p_y] = 0$. This is what makes 3 dimensional physics more interesting. In chapter 6, we will find that,

$$\mathcal{H} \quad \text{generator of time translation} \tag{322}$$

$$\vec{p}(\vec{k}) \quad \text{generator of space translation} \tag{323}$$

$$\vec{x} \quad \text{generator of boosts} \tag{324}$$

$$\vec{L} \quad \text{generator of rotations} \tag{325}$$

Note that translations commute while rotations don't.

X. LECTURE 10

Last time, we showed that the different components of angular momentum do not commute:

$$[L_x, L_y] = i\hbar L_z \tag{326}$$

Note that the cyclic permutations do apply here. This fact reflects the fact that rotations do not commute (while translations and boosts do):

$$[x, y] = 0 \tag{327}$$

$$[p_x, p_y] = 0 \tag{328}$$

$$[x, p_x] = i\hbar \tag{329}$$

This last equation here indicates that translations do not commute with the corresponding boosts in QM. When we have:

$$[A, B] \neq 0 \tag{330}$$

This means that A and B are incompatible: we can't know both of them at the same time. There's this phase called Complete Set of Commuting Operators (CSCO). If we're trying to describe some state in terms of its eigenvalues, we need a complete set of commuting operators. We're developing the CSCO for the central potential (Hydrogen Atom) right now. Pick,

$$[L^2, L_x] = 0 \tag{331}$$

$$[L_x^2 + L_y^2 + L_z^2, L_x] = [L_x^2, L_x] + [L_y^2, L_x] + [L_z^2, L_x] \tag{332}$$

Recall that,

$$[AB, C] = A[B, C] + [A, C]B \tag{333}$$

then,

$$[L_x^2, L_x] + [L_y^2, L_x] + [L_z^2, L_x] = 0 + L_y[L_y, L_x] + [L_y, L_x]L_y + L_z[L_z, L_x] + [L_z, L_x]L_z \tag{334}$$

$$= L_y(-i\hbar L_z) + (-i\hbar L_z)L_y + L_z(i\hbar L_y) + i\hbar L_y L_z = 0 \tag{335}$$

So, we have:

$$[L^2, L_x] = 0 \tag{336}$$

and for L_y, L_z. We have already chosen to know:

$$L^2 \quad \rightarrow \quad \hbar l(l+1) \tag{337}$$

$$L_z \quad \rightarrow \quad m\hbar \tag{338}$$

These things came up in the spherical harmonics ($Y_l^m \theta, \phi$). Now we look for simultaneous eigenstates of (pretending we don't know the eigenvalue):

$$L^2 f = \lambda f \tag{339}$$

$$L_z f = \mu f \tag{340}$$

In this new approach, we reuse the tricks we developed for the Quantum simple harmonics oscillator. We define:

$$L_\pm = L_x \pm i L_y \tag{341}$$

We can compare this with

$$a_\pm \equiv \frac{1}{\sqrt{2\hbar\omega m}}(m\omega x \mp ip) \tag{342}$$

$$= \frac{1}{\sqrt{2}}\left(\sqrt{\frac{m\omega}{\hbar}}x \mp \sqrt{\frac{\hbar}{m\omega}}\frac{d}{dx}\right) \tag{343}$$

This form of $a_\pm$ reveals the symmetry. Set this aside, now we calculate the commutator of the ladder operator with the z component of the angular momentum:

$$[L_z, L_\pm] = [L_z, L_x] \pm i\hbar[L_x, L_y] \tag{344}$$

$$= i\hbar L_y \pm i\hbar(-iL_x) \tag{345}$$

$$= \pm\hbar(L_x \pm iL_y) = \pm\hbar L\pm \tag{346}$$

Another commutator relation we need is:

$$[L^2, L_\pm] = [L^2, L_x] \pm i[L^2, L_y] = 0 \tag{347}$$

Si if we have,

$$L^2 f = \lambda f \tag{348}$$

$$L_z f = \mu f \tag{349}$$

then $L_\pm f$ is also an eigenstate of L^2 and L_z. Then,

$$L^2(L_\pm f) = L_\pm L^2 f = L_\pm \lambda f = \lambda(L_\pm f) \tag{350}$$

Here the λ is the same eigenvalue. Now, the magic comes in when we apply $Lz(L_{\pm}f)$: they don't commute!

$$[L_z, L_{\pm}] = L_z L_{\pm} - L_{\pm} L_z = \pm\hbar L_{\pm} \tag{351}$$

and

$$L_z L_{\pm} = \pm\hbar L_{\pm} + L_{\pm} L_z \tag{352}$$

$$L_z L_{\pm} f = (\pm\hbar L_{\pm} + L_{\pm} L_z) f \tag{353}$$

$$= \pm\hbar L_{\pm} f + L_{\pm}\mu f = (\pm\hbar + \mu)(L_{\pm} f) \tag{354}$$

Note that here, we arrived at a new eigevalue for L_z. So, we have showed that the $L_{\pm}$ are ladder operators: they change L_z by ± 1 in units of $\hbar$. Key fact: without changing L^2 (This is particularly useful!).

However, we cannot have $L_z^2 > L^2 = L_x^2 + L_y^2 + L_z^2$. (because L_i are observables and thus are real values, therefore, $L_i^2 \geq 0$). So, the ladder must have a top rung or else it would voilate this condition. We require:

$$L_+ f_{\text{top}} = 0 \tag{355}$$

Say (this l here is chosen in anticipation of the result),

$$L_z f_+ = \hbar l f_+ \tag{356}$$

and we also have:

$$L^2 f_+ = \lambda f_+ \tag{357}$$

Now, we want to write L^2 in terms of L_+. We write out:

$$L_{\pm} L_{\mp} = (L_x \pm i L_y)(L_x \mp i L_y) \tag{358}$$

$$= L_x^2 + L_y^2 \pm i(L_y L_x - L_x L_y) \tag{359}$$

$$= L_x^2 + L_y^2 \pm i(-i\hbar L_z) = L_x^2 + L_y^2 \pm \hbar L_z \tag{360}$$

So, this means that:

$$L^2 = L_{\pm} L_{\mp} \mp \hbar L_z + L_z^2 \tag{361}$$

we are choosing signs to exploit equation [355]. So,

$$L^2 = L_{\pm} L_{\mp} \mp \hbar L_z + L_z^2 \tag{362}$$

$$L^2 f_t = (L_- L_+ + \hbar L_z + L_z^2) f_+ \qquad = (\hbar L_z + L_z^2) f t \tag{363}$$

$$= (\hbar\hbar l + (\hbar l)^2) f_+ \tag{364}$$

$$= \hbar^2 l(l+1) f_+ \tag{365}$$

We apply the same argument to the bottom rung:

$$L_- f_b = 0 \quad L_z f_b = \hbar \bar{l} f_b \tag{366}$$

and

$$L^2 f_b = (l_+ l_- - \hbar L_z + L_z^2) f_b \tag{367}$$
$$= (-\hbar(t\bar{l}) + (\hbar\bar{l})^2) f_b \tag{368}$$
$$= \hbar(-\bar{l} + \bar{l}^2) f_b \tag{369}$$
$$= \hbar^2 \bar{l}(\bar{l} - 1) f_b \tag{370}$$

$$L^2 f_+ = \lambda f_+ \tag{371}$$
$$L^2 f_b = \lambda f_b \tag{372}$$

here, λ is the same eigenvalue. What this means is that:

$$\frac{\lambda}{\hbar^2} = l(l+1) = \bar{l}(\bar{l} - 1) \tag{373}$$
$$\bar{l} = l + 1 \tag{374}$$
$$\bar{l} = -l \tag{375}$$

But we know that $\bar{l} = l + 1$ which makes no sense! (implies that the bottom rung is above the top rung!) So that upshot is:

$$L_z f = \hbar m f \quad \text{with} \quad m \in [-l, l] \quad \text{in integer}(2N) \text{ steps} \tag{376}$$

and that:

$$L^2 f_l^m = \hbar^2 l(l+1) f_l^m \tag{377}$$

For N here is an integer and $l = N/s$. We have:

$$N \text{ even :} \quad l \in \text{ integer} \tag{378}$$
$$N \text{ odd :} \quad l \in \text{ half-integer} \tag{379}$$
$$\tag{380}$$

XI. LECTURE 11

So, we're talking about angular momentum now. We mentioned the complete set of commuting observables (CSCO). We were also discussing these eigenvalue equations:

$$L^2 f_l^m = \hbar^2 l(l+1) f_l^m \tag{381}$$

$$L_z f_l^m = m\hbar f_l^m \tag{382}$$

$$L_\pm f_l^m \propto f_l^{m\pm 1} \quad \text{where } f \text{ is the eigenfunction} \tag{383}$$

