Period and Amplitude of Sine and Cosine

We are going to graph some variations of the sine and cosine functions.

For example, consider the following function:

$$f(\theta) = 2\sin(\theta)$$

What is its graph?

To figure this out, remember our work on . . .

transformations of functions

Remember, for example, what happened when we tried to graph

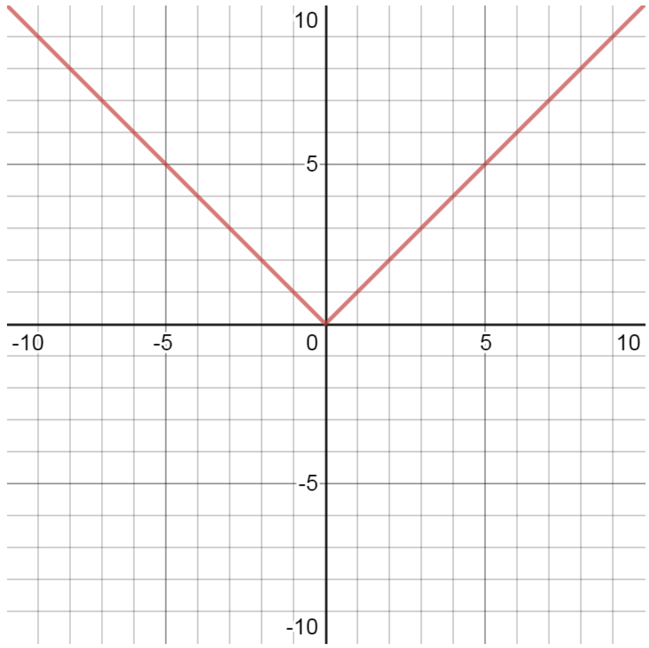
$$h(x) = 2|x| \text{ ?}$$

We decided that the constant multiplier of the basic function $y = |x|$ resulted in a

vertical stretch

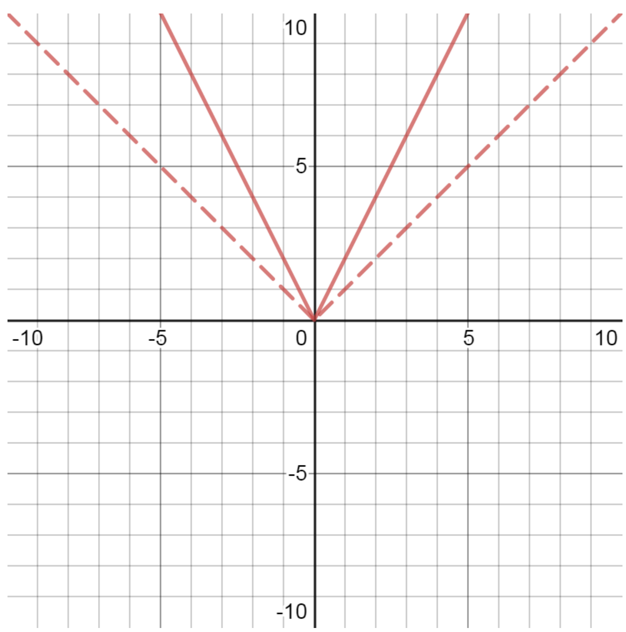
where the graph of $y = |x|$:

$y = |x|$



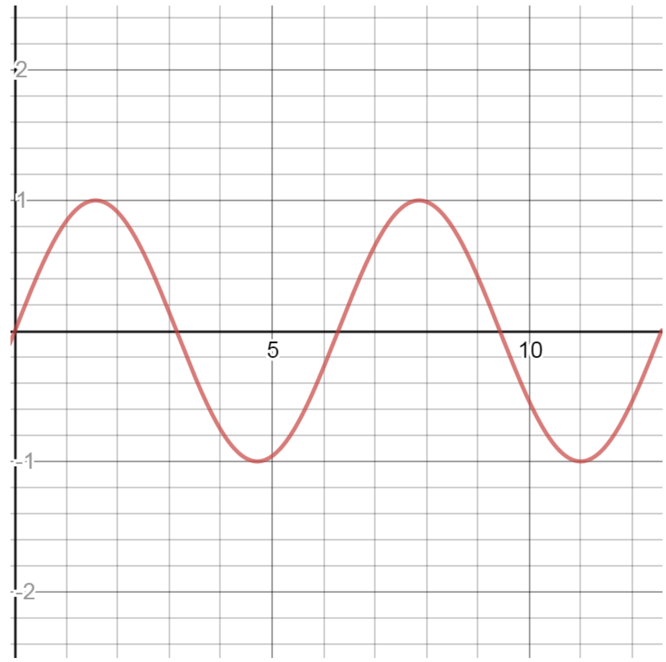
Became stretched upward away from the x-axis when multiplied by 2:

$y = 2|x|$



So to graph $f(\theta) = 2\sin(\theta)$ We start with our standard graph of sine:

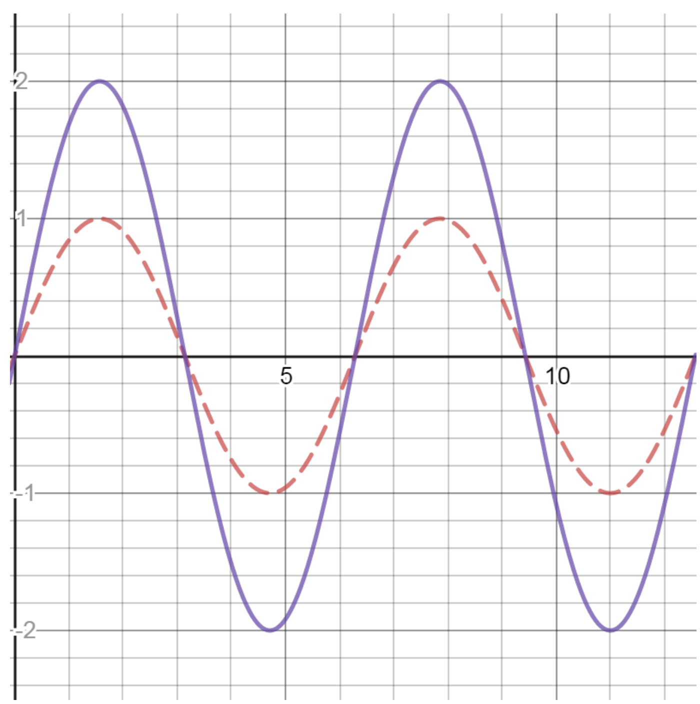
$f(\theta) = \sin(\theta)$



And then apply a **vertical stretch**:

$f(\theta) = 2\sin(\theta)$

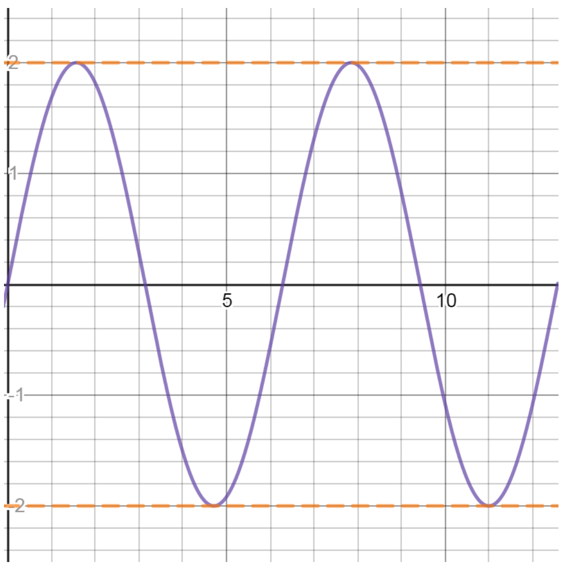
$f(\theta) = \sin(\theta)$

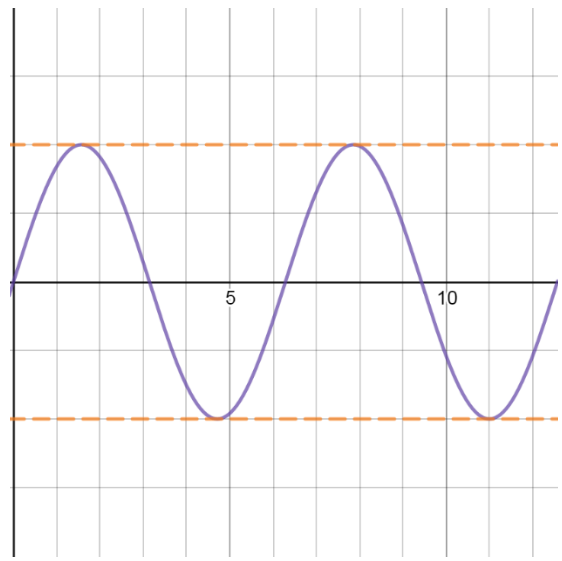


Note:

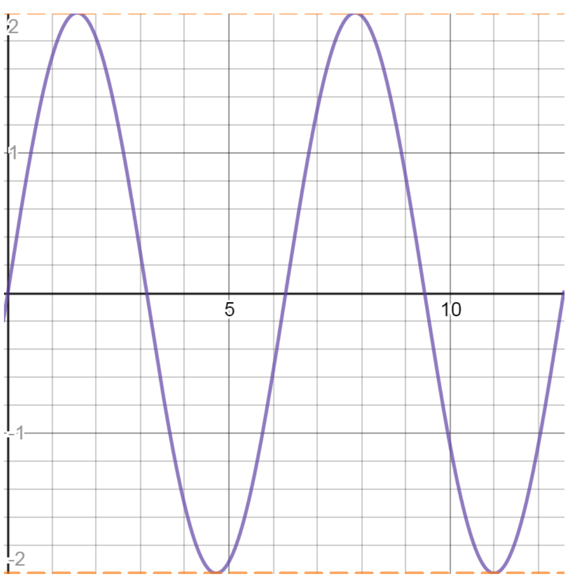
*The visual effect of the vertical stretch depends on the **scaling**!*

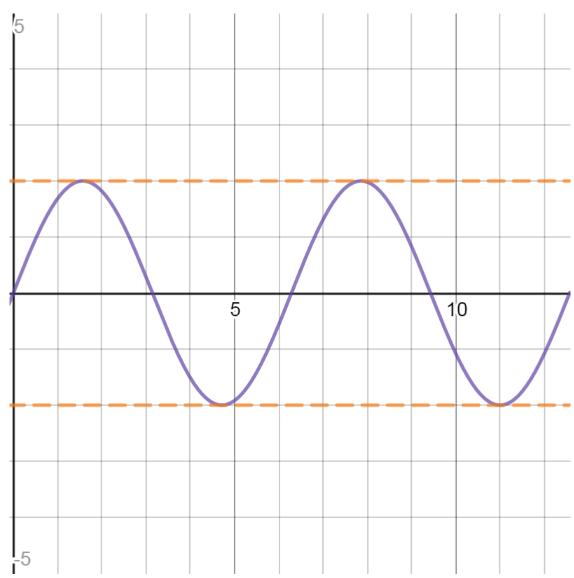
We could draw the graph of $f(\theta) = 2\sin(\theta)$ in several ways:





All of these are correct!





The point is that they all show the maximum and minimum of the graph at

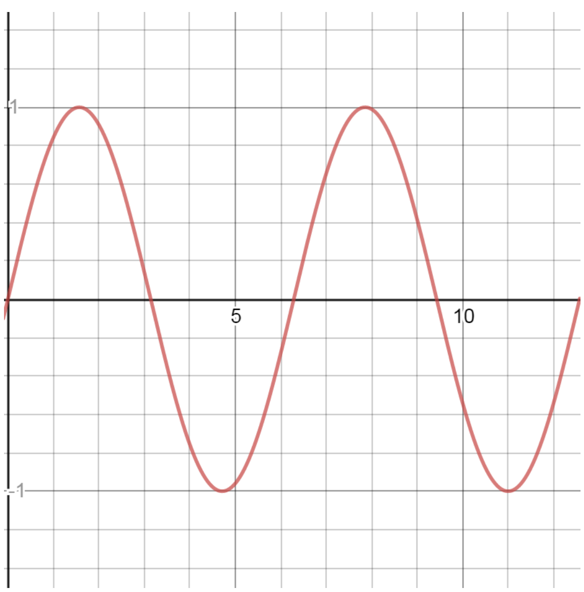
$$\pm 2$$

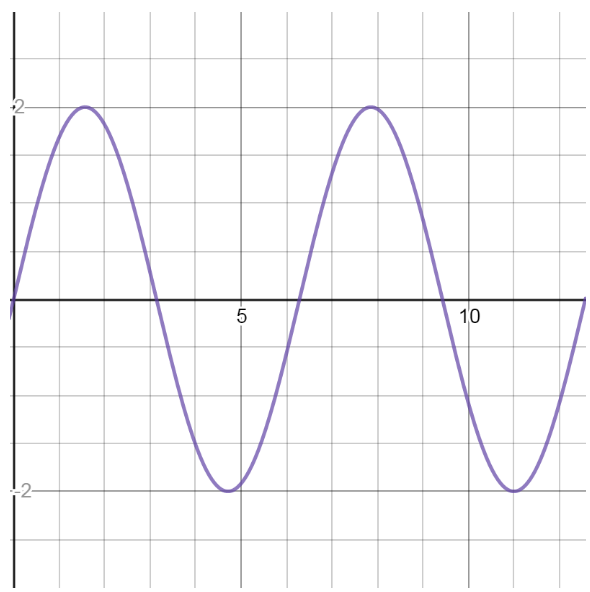
These are the dotted lines.

So, as long as you correctly show the scaling on the y-axis . . .

. . . you don't have to actually stretch the graph!

Put a different way, these next two graphs may **look** the same:

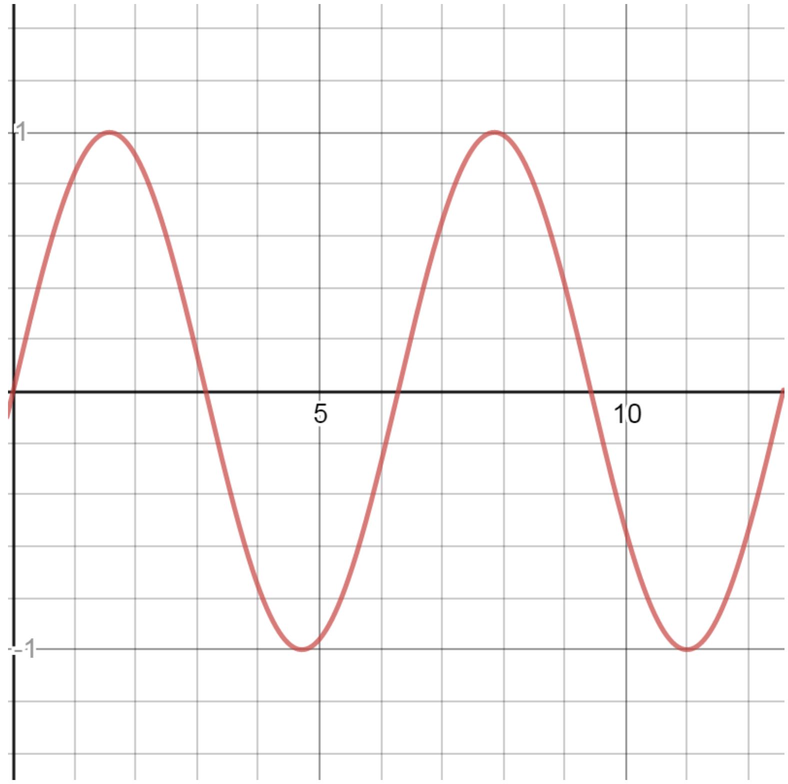


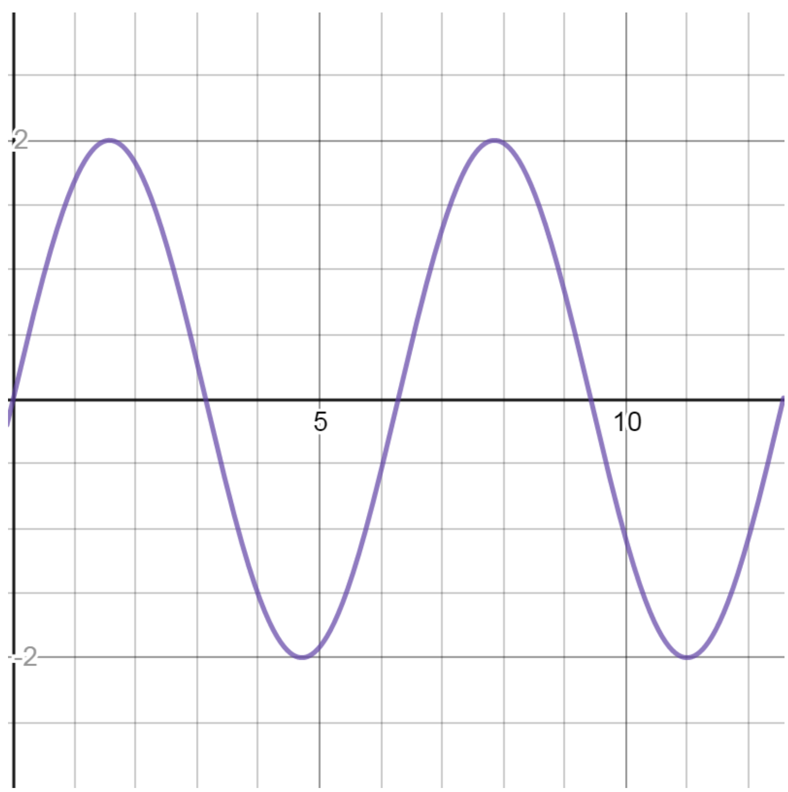


But if you look carefully, you see that they are not!

