In[99]:=
```
$Assumptions = {n ∈ Integers}
```
Out[99]= $\{n \in \mathbb{Z}\}$

In[37]:=
```
boxcar[x_] := UnitStep[x] - UnitStep[x - 1];
Λ[x_] := UnitTriangle[x];
```

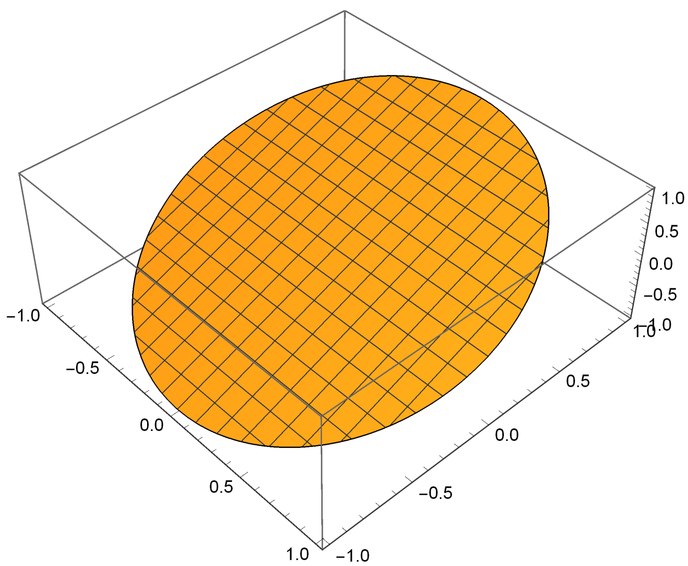
Metal Disc at Thermal Equilibrium

(1)

In[∘]:=
```
u21[r_, θ_] := r * Sin[θ]
```

In[∘]:=
```
Plot3D[u21[Norm[{x, y}], ArcTan[x, y]], {x, -1, 1}, {y, -1, 1}, PlotLegends → Automatic,
 Exclusions → None, RegionFunction → Function[{x, y}, x^2 + y^2 ≤ 1]]
```

Out[∘]=

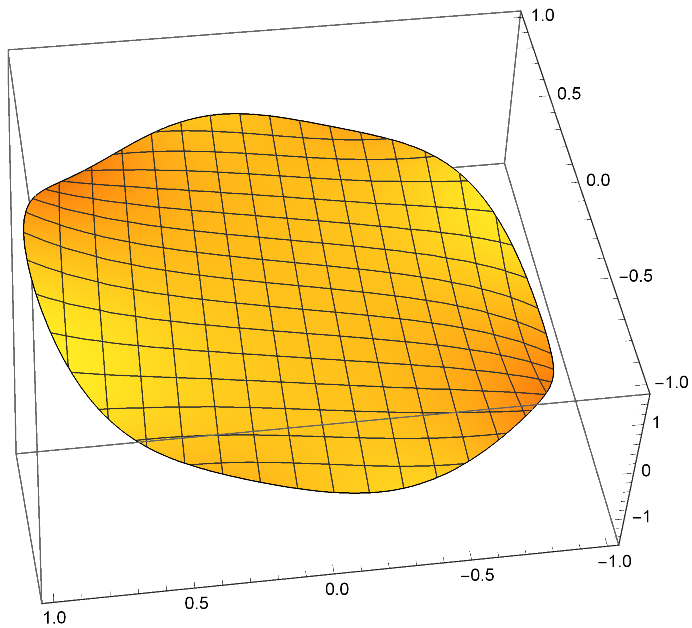


(2)

In[∘]:=
```
u22[r_, θ_] := r * Sin[θ] - 1/3 r^3 Sin[3 θ] + 1/5 r^5 Sin[5 θ]
```

In[ ]:=
```
Plot3D[u22[Norm[{x, y}], ArcTan[x, y]], {x, -1, 1}, {y, -1, 1}, PlotLegends → Automatic,
  Exclusions → None, RegionFunction → Function[{x, y}, x^2 + y^2 ≤ 1]]
```

