Electricity and Magnetism Synopsis

Qingmu "Josh" Deng

May 2019

1 Electrostatics

1.1 Coloumb's Law

Given a source charge q at position $\mathbf{r}'$ and a field charge Q at $\mathbf{r}$

$$\mathbf{F}(\mathbf{r}) = \frac{1}{4\pi\epsilon_0}\frac{qQ}{\mathscr{r}^2}\hat{\boldsymbol{\mathscr{r}}} \tag{1}$$

where where $\epsilon_0 = 8.85 \times 10^{-12}\frac{C}{Nm^2}$ is the permitivity of free space. $\boldsymbol{\mathscr{r}}$ is the displacement vector between Q and q, pointing from q to Q. That is,

$$\boldsymbol{\mathscr{r}} = \mathbf{r} - \mathbf{r}' \tag{2}$$

1.2 Electric Field

$$\mathbf{E}(\mathbf{r}) = \frac{1}{4\pi\epsilon_0}\frac{q}{\mathscr{r}^2}\hat{\boldsymbol{\mathscr{r}}} = \frac{1}{4\pi\epsilon_0}\iiint \frac{\hat{\boldsymbol{\mathscr{r}}}}{\mathscr{r}^2}\rho(\mathbf{r}')dV' \tag{3}$$

Electric field follows superposition, that is, the electric field at one given point is equal to the sum of individual electric field caused by each quantized charges.

1.2.1 Electric Flux and Gauss's Law

$$\Phi = \oiint_S \mathbf{E} \cdot d\mathbf{a} = \frac{Q}{\epsilon_0} \tag{4}$$

$$\Phi = \iiint_v \nabla \cdot \mathbf{E}\, d\tau' = \frac{1}{\epsilon_0}\iiint_v \rho\, d\tau \tag{5}$$

$$\nabla \cdot \mathbf{E} = \frac{\rho}{\epsilon_0} \tag{6}$$

Gauss's law states that the electric flux across an enclosed surface is proportional to the amount of charge in the enclosed volume.

1.2.2 Divergence of the Electric Field

$$\nabla \cdot \mathbf{E} = \frac{1}{4\pi\epsilon_0} \iiint (\nabla \cdot \frac{\hat{\boldsymbol{\mathscr{r}}}}{\boldsymbol{\mathscr{r}}^2}) \rho(\mathbf{r'})\, d\tau' \tag{7}$$

$$\nabla \cdot \frac{\hat{\boldsymbol{\mathscr{r}}}}{\boldsymbol{\mathscr{r}}^2} = 4\pi\sigma^3(\boldsymbol{\mathscr{r}}) \tag{8}$$

$$\nabla \cdot \mathbf{E} = \frac{1}{4\pi\epsilon_0} \iiint 4\pi\sigma^3(\mathbf{r} - \mathbf{r'}) \rho(\mathbf{r'})\, d\tau' \tag{9}$$

$$\nabla \cdot \mathbf{E} = \frac{\rho(\mathbf{r})}{\epsilon_0} \tag{10}$$

1.2.3 Curl of the Electric Field

Since we know

$$\mathbf{E} = -\nabla V \tag{11}$$

the curl of an electric field would give us

$$\nabla \times \mathbf{E} = \nabla \times (-\nabla V) = 0 \tag{12}$$

1.3 Electric Potential

$$V(\mathbf{r}) = -\int_O^{\mathbf{r}} \mathbf{E} \cdot d\mathbf{l} \tag{13}$$

The potential difference between point **a** and **b** is give by

$$\begin{aligned} V(\mathbf{b}) - V(\mathbf{a}) &= -\int_O^{\mathbf{b}} \mathbf{E} \cdot d\mathbf{l} + \int_O^{\mathbf{a}} \mathbf{E} \cdot d\mathbf{l} && (14) \\ &= -\int_{\mathbf{a}}^{\mathbf{b}} \mathbf{E} \cdot d\mathbf{l} && (15) \end{aligned}$$

According to the fundamental theorem for gradient

$$V(\mathbf{b}) - V(\mathbf{a}) = \int (\nabla \cdot V) \cdot d\mathbf{l} \tag{16}$$

We can thus show that

$$\nabla V = -\mathbf{E} \tag{17}$$

Poisson's Equation is thus given by:

$$\nabla^2 V = -\frac{\rho}{\epsilon_0} \tag{18}$$

In vacuum, Poisson's Equation becomes Laplace's equation:

$$\nabla^2 V = 0 \tag{19}$$

1.4 Work and Energy

The work done on a charge q by an electric field is given by

$$W = \int_b^a q\mathbf{E} \cdot d\mathbf{l} = q(V(\mathbf{b}) - V(\mathbf{a})) \tag{20}$$

The work it takes to amass n number of discrete charges is

$$W = \frac{1}{2}\sum_{i=1}^{n} q_i V(\mathbf{r}_i) \tag{21}$$

The analogy in the continuous time domain is simply

$$W = \frac{1}{2}\int \rho V \, d\tau \tag{22}$$

By recognizing the fact that $\rho = \epsilon_0 \nabla \cdot \mathbf{E}$ and integration by parts, we can turn the above equation into the following clever form:

$$W = \frac{\epsilon_0}{2}\int_{all \;\; space} E^2 \, d\tau \tag{23}$$

1.5 Electrostatics in Materials

1.5.1 Conductors

An ideal conductor is equipotential throughout. When situated in an electric field, a conductor would produce an equal but opposite electric within to maintain zero potential. Thanks to Gauss's law, we are also able to derive the boundary condition for a charged conductor whose electric field immediately outside of its surface is $\frac{\sigma}{\epsilon_0}$.

1.5.2 Capacitors

Suppose we have two conductors and have +Q on one and -Q on the other. The potential difference between them is

$$V = V_+ - V_- = -\int_-^+ \mathbf{E} \cdot d\mathbf{l} \tag{24}$$

where the electric field is given by Coloumb's law:

$$\mathbf{E} = \frac{1}{4\pi\epsilon_0}\int \frac{\rho}{\mathscr{r}^2}\hat{\mathscr{r}} \, d\tau \tag{25}$$

Since the electric field is proportional to the amount of charges, so it potential difference. The constant of proportionality, called capacitance, is defined as

$$C = \frac{Q}{V} \tag{26}$$

Given two parallel plates of area A and a distance of separation d, we put a positive charge +Q on the top one and a negative charge -Q on the bottom one. By the definition of the potential, we have the relationship:

$$V = Ed \tag{27}$$

The electric is given by the Gauss's law as:

$$E = \frac{Q}{\epsilon_0 A} \tag{28}$$

The capacitance of such a paralell plate capacitor is therefore

$$C = \frac{Q}{V} = \frac{Q}{\frac{Qd}{\epsilon_0 A}} \tag{29}$$

$$C = \frac{\epsilon_0 A}{d} \tag{30}$$

1.5.3 Dielectrics

Dielectrics, unlike conductors having unlimited supplies of electrons, have electrons that can only align within a given atom or molecules. When a neutral atom, one that might make up a dielectric material, is subject to an electric field, the positively charged region would move in the direction of the electric field while the negatively charged would move in the opposite direction of the field. The dipole moment of such a physical dipole is defined as

$$\mathbf{p} = q\mathbf{r'}_+ - q\mathbf{r'}_- = q(\mathbf{r'}_+ - \mathbf{r'}_-) = q\mathbf{d} \tag{31}$$

where $\mathbf{d}$ is the vector from the negative charge to the positive one.

A whole chunk of material made up of such atoms or molecules would, by the same token, be polarized. The dipole would in the end line with the electric field due to the torque $\mathbf{N}$exerted on the dipole.

$$\begin{aligned}
\mathbf{N} &= (\mathbf{r}_+ \times \mathbf{F}_+) + (\mathbf{r}_- \times \mathbf{F}_-) && (32)\\
&= (\frac{\mathbf{d}_+}{2} \times q\mathbf{E}) + (\frac{-\mathbf{d}_-}{2} \times -q\mathbf{E}) && (33)\\
&= q\mathbf{d} \times \mathbf{E} && (34)\\
&= \mathbf{p} \times \mathbf{E} && (35)
\end{aligned}$$

Polarization $\mathbf{P}$, dipole moment per unit volume, is a good way to describe the extent of such phenomenons.

1.5.4 Bound Charges

The potential for a single dipole is

$$V(\mathbf{r}) = \frac{1}{4\pi\epsilon_0} \frac{\mathbf{p} \cdot \hat{\boldsymbol{\imath}}}{\boldsymbol{\imath}^2} \tag{36}$$

For a volume charge distribution,

$$V(\mathbf{r}) = \frac{1}{4\pi\epsilon_0} \iiint_V \frac{\mathbf{P}(\mathbf{r'}) \cdot \hat{\boldsymbol{\imath}}}{\boldsymbol{\imath}^2}\, d\tau \tag{37}$$

Given that $\nabla' \frac{1}{\boldsymbol{\imath}} = \frac{\hat{\boldsymbol{\imath}}}{\boldsymbol{\imath}^2}$

$$\begin{aligned}
V(\mathbf{r}) &= \frac{1}{4\pi\epsilon_0} \iiint_V \mathbf{P}(\mathbf{r'}) \cdot (\nabla' \frac{1}{\boldsymbol{\imath}})\, d\tau && (38)\\
&= \frac{1}{4\pi\epsilon_0} [\iiint_V \nabla' \cdot \frac{\mathbf{P}(\mathbf{r'})}{\boldsymbol{\imath}}\, d\tau + \iiint_V \frac{1}{\boldsymbol{\imath}}(\nabla' \cdot \mathbf{P}(\mathbf{r'}))\, d\tau] && (39)\\
&= \frac{1}{4\pi\epsilon_0} [\oiint_S \frac{1}{\boldsymbol{\imath}}\mathbf{P}(\mathbf{r'}) \cdot\, d\mathbf{a} + \iiint_V \frac{1}{\boldsymbol{\imath}}(\nabla' \cdot \mathbf{P}(\mathbf{r'}))\, d\tau] && (40)
\end{aligned}$$

The first term is like a surface charge, and the second term is like that of volume charge.

$$\sigma_b = \mathbf{P} \cdot \hat{n} \tag{41}$$
$$\rho_b = -\nabla \cdot \mathbf{P} \tag{42}$$

1.5.5 Electric Displacement

Gauss's law in the presence of dielectrics can be written as

$$\epsilon_0 \nabla \cdot \mathbf{E} = \rho = \rho_b + \rho_f = -\nabla \cdot \mathbf{P} + \rho_F \tag{43}$$

where ρ_f is the amount of free charges bounded in the dielectrics. Through some manipulation.

$$\nabla \cdot (\epsilon_0 \mathbf{E} + \mathbf{P}) = \rho_f \tag{44}$$

The electric displacement $\mathbf{D} = \epsilon_0 \mathbf{E} + \mathbf{P}$ allows the equations to be re-written to

$$\nabla \cdot \mathbf{D} = \rho_f \tag{45}$$
$$or \ \oiint \mathbf{D}\, d\mathbf{a} = Q_{f_{enc}} \tag{46}$$

