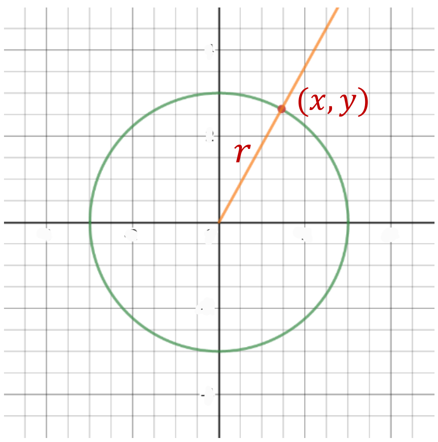
Evaluating angles of Trigonometric Functions

These periodic functions, sine and cosine, are more specifically called “trigonometric functions.” There are six of them in total:



$$\sin(\theta) = \frac{y}{r} \qquad \csc(\theta) = \frac{r}{y} = \frac{1}{\sin(\theta)}$$

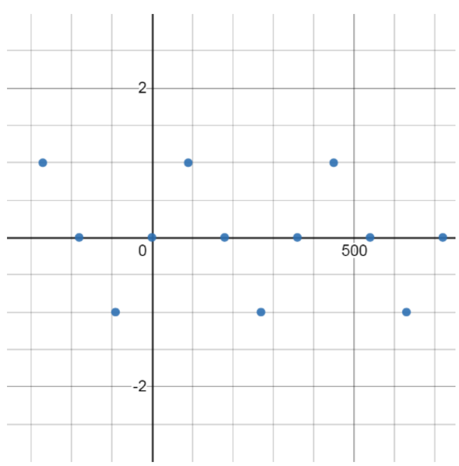
$$\cos(\theta) = \frac{x}{r} \qquad \sec(\theta) = \frac{r}{x} = \frac{1}{\cos(\theta)}$$

$$\tan(\theta) = \frac{y}{x} \qquad \cot(\theta) = \frac{x}{y} = \frac{1}{\tan(\theta)}$$

For now, let’s keep working with $g(\theta) = \sin(\theta)$ so as to get a better sense of its graph.

So far, we have a good idea of its periodicity, but would like to know how to see its shape better:

$g(\theta) = \sin(\theta)$



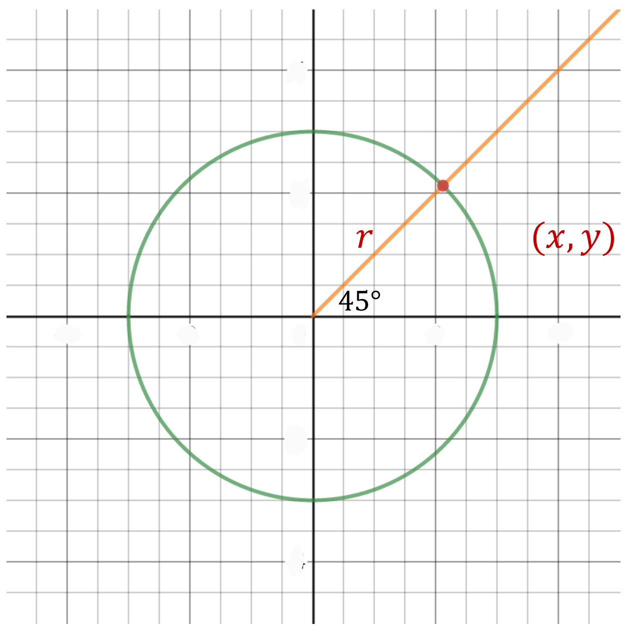
We can calculate a few of these points using basic geometry.

Consider

$$\sin(45°)$$

How can we figure this out?