Out[ ]=

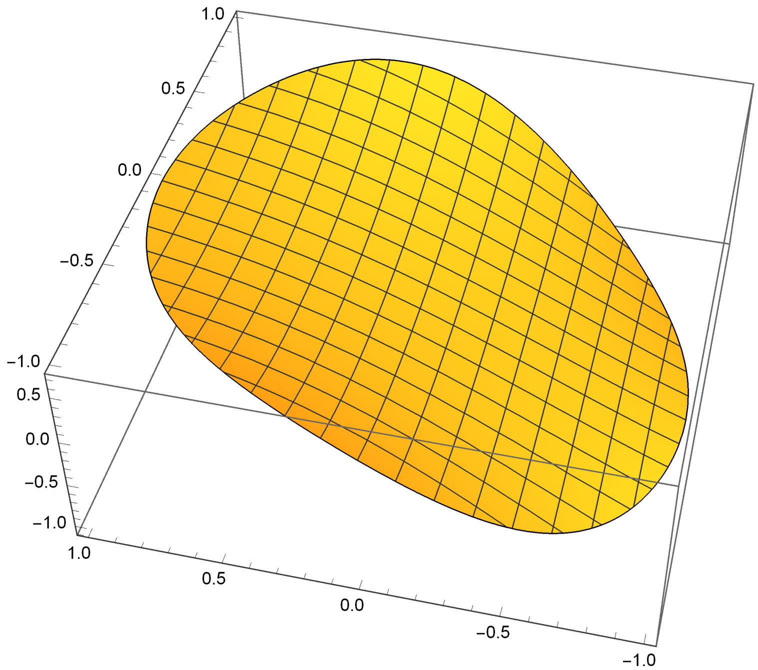


(3)

In[ ]:=
$$u23[r\_, \theta\_] := r * Sin[\theta] + \frac{1}{4} r^2 Cos[2\theta] + \frac{1}{9} r^3 Sin[3\theta] + \frac{1}{16} r^4 Cos[4\theta]$$

In[ ]:=
```
Plot3D[u23[Norm[{x, y}], ArcTan[x, y]], {x, -1, 1}, {y, -1, 1}, PlotLegends → Automatic,
  Exclusions → None, RegionFunction → Function[{x, y}, x^2 + y^2 ≤ 1]]
```

Out[ ]=



(4)

In[ ]:= $\int_0^{2\,\mathrm{Pi}} (\mathrm{Sin}[n\,\theta])^{\wedge}2\,d\theta$

Out[ ]= π

In[ ]:= $\int_0^{2\,\mathrm{Pi}} (\mathrm{Cos}[n\,\theta])^{\wedge}2\,d\theta$

Out[ ]= π

In[5]:= $\mathrm{A024} = \int_0^{2\,\mathrm{Pi}} \mathrm{boxcar}[0]\,d\theta$

Out[5]= $2\,\pi$

In[6]:= $\frac{1}{\mathrm{Pi}}$ Integrate[boxcar[Pi θ] Cos[n θ], {θ, 0, 2 Pi}]

Out[6]= $\frac{\mathrm{Sin}\left[\frac{n}{\pi}\right]}{n\,\pi}$

In[7]:= $\mathrm{An24}[n\_] := \frac{\mathrm{Sin}\left[\frac{n}{\pi}\right]}{n\,\pi}$

In[9]:= $\frac{1}{\text{Pi}}$ Integrate[boxcar[Pi θ] Sin[n θ], {θ, 0, 2 Pi}]

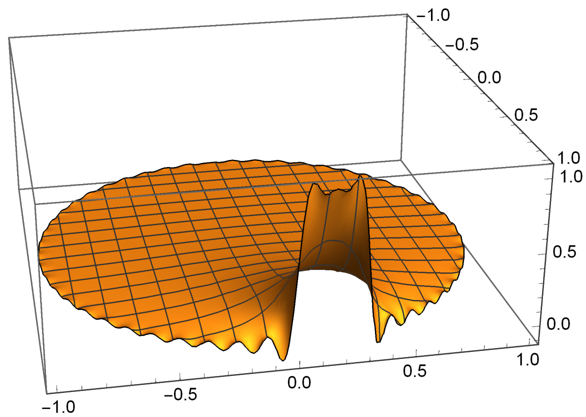
Out[9]= $\frac{1 - \text{Cos}\left[\frac{n}{\pi}\right]}{n\,\pi}$

In[10]:= Bn24[n_] := $\frac{1 - \text{Cos}\left[\frac{n}{\pi}\right]}{n\,\pi}$

In[12]:= u24[R_, k_, r_, θ_] := $\sum_{n=1}^{k} \left(\frac{r}{R}\text{^}n\right) \left(\text{An24[n] Cos[n θ] + Bn24[n] Sin[n θ]}\right) + \frac{1}{2\,\text{Pi^2}}$