1.5.6 Linear Dielectrics

Provided a not too strong electric field, if a piece of dielectrics follows

$$\mathbf{P} = \epsilon_0 \mathcal{X}_e \mathbf{E} \tag{47}$$

where $\mathcal{X}$ is the electric susceptibility of the medium. The electric displacement can then be written to

$$\mathbf{D} = \epsilon_0 \mathbf{E} + \mathbf{P} \tag{48}$$
$$= \epsilon_0 \mathbf{E} + \epsilon_0 \mathcal{X}_e \mathbf{E} \tag{49}$$
$$= \epsilon_0 (1 + \mathcal{X}_e) \mathbf{E} \tag{50}$$
$$= \epsilon_0 \epsilon_r \mathbf{E} \tag{51}$$
$$\mathbf{D} = \epsilon \mathbf{E} \tag{52}$$

where $\epsilon = \epsilon_0 \epsilon_r$ is called the permitivity of the material and $\epsilon_r = (1 + \mathcal{X}_e) = \frac{\epsilon}{\epsilon_0}$ is the relative permitivity, or dielectric constant. If a space is homogeneously filled with a dielectric, giving us

$$\nabla \cdot \mathbf{D} = \rho \qquad \qquad \nabla \times \mathbf{D} = 0 \tag{53}$$

then we can determine the electric displacement quite easilyas if the dielectrics are not there.

$$\mathbf{D} = \epsilon_0 \mathbf{E}_{vac} \tag{54}$$
$$\mathbf{E} = \frac{1}{\epsilon} \mathbf{D} = \frac{1}{\epsilon_r} \mathbf{E}_{vac} \tag{55}$$

If we fill the vacuum between two plates of a parallel capacitor of capacitance $C = \frac{Q}{V}$ with a dielectrics with dielectric constant ϵ_r, the dielectrics will reduce $\mathbf{E}$ by a factor of $\frac{1}{\epsilon_r}$ and V accordingly. The capacitance of the capacitor is thus multiplied by a factor of $\epsilon_{\mathbf{r}}$.

$$C = \epsilon_r C_{vac} \tag{56}$$

2 Magnetostatics

In magnetostatics, similar to stationary charges in electrostatics, steady current is indispensable. Steady current implies:

$$\frac{\partial \rho}{\partial t} = 0 \qquad\qquad \frac{\partial \mathbf{J}}{\partial t} = 0 \tag{57}$$

where ρ is the charge density and $\mathbf{J}$ is the current density.

2.1 Biot-Savart Law under Constant Current

$$\mathbf{B} = \frac{\mu_0}{4\pi} \int \frac{\mathbf{I} \times \hat{\mathbf{\mathscr{r}}}}{\mathscr{r}^2} \, dl' \tag{58}$$

$$= \frac{\mu_0}{4\pi} I \int \frac{d\mathbf{l'} \times \hat{\mathbf{\mathscr{r}}}}{\mathscr{r}^2} \tag{59}$$

where $\mu_0 = 4\pi \times 10^{-7} N/A^2$ is the permeability of free space.

2.2 Lorentz Force Law

Lorentz force law states that the force on a charge Q is

$$\mathbf{F} = Q(\mathbf{E} + \mathbf{v} \times \mathbf{B}) \tag{60}$$

where the magnetic forces is

$$\mathbf{F}_{mag} = Q(\mathbf{v} \times \mathbf{B}) \tag{61}$$

$\mathbf{v}$ is the velocity of charges. That is, only moving charges would feel the force of a magnetic field in a direction perpendicular to both the magnetic field and the direction of charge movement.

2.3 Ampere's Law

$$\oint \mathbf{B} \cdot d\mathbf{l} = \mu_0 I_{enc} \tag{62}$$

Ampere's Law states that the circulation of a magnetic field around a closed loop is proportional to the amount of current enclosed by the loop. By the fundamental theorem of calculus, we also obtain

$$\oint \mathbf{B} \cdot d\mathbf{l} = \iint (\nabla \times \mathbf{B}) \cdot d\mathbf{a} \tag{63}$$

$$u_0 I_{enc} = \mu_0 \iint \mathbf{J} \cdot d\mathbf{a} \tag{64}$$

$$\tag{65}$$

Ampere's law in differential form is thus

$$\nabla \times \mathbf{B} = \mu_0 \mathbf{J} \tag{66}$$

2.4 Divergence and Curl of Magnetic Field

The Biot-Savart law gives

$$\mathbf{B}(\mathbf{r}) = \frac{\mu_0}{4\pi} \iiint \frac{\mathbf{J}(\mathbf{r}') \times \hat{\boldsymbol{\mathscr{r}}}}{\mathscr{r}^2} \, d\tau' \tag{67}$$

Taking the divergence

$$\nabla \cdot \mathbf{B}(\mathbf{r}) = \frac{\mu_0}{4\pi} \iiint \nabla \cdot (\frac{\mathbf{J}(\mathbf{r}') \times \hat{\boldsymbol{\mathscr{r}}}}{\mathscr{r}^2}) \, d\tau' \tag{68}$$

$$\nabla \cdot (\frac{\mathbf{J}(\mathbf{r}') \times \hat{\boldsymbol{\mathscr{r}}}}{\mathscr{r}^2}) = \frac{\boldsymbol{\mathscr{r}}}{\mathscr{r}^2} \cdot (\nabla \times \mathbf{J}(\mathbf{r}')) - \mathbf{J}(\mathbf{r}') \cdot (\nabla \times \frac{\boldsymbol{\mathscr{r}}}{\mathscr{r}^2}) \tag{69}$$

However, both terms are zero because 1. J is solely a function of primed(source) variable and 2.$\mathbf{J}(\mathbf{r}')$ is just zero. Therefore,

$$\nabla \cdot \mathbf{B} = 0 \tag{70}$$

We can apply the same strategy for seeking the curl of magnetic field.

$$\nabla \times \mathbf{B}(\mathbf{r}) = \frac{\mu_0}{4\pi} \iiint \nabla \times (\mathbf{J}(\mathbf{r}') \times \frac{\hat{\boldsymbol{\mathscr{r}}}}{\mathscr{r}^2}) \, d\tau' \tag{71}$$

where

$$\nabla \times \mathbf{J} \times \frac{\hat{\boldsymbol{\mathscr{r}}}}{\mathscr{r}^2} = \mathbf{J}(\nabla \cdot \frac{\hat{\boldsymbol{\mathscr{r}}}}{\mathscr{r}^2}) - \frac{\hat{\boldsymbol{\mathscr{r}}}}{\mathscr{r}^2}(\nabla \cdot \mathbf{J}) \tag{72}$$

$$= \mathbf{J} 4\pi\sigma^3(\boldsymbol{\mathscr{r}}) - 0 \tag{73}$$

$$\nabla \times \mathbf{B}(\mathbf{r}) = \frac{\mu_0}{4\pi} \iiint \mathbf{J}(\mathbf{r}') 4\pi\sigma^3(\boldsymbol{\mathscr{r}}) \, d\tau' = \mu_0 \mathbf{J}(\mathbf{r}') \tag{74}$$

3 Time Dependency

3.1 Time Independent Maxwell's Equations

$$Gauss's\ Law : \nabla \cdot \mathbf{E} = \frac{\rho}{\epsilon_0} \qquad \nabla \cdot \mathbf{B} = 0 \tag{75}$$

$$\nabla \times \mathbf{E} = 0 \qquad Ampere's\ Law : \nabla \times \mathbf{B} = \mu_0 \mathbf{J} \tag{76}$$

3.2 Electromotive Force

3.2.1 Ohm's Law

The amount of current flow is proportional to the amount of forces per unit charge$\mathbf{f}$.

$$\mathbf{J} = \sigma \mathbf{f} \tag{77}$$

where σ stands for the conductivity of a material. Its reciprocal, $\rho = \frac{1}{\sigma}$, is the resistivity of the material.The force per unit charge is usually described by

$$\mathbf{f} = \sigma(\mathbf{E} + \mathbf{v} \times \mathbf{B}) \tag{78}$$

However, the magnetic contribution is usually too small to be accounted for, Thus,

$$\mathbf{J} = \sigma \mathbf{E} \tag{79}$$

which is also known as Ohm's Law.

3.2.2 Electromotive Force

When a circuit is hooked up to a battery, a current flows through the circuit. While it is obvious that the electrochemical process in the battery would result in a current flow in the battery, it is less clear what is pushing electrons through the conductive materials outside of the batteries. The answer is that there are two forces that are driving current around the loop. The first one is the source, $\mathbf{f}_s$, and the other, the electrostatic force, $\mathbf{E}$.

$$\mathbf{f} = \mathbf{f}_s + \mathbf{E} \tag{80}$$

The electromotance, or electromotive force(emf), of the circuit, defined as the integral of a force per unit charge around a closed loop, is given by

$$\varepsilon = \oint \mathbf{f} \cdot d\mathbf{l} = \oint \mathbf{f}_s \cdot d\mathbf{l} \tag{81}$$

$$\oint \mathbf{E} \cdot d\mathbf{l} = 0 \tag{82}$$

For an ideal source of emf, the net force on the charge is zero, so $\mathbf{E} = -\mathbf{f}_0$

$$V = -\int_a^b \mathbf{E} \cdot d\mathbf{l} = \int_a^b \mathbf{f}_s \, d\mathbf{l} \tag{83}$$

Thanks to the fact that $\mathbf{f}_s$ is zero outside of the source, we can extend the integral to the entire loop

$$V = \int_a^b \mathbf{f}_s \, d\mathbf{l} = \oint \mathbf{f}_s \, d\mathbf{l} = \varepsilon \tag{84}$$

3.2.3 Motional EMF

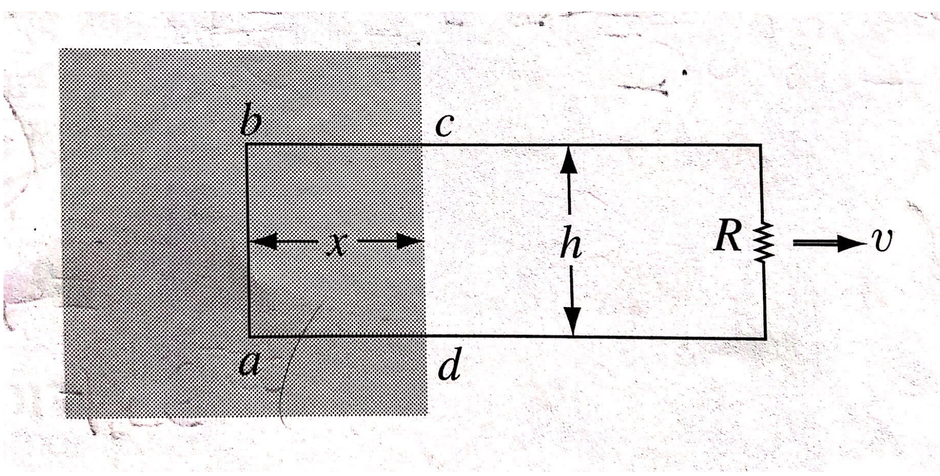

Figure 1: Motional EMF

As the loop of wire is pulled to the right through a magnetic field point into the page, the magnetic force produces an emf that drives current around the loop.

$$\varepsilon = \oint \mathbf{f}_{mag} \cdot d\mathbf{l} = \oint (\vec{\mathbf{v}} \times \mathbf{B}) \cdot d\mathbf{l} = vBh \tag{85}$$

A different of expressing emf comes with magnetic flux.

$$\Phi = \iint \mathbf{B} \cdot d\mathbf{a} \tag{86}$$
$$= Bhx \tag{87}$$

As the loop moves, the flux changes accordingly.

$$\frac{d\Phi}{dt} = Bh\frac{dx}{dt} = -Bhv \tag{88}$$

$$\varepsilon = -\frac{d\Phi}{dt} \tag{89}$$

3.3 Faraday's Experiment and Faraday's Law

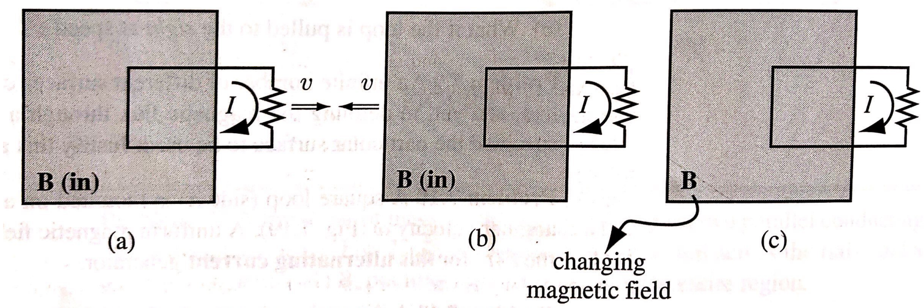

Figure 2: Motional EMF

Michael Faraday conducted a series of experiment in 1831.

