6(3)a

```
In[ ]:= l2conv[n_, p_] := 2^(1/p) ((n^(1+p+2 p α))/(1 + p))^(1/p);

        l2box[n_, p_] := n^(1/p+α);

In[ ]:= l2conv[n, 2]
```

$$\text{Out[ ]=}\ \sqrt{\frac{2}{3}}\ \sqrt{n^{3+4\,\alpha}}$$

```
In[ ]:= l2box[n, 2] * l2box[n, 2]
```

$$\text{Out[ ]=}\ n^{1+2\,\alpha}$$

7 Heat Kernel

```
In[ ]:= h[t_, x_] := (t^-.5) * gaussian[x * t^(-.5)];
```

(a)

```
In[ ]:= FullSimplify[∫_(x-1)^x h[t, y] dy]
```

$$\text{Out[ ]=}\ \frac{0.5\left(-\frac{(-1+x)\,\text{Erf}\left[\frac{\sqrt{\frac{(-1+x)^2}{t^{1.}}}}{\sqrt{2}}\right]}{\sqrt{\frac{(-1+x)^2}{t^{1.}}}}+\frac{x\,\text{Erf}\left[\frac{\sqrt{\frac{x^2}{t^{1.}}}}{\sqrt{2}}\right]}{\sqrt{\frac{x^2}{t^{1.}}}}\right)}{t^{0.5}}$$

(b)

```
In[ ]:= FullSimplify[∫_(-∞)^∞ h[t, y] Sin[5 * (x - y)] dy]
```

$$\text{Out[ ]=}\ \text{ConditionalExpression}\left[1.\,e^{-\frac{25\,t^{1.}}{2}}\,\text{Sin}[5\,x],\ \text{Re}\left[t^{1.}\right] > 0\right]$$

(c)

In[•]:=
```
FullSimplify[∫_(x-1)^x h[t, y] Sin[5 * (x - y)] dy]
```

Out[•]=
$$e^{-12.5\, t^{1.}-(0.+5.\,i)\,x}\left(-0.25\,\text{Erfi}\left[3.53553\,t^{0.5}+\frac{(0.+0.707107\,i)\,(-1.000000000000000+x)}{t^{0.5}}\right]+e^{(0.+10.\,i)\,x}\left((0.-0.25\,i)\,\text{Erf}\left[\frac{0.707107-(0.+3.53553\,i)\,t^{1.}-0.707107\,x}{t^{0.5}}\right]+0.25\,\text{Erfi}\left[3.53553\,t^{0.5}-\frac{(0.+0.707107\,i)\,x}{t^{0.5}}\right]\right)+0.25\,\text{Erfi}\left[3.53553\,t^{0.5}+\frac{(0.+0.707107\,i)\,x}{t^{0.5}}\right]\right)$$

(d)

In[•]:=
```
FullSimplify[∫_(-∞)^∞ h[t, y] estep[x - y] dy]
```

Out[•]=
$$\text{ConditionalExpression}\left[\frac{1}{t^{0.5}\sqrt{\frac{(t^{1.}-x)^2}{t^{1.}}}},\ e^{\frac{t^{1.}}{2}-x}\left(0.5\,t^{0.5}\sqrt{t^{1.}-2.\,x+\frac{x^2}{t^{1.}}}+(-0.5\,t^{1.}+0.5\,x)\,\text{Erf}\left[\frac{\sqrt{\frac{(t^{1.}-x)^2}{t^{1.}}}}{\sqrt{2}}\right]\right),\ \text{Re}\left[t^{1.}\right]\geq 0\right]$$

(e)

In[•]:=
```
tri75[x_] := Piecewise[{{tri[x - 1], x ≥ 0}, {-1 * tri[x + 1], x < 0}}]
```

In[•]:=
```
FullSimplify[∫_(-∞)^∞ h[t, y] tri75[x - y] dy]
```

Out[•]=
$$\left(-0.797885\,e^{-\frac{0.5\,(1.-1.\,x)^2}{t}}+0.398942\,e^{-\frac{0.5\,(2.-1.\,x)^2}{t}}+0.797885\,e^{-\frac{0.5\,(1.+1.\,x)^2}{t}}-0.398942\,e^{-\frac{0.5\,(2.+1.\,x)^2}{t}}\right)\sqrt{t}+(-1.+0.5\,x)\,\text{Erf}\left[\frac{0.707107\,(-2.+x)}{\sqrt{t}}\right]+(1.-1.\,x)\,\text{Erf}\left[\frac{0.707107\,(-1.+x)}{\sqrt{t}}\right]+(1.+1.\,x)\,\text{Erf}\left[\frac{0.707107\,(1.+x)}{\sqrt{t}}\right]-1.\,\text{Erf}\left[\frac{0.707107\,(2.+x)}{\sqrt{t}}\right]-0.5\,x\,\text{Erf}\left[\frac{0.707107\,(2.+x)}{\sqrt{t}}\right]$$

