QingheGao
12589896
Michael van der Werve
10565256
FutongHan
12581135

Assignment 3
The finite difference Method

2020/03/30

Abstract

In this report, we explore the background of the Partial Differential Equation (PDE) approach by deriving the Forward Time Centred in Space (FTCS) scheme using Finite Difference (FD) approximation and compare it to the Crank-Nicolson (CN) scheme for the same equation. We show that these schemes perform well for a given set of market parameters on a European Call under varying values of the underlying asset. We have shown that both schemes converge to the analytical value, but that the CN scheme significantly outperforms the FTCS scheme. Additionally, the FTCS scheme is shown to be only conditionally stable, whereas CN is unconditionally stable. Under their optimal parameters, estimates for delta are very close to the analytical value, where CN is equal to the analytical value under enough grid points. Both schemes fail under a very large tail of S_0.

1 Introduction

The finite difference method is an effective method to evaluate option prices. In this report, we first derived the matrices of FTCS and CN schemes, including boundary conditions, to get the European call option price. And then, we compared the results of FTCS and CN schemes with analytical value in different S_0. Furthermore, the convergence of the two schemes was measured by increasing ΔX. Furthermore, we optimize the mesh size $(\Delta X, \Delta\tau)$ of two methods to improve accuracy. Finally, we used the optimal parameter to plot the delta as a function of the underlying S_0 and compared with the analytical delta in three different S_0.

2 Method and Theory

2.1 Background of PDE approach

First, we recall the underlying stock dynamics:

$$dS_t = \mu S_t dt + \sigma S_t dW_t \tag{1}$$

And we apply Ito calculus to equation to evaluate option price :

$$dV_t = (\mu S_t \frac{\partial V}{\partial S} + \frac{\partial V}{\partial t} + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 V}{\partial S^2})dt + \sigma S \frac{\partial V}{\partial S} dW_t \tag{2}$$

And we have the portfolio value: :

$$\Pi_t = -V_t + \frac{\partial V}{\partial S} S_t \tag{3}$$

Then, we assume there is a risk-free rate r:

$$d\Pi_t = -dV_t + \frac{\partial V}{\partial S} dS_t \tag{4}$$

And we apply equation 2 and 3 into equation 4:

$$d\Pi_t = (-\frac{\partial V}{\partial t} - \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 V}{\partial S^2})dt \tag{5}$$

Since we have $d\Pi_t = r\Pi_t dt$, then:

$$\frac{\partial V}{\partial t} + rS\frac{\partial V}{\partial S} + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 V}{\partial S^2} = rV \tag{6}$$

And in order to get the a PDE with constant coefficients, we use $X = \ln S$ and $\frac{\partial V}{\partial t} = -\frac{\partial V}{\partial \tau}$. Then we get:

$$\frac{\partial V}{\partial \tau} = (r - \frac{1}{2}\sigma^2)\frac{\partial V}{\partial X} + \frac{1}{2}\sigma^2 \frac{\partial^2 V}{\partial X^2} - rV \tag{7}$$

2.2 FTCS (Forward Time Centred in Space)

First we want to introduce FTCS (Forward Time Centred in Space) method. Now we can use Taylor expansion to discrete the equation 7.

First, recall Taylor expansion:

$$\begin{aligned}
V(X,\tau+\Delta\tau) &= V(x,\tau)+\Delta\tau\frac{\partial V}{\partial\tau}+O(\Delta\tau^2)\\
V(X+\Delta X,\tau) &= V(x,\tau)+\Delta X\frac{\partial V}{\partial X}+\frac{1}{2}\Delta X^2\frac{\partial^2 V}{\partial X^2}+O(\Delta X^3)\\
V(X-\Delta X,\tau) &= V(x,\tau)-\Delta X\frac{\partial V}{\partial X}+\frac{1}{2}\Delta X^2\frac{\partial^2 V}{\partial X^2}+O(\Delta X^3)
\end{aligned}$$

Using Taylor expansion we get:

$$\begin{aligned}
\frac{V_i^{n+1}-V_i^n}{\Delta\tau} &= (r-\frac{1}{2}\sigma^2)\frac{V_{i+1}{}^n-V_{i-1}{}^n}{2\Delta X}+\frac{1}{2}\sigma^2\frac{V_{i+1}{}^n+V_{i-1}{}^n-2V_i^n}{\Delta X^2}-rV_i^n\\
V_i^{n+1} &= V_i^n+(r-\frac{1}{2}\sigma^2)\frac{\Delta\tau}{2\Delta X}(V_{i+1}{}^n-V_{i-1}{}^n)+\frac{1}{2}\sigma^2\frac{\Delta\tau}{\Delta X^2}(V_{i+1}{}^n+V_{i-1}{}^n-2V_i^n)-r\Delta\tau V_i^n
\end{aligned} \tag{8}$$

where n denotes time points and i denotes intervals of X. Then we continue to rearrange equation 7:

$$\begin{aligned}
V_i^{n+1} &= \alpha V_{i-1}{}^n+\beta V_i^n+\gamma V_{i+1}{}^n\\
\alpha &= -(r-\frac{1}{2}\sigma^2)\frac{\Delta\tau}{2\Delta X}+\frac{1}{2}\sigma^2\frac{\Delta\tau}{\Delta X^2}\\
\beta &= 1-\sigma^2\frac{\Delta\tau}{\Delta X^2}-r\Delta\tau\\
\gamma &= (r-\frac{1}{2}\sigma^2)\frac{\Delta\tau}{2\Delta X}+\frac{1}{2}\sigma^2\frac{\Delta\tau}{\Delta X^2}
\end{aligned} \tag{9}$$

2.3 Crank-Nicolson scheme

Crank-Nicolson scheme is a method that can improve the accuracy of the solution because the accuracy of time and space are both second order because it takes an average of the backward and forward scheme. Figure 1 shows how we Crank-Nicolson scheme works.