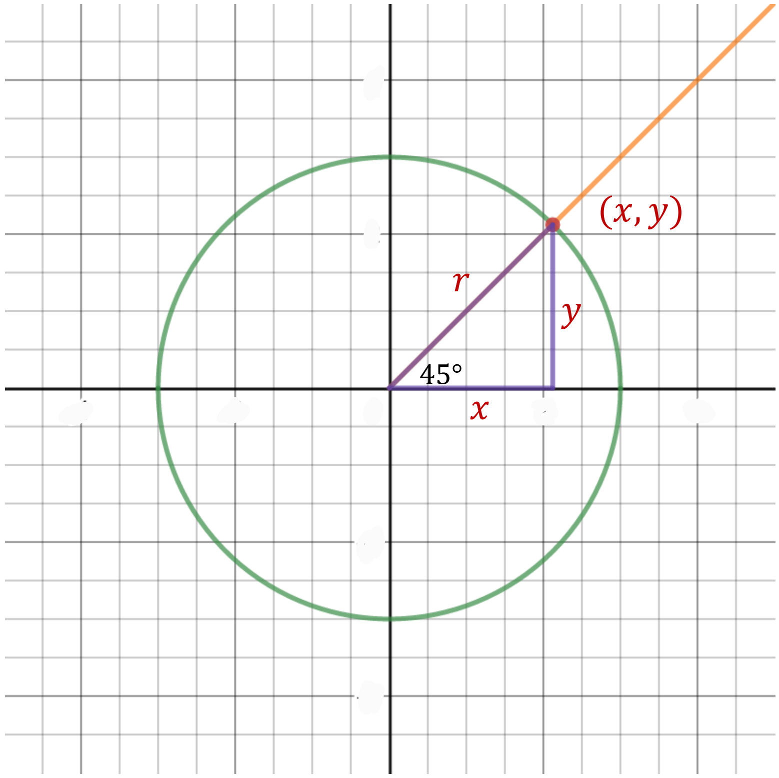
Let's look at the diagram:



We are trying to find

$$\sin(45°) = \frac{y}{r}$$

To figure out that fraction, notice that both y and r can be seen as the sides of a right triangle:



And also note that since $\theta = 45°$, the two legs of the triangle, x and y, are equal:

$$x = y$$

Let's use the Pythagorean theorem:

$$x^2 + y^2 = r^2$$

And substitute in y instead of x:

$$y^2 + y^2 = r^2$$

And now solve for y:

$$y^2 + y^2 = r^2$$
$$2y^2 = r^2$$

Taking the square root of both sides we get:

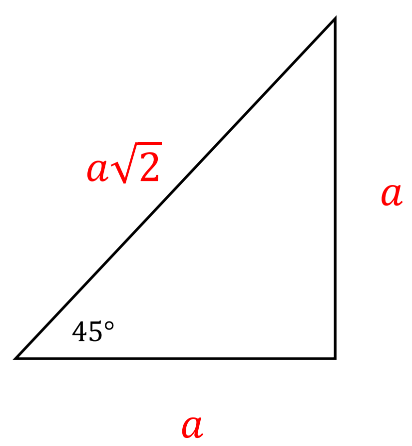
$$\sqrt{2}y = r$$

Note that I am not doing $\pm$ here because based on where we are in the diagram (known as “quadrant I”), all the numbers have to be positive.

Note also that we could have done the same process with x instead of y, which would have resulted in the equation:

$$\sqrt{2}x = r$$

This result tells us that in a 45-45-90 right triangle, the hypotenuse r is equal to $\sqrt{2}$ times the length of either side:



What this means is that

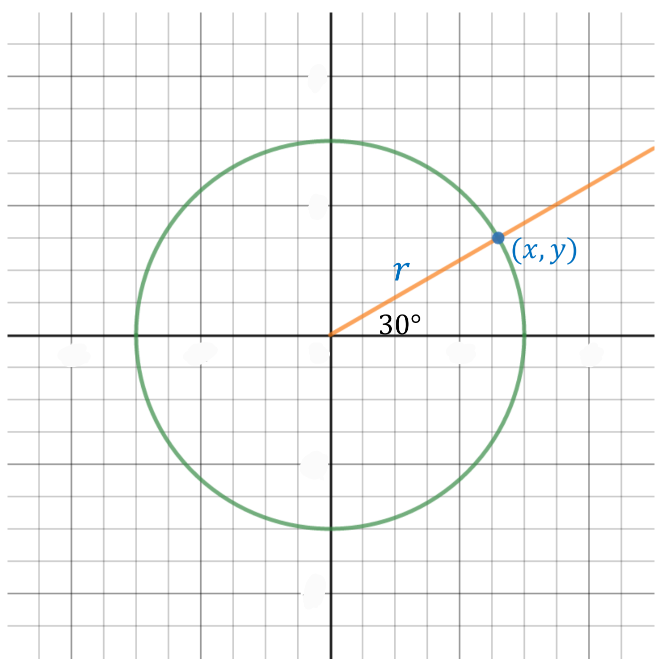
$$\sin(45°) = \frac{y}{r} = \frac{a}{a\sqrt{2}} = \frac{1}{\sqrt{2}}$$

