More solving trigonometric equations

We learned that solving trigonometric equation involves an . . .

algebra part . . .

and

. . . **a trigonometry part**

Let's practice these again with another problem.

Solve:

$$\sin x + \sqrt{2} = -\sin x$$

First we need to do algebra . . . . . . to solve for the $\sin x$.

$$\sin x + \sqrt{2} = -\sin x$$

$$2\sin x + \sqrt{2} = 0$$

$$2\sin x = -\sqrt{2}$$

$$\sin x = -\frac{\sqrt{2}}{2}$$

We have taken the algebra route as far as it can go!

Now we need to figure out

$$\sin x = -\frac{\sqrt{2}}{2}$$

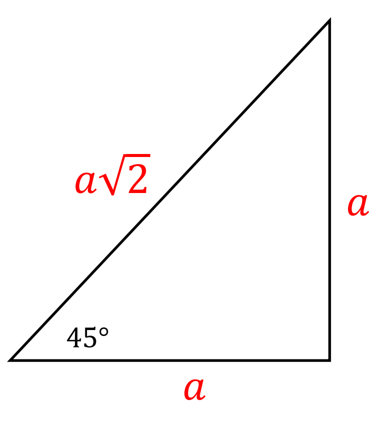
In this case, x is an angle. What angle(s) have a sine equal to $-\frac{\sqrt{2}}{2}$?

We are trying to figure out

$$\sin\theta = \frac{y}{r} = -\frac{\sqrt{2}}{2}$$

Can you think of a triangle that has sides of that ratio?

We definitely saw a triangle like that:



$$\frac{a}{a\sqrt{2}} = \frac{1}{\sqrt{2}} = \frac{\sqrt{2}}{2}$$

Note: here it's important to see that when you rationalize the denominator,

$$\frac{1}{\sqrt{2}} = \frac{\sqrt{2}}{2}$$

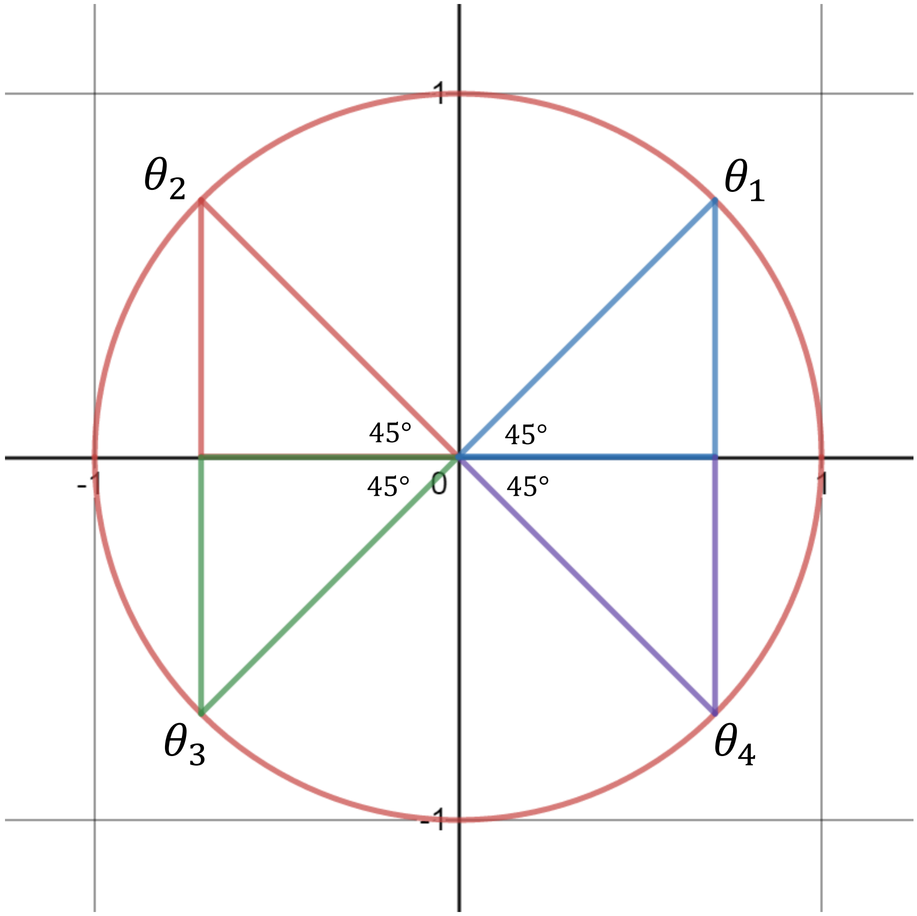
If we used the reference angle method to learn our trigonometric functions, we might have also memorized this ratio in the table:

