Olin College of Engineering
Partial Differential Equations, 2019

Homework 9 Problems
Due Friday, April 5

(1) Review

(a) Solve $u_{tt} = 4u_{xx}$ on $0 < x < 4$ with $u = 0$ at the boundary and initial condition $u_t = \Lambda(x-2)$ and $u = 0$.

(b) Solve $u_t = 5u_{xx}$ on $0 < x < 10$ with $u_x = 0$ at the boundary and initial condition $u = \Lambda(x-2)$.

(2) Reading and Learning Read the following:

(a) The Poisson Kernel

Laplace's Equation on the disc is

$$u_{rr} + \frac{1}{r}u_r + \frac{1}{r^2}u_{\theta\theta} = 0.$$

Its solution is of the form

$$u = \sum_n \left(\frac{r}{R}\right)^n (a_n \cos(n\theta) + b_n \sin(n\theta))$$

where the Fourier coefficients a_n and b_n are given by the Fourier Series of the boundary condition $u(R, \theta)$.

When a unit of heat energy is distributed over the small region $0 \leq \theta \leq \frac{1}{N}$ the boundary condition becomes $u(R, \theta) = NB(N\theta)$.

The corresponding Fourier coefficients are

$$a_0 = \frac{1}{2\pi} \quad a_n = \frac{1}{\pi}\frac{N}{n}\sin(\frac{n}{N}) \qquad b_n = \frac{1}{\pi}\frac{N}{n}\left(1 - \cos(\frac{n}{N})\right)$$

When $n << N$ we have the approximation

$$a_0 = \frac{1}{2\pi} \quad a_n \approx \frac{1}{\pi} \quad b_n \approx \frac{1}{2\pi}\frac{n}{N} \approx 0$$

Note that in this approximation the dependence on N has vanished which is auspicious for taking limits.

For $N >> 1$ we have the approximation

$$u \approx \frac{1}{\pi}\left(\frac{1}{2} + \sum_n \left(\frac{r}{R}\right)^n \cos(n\theta)\right) = \frac{1}{\pi}\left(\frac{1}{2} + \mathrm{Re}\left(\sum_n \left(\frac{r}{R}e^{i\theta}\right)^n\right)\right) = \frac{1}{\pi}\left(\frac{1}{2} + \mathrm{Re}\left(\frac{1}{1 - \frac{r}{R}e^{i\theta}}\right)\right)$$

where $\mathrm{Re}(z)$ denotes the real part of a complex number z.

After some algebra, this simplifies to

$$u \approx \frac{1}{2\pi} \frac{1 - \left(\frac{r}{R}\right)^2}{1 + \left(\frac{r}{R}\right)^2 - 2\frac{r}{R}\cos(\theta)}$$

The function

$$P(r,\theta) = \frac{1 - r^2}{1 + r^2 - 2r\cos(\theta)}$$

is called the **Poisson Kernel** and can be thought of as the (spatial) response of a metal disc to a (sustained) impulse of heat localized on the boundary.

(b) WHY ARE EIGENFUNCTIONS ORTHOGONAL?

The key fact from linear algebra that motivates and justifies the development of the singular value decomposition is that symmetric matrices have orthogonal eigenvectors.

First it is useful to know that the two characterizations of a symmetric matrix below are equivalent:

- $A = A^T$
- $A\mathbf{u} \cdot \mathbf{w} = \mathbf{u} \cdot A\mathbf{w}$ for any vectors $\mathbf{u}$ and $\mathbf{w}$ (that A can legally multiply)

Here is a proof that symmetric matrices have orthogonal eigenvectors.

Let $\mathbf{v}_1$ and $\mathbf{v}_2$ be eigenvectors of A with distinct eigenvalues. We want to show that $\mathbf{v}_1 \cdot \mathbf{v}_2 = 0$.

$$\mathbf{v}_1 \cdot \mathbf{v_2} = \frac{1}{\lambda_1} A\mathbf{v}_1 \cdot \mathbf{v_2} = \frac{1}{\lambda_1} \mathbf{v_1} \cdot A\mathbf{v}_2 = \frac{\lambda_2}{\lambda_1} \mathbf{v}_1 \cdot \mathbf{v}_2$$

Thus

$$(1 - \frac{\lambda_2}{\lambda_1})(\mathbf{v}_1 \cdot \mathbf{v_2}) = 0$$

Thus

$$\mathbf{v}_1 \cdot \mathbf{v}_2 = 0$$

The same proof works if we replace matrices with differential operators and vectors with functions.