The first function is $g(\theta) = \sin(\theta)$; the second is $h(\theta) = 2\sin(\theta)$.

The difference can only be seen by checking the scale on the y-axis:


1
-1
5
10


2
-2
5
10

The coefficient multiplying the sine or cosine function is called the

amplitude

The **amplitude** of $p(\theta) = 3\sin(\theta)$

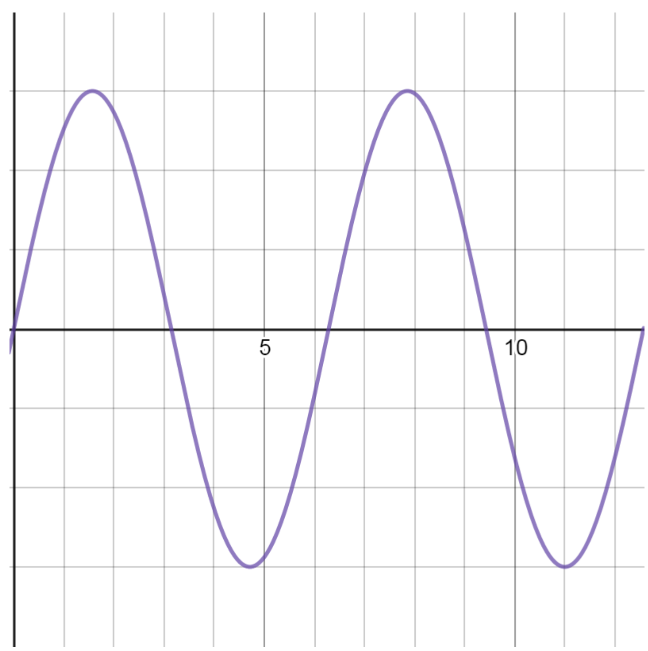
is **3**.

The amplitude of $q(\theta) = \frac{1}{2}\cos(\theta)$

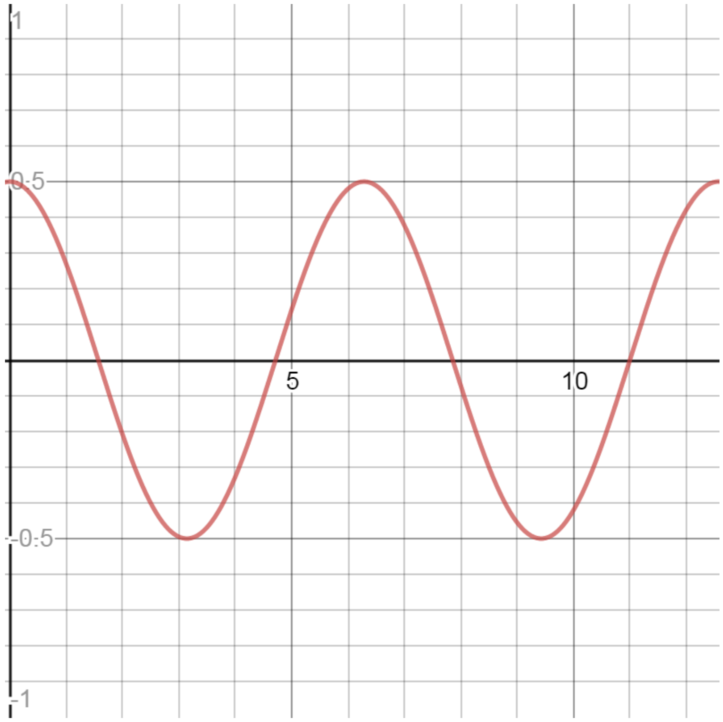
is $\frac{1}{2}$.

Here are the graphs of these two functions:

$p(\theta) = 3\sin(\theta)$



$$q(\theta) = \frac{1}{2}\cos(\theta)$$



Note again that these graphs are different versions of the basic sine and cosine graphs we already memorized!

To correctly adjust for different amplitudes than 1 . . .

. . . the scale of the y-axis just needs to be adjusted!!!

Now, let's take a look at the following function:

$$g(\theta) = \sin{(\pi\theta)}$$

Something different is happening here!

The previous examples involved a vertical stretch . . .

. . . because a **constant** was being **multiplied** to . . .

. . . the ***y-values*** of the function.

But here, a constant is being multiplied to . . . the x-values:

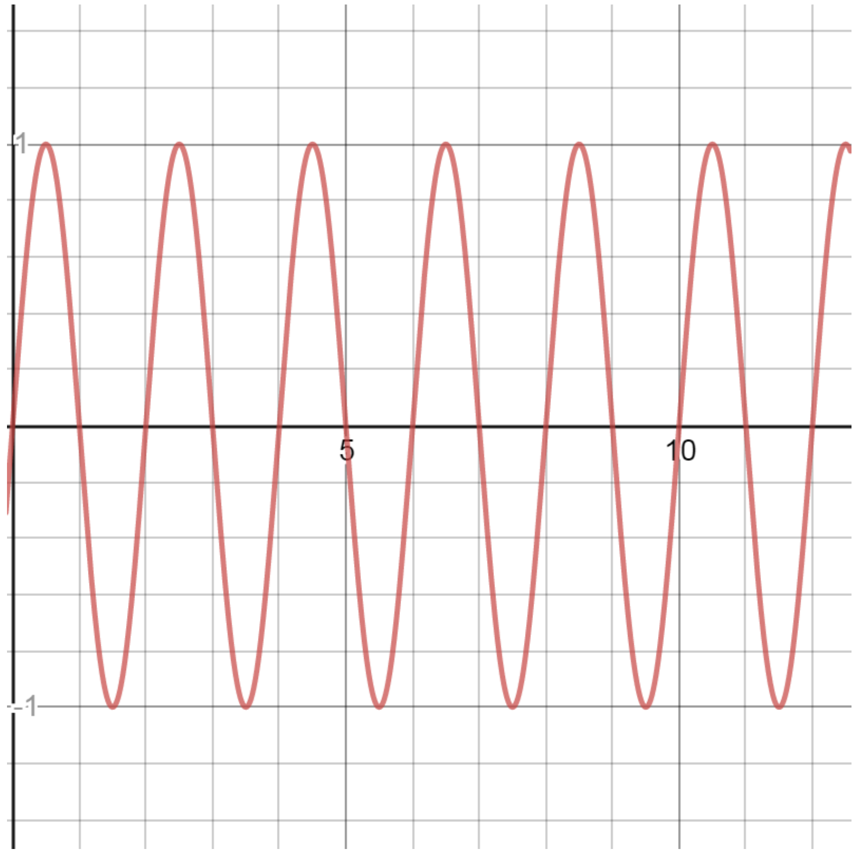
$$g(\theta) = \sin{(\pi\theta)}$$

So we can expect something like a . . .

horizontal stretch!

Let's look at the graph to see if this is so . . .

$g(\theta) = \sin(\pi\theta)$



This is the same scale that we have been using . . .

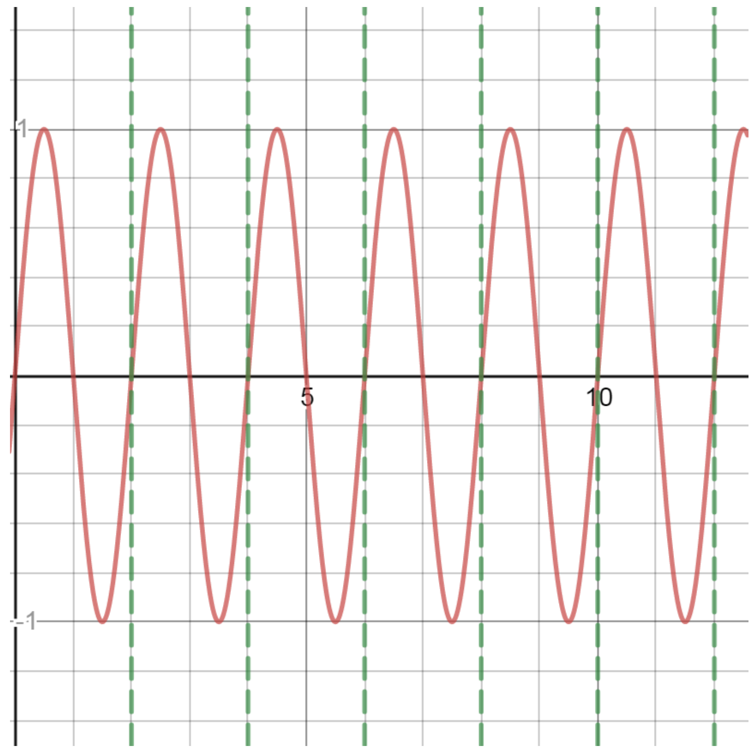
. . . so something is definitely happening with the **horizontal**!