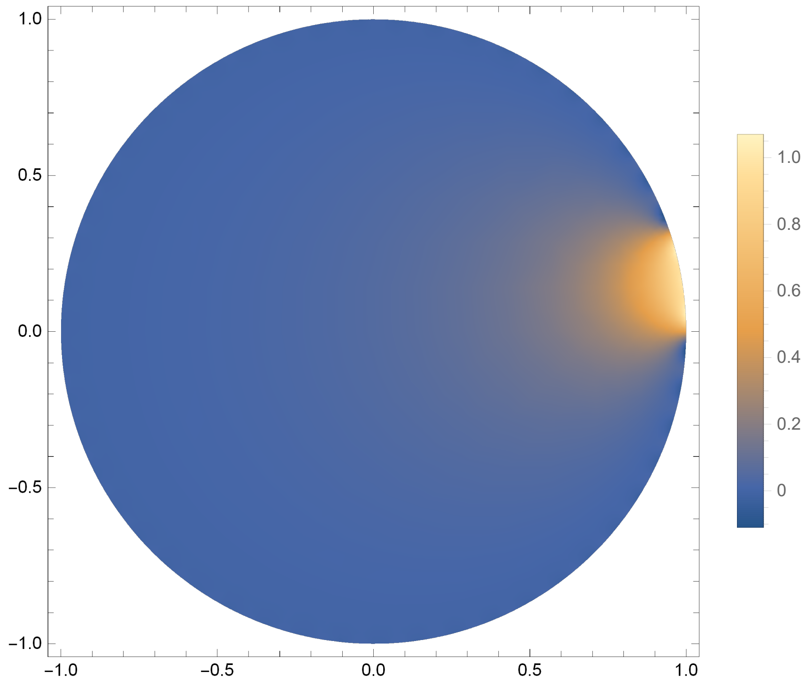
```
In[16]:= Plot3D[u24[1, 50, Sqrt[x^2 + y^2], ArcTan[x, y]],
  {x, -1, 1}, {y, -1, 1}, PlotLegends → Automatic, Exclusions → None,
  PlotRange → Full, RegionFunction → Function[{x, y}, x^2 + y^2 ≤ 1]]
```

Out[16]=



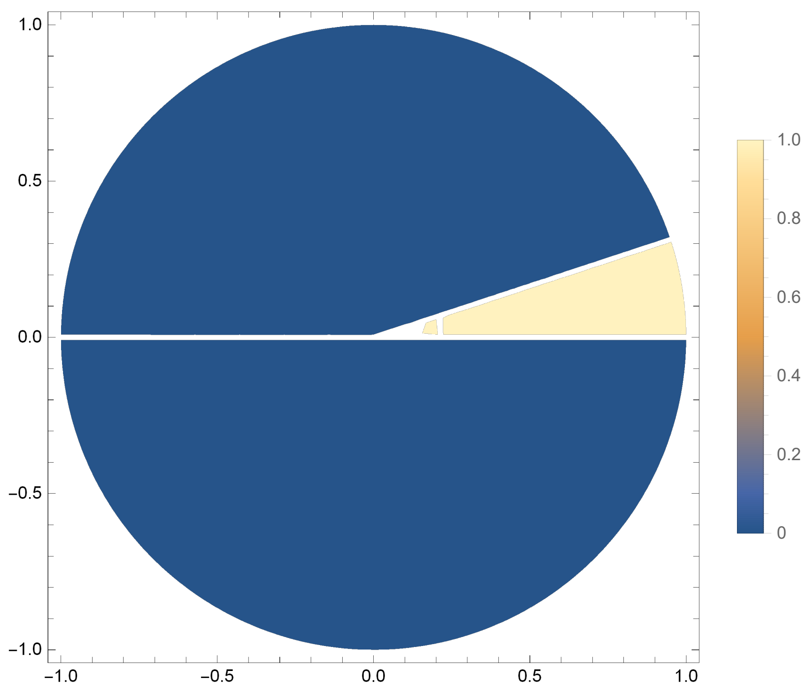
In[15]:=
```
DensityPlot[u24[1, 50, Sqrt[x^2 + y^2], ArcTan[x, y]],
 {x, -1, 1}, {y, -1, 1}, PlotLegends → Automatic, Exclusions → None,
 PlotRange → Full, RegionFunction → Function[{x, y}, x^2 + y^2 ≤ 1]]
```

Out[15]=



In[17]:=
```
DensityPlot[boxcar[Pi ArcTan[x, y]], {x, -1, 1}, {y, -1, 1},
 RegionFunction → Function[{x, y}, x^2 + y^2 ≤ 1], PlotLegends → Automatic]
```

Out[17]=



(4)

In[ ]:= $\int_{0}^{2\,Pi} (\text{Sin}[n\,\theta])^{\wedge}2\,d\theta$

Out[ ]= π

In[ ]:= $\int_{0}^{2\,Pi} (\text{Cos}[n\,\theta])^{\wedge}2\,d\theta$

Out[ ]= π

In[25]:= $\text{A025} = \frac{1}{2\,Pi} \int_{-Pi}^{Pi} \Lambda[\theta]\,d\theta$

Out[25]= 1

In[26]:= $\frac{1}{Pi}$ Integrate[Λ[Pi θ] Cos[n θ], {θ, -Pi, Pi}]

Out[26]= $-\frac{2\left(-\pi + \pi\,\text{Cos}\left[\frac{n}{\pi}\right]\right)}{n^2\,\pi}$