- He pulled a loop of wire to the right through a magnetic field. A current flowed in the loop
- He moved the magnet to the left, holding the loop still. Again, a current flowed in the loop.
- With both the loop and the magnet at rest, he changed the strength of the field using an electromagnet. Once a again, current flowed in the loop.

The first experiment can be explained by the Lorentz force law since the charges inside the wire loop physically moves and thus perceives the magnetic field. It is an example of motional EMF where.

$$\varepsilon = -\frac{\partial\Phi}{\partial t} \tag{90}$$

However, for Experiment 2 and 3, there is no moving charges. Faraday's conclusion is that a changing magnetic field induces an electric field.

$$\varepsilon = \oint \mathbf{E} \cdot d\mathbf{l} = -\frac{\partial\Phi}{\partial t} \tag{91}$$

$$\iint \nabla \times \mathbf{E}\, d\mathbf{a} = -\iint \frac{\partial \mathbf{B}}{\partial t} \cdot d\mathbf{a} \tag{92}$$

$$\nabla \times \mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t} \tag{93}$$

As the above equation shows, an electric field curls a magnetic field that changes in time. Lenz's Law help clarifying the behavior of magnetically induces currents by stating that the induced current will flow in such a direction that the flux it produces tends to cancel the change. The Maxwell's equations are now in the form:

$$Gauss's\ Law : \nabla \cdot \mathbf{E} = \frac{\rho}{\epsilon_0} \qquad \nabla \cdot \mathbf{B} = 0 \tag{94}$$

$$Faraday's\ Law : \nabla \times \mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t} \qquad Ampere's\ Law : \nabla \times \mathbf{B} = \mu_0 \mathbf{J} \tag{95}$$

3.4 Maxwell's Fix to Ampere's Law

If we take the divergence of the magnetic field, we would receive

$$\nabla \cdot (\nabla \times \mathbf{B}) = \mu_0 (\nabla \cdot \mathbf{J}) \tag{96}$$

In the realm of magnetostatics with steady current, the divergence of $\mathbf{J}$ is indeed zero, but it is for sure wrong for anythign beyond. The integral form of Ampere's Law

$$\oint \mathbf{B} \cdot d\mathbf{l} = \mu_0 I_{enc} = \mu_0 \iint \mathbf{J} \cdot d\mathbf{a} \tag{97}$$

implies surface independence. However, in the case of a constant current flowing through a parallel capacitor, the surface that wraps in between the empty space between two plates, the calculation would yield zero current density J and thus zero magnetic field where as a surface that intersects with a piece of current carrying wire would give a magnetic field.

The important realization is that as current flows through a capacitor to charge it, more and more positive charges would accumulate on one of the capacitor plates, and the other one would have more and more positive charges leaving. As more positive charges accumulate on one of the plates, the plate would have a stronger electric field that is able to push positive charges on the other plate off. Given a constant current I across a parallel plate capacitor:

Math is kinda wrong here

$$Q = It \tag{98}$$

$$\mathbf{E} \cdot \mathbf{A} = \frac{Q}{\epsilon_0} = \frac{It}{\epsilon_0} \tag{99}$$

$$\Phi_{electric} = \mathbf{E} \cdot \mathbf{A} = \frac{It}{\epsilon_0} \tag{100}$$

$$\frac{\partial \Phi_e}{\partial t} = \frac{I}{\epsilon_0} \tag{101}$$

$$\iint \mathbf{J}_{field} \cdot d\mathbf{a} = I = \epsilon_0 \frac{\partial \Phi_e}{\partial t} = \epsilon_0 \frac{\partial}{\partial t} \iint \mathbf{E} \cdot d\mathbf{a} \tag{102}$$

Thus, the current term in Ampere's Law could be split into a physical charge movement term and a current due to electric field term.

$$\nabla \times \mathbf{B} = \mu_0 (\mathbf{J}_{wire} + \mathbf{J}_{field})$$
$$\nabla \times \mathbf{B} = \mu_0 \mathbf{J}_{wire} + \mu_0 \epsilon_0 \frac{\partial \mathbf{E}}{\partial t} \tag{103}$$

3.5 Time Dependent Maxwell Equations

$$Gauss's\ Law : \nabla \cdot \mathbf{E} = \frac{\rho}{\epsilon_0} \qquad Gauss's\ Law\ for\ Magnetism : \nabla \cdot \mathbf{B} = 0 \tag{104}$$

$$Faraday's\ Law : \nabla \times \mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t} \qquad Ampere's\ Law : \nabla \times \mathbf{B} = \mu_0 \mathbf{J} + \mu_0 \epsilon_0 \frac{\partial \mathbf{E}}{\partial t} \tag{105}$$

For Partial Differential Equation SP19's final project, I will be continuing the study of electricity and magnetism, especially electromagnetic waves, from the semester before. The time dependent Maxwell Equations are two coupled transport equations that would result in two wave equations: one for the electric field, and the other for the magnetic field. The derivation of the wave equations will be presented, and their implications in terms of absorption and dispersion and potential applications in wave guides will also be presented. This document will be a synthesis of David Griffths's *Introduction to Electrodynamics* Chapter 9.

4 Conservation in Electrodynamics

4.1 The Continuity Equation

The **global** conservation law of charge states that the total charge in the universe is constant. The **local** conservation charge is more subtle. Given that the charge in a volume V with charge density $\rho(\mathbf{r}, t)$,

$$Q(t) = \int_V \rho(\mathbf{r}, t) d\tau \tag{106}$$

according to the local conservation charge, the current flowing out through the boundary surface S is

$$\frac{dQ}{dt} = -\oint_S \mathbf{J} \cdot d\mathbf{a} \tag{107}$$

Invoking the divergence theorem and plugging in Equation (106)

$$\int_V \frac{\partial \rho}{\partial t} d\tau = -\int_V \nabla \cdot \mathbf{J} d\tau \tag{108}$$

Thus, we derive the continuity equation - the precise statement of local conservation of charge.

$$\frac{\partial \rho}{\partial t} = -\nabla \cdot \mathbf{J} \tag{109}$$

4.2 Poyting's Theorem

From Equation (23), we know that the work necessary to assemble a static distribution against the Coulomb repulsion of like charges) is

$$W_e = \frac{\epsilon_0}{2} \int E^2 d\tau \tag{110}$$

A similar equation follows for the work required to get currents going against the back EMF is

$$W_m = \frac{1}{2\mu_0} \int B^2 d\tau \tag{111}$$

Therefore, the total energy store in electromagnetic fields per unit volume is

$$u = \frac{1}{2}\Big(\epsilon_0 E^2 + \frac{1}{\mu_0}B^2\Big) \tag{112}$$

Suppose we have some charge and current configuration which, at time t, produce fields $\mathbf{E}$ and $\mathbf{B}$. In the next instant, dt, the charge move around a bit. According to Lorentz force law in Equation 60, the work dW done on a charge q is

$$dW = \mathbf{F}\cdot d\mathbf{l} = q(\mathbf{E} + \mathbf{v}\times\mathbf{B})\cdot\mathbf{v}dt = q\mathbf{E}\cdot\mathbf{v}dt \tag{113}$$

In terms of the charge and current densities, $q \to \rho\tau$ and $\rho\mathbf{v} \to \mathbf{J}$, so the rate at which work is done on all the charges in a volume in V is

$$\frac{dW}{dt} = \int_V \rho d\tau \mathbf{E}\cdot\mathbf{v} = \int_V \mathbf{E}\cdot\mathbf{J}d\tau \tag{114}$$

$\mathbf{E}\cdot\mathbf{J}$ can then be interpreted as the work done per unit time, per unit volume - or, the power per unit volume. We can replace $\mathbf{J}$ with electric fields and magnetic fields with Equation (103).

$$\mathbf{E}\cdot\mathbf{J} = \frac{1}{\mu_0}\mathbf{E}\cdot(\nabla\times\mathbf{B}) - \epsilon_0\mathbf{E}\cdot\frac{\mathbf{E}}{dt} \tag{115}$$

From product rule 6 in [1],

$$\mathbf{E}\cdot(\nabla\times\mathbf{B}) = \mathbf{B}\cdot(\nabla\times\mathbf{E}) - \mathbf{E}\cdot(\nabla\times\mathbf{B}) \tag{116}$$

Using Farady's law in Equation (93),

$$\mathbf{E}\cdot(\nabla\times\mathbf{B}) = -\mathbf{B}\cdot\frac{\partial\mathbf{B}}{\partial t} - \nabla\cdot(\mathbf{E}\times\mathbf{B}) \tag{117}$$

Meanwhile, by the reverse chain rule of derivatives, we have

$$\mathbf{B}\cdot\frac{\partial\mathbf{B}}{\partial t} = \frac{1}{2}\frac{\partial}{\partial t}(B^2) \qquad\qquad \mathbf{E}\cdot\frac{\partial\mathbf{E}}{\partial t} = \frac{1}{2}\frac{\partial}{\partial t}(E^2) \tag{118}$$

Then,

$$\mathbf{E}\cdot\mathbf{B} = -\frac{1}{2}\frac{\partial}{\partial t}\Big(\epsilon_0 E^2 + \frac{1}{\mu_0}B^2\Big) - \frac{1}{\mu_0}\nabla\cdot(\mathbf{E}\times\mathbf{B}) \tag{119}$$

Bringing back the integrals gives us

$$\frac{dW}{dt} = -\frac{d}{dt}\int_V \frac{1}{2}\Big(\epsilon_0 E^2 + \frac{1}{\mu_0}B^2\Big)d\tau - \frac{1}{\mu_0}\int_V \nabla\cdot(\mathbf{E}\times\mathbf{B})d\tau \tag{120}$$

Invoking the divergence theorem again,

$$\frac{dW}{dt} = -\frac{d}{dt}\int_V \frac{1}{2}\Big(\epsilon_0 E^2 + \frac{1}{\mu_0}B^2\Big)d\tau - \frac{1}{\mu_0}\oint_S (\mathbf{E}\times\mathbf{B})\cdot d\mathbf{a} \tag{121}$$

where the surface S completely bounds the volume V. This is **Poynting's Theorem**: the work done on the charges by the electromagnetic force is equal to the decrese in energy remaining in the fields(the volume integral term), less the energy that flowed out through the surface(the surface integral term). The energy per unit time, per unit area, transported by the eletromagnetic fields is the **Poynting vector**:

$$\mathbf{S} = \frac{1}{\mu_0}(\mathbf{E}\times\mathbf{B}) \tag{122}$$

The Poynting vector points in the direction in which the energy and, accordingly, the wave travel.