We also know that n takes integer steps from $[-l, l]$. Thus, n has $N = 2l$ values. Also, $l = \frac{N}{2}$ is integer for even l and is a fraction for odd l. We can draw a sphere which correlates different components of L. We note that $L^2 > L_z^2$ and $L^2 = L_x^2 + L_y^2 + L_z^2$. Also, we cannot have $\vec{L} = \begin{pmatrix} 0 \\ 0 \\ L_z \end{pmatrix}$. This is $[L_x, L_y] = i\hbar L_z$ and that it is not possible to simultaneously know more than one component at a time. Generally, since $[A, B] \neq 0$, we say that A and B are incompatible. So that's mostly what we want to do with the algebraic theory of angular momentum. Now, we get into the discussion of the eigenfunction of angular momentum. We know that:

$$\vec{L} = \vec{r} \times \vec{p} \quad p = \frac{\hbar}{i}\vec{\nabla} \tag{384}$$

$$= -i\hbar\vec{r} \times \vec{\nabla} \tag{385}$$

In Spherical coordinates, we have:

$$\vec{\nabla} = \hat{r}\frac{\partial}{\partial r} + \hat{\theta}\frac{1}{r}\frac{\partial}{\partial \theta} + \hat{\phi}\frac{1}{\sin\theta}\frac{\partial}{\partial \phi} \tag{386}$$

We work out the vectors:

$$\hat{r} \times \hat{r} = 0 \tag{387}$$

$$\hat{r} \times \hat{\theta} = \hat{\phi} \tag{388}$$

$$\hat{r} \times \hat{\phi} = -\hat{\theta} \tag{389}$$

So, we have:

$$\vec{L} = \left(\hat{\phi}\frac{\partial}{\partial \theta} - \hat{\theta}\frac{1}{\sin\theta}\frac{\partial}{\partial \phi}\right)(-i\hbar) \tag{390}$$

$$\hat{\theta} = \cos\theta\cos\phi\hat{x} + \cos\theta\sin\phi\hat{y} - \sin\theta\hat{z} \tag{391}$$

$$\hat{\phi} = -\sin\phi\hat{x} + \cos\phi\hat{y} \tag{392}$$

We see immediately:

$$L_z = -i\hbar \frac{\partial}{\partial \phi} \tag{393}$$

This means that our eigenfunction, $Y_l^m \propto e^{im\phi}$. Here, we're going to skip some algebra, but we can calculate L_x and L_y and arrive at:

$$L^2 = -\hbar^2 \left[\frac{1}{\sin\theta} \frac{\partial}{\partial\theta} \left(\sin\theta \frac{\partial}{\partial\theta} \right) + \frac{1}{\sin^2\theta} \frac{\partial^2}{\partial\phi^2} \right] \tag{394}$$

and,

$$L^2 f_l^m(\theta, \phi) = \hbar l(l+1) f_l^m(\theta, \phi) \tag{395}$$

where this is just the angular equation that we have already solved:

$$f_l^m(\theta, \phi) = Y_l^m(\theta, \phi) \tag{396}$$

For a central potential, by separation of variables, we have:

$$L^2\psi = \hbar^2 l(l+1)\psi \tag{397}$$

$$L_z\psi = \hbar m\psi \tag{398}$$

$$\mathcal{H}\psi = \left(\frac{p^2}{2m} + v(r) \right) \psi = \frac{1}{2mr^2} \left[-\hbar^2 \frac{\partial}{\partial r} r^2 \frac{\partial}{\partial r} + L^2 \right] \psi + V\psi = E\psi \tag{399}$$

For $V(r) = \frac{-e^2}{4\pi\epsilon_0 r}$, we get the Hydrogen Atom. So, last week, we found that:

$$L^2 = L_\pm L_\mp \mp \hbar L_z + L_z^2 \tag{400}$$

$$L_\pm L_\mp = L^2 - L_z^2 \pm \hbar L_z \tag{401}$$

We now switch to more awesome(Dirac) notations:

$$L^2|lm\rangle = \hbar^2 l(l+1)|lm\rangle \tag{402}$$

$$L_z|lm\rangle = \hbar m|lm\rangle \tag{403}$$

$$L_+|lm\rangle = c_+|l(m+1)\rangle \tag{404}$$

$$L_-|lm\rangle = c_-|l(m-1)\rangle \tag{405}$$

Then,

$$\langle l'm'|L_-L_+|lm\rangle = \langle l'm'|(L^2 - \hbar L_z - L_z^2)|lm\rangle \tag{406}$$

$$= (\hbar^2 l(l+1) - \hbar^2 m - \hbar^2 m^2)\langle l'm'|lm\rangle \tag{407}$$

$$= \hbar(l(l+1) - m(m+1))\delta_{ll'}\delta_{mm'} \tag{408}$$

Note that $(L_+)^\dagger = L_-$ and $L_\pm = L_x \pm iL_y$. So say $l' = l$, $m' = m$, then,

$$\Big|L_+|lm\rangle\Big|^2 = \langle lm|L_-L_+|lm\rangle = \hbar^2(l(l+1) - m(m+1)) \tag{409}$$

$$= \Big|c_+\Big|^2\langle l(m+1)|l(m+1)\rangle \tag{410}$$

$$= \Big|c_+\Big|^2 \tag{411}$$

we have:

$$c_+ = \hbar\sqrt{l(l+1) - m(m+1)} \tag{412}$$

and,

$$L_\pm|lm\rangle = \hbar\sqrt{l(l+1) - m(m+1)}|l(m+1)\rangle \tag{413}$$

XII. LECTURE 12

In QM, we often use J as the general angular momentum, so we switch to that notation:

$$[J_x, J_y] = i\hbar J_z \tag{414}$$

$$[J_i, J_j] = i\hbar\epsilon_{i,j,k}J_k \tag{415}$$

here, we are using the Einstein notation where repeated indexes are summed over. We also have:

$$J^2|jm\rangle = \hbar^2 j(j+1)|jm\rangle \tag{416}$$

$$J_z|im\rangle = \hbar m|jm\rangle \tag{417}$$

$$J_\pm|jm\rangle = (J_x \pm iJ_j)|jm\rangle = \hbar\sqrt{j(j+1) - m(m\pm 1)}|j(m+1)\rangle \tag{418}$$

For $J = L$, we have:

$$\langle\theta,\phi|jm\rangle = Y_l^m(\theta,\phi) \tag{419}$$

$$\langle x|\psi\rangle = \psi(x) \tag{420}$$

The ladder operators change J_z by ± 1 without changing J.(twist) Since we know that for $Y_l^m(\theta,\phi) = Y_l^m(\theta,\phi + 2\pi)$. And that

$$L_z = -i\hbar\frac{\partial}{\partial\phi} \quad \rightarrow \quad Y_l^m(\theta,\phi) \propto e^{im\phi} \tag{421}$$

THe algebraic theory of angular momentum allows that $2l = N$ where N is an integer, so that:

$$e^{im\phi} = e^{im(\phi+2\pi)} \tag{422}$$

$$1 = e^{2\pi m} \rightarrow m \in \text{Integer} \tag{423}$$

Nature does in fact take advantage of that and uses the half integer: the only way to do this is to have no θ, ϕ dependence in the $|jm\rangle$.

Now, we look at an "internal" quantum number which is called the Spin. Spin $\frac{1}{2}$ is our first example:

$$\begin{pmatrix} u \\ d \end{pmatrix} \begin{pmatrix} c \\ s \end{pmatrix} \begin{pmatrix} t \\ b \end{pmatrix} \begin{pmatrix} e \\ \nu_e \end{pmatrix} \begin{pmatrix} \mu \\ \nu_\mu \end{pmatrix} \begin{pmatrix} z \\ \nu_z \end{pmatrix} \tag{424}$$

We also have:

$$p = uud \tag{425}$$

$$n = udd \tag{426}$$

Thus, atoms are made up by spin-$\frac{1}{2}$ particles. Notation:

$$\begin{aligned} \text{Spin :} \quad & S && (427)\\ \text{State :} \quad & |\frac{1}{2} \pm \frac{1}{2}\rangle && (428)\\ \text{Spin-up :} \quad & |\frac{1}{2} + \frac{1}{2}\rangle = |+\rangle = |\uparrow\rangle && (429)\\ \text{Spin-down :} \quad & |\frac{1}{2} - \frac{1}{2}\rangle = |-\rangle = |\downarrow\rangle && (430)\\ & && (431) \end{aligned}$$

When applying S to states:

$$\begin{aligned} S^2|\pm\rangle &= \hbar^2 \frac{1}{2}(1 + \frac{1}{2})|\pm\rangle = \frac{3}{4}\hbar^2|\pm\rangle && (432)\\ S_+|pm\rangle &= \pm\frac{\hbar}{2}|\pm\rangle && (433)\\ S_+|+\rangle &= \hbar\sqrt{\frac{1}{2}\frac{3}{2} - \frac{1}{2}\frac{3}{2}}|+\rangle = 0 && (434)\\ S_-|+\rangle &= \hbar\sqrt{\frac{3}{4} - \frac{1}{2}\left(\frac{1}{2} - 1\right)}|-\rangle = \hbar|-\rangle && (435)\\ S_+|-\rangle &= \hbar|+\rangle && (436)\\ S_-|-\rangle &= 0 && (437) \end{aligned}$$

For

$$a = \begin{pmatrix} a_1 1 & a_1 2 \\ a_2 1 & a_2 2 \end{pmatrix} \tag{438}$$

we got:

$$\langle i|a|j\rangle = a_{ij} \tag{439}$$

For the explicit form of the operator S:

$$S_+ = \begin{pmatrix} 0 & \hbar \\ 0 & 0 \end{pmatrix} = \hbar \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \quad S_- = \hbar \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \tag{440}$$