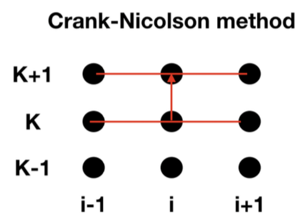


Figure 1: Crank-Nicolson scheme

First we want to use Taylor expansion to show Crank-Nicolson scheme is second order in space. For $\frac{\partial V}{\partial X}$ item, recall the forward Euler method:

$$\begin{aligned}
V(X+\Delta X,\tau) &= V(x,\tau)+\Delta X\frac{\partial V}{\partial X}+\frac{1}{2}\Delta X^2\frac{\partial^2 V}{\partial X^2}+\frac{1}{6}\Delta X^3\frac{\partial^3 V}{\partial X^3}+O(\Delta X^4)\\
V(X-\Delta X,\tau) &= V(x,\tau)-\Delta X\frac{\partial V}{\partial X}+\frac{1}{2}\Delta X^2\frac{\partial^2 V}{\partial X^2}-\frac{1}{6}\Delta X^3\frac{\partial^3 V}{\partial X^2}+O(\Delta X^4)
\end{aligned}$$

Minus

$$\begin{aligned}
\frac{V(X+\Delta X,\tau)-V(X-\Delta X,\tau)}{2\Delta X} &= \frac{\partial V}{\partial X}+\frac{1}{6}\Delta X^2\frac{\partial^3 V}{\partial X^3}\\
&= \frac{\partial V}{\partial X}+O(\Delta X^2)
\end{aligned}$$

Using same process to get backward:

$$\frac{V(X+\Delta X,\tau+\Delta\tau)-V(X-\Delta X,\tau+\Delta\tau)}{2\Delta X} = \frac{\partial V}{\partial X}+O(\Delta X^2) \tag{10}$$

We can see when we take the average of the forward and backward scheme to get the Crank-Nicolson scheme, the $\frac{\partial V}{\partial X}$ item is second-order($O(\Delta X^2)$) accuracy.

For $\frac{\partial^2 V}{\partial X^2}$ item, using same process:

$$\begin{aligned}\frac{V(X+\Delta X,\tau)+V(X-\Delta X,\tau)-2V(X,\tau)}{\Delta X^2} &= \frac{\partial^2 V}{\partial X^2}+\frac{1}{12}\Delta X^2\frac{\partial^4 V}{\partial X^4}\\ &= \frac{\partial^2 V}{\partial X^2}+O(\Delta X^2)\end{aligned} \tag{11}$$

Using same process to get backward :

$$\frac{V(X+\Delta X,\tau+\Delta\tau)+V(X-\Delta X,\tau+\Delta\tau)-2V(X,\tau+\Delta\tau)}{\Delta X^2} = \frac{\partial^2 V}{\partial X^2}+O(\Delta X^2)$$

We can see that for $\frac{\partial^2 V}{\partial X^2}$ item, it is also second-order accuracy. Now we can get the conclusion Crank-Nicolson scheme is second order in space.

Then we finally work on the Crank-Nicolson scheme.

First, we use the forward Euler method (equation 8):

$$\frac{V_i{}^{n+1}-V_i{}^n}{\Delta\tau} = (r-\frac{1}{2}\sigma^2)\frac{V_{i+1}{}^n-V_{i-1}{}^n}{2\Delta X}+\frac{1}{2}\sigma^2\frac{V_{i+1}{}^n+V_{i-1}{}^n-2V_i{}^n}{\Delta X^2}-rV_i{}^n$$

And then we used the backward Euler method:

$$\frac{V_i{}^{n+1}-V_i{}^n}{\Delta\tau} = (r-\frac{1}{2}\sigma^2)\frac{V_{i+1}{}^{n+1}-V_{i-1}{}^{n+1}}{2\Delta X}+\frac{1}{2}\sigma^2\frac{V_{i+1}{}^{n+1}+V_{i-1}{}^{n+1}-2V_i{}^{n+1}}{\Delta X^2}-rV_i{}^{n+1}$$

Then we add the forward Euler method and backward Euler method :

$$\begin{aligned}\frac{V_i{}^{n+1}-V_i{}^n}{\Delta\tau} &= (r-\frac{1}{2}\sigma^2)\frac{1}{4\Delta X}(V_{i+1}{}^n-V_{i-1}{}^n+V_{i+1}{}^{n+1}-V_{i-1}{}^{n+1})\\ &+\frac{\sigma^2}{4\Delta X^2}(V_{i+1}{}^n+V_{i-1}{}^n-2V_i{}^n+V_{i+1}{}^{n+1}+V_{i-1}{}^{n+1}-2V_i{}^{n+1})-\frac{r}{2}(V_i{}^n+V_i{}^{n+1})\end{aligned} \tag{12}$$

