1 Question 1

1.1

For u, we have:

$$\begin{aligned}\frac{\partial u}{\partial t} &= D_u\Delta^2 u + f(u,v)\\ \frac{\partial u}{\partial t} &= D_u\Delta^2 u + \lambda u + u^3 - k_u - \delta v\\ \frac{u_i^{k+1}-u_i^k}{\Delta t} &= D_u\frac{u_{i+1}^k + u_{i-1}^k - 2u_i^k}{\Delta x^2} + \lambda u_i^k + (u_i^k)^3 - k_u - \sigma v_i^k\\ u_i^{k+1} &= u_i^k + \frac{D_u\Delta t}{\Delta x^2}(u_{i+1}^k + u_{i-1}^k - 2u_i^k) + \lambda u_i^k + (u_i^k)^3 - k_u - \sigma v_i^k\end{aligned} \tag{1}$$

For v, we have:

$$\begin{aligned}\frac{\partial v}{\partial t} &= D_v\Delta^2 v + f(u,v)\\ \frac{\partial v}{\partial t} &= D_v\Delta^2 v + \frac{u-v}{\tau}\\ \frac{v_i^{k+1}-v_i^k}{\Delta t} &= D_v\frac{v_{i+1}^k + v_{i-1}^k - 2v_i^k}{\Delta x^2} + \frac{u_i^k - v_i^k}{\tau}\\ v_i^{k+1} &= v_i^k + \frac{D_v\Delta t}{\Delta x^2}(v_{i+1}^k + v_{i-1}^k - 2v_i^k) + \frac{u_i^k - v_i^k}{\tau}\end{aligned} \tag{2}$$

Thus, Where $x_i = i\delta x$, $i = 1,2,3\ldots n$ and $\Delta x = 1/n$. $t_k = k\Delta t$ and $k = 1,2,\ldots$.

$$\begin{aligned}u_i^{k+1} &= u_i^k + \frac{D_u\Delta t}{\Delta x^2}(u_{i+1}^k + u_{i-1}^k - 2u_i^k) + \lambda u_i^k + (u_i^k)^3 - k_u - \sigma v_i^k\\ v_i^{k+1} &= v_i^k + \frac{D_v\Delta t}{\Delta x^2}(v_{i+1}^k + v_{i-1}^k - 2v_i^k) + \frac{u_i^k - v_i^k}{\tau}\end{aligned} \tag{3}$$

1.2

We can plot $u - v$ phase portrait. For example as diagram shows, the straight line is v nullcline and on the left of straight line $\frac{dv}{dt} < 0$,n the right of straight line $\frac{dv}{dt} > 0$. The blue curve is u nullcline. Above of this nullcline $\frac{du}{dt} < 0$ and below it is positive. And the black line is the result curve. We can see once the curve goes beyond the right knee of the u-nullcline($1 \le x \le 2$), u decays fast so the result curve move from right to left. While moving left, the curve crosses the v-nullcline from right to left. From this point on, while both u and v diminish, and then the evolution of v dominates and the curve becomes horizontal once again.

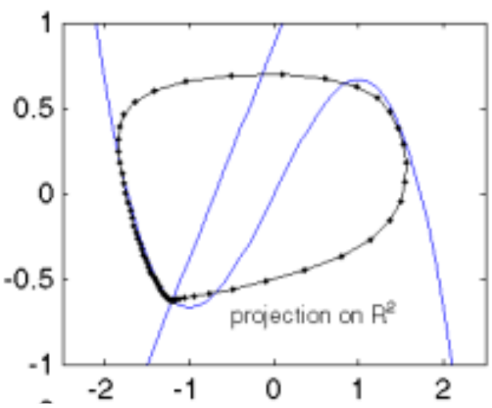


Figure 1: Phase portrait of $u - v$.

2 Question 2

2.1

The problem domain shows as diagram. And initial values show as x coordinates and $0 \leq x \leq 1$. And $t \geq 0$.