Applying them:

$$\langle +|S_+|-\rangle = \hbar\langle +|+\rangle \tag{441}$$

$$\langle -|S_+|+\rangle = 0 \tag{442}$$

$$\langle -|S_+|-\rangle = \hbar\langle -|+\rangle = 0 \tag{443}$$

for

$$|+\rangle = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \quad |-\rangle = \begin{pmatrix} 0 \\ 1 \end{pmatrix} \tag{444}$$

And:

$$S_+|-\rangle = \hbar \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} 0 \\ 1 \end{pmatrix} = \hbar \begin{pmatrix} 1 \\ 0 \end{pmatrix} = \hbar|+\rangle \tag{445}$$

$$S^2 = \frac{3}{4}\hbar^2 \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \tag{446}$$

$$S_z = \frac{\hbar}{2} \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \tag{447}$$

$$S_x = \frac{\hbar}{2} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \tag{448}$$

$$S_y = \frac{\hbar}{2} \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} \tag{449}$$

$$\tag{450}$$

and also:

$$S_+ = S_x + iS_y \tag{451}$$

$$S_- = S_x - iS_y \tag{452}$$

$$S_x = \frac{S_+ + S_-}{2} \tag{453}$$

$$S_y = \frac{S_+ - S_-}{2i} \tag{454}$$

Generally, we have:

$$\vec{S} = \frac{\hbar}{2}\vec{\nabla}, \quad \vec{\nabla} = (\sigma_x, \sigma_y, \sigma_z) = \left(\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \right) \tag{455}$$

Here, S are obviously all Hermitian and $S_\pm$ are obviously not Hermitian ($S_\pm \equiv S_x \pm iS_y$). In fact, some authors(Sakurai) actually start with spin as their first chapter of QM since it is the simplest quantum mechanical property. We have closure:

$$\mathbb{1} = \sum |\pm\rangle\langle\pm| = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \begin{pmatrix} 1 & 0 \end{pmatrix} + \begin{pmatrix} 1 & 0 \end{pmatrix} \begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \tag{456}$$

and we compare this to:

$$\mathbb{1} = \int dx |x\rangle\langle x| \tag{457}$$

$$= \int dp |p\rangle\langle p| \tag{458}$$

Other facts:

$$\sigma_x^2 = \sigma_y^2 = \sigma_z^2 = \mathbb{1} \tag{459}$$

$$\sigma_x\sigma_y = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} = \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix} = i\sigma_z \tag{460}$$

$$\text{(The permutation thing works here)} \tag{461}$$

Also, the eigenvalues of all the σ_i are ± 1. We note that σ_z is special that it's diagonal and its eigenvectors are:

$$|+\rangle = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \quad |-\rangle = \begin{pmatrix} 0 \\ 1 \end{pmatrix} \tag{462}$$

You can also treat the σ_i like vector components:

$$\hat{r} = \sin\theta\cos\phi\hat{x} + \sin\theta\sin\phi\hat{y} + \cos\theta\hat{z} \tag{463}$$

$$\vec{S_r} = \begin{pmatrix} \cos\theta & \sin\theta e^{-i\phi} \\ \sin\theta e^{-\phi} & -\cos\theta \end{pmatrix} \tag{464}$$

Now, let's talk about the physics behind all this. One of the first experiments that demonstrated this is the Stern-Gerlach experiment (1922). (insert story about smoking cigar)

XIII. LECTURE 13

We were talking about spin: in H-atom, if we want to describe an electron state, we need $|nlm_lm_s\rangle$, four quantum number. And if we want to write it in eigenstate: $\langle\vec{r}|nlm_lm_s\rangle = R_{nl}(\vec{r})Y_l^m(\theta,\phi)|x\rangle$. And we previously had shown that:

$$\vec{S} = \frac{\hbar}{2}\vec{\nabla} \tag{465}$$

and

$$\sigma_x = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \sigma_y = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad \sigma_z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \tag{466}$$

and

$$\sigma_i\sigma_j = \mathbb{1}\delta_{ij} + i\epsilon_{ijk}\sigma_k \tag{467}$$

Last time, we talked about the Stern-Garlach equation which is in 1922. Then, in 1929, deBroglie wrote his PHD thesis. In 1926, Schrodinger wrote down the Schrodinger equation. And in 1928, Dirac came up with the Dirac equation. During 1925, Ralph Kronig was 20 and working on his PHD. Pauli stated that electrons has classically indescribable values. Lande proposed the idea that electrons are rotating on the axis and spoke to Heisenberg, Pauli, Bohr who rejected this idea.

He used the classical electron radius:

$$\frac{e^2}{4\pi\epsilon_0 r_e} = mc^2 \tag{468}$$

We have angular momentum:

$$S \to \frac{\hbar}{2} \to r \times p \to rmv \to \frac{e^2}{4\pi\epsilon_0 mc^2}mv \tag{469}$$

that is:

$$\frac{4\pi\epsilon_0}{e^2}\frac{\hbar c}{2} \to \frac{v}{c} \tag{470}$$

This suggests that:

$$\frac{v}{c} \to \frac{1}{2\alpha} \to \frac{137}{2}\rangle\rangle 1 \tag{471}$$

This is basically saying that to pack $\frac{\hbar}{2}$ amount of angular momentum, it must be spinning really fast (faster than the speed of light).

Then, we have Goudsmit and Uhlenbeck who proposed the same idea where Lorentz tried to stopped but failed.

Then, we have Schrodinger who propsed:

$$\mathcal{H} = \frac{p^2}{2m} + V \tag{472}$$

where

$$\mathcal{H} \to i\hbar \frac{\partial}{\partial t} \tag{473}$$

$$\vec{p} = -i\hbar \frac{\partial}{\partial \vec{r}} \tag{474}$$

And the non-relatvistic Schrodinger equation reads:

$$i\hbar \frac{\partial \psi}{\partial t} = -\frac{\hbar}{2m} \frac{\partial^2}{\partial \vec{r}^2} \psi + V\psi \tag{475}$$

Then Dirac tried to work out a relativistic version of the Schrodinger equation (1928):

$$E^2 - p^2c^2 = m^2c^4 \tag{476}$$

$$-\frac{1}{c^2} \frac{\partial^2 \psi}{\partial t^2} + \frac{\partial^2 \psi}{\partial \vec{r}^2} = \left(\frac{mc}{\hbar}\right)^2 \psi \tag{477}$$

This is called the Klein-Gorden equation, and the $\left(\frac{mc}{\hbar}\right)^2$ term adds dispersion. It is difficult to work with since this equation is 2nd-order in time. We know that $|\psi|^2 \propto$ probability, but the second time derivative allows $|\psi|^2$ to be negative. The Schrodinger equation has one time derivative but two spacial derivative(inconsistent with relativity). Dirac attempted to look for a 1st-order equation.

We use the notation we had introduced from the first lecture:

$$\begin{pmatrix} E/C & \vec{p} \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \begin{pmatrix} E/c \\ \vec{p} \end{pmatrix} = P^\mu P_\mu = m^2 \quad \text{(This is our invariant)} \tag{478}$$

and we can rearrange this:

$$P^\mu P_\mu - m^2 = 0 \quad \to \quad (p-m)(p+m) = 0 \tag{479}$$

$$p - m = 0 \quad \text{or} \quad p + m = 0 \tag{480}$$

This would ensure the relativity result. This works great for $\vec{p} = 0$, in that case, we have:

$$E^2 - m^2c^4 = 0 \quad \to \quad E = \pm mc^2 \tag{481}$$

We could argue that Einstein could have predicted antimatter since the energy-momentum relation is quadratic, but he didn't lol.

To convert a 2nd order equation to 1st order equation: we introduce components (coupled 1st-order equations). One example is that from our wave equation (from the Maxwell equations):

$$\nabla \times H = J + \frac{\partial D}{\partial t} \tag{482}$$

$$\nabla \times B = \mu_0 \epsilon_0 \frac{\partial E}{\partial t} \tag{483}$$

$$\frac{1}{c^2}\frac{\partial^2 E}{\partial t^2} - \frac{\partial^2 E}{\partial x^2} = 0 \quad \rightarrow \quad \nabla \times E = -\frac{\partial B}{\partial t} \tag{484}$$

Another instance, from Lagrange's equation of motion:

$$\frac{d}{dt}\frac{\partial \mathcal{L}}{\partial \dot{q}_j} - \frac{\partial \mathcal{L}}{\partial q_j} = 0 \tag{485}$$

Then, we have,

$$\frac{\partial q_j}{\partial t} = \frac{\partial \mathcal{H}}{\partial p_j} = \frac{\partial \omega}{\partial k_j} \tag{486}$$

$$-\frac{\partial p_j}{\partial t} = \frac{\partial \mathcal{H}}{\partial q_i} \tag{487}$$