Rearrange the equation, we get:

$$\begin{aligned}V_i{}^{n+1} &= V_i{}^n+(r-\frac{1}{2}\sigma^2)\frac{\Delta\tau}{4\Delta X}(V_{i+1}{}^n-V_{i-1}{}^n+V_{i+1}{}^{n+1}-V_{i-1}{}^{n+1})\\ &+\frac{\sigma^2\Delta\tau}{4\Delta X^2}(V_{i+1}{}^n+V_{i-1}{}^n-2V_i{}^n+V_{i+1}{}^{n+1}+V_{i-1}{}^{n+1}-2V_i{}^{n+1})-\frac{r\Delta\tau}{2}(V_i{}^n+V_i{}^{n+1})\end{aligned}$$

Furthermore, we use equation 12 to get:

$$\begin{aligned}&(-(r-\frac{1}{2}\sigma^2)\frac{\Delta\tau}{4\Delta X}-\frac{\sigma^2\Delta\tau}{4\Delta X^2})V_{i+1}{}^{n+1}+(1+\frac{\sigma^2\Delta\tau}{2\Delta X^2}+\frac{r\Delta\tau}{2})V_i{}^{n+1}+((r-\frac{1}{2}\sigma^2)\frac{\Delta\tau}{4\Delta X}-\frac{\sigma^2\Delta\tau}{4\Delta X^2})V_{i-1}{}^{n+1}=\\ &((r-\frac{1}{2}\sigma^2)\frac{\Delta\tau}{4\Delta X}+\frac{\sigma^2\Delta\tau}{4\Delta X^2})V_{i+1}{}^n+(1-\frac{\sigma^2\Delta\tau}{2\Delta X^2}-\frac{r\Delta\tau}{2})V_i{}^n+(-(r-\frac{1}{2}\sigma^2)\frac{\Delta\tau}{4\Delta X}+\frac{\sigma^2\Delta\tau}{4\Delta X^2})V_{i-1}{}^n\end{aligned} \tag{13}$$

Then we have:

$$\begin{aligned}\gamma_c V_{i+1}{}^{n+1}+(1+\beta_c)V_i{}^{n+1}+\alpha_c V_{i-1}{}^{n+1} &= -\gamma_c V_{i+1}{}^n+(1-\beta_c)V_i{}^n-\alpha_c V_{i-1}{}^n\\ \gamma_c &= (r-\frac{1}{2}\sigma^2)\frac{\Delta\tau}{4\Delta X}+\frac{\sigma^2\Delta\tau}{4\Delta X^2})\\ \beta_c &= \frac{\sigma^2\Delta\tau}{2\Delta X^2}+\frac{r\Delta\tau}{2}\\ \alpha_c &= (r-\frac{1}{2}\sigma^2)\frac{\Delta\tau}{4\Delta X}-\frac{\sigma^2\Delta\tau}{4\Delta X^2}\end{aligned} \tag{14}$$

2.4 FD-Schemes for European call

2.4.1 FTCS

Now we want to apply the FTCS scheme and the Crank-Nicolson scheme in the form of matrix vector notation as:

$$B\vec{V}^{n+1} = A\vec{V}^n. \tag{15}$$

Besides, we need to include boundary in the equation 15. A boundary condition we use in this report is that B and A are matrix without zero diagonals. For FTCS scheme, the equation is: $V(0,t) = 0$ and $V(X_{max},t) = e^X - Ee^{-r(T-t)}$. Transforming it into the FTCS scheme, we have $V_0^n = 0, V_N^n = Ne^{\Delta X} - Ee^{-rn\Delta t}$.

Thus the equation 15 can be rewritten by equation 9 and boundary condition.

$$B\vec{V}^{n+1} = A\vec{V}^{n} + k^{n}. \tag{16}$$

where k vector contains the boundary condition, B is identity matrix. A is sparse matrix of V^n, V^{n+1} contains the vector of V for next time step and V^n contains the vector of V for this time step.

The matrix will become:

$$B\begin{bmatrix} V_1^{n+1} \\ V_2^{n+1} \\ V_3^{n+1} \\ \vdots \\ V_{N-2}^{n+1} \\ V_{N-1}^{n+1} \end{bmatrix} = \begin{bmatrix} \beta & \gamma & 0 & \dots & 0 & 0 \\ \alpha & \beta & \gamma & \dots & 0 & 0 \\ 0 & \alpha & \beta & \gamma & \dots & 0 \\ \dots & \dots & \dots & \dots & \dots & \dots \\ \dots & \dots & \dots & \alpha & \beta & \gamma \\ 0 & 0 & \dots & 0 & \alpha & \beta \end{bmatrix} \begin{bmatrix} V_1^{n} \\ V_2^{n} \\ V_3^{n} \\ \dots \\ V_{N-2}^{n} \\ V_{N-1}^{n} \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ 0 \\ \vdots \\ 0 \\ \gamma V_N^n \end{bmatrix} \tag{17}$$

2.4.2 Crank-Nicolson scheme

For Crank-Nicolson scheme, from equation 14 we have $B\vec{V}^{n+1} = A\vec{V}^{n}$, but we know this equation doesn't consider the boundary condition and we need to include the boundary condition into this equation. The boundary condition we use like $V(0,t) = 0$ and $V(X_{max}, t) = e^X - Ee^{-r(T-t)}$. Transforming it into the Crank-Nicolson scheme, we have $V_0^{n+1} = 0, V_N^{n+1} = Ne^{\Delta X} - Ee^{-r(n+1)\Delta t}$. And $V_0^n = 0, V_N^n = Ne^{\Delta X} - Ee^{-rn\Delta t}$. Thus the equation will become:

$$B\vec{V}^{n+1} + k^{n+1} = A\vec{V}^{n} + k^{n} \tag{18}$$

where k vector contains boundary condition, B is sparse matrix of V^{n+1},A is sparse matrix of V^n, and V^{n+1} contains vector of V for next time step and V^n contains vector of V for this time step.