8. Visualizing solution to the heat equation on the line

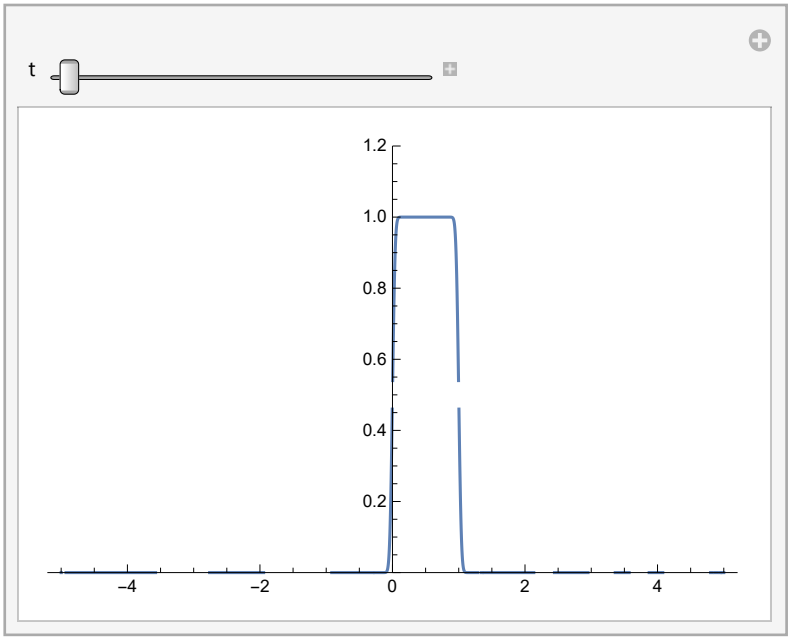
In[ ]:=
```
u71[t_, x_] = 0.5` (-((-1+x) Erf[Sqrt[(-1+x)^2/t^1.`]/Sqrt[2]]/Sqrt[(-1+x)^2/t^1.`]) + x Erf[Sqrt[x^2/t^1.`]/Sqrt[2]]/Sqrt[x^2/t^1.`])/t^0.5`;
```

In[ ]:=
```
FullSimplify[0.5` (-((-1+x) Erf[Sqrt[(-1+x)^2/t^1.`]/Sqrt[2]]/Sqrt[(-1+x)^2/t^1.`]) + x Erf[Sqrt[x^2/t^1.`]/Sqrt[2]]/Sqrt[x^2/t^1.`])/t^0.5`]
```

Out[ ]=
$$\frac{0.5\left(-\frac{(-1+x)\,\mathrm{Erf}\left[\frac{\sqrt{\frac{(-1+x)^2}{t^{1.}}}}{\sqrt{2}}\right]}{\sqrt{\frac{(-1+x)^2}{t^{1.}}}}+\frac{x\,\mathrm{Erf}\left[\frac{\sqrt{\frac{x^2}{t^{1.}}}}{\sqrt{2}}\right]}{\sqrt{\frac{x^2}{t^{1.}}}}\right)}{t^{0.5}}$$

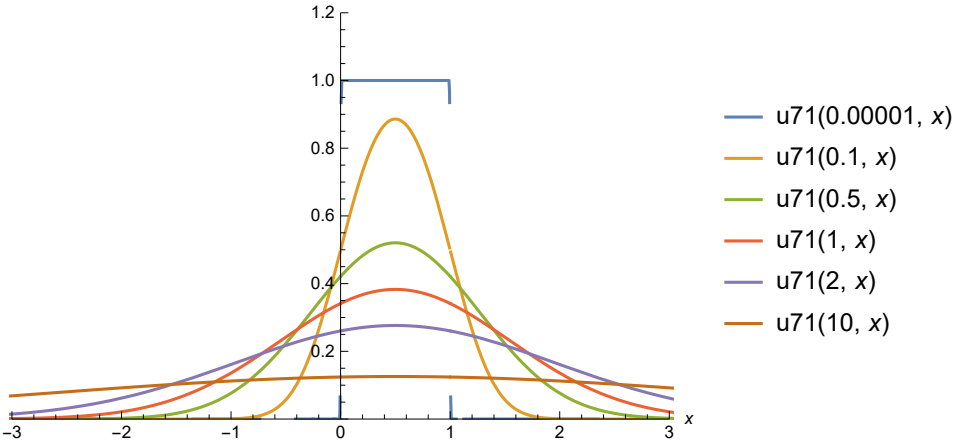
In[ ]:=
```
Manipulate[Plot[u71[t, x], {x, -5, 5}, PlotRange → {0, 1.2}], {t, 0.001, 5}]
```