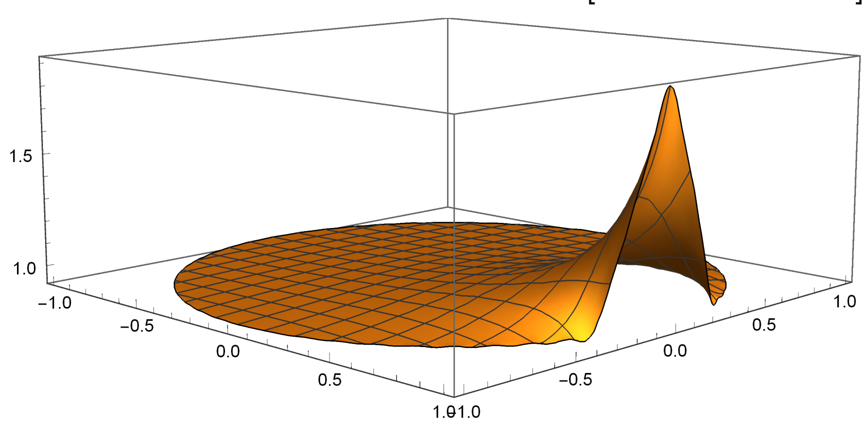
In[27]:= $\text{An25}[n\_] := -\frac{2\left(-\pi + \pi\,\text{Cos}\left[\frac{n}{\pi}\right]\right)}{n^2\,\pi}$

In[28]:= Bn25[n_] := 0

In[32]:= $\text{u25}[R\_, k\_, r\_, \theta\_] := \sum_{n=1}^{k} \left(\frac{r}{R}^{\wedge}n\right) (\text{An25}[n]\,\text{Cos}[n\,\theta]) + 1$

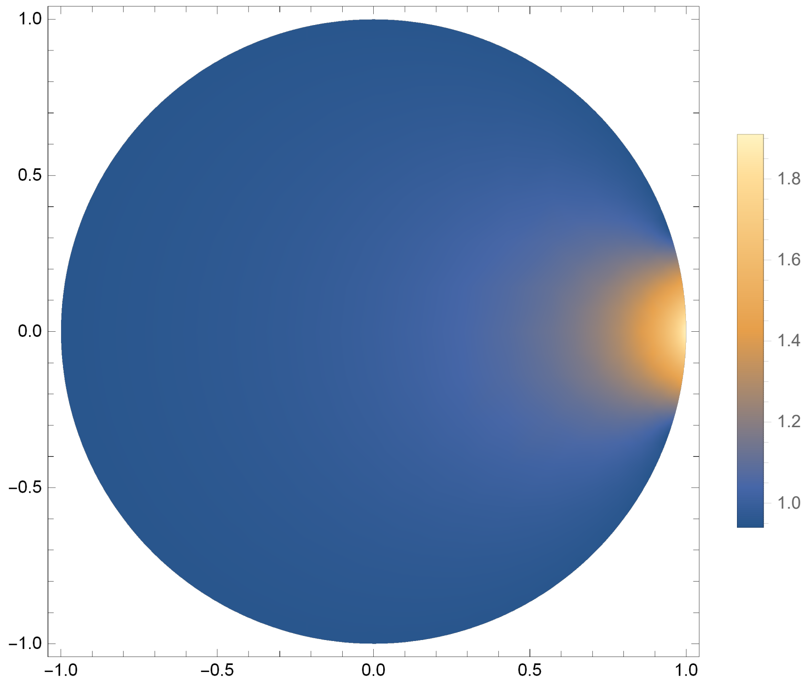
In[33]:= Plot3D[u25[1, 50, Sqrt[x^2 + y^2], ArcTan[x, y]],
{x, -1, 1}, {y, -1, 1}, PlotLegends → Automatic, Exclusions → None,
PlotRange → Full, RegionFunction → Function[{x, y}, $x^2 + y^2 \le 1$]]

