Olin College of Engineering
Partial Differential Equations, 2019

Homework 8 Problems
Due Friday, March 29

1. Fourier's Method for the Heat Equation

(1) Solve the heat equation $u_t = 2u_{xx}$ on the domain $0 < x < 5$ with boundary conditions $u = 0$ at $x = 0$ and $u_x = 0$ at $x = 5$ and initial condition $u = x^2(x-5)^2$ at $t = 0$.

(2) Solve the heat equation $u_t = 5u_{xx}$ on the domain $-1 < x < 2$ with boundary conditions $u_x = 0$ at $x = -1$ and $u_x = 0$ at $x = 2$ and initial condition $u = B(x)$ at $t = 0$.

2. The Metal Disc at Thermal Equilibrium

The heat equation in two dimensions is $u_t = u_{xx} + u_{yy}$. For a metal disc of radius R with temperature fixed at the boundary, the boundary condition is of the form $u = f$ for $x^2 + y^2 = R^2$. After changing to polar coordinates the equation becomes $u_t = u_{rr} + \frac{1}{r}u_r + \frac{1}{r^2}u_{\theta\theta}$ with the boundary condition taking the reasonable form $u(R,\theta) = f(\theta)$. At thermal equilibrium we have $u_t = 0$. Thus

$$\begin{cases} u_{rr} + \frac{1}{r}u_r + \frac{1}{r^2}u_{\theta\theta} = 0 \\ u(R,\theta) = f(\theta) \end{cases}$$

You have already found the separated solutions to be $r^n\cos(n\theta)$ and $r^n\sin(n\theta)$.

Write down the solution u for each of the choices of f below. Also plot the corresponding solution.

(1) $f(\theta) = \sin(\theta)$

(2) $f(\theta) = \sin(\theta) - \frac{1}{3}\sin(3\theta) + \frac{1}{5}\sin(5\theta)$

(3) $f(\theta) = \sin(\theta) + \frac{1}{4}\cos(2\theta) + \frac{1}{9}\sin(3\theta) + \frac{1}{16}\cos(4\theta)$

(4) $f(\theta) = B(\pi\theta)$

(5) $f(\theta) = \Lambda(\pi\theta)$

3. The Piano String Revisited

Consider the model below with five parameters: damping (β), stiffness (α), hammer width (δ), hammer center (x_0), and sonic speed in the wire (c).

$$\begin{cases} u_{tt} + \beta u_t = c^2(u_{xx} - \alpha u_{xxxx}) & 0 < x < 1 \\ u = u_{xx} = 0 & x = 0, 1 \\ u = 0 & t = 0 \\ u_t(0, x) = f(x; \delta, x_0) & t = 0 \end{cases}$$

with

$$f(x; \delta, x_0) = \begin{cases} \frac{1}{2} + \cos(\frac{\pi(x - x_0)}{2\delta}) & |x - x_0| < \delta \\ 0 & \text{else} \end{cases}$$

The boundary condition $u_{xx} = 0$ signifies that to the extent we are modeling the piano wire as a beam we should not regard the beam as clamped but rather pinned.

To solve this problem for $u(t, x)$ one must find the spatial eigenfunctions $X_n(x)$, the temporal eigenfunctions $T_n(t)$ and the Fourier coefficients b_n so that the solution is given by

$$u(t, x) = \sum_n b_n T_n(t) X_n(x)$$

A modification of the work you did last week leads to the spatial eigenfunctions

$$X_n(x) = \sin(\omega_n x) \text{ with } \omega_n = \frac{n\pi}{L}\sqrt{1 + \alpha\left(\frac{n\pi}{L}\right)^2},$$

the temporal eigenfunctions

$$T_n(t) = e^{-\frac{\beta}{2}t}\sin(\eta_n t) \text{ with } \eta_n = c\omega_n\sqrt{1 - \left(\frac{\beta^2}{\left(\frac{2cn\pi}{L}\right)^2\left(1 + \alpha\left(\frac{n\pi}{L}\right)^2\right)}\right)}$$

and the Fourier coefficients

$$b_n = \frac{\delta\sin(\omega_n x_0)}{\eta_n}\left[\frac{\sin(\delta\omega_n)}{\delta\omega_n} + \frac{2\pi\cos(\delta\omega_n)}{\pi^2 - (2\delta\omega_n)^2}\right]$$

Typical parameters for a piano might be $c \approx 10^2$, $\beta \approx 10^0$ and $\alpha \approx 10^{-3}$, justifying the approximation $\alpha = \beta = 0$.

(1) In the usual (twelve tone equal temperament chromatic) musical scale the **fundamental** η_1 and the **overtone** η_n are said to be **concordant** or **in harmony** if $12\log_2\left(\frac{\eta_n}{\eta_1}\right)$ is close to an integer and said to be **discordant** otherwise.

  (a) In the limiting case $\alpha = \beta = 0$, which of the first fifteen overtones are most discordant.

  (b) In the limiting case $\alpha = \beta = 0$, come up with a solution to the following design problem: find hammer parameters δ and x_0 that minimize the energy (b_n^2) in the discordant overtones without disrupting the timbre of the instrument ($b_n \sim \frac{1}{n^2}$ for concordant overtones).

  (c) What changes when the piano wire has significant bending stiffness ($10^{-1} < \alpha < 10$)?

(2) The music box satisfies the same equation, except now $\alpha > 1$, the boundary conditions are for a cantilevered beam ($u = u_x = 0$ at $x = 0$ and $u_{xx} = u_{xxx} = 0$ at $x = 1$) and the initial displacement is given by solving $u_{xxxx} + f(x) = 0$ with $x_0 + \delta = 1$. Make a prediction for how the solution (the temporal and spatial eigenfunctions, the fundamental and its overtones, the corresponding Fourier coefficients) differ between the piano and the music box. Write down the computations you would have to do (numerically) in order to test you predictions. If you have the time to do so efficiently, numerically solve these equations and numerically test your predictions.