Out[ ]=



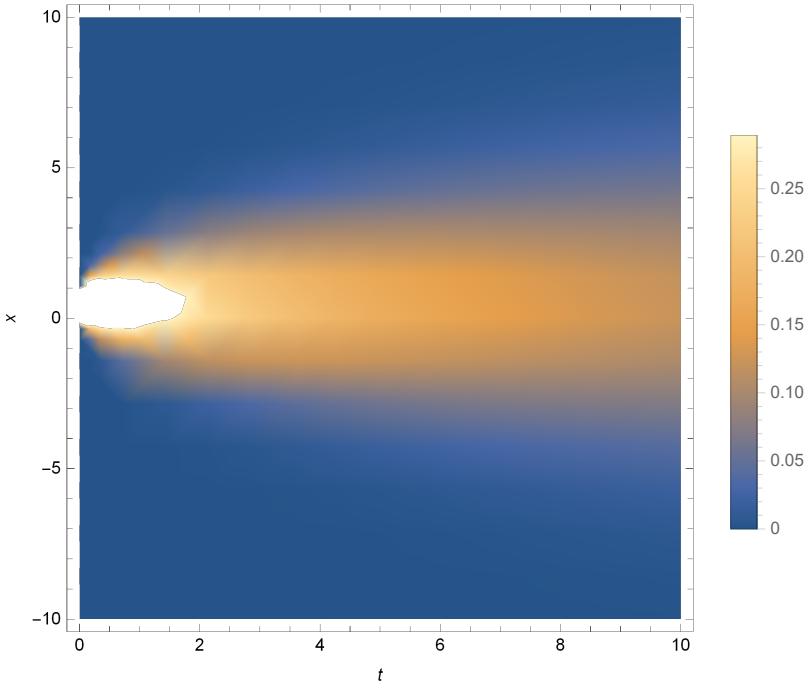
In[ ]:=
```
Plot[{u71[0.00001, x], u71[0.1, x], u71[.5, x], u71[1, x], u71[2, x], u71[10, x]},
 {x, -5, 10}, PlotRange → {{-3, 3}, {0, 1.2}},
 PlotLegends → "Expressions", AxesLabel → Automatic]
```

Out[ ]=



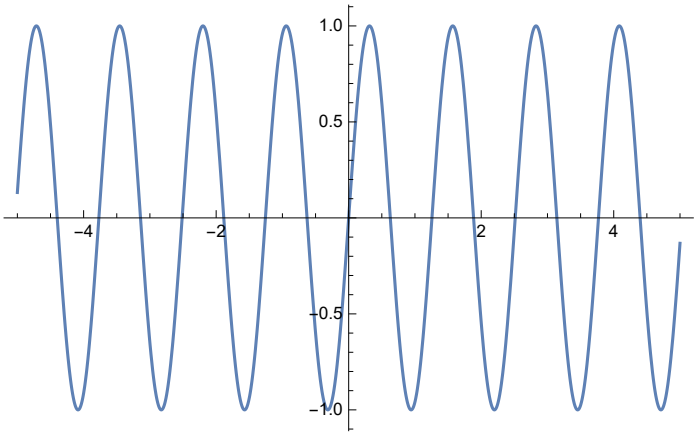
In[ ]:=
```
DensityPlot[u71[t, x], {t, 0, 10}, {x, -10, 10},
 PlotLegends → Automatic, FrameLabel → Automatic]
```

Out[ ]=



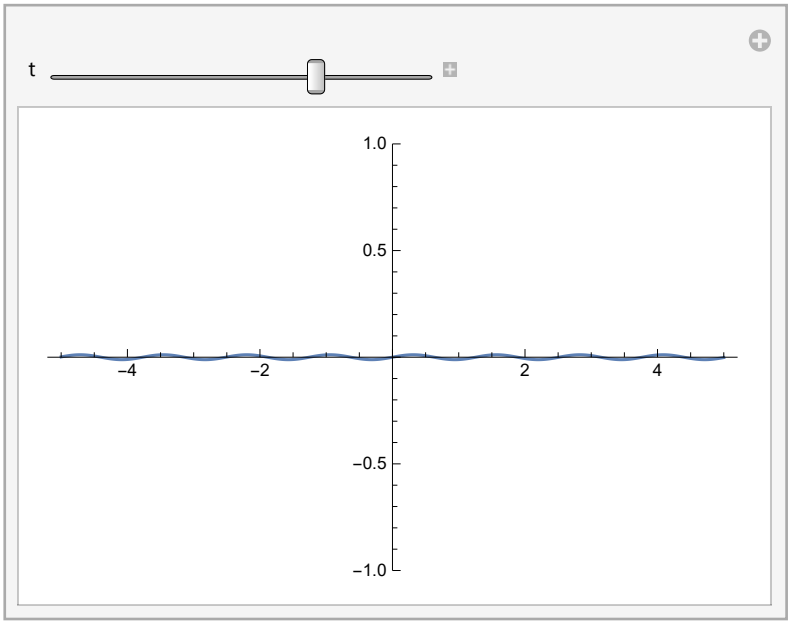
In[ ]:=
```
u72[t_, x_] := 1.` ℯ^(-25 t^1.`/2) Sin[5 x];
Plot[u72[0.000001, x], {x, -5, 5}]
```