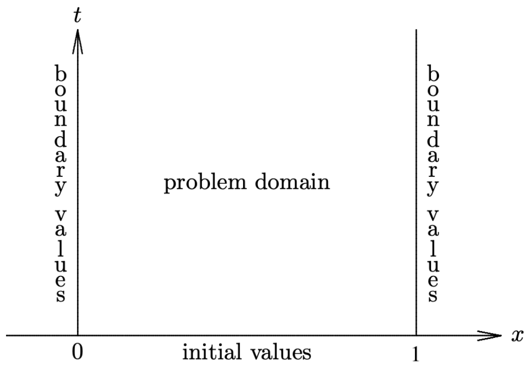


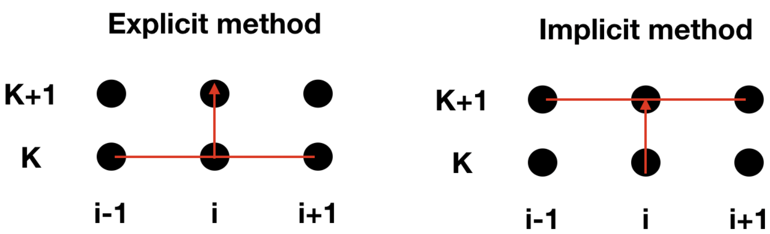
Figure 2: initial values is x coordinates and $0 \leq x \leq 1$. And $t \geq 0$.

2.2

Explicit schemes is that solution values at this time step(t_{k+1}) only depend on values that are available from the previous time step(t_k). While implicit schemes also uses the information at this time step(t_{k+1}) to get the solution value at t_{k+1}.

2.3

Stencils of heat equation for both the explicit and implicit scheme show as follow. K and $K+1$ means time step. i means grids of x. Red line means the grids were used to calculate the solution value. And red arrow line means solution value we want to calculate.



(a) Explicit method (b) Implicit method

Figure 3: Heat equation

2.4

Explicit equation:

$$
\begin{aligned}
u_t &= c u_{xx} \\
\frac{{u_i}^{k+1} - {u_i}^k}{\Delta t} &= c \frac{{u_{i+1}}^k + {u_{i-1}}^k - 2{u_i}^k}{(\Delta x)^2} \\
{u_i}^{k+1} &= u_i^k + \frac{c\Delta t}{(\Delta x)^2}({u_{i+1}}^k + {u_{i-1}}^k - 2{u_i}^k)
\end{aligned} \tag{4}
$$

Implicit equation:

$$
\begin{aligned}
u_t &= c u_{xx} \\
\frac{{u_i}^{k+1} - {u_i}^k}{\Delta t} &= c \frac{{u_{i+1}}^{k+1} + {u_{i-1}}^{k+1} - 2{u_i}^{k+1}}{(\Delta x)^2} \\
{u_i}^{k+1} &= u_i^k + \frac{c\Delta t}{(\Delta x)^2}({u_{i+1}}^{k+1} + {u_{i-1}}^{k+1} - 2{u_i}^{k+1})
\end{aligned} \tag{5}
$$

Where $x_i = i\delta x$, $i = 1, 2, 3 \ldots n$ and $\Delta x = 1/n$. $t_k = k\Delta t$ and $k = 1, 2, \ldots$.

2.5

First we consider first order Taylor expansion:

$$u_j{}^{k+1} = u_j{}^k + \Delta t(u_t) + O(\Delta t^2) \tag{6}$$

and substitute into the approximation of the time derivative:

$$u_t \cong \frac{u_j{}^{k+1} - u_j{}^k}{\Delta t} \tag{7}$$

we have:

$$u_t \cong \frac{\Delta t(u_t) + O(\Delta t^2)}{\Delta t} \tag{8}$$

And the consistency show when Δt approaches 0:

$$\lim_{\Delta t \to 0} [\frac{1}{\Delta t}(u_t + O(\Delta t^2))] = u_t \tag{9}$$

2.6

First we have the solution: $u_j^n = \epsilon^n e^{ikj\Delta x}$. And k is real spatial wave number, ϵ is the amplification factor, n is time steps, j is intervals of x and i the imaginary number.

