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Assignment 2
Monte Carlo (MC) Methods in Finance

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1 Abstract

Monte Carlo method can be applied into the evaluation of option price. First, Monte Carlo method was used to analyze basic European put option price with changing paths, volatility and strike price. Then We used Monte Carlo method to estimate of δ of European put option price and digital option price. And for digital option price, a Likelihood ratio method was used to get more accurate estimate of δ. Then, we use Monte Carlo method to analyze Asian option price and apply control variate technique to decrease the variance of MC to get accurate estimation.

2 Introduction

Monte Carlo (MC) is an effective method to evaluate stochastic process. And it can also be used into evaluation of option price. In this report, first we use the MC method to evaluate basic European option price, which mainly about exploring convergence, varying strike price and varying volatility. And the results will compare with analytical solution from Black-Scholes formula and binary tree used in assignment 1.

Furthermore, we used MC method to estimate the sensitivities. Bump-and-revalue method was used to get the hedge parameter σ. Same seed and different seeds were discussed respectively. And we also evaluate δ in digital option. A Likelihood ratio method was used to get more accurate estimate than MC method for digital option.

Finally, we use MC method to analyze convergence of Asian option price base on geometric averages. Furthermore, we used control variate technique to decrease variance of MC method to get accurate result of Asian option price base on arithmetic averages. And we change number of time pints, paths, strike price and volatility to evaluate the Asian option price.

3 Method and Theory

3.1 Basic Option Valuation

First recall Black-Scholes put option price formula:

$$\begin{aligned} P_t &= N(-d_2)Ke^{-r(T-t)} - N(-d_1)S_t \\ d_1 &= \frac{1}{\sigma\sqrt{T-t}}\left[\ln(\frac{S_t}{K}) + (r+\frac{\sigma^2}{2})(T-t)\right] \\ d_2 &= d_1 - \sigma\sqrt{T-t} \end{aligned} \tag{1}$$

where $N(x) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{x} e^{-\frac{1}{2}x^2}dx$. The hedge parameter for put option Δt ($t < T$) is then

$$\Delta t = \frac{\partial C}{\partial S} = -N(-d_1) \tag{2}$$

MC method can be used to evaluate option price. The steps show as follow:

- Using equation 3 to generate N paths of stock price S_T
- Get $C_1, C_2, \ldots, C_N$ option price.According to Law of Large Numbers, when N is large enough MC estimate will converge to the true value.
- Using equation 4 to get first estimate of call option $\bar{C}_1$.
- Repeat step 1,2,3 to generate $\bar{C}_1, \bar{C}_2 \ldots \bar{C}_m$. Estimate of MC follows normal distribution according to Central Limit Theorem.

$$S_T = S_0 e^{(r-0.5\sigma^2)T+\sigma\sqrt{T}Z} \tag{3}$$

$$e^{-rT}E(MAX(S_T-K,0)) \approx e^{-rT}\frac{1}{N}(\sum_{i=1}^{N} MAX(S_i-K,0)) \tag{4}$$

As a note, since $\bar{C}$ follows a normal distribution, the 95% confidence interval can be written like $[\bar{C} - 1.96\frac{\sigma_{\bar{C}}}{\sqrt{m}}, \bar{C} + 1.96\frac{\sigma_{\bar{C}}}{\sqrt{m}}]$. The standard error was measured by $\frac{\sigma_{\bar{C}}}{\sqrt{m}}$.

In the report, strike price $K = 99$, stock price $S_0 = 100$, volatility $\sigma = 0.2$, risk-free interest rate $r = 0.06$ and time length $T = 1$.

3.2 Estimation of Sensitivities in MC method

3.2.1 Bump-and-revalue method

Furthermore, MC method was used to analyze sensitivities of option price. The hedge parameter δ can be estimated by bump-and-revalue method. The steps of method show as follow:

- Using same seeds or different seeds in equation 3 to generate N paths of S_{T1}, S_{T2} with the initial stock price S_0 and $S_0 + \epsilon$.
- Using MC method to calculate mean option price, wich are $\bar{V}(S_0) = \sum_{i1=1}^{N} MAX(S_{i1} - K, 0))$ and $\bar{V}(S_0 + \epsilon) = \sum_{i2=1}^{N} MAX(S_{i2} - K, 0))$.
- Calculate first estimate δ:$\bar{\delta} = e^{-rT}\frac{\bar{V}(S_0+\epsilon)-\bar{V}(S_0)}{\epsilon}$
- Repeat step 3 to generate $\bar{\delta_1}, \bar{\delta_2} \ldots \bar{\delta_m}$. Estimate of MC follows a normal distribution according to Central Limit Theorem.

3.2.2 Digital option price

A digital option is also called binary option. Digital options work as follows.

$$\text{Option price} = \begin{cases} 1 \text{ if } S_T > K \\ 0 \text{ if } S_T \leqslant K \end{cases} \tag{5}$$

Besides, the Black-Scholes formula is

$$\begin{aligned} C_t &= N'(d_2)e^{-rT} \\ d_2 &= \frac{1}{\sigma\sqrt{T}}\left[\ln(\frac{S_t}{K}) + (r - \frac{\sigma^2}{2})T\right] \end{aligned} \tag{6}$$

where $N'(x)$ is probability density function of normal distribution. Then the hedge parameter for call option δ:

$$\delta = \frac{\partial C}{\partial S} = \frac{e^{-rT}N'(d_2)}{S_0\sigma\sqrt{T}} \tag{7}$$

To evaluate the δ of a digital option, we can use the MC method as introduced before. Another method is the Likelihood ratio method. The theory of the Likelihood ratio method is as follows.