Out[ ]=



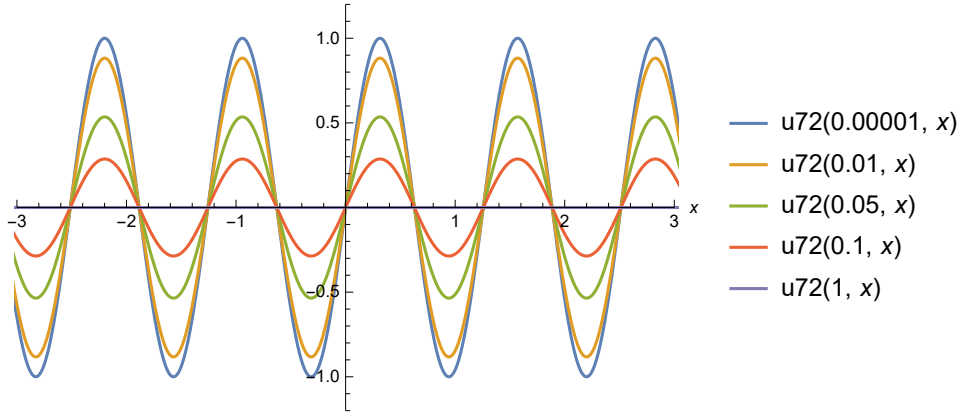
In[ ]:=
```
Manipulate[Plot[u72[t, x], {x, -5, 5}, PlotRange → {-1, 1}], {t, 0.000, .5}]
```

Out[ ]=



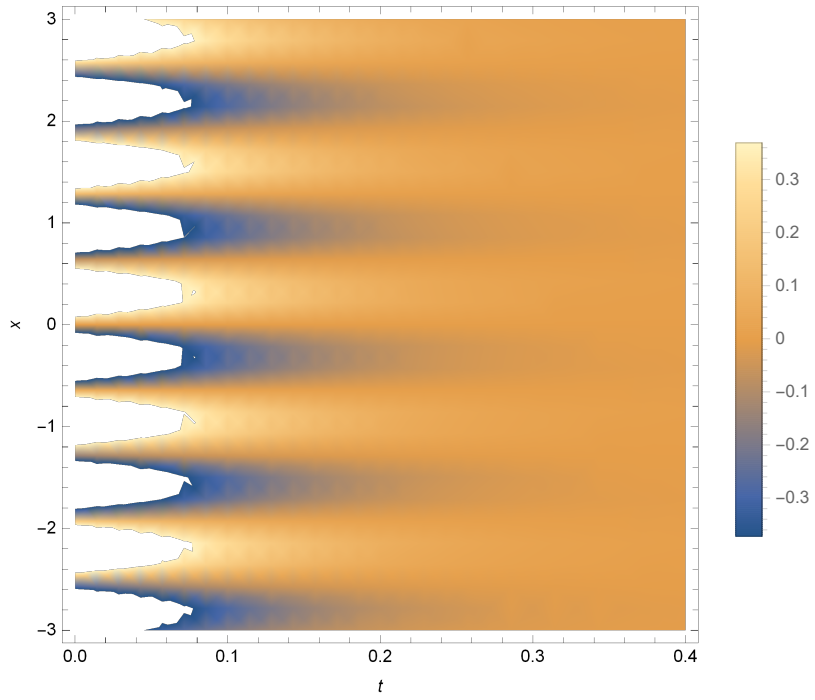
In[ ]:=
```
Plot[{u72[0.00001, x], u72[0.01, x], u72[.05, x], u72[.1, x], u72[1, x]}, {x, -5, 10},
 PlotRange → {{-3, 3}, {-1.2, 1.2}}, PlotLegends → "Expressions", AxesLabel → Automatic]
```

Out[ ]=



In[ ]:=
```
DensityPlot[u72[t, x], {t, 0, .4}, {x, -3, 3},
 PlotLegends → Automatic, PerformanceGoal → "Quality", FrameLabel → Automatic]
```