Out[33]=



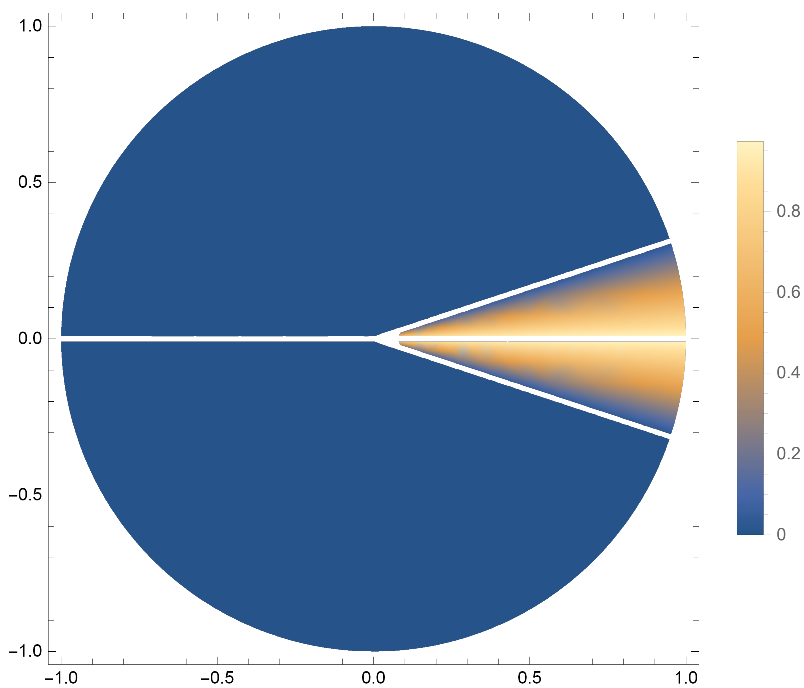
In[34]:=
```
DensityPlot[u25[1, 50, Sqrt[x^2 + y^2], ArcTan[x, y]],
  {x, -1, 1}, {y, -1, 1}, PlotLegends → Automatic, Exclusions → None,
  PlotRange → Full, RegionFunction → Function[{x, y}, x^2 + y^2 ≤ 1]]
```

Out[34]=



In[40]:=
```
DensityPlot[Λ[Pi ArcTan[x, y]], {x, -1, 1}, {y, -1, 1},
  RegionFunction → Function[{x, y}, x^2 + y^2 ≤ 1], PlotLegends → Automatic]
```

Out[40]=



The Piano String Revisited

(a) Find the most Discordant

In[154]:=
```
α = 0; β = 0; c = 10^2;
```

In[62]:=
$$\omega[n\_, L\_] := \frac{n\,\mathrm{Pi}}{L}\,\mathrm{Sqrt}\left[1 + \alpha\left(\frac{n\,\mathrm{Pi}}{L}\right)\mathrm{\hat{}}2\right]$$

In[63]:=
$$\eta[n\_, L\_] := c\,\omega[n, L]\sqrt{\left(1 - \frac{\beta\mathrm{\hat{}}2}{\left(\frac{2\,c\,n\,\mathrm{Pi}}{L}\right)\mathrm{\hat{}}2\left(1 + \alpha\left(\frac{n\,\mathrm{Pi}}{L}\right)\mathrm{\hat{}}2\right)}\right)}$$

In[64]:=
```
η1 = η[1, L]
```

Out[64]=
$$\frac{100\,\pi}{L}$$

In[75]:=
$$b[\delta\_, x0\_, n\_, L\_] := \frac{\delta\,\mathrm{Sin}[\omega[n, L]\,x0]}{\eta[n, L]}\left(\frac{\mathrm{Sin}[\delta\,\omega[n, L]]}{\delta\,\omega[n, L]} + \frac{2\,\mathrm{Pi}\,\mathrm{Cos}[\delta\,\omega[n, L]]}{\pi\mathrm{\hat{}}2 - (2\,\delta\,\omega[n, L])}\right)$$

In[81]:=
$$\mathrm{concor}[n\_, L\_] := 12\,\mathrm{Log}\left[2, \frac{\eta[n, L]}{\eta[1, L]}\right]$$

In[92]:=
```
Table[{i, N[concor[i, 1]]}, {i, 1, 16}]
```

Out[92]=
```
{{1, 0.}, {2, 12.}, {3, 19.0196}, {4, 24.}, {5, 27.8631}, {6, 31.0196},
 {7, 33.6883}, {8, 36.}, {9, 38.0391}, {10, 39.8631}, {11, 41.5132},
 {12, 43.0196}, {13, 44.4053}, {14, 45.6883}, {15, 46.8827}, {16, 48.}}
```

- the 11th overtone is the most discordant

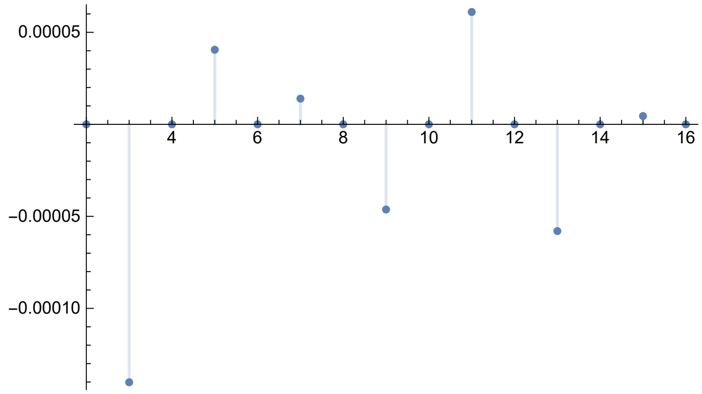
(b) Find Hammer parameters that minimize the energy in the discordant overtones