To investigate further, let's look into the

period

of this graph!

$$g(\theta) = \sin(\pi\theta)$$



Do you see that the period of this function is 2?

A ***horizontal stretch*** . . .

(or in this case, a ***shrink***)

. . . affects the **period**!

So instead of thinking about this as a horizontal stretch (or shrink) . . .

. . . we are going to just figure out the period!

It must be based on the coefficient of θ.

$$g(\theta) = \sin(\pi\theta)$$

Our period of this function was

$$2$$

Notice that

$$\frac{2\pi}{\pi} = 2$$

And notice that for our basic sine function:

$$f(\theta) = \sin(\theta)$$

The period is

$$\frac{2\pi}{1} = 2\pi$$

Here is how the coefficient of θ . . .

. . . in a trigonometric function . . .

. . . can tell us the period:

If

$$f(\theta) = \sin(B * \theta)$$

or

$$f(\theta) = \cos(B * \theta)$$

Then

$$\textbf{period} = \frac{2\pi}{B}$$

Let's do a couple of problems!

First, an applied example:

The following function describes the blood pressure of a person:

$$P(t) = 100 - 20\cos\left(\frac{5\pi t}{3}\right)$$

where

$$P(t) = the\ blood\ pressure\ after\ t\ seconds$$

First, does this function make general sense?

We know that our heart beats according to a regular pattern, and that every time the human heart pumps blood, that means the blood pressure is increasing.

So yes, it does make sense that blood pressure would go slightly up and down over time, in a periodic way . . . a cosine function!

Here's the problem:

How many times does the heart beat in a minute?

Think about what this question is asking. We know that the cosine function, $P(t)$, goes through a full cycle every time the heart beats.

And the length of a full cycle, here, is the ***amount of time*** this takes to happen.

If we can figure out the length of time for a full heartbeat cycle . . .

. . . we can figure out how many happen in a minute!

In other words, we need to figure out the . . . **period**.

The period is the time for a full cycle of the heartbeat.

Here is the function again:

$$P(t) = 100 - 20\cos\left(\frac{5\pi t}{3}\right)$$

This function also has a vertical shift $(+100)$ which we have not covered . . .

. . . but this is not relevant to the t variable, or the period . . .

. . . so we won't cover it here.

We know that the period obeys the formula:

$$\textbf{period} = \frac{2\pi}{B}$$

What is $\boldsymbol{B}$ for this function?

It's the coefficient of t, which is

$$\frac{5\pi}{3}$$

So we have that

$$period = \frac{2\pi}{\frac{5\pi}{3}}$$

$$= \frac{6}{5}$$

What does this mean?

This means that every $\frac{6}{5}$ seconds (or 1.2 seconds) . . .

. . . the heart completes a full cycle of beating.

How many times does this happen in one minute? That's easy.

One full minute contains 60 seconds.

So the number of times $\frac{6}{5}$ seconds divides into 60 seconds is

$$60 \div \frac{6}{5} = 50$$

So the human heart beats approximately 50 times per minute.

Let's do some graphing problems now, which are the types of problems you will be tested on:

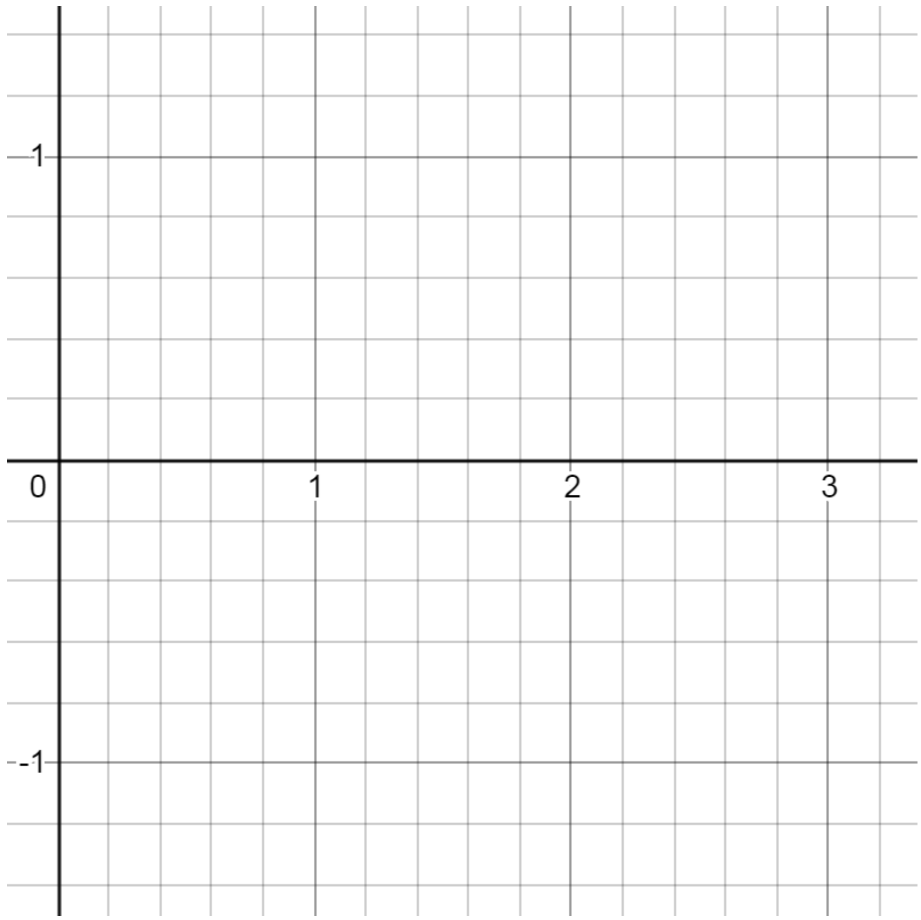
Graph two periods of the function

$$p(\theta) = \cos(4\theta)$$

First, we find the period of this function. In this case the coefficient is 4, so

$$period = \frac{2\pi}{4} = \frac{\pi}{2}$$

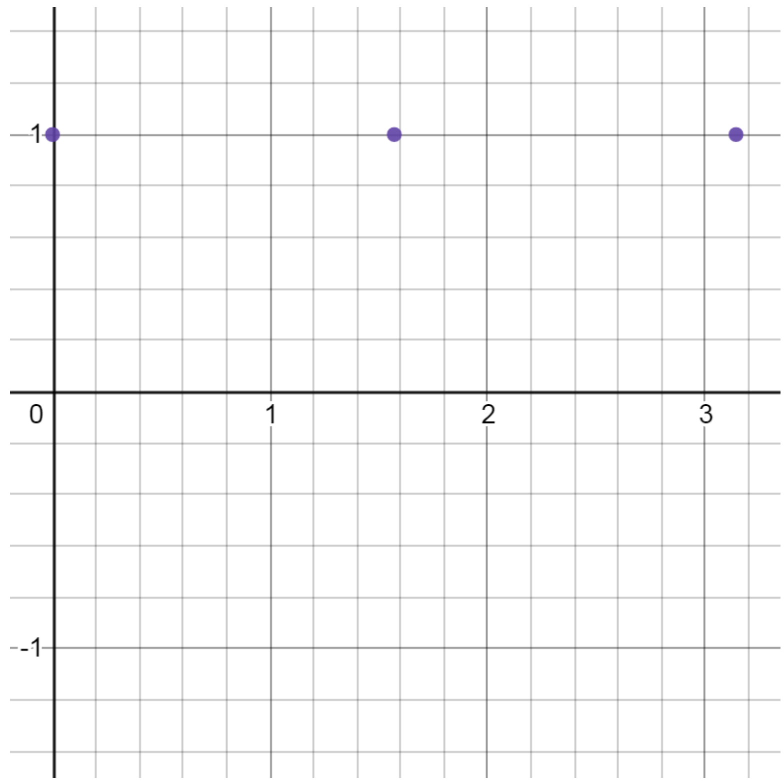
When we construct our graph, we have to first scale our graph so that it includes 2 periods:



As well as making just enough space on the y-axis scale to include the amplitude, which is ± 1.

Next, let's plot the major points for the cosine function as a guide to constructing our graph. Remember . . .

cosine starts and ends its period at its maximum



Here, I have only plotted the starting and ending points for two periods:

$$(0,1) \qquad \left(\frac{\pi}{2},1\right) \qquad (\pi,1)$$

Now, let's plot the minimum points of the function!