and

$$\cos(45°) = \frac{x}{r} = \frac{a}{a\sqrt{2}} = \frac{1}{\sqrt{2}}$$

Now let's look at **two more** important angles: 30° and 60°.

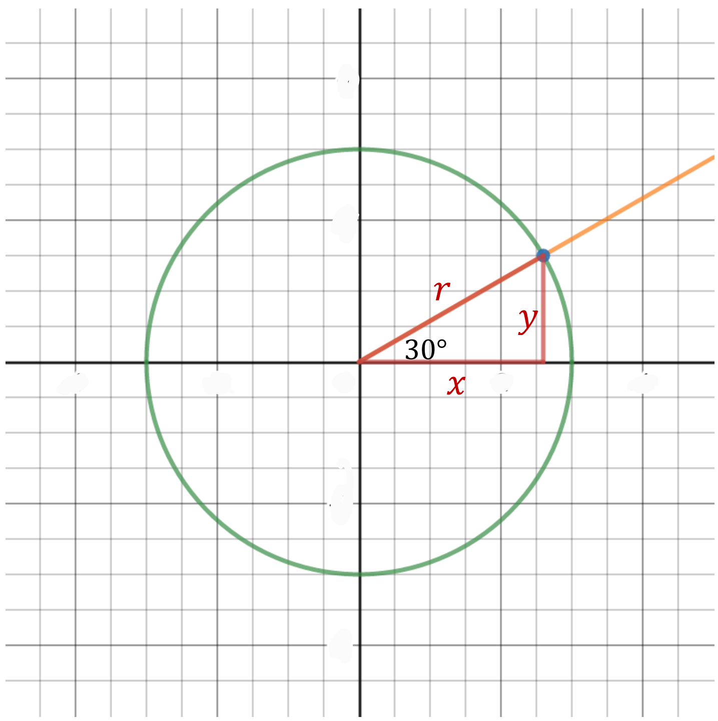
To begin, let's try to find $\sin(30°)$:



Again, we need to find the ratio:

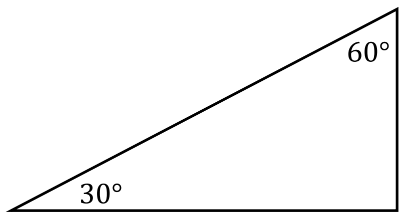
$$\sin(30°) = \frac{y}{r}$$

Which depends on the ratio of the sides of this 30°-60°-90° triangle:

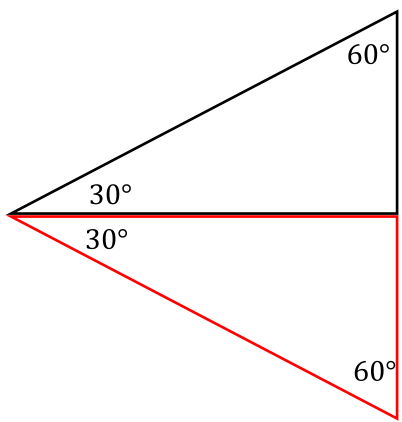


Just as with the 45°-45°-90° triangle, we can use geometry and the Pythagorean Theorem to figure out the ratios of these sides.