In[155]:=
```
Table[{i, N[b[.1, .5, i, 1]]}, {i, 1, 16}]
```

Out[155]=
$\{\{1, 0.000518927\}, \{2, 2.97366\times10^{-20}\}, \{3, -0.000140155\}, \{4, -1.98956\times10^{-20}\},$
$\{5, 0.0000405285\}, \{6, 3.62994\times10^{-21}\}, \{7, 0.0000139654\}, \{8, 1.5899\times10^{-20}\},$
$\{9, -0.0000462791\}, \{10, -3.41468\times10^{-20}\}, \{11, 0.0000610438\}, \{12, 4.55648\times10^{-20}\},$
$\{13, -0.0000579978\}, \{14, -3.94789\times10^{-20}\}, \{15, 4.50316\times10^{-6}\}, \{16, 2.09929\times10^{-19}\}\}$

In[156]:= `DiscretePlot[(b[.1, .5, i, 1]), {i, 2, 16}, PlotRange → Full]`

Out[156]=



In[166]:= $\texttt{NSolve}\left[\left\{\texttt{b}[\delta, \texttt{x0}, 2, 1] == \frac{1}{4}, \texttt{b}[\delta, \texttt{x0}, 4, 1] == \frac{1}{16}\right\}, \{\delta, \texttt{x0}\}\right]$

Out[166]= `$Aborted`

Pl

In[160]:= $(\texttt{b}[\delta, \texttt{x0}, 7, 1])\texttt{\^{}}2 + (\texttt{b}[\delta, \texttt{x0}, 11, 1])\texttt{\^{}}2 + (\texttt{b}[\delta, \texttt{x0}, 13, 1])\texttt{\^{}}2$

Out[160]=
$$\frac{\delta^2\,\text{Sin}[7\,\pi\,\text{x0}]^2\left(\frac{2\,\pi\,\text{Cos}[7\,\pi\,\delta]}{\pi^2-14\,\pi\,\delta}+\frac{\text{Sin}[7\,\pi\,\delta]}{7\,\pi\,\delta}\right)^2}{490\,000\,\pi^2}+\frac{\delta^2\,\text{Sin}[11\,\pi\,\text{x0}]^2\left(\frac{2\,\pi\,\text{Cos}[11\,\pi\,\delta]}{\pi^2-22\,\pi\,\delta}+\frac{\text{Sin}[11\,\pi\,\delta]}{11\,\pi\,\delta}\right)^2}{1\,210\,000\,\pi^2}+\frac{\delta^2\,\text{Sin}[13\,\pi\,\text{x0}]^2\left(\frac{2\,\pi\,\text{Cos}[13\,\pi\,\delta]}{\pi^2-26\,\pi\,\delta}+\frac{\text{Sin}[13\,\pi\,\delta]}{13\,\pi\,\delta}\right)^2}{1\,690\,000\,\pi^2}$$

In[151]:= `Simplify[%150]`

Out[151]=
$$\frac{1}{10\,020\,010\,000\,\pi^2}\,\delta^2\left(20\,449\,\text{Sin}[7\,\pi\,\text{x0}]^2\left(\frac{2\,\text{Cos}[7\,\pi\,\delta]}{\pi-14\,\delta}+\frac{\text{Sin}[7\,\pi\,\delta]}{7\,\pi\,\delta}\right)^2+8281\,\text{Sin}[11\,\pi\,\text{x0}]^2\left(\frac{2\,\text{Cos}[11\,\pi\,\delta]}{\pi-22\,\delta}+\frac{\text{Sin}[11\,\pi\,\delta]}{11\,\pi\,\delta}\right)^2+5929\,\text{Sin}[13\,\pi\,\text{x0}]^2\left(\frac{2\,\text{Cos}[13\,\pi\,\delta]}{\pi-26\,\delta}+\frac{\text{Sin}[13\,\pi\,\delta]}{13\,\pi\,\delta}\right)^2\right)$$