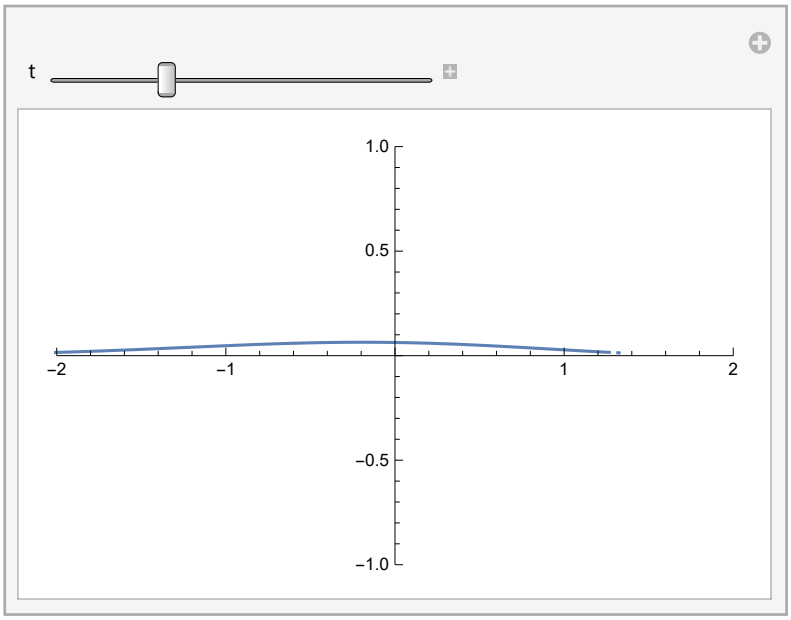
Out[ ]=



In[ ]:=
```
u73[t_, x_] = e^(-12.5` t^1.` - (0.` + 5.` i) x) (-0.25` Erfi[3.5355339059327373` t^0.5` +
    1/t^0.5` (0.` + 0.7071067811865475` i) (-1.` 15.954589770191003 + x)] +
  e^((0.` + 10.` i) x) ((0.` - 0.25` i) Erf[1/t^0.5` (0.7071067811865475` -
      (0.` + 3.5355339059327373` i) t^1.` - 0.7071067811865475` x)] +
    0.25` Erfi[3.5355339059327373` t^0.5` - ((0.` + 0.7071067811865475` i) x)/t^0.5`]) +
  0.25` Erfi[3.5355339059327373` t^0.5` + ((0.` + 0.7071067811865475` i) x)/t^0.5`]);
```

In[ ]:=
```
Manipulate[Plot[u73[t, x], {x, -5, 5}, PlotRange → {{-2, 2}, {-1, 1}}], {t, 0.0001, 5}]
```

Out[ ]=



General: Exp[−179994. − 29.9991 i] is too small to represent as a normalized machine number; precision may be lost.

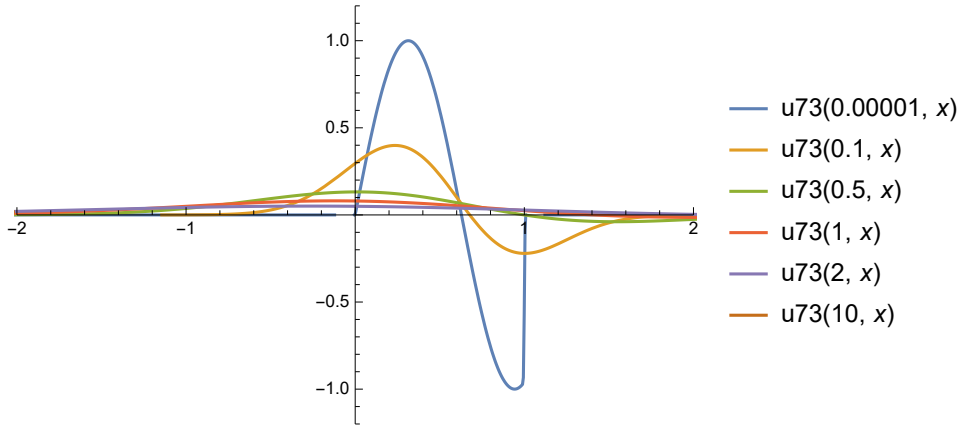
General: Exp[−179994. + 29.9991 i] is too small to represent as a normalized machine number; precision may be lost.

General: Exp[−124996. + 24.9991 i] is too small to represent as a normalized machine number; precision may be lost.

General: Further output of General::munfl will be suppressed during this calculation.