And the matrix of equation 18 like:

$$\begin{bmatrix} 1+\beta_c & \gamma_c & 0 & \dots & 0 & 0 \\ \alpha_c & 1+\beta_c & \gamma_c & \dots & 0 & 0 \\ 0 & \alpha_c & 1+\beta_c & \gamma_c & \dots & 0 \\ \dots & \dots & \dots & \dots & \dots & \dots \\ \dots & \dots & \dots & \alpha_c & 1+\beta_c & \gamma_c \\ 0 & 0 & \dots & 0 & \alpha_c & 1+\beta_c \end{bmatrix} \begin{bmatrix} V_1^{n+1} \\ V_2^{n+1} \\ V_3^{n+1} \\ \vdots \\ V_{N-2}^{n+1} \\ V_{N-1}^{n+1} \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ 0 \\ \dots \\ 0 \\ \gamma_c V_N^{n+1} \end{bmatrix} = \tag{19}$$

$$\begin{bmatrix} 1-\beta_c & -\gamma_c & 0 & \dots & 0 & 0 \\ -\alpha_c & 1-\beta_c & -\gamma_c & \dots & 0 & 0 \\ 0 & -\alpha_c & 1-\beta_c & -\gamma_c & \dots & 0 \\ \dots & \dots & \dots & \dots & \dots & \dots \\ \dots & \dots & \dots & -\alpha_c & 1-\beta_c & -\gamma_c \\ 0 & 0 & \dots & 0 & -\alpha_c & 1-\beta_c \end{bmatrix} \begin{bmatrix} V_1^{n} \\ V_2^{n} \\ V_3^{n} \\ \vdots \\ V_{N-2}^{n} \\ V_{N-1}^{n} \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ 0 \\ \dots \\ 0 \\ \gamma_c V_N^{n} \end{bmatrix} \tag{20}$$

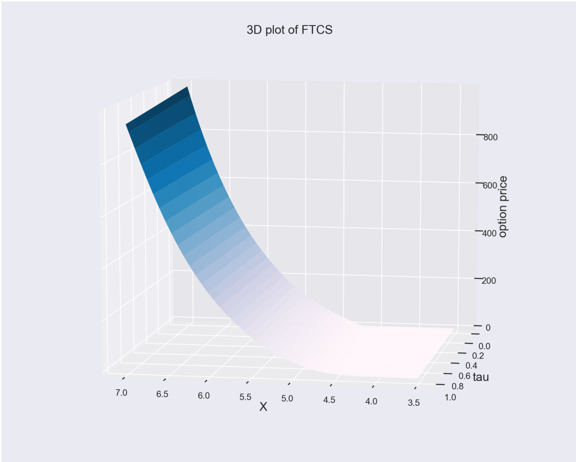
Then we can use matrix to solve this problem by:

$$\vec{V}^{n+1} = B^{-1}(A\vec{V}^{n} + k^{n} - k^{n+1}) \tag{21}$$

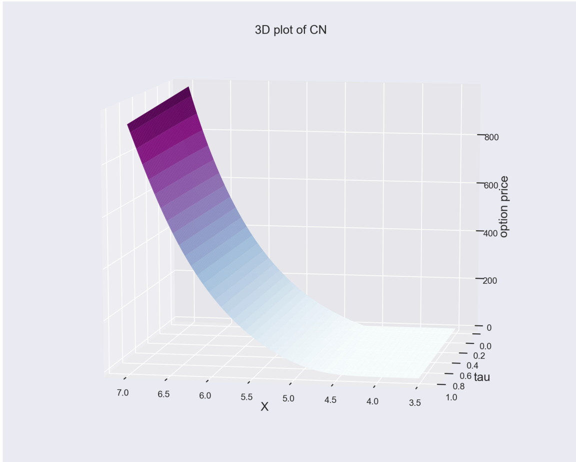
3 Results and Discussion

3.1 Result of two schemes

Figure 2 show the grid values over varying X values and τ values, for 1000 points for τ and 1000 points of X. In the figure, the results of the FTCS and CN method are nicely shown, and there is no visual difference in the curves. However, when numerically comparing them, Table 1 nicely show the final option values for FTCS and CN versus the analytical value from the Black-Scholes model. For all different S_0, the CN method gives a closer value for the option. Both of the schemes overestimate the option price under these parameters. The magnitude of the estimation differences has been visualized in Figure 3. This difference shows that the CN scheme significantly outperforms the FTCS scheme, with the most significant being the at-the-money option.



(a) 3D plot of FTCS. $\Delta\tau = 0.001, \Delta X = 0.012$. And range of X is from 3.5 to 7. $r = 0.04, T = 1, \sigma = 0.3 K = 110$.