In[183]:= `NMinimize[{%151, 0 ≤ x0 ≤ 1 && 0 ≤ δ ≤ x0}, {x0, δ}]`

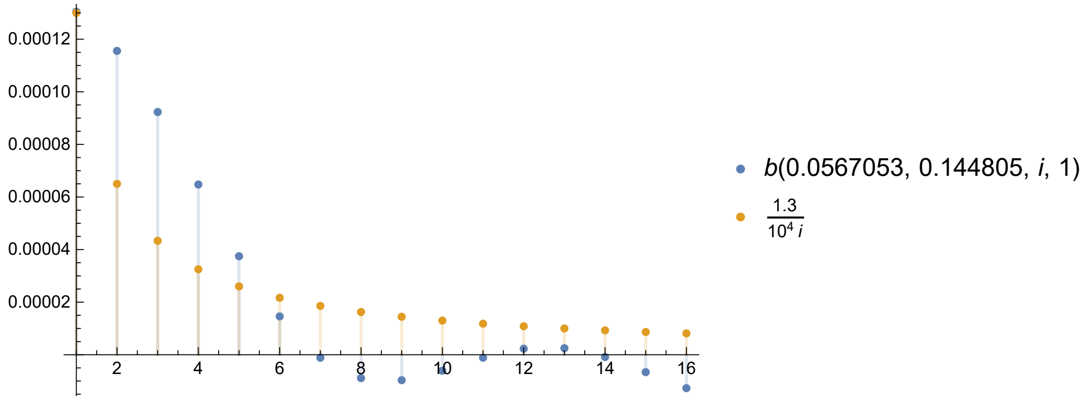
Out[183]= $\{8.75224\times10^{-12}, \{\text{x0}\to0.144805, \delta\to0.0567053\}\}$

In[184]:= `Table[{i, N[b[0.05670530907460242`, 0.14480457808512992`, i, 1]]}, {i, 1, 16}]`

Out[184]= $\{\{1, 0.000130442\}, \{2, 0.000115559\}, \{3, 0.0000923131\}, \{4, 0.0000647492\}, \{5, 0.0000374696\}, \{6, 0.0000145931\}, \{7, -1.13849\times10^{-6}\}, \{8, -8.87796\times10^{-6}\}, \{9, -9.68782\times10^{-6}\}, \{10, -6.0663\times10^{-6}\}, \{11, -1.1307\times10^{-6}\}, \{12, 2.30315\times10^{-6}\}, \{13, 2.48548\times10^{-6}\}, \{14, -8.65466\times10^{-7}\}, \{15, -6.61986\times10^{-6}\}, \{16, -0.0000127108\}\}$

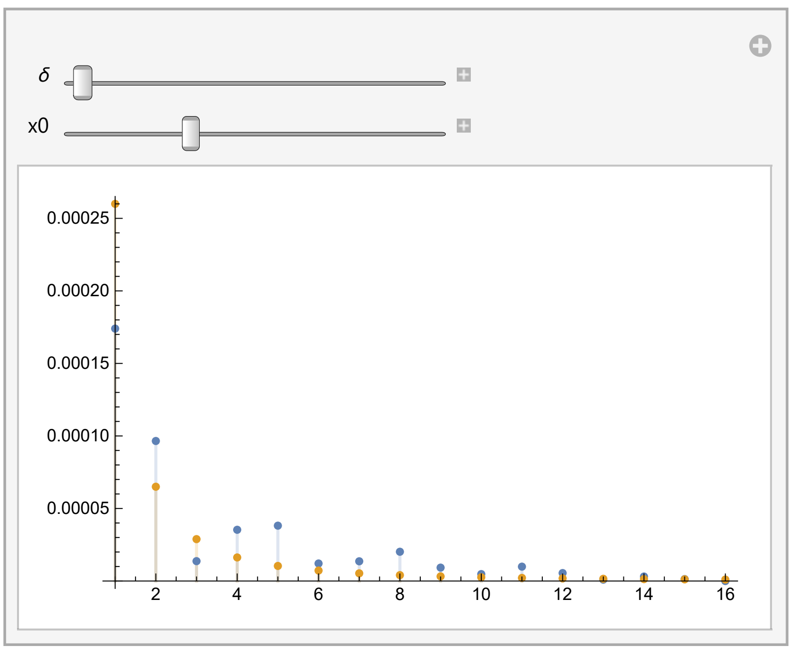
In[201]:=
```
DiscretePlot[{b[0.05670530907460242`, 0.14480457808512992`, i, 1], 1.3/(10^4 i)},
  {i, 1, 16}, PlotRange → Full, PlotLegends → "Expressions"]
```

Out[201]=



In[202]:=
```
Manipulate[DiscretePlot[{Abs[b[δ, x0, i, 1]], 2.6/(10^4 i^2)}, {i, 1, 16}, PlotRange → Full],
  {δ, 0.04, 0.07}, {x0, 0, 1}]
```

Out[202]=



(c) What changes when the piano wire has significant bending stiffness?

