EE364a Homework # 6

8.8 *Intersection and containment of polyhedra.* Let $\mathcal{P}_1$ and $\mathcal{P}_2$ be two polyhedra defined as

$$\mathcal{P}_1 = \{x \mid Ax \preceq b\}, \qquad \mathcal{P}_2 = \{x \mid Fx \preceq g\},$$

with $A \in \mathbf{R}^{m\times n}$, $b \in \mathbf{R}^m$, $F \in \mathbf{R}^{p\times n}$, $g \in \mathbf{R}^p$. Explain how to carry out the tasks below by solving one or a modest number of LP feasibility problems.

(a) Find a point in the intersection $\mathcal{P}_1 \cap \mathcal{P}_2$.

(b) Determine whether $\mathcal{P}_1 \subseteq P_2$.

Repeat the question for two polyhedra defined as

$$\mathcal{P}_1 = \mathbf{conv}\{v_1, \ldots, v_K\}, \qquad \mathcal{P}_2 = \mathbf{conv}\{w_1, \ldots, w_L\},$$

with $v_1, \ldots, v_K, w_1, \ldots, w_L \in \mathbf{R}^n$.

A6.3 *Approximation with trigonometric polynomials.* Suppose $y : \mathbf{R} \to \mathbf{R}$ is a 2π-periodic function. We will approximate y with the trigonometric polynomial

$$f(t) = \sum_{k=0}^{K} a_k \cos(kt) + \sum_{k=1}^{K} b_k \sin(kt).$$

We consider two approximations: one that minimizes the L_2-norm of the error, defined as

$$\|f - y\|_2 = \left(\int_{-\pi}^{\pi} (f(t) - y(t))^2 \, dt\right)^{1/2},$$

and one that minimizes the L_1-norm of the error, defined as

$$\|f - y\|_1 = \int_{-\pi}^{\pi} |f(t) - y(t)| \, dt.$$

The L_2 approximation is of course given by the (truncated) Fourier expansion of y.

To find an L_1 approximation, we discretize t at $2N$ points,

$$t_i = -\pi + i\pi/N, \quad i = 1, \ldots, 2N,$$

and approximate the L_1 norm as

$$\|f - y\|_1 \approx (\pi/N) \sum_{i=1}^{2N} |f(t_i) - y(t_i)|.$$

(A standard rule of thumb is to take N at least 10 times larger than K.) The L_1 approximation (or really, an approximation of the L_1 approximation) can now be found by solving the (finite-dimensional) convex problem, which can be converted to an LP.

We consider a specific case, where y is a 2π-periodic square-wave, defined for $-\pi \leq t \leq \pi$ as

$$y(t) = \begin{cases} 1 & |t| \leq \pi/2 \\ 0 & \text{otherwise.} \end{cases}$$

(The graph of y over a few cycles explains the name 'square-wave'.)

Find the optimal L_2 approximation and (discretized) L_1 optimal approximation for $K = 10$. You can find the L_2 optimal approximation analytically, or by solving a least-squares problem associated with the discretized version of the problem. Since y is even, you can take the sine coefficients in your approximations to be zero. Show y and the two approximations on a single plot.

In addition, plot a histogram of the residuals (*i.e.*, the numbers $f(t_i)-y(t_i)$) for the two approximations. Use the same horizontal axis range, so the two residual distributions can easily be compared. Make some brief comments about what you see.

A6.12 *Least-squares with some permuted measurements.* We want to estimate a vector $x \in \mathbf{R}^n$, given some linear measurements of x corrupted with Gaussian noise. Here's the catch: some of the measurements have been *permuted*.

More precisely, our measurement vector $y \in \mathbf{R}^m$ has the form

$$y = P(Ax + v),$$

where v_i are IID $\mathcal{N}(0,1)$ measurement noises, $x \in \mathbf{R}^n$ is the vector of parameters we wish to estimate, and $P \in \mathbf{R}^{m\times m}$ is a permutation matrix. (This means that each row and column of P has exactly one entry equal to one, and the remaining $m-1$ entries zero.) We assume that $m > n$ and that at most k of the measurements are permuted; *i.e.*, $Pe_i \neq e_i$ for no more than k indices i. We are interested in the case when $k < m$ (*e.g.* $k = 0.4m$); that is, only *some* of the measurements have been permuted. We want to estimate x and P.

Once we make a guess $\hat{P}$ for P, we can get the maximum likelihood estimate of x by minimizing $\|Ax - \hat{P}^T y\|_2$. The residual $A\hat{x} - \hat{P}^T y$ is then our guess of what v is, and should be consistent with being a sample of a $\mathcal{N}(0, I)$ vector.

In principle, we can find the maximum likelihood estimate of x and P by solving a set of $\binom{m}{k}(k! - 1)$ least-squares problems, and choosing one that has minimum residual. But this is not practical unless m and k are both very small.

Describe a *heuristic* method for approximately solving this problem, using convex optimization. (There are many different approaches which work quite well.)

You might find the following fact useful. The solution to

$$\text{minimize} \quad \|Ax - P^T y\|_2$$

1. find a matrix P that orders Ax and y by element magnitude
2. Optimize over x

1. Sort Ax and y and calculate residue
2. find

over $P \in \mathbf{R}^{m\times m}$ a permutation matrix, is the permutation that matches the smallest entry in y with the smallest entry in Ax, does the same for the second smallest entries and so forth.

Carry out your method on the data in `ls_perm_meas_data.*`. Give your estimate of the permuted indices. The data file includes the true permutation matrix and value of x (which of course you cannot use in forming your estimate). Compare the estimate of x you get after your guessed permutation with the estimate obtained assuming $P = I$.

Remark. This problem comes up in several applications. In target tracking, we get multiple noisy measurements of a set of targets, and then guess which targets are the same in the different sets of measurements. If some of our guesses are wrong (*i.e.*, our target association is wrong) we have the present problem. In vision systems the problem arises when we have multiple camera views of a scene, which give us noisy measurements of a set of features. A feature correspondence algorithm guesses which features in one view correspond to features in other views. If we make some feature correspondence errors, we have the present problem.

A6.19 *Colorization with total variation regularization.* A $m \times n$ color image is represented as three matrices of intensities $R, G, B \in \mathbf{R}^{m\times n}$, with entries in $[0, 1]$, representing the red, green, and blue pixel intensities, respectively. A color image is converted to a monochrome image, represented as one matrix $M \in \mathbf{R}^{m\times n}$, using

$$M = 0.299R + 0.587G + 0.114B.$$

(These weights come from different perceived brightness of the three primary colors.)

In *colorization*, we are given M, the monochrome version of an image, and the color values of *some* of the pixels; we are to guess its color version, *i.e.*, the matrices R, G, B. Of course that's a very underdetermined problem. A very simple technique is to minimize the total variation of (R, G, B), defined as

$$\mathbf{tv}(R, G, B) = \sum_{i=1}^{m-1}\sum_{j=1}^{n-1} \left\| \begin{bmatrix} R_{ij} - R_{i,j+1} \\ G_{ij} - G_{i,j+1} \\ B_{ij} - B_{i,j+1} \\ R_{ij} - R_{i+1,j} \\ G_{ij} - G_{i+1,j} \\ B_{ij} - B_{i+1,j} \end{bmatrix} \right\|_2,$$

subject to consistency with the given monochrome image, the known ranges of the entries of (R, G, B) (*i.e.*, in $[0, 1]$), and the given color entries. Note that the sum above is of the norm of 6-vectors, and not the norm-squared. (The 6-vector is an approximation of the spatial gradient of (R, G, B).)