θ	$0°$	$30°$	$\mathbf{45°}$	$60°$	$90°$	$180°$	$270°$
$\sin(\theta)$	0	$\frac{1}{2}$	$\mathbf{\frac{1}{\sqrt{2}}}$	$\frac{\sqrt{3}}{2}$	1	0	-1
$\cos(\theta)$	1	$\frac{\sqrt{3}}{2}$	$\frac{1}{\sqrt{2}}$	$\frac{1}{2}$	0	-1	0
$\tan(\theta)$	0	$\frac{\sqrt{3}}{3}$	1	$\sqrt{3}$	$undef.$	0	$undef.$

So we know we are dealing with a 45-45-90 triangle!!

But where in the unit circle will this show up?

Remember, there are **four** 45-45-90 triangles in the unit circle:



The question is . . .

Which of these angles has a sine of

$$-\frac{\sqrt{2}}{2} \quad ?$$

Well, we know that

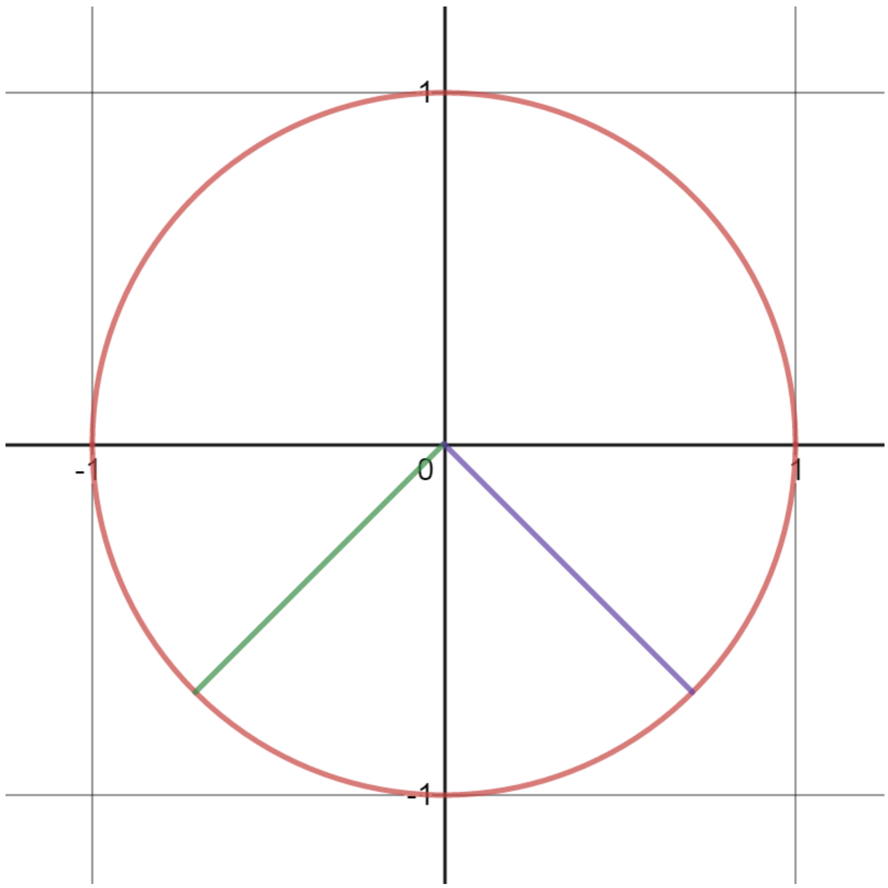
$$\sin\theta = \frac{y}{r}$$

and r is never negative . . .

So if the sine is negative . . .

y must be negative!