Let’s look at a general 30-60-90 triangle:

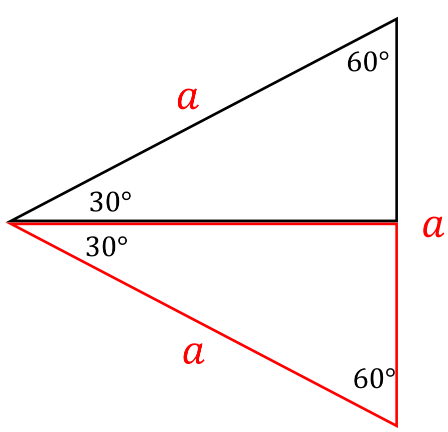


Now, we will draw another 30-60-90 triangle of the same size**, reflected below**:

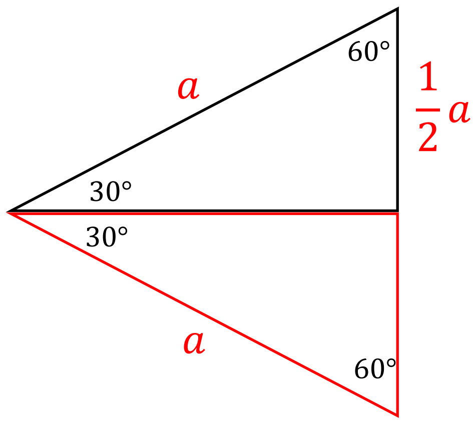


Notice that the angles inside the **larger** triangle are all $60°$.

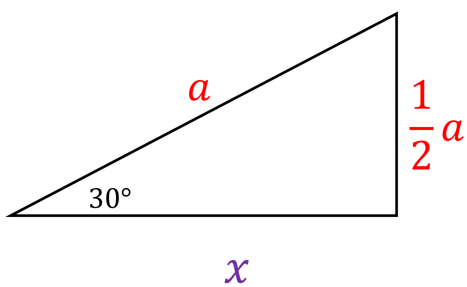
Since the interior angles are equal, that means the ***sides must be equal***!!!



Which means that the shortest side of the top triangle must be half of the hypotenuse!!



Now that we know two of the sides, we can now go back and use the Pythagorean Theorem to find the third side:



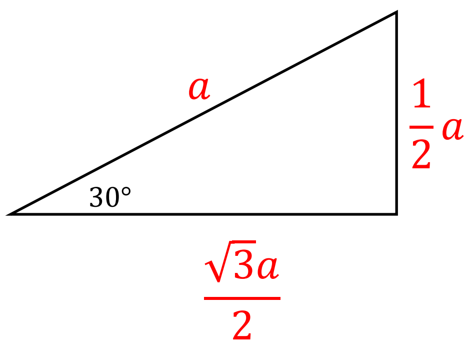
$$x^2 + \left(\frac{1}{2}a\right)^2 = a^2$$

$$x^2 + \frac{a^2}{4} = a^2$$

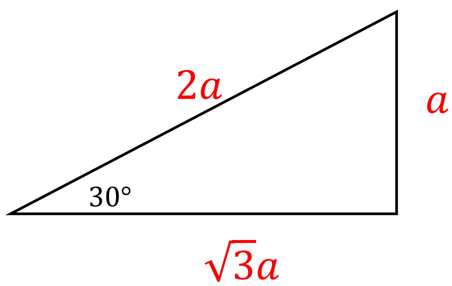
$$x^2 = \frac{4a^2}{4} - \frac{a^2}{4} = \frac{3a^2}{4}$$

$$x = \frac{\sqrt{3}a}{2}$$

So now we know the ratio of the sides of a 30-60-90 triangle:



Which if we multiply all sides by 2 gives us

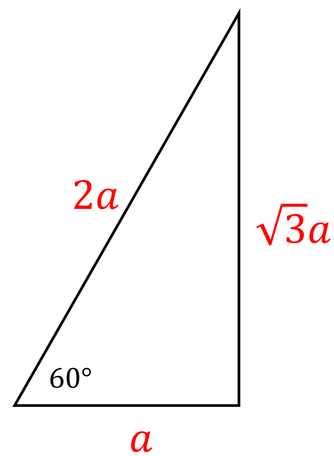


We can now state both the sine and the cosine of 30°:

$$\sin(30°) = \frac{y}{r} = \frac{a}{2a} = \frac{1}{2}$$

$$\cos(30°) = \frac{x}{r} = \frac{\sqrt{3}a}{2a} = \frac{\sqrt{3}}{2}$$

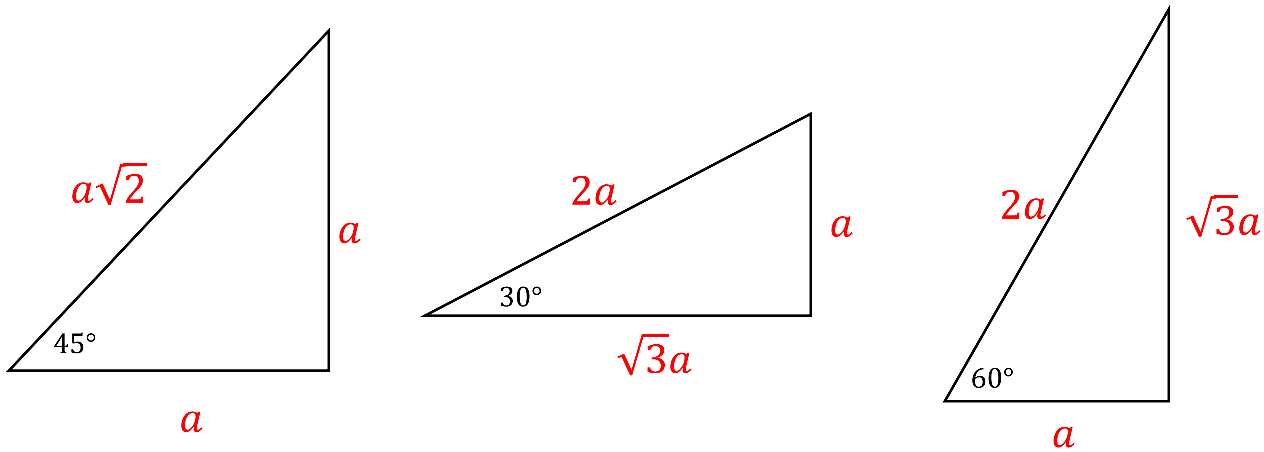
Also note that we can see the sine cosine of 60°:



$$\sin(60°) = \frac{y}{r} = \frac{\sqrt{3}a}{2a} = \frac{\sqrt{3}}{2}$$

$$\cos(30°) = \frac{x}{r} = \frac{a}{2a} = \frac{1}{2}$$

Let's use these two types of special triangles to figure out some other trigonometric function values:



$$\tan(45°) = \frac{y}{x} = \frac{a}{a} = 1$$

$$\sec(30°) = \frac{r}{x} = \frac{2a}{\sqrt{3}a} = \frac{2}{\sqrt{3}}$$

$$\csc(45°) = \frac{r}{y} = \frac{a\sqrt{2}}{a} = \sqrt{2}$$

$$\cot(60°) = \frac{x}{y} = \frac{\sqrt{3}a}{a} = \sqrt{3}$$

All of the above problems were worked out using the **ratios of the sides of special triangles**, combined with the **formulas for the trigonometric functions.**

Some people prefer to ***memorize*** the trig values instead:

θ	$0°$	$30°$	$45°$	$60°$	$90°$	$180°$	$270°$
$\sin(\theta)$	0	$\frac{1}{2}$	$\frac{1}{\sqrt{2}}$	$\frac{\sqrt{3}}{2}$	1	0	-1
$\cos(\theta)$	1	$\frac{\sqrt{3}}{2}$	$\frac{1}{\sqrt{2}}$	$\frac{1}{2}$	0	-1	0
$\tan(\theta)$	0	$\frac{\sqrt{3}}{3}$	1	$\sqrt{3}$	$undef.$	0	$undef.$

There are some things worth noting here . . .