First, we have call option.

$$\begin{aligned} C =& e^{-rT}E(MAX(S_T-K,0)) \\ &\text{where log-normal density of } S_T: \\ g(x) &= \frac{x}{x\sigma\sqrt{2\pi T}}e^{-0.5(\ln(\frac{x}{S_0})-(r-\frac{\sigma^2}{2}))} \\ \text{Then:}\frac{\partial \ln g(x)}{\partial S_0} &= \frac{Z}{S_0\sigma\sqrt{T}} \\ \text{Finally:}\delta = \frac{dC}{dS_0} &= E(e^{-rT}MAX(S_T-K,0)\frac{Z}{S_0\sigma\sqrt{T}}) \\ \text{For digital option price : } \delta &= E(e^{-rT}1_{S_T>K}\frac{Z}{S_0\sigma\sqrt{T}}) \end{aligned} \tag{8}$$

3.3 Variance Reduction

3.3.1 Derivation of Analytical Asian option price

In order to get the analytical Asian option price, first recall the geometric average

$$\widetilde{A} = \left(\prod_{i=1}^{N} S_i\right)^{\frac{1}{N}} \tag{9}$$

then we have:

$$\begin{aligned}
\ln \frac{\widetilde{A_N}}{S_0} &= \ln \frac{\left(\prod_{i=1}^{N} S_i\right)^{\frac{1}{N}}}{S_0} \\
&= \ln \frac{\left(\frac{S_N}{S_{N-1}}\left(\frac{S_{N-1}}{S_{N-2}}\right)^2\left(\frac{S_{N-2}}{S_{N-3}}\right)^3\cdots\left(\frac{S_1}{S_0}\right)^N S_0^N\right)^{\frac{1}{N}}}{S_0} \\
&= \ln \left(\frac{S_N}{S_{N-1}}\right)^{\frac{1}{N}}\left(\frac{S_{N-1}}{S_{N-2}}\right)^{\frac{2}{N}}\left(\frac{S_{N-2}}{S_{N-3}}\right)^{\frac{3}{N}}\cdots\left(\frac{S_2}{S_1}\right)^{\frac{N-1}{N}}\left(\frac{S_1}{S_0}\right)^{\frac{N}{N}} \\
&= \frac{1}{N}\ln\frac{S_N}{S_{N-1}} + \frac{2}{N}\ln\frac{S_{N-1}}{S_{N-2}} + \cdots + \frac{N-1}{N}\ln\frac{S_2}{S_1} + \frac{N}{N}\ln\frac{S_1}{S_0}
\end{aligned} \tag{10}$$

Recall we have:$S_i = S_{i-1}e^{(r-0.5\sigma^2)\frac{T}{N}+\sigma Z_i}$, where $i \in 1, 2, \ldots, N$ and $Z_i \sim \mathbb{N}(0, \frac{T}{N})$. Then: and then we got:

$$\begin{aligned}
\ln \frac{S_i}{S_{i-1}} &= (r-0.5\sigma^2)\frac{T}{N} + \sigma Z_i \\
&\sim \mathbb{N}((r-0.5\sigma^2)\frac{T}{N}, \sigma^2\frac{T}{N})
\end{aligned} \tag{11}$$

Then, we can get use equation 10 to get:

$$\begin{aligned}
\ln \frac{\widetilde{A_N}}{S_0} &\sim \mathbb{N}(\text{Mean}, \text{Variance}) \\
\text{Mean} &= \frac{1}{N}(1+2+\cdots+N)(r-0.5\sigma^2)\frac{T}{N} \\
&= \frac{1}{N}\frac{N(N+1)}{2}(r-0.5\sigma^2)\frac{T}{N} \\
&= (r-0.5\sigma^2)\frac{(N+1)}{2N}T \\
\text{Variance} &= \frac{1}{N^2}(1^2+2^2+\cdots+N^2)\sigma^2\frac{T}{N} \\
&= \frac{1}{N^2}\frac{N(N+1)(2N+1)}{6}\sigma^2\frac{T}{N} \\
&= T\frac{(N+1)(2N+1)}{6N^2}\sigma^2 \\
\text{Then}: \ln \frac{\widetilde{A_N}}{S_0} &\sim \mathbb{N}((r-0.5\sigma^2)\frac{(N+1)}{2N}T, T\frac{(N+1)(2N+1)}{6N^2}\sigma^2) \\
&\overset{N\to\infty}{\to} \mathbb{N}(\frac{T}{2}(r-0.5\sigma^2), \frac{T}{3}\sigma^2)
\end{aligned} \tag{12}$$

Thus, we can get the call option price. Note we use $-Z$:

$$\begin{aligned}
V_0 &= e^{-rT}\mathbb{E}[(A_N - K)^+] \\
&= e^{-rT}\mathbb{E}[(S_0 e^{\frac{1}{2}(r-0.5\sigma^2)T-\sigma\sqrt{\frac{T}{3}}Z} - K)^+] \\
&= \frac{e^{-rT}}{\sqrt{2\pi}}\int_{-\infty}^{+\infty}(S_0 e^{\frac{1}{2}(r-0.5\sigma^2)T-\sigma\sqrt{\frac{T}{3}}x} - K)^+ e^{-\frac{1}{2}x^2}dx
\end{aligned} \tag{13}$$

Since we only need the non-zero part: $S_0 e^{\frac{1}{2}(r-0.5\sigma^2)T-\sigma\sqrt{\frac{T}{3}}x} - K > 0$, then we have $x < \frac{\ln\frac{S_0}{K}+\frac{1}{2}(r-0.5\sigma^2)T}{\sigma\sqrt{\frac{T}{3}}x} :=$

d_- Thus, we can rewrite equation 13:

$$\begin{aligned} V_0 &= \frac{e^{-rT}}{\sqrt{2\pi}} \int_{-\infty}^{d-} S_0 e^{\frac{1}{2}(r-0.5\sigma^2)T - \sigma\sqrt{\frac{T}{3}}x} - Ke^{-\frac{1}{2}x^2} dx \\ &= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{d-} e^{-rT} S_0 e^{\frac{1}{2}(r-0.5\sigma^2)T - \sigma\sqrt{\frac{T}{3}}x} e^{-\frac{1}{2}x^2} dx - \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{d-} e^{-rT} Ke^{-\frac{1}{2}x^2} dx \end{aligned} \tag{14}$$