Here, we converted a second-order equation to two 1st order equations. Then,

$$p^\mu p_\mu = (\beta^\kappa p_\kappa + m)(\gamma^\lambda p_\lambda - m) = 0 \tag{488}$$

$$= B^\kappa p_\kappa \gamma^\lambda p_\lambda + m(\beta^\kappa - \gamma^\kappa)p_k - m^2 = 0 \tag{489}$$

$$= \gamma^\kappa p_\kappa \gamma^\lambda p_\lambda - m^2 = 0 \tag{490}$$

$$= (\gamma^0 p^0 + \gamma^1 p^1 + \gamma^2 p^2 + \gamma^3 p^3)(\gamma^0 p^0 + \gamma^1 p^1 + \gamma^2 p^2 + \gamma^3 p^3) - m^2 \tag{491}$$

$$= {\gamma^0}^2 p_0^2 + {\gamma^1}^2 p_2^2 + {\gamma^2}^2 p_2^2 + {\gamma^3}^2 \gamma_3^2 + (\gamma^0\gamma^1 + \gamma^1\gamma^0)p_0 p_1 + (\gamma^1\gamma^2 + \gamma^2\gamma^1)p_1 p_2 \tag{492}$$

$$+ (\gamma^2\gamma^3 + \gamma^3\gamma^2)p_2 p_3 + (\gamma^0\gamma^3 + \gamma^3\gamma^0)p_0 p_3 + (\gamma^0\gamma^2 + \gamma^2\gamma^0)p_0 p_2 + (\gamma^1\gamma^3 + \gamma^3\gamma^1)p_1 p_3 - m^2 \tag{493}$$

And since:

$$\gamma^{02} = 1, \quad (\gamma^{1,2,3})^2 = -1, \quad , \gamma^0 = 1, \quad \gamma^{1,2,3} = i \tag{494}$$

Then,

$$\gamma^\nu \gamma^\mu + \gamma^\mu \gamma^\nu = 0 \quad \text{for} \quad \nu \neq \mu \tag{495}$$

We choose one choice that works:

$$\gamma_0 = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix} = \begin{pmatrix} \mathbb{1} & 0 \\ 0 & -\mathbb{1} \end{pmatrix}, \quad \gamma_i = \begin{pmatrix} 0 & \sigma_i \\ -\sigma_i & 0 \end{pmatrix} \tag{496}$$

Then,

$$P^\mu P_\mu - m^2 = (\gamma^k p_k + m)(\gamma^\lambda P_\lambda - m) = 0 \tag{497}$$

$$\gamma^\lambda - m = 0 \to i\hbar\gamma^\lambda \partial_\lambda \psi - mc\psi = 0 \tag{498}$$

with $P_\lambda \to i\hbar\partial_\lambda$ where γ is a 4×4 matrix. This is the Dirac Equation. This is actually 4 equations because ψ has 4 components. There are 2 bits of the 4×4 components: we have matter/antimatter and spin up/spin down.

XIV. LECTURE 14

We were talking about spin. There is a magnetic moment that is proportional to spin:

$$\vec{\mu} \propto \vec{S} \tag{499}$$

or

$$\vec{\mu} = I\vec{\alpha} = qf \cdot \pi r^2 \tag{500}$$

$$= \frac{qv}{2\pi r}\pi r = \frac{qvmr}{2m} = \frac{q}{2m}\vec{J} \tag{501}$$

classical expectation for:

$$|c| = \hbar \tag{502}$$

$$|s| = \frac{\hbar}{2} \tag{503}$$

Now, we discuss spin in an electric field. Classically, we have:

$$\vec{J} = \vec{\mu} \times \vec{B} \tag{504}$$

$$\mathcal{H} = -\vec{\mu} \cdot \vec{B} = -\gamma \vec{S} \cdot \vec{B} = -\gamma B S_z = -\frac{\gamma B \hbar}{2}\sigma_z \tag{505}$$

That is $\hat{z}$ is always our favorite direction for spin. We take $\vec{B} = |B|\hat{z}$.

We know that:

$$|\psi(t)\rangle = \sum c_n|\varphi_n\rangle e^{\frac{-i\hbar t}{\hbar}} \tag{506}$$

$$|\psi(t=0)\rangle = a|+\rangle + b|-\rangle = \begin{pmatrix} a \\ b \end{pmatrix} \tag{507}$$

For convenience, we choose $a, b \in$ Reals. We continue:

$$|\psi(t)\rangle = a|+\rangle e^{-i\omega_1 t} + b|i\rangle e^{-i\omega_2 t} \tag{508}$$

We have:

$$\langle \vec{S} \rangle = \langle (S_x, S_y, S_z) \rangle \tag{509}$$

We write this in matrix form:

$$|\psi(t)\rangle = \begin{pmatrix} ae^{i\gamma Bt/2} \\ be^{-i\gamma Bt/2} \end{pmatrix} = \begin{pmatrix} \cos(\alpha/2)e^{i\gamma Bt/2} \\ \sin(\alpha/2)e^{-i\gamma Bt/2} \end{pmatrix} \tag{510}$$

$$\langle \psi|S_x|\psi\rangle = \frac{\hbar}{2} \begin{pmatrix} \cos(\alpha/2)e^{i\gamma Bt/2} & \sin(\alpha/2)e^{-i\gamma Bt/2} \end{pmatrix} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} \cos(\alpha/2)e^{i\gamma Bt/2} \\ \sin(\alpha/2)e^{-i\gamma Bt/2} \end{pmatrix} \begin{pmatrix} \cos(\alpha/2)e^{-i\gamma Bt/2} \\ \sin(\alpha/2)e^{i\gamma Bt/2} \end{pmatrix} \tag{511}$$

So we have:

$$\langle S_x \rangle = \frac{\hbar}{2} \sin\frac{\alpha}{2} \cos\frac{\alpha}{2} (e^{-\gamma Bt} + e^{\gamma Bt}) \tag{512}$$

$$= \frac{\hbar}{2} \sin\alpha \cos\gamma Bt \tag{513}$$

Similarly, we have

$$\langle S_y \rangle = -\frac{\hbar}{2} \sin\alpha \sin\gamma Bt \tag{514}$$

$$\langle S_z \rangle = \frac{\hbar}{2} \cos\alpha \tag{515}$$

Then, in a spherical picture, the state is precising on the clockwise orientation on the X-Y plane. This is the foundation of magnetic resonance. While this is a static magnetic field, we can apply a time-varying field that can perform a spin flip.

Now, let's talk about measurement. (Insert variations of Stern-Gerloch experiment) (Insert Density Operator)

XV. LECTURE 15

Today we're talking about the addition of angular momenta. We start with a quick example:

$$\begin{pmatrix} u \\ d \end{pmatrix} \begin{pmatrix} c \\ s \end{pmatrix} \begin{pmatrix} t \\ b \end{pmatrix} \quad \begin{pmatrix} e \\ \nu_e \end{pmatrix} \begin{pmatrix} \mu \\ \nu_\mu \end{pmatrix} \begin{pmatrix} z \\ \nu_z \end{pmatrix} \tag{516}$$

These are all fermions with spin $\frac{1}{2}$. Protons and neutrons:

$$p = uud \tag{517}$$

$$n = udd \tag{518}$$

are also spin $\frac{1}{2}$ particles. We can consider:

$$p = |\uparrow\rangle \tag{519}$$

$$n = |\downarrow\rangle \tag{520}$$

in the isospin space. Theoretically, we can rotate a spin up particle into a spin down particle, but this cannot be done rotating the coordinate system. We can only describe 2 state system using our 2×2 Pauli matrices + $\mathbb{1}$:

$$A = a\mathbb{1} + a\sigma_x + b\sigma_y + c\sigma_z \tag{521}$$

If A is an 2×2 matrix. Also if and only if $a, b, c, d \in$ real, then $A = A^\dagger$. [Insert ammonia example].

Now, we talk about the addition of angular momenta. We use J for general momenta:

$$J \equiv J_1 + J_2 \tag{522}$$

Classically, this is simply vector addition, and different system we might find J is conserved but not J_1 and J_2. We've seen recently the magnetic moments are proportional to spins:

$$\vec{\mu} \propto \vec{S} \tag{523}$$

We know that the Hamiltonian:

$$\mathcal{H} = -\vec{\mu} \cdot \vec{B} = \vec{S_1} \cdot \vec{S_2} \tag{524}$$

We can see this from the following. We first define

$$\vec{S} = \vec{S_1} + \vec{S_2} \tag{525}$$

and we square this:

$$S^2 = (S_1 + S_2)^2 = S_1^2 + S_2^2 + 2\vec{S_1}\vec{S_2} \tag{526}$$

Thus, we have

$$\mathcal{H} = \vec{S_1} \cdot \vec{S_2} = \frac{S^2 - S_1^2 - S_2^2}{2} \tag{527}$$