Carry out this method on the data given in `image_colorization_data.*`. The file loads `flower.png` and provides the monochrome version of the image, `M`, along with vectors of known color intensities, `R_known`, `G_known`, and `B_known`, and `known_ind`, the

A8.16 *Fitting a sphere to data.* Consider the problem of fitting a sphere $\{x \in \mathbf{R}^n \mid \|x-c\|_2 = r\}$ to m points $u_1, \ldots, u_m \in \mathbf{R}^n$, by minimizing the loss function

$$\sum_{i=1}^{m} \left(\|u_i - c\|_2^2 - r^2\right)^2$$

$(u_j - c_j)^2$

$u_j^2 - 2u_j c_j + c_j^2$

over the variables $c \in \mathbf{R}^n$, $r \in \mathbf{R}$.

(a) Explain how to solve this problem using convex optimization. *Hint.* Consider the change of variables from (c, r) to (c, t), with $t = r^2 - \|c\|_2^2$. You'll need to argue that you can recover $r^\star$ from $t^\star$ once you solve the problem with these transformed variables.

$t = r^2 - \|c\|_2^2$

$r^2 = t + \|c\|_2^2$

minimize c, r $\quad \sum_{i=1}^{m} \left(\|u_i - c\|_2^2 - r^2\right)^2$

apply change of variable

minimize $\quad \sum_{i=1}^{m} \left(\|u_i - c\|_2^2 - t - \|c\|_2^2\right)^2$

expand inside:

$u_0^2 - 2u_0 c_0 + c_0^2$

$+ u_1^2 - 2u_1 c_1 + c_1^2$

$- t - (c_0^2 + c_1^2)$

the square term in c cancels out

minimize $\quad \sum_{i=1}^{m} \left(u_{i0}^2 - 2u_{i0}c_0 + u_{i1}^2 - 2u_{i1}c_1 - t\right)$

recover $\quad r = \sqrt{t + \|c\|_2^2}$

Suppose t^* and C^* are such that

$$t^* + \|C^*\|_2 < 0$$

we know, $\|C^*\|_2 \geq 0$

then $t^* < 0 \leq \|C^*\|_2$

$$\sum \left(\|u_i - c\|_2^2 - \left(t^* + \|C^*\|_2\right)\right)^2$$

then

$$\|u_i - c\|_2^2 - \left(t^* + \|C^*\|_2\right) \leq$$

Suppose t^* and C^* are such that

$$t^* + \|C^*\|_2 < 0$$

we know, $\|C^*\|_2 \geq 0$

then $t^* < 0 \leq \|C^*\|_2$

$$\sum \left(\|u_i - C\|_2^2 - (t^* + \|C^*\|_2)\right)^2$$

then

$$\|u_i - C^*\|_2^2 - (t^* + \|C^*\|_2) \geq \|u_i - C^*\|_2^2 - \|C^*\|_2$$

then we can obtain a lower optimal objective value by setting $t^* = 0$,

this contradicts the assumption that $t^* + \|C^*\|_2 < 0$

thus we must have $t^* + \|C^*\|_2 \geq 0$

indices of the pixels with known values. If `R` denotes the red channel of an image, then `R(known_ind)` returns the known red color intensities in Matlab, and `R[known_ind]` returns the same in Python and Julia. The file also creates an image, `flower_given.png`, that is monochrome, with the known pixels colored.

The `tv` function, invoked as `tv(R,G,B)`, gives the total variation. CVXPY has the `tv` function built-in, but CVX and CVX.jl do not, so we have provided the files `tv.m` and `tv.jl` which contain implementations for you to use.

In Python and Julia we have also provided the function `save_img(filename,R,G,B)` which writes the image defined by the matrices `R`, `G`, `B`, to the file `filename`. To view an image in Matlab use the `imshow` function.

The problem instance is a small image, 75×75, so the solve time is reasonable, say, under ten seconds or so in CVX or CVXPY, and around 60 seconds in Julia.

Report your optimal objective value and, if you have access to a color printer, attach your reconstructed image. If you don't have access to a color printer, it's OK to just give the optimal objective value.

A8.16 *Fitting a sphere to data.* Consider the problem of fitting a sphere $\{x \in \mathbf{R}^n \mid \|x-c\|_2 = r\}$ to m points $u_1, \ldots, u_m \in \mathbf{R}^n$, by minimizing the loss function

$$\sum_{i=1}^{m} \left(\|u_i - c\|_2^2 - r^2\right)^2$$

over the variables $c \in \mathbf{R}^n$, $r \in \mathbf{R}$.

(a) Explain how to solve this problem using convex optimization. *Hint.* Consider the change of variables from (c, r) to (c, t), with $t = r^2 - \|c\|_2^2$. You'll need to argue that you can recover $r^\star$ from $t^\star$ once you solve the problem with these transformed variables.

(b) Use your method to solve the problem instance with data given in the file `sphere_fit_data.*`, with $n = 2$. This file creates u_i as a $2 \times m$ matrix `U`. Plot the fitted circle and the data points.

A8.27 *Minimum volume ellipsoid that contains points and is inside a polyhedron.* We seek the minimum volume ellipsoid in $\mathbf{R}^n$, centered at 0, that contains the points $x_1, \ldots, x_K \in \mathbf{R}^n$, and is itself contained in (*i.e.*, a subset of) a polyhedron $\mathcal{P} = \{x \mid Ax \preceq b\}$, where $A \in \mathbf{R}^{m \times n}$ and $b \in \mathbf{R}^m$. This combines two of the extremal volume problems studied in the book. The data are the points x_i, and the matrix A and vector b that define the polyhedron. You can assume that $b \succeq 0$, which means $0 \in \mathcal{P}$.

Explain how to use convex optimization to find this ellipsoid, or to determine that no such ellipsoid exists. Be sure to explain how you parametrize the ellipsoid, how the constraints on the ellipsoid are expressed in your problem, and why the problem you propose is convex.

A8.27 *Minimum volume ellipsoid that contains points and is inside a polyhedron.* We seek the minimum volume ellipsoid in $\mathbf{R}^n$, centered at 0, that contains the points $x_1, \ldots, x_K \in \mathbf{R}^n$, and is itself contained in (*i.e.*, a subset of) a polyhedron $\mathcal{P} = \{x \mid Ax \preceq b\}$, where $A \in \mathbf{R}^{m \times n}$ and $b \in \mathbf{R}^m$. This combines two of the extremal volume problems studied in the book. The data are the points x_i, and the matrix A and vector b that define the polyhedron. You can assume that $b \succeq 0$, which means $0 \in \mathcal{P}$.

Explain how to use convex optimization to find this ellipsoid, or to determine that no such ellipsoid exists. Be sure to explain how you parametrize the ellipsoid, how the constraints on the ellipsoid are expressed in your problem, and why the problem you propose is convex.