5 Electromagnetic Waves

5.1 Waves in One Dimension

5.1.1 The Origin of Wave Equation and the Representation of its Solutions

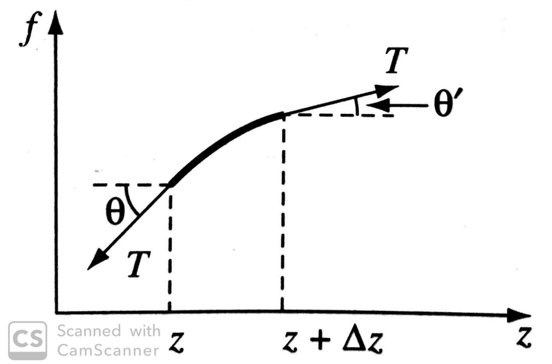

Figure 3: String displacement results in wave motion.

Imagine wave motions on a string. Why would a streched string support wave motion? Let us view it in the light of Newton's second law. With the string under the tension T, its displacement from the equilibrium would produce a net transverse force for the segment between z and $z + \Delta z$

$$\Delta F = T \sin\theta' - T \sin\theta \tag{123}$$

where $\theta^{'}$ is the angle the string makes with the z direction at point $z+\Delta z$, and θ us the corresponding angle at point z. For a very small segment, we can approximate that the distortion is very small as well and sine as tangent.

$$\Delta F \approx T(\tan\theta' - \tan\theta) = T(\frac{\partial f}{\partial z}\Big|_{z+\Delta z} - \frac{\partial f}{\partial z}\Big|_{z}) \approx T\frac{\partial^2 f}{\partial z^2} \tag{124}$$

If the mass per unit length is μ, Newton's second law suggest

$$\Delta F = \mu(\Delta z)\frac{\partial^2 f}{\partial t^2} \tag{125}$$

Therefore, we have the classical one dimensional wave equation as

$$\frac{\partial^2 f}{\partial z^2} = \frac{1}{v^2}\frac{\partial^2 f}{\partial t^2} \tag{126}$$

where v, the speed of speed, is

$$v = \sqrt{\frac{T}{\mu}} \tag{127}$$

The solution of the wave equation admits solutions of the forms

$$g(z,t) = g(z - vt) + h(z + vt) \tag{128}$$

because the wave equation involves the square of v. The wave equation is also linear, which suggests any linear combination of the solutions is still a solution.

Sinusoidal waves are the mode shapes of waves on a one dimensional string. A typical sinusoidal wave could be the following:

$$f(z,t) = A\cos[k(z - vt) + \delta] \tag{129}$$

A is the amplitude of the wave; the argument of the cosine is the phase, and δ is called phase constant. If $\delta = 0$, then the central maximum passes the origin at time $t = 0$; generally, δ/k is the distance by which the central maximum and the whole wave is delayed. k is the wave number. It is related the wavelength λ, or the "spatial period", by

$$\lambda = \frac{2\pi}{k} \tag{130}$$

As time passes, the wave shifts to the right at speed v. The (temporal) period for completing one cycle is given by

$$T = \frac{2\pi}{kv} \tag{131}$$

The frequency f, the number of oscillation per unit time, is

$$f = \frac{1}{T} = \frac{kv}{2\pi} = \frac{v}{\lambda} \tag{132}$$

The angular frequency ω, the number of radians per unit time, is

$$\omega = 2\pi f = kv \tag{133}$$

To simplify the algebra, we will write the sinusoidal wave as a complex exponential:

$$f(z,t) = A\cos[k(z - vt) + \delta] = Re\left[Ae^{i(kz-\omega t+\delta)}\right] \tag{134}$$

This leads us to the definition of the complex wave function

$$\tilde{f}(z,t) \equiv \tilde{A}e^{i(kz-\omega t)} \tag{135}$$

with the complex amplitude $\tilde{A} \equiv Ae^{i\delta}$. Thus,

$$f(z,t) = Re[\tilde{f}(z,t)] \tag{136}$$

5.1.2 Boundary Conditions

Suppose, instead of an infinitely long string stretching across the whole z axis, we have two pieces of infinite strings knot together at $z = 0$. While the tension is the same for both strings, we assume the mass per unit length is not, and hence wave velocities v_1 and v_2 are also different. We generate an **incident** wave that comes from the left:

$$\tilde{f}_I(z,t) = \tilde{A}_I e^{i(k_1 z-\omega t)} \tag{137}$$

When the wave hit the knot, a **reflected** wave will be generated traveling to the left:

$$\tilde{f}_R(z,t) = \tilde{A}_R e^{i(-k_1 z-\omega t)} \tag{138}$$

Part of the incident wave will also continue onto the second string, giving rise to a **transmitted** wave:

$$\tilde{f}_T(z,t) = \tilde{A}_T e^{i(k_2 z-\omega t)} \tag{139}$$

The (temporal) angular frequency ω is preserved across the whole medium. Therefore, we can derive the following chain of equality between wave length, wave numbers, and wave velocity.

$$\omega_1 = \omega_2 = \omega \tag{140}$$
$$k_1 v_1 = k_2 v_2 \tag{141}$$
$$\frac{v_2}{v_1} = \frac{k_1}{k_2} = \frac{2\pi/\lambda_1}{2\pi/\lambda_2} = \frac{\lambda_2}{\lambda_1} \tag{142}$$

With all the above waves added together, the net disturbance on the string is given as

$$\tilde{f}(z,t) = \begin{cases} \tilde{A}_I e^{i(k_1 z - \omega t)} + \tilde{A}_R e^{i(-k_1 z - \omega t)} & z < 0 \\ \tilde{A}_T e^{i(k_2 z - \omega t)} & z > 0 \end{cases} \tag{143}$$

with the boundary condition at $z = 0$

$$\tilde{f}(0^-, t) = \tilde{f}(0^+, t) \qquad \frac{\partial \tilde{f}}{\partial z}\bigg|_{0^-} = \frac{\partial \tilde{f}}{\partial z}\bigg|_{0^+} \tag{144}$$

since the knot at $z = 0$ ensures that the string must be continuous and, with negligible masses, the first derivative must also be continuous. Taking the derivative of $\tilde{T}$ with respect to z:

$$\frac{\partial \tilde{f}}{\partial z}(z,t) = \begin{cases} k_1 \tilde{A}_I e^{i(k_1 z - \omega t)} + k_1 \tilde{A}_R e^{i(-k_1 z - \omega t)} & z < 0 \\ k_2 \tilde{A}_T e^{i(k_2 z - \omega t)} & z > 0 \end{cases} \tag{145}$$

Plugging in $z = 0$ for the boundary conditions

$$\tilde{A}_I + \tilde{A}_R = \tilde{A}_T \tag{146}$$
$$k_1 \tilde{A}_I - k_1 \tilde{A}_R = k_2 \tilde{A}_T \tag{147}$$

Plugging Equation (146) in terms of $\tilde{A}_T$ into Equation (147):

$$k_1 \tilde{A}_I - k_1 \tilde{A}_R = k_2 \tilde{A}_I + k_2 \tilde{A}_R \tag{148}$$
$$(k_1 - k_2)\tilde{A}_I = (k_1 + k_2)\tilde{A}_R \tag{149}$$
$$\tilde{A}_R = \frac{k_1 - k_2}{k_1 + k_2}\tilde{A}_I = \frac{k_1 - \frac{v_1}{v_2}k_1}{k_1 + \frac{v_1}{v_2}k_1}\tilde{A}_I = \frac{v_2 - v_1}{v_1 + v_2}\tilde{A}_I \tag{150}$$

where the last step is invoking Equation (142). Deriving a similar equation for $\tilde{A}_T$, we get:

$$\tilde{A}_T = \frac{2k_1}{k1 + k2}\tilde{A}_I = \frac{2v_2}{v_1 + v_2}\tilde{A}_I \tag{151}$$

The transmitted wave will always be in phase with the incident wave because v_1 and v_2 are only dependent on the properties of the strings and the relationship $v = \sqrt{T/\mu}$. By the same token, if the second string is lighter, $\mu_2 < \mu_1$,$v_2 > v_1$ and the reflected wave will be in phase with the incident wave; if the second string is heavier, $\mu_2 > \mu_1$,$v_2 < v_1$, and a negative sign is introduced in Equation (150). The reflected will therefore be 180 degrees phase shifted from the incident.

5.2 Electromagnetic Waves in Vacuum

One way to simplify the cross couple the equations is to situate ourselves in a vacuum, which is absent of any charge or current. The Maxwell equations then become the following:

$$\begin{aligned} &Gauss's\ Law : \nabla \cdot \mathbf{E} = 0 && Gauss's\ Law\ for\ Magnetism : \nabla \cdot \mathbf{B} = 0 \\ &Faraday's\ Law : \nabla \times \mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t} && Ampere's\ Law : \nabla \times \mathbf{B} = \mu_0 \epsilon_0 \frac{\partial \mathbf{E}}{\partial t} \end{aligned} \tag{152}$$

This simplification lets us see the interesting relationship between magnetic and electric fields. A magnetic field that changes over time results in an electric field that curls around it while an electric field that changes over time would result in a magnetic field curling around it. By the way triple cross product works, we also have the relationship

$$\nabla \times (\nabla \times \mathbf{A}) = \nabla(\nabla \cdot \mathbf{A}) - \nabla^2 \mathbf{A} \tag{153}$$

Applying curl to both sides of Faraday's Law,

$$\nabla \times (\nabla \times \mathbf{E}) = -\frac{\partial}{\partial t}(\nabla \times \mathbf{B}) \tag{154}$$

$$\nabla(\nabla \cdot \mathbf{E}) - \nabla^2 \mathbf{E} = -\frac{\partial}{\partial t}(\mu_0 \epsilon_0 \frac{\partial \mathbf{E}}{\partial t}) \tag{155}$$

$$\nabla^2 \mathbf{E} = \mu_0 \epsilon_0 \frac{\partial^2 \mathbf{E}}{\partial t^2} \tag{156}$$