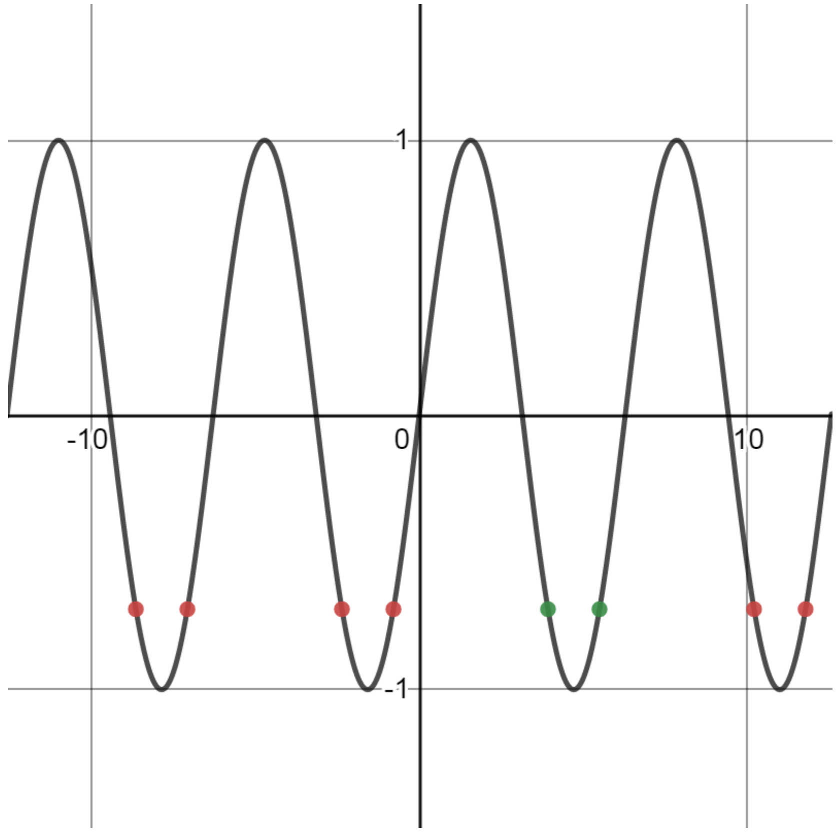
So we are looking at these two angles:



Which are $225°$ and $315°$, or $\frac{5\pi}{4}$ and $\frac{7\pi}{4}$

But every complete cycle, the sine returns to the same place:

$y = \sin\theta$



As with the previous example, all of these other solutions are . . .

. . . one period away . . .

. . . two periods away . . .

. . . negative six periods away . . .

. . . and so on.

Therefore our solution to this equation is

$$\frac{5\pi}{4} + 2\pi * k$$

$$\frac{7\pi}{4} + 2\pi * k$$

One last problem!!!

Solve on $[0,2\pi)$:

$$2\sin^2\theta + 3\cos\theta - 3 = 0$$

Okay, so first notice something about the way this problem is asked:

Solve on $[\mathbf{0}, \mathbf{2\pi})$

This means we don't have to find the **general solution** . . .

. . . so we ***don't*** have to add $2\pi * k$!

We just need to find any angle in between 0° and 360° . . .

. . . or between 0 and 2π radians.

Okay, now to the solving part:

$$2\sin^2\theta + 3\cos\theta - 3 = 0$$

This one seems pretty tricky . . . we have both sine **and** cosine!!

How are we going to get the angle . . .

. . . when every angle . . .

. . . has different sines and cosines?

Maybe we can do something algebraically????

We did learn some things about the **squares** of trigonometric functions . . .

Remember this?

$$\sin^2\theta + \cos^2\theta = 1$$

This was called the **Pythagorean Identity** . . .

. . . and we used it to convert between sine and cosine!

In particular, we found that

$$\sin^2\theta = 1 - \cos^2\theta$$

We can substitute this into our equation!

Instead of

$$2\sin^2\theta + 3\cos\theta - 3 = 0$$

We get

$$2(1 - \cos^2\theta) + 3\cos\theta - 3 = 0$$

Which if we multiply out gives us

$$2 - 2\cos^2\theta + 3\cos\theta - 3 = 0$$

$$-2\cos^2\theta + 3\cos\theta - 1 = 0$$

And so we can start with a positive coefficient, let's multiply by -1:

$$2\cos^2\theta - 3\cos\theta + 1 = 0$$

Whew! Now at least we have the equation **only in cosine**!

But we still have work to do . . .

We need to solve for $\boldsymbol{\cos x}$ . . .

But we have both $\boldsymbol{\cos\theta}$ and $\boldsymbol{\cos^2\theta}$ . . . in the same equation.