Spse we parameterize the ellipsoid as a set of vector whose image under affine transform is the unit circle

that is,

$$\mathcal{E} = \{x \mid \|Px + q\|_2 \leq 1\}$$

because we are centered at zero

$$q = 0$$

$$\mathcal{E} = \{x \mid \|Px\|_2 \leq 1\}$$

$$= \{x \mid \|Px\|_2^2 \leq 1\}$$

$$= \{x \mid x^T P^T P x \leq 1\}$$

The problem we are interested in can then be formulated as

minimize $\log\det P^{-1}$

s.t. ① $\|Px_i\|_2 \leq 1$ for $x_i \in \{x_1, \ldots, x_k\}$

② $A^T v \leq b$ for all $v \in \{v \mid v^T P^T P x \leq 1\}$

② is not tractable because there is infinitely many v

② can be expanded as

$$a_i^T v \le b_i \quad \text{for } i \in \{1, \dots, m\}$$
$$\text{and for all } v \in \{v \mid v^T P P v \le 1\}$$

$$\sup_v \{a_i^T v \mid v^T P P v \le 1\} \le b_i$$

$$a_i v = a_i^T I v$$
$$= a_i^T P P^{-1} v$$

let $u = P^{-1} v$, $u^T u \le 1$

$$\sup_u \{a_i^T P u \mid \|u\|_2^2 \le 1\} = \|a_i^T P\|_2 \le b_i$$

then

maximize $\log\det P$

s.t. ① $\|P x_i\|_2 \le 1$ for $x_i \in \{x_1, \dots, x_k\}$

② $\|a_i^T P\|_2 \le b_i$ for all $i \in \{1, \dots, m\}$

Alternatively,

$$\mathcal{E} = \{Bu + d \mid \|u\|_2 \leq 1\}$$

minimize $\log \det$

A17.4 *Bounding portfolio risk with incomplete covariance information.* Consider the following instance of the problem described in §4.6, on p171–173 of *Convex Optimization*. We suppose that Σ_{ii}, which are the squares of the price volatilities of the assets, are known. For the off-diagonal entries of Σ, all we know is the sign (or, in some cases, nothing at all). For example, we might be given that $\Sigma_{12} \geq 0$, $\Sigma_{23} \leq 0$, etc. This means that we do not know the correlation between p_1 and p_2, but we do know that they are nonnegatively correlated (*i.e.*, the prices of assets 1 and 2 tend to rise or fall together).

Compute σ^2_{wc}, the worst-case variance of the portfolio return, for the specific case

$$x = \begin{bmatrix} 0.1 \\ 0.2 \\ -0.05 \\ 0.1 \end{bmatrix}, \quad \Sigma = \begin{bmatrix} 0.2 & + & + & \pm \\ + & 0.1 & - & - \\ + & - & 0.3 & + \\ \pm & - & + & 0.1 \end{bmatrix},$$

where a "+" entry means that the element is nonnegative, a "−" means the entry is nonpositive, and "±" means we don't know anything about the entry. (The negative value in x represents a *short position*: you sold stocks that you didn't have, but must produce at the end of the investment period.) In addition to σ^2_{wc}, give the covariance matrix Σ_{wc} associated with the maximum risk. Compare the worst-case risk with the risk obtained when Σ is diagonal.

HW6

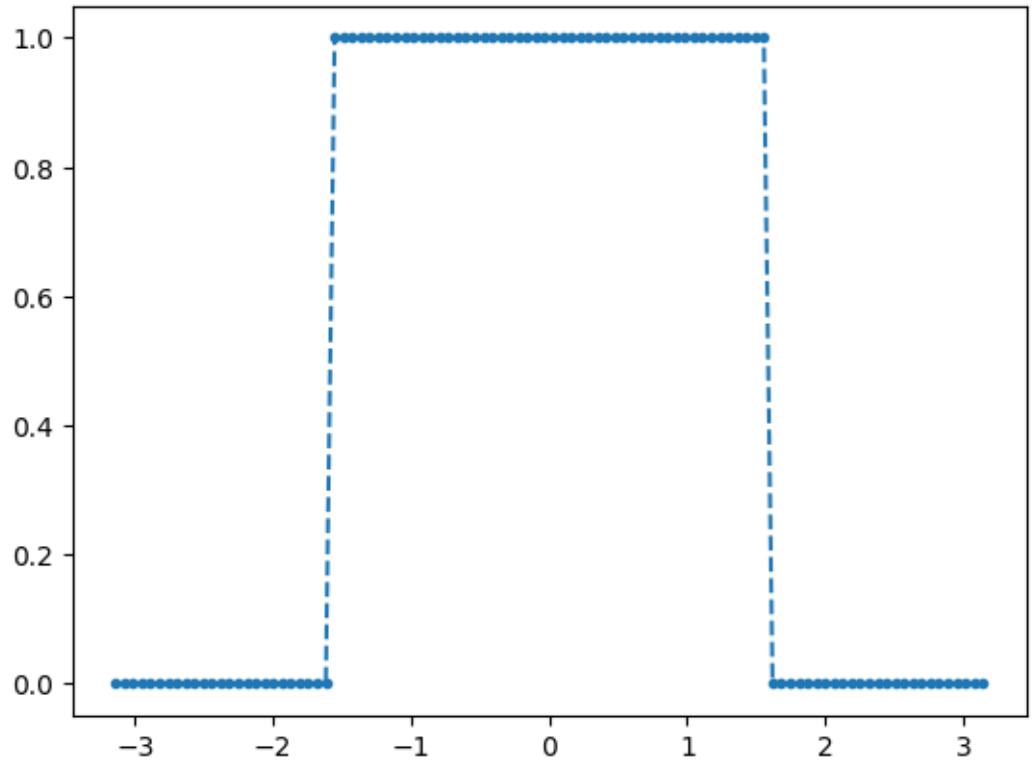
August 9, 2024

[18]:
```
import cvxpy as cp
import numpy as np
import matplotlib.pyplot as plt
import matplotlib.image as mpimg
```

0.0.1 A6.3

[19]:
```
# Set up Problem
start = -np.pi
stop = np.pi
num = 100
T = np.linspace(start, stop, num)

y = lambda x: np.array( np.abs(x)<= np.pi/2 ).astype(int)

plt.plot(T, y(T),'.--')
```

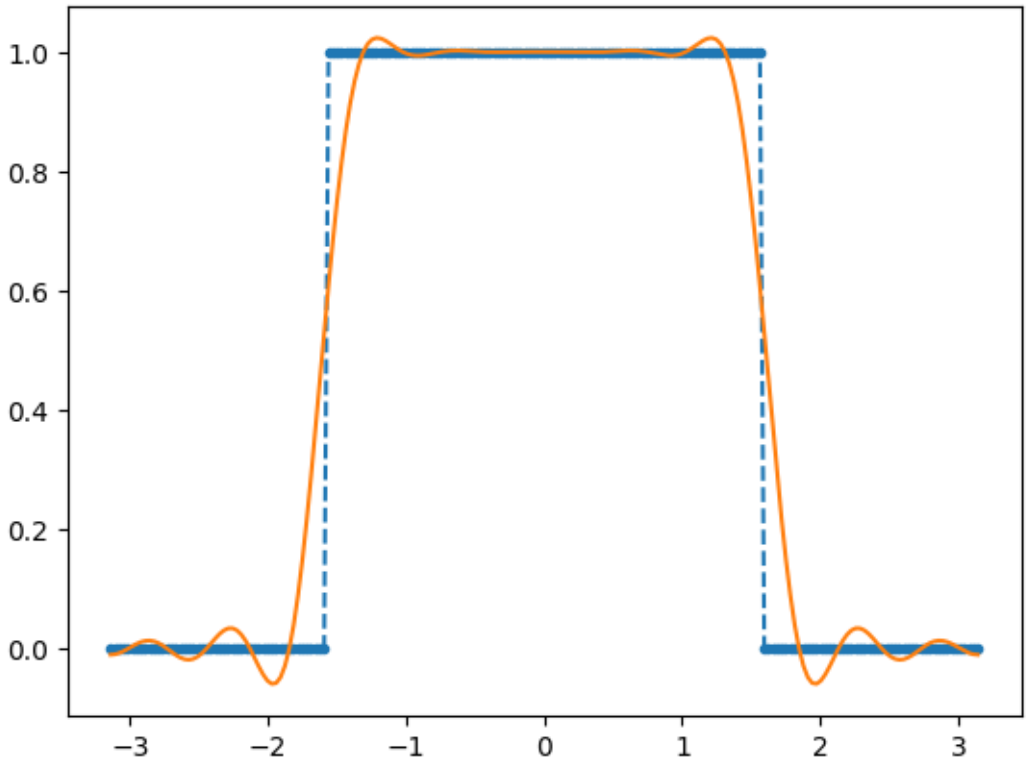
[19]:
```
[<matplotlib.lines.Line2D at 0x15ac80ad0>]
```