Now we work out how this addition works. In our J_1, J_2 state space,

$$|J_1 m_1 J_2 m_2\rangle = |J_1 m_1\rangle \otimes |J_1 m_2\rangle \tag{528}$$

where $\otimes$ denote the tensor product. Let's show how the tensor product in the matrix representation. We know that

$$[J_{1i}, J_{1j}] = i\hbar\epsilon_{ijk}J_k \tag{529}$$

$$[J_1^2, \vec{J}] = 0 \tag{530}$$

$$[\vec{J_1}, \vec{J_2}] = 0 \tag{531}$$

We write,

$$|s_1 m_1 s_2 m_2\rangle = |m_1 m_2\rangle \text{ for short} \tag{532}$$

$$= |s_1 m_1\rangle \otimes |s_2 m_2\rangle \;\rightarrow\; 2 \otimes 2 = 3 \oplus 1 \tag{533}$$

$$s_1^2|m_1 m_2\rangle = s_2^2|m_1 m_2\rangle = \frac{3}{4}\hbar^2|m_1 m_2\rangle = \hbar^2 s(s+1)|m_1 m_2\rangle \tag{534}$$

$$s_{1z}|m_1 m_2\rangle = m_1\hbar|m_1 m_2\rangle \tag{535}$$

$$s_{2z}|m_1 m_2\rangle = m_2\hbar|m_1 m_2\rangle \tag{536}$$

Our total spin,

$$\vec{S} = \vec{S_1} + \vec{S_2} \tag{537}$$

is an angular momentum too. We show this by:

$$[S_x, S_y] = [S_{1x} + S_{2x}, S_{1y} + S_{2y}] \tag{538}$$

$$= [S_{1x}, S_{1y}] + [S_{2x}, S_{2y}] \tag{539}$$

$$= i\hbar S_{1z} + i\hbar S_{1z} \tag{540}$$

$$= i\hbar S_z \tag{541}$$

Since the total obeys the usual angular momentum commutation relations, S is indeed an an angular momenta. We have

$$[\vec{S_1}^2, \vec{S_1}] = [\vec{S_2}^2, \vec{S_2}] = [\vec{S}^2, \vec{S}] = 0 \tag{542}$$

Since $\vec{S} \equiv \vec{S_1} + \vec{S_2}$, we have:

$$[\vec{S_1}^2, \vec{S}] = [\vec{S_2}^2, \vec{S}] = 0 \tag{543}$$

$$[S_{1z}, S_z] = [S_{1z}, S_{1z} + S_{2z}] = 0 = [S_{2z}, S_z] \tag{544}$$

It seems like everything commute with everything, however that is not true:

$$[\vec{S}^2, S_{1z}] \neq 0 \neq [\vec{S}^2, S_{2z}] \tag{545}$$

$$[S_1^2 + S_2^2 + 2\vec{S_1}\vec{S_2}, S_{12}] = 2[S_{1x}S_{2x} + S_{1y}S_{2y} + S_{1z}S_{2z}, S_{1z}] \tag{546}$$

$$= 2i\hbar(-S_{1y}S_{2x} + S_{1x}S_{2y}) \tag{547}$$

$$= -[S^2, S_{2z}] \tag{548}$$

Now, back to our original complete set of commuting operators (CSCO) for $|S_1 m_1 S_2 m_2\rangle$ in $S_1^2, S_{1z}, S_2^2, S_{2z}$. We had had just shown that S_1^2, S_2^2, S^2, S_z all commute with each other (they are also CSCO, we're going to show this now):

$$\vec{S} \equiv \vec{S_1} + \vec{S_2} \tag{549}$$

$$|S_1 S_2 S m\rangle \equiv |SM\rangle \tag{550}$$

$$S_1^2|SM\rangle = S_2^2|SM\rangle = \frac{3}{4}\hbar^2|SM\rangle \tag{551}$$

$$S^2|SM\rangle = \hbar^2 S(S+1)|SM\rangle \tag{552}$$

$$S_z|SM\rangle = \hbar M|SM\rangle \tag{553}$$

We can immediately write down $S_z \equiv S_{1z} + S_{2z}$. Then it follows that:

$$S_z|m_1 m_2\rangle = (S_{1z} + S_{2z})|m_1 m_2\rangle \tag{554}$$

$$= \hbar(m_1 + m_2)|m_1 m_2\rangle \tag{555}$$

For the basis:

$$|m_1 m_2\rangle : \{|++\rangle |+-\rangle |-+\rangle |--\rangle\} \tag{556}$$

we have $m_1 + m_2 = 1, 0, -1$. So:

$$S_z = \hbar \begin{pmatrix} i & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix} \tag{557}$$

Nor, we need S^2:

$$S^2 = (S_1 + S_2)^2 = S_1^2 + S_2^2 + 2(S_{1x}S_{2x} + S_{1y}S_{2y} + S_{1z}S_{2z}) \tag{558}$$

$$S_{1+}S_{2-} + S_{-1}S_{2+} = (S_{1x} + iS_{1y})(S_{2x} - iS_{2y}) + (S_{1x} - iS_{1y})(S_{2x} + iS_{2y}) \tag{559}$$

$$= S_{1x}S_{2x} + S_{1y}S_{2y} - iS_{1x}S_{2y} + iS_{1y}S_{2x} \tag{560}$$

$$S_{1x}S_{2x} + S_{1y}S_{2y} + iS_{1x}S_{2y} - iS_{1y}S_{2x} \tag{561}$$

$$= 2(S_{1x}S_{2x} + S_{1y}S_{2y}) \tag{562}$$

Then, we have:

$$S^2 = S_1^2 S_2^2 + S_{1+}S_{2-} + S_{1-}S_{2+} + 2S_{12}S_{2z} \tag{563}$$

$$S_{+-}|+_1\rangle \hbar \sqrt{S(S+1) - m(m-1)}|s\, m-1\rangle \tag{564}$$

$$= \hbar \sqrt{\frac{1}{2}\frac{3}{2} - \frac{1}{2}\left(\frac{1}{2} - 1\right)}|s\, m-1\rangle = \hbar |-_-\rangle \tag{565}$$

$$= S^2|++\rangle = \hbar^2 \left\{\frac{3}{4} + \frac{3}{4} + 2\frac{1}{2}\frac{1}{2}\right\}|++\rangle = 2\hbar|++\rangle \tag{566}$$

$$S^2|+-\rangle = \hbar^2 \left[\left(\frac{3}{4} + \frac{3}{4} - 2\frac{1}{2}\frac{1}{2}\right)|+-\rangle + |-+\rangle\right] \tag{567}$$

$$= \hbar^2(|+-\rangle + |-+\rangle) \tag{568}$$

$$S^2|--\rangle = 2\hbar^2|--\rangle \tag{569}$$

$$S^2|-+\rangle = \hbar^2(|-+\rangle + |+-\rangle) \tag{570}$$

$$S^2 = \hbar^2 \begin{pmatrix} 2 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 2 \end{pmatrix} \quad \text{in the order:} \quad |++\rangle, |+-\rangle, |-+\rangle, |--\rangle \tag{571}$$

XVI. LECTURE 16

We continue our discussion on addition of angular momentum:

$$|\pm\pm\rangle = |m_1 m_2\rangle\{|++\rangle|+-\rangle|-+\rangle|--\rangle\} \tag{572}$$

Last time we had:

$$S_z|m_1m_2\rangle = \hbar \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix} |m_1m_2\rangle \tag{573}$$

$$S^2|m_1m_2\rangle = \hbar^2 \begin{pmatrix} 2 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 2 \end{pmatrix} |m_1m_2\rangle \tag{574}$$

$$\tag{575}$$

and we defined that:

$$\vec{S} \equiv= \vec{S_1} + \vec{S_2} \tag{576}$$

We have to diagonalize S^2 in the $|+-\rangle, |-+\rangle$ subspace:

$$0 = \det \begin{pmatrix} 1-\lambda & 1 \\ 1 & 1-\lambda \end{pmatrix} = (1-\lambda)^2 - 1 = 1 - 2\lambda + \lambda^2 - 1 = \lambda(\lambda - 2) \tag{577}$$

So, we have eigenvalues to be $2\hbar^2$ and 0. Now we find the eigenvectors:

$$\begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} 1 \\ 1 \end{pmatrix} = \begin{pmatrix} 2 \\ 2 \end{pmatrix} = 2 \begin{pmatrix} 1 \\ 1 \end{pmatrix} \tag{578}$$

$$\begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} 1 \\ -1 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} = 0 \tag{579}$$

In this basis, we have $\frac{|+-\rangle+|-+\rangle}{\sqrt{2}}$ and $\frac{|+-\rangle-|-+\rangle}{\sqrt{2}}$. And $2\hbar$ corresponds to $S = 1$ and 0 eigenvalue corresponds to $S = 0$. To see the correlation:

$$SM \to |m_1m_2\rangle \tag{580}$$

$$|11\rangle \to |++\rangle \tag{581}$$

$$|10\rangle \to \frac{|+-\rangle + |-+\rangle}{\sqrt{2}} \tag{582}$$

$$|1-1\rangle \to |--\rangle \tag{583}$$

$$|00\rangle = \frac{|+-\rangle - |-+\rangle}{\sqrt{2}} \tag{584}$$

The first three states on the left are called the triplet state and the last one($|00\rangle$) is called the singlet state. We can see that the triplet state are symmetric if we exchange the two particles while the singlet state is anti-symmetric when we switch particles. Note that all state on the right are orthonormal. Now, we are starting to see some new physics that has something to do with tensor product.

$|11\rangle$ and $|1-1\rangle$ can be written $|+_1\rangle \otimes |+_2\rangle$ and$|-_1\rangle \otimes |-_2\rangle$ respectively. While $|10\rangle$ and $|00\rangle$ cannot be written as a product of single particle w.f.s. We say that $|10\rangle$ and $|10\rangle$ represent "entangled states". If we measure one of the particle, the other particle automatically collapse to a designated state. Before we get into the EPR paradox, we are going to talk about more practical issues.