(b) 3D plot of CN. $\Delta\tau = 0.001, \Delta X = 0.012$. And range of X is from 3.5 to 7. $r = 0.04, T = 1, \sigma = 0.3 K = 110$.

Figure 2: Results of FTCS and CN

	$S_0 = 100$	$S_0 = 110$	$S_0 = 120$
Black-Scholes	9.6253	15.1286	21.7888
FTCS	9.6283	15.1316	21.7935
CN	9.6269	15.1299	21.7918

Table 1: Results of option price from FTCS and CN scheme versus analytical Black-Scholes value.

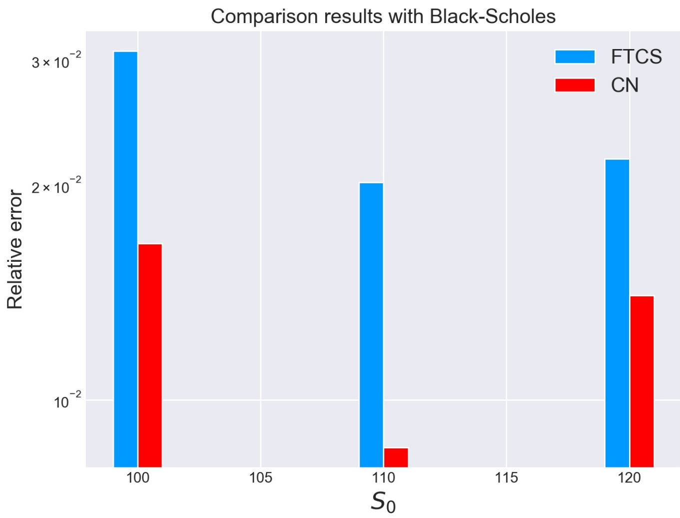


Figure 3: Relative error of estimate option price of two schemes with analytical Black-Scholes.

3.2 Convergence for two schemes in increasing number of space

Regarding the convergence, Figure 4 shows that the FTCS scheme has a strange convergence pattern. The magnitude of the error has to do with the conditional instability for the cases when ΔX is significantly smaller than τ, but the pattern itself is not so easily explained. Note that this pattern would disappear by taking a larger sampling interval such as 100, in which case the pattern would never present itself. Now, there may be multiple reasons for this pattern. A large part of the pattern is simply a sampling issue, having to do with the fact that a step size of 10 was used in the N_X range. This effect can be seen in Figure 5, which is sampled in every step. It shows very rapid oscillations and a converging pattern. When sampling this at 10 point intervals, the oscillations disappear, and a new wave emerges, which is simply a sampling artifact. However, even when correcting for this, the value seems to have lower variance in oscillations around specific points. These points show up as peaks in Figure 4, since the sampling error will be lower there due to lower variance. Of course, this pattern could also be attributed to the matrix being ill-conditioned at these points, meaning no specific meaning can be assigned to them as it is unstable.

Also interesting about the scheme is that the final option price it seems to converge to is slightly positively biased, where the option price is slightly higher than the actual value obtained from the Black-Scholes formula.

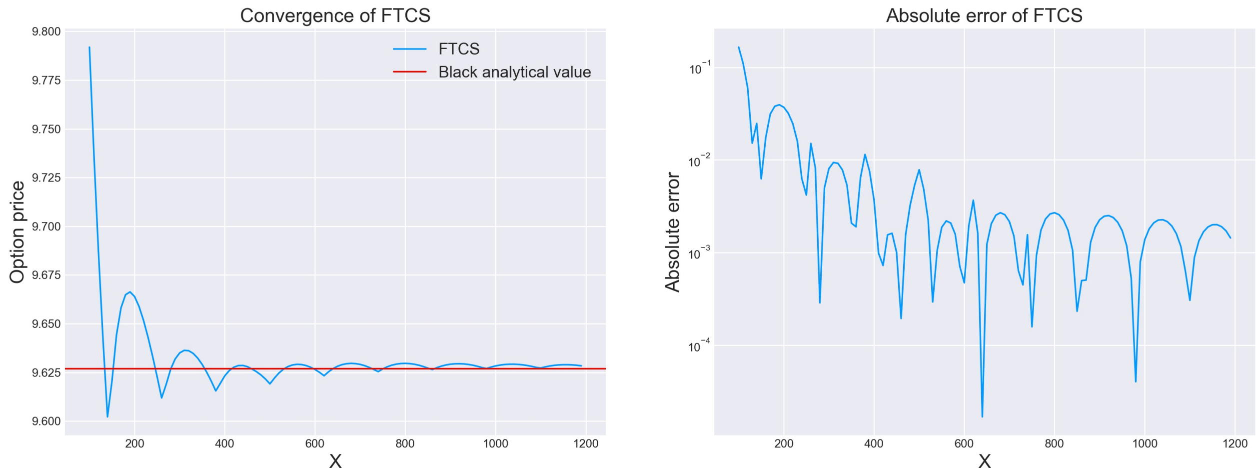


Figure 4: Convergence of FTCS in increasing number of space. ΔX is range from 100 to 1200. $\Delta\tau$ is 1000. $S_0 = 100$. $r = 0.04, T = 1, \sigma = 0.3, K = 110$. And red line is analytical value is 9.6253.