As an example consider the PDE $u_t = u_{xx}$ with boundary conditions $u = 0$ at $x = 0, 1$. The associated eigenvalue problem is $X'' + \lambda X = 0$ with the boundary conditions $X = 0$ at $x = 0, 1$. In other words, we are considering the operator $L = \frac{d^2}{dx^2}$ restricted to functions satisfying the boundary conditions.

The dot product of two functions f and g now becomes $f \cdot g = \int_0^1 f(x)g(x)dx$. (If you look at other sources, dot products for functions are usually written as brackets: $\langle f, g \rangle$.)

Below is a proof that L is symmetric. In other words $Lf \cdot g = f \cdot Lg$. In other words $f''g = fg''$.

$$(f'g - fg')' = f''g + f'g' - f'g' - fg'' = f''g - fg''$$

Thus

$$0 = f'g - fg'\Big|_0^1 = \int_0^1 (f'g - fg')'dx = \int_0^1 (f''g - fg'')dx = Lf \cdot g - f \cdot Lg$$

(c) Convergence of Fourier Series

Consider a function $f(x) = \sum_{n=0}^{\infty} a_n \cos(nx) + b_n \sin(nx)$ Its Fourier coefficients are

$$a_0 = \frac{1}{2\pi}\int_0^{2\pi} f(y)dy \quad a_n = \frac{1}{\pi}\int_0^{2\pi} f(y)\cos(ny)dy \quad b_n = \frac{1}{\pi}\int_0^{2\pi} f(y)\sin(ny)dy$$

If we pass f through a low pass filter that retains only the first N frequencies we get

$$f_N(x) = \sum_{n=0}^{N} a_n \cos(nx) + b_n \sin(nx)$$

Substitute the Fourier coefficients back into the formula for f_N to get

$$f_N(x) = \frac{1}{2\pi}\left[\int_0^{2\pi} f(y)dy + 2\sum_{n=1}^{N}\left(\int_0^{2\pi}(\cos(ny)f(y)dy\right)\cos(nx) + \left(\int_0^{2\pi}\sin(ny)f(y)dy\right)\sin(nx)\right]$$

which rearranges as

$$f_N(x) = \frac{1}{2\pi}\left[\int_0^{2\pi} f(y)dy + 2\sum_{n=1}^{N}\int_0^{2\pi}(\cos(ny)\cos(nx) + \sin(ny)\sin(nx))\,f(y)dy\right]$$

or more cleanly as

$$f_N(x) = \frac{1}{2\pi}\int_0^{2\pi}\left[1 + 2\sum_{n=1}^{\infty}\cos(n(x-y))\right]f(y)dy$$

In other words

$$f_N = D_N * f$$

where $D_N(x) = 1 + 2\sum_{n=1}^{\infty}\cos(nx)$ is the **Dirichlet kernel**.

Amazingly, the sum has a simple formula

$$D_N(x) = \frac{\sin((n+\frac{1}{2})x)}{\sin(\frac{x}{2})}$$

(3) Evidence and Theory

Gather evidence to support each claim below. Examples of evidence include patterns extracted from examples, logical arguments, computations and derivations, etc. This will be graded based on effort and completion. This is for you. Evidence could be as simple as "I did an example with specific functions and it worked." Evidence can also be more complicated and incorporate logic, facts from other branches of math, symbolic computations and derivations. Do your best, given reasonable time constraints, to find compelling evidence.

(a) All of the skipped steps and approximations in all of the readings are justified

(b) The general solution to the equation

$$u_{rr} + \frac{1}{r}u_r + \frac{1}{r^2}u_{\theta\theta} = 0 \text{ on } r < R \text{ with } u(R,\theta) = f(\theta) \text{ on } r = R$$

is

$$u(r,\theta) = \frac{R^2 - r^2}{2\pi}\int_0^{2\pi}\frac{f(\phi)}{R^2 + r^2 - 2rR\cos(\theta - \phi)}d\phi$$

(c) The Poisson Kernel filters out high frequencies more aggressively as you approach the center of the disc.

(d) If f is continuous then $(D_N * f)(x) \to f(x)$ as $N \to \infty$. In fact $\|D_N * f - f\|_2 \to 0$ as well.

(e) If f is piecewise continuous with a jump discontinuity at x_0 then $(D_N * f)(x_0) = \frac{1}{2}(f(x_0+) + f(x_0-))$

(f) The eigenfunctions of the cantilevered beam are orthogonal. The relevant boundary conditions are $u_{xx} = u_{xxx} = 0$ on the free end and $u = u_x = 0$ on the pinned end.