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Assignment 1
Black-Scholes Model and Binomial Tree Methods

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1 Abstract

In this report, we price the option by Binomial tree model and Black-Scholes formula with four conditions: No arbitrage, no dividends, no additional cost, the risk-neutral rate is constant. Both the European option and the American option are considered in our assignment. We analysis the call option and put option and then show the variety of the option price. Also, we validate if the dynamic delta hedging strategy can be used to construct a portfolio that replicates the value of a European call option.

2 Introduction

An option is one of the financial derivatives, which is a contract that gives the buyer a right to buy($call$) or sell(put) an underlying asset at the strike price(K) on a specified day. Furthermore, an American option is that the holder can exercise the option at any time before the expiration, while a European option can only be exercised at the expiration date.[1]

In task 1, we used a binomial tree model to estimate the price of the option. We consider a European call option on a non-dividend-paying stock with a maturity of one year and a strike price of 99 euro. Assuming that the one-year interest rate is 6 percent, and the current price of the stock is 20 percent. Taking a tree with 50 steps, and we experiment with different values by changing the volatility. Also, we compute the convergence rate for different N, computational complexity, and the hedge parameter for the binomial tree model. Lastly, we changed the European option to an American option to make a comparative analysis.

In task 2, we considered the same condition, and we perform hedging simulation where the volatility in the stock price process is matching the volatility used in the option valuation. Vary the frequency of the hedge adjustment, and we found different results.

3 Method and Theory

3.1 Notation

Table 1: Notation and Parameter descriptions

Notation	Description	Notation	Description
S_0	The stock price at $t = 0$	$S_{i,j}$	Stock price at $[i, j]$ node.
$[i, j]$	j^{th} node at time $i^{\delta t}$ in tree	$f_{i,j}$	Option price at $[i, j]$ node.
K	Strike price	T	Expiry date / maturity
N	number of intervals in T	δt	Length of each N
r	One year interest rate	σ_S	Volatility of stock price
Δ	Hedge parameter	u, d	up/down movement of stock price
p	Risk-neutral probability of stock price move up	C_t	At time t, the price of a European call option
B_t	At time t borrowed money from bank	$N(x)$	CDF of the normal distribution
σ_{est}	Estimated volatility of the option price		

3.2 Binomial tree model

A binomial tree is a model to calculate the price of an option. As figure 1 shows, the binomial tree model assumes the stock price can only move up or down after a time length of δt, which has two possible prices S_0u or S_0d.

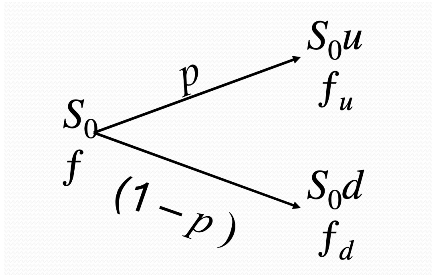


Figure 1: Basic process of Binomial tree model. And the time length is δt

Then, the expectation can be calculated:

$$E(S_T) = pS_o u + (1-p)S_o d \tag{1}$$

And since the assumption is no-arbitrage the expectation can be rewritten like:

$$E(S_T) = S_0 e^{rt} \tag{2}$$

Then:

$$S_0 e^{rt} = pS_o u + (1-p)S_o d \tag{3}$$

$$p = \frac{e^{rt} - d}{u - d} \tag{4}$$

where $u = e^{\sigma \delta t}$ and $d = 1/u$.

In the Binomial tree model, the stock price at $[i, j]$ node can be described like $S_{i,j} = S_0 u^j d^{i-j}$. And the option price for each node can be described as:

$$f_{i,j} = e^{-rt}(pf_{i+1,j+1} + (1-p)f_{i+1,j}) \tag{5}$$

Then, using final nodes of the option price to works back through(equation 6), the tree can get every node's option price. The hedge parameter(Δ) can be calculated from the nodes at time δt like:

$$\Delta = \frac{\Delta f}{\Delta S} \tag{6}$$

In this report, the default parameter are for an European call option, with $K = 99$, $S_0 = 100$, $\sigma = 20\%$, $r = 0.06$, $N = 50$, and $T = 1$ year.