And for explicit:

$$\begin{aligned}
u_i{}^{k+1} &= u_i^k + \frac{c\Delta t}{(\Delta x)^2}(u_{i+1}{}^k + u_{i-1}{}^k - 2u_i{}^k) \\
\epsilon^{n+1} e^{ikj\Delta x} &= \epsilon^n e^{ikj\Delta x} + \frac{c\Delta t}{(\Delta x)^2}(\epsilon^n e^{ik(j+1)\Delta x} + \epsilon^n e^{ik(j-1)\Delta x} - 2\epsilon^n e^{ikj\Delta x}) \\
&\text{divide by } \epsilon^n e^{ikj\Delta x} \\
\epsilon &= 1 + \frac{c\Delta t}{(\Delta x)^2}(e^{ik\Delta x} + e^{-ik\Delta x} - 2) \\
&= 1 + \frac{c\Delta t}{(\Delta x)^2}(2cos(k\Delta x) - 2) \\
&= 1 - \frac{4c\Delta t}{(\Delta x)^2}(sin(\frac{1}{2}k\Delta x))^2
\end{aligned} \tag{10}$$

And we know if we want to be stable, we need $\epsilon \leq 1$:

$$\begin{aligned}
\left|1 - \frac{4c\Delta t}{(\Delta x)^2}(sin(\frac{1}{2}k\Delta x))^2\right| &\leq 1 \\
-1 \leq 1 - \frac{4c\Delta t}{(\Delta x)^2}(sin(\frac{1}{2}k\Delta x))^2 &\leq 1 \\
0 \leq \frac{4c\Delta t}{(\Delta x)^2}(sin(\frac{1}{2}k\Delta x))^2 &\leq 2 \\
0 \leq \frac{c\Delta t}{(\Delta x)^2}(sin(\frac{1}{2}k\Delta x))^2 &\leq \frac{1}{2}
\end{aligned} \tag{11}$$

And we all know $0 \leq (sin(\frac{1}{2}k\Delta x))^2 \leq 1$. Thus if we want to be stable only:

$$\Delta t \leq \frac{(\Delta x)^2}{2c} \tag{12}$$

And for Implicit:

$$u_i{}^{k+1} = u_i^k + \frac{c\Delta t}{(\Delta x)^2}(u_{i+1}{}^{k+1} + u_{i-1}{}^{k+1} - 2u_i{}^{k+1})$$
$$\epsilon^{n+1}e^{ikj\Delta x} = \epsilon^n e^{ikj\Delta x} + \frac{c\Delta t}{(\Delta x)^2}(\epsilon^{n+1}e^{ik(j+1)\Delta x} + \epsilon^{n+1}e^{ik(j-1)\Delta x} - 2\epsilon^{n+1}e^{ikj\Delta x})$$

divide by $\epsilon^n e^{ikj\Delta x}$

$$\begin{aligned}
\epsilon &= 1 + \frac{c\Delta t}{(\Delta x)^2}(\epsilon e^{ik\Delta x} + \epsilon e^{-ik\Delta x} - 2\epsilon) \\
&= 1 + \frac{c\Delta t}{(\Delta x)^2}\epsilon(2cos(k\Delta x) - 2) \\
&= 1 - \frac{4c\Delta t}{(\Delta x)^2}\epsilon(sin(\frac{1}{2}k\Delta x))^2 \\
1 &= \epsilon + \frac{4c\Delta t}{(\Delta x)^2}\epsilon(sin(\frac{1}{2}k\Delta x))^2 \\
\epsilon &= \frac{1}{1 + \frac{4c\Delta t}{(\Delta x)^2}\epsilon(sin(\frac{1}{2}k\Delta x))^2}
\end{aligned} \tag{13}$$

And from this we can get for $\forall \Delta t, \Delta x$, $|\epsilon| \leq 1$. Thus implicit scheme is unconditionally stable. And explicit scheme is conditionally stable only when $\Delta t \leq \frac{(\Delta x)^2}{2c}$.