We use $N(x) := \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{u} e^{-\frac{1}{2}u^2} du$ as standard normal cumulative distribution function. Then, the right part of equation 14 can be rewritten as:

$$\frac{1}{\sqrt{2\pi}} \int_{-\infty}^{d_-} e^{-rT} Ke^{-\frac{1}{2}x^2} dx = e^{-rT} KN(d_-) \tag{15}$$

Then the left part of equation 14, and define $y = x + \sigma\sqrt{\frac{T}{3}}$:

$$\begin{aligned} &\frac{1}{\sqrt{2\pi}} \int_{-\infty}^{d_-} e^{-rT} S_0 e^{\frac{1}{2}(r-0.5\sigma^2)T - \sigma\sqrt{\frac{T}{3}}x} e^{-\frac{1}{2}x^2} dx \\ &= \frac{S_0}{\sqrt{2\pi}} \int_{-\infty}^{d_-} e^{-rT} e^{\frac{1}{2}(r-0.5\sigma^2)T - \sigma\sqrt{\frac{T}{3}}x} e^{-\frac{1}{2}x^2} dx \\ &= \frac{S_0 e^{\frac{-rT}{2} - \frac{\sigma^2 T}{12}}}{\sqrt{2\pi}} \int_{-\infty}^{d_-} e^{-\frac{1}{2}(x+\sigma\sqrt{\frac{T}{3}})^2} dx \\ &= \frac{S_0 e^{\frac{-rT}{2} - \frac{\sigma^2 T}{12}}}{\sqrt{2\pi}} \int_{-\infty}^{d_-(+\sigma\sqrt{\frac{T}{3}})} e^{-\frac{1}{2}(y)^2} dy \\ &= S_0 e^{-\frac{1}{2}(rT + \frac{1}{6}\sigma^2 T)} N(d_+) \end{aligned} \tag{16}$$

Finally, we get the Black-Scholes formula for Asian call price:

$$\begin{aligned} V_0 &= S_0 e^{-\frac{1}{2}(r+\frac{1}{6}\sigma^2)T} N(d_+) - e^{-rT} KN(d_-) \\ d_- &= \frac{\ln\frac{S_0}{K} + \frac{1}{2}(r-0.5\sigma^2)T}{\sigma\sqrt{\frac{T}{3}}x} \\ d_+ &= d_-(+\sigma\sqrt{\frac{T}{3}}) \end{aligned} \tag{17}$$

Using same process we can get put option price:

$$\begin{aligned} P_0 &= e^{-rT} KN(-d_-) - e^{-\frac{1}{2}(r+\frac{1}{6}\sigma^2)T} N(-d_+) \\ d_- &= \frac{\ln\frac{S_0}{K} + \frac{1}{2}(r-0.5\sigma^2)T}{\sigma\sqrt{\frac{T}{3}}x} \\ d_+ &= d_-(+\sigma\sqrt{\frac{T}{3}}) \end{aligned} \tag{18}$$

3.3.2 MC method for Asian option price

Then, we use MC method to obtain Asian call price and the steps show as follow:

- Generate stock price at each time points, $S_i = S_{i-1} e^{(r-0.5\sigma^2)dt + \sigma\sqrt{t}Z}$, where $i \in 1, 2, 3 \ldots m$ and $Z \sim \mathbb{N}(0, 1)$. And then computing the payoff, discount it. Get one call option price C_1.
- Repeat step 1 n times and get $C_1, C_2, \ldots C_n$.
- Compute the average $\bar{C_n} = \sum_{i=1}^{n} C_i$. According to the big number theory, when n is large enough $\bar{C_n}$ is a great estimator of Asian call option price.

3.3.3 Control variate technique

In order to decrease variance of MC method, control variate technique was used to calculate Asian call option price based on the arithmetic average.

- $\tilde{C_A}$ Control variate estimate of Asian call option price based on arithmetic average.

- $\hat{C_A}$ MC estimate of Asian call option price based on arithmetic average.
- $\hat{C_B}$ MC estimate of Asian call option price based on geometric average.
- C_B Analytical solution of Asian call option price based on geometric average.
- σ_A stander error of $\hat{C_A}$.
- σ_B stander error of $\hat{C_B}$.
- ρ correlation between $\hat{C_A}$ and $\hat{C_B}$.
- β Control variate parameter.

Then we have: $\tilde{C_A} = \hat{C_A} - \beta(\hat{C_B} - C_B)$. Additionally, The expectation of $\tilde{C_A}$ is unbiased because $E(\tilde{C_A}) = E(\hat{C_A} - \beta(\hat{C_B} - C_B)) = E(\hat{C_A}) = C_A$.

Additionally, the variance:

$$\begin{aligned} \text{Variance} &= Var(\hat{C_A} - \beta(\hat{C_B} - C_B)) \\ &= Var(\hat{C_A} - \beta\hat{C_B}) \\ &= \sigma_A^2 + \beta^2\sigma_B^2 - 2\beta\rho\sigma_A\sigma_B \end{aligned} \tag{19}$$

Since $\rho > 0$, when we choose $\beta = \frac{\sigma^A}{\sigma^B}\rho$ can minimizes variance of equation 19. In this way, we got:

$$\tilde{C_A} = \hat{C_A} - \frac{\sigma^A}{\sigma^B}\rho(\hat{C_B} - C_B) \tag{20}$$

4 Results and Discussion

4.1 Basic Option Valuation

The analytical value of a put option is 4.7789, displayed as a horizontal line in Figure 1. The Figure clearly shows the MC converging on the analytical value, and always being in the 95% confidence interval. Figure 1 also indicates the standard error exponentially decreasing as the number of paths linearly increases, especially at the start. This decrease shows that the method is accurate, both in estimating the actual value under a large number of paths and indicating the confidence interval.

The final estimate of the option price is 4.775, with a 95% confidence interval of $[4.767, 4.783]$ and a standard error of 0.004. This result means that the analytical value is far within the confidence interval. In comparison, the binomial tree for a European put option in the first assignment had a value of 4.781. This value was overpriced and shows that the MC estimate is a better estimate than the binomial tree model(4.781).