Applying curl to both sides of Ampere's Law,

$$\nabla \times (\nabla \times \mathbf{B}) = \mu_0 \epsilon_0 \frac{\partial}{\partial t}(\nabla \times \mathbf{E}) \tag{157}$$

$$\nabla(\nabla \cdot \mathbf{B}) - \nabla^2 \mathbf{B} = -\mu_0 \epsilon_0 \frac{\partial}{\partial t}(\frac{\partial \mathbf{B}}{\partial t}) \tag{158}$$

$$\nabla^2 \mathbf{B} = \mu_0 \epsilon_0 \frac{\partial^2 \mathbf{B}}{\partial t^2} \tag{159}$$

What we ultimately receives is the wave equations which describes the behavior of electric and magnetic field moving through space.

$$\nabla^2 \mathbf{E} = \mu_0 \epsilon_0 \frac{\partial^2 \mathbf{E}}{\partial t^2} \qquad \nabla^2 \mathbf{B} = \mu_0 \epsilon_0 \frac{\partial^2 \mathbf{B}}{\partial t^2} \tag{160}$$

We will assume that we are far away enough from the source of the electromagnetic waves that the waves are perfect plane waves without any complications from non-Cartesian coordinate systems. The solution to the electric wave equation would have the form:

$$\tilde{\mathbf{E}}(z,t) = \tilde{\mathbf{E}}_0 e^{i(kz-\omega t)} \tag{161}$$

kz can be further generalized by the vector dot product $\mathbf{k} \cdot r$, where $\mathbf{k}$ is the **propagation/wave vector** pointing in the direction of travel. Since different frequencies corresponds to different colors in the visible light spectrum, when we focus on a particular frequency ω, we call the wave **monochromatic**. Assuming a solution for the wave equation for electric field take the real part of the generalized solution:

$$\mathbf{E}(\mathbf{r},t) = \mathbf{E}_0 \cos(\mathbf{k} \cdot \mathbf{r} - \omega t) \tag{162}$$

To satisfy Gauss's Law in vacuum,

$$\nabla \cdot \mathbf{E}(\mathbf{r}, t) = -\mathbf{k} \cdot \mathbf{E}_0 \sin(\mathbf{k} \cdot \mathbf{r} - \omega t) = 0 \tag{163}$$

we would need $\mathbf{k} \cdot \mathbf{E}_0$ to be zero, which suggests that the direction in which the electric field oscillate is perpendicular to the direction of wave propagation. To find the corresponding magnetic field, we employ Faraday's Law where we first take the curl of the electric field and then integrate over time to get magnetic field.

$$\nabla \times (\mathbf{E}_0 \cos(\mathbf{k} \cdot \mathbf{r} - \omega t)) = (\mathbf{k} \times \mathbf{E}_0) \sin(\mathbf{k} \cdot \mathbf{r} - \omega t) = -\frac{\partial \mathbf{B}}{\partial t} \tag{164}$$

$$\mathbf{B} = -\int (\mathbf{k} \times \mathbf{E}_0) \sin(\mathbf{k} \cdot \mathbf{r} - \omega t)\, dt \tag{165}$$

$$\mathbf{B} = \frac{\mathbf{k} \times \mathbf{E}_0}{\omega} \cos(\mathbf{k} \cdot \mathbf{r} - \omega t) = \frac{\mathbf{k} \times \mathbf{E}}{\omega} \tag{166}$$

The cross product also suggests that the direction in which magnetic field oscillates in is perpendicular to both the direction of propagation and the direction in which electric field oscillates in. The ratio between E and B is given as

$$\frac{E}{B} = \frac{E}{\frac{kE}{\omega}} = \frac{\omega}{k} = \frac{1}{v} \tag{167}$$

In the case of a monochromatic plane wave in vacuum, we have

$$B^2 = \frac{1}{c^2} E^2 = \epsilon_0 \mu_0 E^2 \tag{168}$$

Combined with Equation (112), we obtained that the magnetic wave has the same amount of energy as the electric wave, and hence

$$u = \epsilon_0 E^2 = \epsilon_0 E_0^2 \cos^2(kz - \omega t + \delta) \tag{169}$$

The energy flux density (energy per unit area, per unit time) transported by the fields traveling in $\hat{\mathbf{z}}$ direction is given by the Poyting vector in Equation (122):

$$\mathbf{S} = \frac{1}{\mu_0} \mathbf{E} \times \mathbf{B} \tag{170}$$

$$= \frac{1}{c\mu_0} E_0^2 \cos^2(kz - \omega t + \delta) \hat{\mathbf{z}} \tag{171}$$

$$= \frac{1}{\mu_0} \sqrt{\frac{\mu_0}{\epsilon_0}} \epsilon_0 E_0^2 \cos^2(kz - \omega t + \delta) \hat{\mathbf{z}} \tag{172}$$

$$= c\epsilon_0 E_0^2 \cos^2(kz - \omega t + \delta) \hat{\mathbf{z}} = cu\hat{\mathbf{z}} \tag{173}$$

The average power per unit area transported by an electromagnetic wave is the **intensity**:

$$\mathbf{I} = \langle \mathbf{S} \rangle = \frac{1}{2} c\epsilon_0 E_0^2 \hat{\mathbf{z}} \tag{174}$$

5.3 Electromagnetic Waves in Matter

5.3.1 Propagation

Inside matters where there is no free charge or free current, the Maxwell's Equation becomes:

$$(i) : \nabla \cdot \mathbf{D} = 0 \qquad (ii) : \nabla \cdot \mathbf{B} = 0$$
$$(iii) : \nabla \times \mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t} \qquad (iv) : \nabla \times \mathbf{H} = \frac{\partial \mathbf{D}}{\partial t}$$

Assume a linear, homogeneous medium

$$\mathbf{D} = \epsilon \mathbf{E}, \quad \mathbf{H} = \frac{1}{\mu}\mathbf{B} \tag{175}$$

which differs from the wave equation in vacuum Equation (152) in the replacement of $\mu_0\epsilon_0$ by $\mu\epsilon$.

$$(i) : \nabla \cdot \mathbf{E} = 0 \qquad (ii) : \nabla \cdot \mathbf{B} = 0 \tag{176}$$
$$(iii) : \nabla \times \mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t} \qquad (iv) : \nabla \times \mathbf{B} = \mu\epsilon\frac{\partial \mathbf{E}}{\partial t}$$

The wave speed in this linear homogeneous medium at a speed

$$v = \frac{1}{\sqrt{\mu\epsilon}} = \sqrt{\frac{1/(\mu_0\epsilon_0)}{\mu\epsilon/(\mu_0\epsilon_0)}} = \frac{c}{n} \tag{177}$$

The **index of refraction** of the medium, n, is

$$n \equiv \sqrt{\frac{\mu\epsilon}{\mu_0\epsilon_0}} \tag{178}$$

For most materials, μ is very close to μ_0, therefore

$$n \approx \sqrt{\epsilon_r} \tag{179}$$

From Section 1.5.6, we know that ϵ_r is almost always greater than 1, which suggest light travels more slowly through matter than through vacuum. Fortunately, all the results we derived for electromagnetic waves in vacuum applies as well with simple symbol swapping.

$$u = \frac{1}{2}\Big(\epsilon E^2 + \frac{1}{\mu}B^2\Big) \tag{180}$$

$$\mathbf{S} = \frac{1}{\mu}(\mathbf{E} \times \mathbf{B}) \tag{181}$$

$$I = \frac{1}{2}\epsilon\mu E_0^2 \tag{182}$$

The boundary conditions at the intersection of two media is given by

$$(i)\epsilon_1 E_1^\perp = \epsilon_2 E_2^\perp \qquad (iii) E_1^\parallel = E_2^\parallel$$
$$(ii) B_1^\perp = B_2^\perp \qquad (iv) \frac{1}{\mu_1}B_1^\parallel = \frac{1}{\mu_2}B_2^\parallel \tag{183}$$

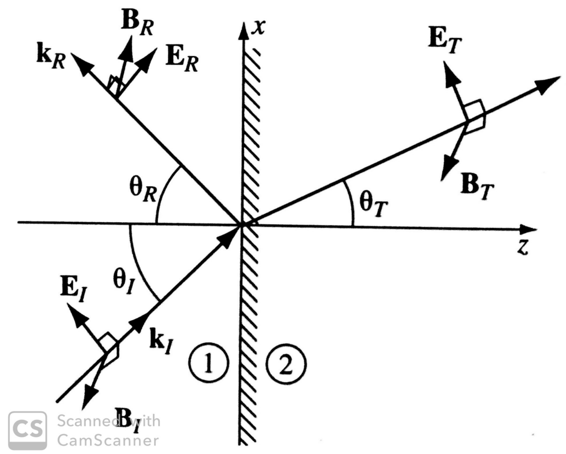

Figure 4: An incidence wave gives rise to a reflected wave and a transmitted wave at the interface of two media.

5.3.2 Reflection and Transmission

Suppose that a monochromatic plane wave polarized in the $x - z$ plane

$$\tilde{\mathbf{E}}_I(\mathbf{r},t) = \tilde{\mathbf{E}}_{0_I} e^{i(\mathbf{k}_I\cdot\mathbf{r}-\omega t)}, \quad \tilde{\mathbf{B}}_I = \frac{1}{v_1}(\hat{\mathbf{k}}_I \times \tilde{\mathbf{E}}_I) \tag{184}$$

approaches from the left and giving rise to a reflected wave,

$$\tilde{\mathbf{E}}_R(\mathbf{r},t) = \tilde{\mathbf{E}}_{0_R} e^{i(\mathbf{k}_R\cdot\mathbf{r}-\omega t)}, \quad \tilde{\mathbf{B}}_R = \frac{1}{v_1}(\hat{\mathbf{k}}_R \times \tilde{\mathbf{E}}_R) \tag{185}$$

and a transmitted wave

$$\tilde{\mathbf{E}}_T(\mathbf{r},t) = \tilde{\mathbf{E}}_{0_T} e^{i(\mathbf{k}_T\cdot\mathbf{r}-\omega t)}, \quad \tilde{\mathbf{B}}_T = \frac{1}{v_2}(\hat{\mathbf{k}}_T \times \tilde{\mathbf{E}}_T) \tag{186}$$

Since the frequency is preserved for all three waves, we have the relationship

$$k_I v_1 = k_R v_1 = k_T v_2 = \omega \tag{187}$$

or, alternatively,

$$k_I = k_R = \frac{v_2}{v_1} k_T = \frac{\sqrt{\mu_1\epsilon_1}}{\sqrt{\mu_2\epsilon_2}} k_T = \frac{\sqrt{\frac{\mu_1\epsilon_1}{\mu_0\epsilon_0}}}{\sqrt{\frac{\mu_2\epsilon_2}{\mu_0\epsilon_0}}} k_T = \frac{n_1}{n_2} k_T$$

(188)

The boundary conditions (Equation (183)) would result in equations of the following form

$$(\ )e^{i(\mathbf{k}_I\cdot\mathbf{r}-\omega t)} + (\ )e^{i(\mathbf{k}_R\cdot\mathbf{r}-\omega t)} = (\ )e^{i(\mathbf{k}_T\cdot\mathbf{r}-\omega t)} \quad when\ z = 0 \tag{189}$$

Regardless of the coefficient, all the exponents must be equal for the boundary condition to hold for the whole $z = 0$ plane.

$$\mathbf{k}_I \cdot \mathbf{r} - \omega t = \mathbf{k}_R \cdot \mathbf{r} - \omega t = \mathbf{k}_T \cdot \mathbf{r} - \omega t \tag{190}$$

This reaffirms that all three waves have the same frequency.

$$\mathbf{k}_I \cdot \mathbf{r} = \mathbf{k}_R \cdot \mathbf{r} = \mathbf{k}_T \cdot \mathbf{r} \tag{191}$$

$$x(\mathbf{k}_I)_x + y(\mathbf{k}_I)_y = x(\mathbf{k}_R)_x + y(\mathbf{k}_R)_y = x(\mathbf{k}_T)_x + y(\mathbf{k}_T)_y \tag{192}$$

Then, all the x and y terms must be equal separately.

$$(\mathbf{k}_I)_y = (\mathbf{k}_R)_y = (\mathbf{k}_T)_y \tag{193}$$

$$(\mathbf{k}_I)_x = (\mathbf{k}_R)_x = (\mathbf{k}_T)_x \tag{194}$$

Since we orient our electromagnetic wave only in the $x-z$ plane, $(\mathbf{k}_I)_y = 0$, so will $(\mathbf{k}_R)_y$ and $(\mathbf{k}_T)_y$ according to Equation (193), which leads us to the **First Law** of geometrical optics: The incident, reflected, and transmitted wave vectors form a plane (called the plane of incidence), which also includes the normal of the surface.