Does that seem familiar???

In algebra, we sometimes had x and $\boldsymbol{x^2}$ . . . in the same equation.

In other words . . .

The equation

$$2\cos^2\theta - 3\cos\theta + 1 = 0$$

looks a bit like

$$2x^2 - 3x + 1 = 0$$

Which we know how to solve . . .

$$2x^2 - 3x + 1 = 0$$

$$(2x - 1)(x - 1) = 0$$

$$2x - 1 = 0 \quad \text{or} \quad x - 1 = 0$$

$$x = \frac{1}{2} \quad \text{or} \quad x = 1$$

. . . by factoring!!!

Let's use the same method to solve our equation:

$$2\cos^2\theta - 3\cos\theta + 1 = 0$$

This equation can factor in exactly the same way . . .

. . . if you think of $\cos\theta$ as x:

$$2\cos^2\theta - 3\cos\theta + 1 = 0$$

can factor into

$$(2\cos\theta - 1)(\cos\theta - 1) = 0$$

which gives us two solutions for $\cos\theta$:

$$2\cos\theta - 1 = 0 \quad \text{and} \quad \cos\theta - 1 = 0$$

becoming

$$\cos\theta = \tfrac{1}{2} \quad \text{and} \quad \cos\theta = 1$$

We're almost there!!!!

We need to look at these equations separately . . .

. . . so first let's look at

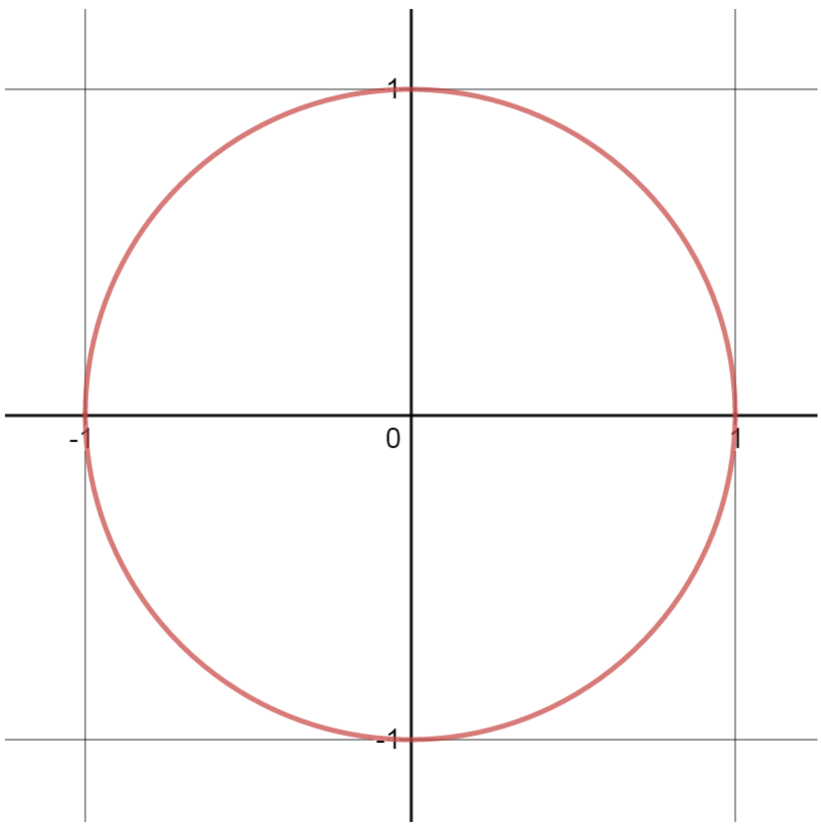
$$\cos\theta = \frac{1}{2}$$

We need to find any and all angles θ . . .

(. . . where $0 \leq \theta < 2\pi$ . . .)