3.3 Black-Scholes model

The Black-Scholes model is a method that can calculate the analytical value of the European call or put option price. In this report, the dynamics of the stock price follows the equation:

$$dS = rSdt + \sigma SdZ \tag{7}$$

And the call option price follows:

$$C_t = N(d_1)S_t - N(d_2)Ke^{-r(T-t)} \tag{8}$$

$$d_1 = \frac{1}{\sigma\sqrt{T-t}}\left[\ln(\frac{S_t}{K}) + (r + \frac{\sigma^2}{2})(T-t)\right] \tag{9}$$

$$d_2 = d_1 - \sigma\sqrt{T-t} \tag{10}$$

where the $N(x)$ is the standard normal cumulative probability distribution function. $N(x) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{x} e^{-\frac{1}{2}z^2}dz$.

Then the hedge parameter Δt ($t < T$):

$$\Delta t = \frac{\partial C}{\partial S} = N(d_1) \tag{11}$$

Furthermore, Black-Scholes is an effective method for the trader to operate the dynamic hedge strategy. The frequency of hedge are various, and in this report, only daily and weekly hedge strategy are discussed. The process of hedge strategy can be described like this:

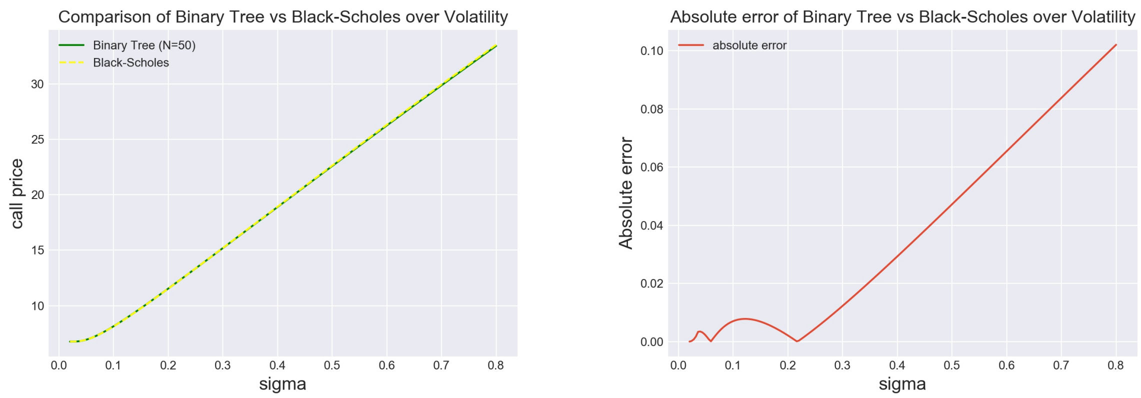
1. Using equation 10 to calculate whole time T stock price.
2. Using equation 7 and 11 to calculate initial option price C_0 and hedge parameter Δ_0. And calculate the borrow money:$B_0 = C_0 - S_0\Delta_0$
3. After a certain time length(daily or weekly), recalculated the hedge parameter Δ_t. Traders will buy or sell $|\Delta_t - \Delta_{t-1}|$ stock. And update the $B_t = B_{t-1} * e^{r\delta t} + (\Delta_t - \Delta_{t-1}) * S_T$
4. Repeat step 3 till expiration date.
5. Calculate final earnings: $S_t * \Delta_{T-1} - max(S_t - k, 0) - B_{T-1} * e^{r\delta t}$

In this report, the default parameters are for an European call option, with $K = 99$, $S_0 = 100$, $\sigma = 20\%$, $r = 0.06$, $N = 50$, and $T = 1$ year.

4 Results and discussion

4.1 Option Valuation

Figure 2 shows the comparison of option price from the binomial tree versus Black-Scholes over different volatility values. It is clear to see that the option price increases as σ increases. That is because when σ is larger, the future values of the price are more uncertain. Consequently, this also means that under a log-normal distribution, there is a larger probability density above the strike price K. In this way, the option price will certainly increase. Additionally, figure 2 shows the results of a binomial tree model compared with the analytical results from the Black-Scholes model (equation 10). It is clear to see that the results of the binomial tree model are close to the analytical result, and absolute errors from different σ are smaller than 0.11. Besides, the absolute error increase as the σ increase. The possible explanation is when the σ is large, a larger N needs to be chosen to converge to the true value. Fixed N will increase the absolute error for comparatively larger σ because the discrete values of the binomial tree are not sufficient to represent the true values that the stock can take on.