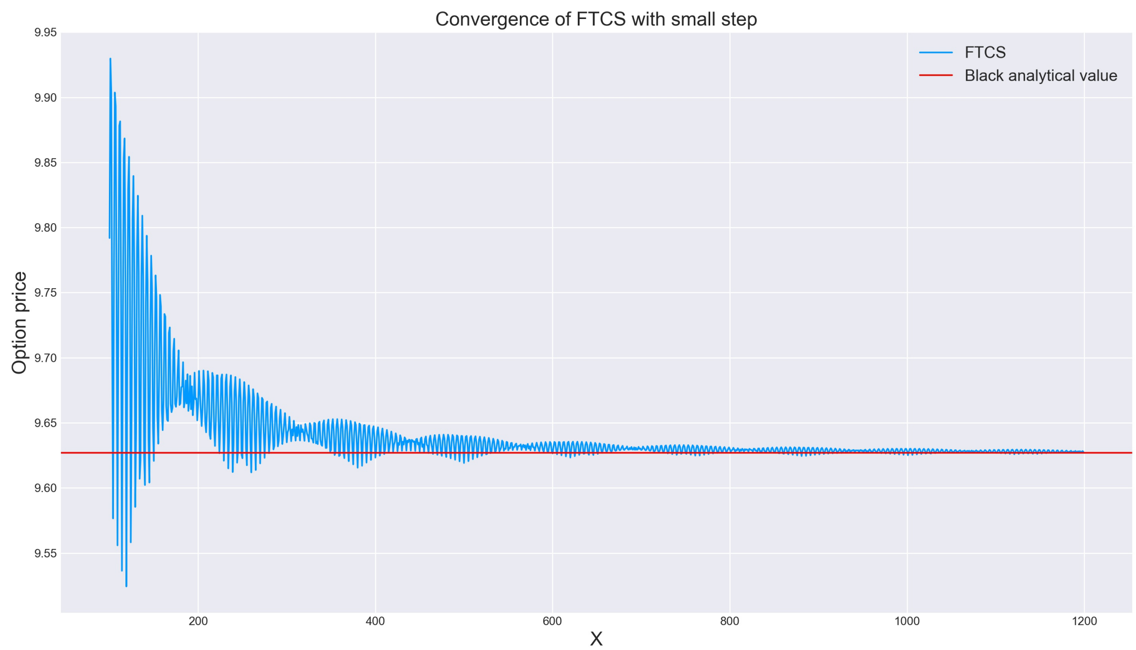


Figure 5: Convergence of FTCS in increasing number of space. ΔX is range from 100 to 1200 and step is 1. $\Delta\tau$ is 1000. $S_0 = 100$. $r = 0.04, T = 1, \sigma = 0.3, K = 110$. And red line is analytical value is 9.6253.

Figure 6 shows the expected pattern of convergence, where there are high initial value and incorrect pricing, and the price gets closer to the analytical value as the number of points in X increases. Interestingly, there is a significant dip right before 2000 grid points for X. This is most likely the point where τ becomes the bottleneck for further improvements, and lack of points in τ will drive the error. Additionally, in comparison with the final option price in the FTCS scheme, the CN scheme seems to *underestimate* the option price. However, this might be an artifact because of the lack of points in τ again.

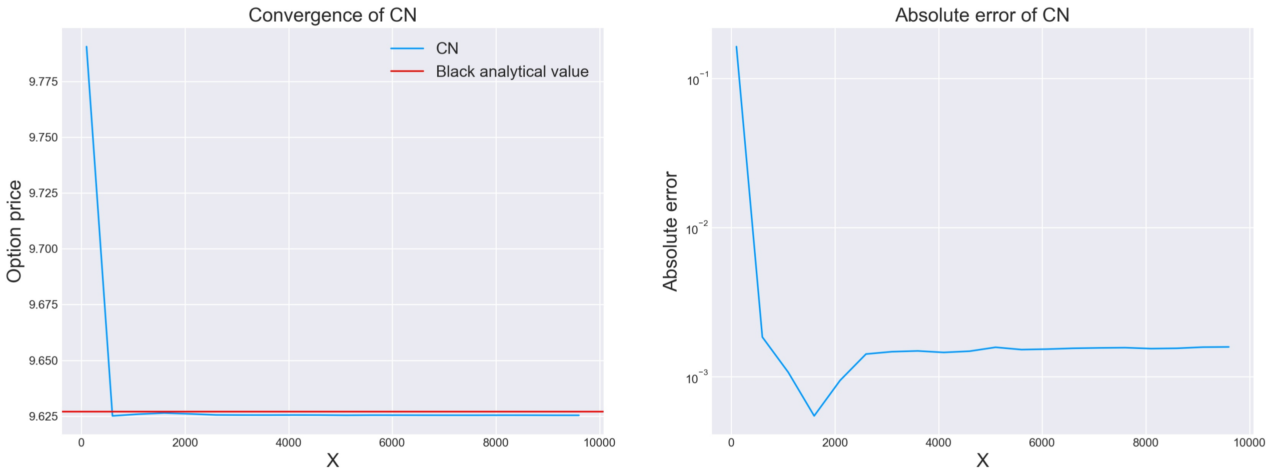


Figure 6: Convergence of CN in increasing number of space. ΔX is range from 100 to 10000. $\Delta\tau$ is 1000. $S_0 = 100$. $r = 0.04, T = 1, \sigma = 0.3, K = 110$. And red line is analytical value is 9.6253.