We've been saying that $2 \otimes 2 = 3 \oplus 1$, where the 3 and 1 represent the triplet and singlet state. We have proton and neutron in an isospin space:

$$|p\rangle = |+\rangle \tag{585}$$

$$|n\rangle = |-\rangle \tag{586}$$

There's $^4_2\mathrm{He} : 2p, 2n$. There's no $2p$ bound state but there is a pn bound state called the deuteron. There's also no $2n$ bound state. Heisenberg postulate that the strong interaction is invariant under rotations in "isospin" space. From Noether theorem, we know that from a symmetry, we have a conserved quantity. Thus, isospin is conserved by the strong interaction. There's a term in the Hamiltonian that looks like $I_1 \cdot I_2$. So:

$$\mathcal{H} \propto \frac{I^2 - I_2^2 - I_1^2}{2} \tag{587}$$

So just based on symmetry, we can predict what kind of nuclei we are expected to see in the periodic table.

Now, we talk about a more general case:

$$J = J_1 + J_2 \tag{588}$$

Say we have $J_1 = 3/2$(4 states) and $J_2 = 1$(3 states). This is going to give us a $4 \otimes 3 = 6 \oplus 4 \oplus 2 = 12$, since J goes from $J_1 + J_2, J_1 + J_2 - 1, J_1 + J_2 - 2 \ldots |J_1 - J_2|$. So that $J = \frac{5}{2}, \frac{3}{2}, \frac{1}{2}$. We have two bases:$|j_1, j_2, m_1, m_2\rangle$ and $|j_1, j_2, j, m\rangle$. Griffths writes:

$$|jm\rangle = \sum_{m1+m2=m} c(j_1, j_2, j, m_1, m_2, m)|m_1 m_2\rangle \tag{589}$$

A better way to look at it is:

$$
\begin{aligned}
|jm\rangle &= \mathbb{1}|jm\rangle && (590)\\
&= \sum_{\text{all}\, m_1,m_2} |m_1m_2\rangle\langle m_1m_2|jm\rangle \quad \text{where} \quad \langle m_1m_2|jm\rangle = 0 \quad \text{for} \quad m_1+m_2 \neq m && (591)\\
&= \sum_{\text{all}\, m_1,m_2} \langle m_1m_2|jm\rangle|m_1m_2\rangle && (592)\\
&= \sum_{m_1+m_2=m} \langle m_1m_2|jm\rangle|m_1m_2\rangle && (593)
\end{aligned}
$$

This is just $\langle j_1j_2m_1m_2|j_1j_2jm\rangle$, which is the Clebsch-Gordon coefficient. Then

$$
\begin{aligned}
|m_1m_2\rangle &= \mathbb{1}|m_1m_2\rangle && (594)\\
&= \sum_{\text{all}\,|jm\rangle} |jm\rangle\langle jm|m_1m_2\rangle && (595)\\
&= \sum_j \langle jm|m_1m_2\rangle|jm\rangle \quad \text{with} \quad m = m_1+m_2 \quad \text{fixed} && (596)
\end{aligned}
$$

This is also the CG coefficients because the CG coefficients are chosen to be real.

Now, a really useful trick with tensor product. We can write

$$S_{1z} = \frac{\hbar}{2}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \tag{597}$$

So, how do we "extend" to the 2-particle state space? The extended S_{1z}:

$$S_{1z} = S_{1z} \otimes \mathbb{1}_2 \tag{598}$$

Let's look at our bases:

$$|++\rangle, |+-\rangle, |-+\rangle, |--\rangle \tag{599}$$

High, High, Low, Low is called the big period. While High, Low, High, Low is called the small period:

$$
\begin{aligned}
S_{1z} &= \frac{\hbar}{2}\begin{pmatrix} \mathbb{1} & 0 \\ 0 & -\mathbb{1} \end{pmatrix} && (600)\\
&= \frac{\hbar}{2}\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 1 & 0 & -1 \end{pmatrix} && (601)
\end{aligned}
$$

Assuming this basis ordering:

$$S_{12} = \begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \end{pmatrix} = \frac{\hbar}{2} \begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \end{pmatrix}, \quad S_{12} = \begin{pmatrix} 0 \\ 0 \\ 1 \\ 0 \end{pmatrix} = \frac{\hbar}{2} \begin{pmatrix} 0 \\ 0 \\ 1 \\ 0 \end{pmatrix} \tag{602}$$

And we have:

$$S_{2z} = \mathbb{1} \otimes S_{2z} = \begin{pmatrix} S_{1z} & 0 \\ 0 & S_{2z} \end{pmatrix} = \frac{\hbar}{2} \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -2 \end{pmatrix} \tag{603}$$

$$S_z = S_{1z} \otimes \mathbb{1} + \mathbb{1} \otimes S_{z2} \tag{604}$$

$$= \frac{\hbar}{2} \begin{pmatrix} 2 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -2 \end{pmatrix} \tag{605}$$

in the basis ordering $|++\rangle, |+-\rangle, |-+\rangle, |--\rangle$.

XVII. LECTURE 17

So we're doing electromagnetic interactions, Griffiths starts with the Lorentz force law:

$$\vec{F} = q(\vec{E} + \vec{V} \times \vec{B}) \tag{606}$$

This cannot be written as:

$$F = -\nabla V \tag{607}$$

We know that classically

$$\mathcal{H} = \frac{(p - qA)^2}{2m} + V(r) \tag{608}$$

where $\vec{A}$ is the vector potential. Regan dislikes this type of reasoning, and think we should be able to derive it.

Now, we dive into the mathematical concept. First, we start with the gauge symmetry (and Noether's theorem) which tells us that symmetry gives us a conserved quantity:

$$\text{translation in } \vec{r} \qquad \vec{p}(\text{or } \vec{R}) \tag{609}$$

$$\text{translation in } \vec{t} \qquad E(\text{or } \vec{f}) \tag{610}$$

$$\text{rotation in } \vec{\theta} \qquad J \tag{611}$$

In quantum mechanics, we know that a global phase factor does not change any physics:

$$|\psi\rangle \to |\psi'\rangle = e^{i\theta}|\psi\rangle \tag{612}$$

We can see that, when we calculate the expectation of an observable:

$$\langle\psi|G|\psi\rangle = \langle\psi'|G|\psi'\rangle \tag{613}$$

$$= \langle\psi|e^{i\theta}Ge^{-i\theta}|\psi\rangle \tag{614}$$

or

$$|\psi\rangle = \sum c_n|\varphi_n\rangle \to p(n) = |c_n|^2 \tag{615}$$

For a probability current (J is not angular momentum here):

$$J = \frac{i\hbar}{2m}\left(\psi\frac{d\psi^*}{dx} - \psi^*\frac{d\psi}{dx}\right) \tag{616}$$

We see that the phase cancels our and we get a conserved current.

$$|\psi'\rangle = e^{i\theta}|\psi\rangle \to (1+i\theta)|\psi\rangle \tag{617}$$

What if our phase(gauge) transformation is local?? Assume physics is invariant under:

$$|\psi\rangle \to |\psi'\rangle = e^{iq\Lambda(t,\vec{r})}|\psi\rangle \tag{618}$$

where we defined $q\Lambda = \theta$. When we make this assumption, we immediatly have a problem which is that when we calculate the expectation value of the momentum:

$$\langle p\rangle = \langle\psi|p|\psi\rangle \tag{619}$$

$$= \langle\psi'|p|\psi'\rangle \tag{620}$$

$$= \langle\psi|e^{iq\Lambda}(-i\nabla)e^{iq\Lambda}|\psi\rangle \tag{621}$$

$$= \langle\psi|e^{-iq\Lambda}[-ie^{iq\Lambda}(iq\nabla\Lambda)|\psi\rangle + e^{iq\Lambda}(-i\nabla|\psi\rangle)] \tag{622}$$

$$= \langle\psi|p|\psi\rangle + \langle\psi|e^{-iq\Lambda}e^{iq\Lambda}q\nabla\Lambda|\psi\rangle \tag{623}$$

$$= \langle\psi|p|\psi\rangle + \langle\psi|q\nabla\Lambda|\psi\rangle \tag{624}$$

$$= \langle\psi|p|\psi\rangle + q\nabla\Lambda \tag{625}$$

We see that our local gauge transformation is not conserved. But now, we insist that we want our gauge transformation to be conserved. **Side Note:**

$$\psi = \psi_0 e^{i(kx-\omega t)} \tag{626}$$

$$\psi = \psi' = e^{ik'x}\psi = \psi_0 e^{i((k+k')x-\omega t)} \tag{627}$$

We define a new momentum (adding a field):

$$\Pi \equiv p - qA \tag{628}$$

and we say that:

$$A \to A' = A + \nabla\Lambda \tag{629}$$

under a gauge transformation. We want to see if our new momentum works:

$$\begin{aligned}
\langle\psi|\Pi|\psi\rangle &= \langle\psi'|\Pi'|\psi'\rangle && (630)\\
&= \langle\psi|e^{-iq\Lambda[p-q(A+\nabla\Lambda)]e^{iq\Lambda}}|\psi\rangle && (631)\\
&= \langle\psi|e^{iq\Lambda}[-i\nabla - qA]e^{iq\Lambda}|\psi\rangle + \langle\psi|e^{iq\Lambda}(-q\nabla\Lambda)e^{iq\Lambda}|\psi\rangle && (632)\\
&= \langle\psi| - i\nabla - qA|\psi\rangle + q\nabla\Lambda - q\nabla\Lambda && (633)\\
&= \langle\psi|p - qA|\psi\rangle && (634)\\
&= \langle\psi|\Pi|\psi\rangle && (635)
\end{aligned}$$