2.7

Accuracy of explicit and implicit schemes is $O(\Delta t) + O(\Delta x^2)$. But Crank-Nickelson method accuracy is $O(\Delta t^2) + O(\Delta x^2)$, which is second-order accuracy in time and space. Because Crank-Nickelson method basically takes the average of explicit and implicit schemes, which is both left hand and right hand are centered at $u_i^{k+\frac{1}{2}}$. In this way, the equation becomes:

$$\begin{aligned}
u_t &= cu_{xx} \\
\frac{u_i{}^{k+1} - u_i{}^k}{\Delta t} &= c(\frac{u_{i+1}{}^{k+1} + u_{i-1}{}^{k+1} - 2u_i{}^{k+1}}{2(\Delta x)^2} + \frac{u_{i+1}{}^k + u_{i-1}{}^k - 2u_i{}^k}{2(\Delta x)^2})
\end{aligned} \tag{14}$$

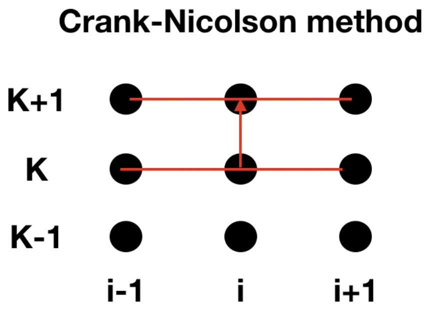


Figure 4: Crank-Nicolson method

Both time and space are second order accuracy. Furthermore, we evaluate the stability using the same process as explicit and implicit. We found out Crank-Nickelson method is also unconditionally stable($\forall \Delta t, \Delta x$, $|\epsilon| \leq 1$). Thus, we improve the accuracy by Crank-Nickelson method.

3 Question 3

3.1

Explicit method for wave equation, K and $K+1$ means time step. i means grids of x. Red line means the grids were used to calculate the solution value. And red arrow line means solution value we want to calculate.

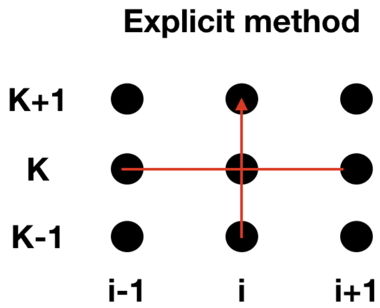


Figure 5: Stencils for explicit method for wave equation

3.2

$$\begin{aligned} u_{tt} &= c u_{xx} \\ \frac{{u_i}^{k+1} + {u_i}^{k-1} - 2{u_i}^k}{(\Delta t)^2} &= c\frac{{u_{i+1}}^k + {u_{i-1}}^k - 2{u_i}^k}{(\Delta x)^2} \\ {u_i}^{k+1} &= 2{u_i}^k - {u_i}^{k-1} + \frac{c(\Delta t)^2}{(\Delta x)^2}({u_{i+1}}^k + {u_{i-1}}^k - 2{u_i}^k) \end{aligned} \tag{15}$$

3.3

For time t:

$$\begin{aligned} u_j^{k+1} &= u_j^k + \Delta t u_t + \frac{1}{2}(\Delta t)^2 u_{tt} + \frac{1}{6}(\Delta t)^3 u_{ttt} + \frac{1}{24}(\Delta t)^4 u_{tttt} + O((\Delta t)^5) \\ u_j^{k-1} &= u_j^k - \Delta t u_t + \frac{1}{2}(\Delta t)^2 u_{tt} - \frac{1}{6}(\Delta t)^3 u_{ttt} + \frac{1}{24}(\Delta t)^4 u_{tttt} + O((\Delta t)^5) \\ &\text{Plus equation:} \\ u_{tt} &= \frac{u_j^{k+1} + u_j^{k-1} - 2u_j^k}{\Delta t^2} - \frac{1}{12}(\Delta t)^2 u_{tttt} + O((\Delta t)^5) \end{aligned} \tag{16}$$

Thus, we can see the time accuracy is $O((\Delta t)^2)$.