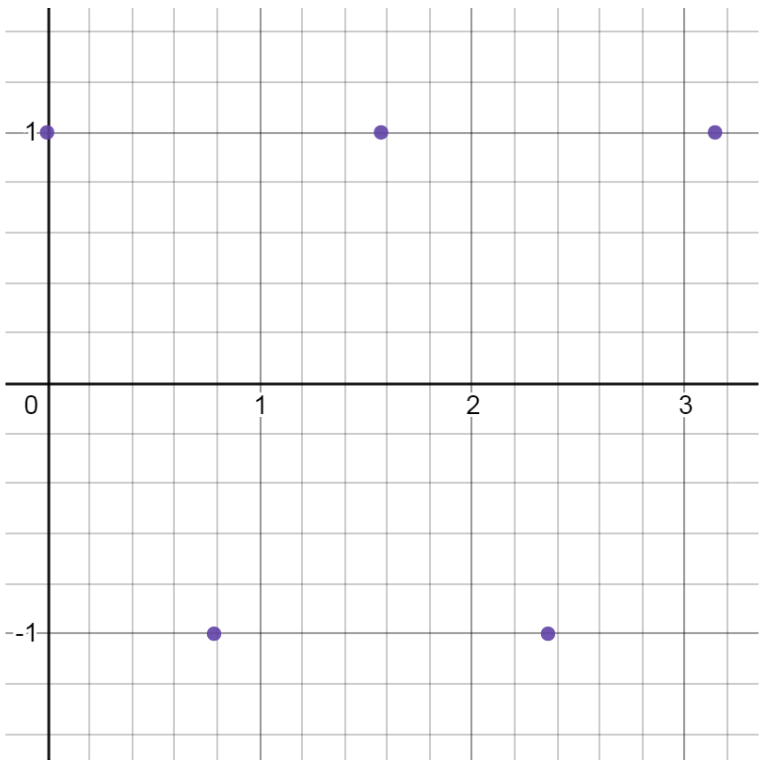
They will happen exactly halfway through the period.

So the first one will happen at

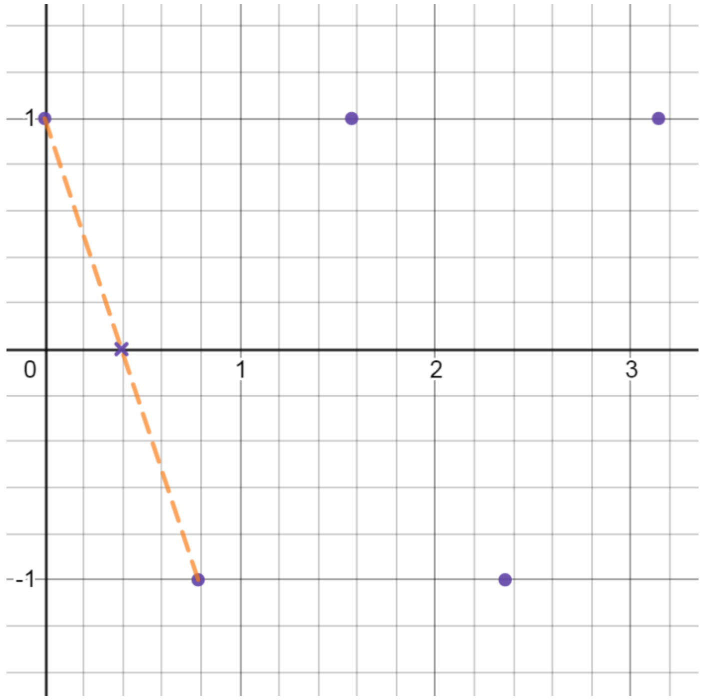
$$\theta = \frac{\pi}{2} \div 2 = \frac{\pi}{4}$$

And the second one at

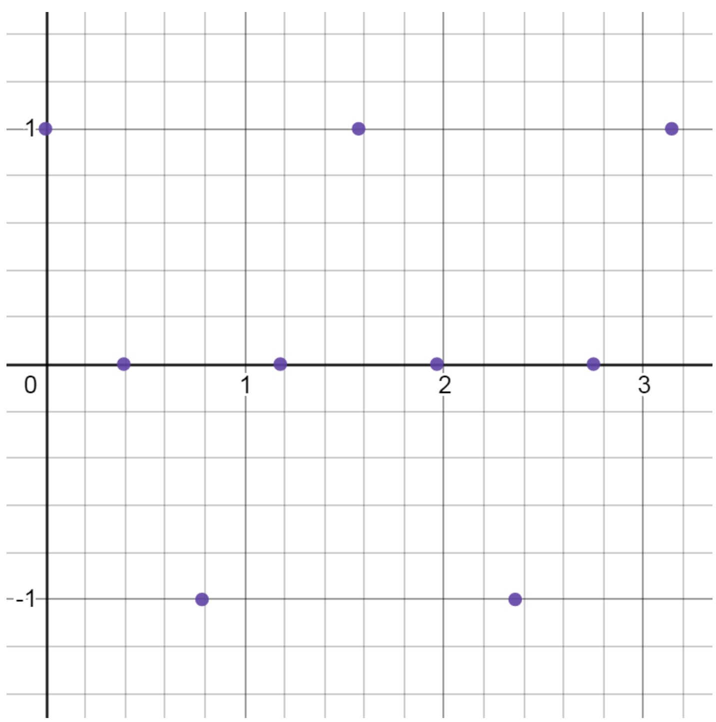
$$\frac{\pi}{2} + \frac{\pi}{4} = \frac{3\pi}{4}$$



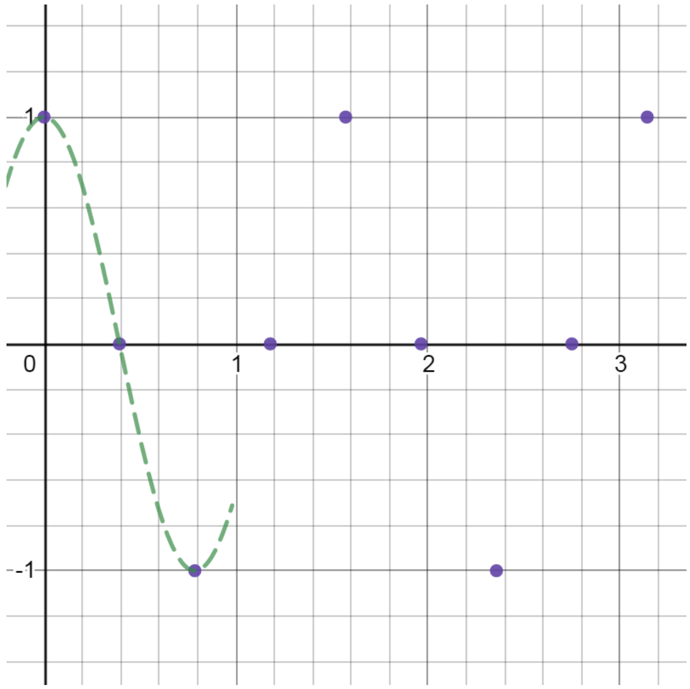
Finally, I know that the graph will cross the x-axis at points directly in between the minimum and maximum. I can either find them by trying to estimate them visually or by connecting a straight line between the points:



Which can used to generate the other x-intercepts:

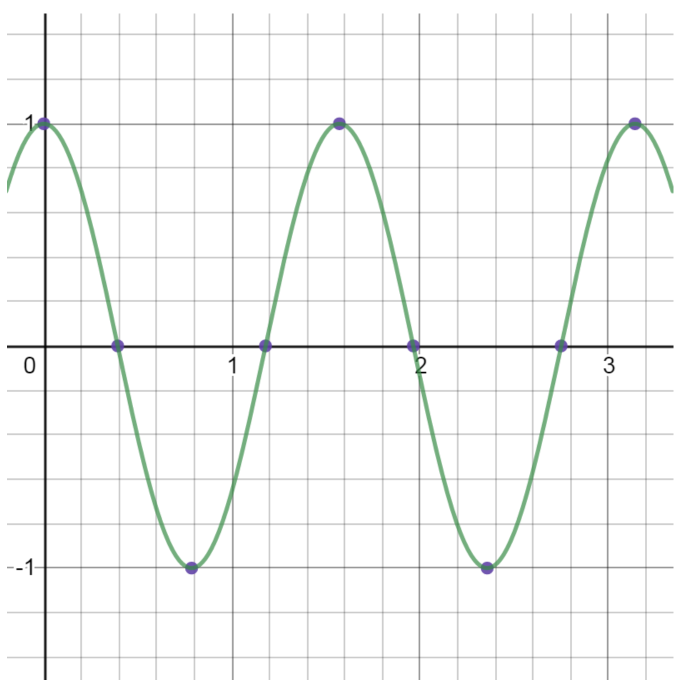


And finally, a curve is drawn (not a series of straight line segments):