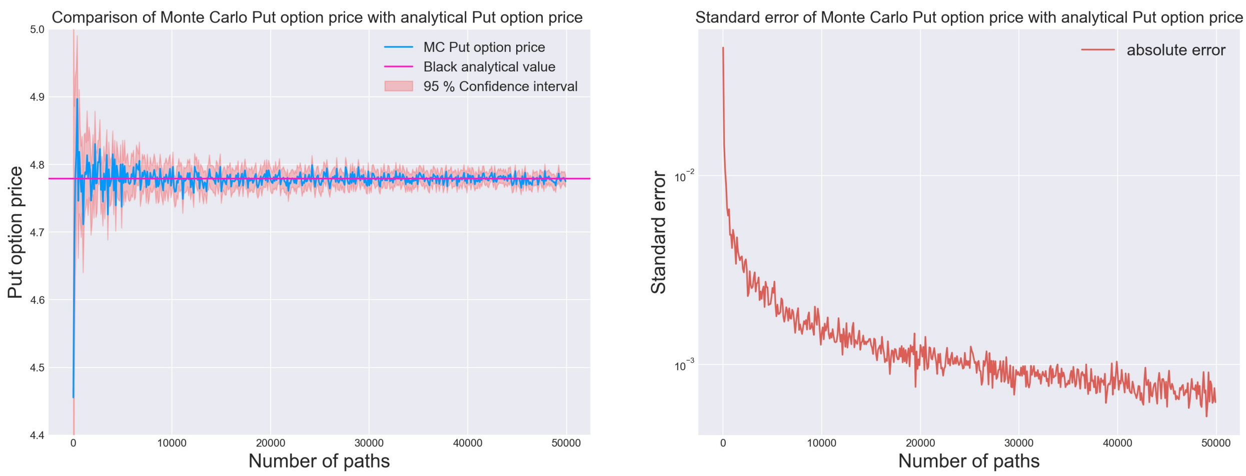

Figure 1: Estimate of MC method for varying number of paths. Range of paths is from 10 to 50000. And each number of paths have 50 realizations to get the mean put option price. Analytical value is 4.7789. Methods of confidence interval and standard error was listed at method part.

Next, Figure 2 shows the price curving upwards as the strike price is further in the money. However, it also indicates that the standard error linearly increases as the strike price gets higher. Initially, this seems surprising since the volatility does not change. However, this is since the intrinsic value of the put option increases as the strike price increases, meaning that the estimation error on the larger value will most likely also be larger. Additionally, as the price is further in the money, the probability of the option not being exercised will essentially become zero, and the intrinsic value increase will make up most of the option price.

Figure 2: Estimate of MC method for varying strike price. Range of strike price is from 90 to 110. For each paths we generates 1000 paths. For each path has 2000 realizations to get the mean value, confidence interval and standard error.

Lastly, Figure 3 shows the same thing happening for volatility. This increase is not surprising since the distribution directly depends on σ. This relation is reflected in the linear relationship between volatility a put option price. The confidence interval is so small that it is barely visible in the figure. The same argument holds, but since the volatility only has a linear relation, the standard error curve slopes downward.

This larger error could be corrected by taking more paths, since the outcomes of the experiment are more distributed for larger σ and further in the money, causing a larger standard error.

Figure 3: Estimate of MC method for varying volatility. Range of volatility is from 0.1 to 0.9. For each paths we generates 1000 paths. For each path has 2000 realizations to get the mean value, confidence interval and standard error.

4.2 Estimation of Sensitivities in MC

Figure 4 shows the δ convergence for a varying number of paths, and the standard error. Figure 5 shows the comparison of delta error between the same and different seeds. The magnitude of the error on the different-seed plot indicates that noise is being sampled, even though it goes down as the number of paths goes up. For the same seed, there does not seem to be a clear relationship between the number of paths and the relative error for different ϵ, making the error reasonably constant over the number of paths. However, as ϵ decreases, the relative error also vastly decreases. Because using different seeds is sampling noise, the figure shows a nearly inverse relation between ϵ and the relative error, making it smaller as ϵ goes up.

The analytical value for the δ of this option is -0.3262. The δ from the binomial tree model of assignment 1 is -0.3274. The equivalent MC estimate for the δ parameter is -0.3261 with a 95% confidence interval of

$[-0.3265, -0.3257]$ for a standard error of 0.0002. This result, again, shows that the MC estimate is a better estimate than the binomial tree model for $N = 50$ in the binomial tree model.

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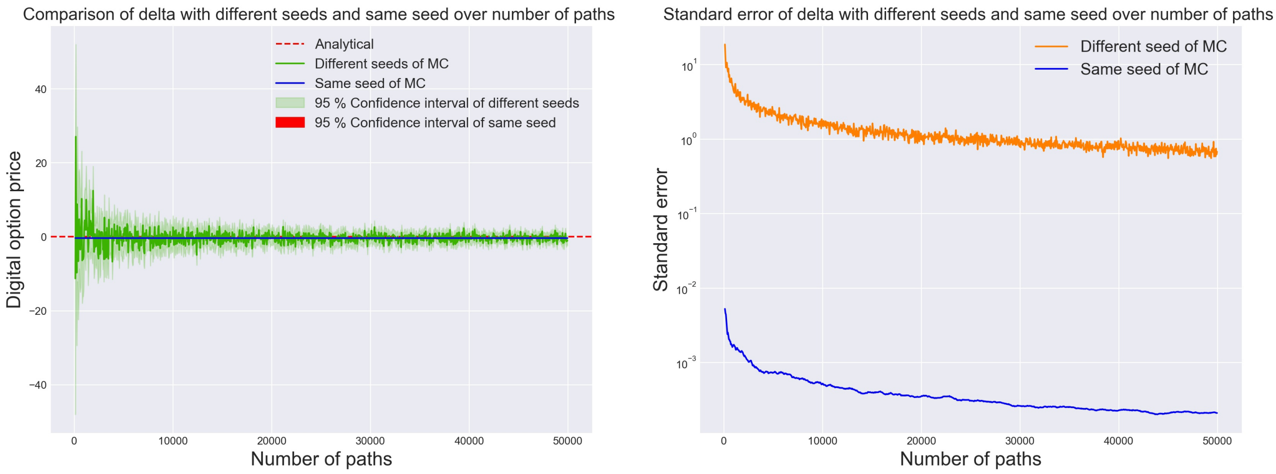

Figure 4: Convergence of δ of MC method. Left plot used different seeds, right plot used same seeds. The range of path is from 100 to 50000. Each path has 50 realizations.