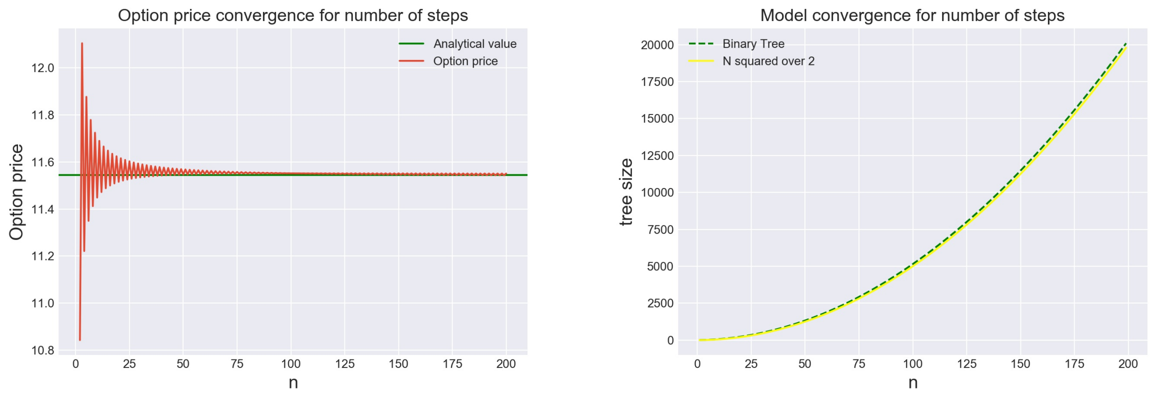
(a) Comparison of binomial tree versus Black-Scholes over Volatility, N=50

(b) Absolute error of binomial tree versus Black-Scholes over Volatility, the absolute error is measured by difference between the binomial tree and Black-Scholes

Figure 2: Global sensitivity analysis-Total order

Moreover, convergence is explored in this report. As figure 3(a) shows, the option price first fluctuates significantly and gradually converges as the N increases. Small values of N cannot converge because firstly, δt is too large in the binomial tree model, and secondly, the prices in the final layer are too coarse. Large N ensure the time interval δt is small enough, which means the u, d in the tree are sufficiently small in each time interval. Thus, the distribution of results from the binomial tree is converging to a log-normal distribution, and then the option price is converging to the analytical value.

Additionally, for the computation complexity the number of calculations required can be described like: $1 + 2 + 3 + 4 + \cdots + n = O(n^2)$. Then, using analytical complexity $(n^2/2)$ compared with the computation complexity of the binomial tree. The results of the binomial tree are close to the analytical results, which means the computational complexity of the binomial tree is on the order of $O(n^2)$.

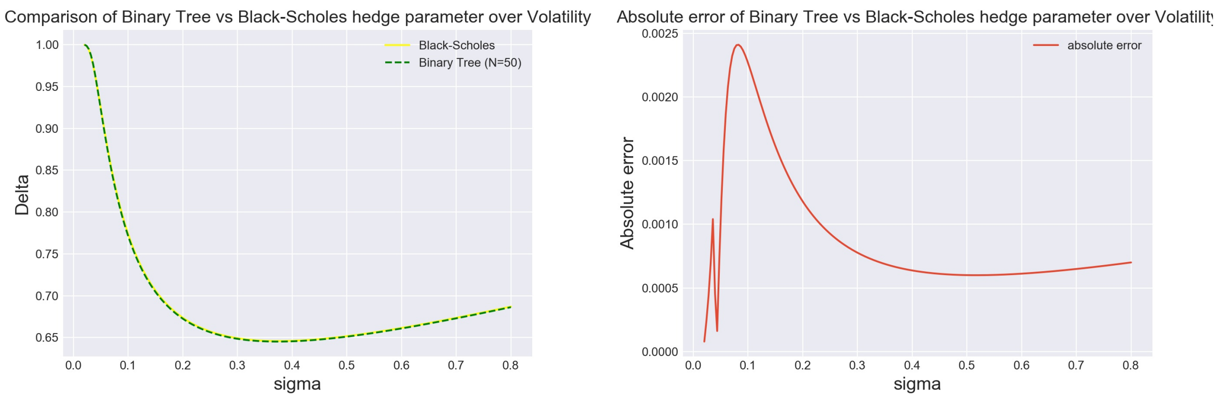


(a) Option price convergence for number of steps, $N = (1, 200)$, step$= 1$. Green line is analytical value $=11.544$