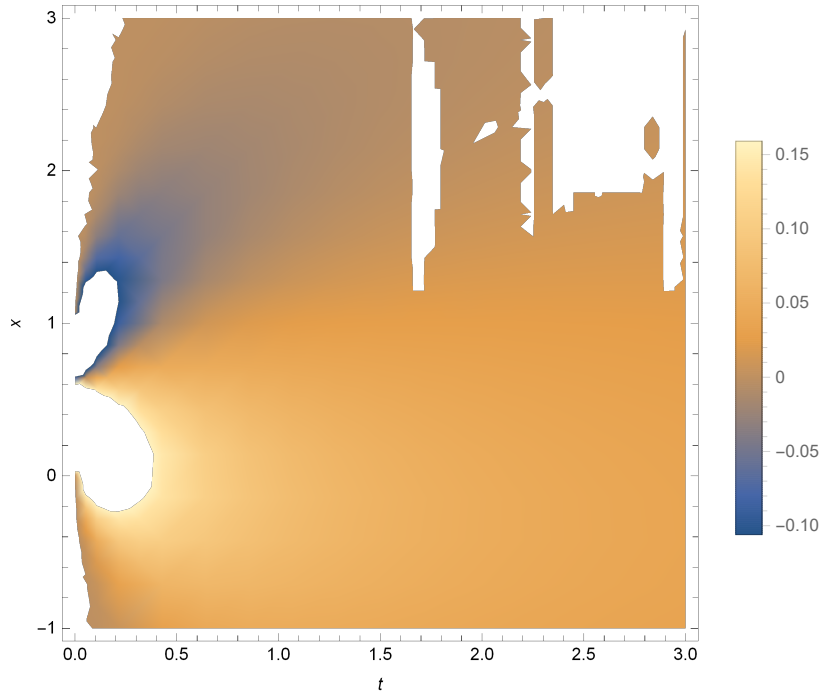
In[•]:=
```
Plot[{u73[0.00001, x], u73[0.1, x], u73[.5, x], u73[1, x], u73[2, x], u73[10, x]},
 {x, -5, 10}, PlotRange → {{-2, 2}, {-1.2, 1.2}}, PlotLegends → "Expressions"]
```

Out[•]=



In[•]:=
```
DensityPlot[u73[t, x], {t, 0, 3}, {x, -1, 3},
 PerformanceGoal → "Quality", PlotLegends → Automatic, FrameLabel → Automatic]
```

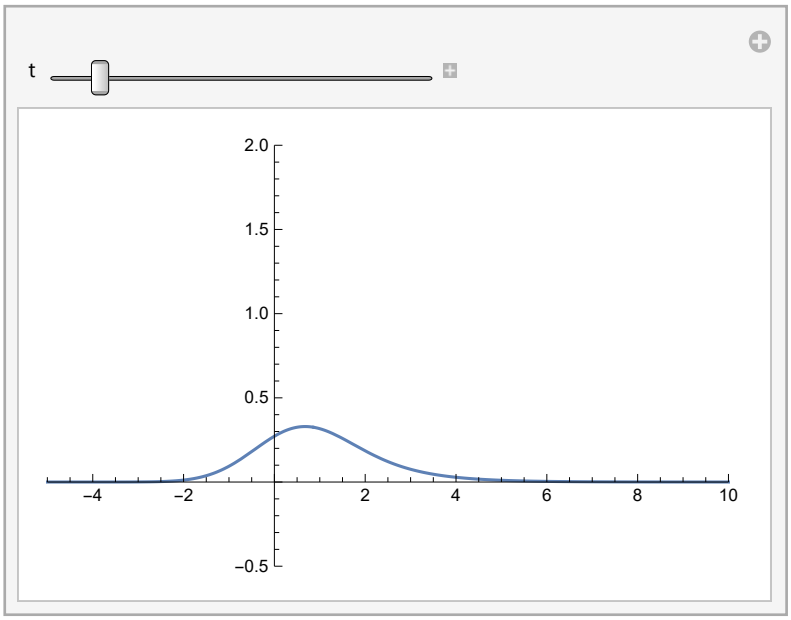
Out[•]=



In[•]:=
```
u74[t_, x_] := (e^(t^1.`/2 - x) (0.5000000000000001` t^0.5` Sqrt[t^1.` - 2.` x + x^2/t^1.`] +
   (-0.5` t^1.` + 0.5` x) Erf[Sqrt[(t^1.` - x)^2/t^1.`]/Sqrt[2]])) / (t^0.5` Sqrt[(t^1.` - x)^2/t^1.`]);
```