In the meantime, Equation (194) suggests that

$$k_I \sin\theta_I = k_R \sin\theta_R = k_I \sin\theta_I \tag{195}$$

Since we know that $k_I = k_R$ from Equation (5.3.2), we derive the **Second Law** of geometrical optics, or the **law of reflection**: the angle of incidence is equal to the angle of reflection.

$$\theta_I = \theta_R \tag{196}$$

Again, Equation (5.3.2) tells us $k_I = \frac{n_1}{n_2} k_T$. We thus derive the **Third Law**, or the **Law of Refraction/Snell's Law**:

$$\frac{\sin\theta_T}{\sin\theta_I} = \frac{n_1}{n_2} \tag{197}$$

The boundary conditions Equation (183) gives us

$$\epsilon_1(-\tilde{E}_{0I}\sin\theta_I + \tilde{E}_{0R}\sin\theta_R) = \epsilon_2(-\tilde{E}_{0T}\sin\theta_T) \tag{198}$$

$$\tilde{E}_{0I}\sin\theta_I + \tilde{E}_{0R}\cos\theta_R) = \tilde{E}_{0T}\sin\theta_T) \tag{199}$$

$$\frac{1}{\mu_1 v_1}(\tilde{E}_{0I} - \tilde{E}_{0R}) = \frac{1}{\mu_2 v_2}\tilde{E}_{0T} \tag{200}$$

which can be simplified to the following two equations

$$\tilde{E}_{0_I} - \tilde{E}_{0_R} \quad = \beta \tilde{E}_{0_T} \tag{201}$$

$$\tilde{E}_{0_I} + \tilde{E}_{0_R} \quad = \alpha \tilde{E}_{0_T} \tag{202}$$

where

$$\beta = \frac{\mu_1 v_1}{\mu_2 v_2} = \frac{\mu_1 n_2}{\mu_2 n_1} \qquad\qquad \alpha = \frac{\cos\theta_T}{\cos\theta_I} \tag{203}$$

Solving for $\tilde{E}_{0_R}$ and $\tilde{E}_{0_T}$ in terms of $\tilde{E}_{0_I}$, α and β

$$\tilde{E}_{0_R} = \frac{\alpha - \beta}{\alpha + \beta}\tilde{E}_{0_I}, \quad \tilde{E}_{0_T} = \frac{2}{\alpha + \beta}\tilde{E}_{0_I} \tag{204}$$

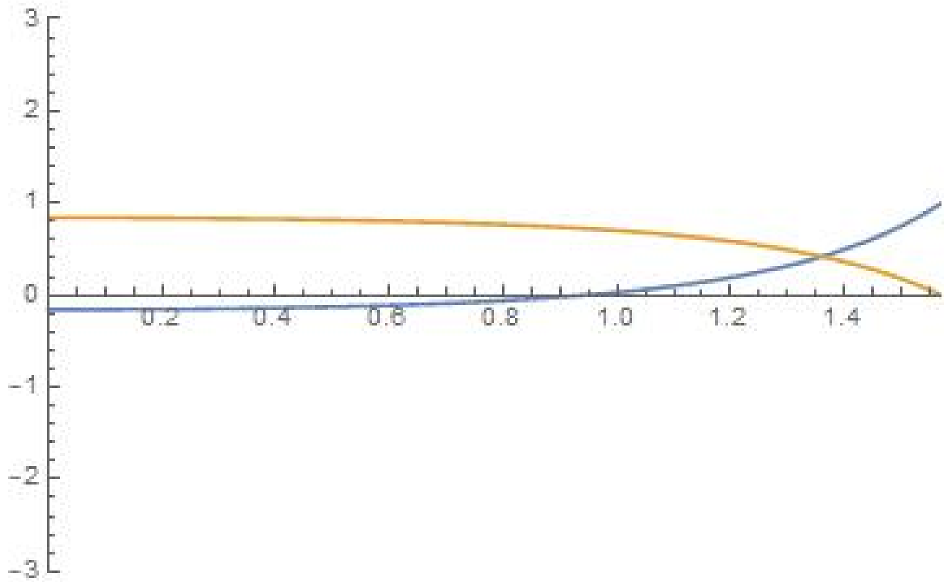

Figure 5: Amount of reflection and refraction that happens when the second medium has a larger index of refraction than the first medium. The blue indicates the reflected wave and the orange is the refracted wave.

5.4 Absorption and Dispersion

5.4.1 Electromagnetic Waves in Conductors

According to Ohm's law in Equation 79, the free current density in a conductor in proportional to the electric field is

$$\mathbf{J}_f = \sigma \mathbf{E} \tag{205}$$

Therefore, the Maxwell's equations for conductors at equilibrium (all accumulated free charge disappears) are

$$\begin{aligned} &(i) : \nabla \cdot \mathbf{E} = 0 && (ii) : \nabla \cdot \mathbf{B} = 0 \\ &(iii) : \nabla \times \mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t} && (iv) : \nabla \times \mathbf{B} = \mu\epsilon \frac{\partial \mathbf{E}}{\partial t} + \mu\sigma \mathbf{E} \end{aligned} \tag{206}$$

Going through a similar derivation from Equation (154) - (159), we would obtain the following wave equations for $\mathbf{E}$ and $\mathbf{B}$.

$$\nabla^2 \mathbf{E} = \mu\epsilon \frac{\partial^2 \mathbf{E}}{\partial t^2} + \mu\sigma \frac{\partial \mathbf{E}}{\partial t} \qquad \nabla^2 \mathbf{B} = \mu\epsilon \frac{\partial^2 \mathbf{B}}{\partial t^2} + \mu\sigma \frac{\partial \mathbf{B}}{\partial t} \tag{207}$$

While the plane wave solutions like the following are still solutions, plugging them into the wave equation results in a compplex wave number.

$$\tilde{\mathbf{E}}(z,t) = \tilde{\mathbf{E}}_0 e^{i(\tilde{k}z - \omega t)} \tag{208}$$

$$\tilde{\mathbf{B}}(z,t) = \tilde{\mathbf{B}}_0 e^{i(\tilde{k}z - \omega t)} \tag{209}$$

$$\tilde{k}^2 = \mu\epsilon\omega^2 + i\mu\sigma\omega \tag{210}$$

Taking the square root of $\tilde{k}$

$$\tilde{k} = k + i\kappa \tag{211}$$

where

$$k \equiv \omega\sqrt{\frac{\epsilon\mu}{2}}\Big[\sqrt{1+(\frac{\sigma}{\epsilon\omega})}+1\Big]^{\frac{1}{2}} \qquad \kappa \equiv \omega\sqrt{\frac{\epsilon\mu}{2}}\Big[\sqrt{1+(\frac{\sigma}{\epsilon\omega})}-1\Big]^{\frac{1}{2}} \tag{212}$$

The imaginary part of $\tilde{k}$ results in attenuation of the waves

$$\tilde{\mathbf{E}}(z,t) = \tilde{\mathbf{E}}_0 e^{-\kappa z} e^{i(kz-\omega t)} \qquad \tilde{\mathbf{B}}(z,t) = \tilde{\mathbf{B}}_0 e^{-\kappa z} e^{i(kz-\omega t)} \tag{213}$$

The **skin depth**, d, to reduce the amplitude by $1/e$ is

$$d \equiv \frac{1}{\kappa} \tag{214}$$

Maxwell's Equation (iii), at the same time, enforces the following relationship between $\tilde{\mathbf{E}}$ and $\tilde{\mathbf{B}}$.

$$\mathbf{B}(z,t) = \frac{\tilde{k}}{\omega}\tilde{E}_0 e^{-\kappa z} e^{i(kz-\omega t)} \tag{215}$$

where

$$\tilde{k} = Ke^{i\phi} \tag{216}$$

$$K = \sqrt{k^2+\kappa^2} = \omega\sqrt{\epsilon\mu\sqrt{1+(\frac{\sigma}{\epsilon\omega})^2}}\phi = \tan^{-1}\frac{\kappa}{k} \tag{217}$$

According to Equation (213) and (215), the complex amplitude are related then by

$$B_0 e^{i\delta_B} = \frac{Ke^{i\phi}}{\omega}E_0 e^{i\delta_E} \tag{218}$$

$$\delta_B - \delta_E = \phi \tag{219}$$

In conclusion, the magnetic field lags behind the electric field by ϕ.

$$\mathbf{E}(z,t) = E_0 e^{-\kappa z}\cos(kz-\omega t+\delta_E)\hat{\mathbf{x}} \tag{220}$$

$$\mathbf{B}(z,t) = B_0 e^{-\kappa z}\cos(kz-\omega t+\delta_B+\phi)\hat{\mathbf{y}} \tag{221}$$

5.4.2 Reflection at a Conducting Surface

In the presence of free charges and currents, we have more general relations at the boundary

$$\begin{aligned} &(i)\epsilon_1 E_1^{\perp} - \epsilon_2 E_2^{\perp} = \sigma_f && (iii)\mathbf{E}_1^{\parallel} - \mathbf{E}_2^{\parallel} = \mathbf{0} \\ &(ii)B_1^{\perp} - B_2^{\perp} = 0 && (iv)\frac{1}{\mu_1}\mathbf{B}_1^{\parallel} - \frac{1}{\mu_2}\mathbf{B}_2^{\parallel} = \mathbf{K}_f \times \hat{\mathbf{n}} \end{aligned} \tag{222}$$

where σ_f is the free surface charge, $\mathbf{K}_f$ is the free surface current, and $\hat{\mathbf{n}}$ is the normal vector at the interface, pointing from Medium 2 to Meidum 1. For ohmic conductors, there can be no free surface current since it would require an infinite electric field at the boundary.