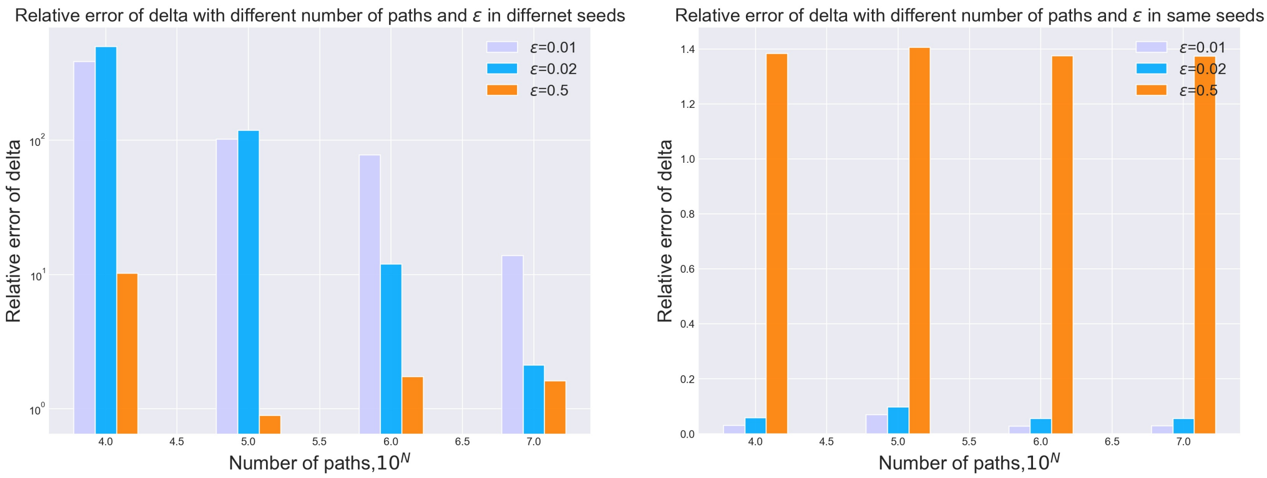

Figure 5: Relative error of δ of MC method. Left plot used different seeds, right plot used same seeds. Each parameter generates 1000 paths and each path has 50 realizations.

Paths	$\epsilon = 0.01$	$\epsilon = 0.02$	$\epsilon = 0.5$
10^4	386.67	501.46	10.27
10^5	102.25	119.05	0.89
10^6	77.99	12.02	1.74
10^7	13.89	2.12	1.62

Table 1: Different seeds of relative error in percent.

Paths	$\epsilon = 0.01$	$\epsilon = 0.02$	$\epsilon = 0.5$
10^4	0.031	0.059	1.383
10^5	0.071	0.097	1.406
10^6	0.028	0.056	1.375
10^7	0.029	0.057	1.374

Table 2: Same seed method of relative error in percent.

For the digital option, Figure 6 shows that the Monde Carlo estimate does not converge correctly to the analytical value, or that a prohibitive amount of paths need to be taken to obtain the true value. This is because the standard error is scaled with $\sqrt{N}$ and so 100 times as many paths only scales the standard error down 10 times. The analytical value is within the 95% confidence interval of the MC estimate, but around the top. The right part of Figure 6 again shows the standard error significantly decreasing as the number of paths increases. The fact that the MC estimate does not converge well is to be expected due to the inherent discontinuous payout in the digital option, as explained in the lecture. Applying the sophisticated methods, we have chosen for the Likelihood Ratio method, which is shown in the same figure. The figure clearly shows the digital option price is almost the same as the analytical value, and that the confidence interval is barely visible. This advantage is also reflected in the plot to the right, where the Likelihood Ratio standard error is a fraction of the MC estimate standard error.

The analytical δ of digital option is 0.0182, the final estimate of Likelihood ratio method is 0.0182, and 95% confidence interval is $[0.01818, 0.01823]$. The standard error for this method is $1.8 * 10^{-5}$. The final estimate of the MC method is 0.0170, with a 95% confidence interval of $[0.0165, 0.0185]$, for a standard error of 0.0008. In this case, the Likelihood ratio method yields an estimate that is equal to the analytical value, with a very narrow confidence interval. This result is in stark contrast to the normal MC estimate.