. . . such that $\cos\theta = \frac{1}{2}$

Let's go back to our unit circle:



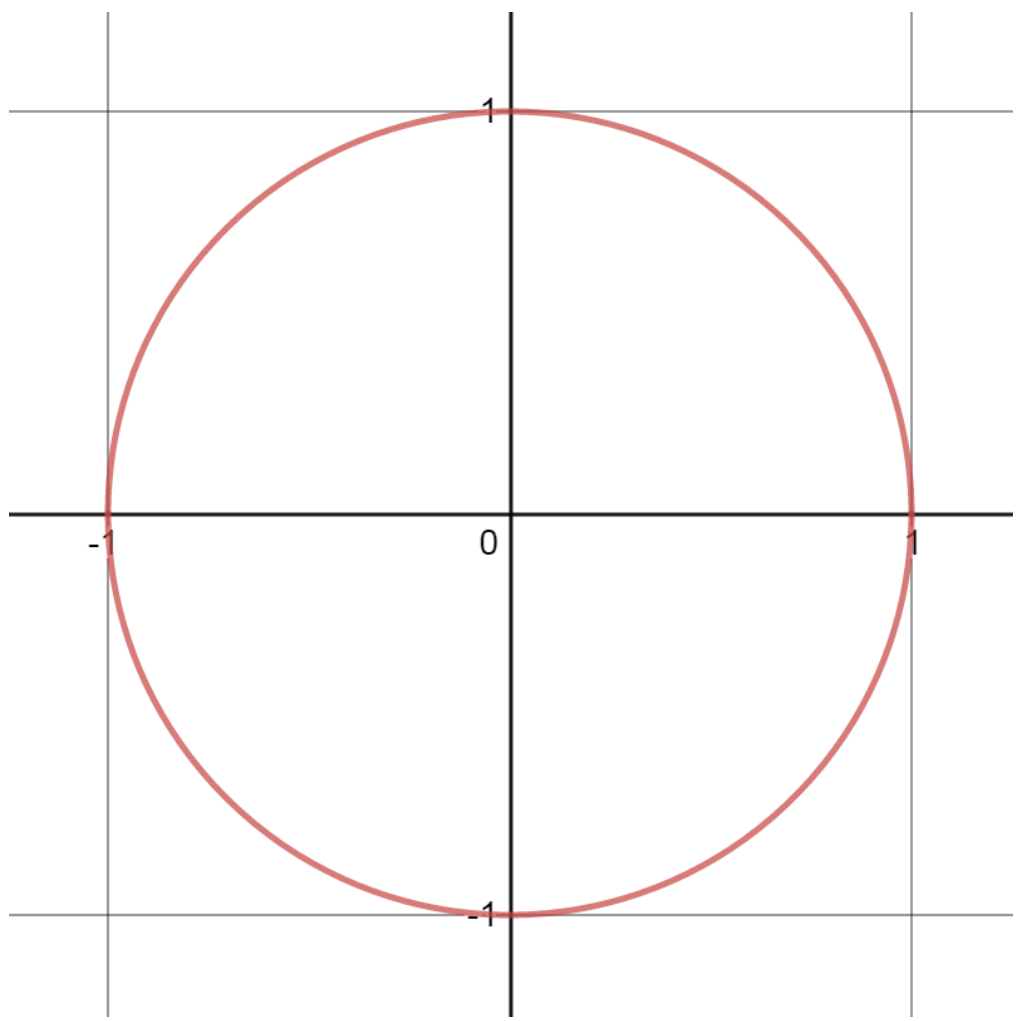
And see if we can figure out which angles give a cosine of $\frac{1}{2}$

Well here's one thing . . . since $\frac{1}{2}$ is positive . . .

. . . and $\cos\theta = \frac{x}{r}$ . . .

. . . we must have that x is positive!

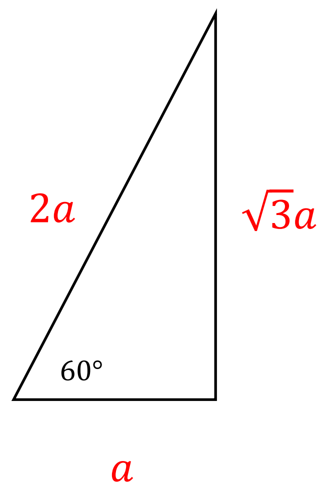
So we are in Quadrant I or Quadrant IV:



Now let’s think about what sort of triangle would have sides in the ratio of:

$$\cos\theta = \frac{x}{r} = \frac{1}{2}$$

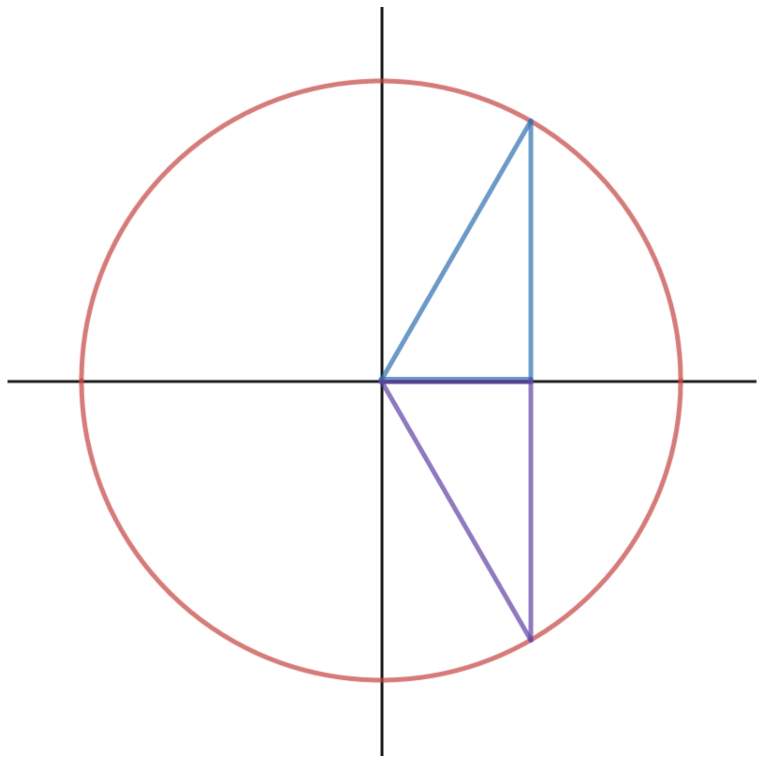
There is one special triangle in particular that comes to mind:



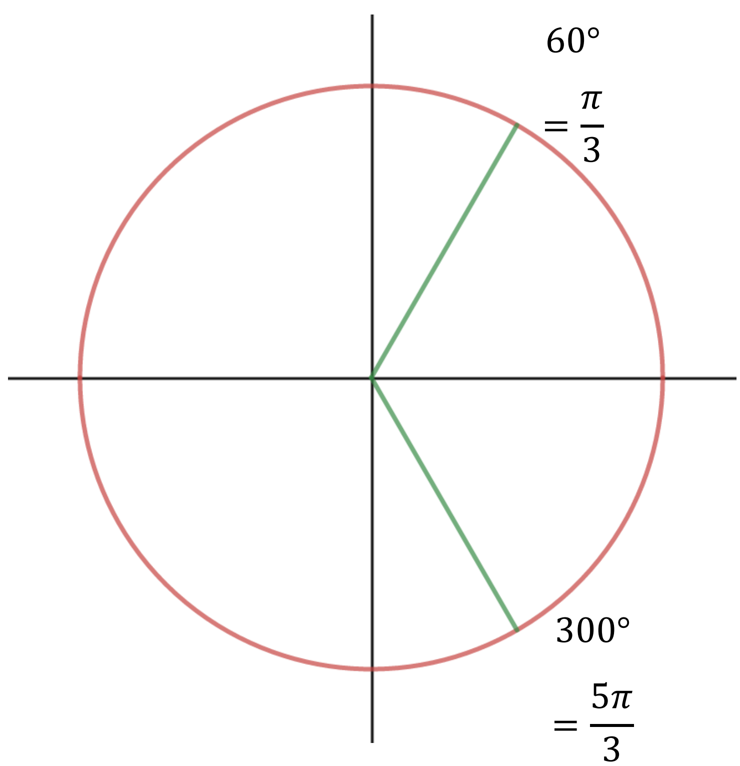
Once again, we would have

$$\cos 60° = \frac{x}{r} = \frac{a}{2a} = \frac{1}{2}$$

These proportions show up twice in the unit circle, in Quadrant I and IV:



Or if we just look at the angles themselves:



We might have also figured this out from our table of reference angles:

θ	$0°$	$30°$	$45°$	$60°$	$90°$	$180°$	$270°$
$\sin(\theta)$	0	$\frac{1}{2}$	$\frac{1}{\sqrt{2}}$	$\frac{\sqrt{3}}{2}$	1	0	-1
$\cos(\theta)$	1	$\frac{\sqrt{3}}{2}$	$\frac{1}{\sqrt{2}}$	$\boldsymbol{\frac{1}{2}}$	0	-1	0
$\tan(\theta)$	0	$\frac{\sqrt{3}}{3}$	1	$\sqrt{3}$	$undef.$	0	$undef.$