[20]:
```
# L1 Norm Approximation
K = 10
N = K * 10

T = np.linspace(start, stop, 2*N)

a1 = cp.Variable(K+1)
# b = cp.Variable(K)

obj = cp.Minimize(
    np.pi/N*cp.norm1(
        cp.vstack(
            [ cp.sum(
                    cp.vstack(
                        [a1[k]*np.cos(k*t) for k in range(K+1)]
                    )
                ) - y(t) for t in T
            ]
        )
    )
)
```

```
constraints = []

prob = cp.Problem(obj, constraints)

prob.solve()

plt.plot(T, y(T),'.--')
plt.plot(T, [ np.sum(
                    # cp.vstack(
                        [a1.value[k]*np.cos(k*t) for k in range(K+1)]
                    # )
                ) for t in T
            ])
```

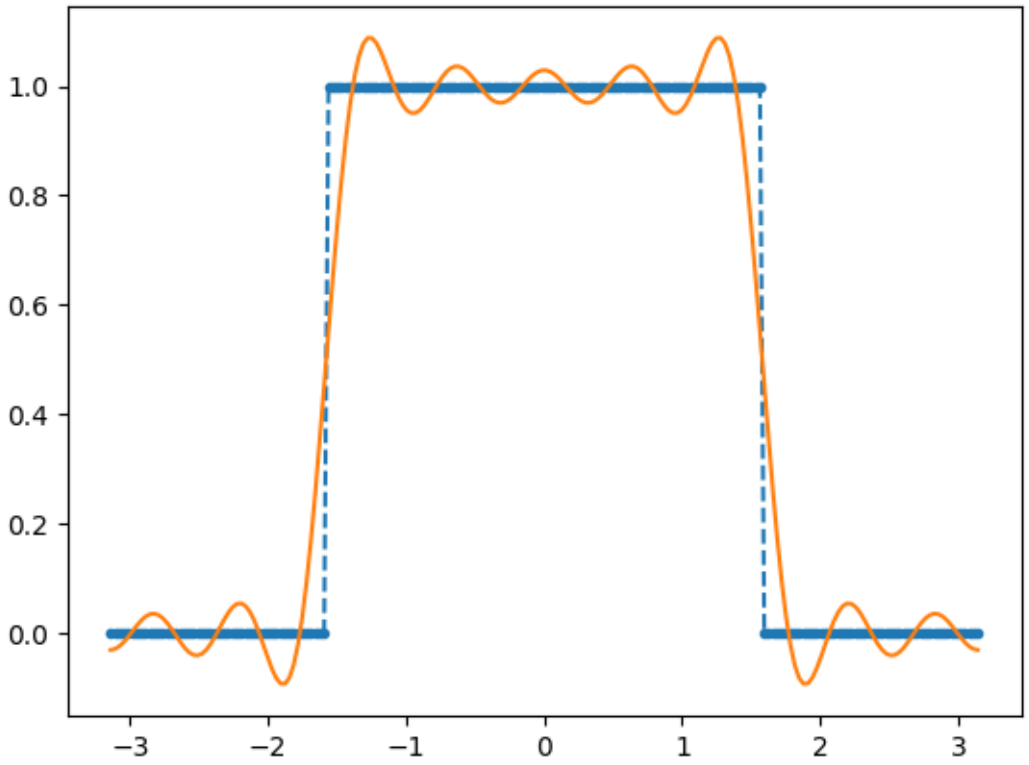
[20]: [<matplotlib.lines.Line2D at 0x3184ef450>]



[21]:
```
# L2 Norm Approximation
K = 10
N = K * 10

T = np.linspace(start, stop, 2*N)
```

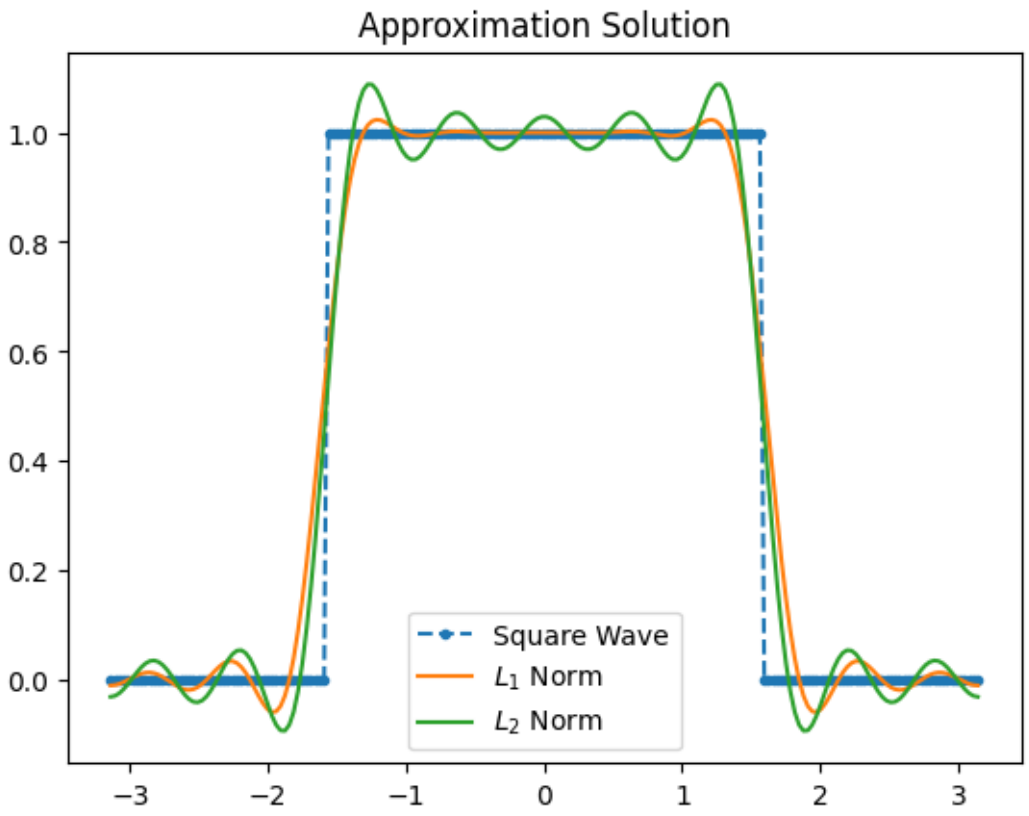
```
a2 = cp.Variable(K+1)

obj = cp.Minimize(
    np.pi/N*cp.norm2(
        cp.vstack(
            [ cp.sum(
                    cp.vstack(
                        [a2[k]*np.cos(k*t) for k in range(K+1)]
                    )
                ) - y(t) for t in T
            ]
        )
    )
)


constraints = []

prob = cp.Problem(obj, constraints)

prob.solve()

plt.plot(T, y(T),'.--')
plt.plot(T, [ np.sum(
                    # cp.vstack(
                        [a2.value[k]*np.cos(k*t) for k in range(K+1)]
                    # )
                ) for t in T
            ])
```

[21]: [<matplotlib.lines.Line2D at 0x17785c050>]