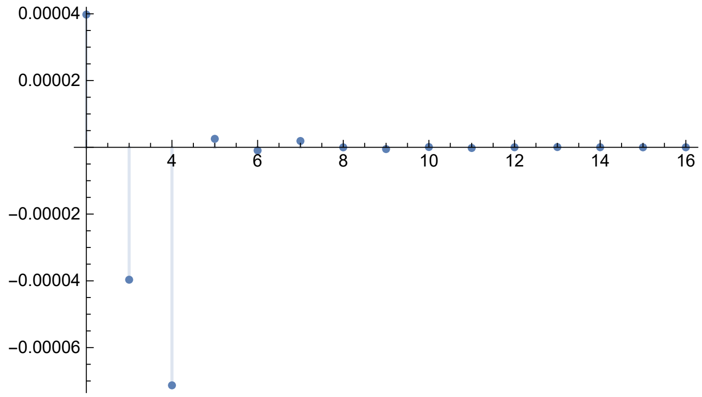
In[124]:=
```
α = 0.1; β = 0; c = 10^2;
```

In[125]:=
```
Table[{i, N[concor[i, 1]]}, {i, 1, 16}]
```

Out[125]=
```
{{1, 0.}, {2, 19.8974}, {3, 32.9056}, {4, 42.4745}, {5, 50.0132}, {6, 56.2224},
 {7, 61.4967}, {8, 66.079}, {9, 70.1288}, {10, 73.7565}, {11, 77.0415},
 {12, 80.0428}, {13, 82.8053}, {14, 85.3642}, {15, 87.7473}, {16, 89.9772}}
```

In[126]:=
```
DiscretePlot[(b[.1, .5, i, 1]), {i, 2, 16}, PlotRange → Full]
```

Out[126]=



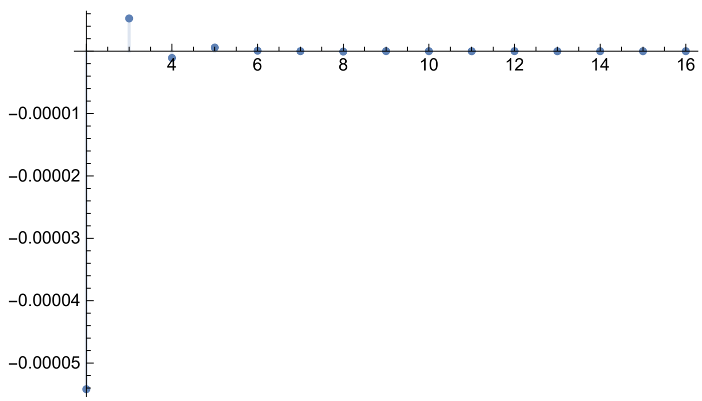
In[128]:=
```
α = 1; β = 0; c = 10^2;
```

In[129]:=
```
Table[{i, N[concor[i, 1]]}, {i, 1, 16}]
```

Out[129]=
```
{{1, 0.}, {2, 23.3811}, {3, 37.3006}, {4, 47.2192}, {5, 54.9259}, {6, 61.228},
 {7, 66.559}, {8, 71.1783}, {9, 75.2536}, {10, 78.8996}, {11, 82.1982},
 {12, 85.2098}, {13, 87.9803}, {14, 90.5456}, {15, 92.9339}, {16, 95.168}}
```

In[130]:=
```
DiscretePlot[(b[.1, .5, i, 1]), {i, 2, 16}, PlotRange → Full]
```

Out[130]=



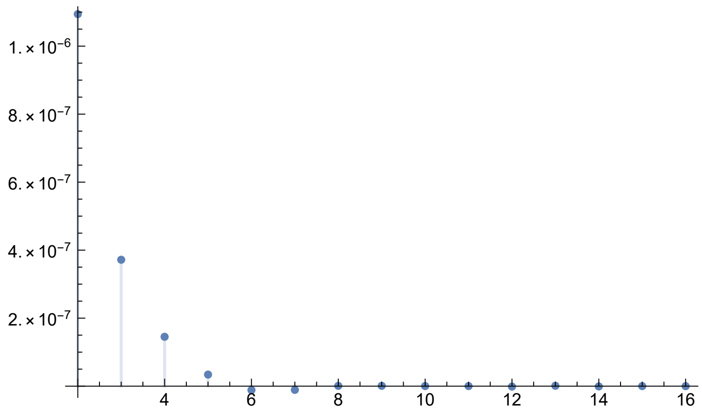
In[131]:=
```
α = 10; β = 0; c = 10^2;
```

In[132]:=
```
Table[{i, N[concor[i, 1]]}, {i, 1, 16}]
```

Out[132]=
```
{{1, 0.}, {2, 23.9346}, {3, 37.9616}, {4, 47.9182}, {5, 55.6425}, {6, 61.9543},
 {7, 67.291}, {8, 71.9141}, {9, 75.992}, {10, 79.6399}, {11, 82.9398},
 {12, 85.9524}, {13, 88.7238}, {14, 91.2897}, {15, 93.6785}, {16, 95.9131}}
```

In[133]:=
```
DiscretePlot[(b[.1, .5, i, 1]), {i, 2, 16}, PlotRange → Full]
```

Out[133]=



With a significant bending stiffness, the PDE now starts to have a beam-equation behavior in addition to a standard wave-equation behavior. The perfectly concordant overtone at 2^n now disappears, and every overtone is somewhat discordant.