3.3 Optimize mesh size

As the optimal points for FTCS we have found that $N_X = 1000$ and $N_\tau = 2000$, at which point the FTCS is conditionally stable. Figure 7 the relative error in relations to the point in X. Note that the range for the points is quite small because the relative error explodes as ΔX gets smaller due to machine precision issues. The same is the case for τ, in which there are two blowups; one for being ill-conditioned, and one for machine precision later. These have been omitted from the graphs since it made a visual inspection of the true effect impossible.

optimal for FTCS:ΔX=1000,$\Delta\tau$=2000(conditionally stable)

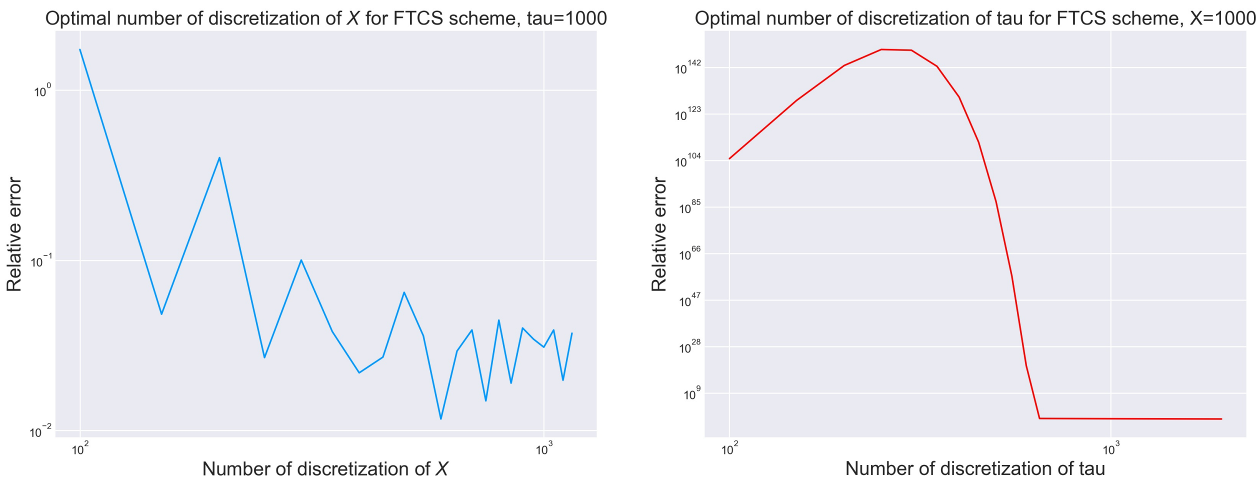


Figure 7: Optimization $\Delta X, \Delta\tau$ of FTCS. ΔX is range from 100 to 1500. $\Delta\tau$ is range from 100 to 2000. $S_0 = 100$. $r = 0.04, T = 1, \sigma = 0.3, K = 110$.

For the CN scheme, the parameters we chose are $N_X = 10000$ and $N_\tau = 2000$. Do note that the CN scheme is unconditionally stable, so more grid points are better, up to the point machine precision issues are hit ($\Delta X < \epsilon$), Apart from that, estimations will improve.

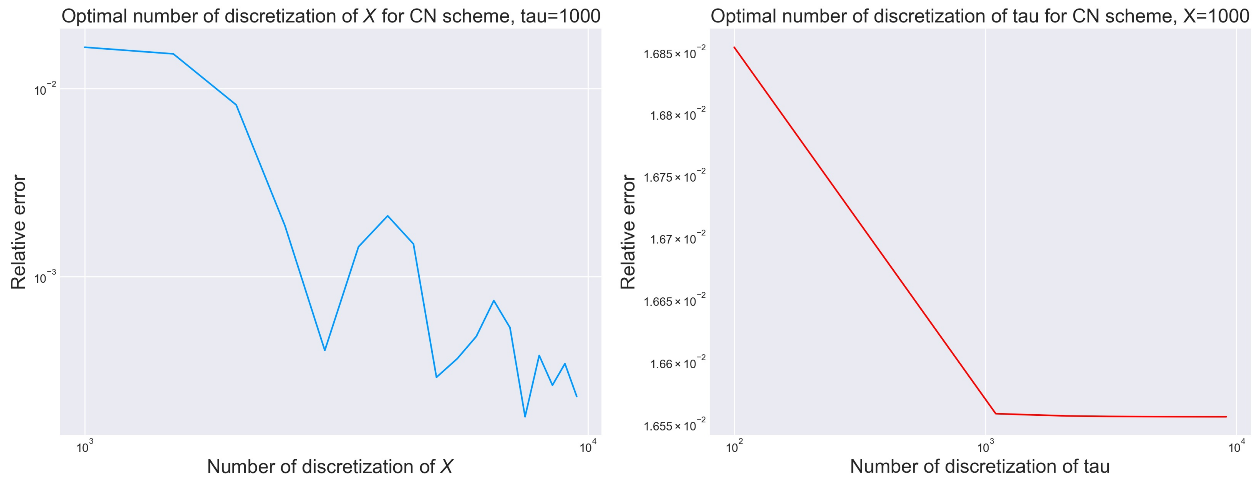
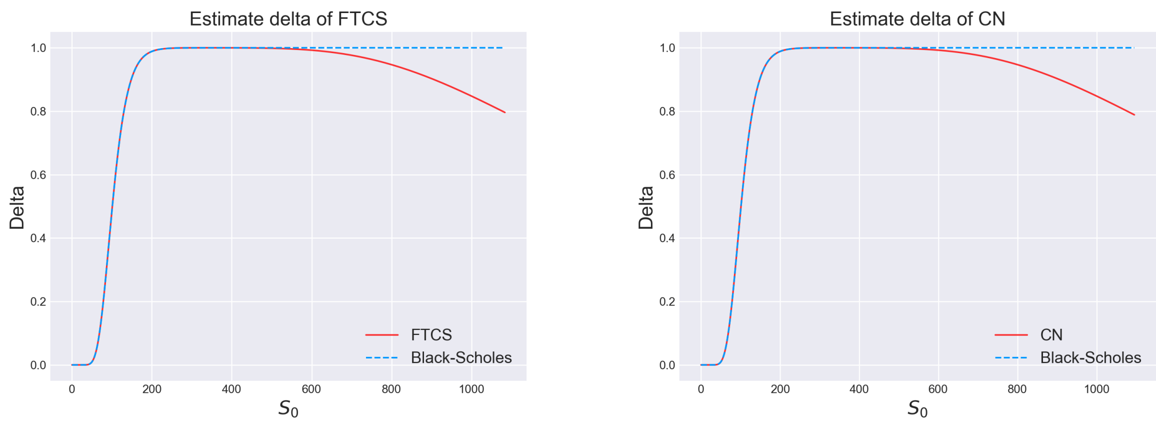