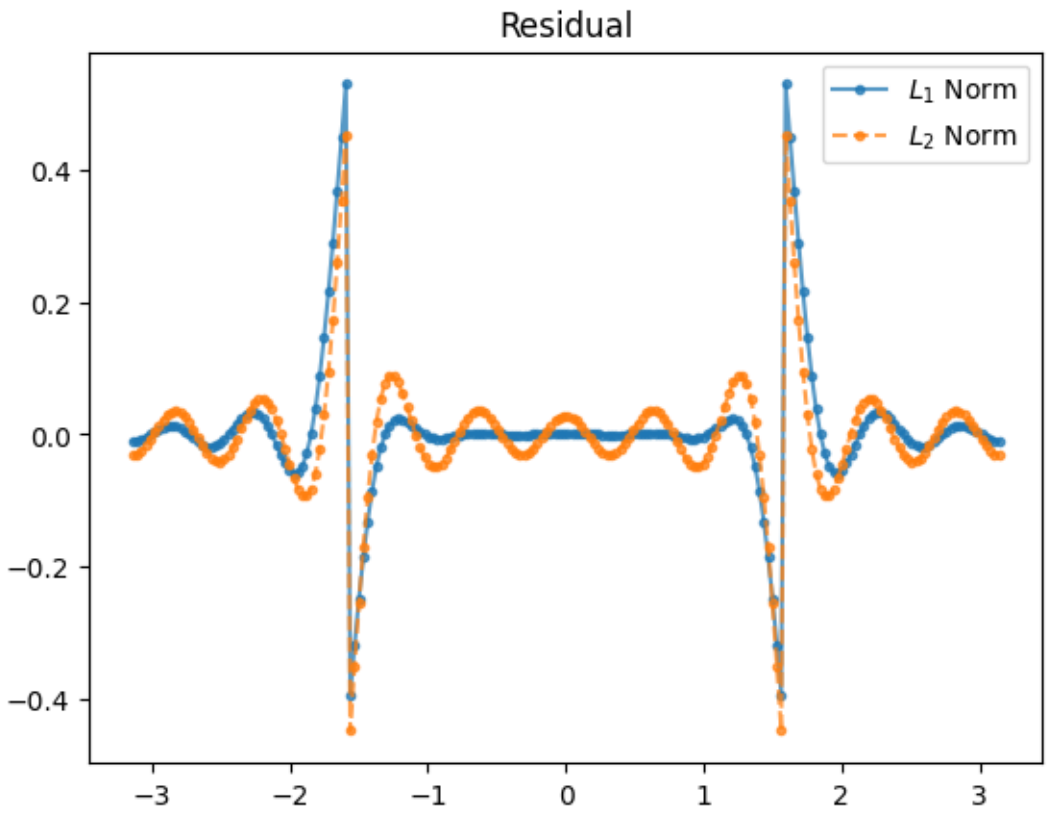
[22]:
```
plt.plot(T, y(T),'.--', label='Square Wave')
plt.plot(T, [ np.sum(
                    # cp.vstack(
                        [a1.value[k]*np.cos(k*t) for k in range(K+1)]
                    # )
                ) for t in T
            ], label='$L_1$ Norm')
plt.plot(T, [ np.sum(
                    # cp.vstack(
                        [a2.value[k]*np.cos(k*t) for k in range(K+1)]
                    # )
                ) for t in T
            ], label='$L_2$ Norm')
plt.legend()
plt.title('Approximation Solution')
```

[22]:
```
Text(0.5, 1.0, 'Approximation Solution')
```



[23]:
```
# plt.plot(T, y(T)-y(T),'.--', label='Square Wave')
plt.plot(T, [ np.sum(
                    # cp.vstack(
                        [a1.value[k]*np.cos(k*t) for k in range(K+1)]
                    # )
                ) - y(t) for t in T
            ],'.-', label='$L_1$ Norm', alpha=3/4)
plt.plot(T, [ np.sum(
                    # cp.vstack(
                        [a2.value[k]*np.cos(k*t) for k in range(K+1)]
                    # )
                ) - y(t) for t in T
            ],'.--', label='$L_2$ Norm', alpha=3/4)
plt.legend()
plt.title('Residual')
```

[23]:
```
Text(0.5, 1.0, 'Residual')
```



- Faster Rise Time for L2 compared to L1
- L1 settle faster than L2

0.0.2 A6.12

We will approximately solve this problem iteratively by the following method: 1. Let the initial objective value take on +inf 2. Make an initial guess that our permutation to be the identity matrix 3. Solve the minimization over x of $||Ax - P^T y||_2$ 4. If the difference between the last and the new objective value is less than 1e-3, we think the solution has converged and exit; otherwise, continue to next steps 5. Based on the optimal x^*, we calculate Ax 6. Find what P^T would reorder y in such a way that Entry i in both Ax and $P^T y$ are ranked the same if we were to sort the two vectors 7. Go to Step 2, solve the Minimization with the new P obtained in Step 6

[8]:
```
np.random.seed(0)
m=100
k=40 # max # permuted measurements
n=20
A=10*np.random.randn(m,n)
x_true=np.random.randn(n,1) # true x value
y_true = A.dot(x_true) + np.random.randn(m,1)
# build permuted indices
```

```
perm_idxs=np.random.permutation(m)
# perm_idxs
perm_idxs=np.sort(perm_idxs[:k])
# perm_idxs
temp_perm=np.random.permutation(k)
# temp_perm
new_pos=np.zeros(k)
for i in range(k):
    new_pos[i] = perm_idxs[temp_perm[i]]
new_pos = new_pos.astype(int)
# true permutation matrix
P=np.identity(m)
P[perm_idxs,:]=P[new_pos,:]
true_perm=[]
for i in range(k):
    # print(perm_idxs[i], new_pos[i])
    if perm_idxs[i] != new_pos[i]:
        true_perm = np.append(true_perm, perm_idxs[i])
y=P.dot(y_true)
# new_pos = None
```

[9]:
```
np.random.seed(1)
x0 = np.random.randn(n)
Ax = A@x0
# Ax.sort(axis=0)

Ax_rank = (Ax).argsort(axis=0).argsort(axis=0)#==y.argsort(axis=0).
 ↪argsort(axis=0)
y_rank = y.argsort(axis=0).argsort(axis=0)
```

[10]:
```
y.shape
```

[10]:
```
(100, 1)
```

[11]:
```
np.random.choice(100, size=100, replace=False)
```

[11]:
```
array([78, 60, 31,  5, 70, 45, 14, 44, 73, 88, 50, 53, 85, 92, 29, 16, 35,
       89, 83, 19, 40, 91, 63, 97, 59, 42, 33, 98, 69, 96, 17, 28, 65, 72,
       81, 66, 18,  4, 47, 39, 36, 30, 20, 51, 48, 75, 77, 86, 11, 54, 79,
       12,  2, 58, 21, 94,  6, 61, 56,  0,  3, 13, 49,  1, 55, 71, 74, 46,
       62, 84, 90, 93, 10, 34, 32,  8, 38, 57, 37, 27, 23, 67,  9, 22, 87,
       95,  7, 25, 99, 64, 15, 82, 41, 80, 52, 26, 76, 43, 24, 68])
```

[12]:
```
import scipy.stats
```

[13]:
```
cost_min = np.inf
x_optimal = None
```

```
P_optimal = None

for i in range(1):

    P_hat = np.eye(m)
    # P_hat = P_hat[np.random.choice(100, size=100, replace=False)]

    x_eye = None

    improvement = np.inf
    cost_last = np.inf

    eps = 1e-3

    while improvement>=eps:

        x = cp.Variable((n,1))

        obj = cp.Minimize(cp.norm2(A@x-P_hat.T@y))

        constraints = []

        prob = cp.Problem(obj, constraints)

        cost = prob.solve()
        improvement = cost_last - cost
        cost_last = cost
        # print(cost)

        x0 = x.value

        # Save the solution to x when we assume P = I
        if (P_hat==np.eye(m)).all():
            x_eye = x0

        Ax = A@x0
        Ax_rank = (Ax).argsort(axis=0).argsort(axis=0)
        y_rank = y.argsort(axis=0).argsort(axis=0)
        # Constrcut a permutation matrix
        P_reorder = []
        for r in Ax_rank:
            i_y, _ = np.where(y_rank==r)

            P_reorder.append(i_y)
            # break

        P_reorder = np.array(P_reorder)
```

```
        # P_reorder.shape

        P_hat = []
        for i_y in P_reorder:
            u_i = np.zeros((m))
            u_i[i_y] = 1

            P_hat.append(u_i)

        P_hat = np.array(P_hat).T
        # break
    print(cost)
    if cost < cost_min:
        cost_min = cost
        x_optimal = x0
        P_optimal = P_hat
```

```
35.272189635474625
```