For space x:

$$\begin{aligned} u_{j+1}^k &= u_j^k + \Delta x u_x + \frac{1}{2}(\Delta x)^2 u_{xx} + \frac{1}{6}(\Delta x)^3 u_{xxx} + \frac{1}{24}(\Delta x)^4 u_{xxxx} + O((\Delta x)^5) \\ u_{j-1}^k &= u_j^k - \Delta x u_x + \frac{1}{2}(\Delta x)^2 u_{xx} - \frac{1}{6}(\Delta x)^3 u_{xxx} + \frac{1}{24}(\Delta x)^4 u_{xxxx} + O((\Delta x)^5) \\ &\text{Plus equation:} \\ u_{xx} &= \frac{u_{j+1}^k + u_{j-1}^k - 2u_j^k}{\Delta x^2} - \frac{1}{12}(\Delta x)^2 u_{xxxx} + O((\Delta x)^5) \end{aligned} \tag{17}$$

Thus, we know accuracy of the fully discrete numerical solution of the wave equation is second order accuracy for both time and space. But it required two successive time steps' data.

3.4

Consider second order Taylor expansion:

$$\begin{aligned} u_{j+1}^k &= u_j^k + \Delta x u_x + \frac{1}{2}\Delta x^2 u_{xx} + \frac{1}{6}\Delta x^3 u_{xxx} + O(\Delta x^4) \\ u_{j-1}^k &= u_j^k - \Delta x u_x + \frac{1}{2}\Delta x^2 u_{xx} - \frac{1}{6}\Delta x^3 u_{xxx} + O(\Delta x^4) \end{aligned} \tag{18}$$

And substitute into the approximation of the spatial derivative:

$$u_t \cong \frac{{u_{i+1}}^k + {u_{i-1}}^k - {u_i}^k}{\Delta x^2}$$
$$u_t \cong \frac{\Delta x^2 u_{xx} + O(\Delta x^4)}{\Delta x^2} \tag{19}$$

Then:

$$\lim_{\Delta x \to 0}[\frac{1}{\Delta x^2}(u_{xx} + O(\Delta x^4))] = u_{xx} \tag{20}$$

Thus, we can get the conclusion approximation for u_{xx} is consistent.

3.5

we still use method for 2.6, we have $u_j^n = \epsilon^n e^{ikj\Delta x}$. Substitute into equation 15:

$${u_i}^{k+1} = 2u_i^k - u_i^{k-1} + \frac{c(\Delta t)^2}{(\Delta x)^2}({u_{i+1}}^k + {u_{i-1}}^k - 2{u_i}^k)$$

$$\epsilon^{n+1} e^{ikj\Delta x} = 2\epsilon^n e^{ikj\Delta x} - 2\epsilon^{n-1} e^{ikj\Delta x} + \frac{c(\Delta t)^2}{(\Delta x)^2}(\epsilon^n e^{ik(j+1)\Delta x} + \epsilon^n e^{ik(j-1)\Delta x} - 2\epsilon^n e^{ikj\Delta x})$$

divide by $\epsilon^n e^{ikj\Delta x}$

$$\epsilon = 2 - \frac{1}{\epsilon} + \frac{c(\Delta t)^2}{(\Delta x)^2}(e^{ik\Delta x} + e^{-ik\Delta x} - 2)$$
$$= 2 - \frac{1}{\epsilon} - \frac{4c(\Delta t)^2}{(\Delta x)^2}(sin(\frac{1}{2}k\Delta x))^2 \tag{21}$$
$$\epsilon^2 = (2 - \frac{4c(\Delta t)^2}{(\Delta x)^2}(sin(\frac{1}{2}k\Delta x))^2)\epsilon - 1$$
$$\epsilon^2 - 2\alpha\epsilon + 1 = 0 : \text{ where, } \alpha = 1 - \frac{2c(\Delta t)^2}{(\Delta x)^2}(sin(\frac{1}{2}k\Delta x))^2)$$