And then completed:

$$p(\theta) = \cos(4\theta)$$



Now let's do a problem that requires adjusting for **period** and **amplitude**:

Graph two full periods of

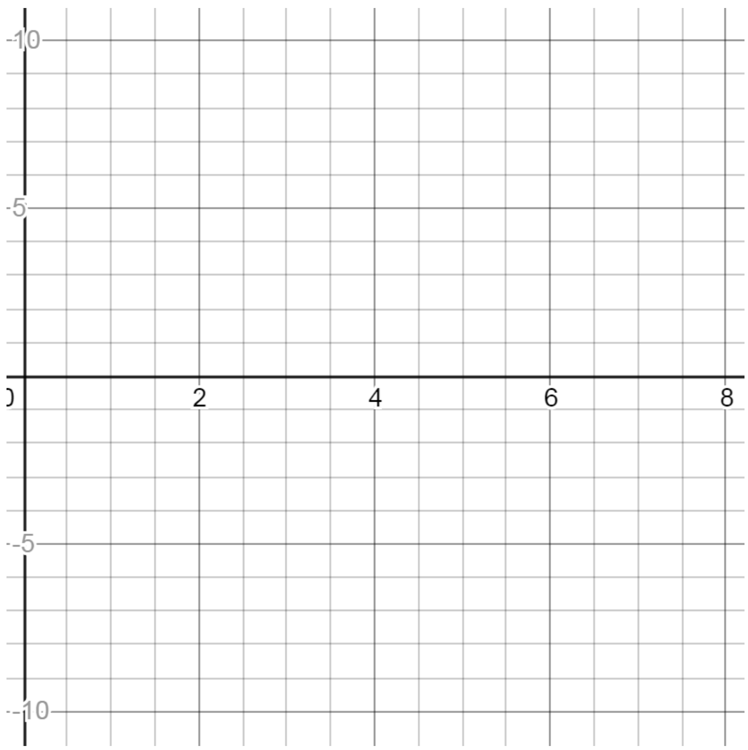
$$h(\theta) = 10\sin\left(\frac{\pi}{2}\theta\right)$$

Seeing the amplitude is easy . . . that's simply 10!

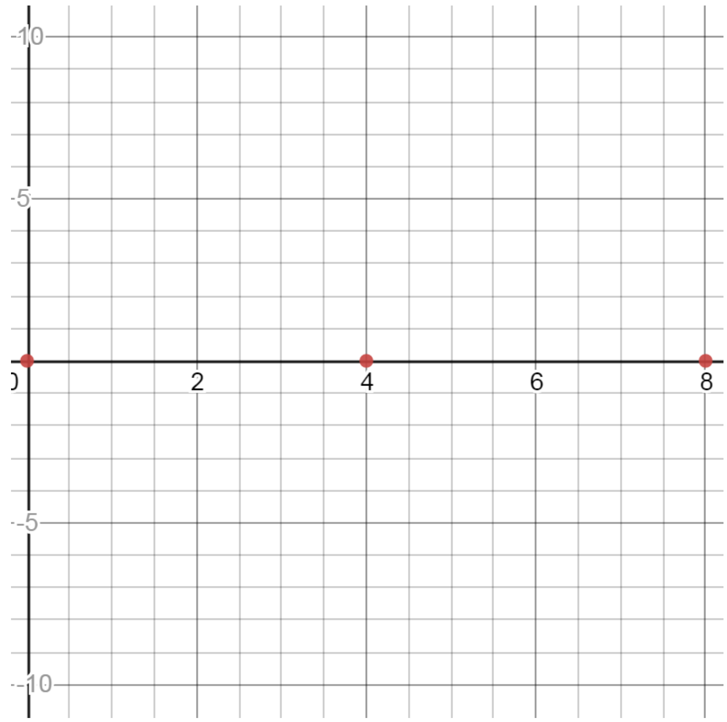
And the period is simple to figure out too:

$$period = \frac{2\pi}{\left(\frac{\pi}{2}\right)} = 4$$

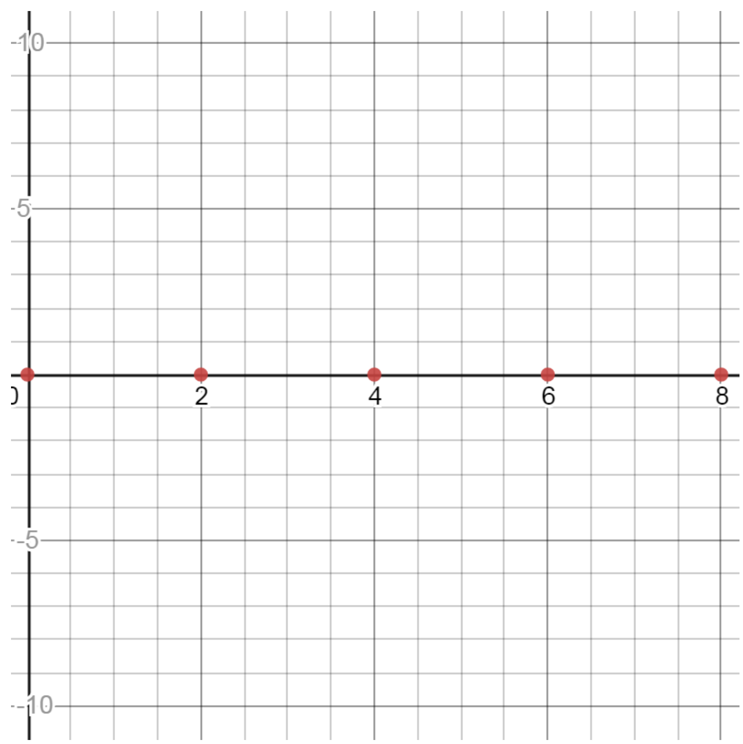
So first let's set up our graph with the correct scale:



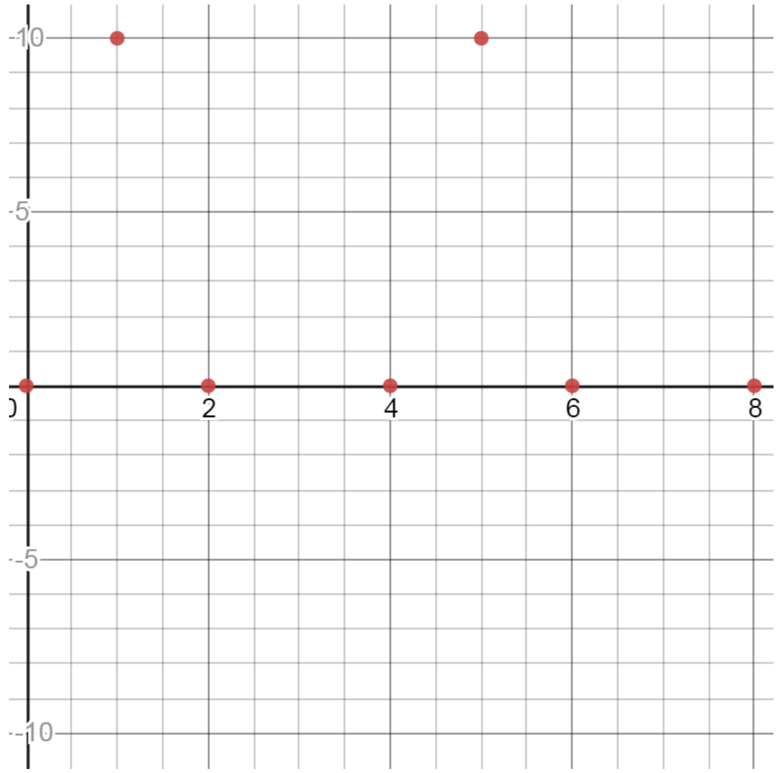
Next, the sine function starts and ends its period on the x-axis . . .