[14]:
```
x_eye
```

[14]:
```
array([[-1.11507836],
       [-1.40642658],
       [ 0.81540648],
       [-0.74677862],
       [-0.37023583],
       [-0.24322015],
       [-0.74092514],
       [-0.44079927],
       [ 0.31571575],
       [ 0.09169404],
       [ 0.35551171],
       [-0.08121674],
       [-0.0261098 ],
       [ 0.51482702],
       [ 0.42091395],
       [-0.00784192],
       [ 0.41527582],
       [ 0.70532982],
       [-2.07503966],
       [ 1.04241053]])
```

[15]:
```
cost_min,x_optimal
```

[15]:
```
(35.272189635474625,
 array([[-1.6293334 ],
        [-2.35203072],
        [ 1.43385102],
```

```
       [-0.75241133],
       [-0.3539387 ],
       [-0.24034968],
       [-1.02804586],
       [-0.47251534],
       [ 0.93358932],
       [ 0.40094005],
       [ 0.50242688],
       [ 0.04686509],
       [ 0.28244187],
       [ 0.40752456],
       [ 0.43346313],
       [-0.46582331],
       [ 0.88061023],
       [ 1.00649178],
       [-2.76513174],
       [ 1.17787041]]))
```

[16]:
```
np.linalg.norm(x_optimal-x_true)
```

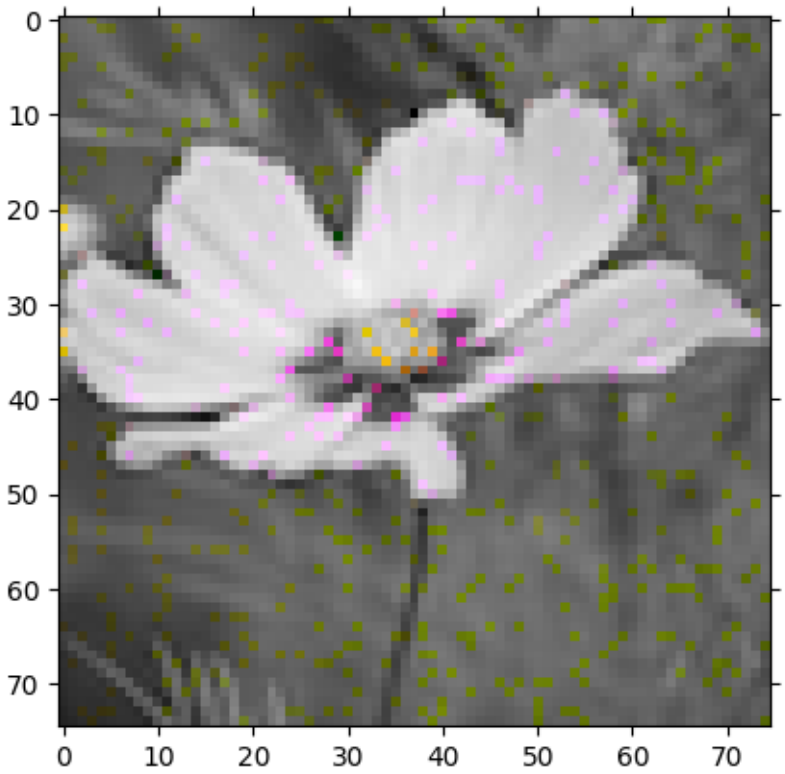
[16]:
```
2.280120529425616
```

[17]:
```
x_true
```

[17]:
```
array([[-1.53292105],
       [-1.71197016],
       [ 0.04613506],
       [-0.95837448],
       [-0.08081161],
       [-0.70385904],
       [-0.7707843 ],
       [-0.48084534],
       [ 0.70358555],
       [ 0.92914515],
       [ 0.37117255],
       [-0.98982255],
       [ 0.64363128],
       [ 0.68889667],
       [ 0.2746472 ],
       [-0.60362044],
       [ 0.70885958],
       [ 0.42281857],
       [-3.11685659],
       [ 0.64445203]])
```

0.0.3 A6.19

[2]:
```
# Setting up the Problem
img = mpimg.imread("../figures/flower.png")
img = img[:,:,0:3]
m,n,_ = img.shape

np.random.seed(5)
known_ind = np.where(np.random.rand(m,n) >= 0.90)
# grayscale image
M = 0.299*img[:,:,0]+0.587*img[:,:,1]+0.114*img[:,:,2]
# known color values
R_known = img[:,:,0]
G_known = img[:,:,1]
B_known = img[:,:,2]
R_known = R_known[known_ind]
G_known = G_known[known_ind]
B_known = B_known[known_ind]

def save_img(filename, R,G,B):
  img = np.stack((np.array(R),np.array(G),np.array(B)), axis=2)
  # turn off ticks and labels of the figure
  plt.tick_params(
    axis='both', which='both', labelleft='off', labelbottom='off',
    bottom='off', top='off', right='off', left='off'
  )
  fig = plt.imshow(img)
  plt.savefig(filename,bbox_inches='tight',pad_inches=0.)

R_given = np.copy(M);
R_given[known_ind] = R_known;
G_given = np.copy(M);
G_given[known_ind] = G_known;
B_given = np.copy(M);
B_given[known_ind] = B_known;
save_img("flower_given.png", R_given, G_given, B_given)
```



[3]:
```
%%time
R_act = cp.Variable((75, 75))
G_act = cp.Variable((75, 75))
B_act = cp.Variable((75, 75))

obj = cp.Minimize(
    cp.tv(*[R_act, G_act, B_act])
)
# +5*cp.norm1(M-0.299*R_act-0.587*G_act-0.114*B_act) # Add Constraint on␣
 ↪Monochrome Reconstruction as penalty

constraints = []
# Constraints on known pixel
constraints += [R_act[known_ind]==R_given[known_ind]]
constraints += [G_act[known_ind]==G_given[known_ind]]
constraints += [B_act[known_ind]==B_given[known_ind]]
# Constraint on the range of value
constraints += [R_act[i, j]>=0 for i in range(75) for j in range(75)]
constraints += [R_act[i, j]<=1 for i in range(75) for j in range(75)]
constraints += [G_act[i, j]>=0 for i in range(75) for j in range(75)]
constraints += [G_act[i, j]<=1 for i in range(75) for j in range(75)]
```

```
constraints += [B_act[i, j]>=0 for i in range(75) for j in range(75)]
constraints += [B_act[i, j]<=1 for i in range(75) for j in range(75)]
# Constraint on Monochrome Reconstruction
constraints += [M-0.299*R_act-0.587*G_act-0.114*B_act==0]

prob = cp.Problem(obj, constraints)

prob.solve('SCS')
```

```
CPU times: user 18.8 s, sys: 293 ms, total: 19.1 s
Wall time: 19.2 s
```