(b) Computational complexity

Figure 3: Model convergence and computational complexity

Furthermore, Figure 4 shows the hedge parameter of the binomial tree model with the Black-Scholes analytical hedge parameter. This parameter traces the analytical values very closely, as can be seen in the absolute error graph. The graph clearly shows a slight bump around $\sigma = 10\%$. Still, curiously the error in the hedge parameter does not seem to be correlated with the absolute error of the option price over varying volatility. This lack of correlation for higher σ is strange because it means that even though the mispricing gets larger as N stays the same for higher volatility, the hedge parameter does not. That means that essentially, the risk does not increase. Conversely, the mean payout is lower since the initial option price lower than the price should have been.



(a) Comparison of binomial tree versus Black-Scholes hedge parameter over Volatility, N=50

(b) Absolute error of binomial tree versus Black-Scholes hedge parameter over Volatility, the absolute error is measured by difference between the binomial tree and Black-Scholes

Figure 4: Global sensitivity analysis-Total order

Figure 5 shows the comparison of European and American call/put options for varying values of sigma. Interestingly, only the put options seem to have a premium for the right to exercise at any moment in time. This lack of premium makes sense because exercising the call does not carry any added benefit. Mathematically, if an American call option is exercised early, $S - K$ is obtained. However, if the call option is sold, C is obtained. Because an option consists of the intrinsic value of the option with $S - K$ and the time value, it must be so that $C > S - K$ and so exercising early on a call option is never beneficial. Thus they must have equal prices. This is different for the put options, since the underlying asset may be sold at any given time. This, in turn, means that the price may reach a level where the payoff $K - S$ is larger than the current option value, and early exercising would be beneficial. To compensate for this, a larger hedge needs to be taken, which in turn requires a larger premium.

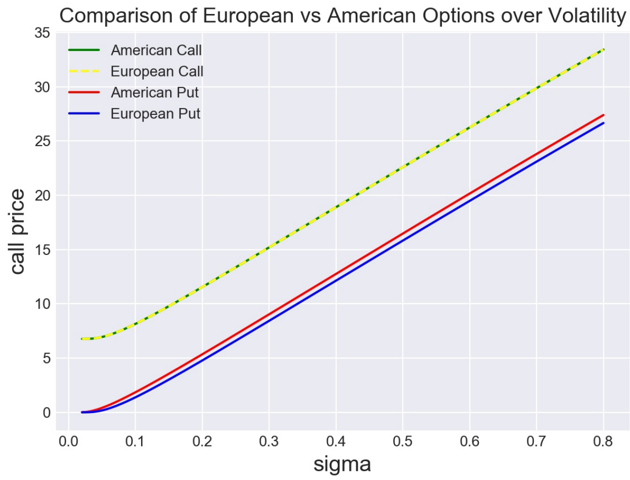


Figure 5: Comparison of European versus American Options over volatility. $N = 50$.