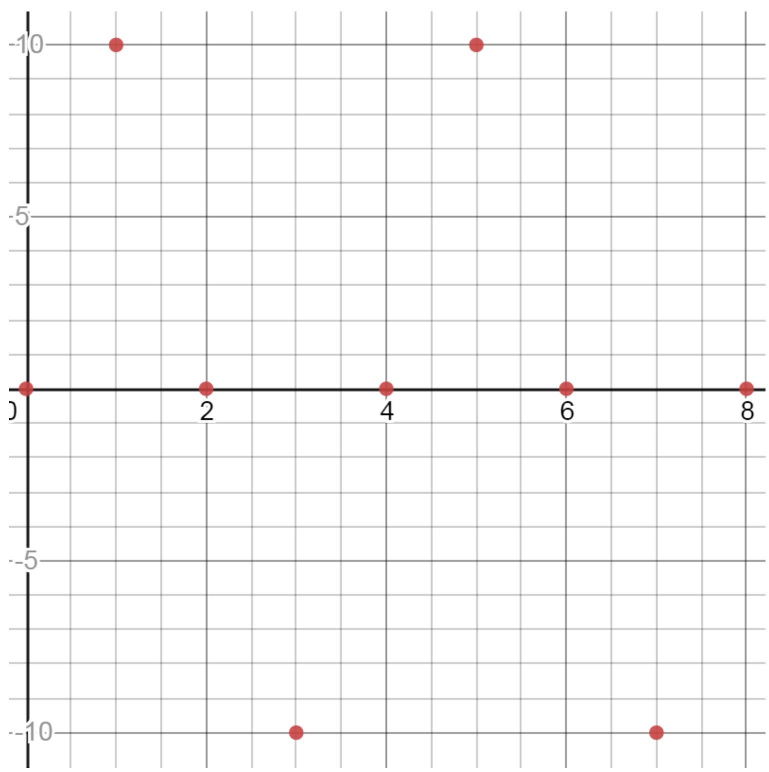
And also crosses the x-axis halfway through its period:



Finally, the sine function goes to its maximum first:



And its minimum second . . .

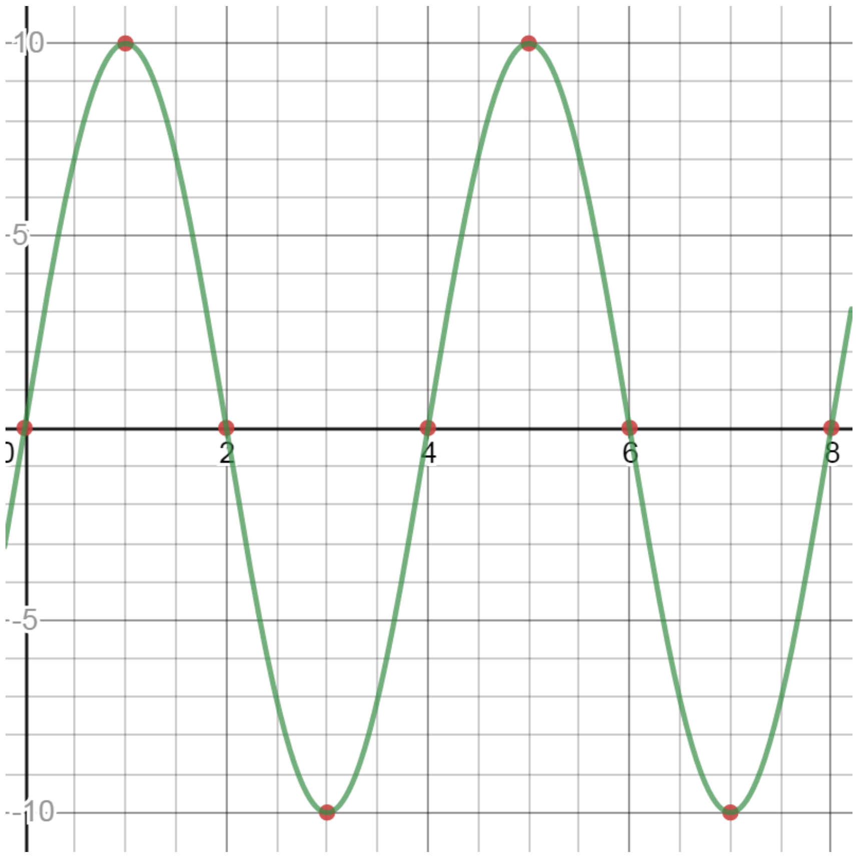


Using these key points as a guide . . .

. . . we then draw a smooth curve . . .

. . . containing all of these points . . .

$$h(\theta) = 10\sin\left(\frac{\pi}{2}\theta\right)$$



One final transformation that we need to consider . . .

Let's look at the function

$$q(\theta) = -\sin(\theta)$$

Can you identify the transformation happening to this function?

There is a negation happening:

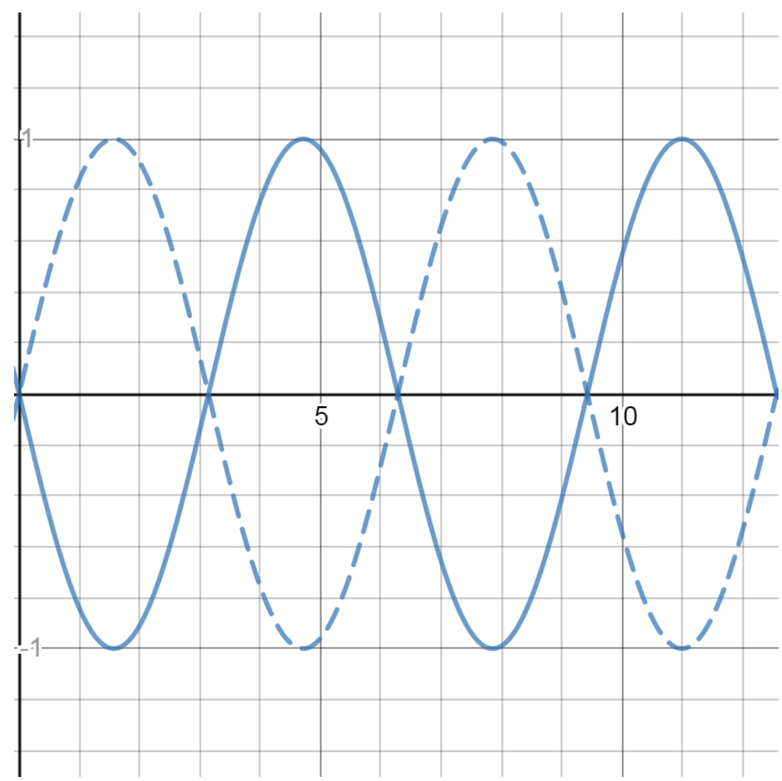
$$q(\theta) = -\sin(\theta)$$

Meaning that we have some sort of reflection . . .

And the negation is happening to the **entire function** . . .

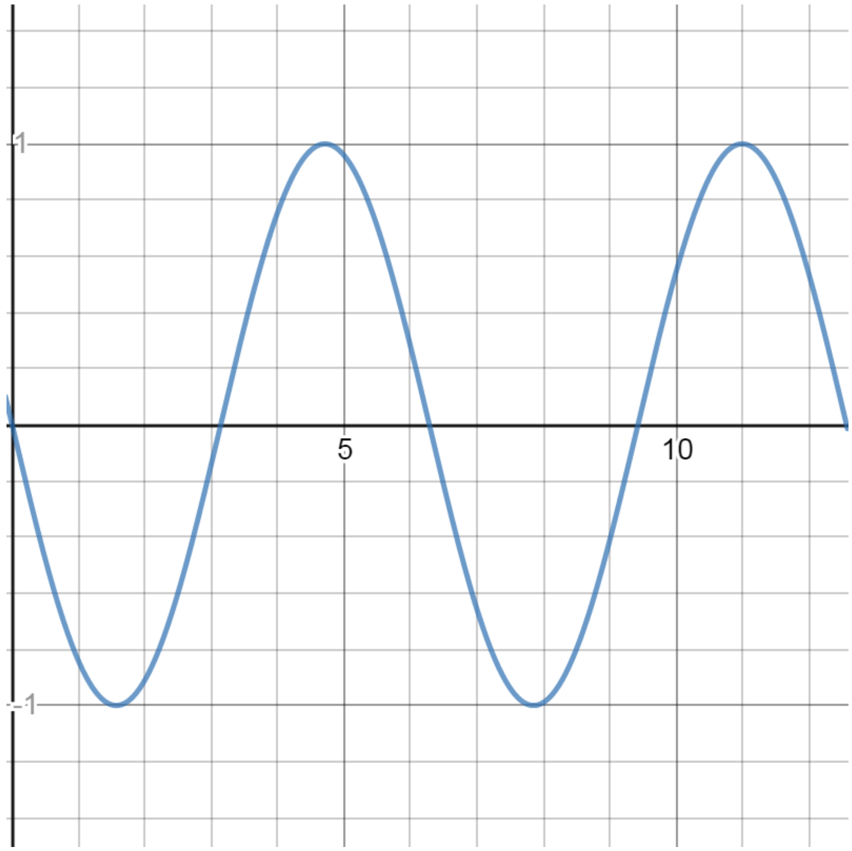
. . . meaning that it is affecting the y-values . . .

. . . giving us a ***vertical reflection!***



And so the final graph looks like:

$q(\theta) = -\sin(\theta)$



Let's put all three of our transformations together for a final problem!

Graph two full periods of the function:

$$f(\theta) = -2\cos\left(\frac{1}{2}\theta\right)$$

First, let's scale the graph according to our amplitude and period.