So when we add $\vec{A}$, we need to give it a kinetic energy term in the Hamiltonian (or Lagrangian). (we need the kinetic energy to be invaraiant under a Lorentz and Gauge transformation) It turns out there's only one kinetic energy term we can write down:

$$-\frac{1}{4}(\partial_\mu A_v - \partial_v A_\mu) \tag{636}$$

Assume variance under a phase transformation:

$$\psi \to \psi' = e^{i\vec{\sigma}\cdot\vec{a}}\psi \tag{637}$$

where $\vec{a}$ is the local spinor and $\vec{\sigma}$ is the 2×2 Pauli matrices. We get the weak interaction. Then, if we assume:

$$\psi \to \psi' = e^{i\vec{\lambda}\cdot\vec{a}}\psi \tag{638}$$

where $\vec{\lambda}$ are 3×3 matrices. This gives us the strong interaction.

There is a four-vector:

$$(\varphi, \vec{A}) \to (\varphi - \frac{\partial \Lambda}{\partial t}, A + \nabla \Lambda) \tag{639}$$

We want to talk a little about what this new momentum is. From before, we said there are bunch of ways to define momentum:

$$p = \int F dt, mv, \gamma mv, \frac{h}{\lambda}, \hbar \vec{k}, -i\hbar \nabla \tag{640}$$

While,

$$\Pi \text{ is the mechanical, covariant, kinetic, mv, observable} \tag{641}$$

$$p \text{ is the canonical, conjugate(to } \vec{r}\text{), total, } -i\hbar\nabla\text{, or the generalized momentum} \tag{642}$$

Under a gauge(phase) transformation, nothing changes. p and A both changes while Π does not.

XVIII. LECTURE 18

Last lecture, we found the necessity of a vector potential from insisting on a local gauge invariance:

$$\nabla \times A = B \tag{643}$$

Today, we talk about the Aharonov-Bohm effect:

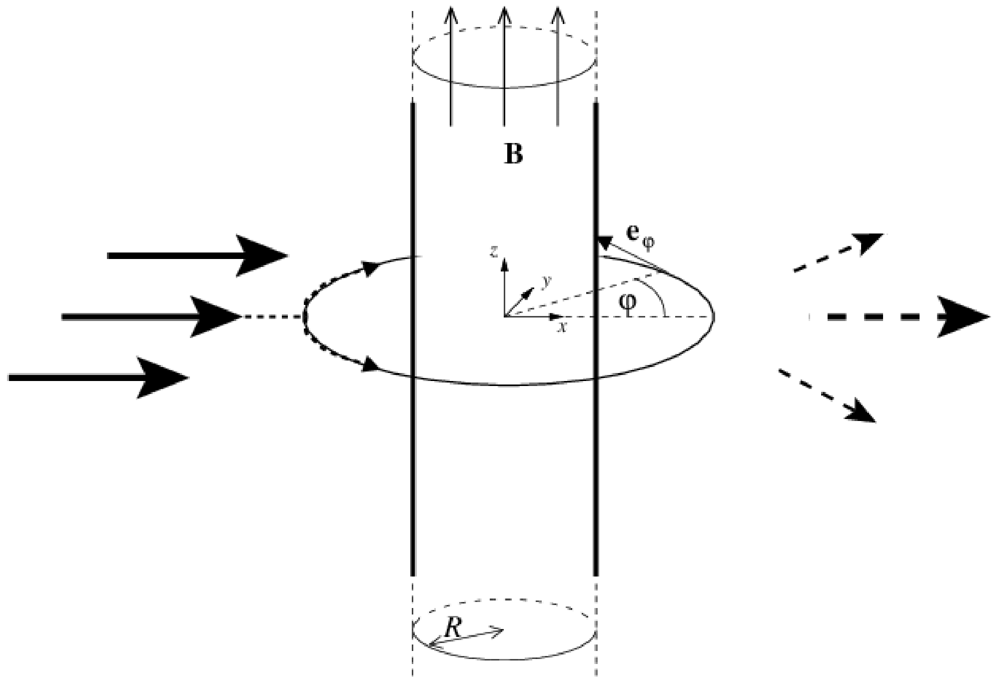

FIG. 1: Aharonov-Bohm effect

We choose:

$$\nabla \cdot A = 0 \tag{644}$$

which is the Coulomb gauge. We first find the vector potential of this thing:

$$\nabla \times H = J + \frac{\partial D}{\partial t} \tag{645}$$

$$\nabla \times B = \mu_0 J \tag{646}$$

$$\int \nabla \times B da = \oint B dl = I_{\text{enclosed}} \tag{647}$$

$$\vec{B} = \frac{\mu_0 I_{\text{enc}}}{2\pi r} \hat{\phi} \tag{648}$$

We know that:

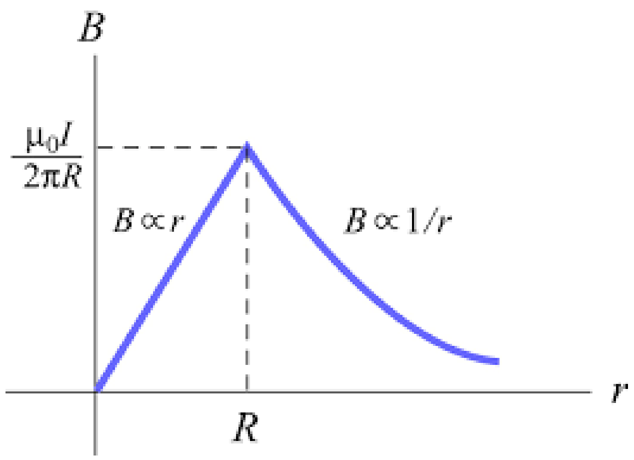

FIG. 2: B-r

The vector potential A also follows this rule. Since we have:

$$A = \Phi \left(\frac{r}{a}\right)^2 \cdot \frac{1}{2\pi r} \tag{649}$$

$$= \frac{\Phi r}{2\pi a^2} \hat{\phi} \quad \text{for}\, r < a \tag{650}$$

$$A = \frac{\Phi}{2\pi r} \hat{\phi} \quad \text{for}\, r > a \tag{651}$$

Now, we're going to make this system quantum mechanical by imagining a wire of radius b that has a quantum mechanical of charge q on a bead. We know that the Hamiltonian is given:

$$\mathcal{H} = \frac{(p - qA)^2}{2m} = \frac{1}{2m}[-\hbar^2 \nabla^2 + q^2 A^2 + 2i\hbar q A \cdot \nabla] \quad \text{show in hw} \tag{652}$$

Here $r = b$ is fixed and z is also fixed by the ring. So our gradient on acts on the ϕ component:

$$\nabla \to \hat{\phi} \frac{1}{b} \frac{d}{d\phi} \tag{653}$$

We write down the time-independent Schrodinger equation:

$$\mathcal{H}\psi = \frac{1}{2m}\left[-\frac{\hbar^2}{b^2}\frac{d^2}{dd\phi^2} + \left(\frac{q\Phi}{2\pi}\right)^2 + \frac{i\hbar q\Phi}{\pi b^2}\frac{d}{d\phi}\right]\psi(\phi) = E\psi \tag{654}$$

We times $-\frac{2mb^2}{\hbar^2}$, bring the E over and rearrange. We end up with:

$$\frac{d^2}{d\phi^2} - \frac{iq\Phi}{\hbar\pi}\frac{d}{d\phi} + \frac{2mb^2E}{\hbar^2} - \left(\frac{q\Phi}{2\pi\hbar}^2\right)\psi(\phi) = 0 \tag{655}$$

This looks like a nasty differential equation, but we can turn this into an algebraic equation with a guess: $\psi(\phi) = \psi_0 e^{i\phi\lambda}$. Now we have:

$$+\lambda^2 - \frac{\lambda q\Phi}{\hbar\pi} - \frac{2mb^2E}{\hbar^2} + \left(\frac{q\Phi}{2\pi\hbar}^2\right) = 0 \tag{656}$$

This root of a quadratic equation $a\lambda^2 + b\lambda + c = 0$:

$$\lambda = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} \tag{657}$$

The λ is thus:

$$\lambda = \frac{1}{2}\left[\frac{q\Phi}{\hbar\pi} \pm \sqrt{\left(\frac{q\Phi}{\hbar\pi}\right)^2 - 4\left[\left(\frac{q\Phi}{2\pi\hbar}\right)^2 - \frac{2mb^2E}{\hbar^2}\right]}\right] \tag{658}$$

We simplify this a bit:

$$\lambda = \frac{q\Phi}{2\pi\hbar} \pm \frac{b}{\hbar}\sqrt{2mE} \tag{659}$$

So we have:

$$\psi = \psi_0 e^{i\lambda\phi} \rightarrow \psi(\phi + 2\pi) = \psi(\phi) \tag{660}$$

and it follows that:

$$e^{i\lambda(\phi + w\pi)} = e^{i\lambda\phi} \tag{661}$$

$$e^{i\lambda 2\pi} = 1 \rightarrow \lambda \tag{662}$$

We call λ $n = 0, \pm 1, \pm 2, \ldots$ We have:

$$n = \frac{q\Phi}{h} + \frac{b}{\hbar}\sqrt{2mE} \tag{663}$$

$$\left(n - \frac{q\Phi}{h}\right)^2 = \left(\frac{b}{\hbar}\right)^2 \cdot 2mE \tag{664}$$

$$E = \frac{1}{2m}\left(\frac{\hbar}{b}\right)^2\left(n - \frac{q\Phi}{h}\right)^2 \tag{665}$$

where $n = 0, \pm 1, \pm 2, \ldots$. So energy depends on the existence of A, but not on A explicitly. The reason is that A is not gauge invariance. While E depends on the flux which is gauge invariance. We see that when the bead and the flux is oriented in the same way, the energy is lower; and vice versa.