Here, again, we would need to understand reference angles enough to know that if

$$\cos 60° = \frac{1}{2}$$

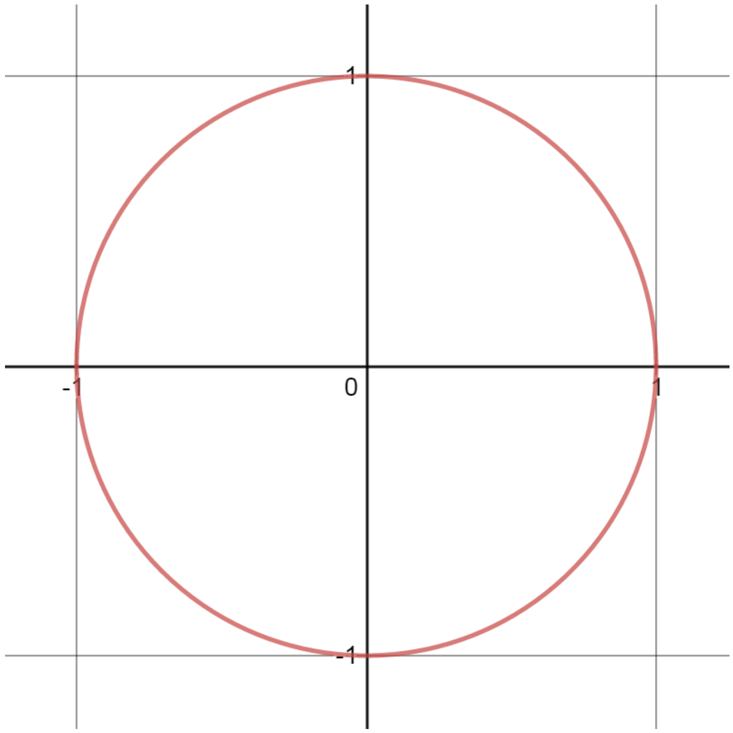
Then the reference angle associated with $60°$, $300°$ also has a cosine of $\frac{1}{2}$.

We are not done!

We also have to solve:

$$\cos\theta = 1$$

Here, we can also use our unit circle



And try to think where

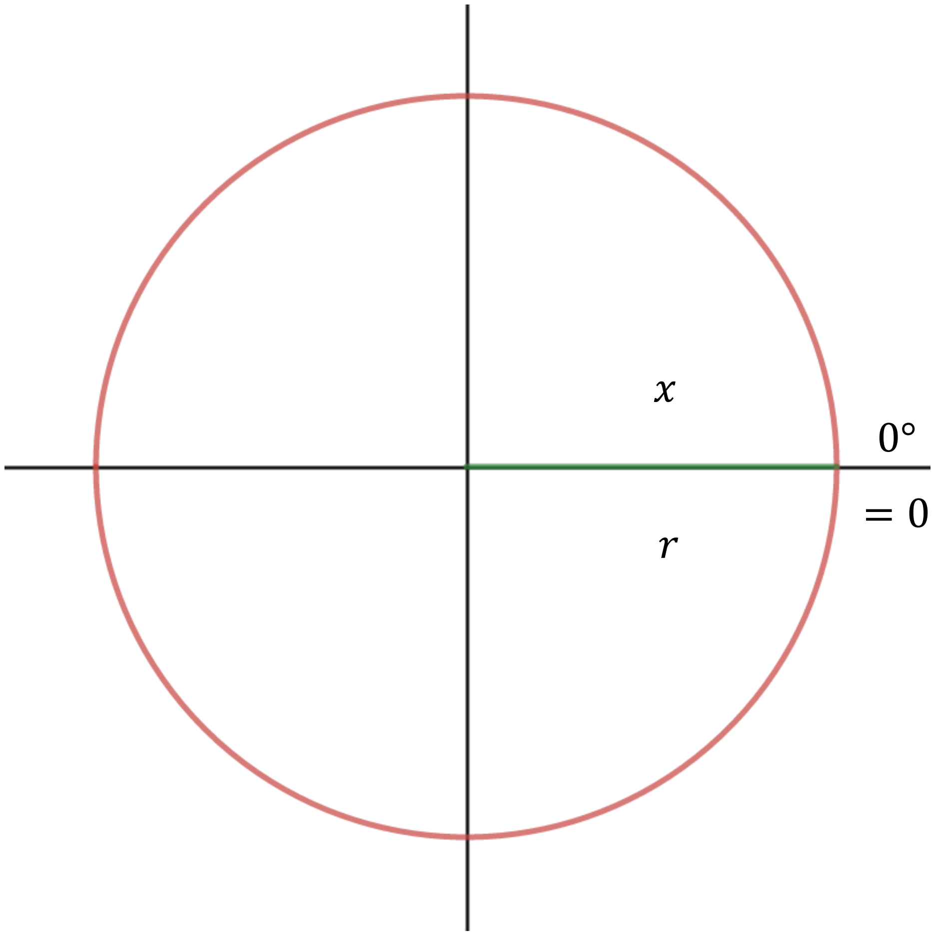
$$\cos\theta = \frac{x}{r} = 1$$

Hey! This would only happen if

$$x = r$$

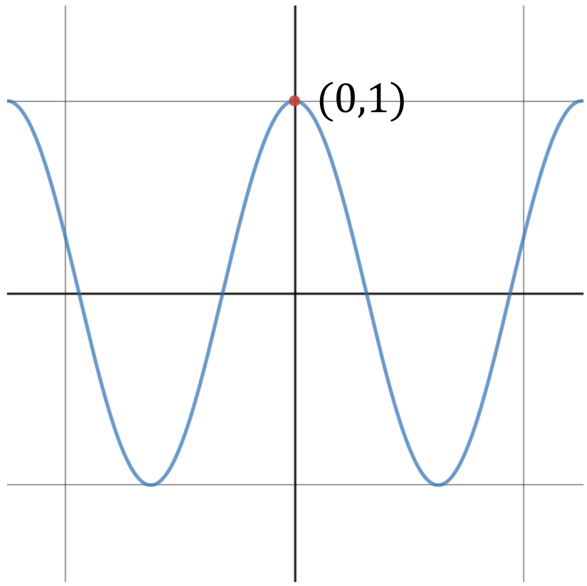
Can you think of a point on the unit circle where $x = r$?

It would only happen here:



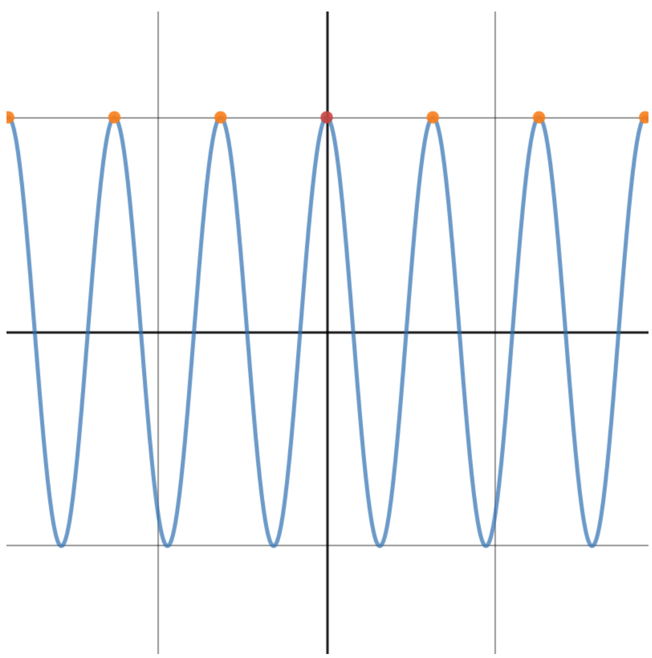
$$\cos 0° = \cos 0 = \frac{x}{r} = \frac{r}{r} = 1$$

We also know this from the graph of $f(\theta) = \cos\theta$:



Here we see that $\cos 0 = 1$.

Again, the cosine function becomes equal to 1 at many other angles . . .



. . . but none of them are within the interval $[0,2\pi)$ . . .

. . . which was the range of solutions that the original problem asked for.

Hence our final answer to the problem:

Solve on $[0,2\pi)$:

$$2\sin^2\theta + 3\cos\theta - 3 = 0$$

is the set of three angles,

$$\left\{0, \frac{\pi}{3}, \frac{5\pi}{3}\right\}$$