(*the vertical reflection would not affect the scale*)

$$f(\theta) = -2\cos\left(\frac{1}{2}\theta\right)$$

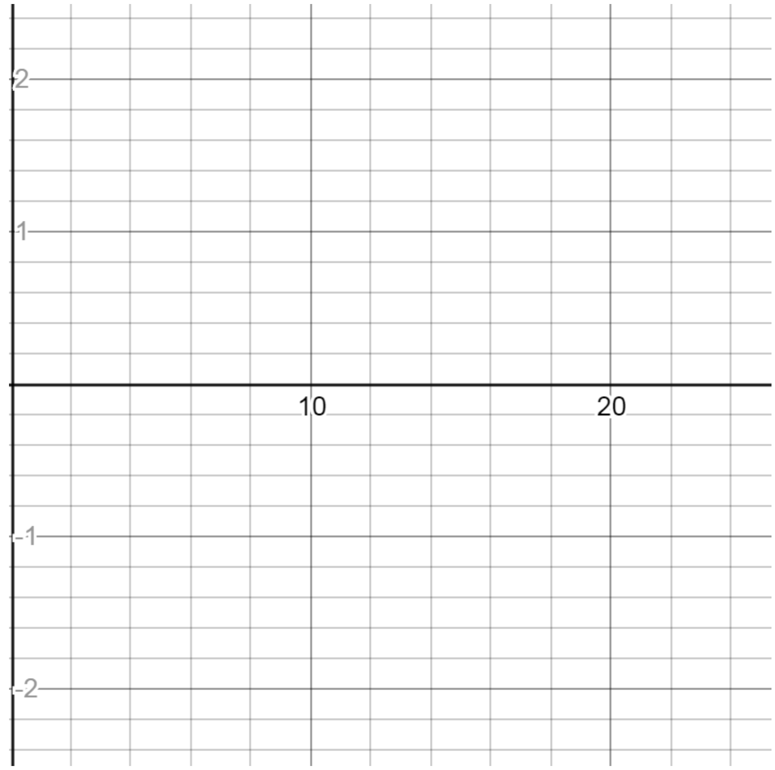
We have that the amplitude is

$$2$$

. . . and the period is given by

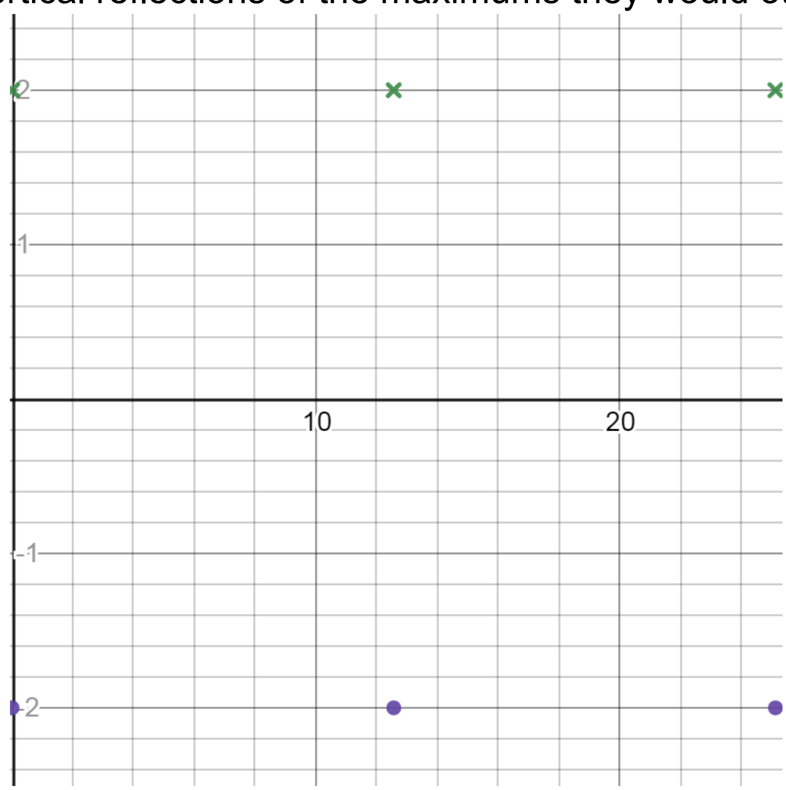
$$period = \frac{2\pi}{\left(\frac{1}{2}\right)} = 4\pi$$

So let's first set up our graph by scaling it according to these numbers:

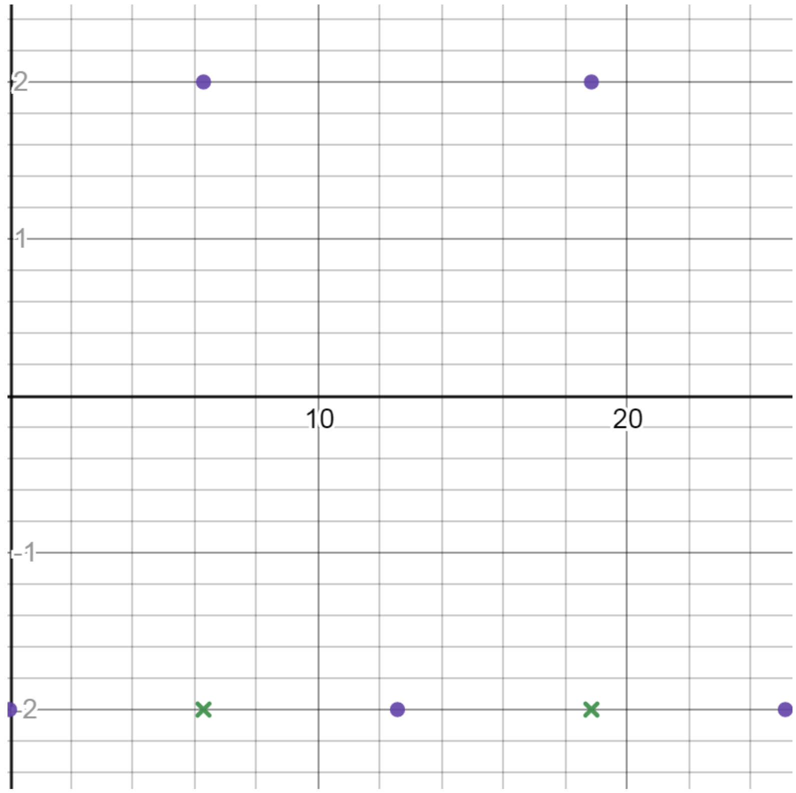


Next let’s plot the minimum points of the function.

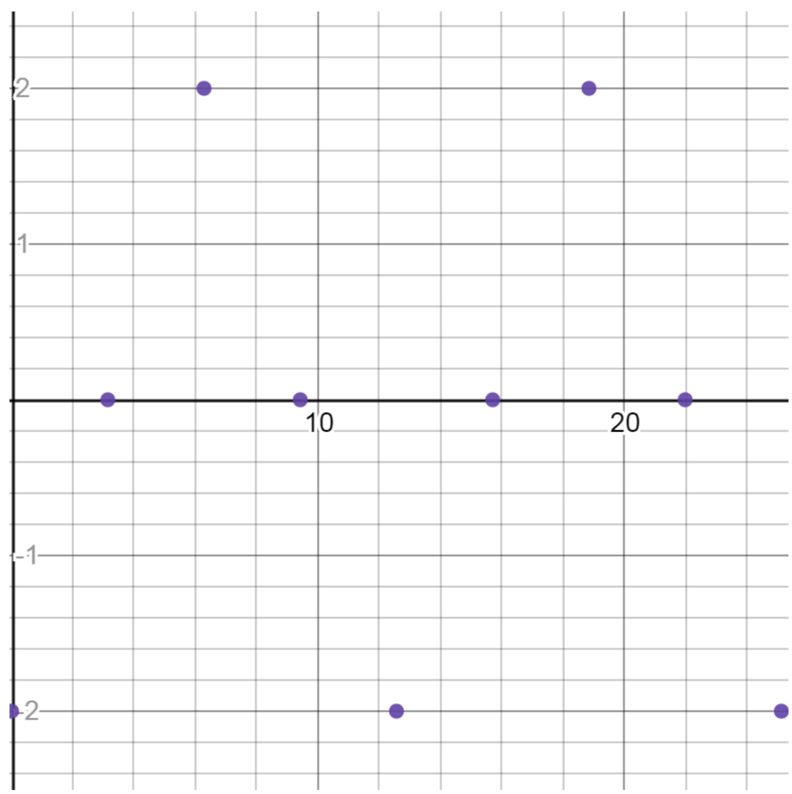
They are vertical reflections of the maximums they would otherwise be:



And the minimums have turned into maximums:



Next let's plot the x-intercepts:



And finally complete the graph with a smooth curve:

$f(\theta) = -2\cos\left(\frac{1}{2}\theta\right)$