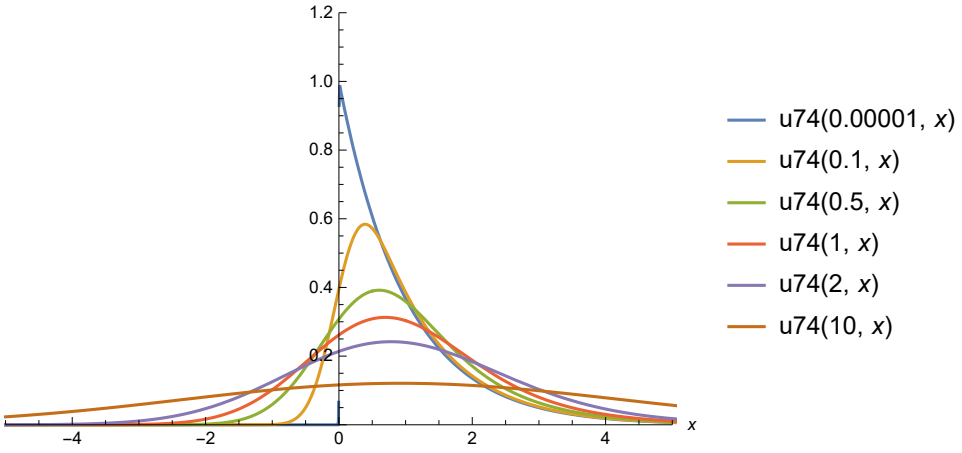
In[ ]:=
```
Manipulate[Plot[u74[t, x], {x, -5, 10}, PlotRange → {{-5, 10}, {-.5, 2}}], {t, 0.0001, 10}]
```

Out[ ]=



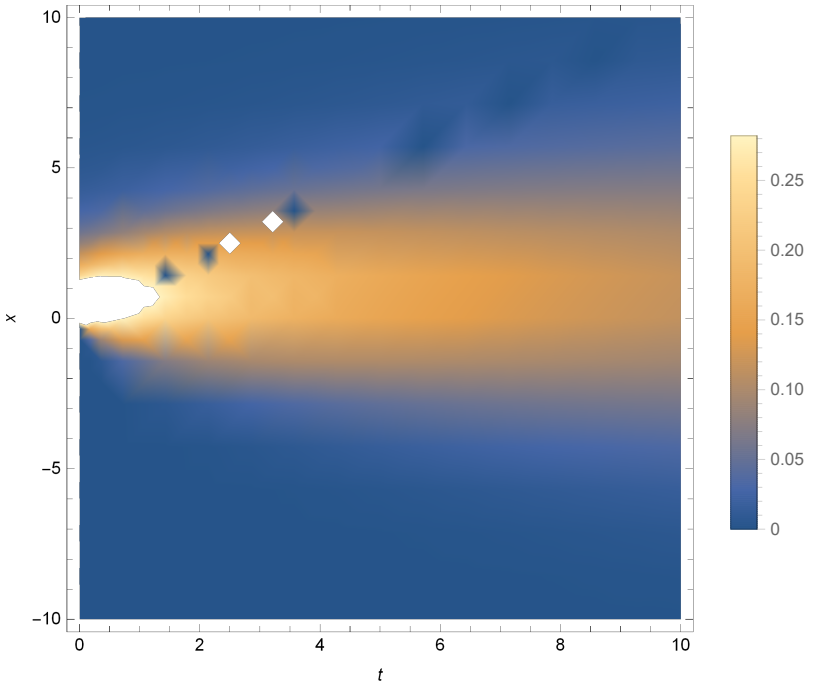
In[ ]:=
```
Plot[{u74[0.00001, x], u74[0.1, x], u74[.5, x], u74[1, x], u74[2, x], u74[10, x]},
 {x, -5, 10}, PlotRange → {{-5, 5}, {0, 1.2}},
 PlotLegends → "Expressions", AxesLabel → Automatic]
```

Out[ ]=



In[•]:=
```
DensityPlot[u74[t, x], {t, 0, 10}, {x, -10, 10},
 PerformanceGoal → "Quality", PlotLegends → Automatic, FrameLabel → Automatic]
```