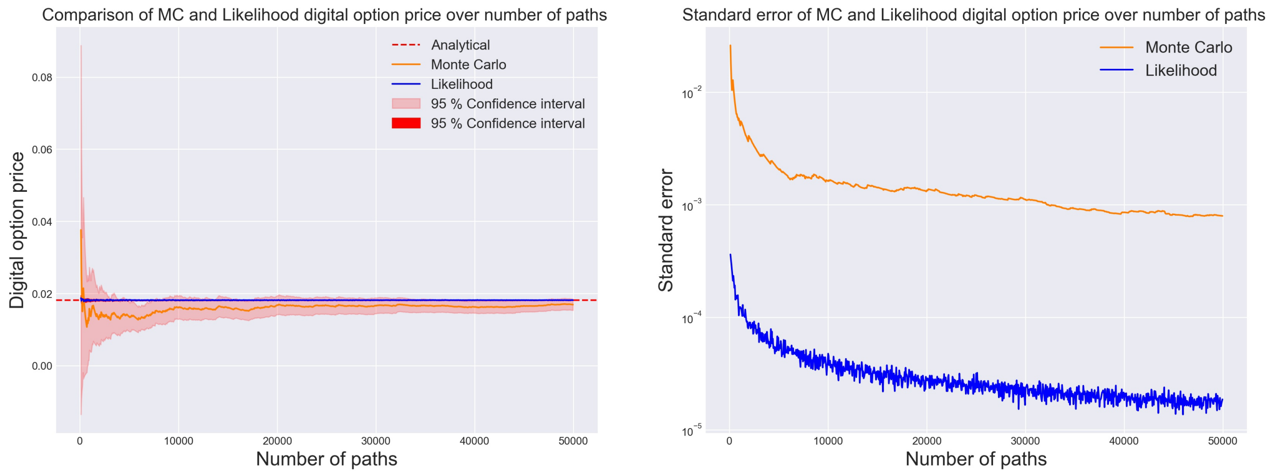

Figure 6: Comparison MC method with Likelihood ratio method. Left plot shows the convergence of two methods. Right plots shows comparison of standard error. The range of path is from 100 to 50000. Each path has 50 realizations.

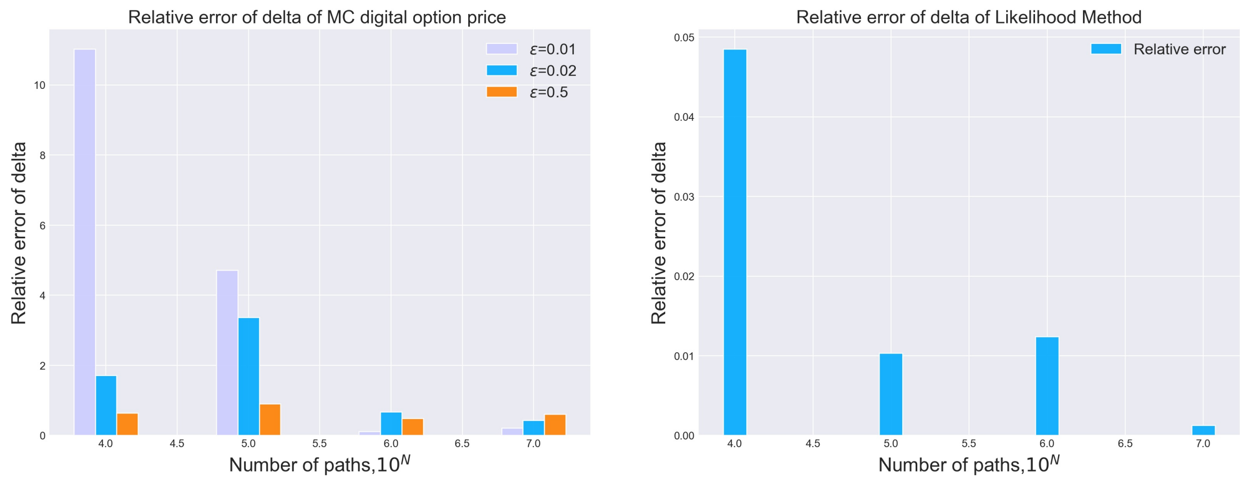

Figure 7: Relative error of δ of MC method and Likelihood method. Left plot is MC method, right plot is Likelihood method. Each parameter generates 1000 paths and each path has 50 realizations.

Table 3: δ-Same seeds MC method of relative error(%)

Paths	$\epsilon = 0.01$	$\epsilon = 0.02$	$\epsilon = 0.5$
10^4	11.03	1.72	0.64
10^5	4.72	3.37	0.91
10^6	0.11	0.67	0.48
10^7	0.21	0.43	0.61

Table 4: δ-Likelihood ratio method of absolute error and relative error(%)

Paths	Absolute error	Relative error
10^4	$8.8 * 10^{-6}$	0.048
10^5	$1.9 * 10^{-6}$	0.010
10^6	$2.3 * 10^{-6}$	0.012
10^7	$2.3 * 10^{-6}$	0.001

4.3 Variance Reduction

The Black-Scholes formula for an Asian call option based on a geometric average was derived in section 3.3.1. Figure 8 shows the convergence of the Asian geometric put option price for the number of paths. This plot clearly shows the MC estimate converging to the analytical price for the option. The analytical option price of geometric Asian put is 6.335. The final estimate of MC is 6.342, with a 95% confidence interval of $[6.336, 6.358]$ and a standard error of 0.008.

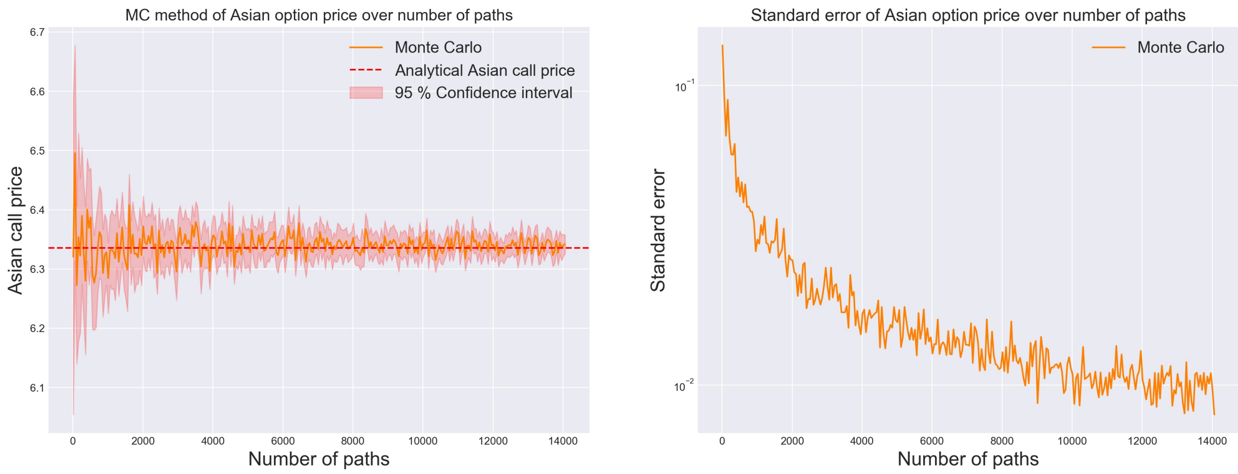

Figure 8: Convergence of MC method of geometric Asian put option price. number of time points is range from 10 to 10000. And number of paths is range from 10 to 10000.