From this we can get the solution of ϵ:

$$\epsilon_{1,2} = \alpha \pm \sqrt{\alpha^2 - 1} \tag{22}$$

We know if we want a stable state, $-1 \leq \epsilon \leq 1$. And we can see that if $\alpha > 1$, at least one ϵ is bigger than 1. Thus, we should choose $\alpha < 1$. And in this way, the solution become: $\epsilon_{1,2} = \alpha \pm i\sqrt{\alpha^2 - 1}$. And Now we know:

$$-1 \leq \alpha \leq 1$$
$$-1 \leq 1 - \frac{2c(\Delta t)^2}{(\Delta x)^2}(sin(\frac{1}{2}k\Delta x))^2) \leq 1 \tag{23}$$
$$0 \leq \frac{c(\Delta t)^2}{(\Delta x)^2}(sin(\frac{1}{2}k\Delta x))^2) \leq 1$$

Finally we have stability of the finite difference scheme:

$$\Delta t \leq \frac{\Delta x}{\sqrt{c}} \tag{24}$$

4 Question 4

First, Use a second order accurate, centred finite difference scheme:

$$u_{xx} \approx \frac{u_{x+\Delta x,y} + u_{x-\Delta x,y} - 2u_{x,y}}{\Delta x^2}$$
$$u_{yy} \approx \frac{u_{x,y+\Delta y} + u_{x,y-\Delta y} - 2u_{x,y}}{\Delta y^2} \tag{25}$$

Substitute into $u_{xx} + u_{yy} = 1 + y$:

$$\frac{u_{x+\Delta x,y} + u_{x-\Delta x,y} - 2u_{x,y}}{\Delta x^2} + \frac{u_{x,y+\Delta y} + u_{x,y-\Delta y} - 2u_{x,y}}{\Delta y^2} \approx = 1 + y \tag{26}$$

And then we have $x = y = \frac{1}{2}$ and $\Delta x = \Delta y = 0.5$:

$$\frac{1 + 0 - 2u_{x,y}}{0.25} + \frac{1 + 0 - 2u_{x,y}}{0.25} \approx 1.5$$
$$u_{x,y} \approx 0.40625 \tag{27}$$

5 Question 5

5.1

First, we have $u_t + cu_x = 0$, which is also $\frac{\partial u}{\partial t} + c\frac{\partial u}{\partial x} = 0$. And with the initial condition:$u(0,x) = f(x)$. We want to seek the exact solution $u(x,t) = X(x)T(t)$. And substitute into partial differential equation:

$$\begin{aligned} X(x)\frac{dT}{dt} + cT(t)\frac{dX}{dx} = 0 \\ \frac{1}{T}\frac{dT}{dt} = -c\frac{1}{X}\frac{dX}{dx} \end{aligned} \tag{28}$$

And left hand only depend on time and right hand only depend on x. Thus they can be equal only if both of them are constant:

$$\begin{aligned} \frac{1}{X}\frac{dX}{dx} = \lambda, \frac{dX}{dx} = X\lambda \\ \frac{1}{T}\frac{dT}{dt} = -c\lambda, \frac{dT}{dt} = -cT\lambda \end{aligned} \tag{29}$$

And then we have solution of x and t:

$$\begin{aligned} X = X_0 e^{\lambda x} \\ T = T_0 e^{-c\lambda T} \end{aligned} \tag{30}$$

and the solution of the advection equation is:

$$u(x,t) = X_0 T_0 e^{\lambda x - c\lambda T} = u_0 e^{\lambda x - c\lambda T} = f(x - ct) \tag{31}$$