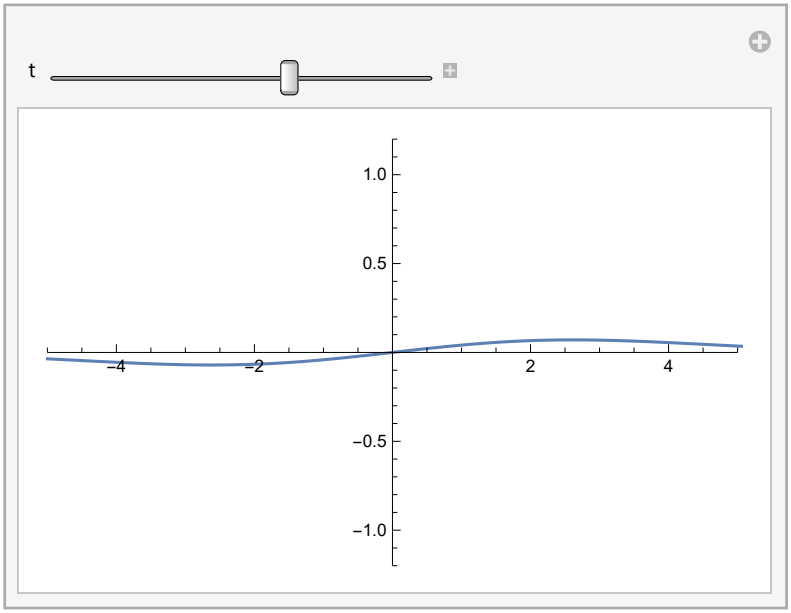
Out[•]=



In[•]:=
```
u75a[t_, x_] :=
 (-0.7978845608028655` e^-(0.5000000000000001` (1.`-1.` x)^2)/t + 0.39894228040143276` e^-(0.5000000000000001` (2.`-1.` x)^2)/t +
  0.7978845608028655` e^-(0.5000000000000001` (1.`+1.` x)^2)/t - 0.39894228040143276` e^-(0.5000000000000001` (2.`+1.` x)^2)/t)
 Sqrt[t] + (-1.0000000000000002` + 0.5000000000000001` x)
 Erf[0.7071067811865476` (-2.` + x)/Sqrt[t]] +
(1.0000000000000002` - 1.0000000000000002` x) Erf[0.7071067811865476` (-1.` + x)/Sqrt[t]] +
(1.0000000000000002` + 1.0000000000000002` x) Erf[0.7071067811865476` (1.` + x)/Sqrt[t]] -
1.0000000000000002` Erf[0.7071067811865476` (2.` + x)/Sqrt[t]] -
0.5000000000000001` x Erf[0.7071067811865476` (2.` + x)/Sqrt[t]];
```

In[•]:=
```
Manipulate[
 Plot[u75a[t, x], {x, -5, 10}, PlotRange → {{-5, 5}, {-1.2, 1.2}}], {t, 0.00001, 10}]
```

Out[•]=



In[•]:=
```
Plot[{u75a[0.00001, x], u75a[0.1, x], u75a[.5, x], u75a[1, x], u75a[2, x], u75a[10, x]},
 {x, -5, 10}, PlotRange → {{-5, 5}, {-1.2, 1.2}},
 PlotLegends → "Expressions", PlotLabel → Automatic]
```

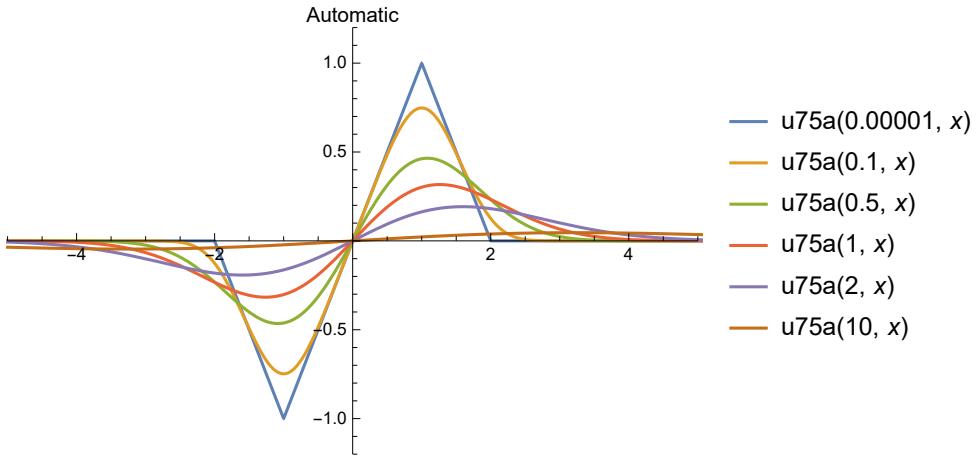
General: Exp[−1.79982 × 10^6] is too small to represent as a normalized machine number; precision may be lost.

General: Exp[−2.44979 × 10^6] is too small to represent as a normalized machine number; precision may be lost.

General: Exp[−799877.] is too small to represent as a normalized machine number; precision may be lost.

General: Further output of General::munfl will be suppressed during this calculation.

Out[•]=



In[ ]:=
```
DensityPlot[u75a[t, x], {t, 0, 100}, {x, -10, 10},
 PlotLegends → Automatic, FrameLabel → Automatic]
```

General: Exp[−8467.8] is too small to represent as a normalized machine number; precision may be lost.

General: Exp[−10077.6] is too small to represent as a normalized machine number; precision may be lost.

General: Exp[−5668.2] is too small to represent as a normalized machine number; precision may be lost.

General: Further output of General::munfl will be suppressed during this calculation.

Out[ ]=