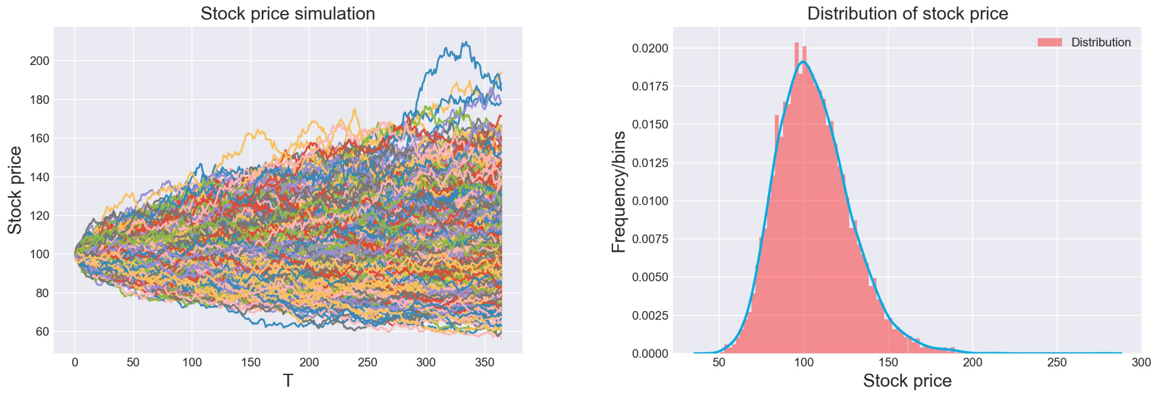
4.2 Hedge simulation

First, using process of dynamic hedge strategy figure 6 shows result of one simulation. In the same stock price, the trajectory of changing hedge are quite similar. While when compared the portfolio in the model and analytical option price, it is clear to see that the weekly hedge have larger absolute error, which is because weekly hedge do few times hedge than the daily hedge and will accumulate the error and risk in a week.

Figure 6: One simulation of hedge. Top three plot are daily hedge and bottom three are weekly hedge. The first column plots are stock priceS_t. Second column plots are hedge parameter. The third parameter are the absolute error between the portfolio and option price

Since single simulation is not convincing, figure 7 shows a plot of 10000 simulations for $S_0 = 100$, $\sigma_S = 20\%$, $r = 0.06$, $N = 50$, and $T = 1$ year, as well as the distribution of the final stock price after one year. This clearly shows that the distribution is log-normal.

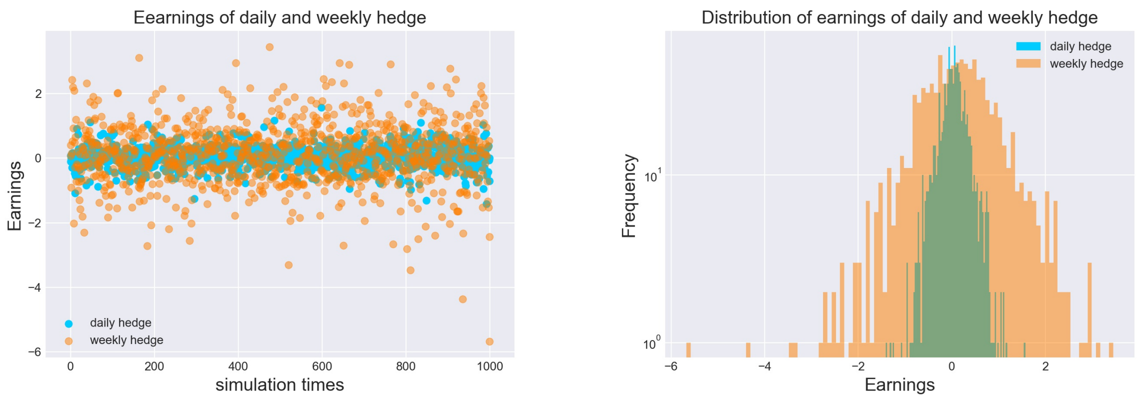
Running these simulations with daily and weekly hedge adjustments is shown for an option with $\sigma_{est} = 20\%$ is shown in Figure 8. This is a matching σ, and in the plots, we see that the earnings seem normally distributed, centered around 0. This centering is desirable since that means that when we do it many times, our average earnings are zero, and our position is risk-less on average. Weekly hedge adjustments show the same pattern; however, the standard deviation is a lot larger than the daily hedge adjustments. These adjustments essentially cause the payments to be more varied, and our positions to be riskier. This effect also makes intuitive sense, since the weekly hedge adjustments take a weekly risk, potentially having to buy the stock at a higher price because of a price surge. However, intuitively, the portfolio value is slightly negative, since the mean increase of the price is positive, and therefore our stock purchase 7 days later is higher on average than a partial stock purchase over every day for 7 days.



(a) 10000 simulation of stock price

(b) Distribution of stock price, the blue line is fitted curve which is log-normal.