5.2

First, recall the Taylor expansion:

$$\begin{aligned} u_{j+1}^k = u_j^k + \Delta x u_x + \frac{1}{2}(\Delta x)^2 u_{xx} + \frac{1}{6}(\Delta x)^3 u_{xxx} + O((\Delta x)^4) \\ u_{j-1}^k = u_j^k - \Delta x u_x + \frac{1}{2}(\Delta x)^2 u_{xx} - \frac{1}{6}(\Delta x)^3 u_{xxx} + O((\Delta x)^4) \\ \text{Minus equation:} \\ u_{j+1}^k - u_{j-1}^k = 2\Delta x u_x + \frac{1}{3}(\Delta x)^3 u_{xxx} \\ u_x = \frac{u_{j+1}^k - u_{j-1}^k}{2\Delta x} - \frac{1}{6}(\Delta x)^2 u_{xxx} \end{aligned} \tag{32}$$

In this way, we can verify central finite difference scheme has an accuracy O(Δx^2).

5.3

FTCS scheme:

$$\begin{aligned} u_t = -cu_x \\ \frac{{u_i}^{k+1} - {u_i}^k}{\Delta t} = -c\frac{{u_{i+1}}^k - {u_{i-1}}^k}{2\Delta x} \\ {u_i}^{k+1} = {u_i}^k - \frac{c\Delta t}{2\Delta x}({u_{i+1}}^k - {u_{i-1}}^k) \end{aligned} \tag{33}$$

5.4

Unconditionally unstable means no matter how small the time step is, the solution will significantly fluctuate and the error will accumulate greatly for each step. Thus, basically we can not use this method to get the solution.

5.5

Still we have $u_j^n = \epsilon^n e^{ikj\Delta x}$. Substitute into equation 33.

$$\begin{aligned}
\epsilon^{n+1} e^{ikj\Delta x} &= \epsilon^n e^{ikj\Delta x} + \frac{c\Delta t}{2\Delta x}(\epsilon^n e^{ik(j+1)\Delta x} - \epsilon^n e^{ik(j-1)\Delta x}) \\
\text{divide by } \epsilon^n e^{ikj\Delta x} & \\
\epsilon &= 1 - \frac{c\Delta t}{2\Delta x}(e^{ik\Delta x} - e^{-ik\Delta x}) \\
&= 1 - \frac{c\Delta t}{2\Delta x} isin(k\Delta x) \\
|\epsilon| &= \sqrt{1 + \frac{c\Delta t}{2\Delta x}(sin(k\Delta x))^2}
\end{aligned} \tag{34}$$

Thus, from equation we can get$\forall \Delta t, \Delta x$, $|\epsilon| \geq 1$. Thus, FTCS is unconditionally unstable.

5.6

For the Lax scheme, we used $C = \frac{c\Delta t}{\Delta x}$

$$\begin{aligned}
{u_i}^{k+1} &= \frac{{u_{i+1}}^k + {u_{i-1}}^k}{2} + \frac{C}{2}({u_{i+1}}^k - {u_{i-1}}^k) \\
\epsilon^{n+1} e^{ikj\Delta x} &= \frac{\epsilon^n e^{ik(j+1)\Delta x} + \epsilon^n e^{ik(j-1)\Delta x}}{2} + \frac{C}{2}(\epsilon^n e^{ik(j+1)\Delta x} - \epsilon^n e^{ik(j-1)\Delta x}) \\
\text{divide by } \epsilon^n e^{ikj\Delta x} & \\
\epsilon &= \frac{e^{ik\Delta x} + e^{-ik\Delta x}}{2} - \frac{C}{2}(e^{ik\Delta x} - e^{-ik\Delta x}) \\
&= cos(kh) - iCsin(kh) \\
|\epsilon| &= \sqrt{cos(kh)^2 + C^2 sin(k\Delta x)^2}
\end{aligned} \tag{35}$$

We know $cos(kh)^2 \leq 1$ and $sin(k\Delta x)^2 \leq 1$. Thus, $|\epsilon| \leq 1$, if $C \leq 1$. It means:

$$\begin{gathered}
\frac{c\Delta t}{\Delta x} \leq 1 \\
\text{or} \\
\Delta t \leq \frac{\Delta x}{c}
\end{gathered} \tag{36}$$

Thus, we get CFL stability condition from equation 36.