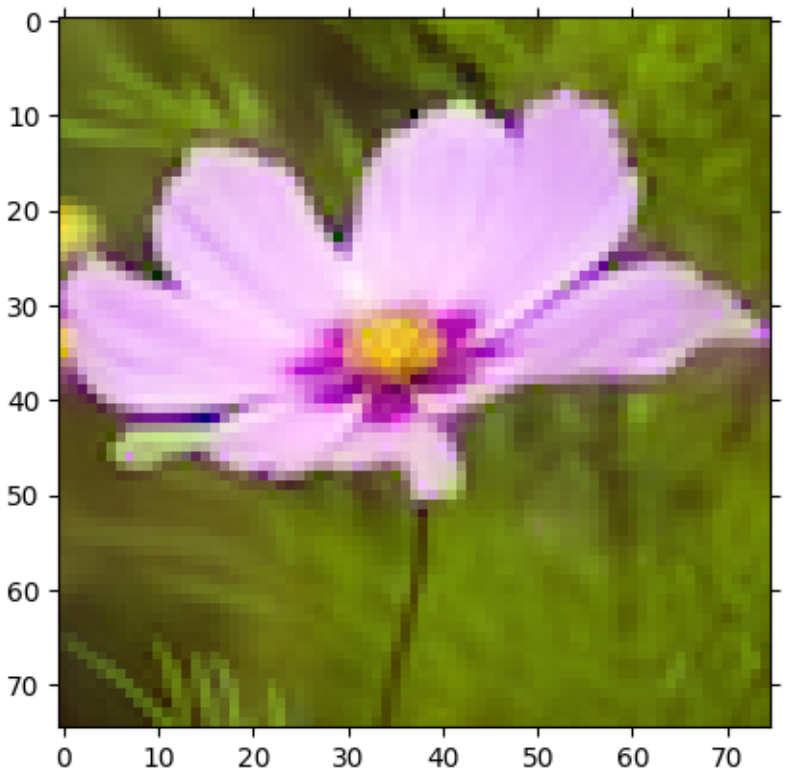
[3]:
```
620.7815937543677
```

[4]:
```
prob.is_dcp()
```

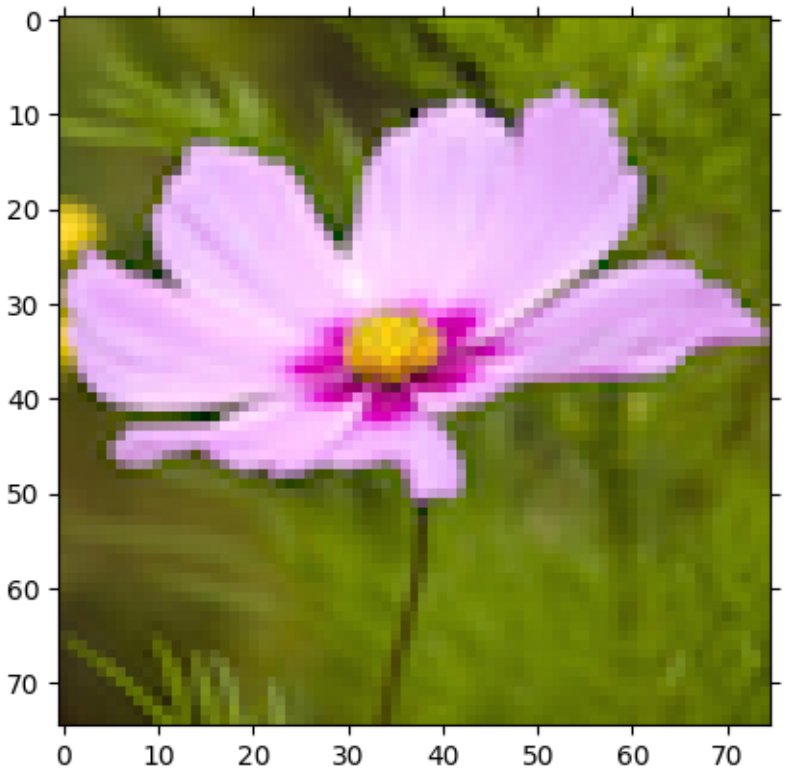
[4]:
```
True
```

[5]:
```
save_img("flower_minimal_tv.png", R_act.value, G_act.value, B_act.value)
```

```
Clipping input data to the valid range for imshow with RGB data ([0..1] for
floats or [0..255] for integers).
```



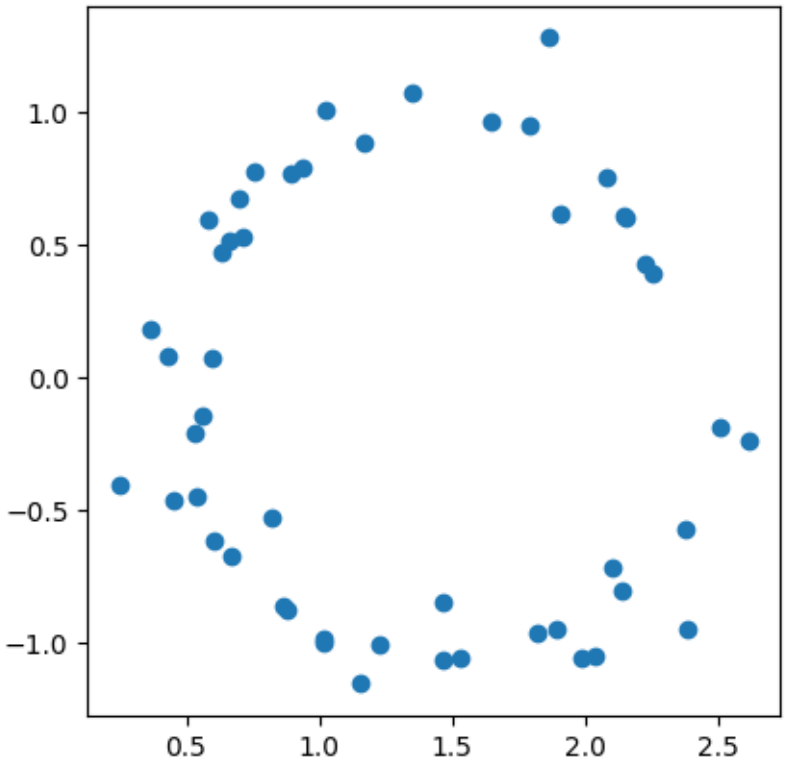
[44]:
```
save_img("flower_original.png", img[:,:,0], img[:,:,1], img[:,:,2])
```