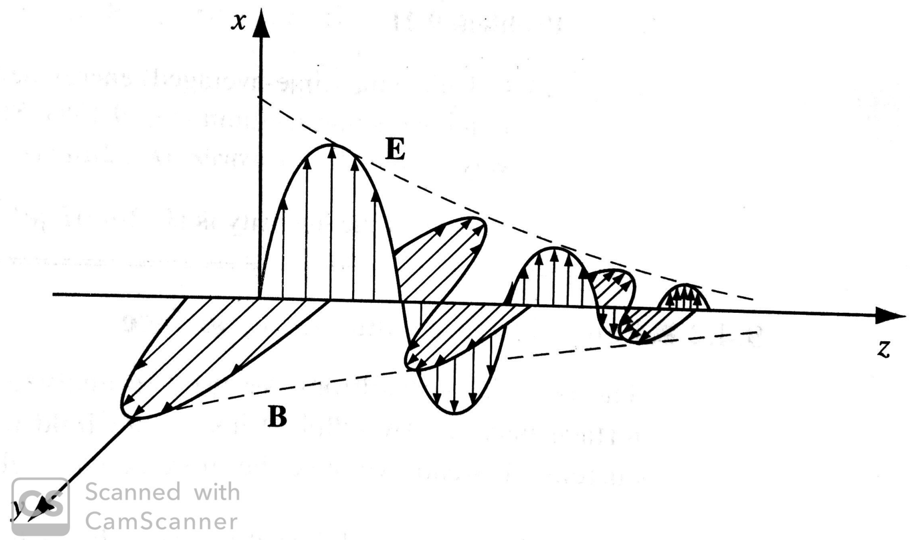

Figure 6: Magnetic wave lags behind in a conductor.

Let the xy plane be the interface between a nonconducting linear Medium 1 and a conductor 2. Suppose that a monochromatic plane wave polarized in the $x-z$ plane

$$\tilde{\mathbf{E}}_I(z,t) = \tilde{E}_{0_I} e^{i(k_1 z - \omega t)}\hat{\mathbf{x}}, \quad \tilde{\mathbf{B}}_I(z,t) = \frac{1}{v_1}\tilde{E}_{0_I} e^{i(k_1 z - \omega t)}\hat{\mathbf{y}} \tag{223}$$

approaches from the left and giving rise to a reflected wave,

$$\tilde{\mathbf{E}}_R(z,t) = \tilde{E}_{0_I} e^{i(-k_1 z - \omega t)}\hat{\mathbf{x}}, \quad \tilde{\mathbf{B}}_R(z,t) = -\frac{1}{v_1}\tilde{E}_{0_I} e^{i(-k_1 z - \omega t)}\hat{\mathbf{y}} \tag{224}$$

and a transmitted wave

$$\tilde{\mathbf{E}}_T(z,t) = \tilde{E}_{0_T} e^{i(-\tilde{k}_2 z - \omega t)}\hat{\mathbf{x}}, \quad \tilde{\mathbf{B}}_T(z,t) = -\frac{\tilde{k}_2}{\omega}\tilde{E}_{0_T} e^{i(-\tilde{k}_2 z - \omega t)}\hat{\mathbf{y}} \tag{225}$$

which is attenuated as it propagates in the conductor. Boundary conditions (iii) and (iv) give the following two equations

$$\tilde{E}_{0_I} + \tilde{E}_{0_R} = \tilde{E}_{0_T} \tag{226}$$

$$\frac{1}{\mu_0 v_1}(\tilde{E}_{0_I} - \tilde{E}_{0_R}) - \frac{\tilde{k}_2}{\mu_2 \omega}\tilde{E}_{0_T} = 0 \tag{227}$$

Solving for $\tilde{E}_{0_R}$ and $\tilde{E}_{0_T}$ in terms of $\tilde{E}_{0_I}$ gives us

$$\tilde{E}_{0_R} = \left(\frac{1-\tilde{\beta}}{1+\tilde{\beta}}\right)\tilde{E}_{0_I}, \quad \tilde{E}_{0_T} = \left(\frac{2}{1+\tilde{\beta}}\right)\tilde{E}_{0_I} \tag{228}$$

where $\tilde{\beta}$ is given as

$$\tilde{\beta} = \frac{\mu_1 v_1}{\mu_2 \omega}\tilde{k}_2 \tag{229}$$

For a perfect conductor with conductivity $\sigma = \infty$, $\tilde{k}_2$ also goes to infinity, Equation (228) becomes

$$\tilde{E}_{0_R} = -\tilde{E}_{0_I}, \quad \tilde{E}_{0_T} = 0 \tag{230}$$

The wave is totally reflected, with a 180° degree phase shift, which is why excellent conductors like silver make good mirrors.

5.4.3 The Frequency Dependence of Permittivity

In this section, we will use a classical model of electrons and develop an *approximation* to the true dispersion in transparent media.

The electrons in a nonconductor are bound to specific molecules, and we will picture each electron as attached to the end of a spring, with force constant k_{spring}:

$$F_{binding} = -k_{spring}x = -m\omega_0^2 x \tag{231}$$

where $\omega_0 = \sqrt{k_{spring}/m}$ is the natural oscillation frequency. Meanwhile, some sort of damping force on the electron can be modeled as

$$F_{damping} = -m\gamma\frac{dx}{dt} \tag{232}$$

In the presence of an electromagnetic wave of frequency ω, polarized in the x direction, the electron would then be subjected to a driving force.

$$F_{driving} = qE = qE_0\cos\omega t \tag{233}$$

According to Newton's second law,

$$m\frac{d^2x}{dt^2} = \sum F = F_{spring} + F_{damping} + F_{driving} \tag{234}$$

$$m\frac{d^2x}{dt^2} + m\gamma\frac{dx}{dt} + m\omega_0^2 x = qE_0\cos\omega t \tag{235}$$

To simplify the algebra by making the equation complex

$$\frac{d^2\tilde{x}}{dt^2} + \gamma\frac{dx}{dt} + \omega_0^2 x = \frac{q}{m}E_0 e^{-\omega t} \tag{236}$$

which suggests that in the steady state, the system would oscillate at the driving frequency

$$\tilde{x}(t) = \tilde{x}_0 e^{-i\omega t} \tag{237}$$

Plugging it into Equation (236), we have

$$\tilde{x}_0 = \frac{q/m}{\omega_0^2 - \omega^2 - i\gamma\omega}E_0 \tag{238}$$

with a resulting dipole moment as the real part of

$$\tilde{p}(t) = q\tilde{x}(t) = \frac{q^2/m}{\omega_0^2 - \omega^2 - i\gamma\omega}E_0 e^{-i\omega t} \tag{239}$$

Let's say there are f_j electrons with natural frequency ω_j and damping γ_j in each molecule. If there are N molecules per unit volume, the polarization per $\mathbf{P}$ is the real part of

$$\tilde{\mathbf{P}} = \frac{Nq^2}{m} \sum_j \frac{f_j}{\omega_j^2 - \omega^2 - i\gamma\omega} \tilde{\mathbf{E}} \tag{240}$$

We then relate $\tilde{\mathbf{P}}$ with $\tilde{\mathbf{E}}$ through a **complex susceptibility**, $\tilde{\chi}_e$:

$$\tilde{\mathbf{P}} = \epsilon_0 \tilde{\chi}_e \tilde{\mathbf{E}} \tag{241}$$

It leads us to conclude that the complex permittivity $\tilde{\epsilon} = \epsilon_0(1 + \tilde{\chi}_e)$.

$$\tilde{\epsilon}_r = \frac{\tilde{\epsilon}}{\epsilon_0} = 1 + \frac{Nq^2}{m\epsilon_0} \sum_j \frac{f_j}{\omega_j^2 - \omega^2 - i\gamma\omega} \tag{242}$$

in a dispersive medium whose wave equation reads:

$$\nabla^2 \tilde{\mathbf{E}} = \tilde{\epsilon}\mu_0 \frac{\partial^2 \tilde{\mathbf{E}}}{\partial t^2} \tag{243}$$

It still admits a plane wave solution:

$$\tilde{\mathbf{E}}(z,t) = \tilde{E}_0 e^{i(\tilde{k}z - \omega t)} \tag{244}$$

$$\tilde{k} \equiv \sqrt{\tilde{\epsilon}\mu_0}\omega = \sqrt{\tilde{\epsilon}_r \epsilon_0 \mu_0}\omega = \frac{\omega}{c}\sqrt{\tilde{\epsilon}_r} \tag{245}$$

Writing $\tilde{k}$ in terms of its real and imaginary parts,

$$\tilde{k} = k + i\kappa \tag{246}$$

we have

$$\tilde{\mathbf{E}}(z,t) = \tilde{E}_0 e^{-\kappa} e^{i(kz - \omega t)} \tag{247}$$

The wave is thus attenuated. Because the intensity is proportional to E^2, the quantity

$$\alpha \equiv 2\kappa \tag{248}$$

is called the **absorption coefficient**. At the same time, the wave velocity is ω/k, giving us the index of refraction

$$n = \frac{ck}{\omega} \tag{249}$$

For gas, the summation in Equation (242) is small so we can approximate the square root with a binomial expansion $\sqrt{1 + \mathcal{E}} \approx 1 + \frac{1}{2}\mathcal{E}$

$$\tilde{k} = \frac{\omega}{c}\sqrt{\tilde{\epsilon}_r} \approx \frac{\omega}{c}\Big[1 + \frac{Nq^2}{2m\epsilon_0} \sum_j \frac{f_j}{\omega_j^2 - \omega^2 - i\gamma_j\omega}\Big] \tag{250}$$

As a result,

$$n = \frac{ck}{\omega} \approx 1 + \frac{Nq^2}{2m\epsilon_0} \sum_j \frac{f_j(\omega_j^2 - \omega^2)}{(\omega_j^2 - \omega^2)^2 + (\gamma_j\omega)^2} \tag{251}$$

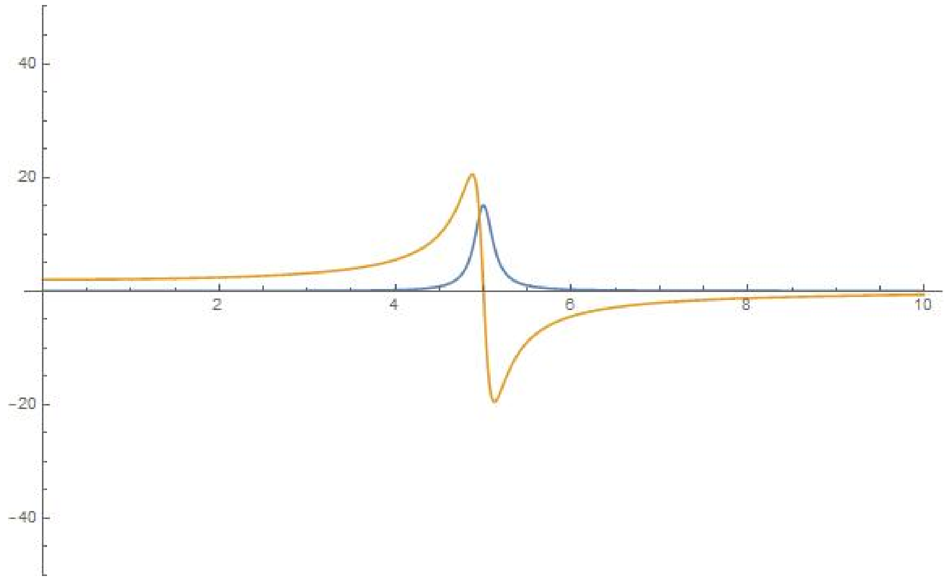

Figure 7: Anomalous Dispersion. The blue line is $\alpha/100$ and the orange line is $n-1$. The peak of α occurs at $\omega = \omega_0$.

and

$$\alpha = 2\kappa \approx \frac{Nq^2\omega^2}{m\epsilon_0 c} \sum_j \frac{f_j \gamma_j}{(\omega_j^2 - \omega^2)^2 + (\gamma_j \omega)^2} \tag{252}$$

Plotting α and $n-1$ against the angular frequency ω shows an increasing index of refraction as the frequency grows higher, as excepted with our daily interactions with optics. Right around the immediate neighborhood of a resonance, however, the index of refraction drops drastically. This behavior is called **anomalous dispersion**. The region of anomalous dispersion also happen to coincides with the region of maximum absorption. The reason is that the electrons are driven at their favorite natural frequencies, so the amplitude is large, and a correspondingly large amount of energy is also dissipated by the damping mechanism.