Now, we do some geometric algebra. Scalars are the first type of math we learned with operations: $+, -, \times, /$. For vector math, we have:

$$(\vec{a} \cdot \vec{b}) = c, \quad \vec{a} \times \vec{b} = \vec{c} \tag{666}$$

Given $\vec{a}$ and c(or $\vec{c}$), we cannot find $\vec{b}$! While for the dot product, we can always add $\vec{a} \perp \vec{a}$ to $\vec{b}$ and still get the same c. For the cross product, we cam aways add $\vec{a} \parallel \vec{a}$ to $\vec{b}$ and still get the same $\vec{c}$. It would be nice to have a "division algebra".

Suppose we have a model for complex numbers. We have $Z = x + iy = |z|e^{i\theta}$. Then:

$$|z|^2 = z^* z = (x - iy)(x + iy) = x^2 + y^2 \tag{667}$$

and say we have $w = u + iv$, then:

$$w^* z = (u - iv)(x + iy) = ux + vy + i(yu - xv) \tag{668}$$

so the first two term is just like the dot product while the last term look just like the cross-product. We can do this in polar coordinate:

$$= |w|e^{-i\phi}|z|e^{i\theta} = |w||z|e^{i(\theta - \phi)} \tag{669}$$

$$= |w||z|[\cos(\theta - \phi) + i\sin(\theta - \phi)] \tag{670}$$

so we see that dot product gives us $\cos$ and cross product gives us $\sin$! This is also a division algebra!

$$w^* z = q \tag{671}$$

given w^*, q, we want to find z. We have:

$$ww^* z = wq \tag{672}$$

$$z = \frac{wq}{|w|^2} \tag{673}$$

and

$$w^{-1} = \frac{w^*}{|w|^2} \tag{674}$$

XIX. REVIEW ON RELATIVITY (GRIFFTHS)

A. Lorentz Transform

In the Lorentz transformation: say we have two inertial frames S and S' with S' moving at uniform velocity v with respect to S:

$$x' = \gamma(x - vt) \tag{675}$$

$$y' = y \tag{676}$$

$$z' = z \tag{677}$$

$$t' = \gamma\left(t - \frac{v}{c^2}x\right) \tag{678}$$

where,

$$\gamma \equiv \frac{1}{\sqrt{1 - v^2/c^2}} \tag{679}$$

The inverse transformation is given:

$$x = \gamma(x' + vt') \tag{680}$$

$$y = y' \tag{681}$$

$$z = z' \tag{682}$$

$$t = \gamma\left(t' + \frac{v}{c^2}x'\right) \tag{683}$$

The Lorentz transformation demonstrate a couple important ideas. First, it's the relativity of simultaneity: If two events occur at the sam time in S, but at different locations, then they do not occur at the same time in S'/ Specifically, if $t_A = t_B$, then

$$t'_A = t'_B + \frac{\gamma v}{c^2}(x_B - x_A) \tag{684}$$

Second is the Lorentz contraction (moving object is shortened by a factor of γ). Third is Time dilation (moving clocks run slow by a factor of γ). The last one is velocity addition (addition of velocity needs to be corrected by a relativistic term).

B. Four Vectors

We define the position-time four-vector x^μ for $\mu = 0, 1, 2, 3$:

$$x^0 = ct \tag{685}$$

$$x^1 = x \tag{686}$$

$$x^2 = y \tag{687}$$

$$x^3 = z \tag{688}$$

The Lorentz transformation on x^μ:

$$x^{0'} = \gamma(x^0 - \beta x^1) \tag{689}$$

$$x^{1'} = \gamma(x^1 - \beta x^0) \tag{690}$$

$$x^{2'} = x^2 \tag{691}$$

$$x^{3'} = x^3 \tag{692}$$

where

$$\beta \equiv \frac{v}{c} \tag{693}$$

Or we can write:

$$x^{\mu'} = \sum_{v=0}^{3} \Lambda_v^\mu x^v \quad (\mu = 0, 1, 2, 3) \tag{694}$$

where,

$$\lambda = \begin{pmatrix} \gamma & -\gamma\beta & 0 & 0 \\ -\gamma\beta & \gamma & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \tag{695}$$

Here, we use Einstein's notation which says that repeated Greak indices (one as subscript, one as superscript) are summed. Then, we have,

$$x^{\mu'} = \Lambda_v^\mu x^v \tag{696}$$

Though the individual coordinates of an event change when we go from S to S', there is a particular combination of them that remains the same:

$$I \equiv (x^0)^2 - (x^1)^2 - (x^3)^2 = (x^{0'})^2 - (x^{1'})^2 - (x^{2'})^2 - (x^{3'})^2 \tag{697}$$

Due to the minus signs, we need to introduce the metric $g_{\mu v}$ whose components can be displayed as a matrix g:

$$g = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix} \tag{698}$$

Now, we can write

$$I = \sum_{\mu=0}^{3} \sum_{v=0}^{3} g_{\mu v} x^{\mu} x^{v} = g_{\mu v} x^{\mu} x^{v} \tag{699}$$

Then, we define the covariant four-vector x_μ (we call the original one contravarient):

$$x_\mu \equiv g_{\mu v} x^{v} \tag{700}$$

Now, we can write:

$$I = x_\mu x^\mu \tag{701}$$

To generalize to all four-vector, we define a four-vector a^μ, as a four-component object that transforms in the same way x^μ does when we go from one inertial system to another, to wit:

$$a^{\mu'} = \Lambda^{\mu}_{v} a^{v} \tag{702}$$

with the same Λ as before. For these contravariant four-vector, we associate a covariant four-vector a_μ:

$$a_\mu = g_{\mu v} a^{v} \tag{703}$$

From covariant to contravariant, we have:

$$a^\mu = g^{\mu v} a_v \tag{704}$$

where $g^{\mu v}$ are elements in g^{-1}. Since $g = g^{01}$, then $g^{\mu v} = g_{\mu v}$. Given any two four-vectors a^μ amd b^μ,

$$a^\mu b_\mu = a_\mu b^\mu = a^0 b^0 - a^1 b^1 - a^2 b^2 - a^3 b^3 \tag{705}$$

is invariant. This is also referred as the scalar product of a and b, we can just write:

$$a \cdot b \equiv a_\mu b^\mu \tag{706}$$

To distinguish between four-vector and the three-dimensional vectors, we put an arrow on top of the three dimensional ones:

$$a \cdot b = a^0 b^0 - \vec{a} \cdot \vec{b} \tag{707}$$

and we also have

$$a^2 \equiv a \cdot a = (a^0)^2 - a^2 \tag{708}$$

We classify our four-vectors according to the sign of a^2:

$$\text{If } a^2 > 0, \quad a^\mu \text{ is called timelike} \tag{709}$$

$$\text{If } a^2 < 0, \quad a^\mu \text{ is called spacelike} \tag{710}$$

$$\text{If } a^2 = 0, \quad a^\mu \text{ is called lightlike} \tag{711}$$

A second-rank tensor $s^{\mu v}$ carries two indices, has $4^2 = 16$ components, and transform with two factors of Λ:

$$s^{\mu v'} = \Lambda^\mu_\kappa \Lambda^v_\sigma S^{\kappa\sigma} \tag{712}$$

a third-rank tensor $t^{\mu v \lambda}$ has three indices, $4^3 = 64$ components, and transforms with three factors of Λ:

$$t^{\mu v \lambda'} = \Lambda^\mu_\kappa \Lambda^v_\sigma \Lambda^\lambda_\tau t^{\kappa\sigma\tau} \tag{713}$$

So that a vector is a tensor of rank 1 and a scalar is a tensor of rank zero. We construct covariant and mixed tensors by lowering indices, for example,

$$S^\mu_v = g_{v\lambda} S^{\mu\lambda}; \quad S_{\mu v} = g_{\mu\kappa} g_{v\lambda} S^{\kappa\lambda} \tag{714}$$