0.0.4 A8.16

[18]:
```
n = 2
m = 50
```

```
U = np.array([[1.824183228637652032e+00, 1.349093690455489103e+00, 6.
 ↪966316403935147727e-01, 7.599387854623529392e-01, 2.388321695850912363e+00,␣
 ↪8.651370608981923116e-01, 1.863922545015865406e+00, 7.
 ↪099743941474848663e-01, 6.005484882320809570e-01, 4.561429569892232472e-01,␣
 ↪5.328296545713475663e-01, 2.138547819234526415e+00, 1.
 ↪906676474276197464e+00, 1.015547309536922516e+00, 8.765948388006337133e-01,␣
 ↪1.648147347399247842e+00, 1.027902202451572045e+00, 2.
 ↪145586297520478691e+00, 1.793440421753045744e+00, 1.020535583041398908e+00,␣
 ↪8.977911075271942654e-01, 1.530480229262339398e+00, 2.
 ↪478088034137528872e-01, 2.617415807793897820e+00, 2.081978553098443374e+00,␣
 ↪1.891226687205936452e+00, 8.222497927065576251e-01, 5.
 ↪803514604868882376e-01, 1.158670193449639063e+00, 6.016685032455900695e-01,␣
 ↪5.605410828151705660e-01, 2.508815467550573164e+00, 2.
 ↪230201413385580977e+00, 1.170848897912992514e+00, 2.256355929901105561e+00,␣
 ↪6.686991510936428629e-01, 2.040269595792217672e+00, 3.
 ↪634166812924328749e-01, 5.418647611079159265e-01, 6.631470058399455692e-01,␣
 ↪4.286142597532469622e-01, 2.155925078996823618e+00, 2.
 ↪379380016960549682e+00, 6.343212414048013947e-01, 1.469076407947448981e+00,␣
 ↪1.225322035289937439e+00, 1.467602887401966871e+00, 9.
 ↪345319187253748883e-01, 1.985592768641736505e+00, 2.106896115090134636e+00],
[-9.644136284187876385e-01, 1.069547315003422927e+00, 6.733229334437943470e-01,␣
 ↪7.788072961810316164e-01, -9.467465278344706636e-01, -8.
 ↪591303443863639311e-01, 1.279527420871080956e+00, 5.314829019311283487e-01,␣
 ↪6.975676079749143499e-02, -4.641873429414754559e-01, -2.
 ↪094571396598311763e-01, -8.003479827938377866e-01, 6.135280782546607137e-01,␣
 ↪-9.961307468791747999e-01, -8.765215480412106297e-01, 9.
 ↪655406812422813179e-01, 1.011230180540185541e+00, 6.105416770440197372e-01,␣
 ↪9.486552370654932620e-01, -9.863592657836954825e-01, 7.
 ↪695327845100754516e-01, -1.060072365810699413e+00, -4.
 ↪041043465424410952e-01, -2.352952920283236105e-01, 7.560391050507236921e-01,␣
 ↪-9.454246095204003053e-01, -5.303145312191936966e-01, 5.
 ↪979590038743245461e-01, -1.154309511133019717e+00, -6.
 ↪123184171955468047e-01, -1.464683782538583889e-01, -1.
 ↪839128688968104386e-01, 4.250070477845909744e-01, 8.861864983476224200e-01,␣
 ↪3.927648421593328276e-01, -6.726102374256350824e-01, -1.
 ↪047252884197514833e+00, 1.825096825995130845e-01, -4.482373962742914886e-01,␣
 ↪5.115625649313135792e-01, 7.846201103116770548e-02, 6.
 ↪006325432819290544e-01, -5.710733714464664157e-01, 4.725559971890586075e-01,␣
 ↪-8.440290321502940118e-01, -1.003920890712479475e+00, -1.
 ↪067089412136528637e+00, 7.909281966910661765e-01, -1.059509163675931065e+00,␣
 ↪-7.136351632325785843e-01]
])

fig,ax =plt.subplots()
ax.scatter(U[0,:],U[1,:])
ax.set_aspect('equal')
```



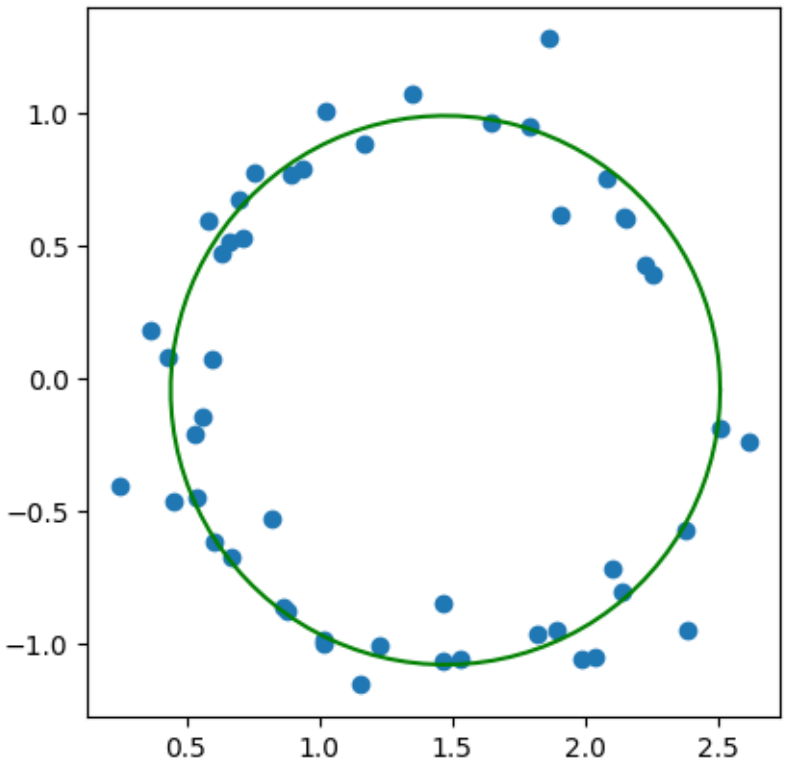
[20]:
```
c = cp.Variable(n)
t = cp.Variable()
# r = cp.Variable()

obj = cp.Minimize(
    cp.sum(
        [
            cp.square( U[0,i]**2+U[1,i]**2-2*U[0,i]*c[0]-2*U[1,i]*c[1] - t )
            for i in range(m)
        ]
    )
)

constraints = []

prob = cp.Problem(obj, constraints)

prob.solve()
```

[20]: 2.9038284259769465

[21]:
```
c.value, t.value
```

[21]: (array([1.47590677, -0.04404696]), array(-1.11063708))

[24]:
```
# Recover R through change of variable
r = np.sqrt(t.value + np.linalg.norm(c.value, 2)**2)

 = np.linspace(0, 2*np.pi, 100)
circle_fit = np.array([r*np.cos( ), r*np.sin( )]).T + c.value

fig,ax =plt.subplots()
ax.scatter(U[0,:],U[1,:])
ax.plot(circle_fit[:,0],circle_fit[:,1],'g-')
ax.set_aspect('equal')
```



0.0.5 A17.4

[22]:
```
x = np.array([[0.1, 0.2, -0.05, 0.1]]).T

Σ = cp.Variable((4,4))

obj = cp.Maximize(x.T@Σ@x)

constraints = []
```

```
constraints += [Σ[0,0]==0.2]
constraints += [Σ[1,1]==0.1]
constraints += [Σ[2,2]==0.3]
constraints += [Σ[3,3]==0.1]
constraints += [Σ[0,1]>=0]
constraints += [Σ[0,2]>=0]
constraints += [Σ[1,0]>=0]
constraints += [Σ[2,0]>=0]
constraints += [Σ[2,3]>=0]
constraints += [Σ[3,2]>=0]
constraints += [Σ[1,2]<=0]
constraints += [Σ[1,3]<=0]
constraints += [Σ[2,1]<=0]
constraints += [Σ[3,1]<=0]
constraints += [Σ>>0]

prob = cp.Problem(obj, constraints)

print(" _wc value: ", prob.solve())
```

```
_wc value:  0.015166264279157526
```

[18]:
```
# Worst Case Covariance Matrix
Σ.value
```

[18]:
```
array([[ 2.00000000e-01,  9.30113522e-02, -3.38481035e-07,
         8.36591757e-02],
       [ 9.30113522e-02,  1.00000002e-01, -1.10631305e-01,
        -3.79171057e-07],
       [-3.38481035e-07, -1.10631305e-01,  3.00000000e-01,
         1.89987725e-02],
       [ 8.36591757e-02, -3.79171056e-07,  1.89987725e-02,
         9.99999991e-02]])
```

[19]:
```
# Diagonal Covariance Matrix
x.T@np.diag([0.2,0.1,0.3,0.1])@x
```

[19]:
```
array([[0.00775]])
```

[24]:
```
!export PATH=/Library/TeX/texbin:$PATH
```

[ ]:
```

```