Figure 8: Optimization $\Delta X, \Delta\tau$ of CN. ΔX is range from 1000 to 10000. $\Delta\tau$ is range from 100 to 10000. $S_0 = 100$. $r = 0.04, T = 1, \sigma = 0.3, K = 110$.

3.4 Delta

Figure 9 shows that for both the FTCS scheme and the CN scheme, the estimated delta is visually indistinguishable from the analytical delta. Table 2 quantifies this and shows that the delta estimation for both FTCS and CN scheme is very close to the true value, but for $S_0 = 110$ and $S_0 = 120$ the CN scheme actually reaches the true value. Figure 10 nicely shows that the CN scheme significantly outperforms the FTCS scheme.

Interestingly, the graph also shows that the delta gets worse again as S_0 increases. This error might be due to machine precision and rounding errors. This result would mean that the schemes would not be very robust against moves that would be too large, such as cases where the price increases tenfold. The estimated delta (and most likely also the price) will be incorrect. It must also be noted that these events are extremely improbable, but possible nevertheless.



(a) FTCS scheme

(b) CN scheme

Figure 9: (a) Estimate Delta of FTCS. $\Delta\tau = 0.0005, \Delta X = 0.001$. And S_0 is range from 10 to 1000. $r = 0.04, T = 1, \sigma = 0.3 K = 110$. (b) Estimate Delta of CN. $\Delta\tau = 0.0005, \Delta X = 0.001$. And range of S_0 is range from 10 to 1000. $r = 0.04, T = 1, \sigma = 0.3 K = 110$.

	$S_0 = 100$	$S_0 = 110$	$S_0 = 120$
Black-Scholes	0.4862	0.6115	0.7168
FTCS	0.4851	0.6101	0.7162
CN	0.4863	0.6115	0.7168

Table 2: Results of delta from FTCS and CN scheme versus analytical Black-Scholes.

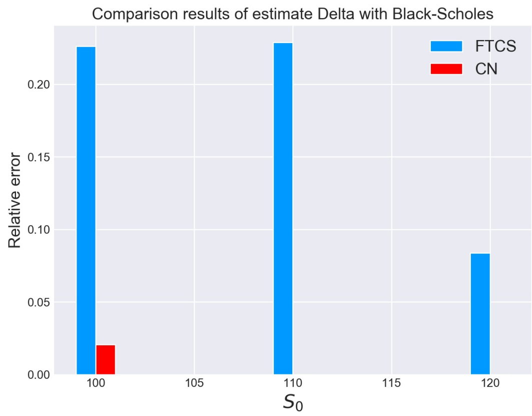


Figure 10: Relative error of estimate Δ of FTCS and CN scheme in comparison to the analytical value.

4 Conclusion

Firstly, we have shown that both the FTCS scheme and the CN scheme make good approximations under their optimal parameters for most of the prices around S_0. We have also shown that the CN scheme approximation is better in all respects than the FTCS scheme, as the error is lower for any combination. This result shows that the CN scheme is an improvement over the original FTCS scheme.

Secondly, we have determined the optimal parameters for the number of points in X and τ for both cases, and have shown that more is better up to the point where the error explodes due to machine precision issues. Additionally, we have shown that the conditional stability of the FTCS scheme makes it inferior to the CN scheme and that the CN scheme converges a lot faster than the FTCS scheme for the same option.

Lastly, we have also shown that the schemes can be used to estimate Δ for a given option and that the CN scheme under the optimal parameters has Δ as the analytical value. Under this case, the delta also becomes incorrect as S_0 increases across a threshold, and calculation errors become more apparent.