5.5 Wave Guides

Let's now consider electromagnetic waves confined to the interior of a hollow pipe, or **wave guide**. We assume the wave guide is a perfect conductor, so $\mathbf{E} = \mathbf{0}$ and $\mathbf{B} = \mathbf{0}$ inside the material itself, and hence the boundary conditions at the inner wall are

$$(i)\mathbf{E}^{\parallel} = \mathbf{0} \qquad (ii)B^{\perp} = 0 \tag{253}$$

Free charges and currents will be induced on the surface to enforce these boundary conditions. The electromangetic waves will have the generic form

$$\tilde{\mathbf{E}}(x, y, z, t) = \tilde{\mathbf{E}}_0(x, y)e^{i(kz-\omega t)} \tag{254}$$

$$\tilde{\mathbf{B}}(x, y, z, t) = \tilde{\mathbf{B}}_0(x, y)e^{i(kz-\omega t)} \tag{255}$$

As we will soon see, the confined waves are not usually transverse; in order to fit the boundary conditions, we will need to include the longitudinal components:

$$\tilde{\mathbf{E}}_0 = E_x\hat{\mathbf{x}} + E_y\hat{\mathbf{y}} + E_z\hat{\mathbf{z}}, \quad \tilde{\mathbf{B}}_0 = B_x\hat{\mathbf{x}} + B_y\hat{\mathbf{y}} + B_z\hat{\mathbf{z}} \tag{256}$$

Plugging them into Farady's Law and Ampere's Law in Equation (152), we obtain

$$(i)\frac{\partial E_y}{\partial x} - \frac{\partial E_x}{\partial y} = I\omega B_z, \qquad (iv)\frac{\partial B_y}{\partial x} - \frac{\partial B_x}{\partial y} = -\frac{i\omega}{c^2}E_z \tag{257}$$

$$(ii)\frac{\partial E_z}{\partial y} - ikE_y = I\omega B_x, \qquad (v)\frac{\partial B_z}{\partial x} - ikB_y = -\frac{i\omega}{c^2}E_x \tag{258}$$

$$(i)ikE_x - \frac{\partial E_z}{\partial x} = I\omega B_y, \qquad (iv)ikB_x - \frac{\partial B_z}{\partial x} = -\frac{i\omega}{c^2}E_y \tag{259}$$

Solving for E_x, E_y, B_x, and B_y:

$$(i)E_x = \frac{i}{(\omega/c)^2 - k^2}\Big(k\frac{\partial E_z}{\partial x} + \omega\frac{\partial B_z}{\partial y}\Big) \tag{260}$$

$$(ii)E_y = \frac{i}{(\omega/c)^2 - k^2}\Big(k\frac{\partial E_z}{\partial y} - \omega\frac{\partial B_z}{\partial x}\Big) \tag{261}$$

$$(iii)B_x = \frac{i}{(\omega/c)^2 - k^2}\Big(k\frac{\partial B_z}{\partial x} - \frac{\omega}{c^2}\frac{\partial E_z}{\partial y}\Big) \tag{262}$$

$$(iv)B_x = \frac{i}{(\omega/c)^2 - k^2}\Big(k\frac{\partial B_z}{\partial y} + \frac{\omega}{c^2}\frac{\partial E_z}{\partial x}\Big) \tag{263}$$

Plugging the last four equations into the remaining Maxwell equation we receive,

$$(i)\Big[\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + (\omega/c)^2 - k^2\Big]E_z = 0 \tag{264}$$

$$(ii)\Big[\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + (\omega/c)^2 - k^2\Big]B_z = 0 \tag{265}$$

If $E_z = 0$, we call these **TE** (transverse electric) **waves**; if $B_z = 0$, they are called **TM** (transverse magnetic) **waves**; if both $E_z = 0$ and $B_z = 0$, we call them **TEM** waves. It turns out that TEM waves cannot occur in a hollow wave guide.

If $E_z = 0$, Gauss's law states

$$\frac{\partial E_x}{\partial x} + \frac{\partial E_y}{y} = 0 \tag{266}$$

and if $B_z = 0$, Faraday's law states

$$\frac{\partial E_y}{\partial x} - \frac{\partial E_x}{\partial y} = 0 \tag{267}$$

Having zero divergence and zero curl, the vector $\tilde{\mathbf{E}}_0$ can therefore be written as the gradient of a scalar potential that satisfies Laplace's equation. However, the surface of the wave guide is equipotential, suggesting the potential is constant throughout, and hence there is no electric field or wave at all.

5.5.1 TE Waves in a Rectangular Wave Guide

Suppose we have a rectanular shaped wave guide, with height a and width b, and we are interested in the propagation of TE waves. We will solve the problem by separation of variable. Let

$$E_z(x,y) = 0 \quad B_z(x,y) = X(x)Y(y) \tag{268}$$

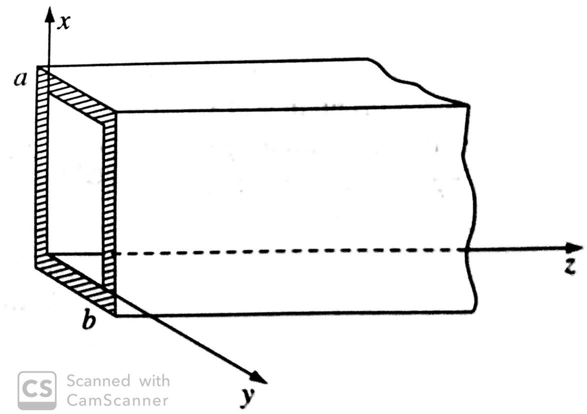

Figure 8: A rectangular TE wave guide.

so that

$$\left[\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\omega^2}{c^2} - k^2\right] B_z = 0 \tag{269}$$

$$YX'' + XY'' + (\frac{\omega^2}{c^2} - k^2)XY = 0 \tag{270}$$

$$\frac{X''}{X} + \frac{Y''}{Y} + \frac{\omega^2}{c^2} - k^2 = 0 \tag{271}$$

$$\tag{272}$$

from which we know that the X term and the Y term are each a constant. Let's set them to

$$\frac{X''}{X} = -k_x^2 \quad \frac{Y''}{Y} = -k_y^2 \tag{273}$$

where

$$-k_x^2 - k_y^2 + \frac{\omega^2}{c^2} - k^2 = 0 \tag{274}$$

The general solution for $X(x)$ is

$$X(x) = A \sin k_x x + B \cos k_x x \tag{275}$$

However, the boundary condition require that B_x vanishes at $x = 0$ and $x = a$. Equation (262) then tells us $\frac{\partial B_z}{\partial x}$ and, therefore, $\frac{dX}{dx}$ goes to zero at $x = 0$ and $x = a$.

$$X' = Ak_x \cos k_x x - Bk_x \sin k_x x \tag{276}$$

So, $A = 0$, and

$$k_x = \frac{m\pi}{a} \tag{277}$$

The same goes for $Y(y)$, with

$$k_y = \frac{n\pi}{b} \tag{278}$$

We conclude that

$$B_z = B_0 \cos(m\pi x/a) \cos(n\pi y/b) \tag{279}$$

which is called the TE_{mn} mode. The wave number k is solved back plugging the known back into Equation (274).

$$k = \sqrt{\frac{\omega^2}{c^2} - \pi(\frac{m^2}{a^2} + \frac{n^2}{b^2})} \tag{280}$$

If the wave number is imaginary, that is

$$\frac{\omega^2}{c^2} < \pi(\frac{m^2}{a^2} + \frac{n^2}{b^2}) \tag{281}$$

$$\omega^2 < c^2\pi^2(\frac{m^2}{a^2} + \frac{n^2}{b^2}) \tag{282}$$

$$\omega < c\pi\sqrt{\frac{m^2}{a^2} + \frac{n^2}{b^2}} \equiv \omega_{mn}, \tag{283}$$

instead of a traveling wave, we have exponentially attenuated fields in Equation (254). For this reason, ω_{mn} is called the **cutoff frequency** for the mode in question. The lowest cutoff frequency for a given wave guide occurs for the mode TE_{10}(assuming $a \geq b$):

$$\omega_{10} = c\pi/a \tag{284}$$

Frequency less than this will not propagate at all.

5.5.2 The Coaxial Transmission Line

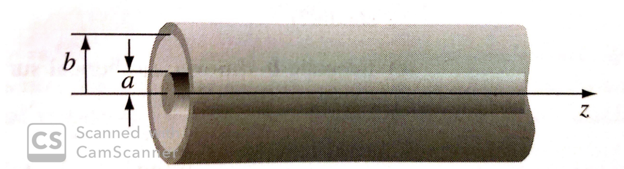

Figure 9: A coaxial transmission line.

While a hollow wave guide cannot support TEM waves, a coaxial transmission line, consisting of a long straight wire of radius a, surrounded by a cylindrical conducting sheath of radius b, does admit TEM waves. The Maxwell's Equations (257)-(259) becomes

$$k = \omega/c \tag{285}$$

$$cB_y = E_x \quad cB_x = -E_y \tag{286}$$

and

$$\frac{\partial E_x}{\partial x} + \frac{\partial E_y}{\partial y} = 0 \qquad \frac{\partial E_y}{\partial x} - \frac{\partial E_x}{\partial y} = 0 \tag{287}$$

$$\frac{\partial B_x}{\partial x} + \frac{\partial B_y}{\partial y} = 0 \qquad \frac{\partial B_y}{\partial x} - \frac{\partial B_x}{\partial y} = 0 \tag{288}$$

These are the equations of electrostatics and magnetostatics. We can borrow the solutions from our earlier exploitation of cylindrical symmetry.

$$\mathbf{E}(s, \phi, z, t) = \frac{A}{s}\cos kz - \omega t\hat{\mathbf{s}} \tag{289}$$

$$\mathbf{B}(s, \phi, z, t) = \frac{A}{cs}\cos kz - \omega t\hat{\phi} \tag{290}$$

References

[1] Griffiths, D. J. (1999). Introduction to electrodynamics. Upper Saddle River, N.J: Prentice Hall.