Now, for the arithmetic average Asian put option, Figure 9 and 10 show the MC estimate and the corresponding control variate estimate with their confidence interval. It nicely shows that, although the analytical value is always within the confidence interval for both varying over time-points and varying over number of paths, the 95% confidence interval for the plain MC method is far wider than the 95% confidence interval for the control variate technique estimation, which is barely visible on the plot. The final estimate of the Asian option price based on arithmetic averages of the MC method is 6.577, with a 95% confidence interval of $[6.551, 6.603]$ and a standard error of 0.0132. For the control variate method, the final estimate is 6.567, with a 95% confidence interval of $[6.566, 6.568]$ and a standard error of 0.0004. This result shows that the control variate technique significantly reduces the standard error of the final estimate. Unfortunately, there is no analytical to compare against, but the significantly smaller confidence interval is enough to demonstrate the viability of the technique since theoretically, there should be no bias introduced from the technique.

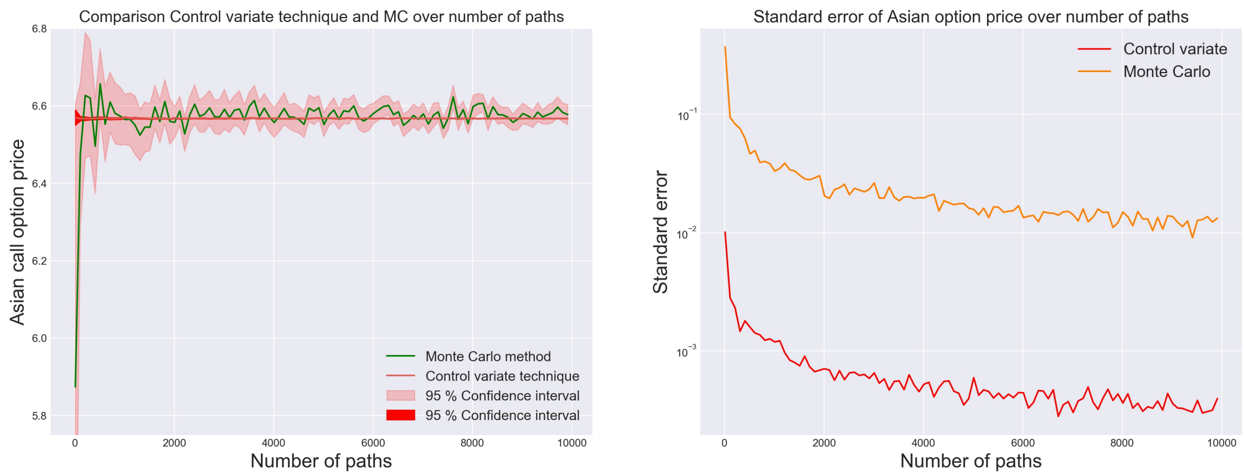

Figure 9: Convergence of MC method of arithmetic Asian put option price. And number of paths ranges from 10 to 10000. Each path has 50 realizations

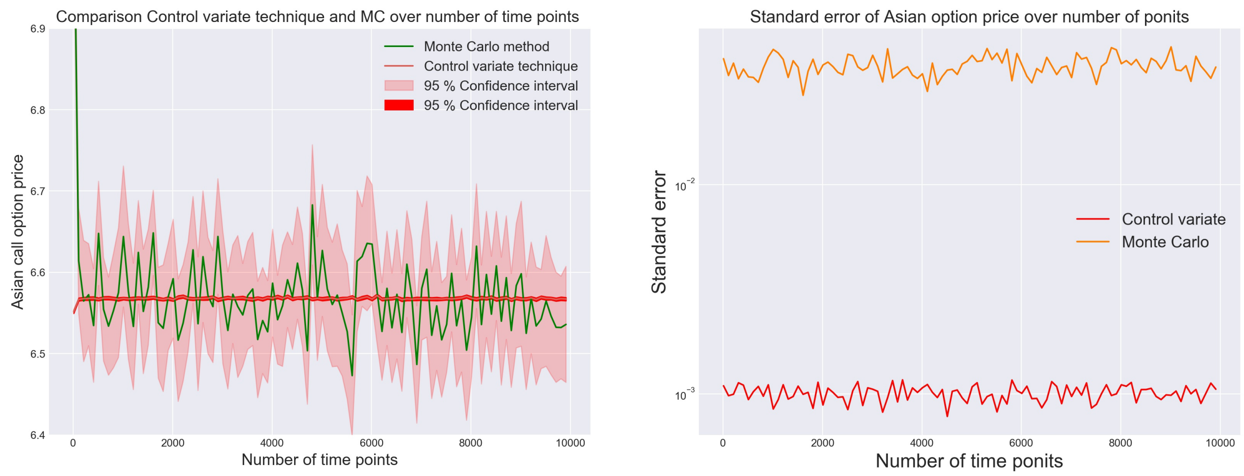

Figure 10: Convergence of MC method of arithmetic Asian put option price. Number of time points ranges from 10 to 10000. Each time point has 1000 paths with 50 realizations.

We have already explained the reasoning behind the control variate in section 3.3.3. The reason it works so well, in this case, is because there is a 0.99 correlation[1] between the geometric and the arithmetic average Asian put option, making the theoretical variance reduction $1 - \rho^2 = 1 - 0.99^2 = 0.0199$, which is more than a factor 50. This reduction makes this estimate much better than the regular MC estimate.