Figure 7: Stock price simulation



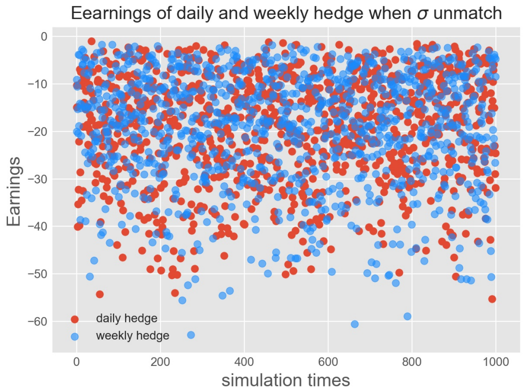
(a)

(b)

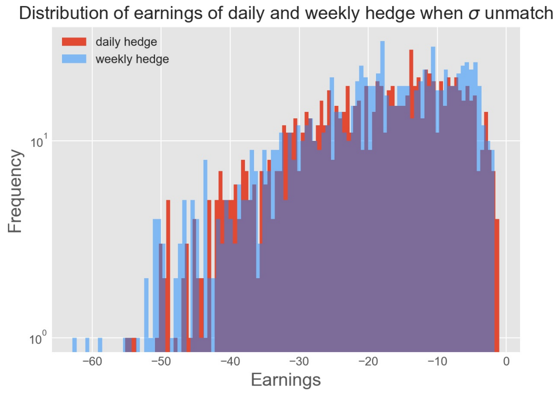
Figure 8: (a) Results comparison of earnings from daily hedge and weekly hedge. The number of simulation 1000. Each scatter point represent one simulation of earnings. (b) Distribution of results comparison of earnings from daily hedge and weekly hedge. The number of simulations is 1000. $\delta_S = \delta_{est} = 0.2$

Figure 9 shows what happens when the estimated volatility σ_{est} does not match the actual price volatility σ_S. In this case, the payout is skewed towards the negative side, with a negative mean payout over all the simulations. This happens because the original option price should have been higher, since the d_1 term grows proportional to $\frac{\sigma}{2}\sqrt{T-t}$, yielding a higher outcome of $N(d_1)$. Additionally, d_2 lowers because $\sigma\sqrt{T-t}$ becomes larger, yielding less probability mass on the right side of the equation. Consequently, because the initial option price is not large enough, the offset becomes negative, since the borrowed amount needs to be larger. Secondly, the distribution gets wider because the payouts get more varied due to many adjustments. Also, because the estimated d_1 is lower than the actual d_1 according to the true σ, not enough is appropriately hedged when compared to the true volatility, making the position risky.

Strangely, there does not seem to be a difference between the distribution of weekly hedges and daily hedges. This lack of difference could have two possible explanations; either the hedge readjustment moments are equally bad, or there is an error in the implementation. When adjusting the hedge, making a daily adjustment can cause a lot of sub-optimal adjustments since the price can vary wildly. Altering the adjustment to a weekly one, the realized variance is wider, but the adjustments are most likely equally bad. However, many fewer of these adjustments are required, and so the final distributions are similar.



(a)



(b)

Figure 9: (a) Results comparison of earnings from daily hedge and weekly hedge when σ does not match the estimated σ. The number of simulations is 1000. And σ_S is 0.7, in Black-Scholes σ_{est} is 0.2. Each scatter point represents one simulation of earnings. (b) Distribution of results comparison of earnings from daily hedge and weekly hedge when σ_S does not match the σ_{est}.

Lastly, varying the true σ_S, Figure 11 shows the distribution of earnings over 1000 simulated runs per σ. The graph clearly shows the distribution flattening (getting a higher variance) and a lower mean. The mean is also plotted in Figure 10, clearly showing the lower average earnings as the actual σ changes in relation to the estimated σ for the option price. Both the reason the variance gets larger, and the mean gets lower as the actual sigma rises, resulting in a flatter distribution, has been discussed previously. However, it is more interesting to look at the first distribution, for a lower actual σ than the estimated one. In that case, the hedge is still too high, since now the volatility is overestimated, but a higher mean since the initial price for the option was higher, resulting in less borrowed funds. So for lower σ as well, the mean is higher, but the variance still wider, resulting in more risky payouts.

Figure 10: Mean earnings from different σ of daily hedge adjustment. The range of stock price σ_S is $(0.1-0.9)$, inclusive, and step size is 0.1. And option price $\sigma_{est}=0.2$, 1000 simulation runs.

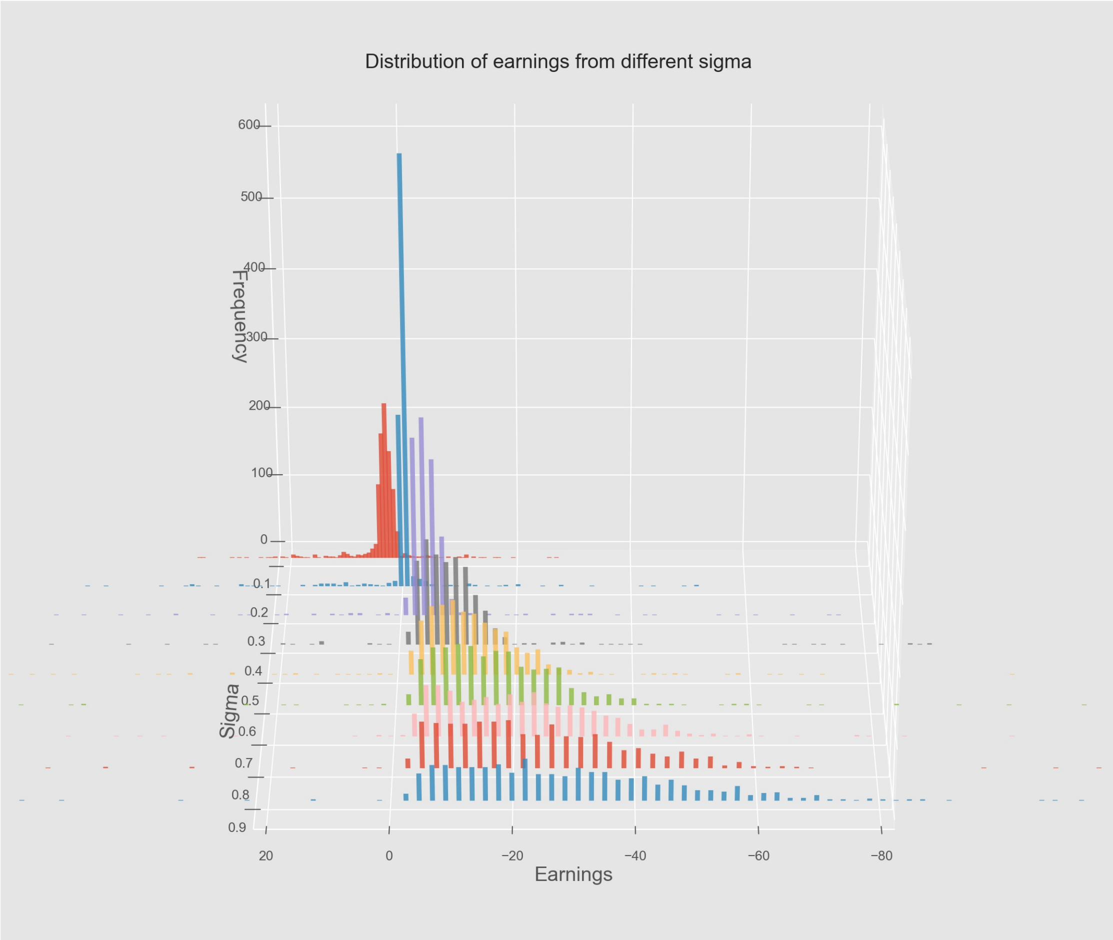


Figure 11: Distribution of earnings from different sigma of daily hedge adjustment. The range of stock price σ_S is $(0.1 - 0.9)$, inclusive, and step size is 0.1. And option prices $\sigma_{est} = 0.2$, 1000 simulation runs for each σ_S

5 Conclusion

In this report, option evaluation and hedge simulation were discussed. The binomial tree model is a useful model to evaluate the option price, and it is close to the analytical option price. Besides, computational complexity is only $O(n^2)$, and the model converges fast. Moreover, for the hedge simulation, daily hedge performs better than weekly hedge adjustments in the match σ, since daily hedge adjustments can reduce the risk in a daily time length. When σ doesn't match, daily and weekly hedge both have negative earnings and perform poorly in reducing risk. Finally, the different σ_s has been discussed; it turns out that the relation between the σ_S and σ_{est} of the option price is linear, with a negative slope and an intercept around the σ_{est} and σ_S being equal.

References

[1] FÜRST, C. J. K. *Simulation approaches to delta hedging in the black-scholes model*. PhD thesis, master thesis, Royal Institute of Technology, Stockholm, Sweden, 2012.