For further experimentation, we have also varied strike price and volatility, shown in Figure 11 and Figure 12, respectively. This plot shows that overall, the lines overlap quite nicely. However, on the standard error plots accompanied with them, it shows that the control variate yields much lower standard errors across both volatility and strike price. This effect is to be expected. However, the variance reduction seems less than expected, especially towards the higher volatilities and strike price, where the curves seem to move towards each other. One reason for this could be that for higher volatility and strike price, the options are more in the money, making the moves in option value come mostly from the intrinsic value instead of the time value. This effect, paired with the possibility that the correlation between the geometric option price and the arithmetic option price breaks down as the option is further in the money or the underlying asset has greater volatility.

[1]Taken from the lecture slides.

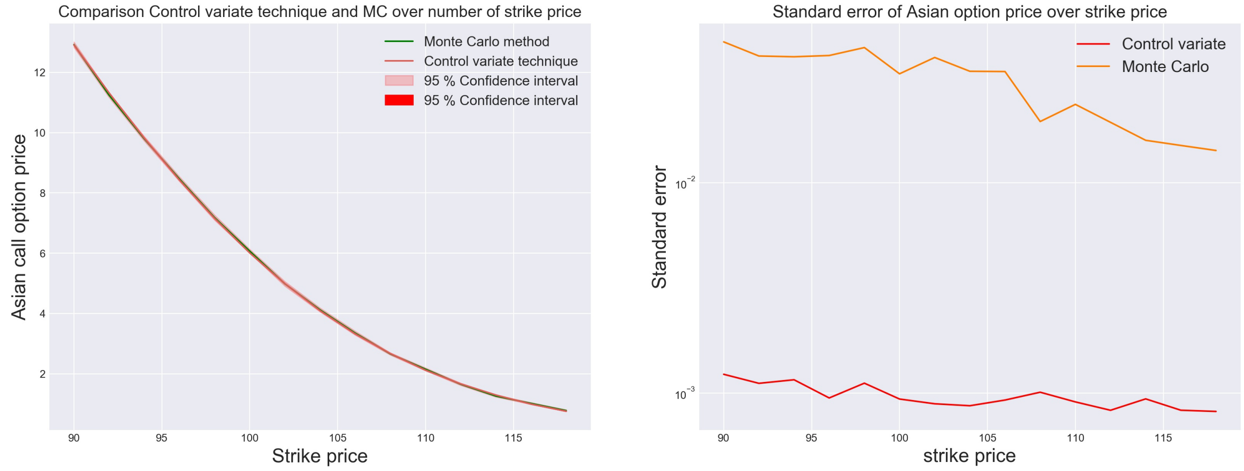

Figure 11: Comparison of Asian option price with MC method and Control variate technique over varying strike price. Left plot strike price from 90 to 120, and right plot shows the standard error. Each point is generated from 50 trials of 1000 paths.

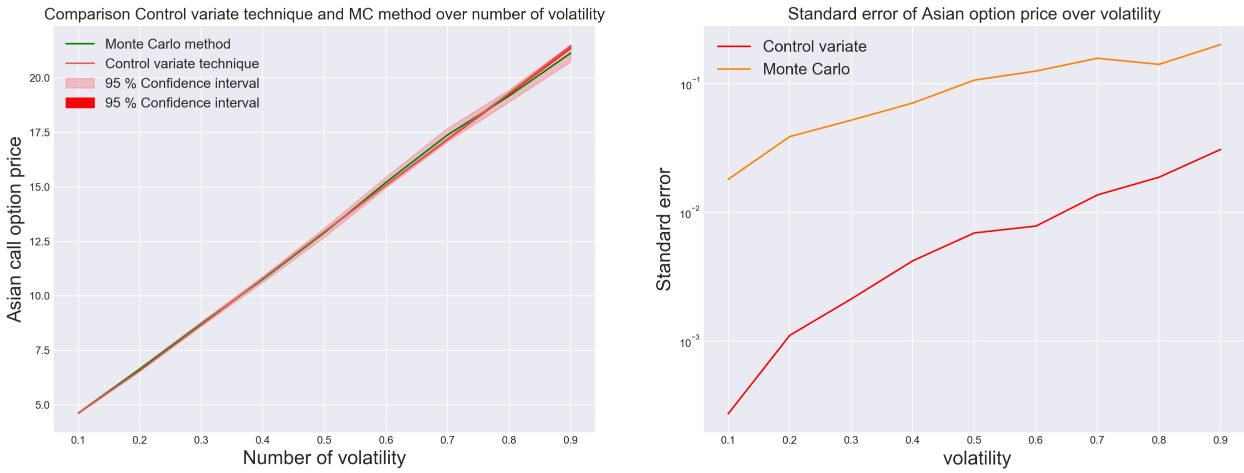

Figure 12: Comparison of Asian option price with MC method and Control variate technique over varying strike price. Left plot strike price from 90 to 120, and right plot shows the standard error. Each point is generated from 50 trials of 1000 paths.

5 Conclusion

First, we have shown that MC is a better method than the binomial tree model in terms of accuracy and that given enough paths, it gives a reasonable estimate close to the analytical value in most cases. It also indicates the uncertainty of the value, giving the confidence interval of the analytical estimate.

Secondly, we have shown that the bump-and-revalue method provides a reasonable estimate of δ, given the payoff is smooth. When the payoff is not smooth, such as with a digital option, the direct MC estimate breaks down and retains a reasonably large confidence interval and large path requirement for convergence. We have used the likelihood ratio method to correct for this, which significantly reduces the standard error. We have also shown that it is required to use the same seed since otherwise, noise is sampled because the paths do not correspond.

Lastly, we have shown a variance reduction technique by first deriving the Black-Scholes formula for geometric Asian put options, and subsequently showing that the MC estimate of this geometric option converges to the analytical value. Then, using the given 0.99 correlation, we have used the control variate method to calculate a more accurate estimate of the arithmetic option price. We have shown that this significantly reduces the standard error and stability of the estimate.

References