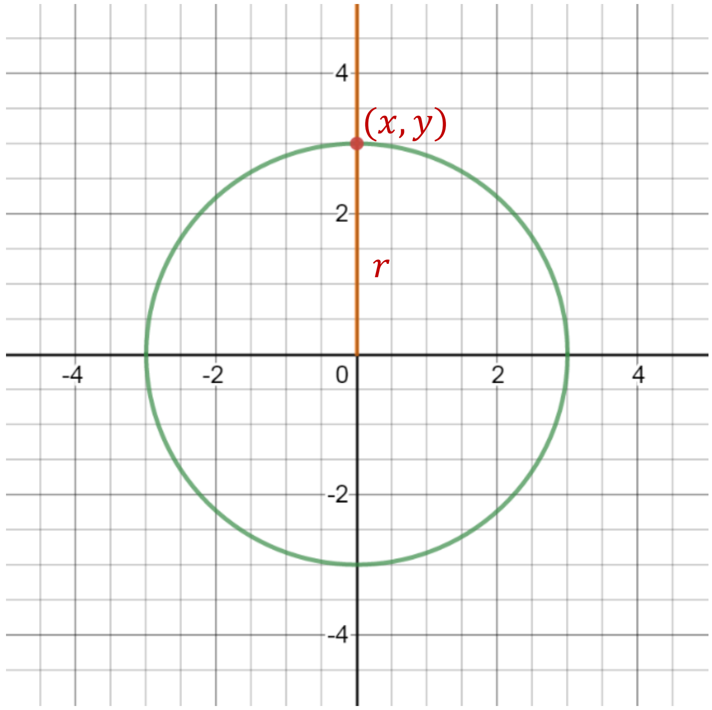
We have that $\tan(90°)$ is undefined because

$$\tan(90°) = \frac{y}{x} = \frac{y}{0}$$

The diagram below shows why this is true:

$$\tan(90°) = \frac{y}{x} = \frac{y}{0}$$

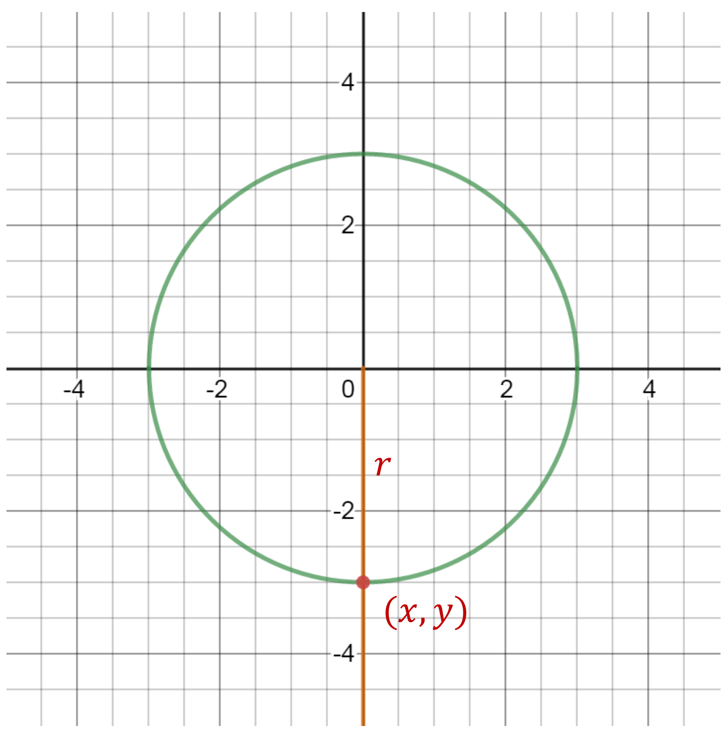
$$= undefined$$



This is also true at 270°:

$$\tan(270°) = \frac{y}{x} = \frac{y}{0}$$

$$= undefined$$



Also note that to find any of the other trigonometric functions, such as secant ($\sec(\theta)$), cosecant ($\csc(\theta)$), and cotangent ($\cot(\theta)$), we can just use the fact that they are the reciprocals of the original three:

Ex.:

$$\sec(60°) = \frac{1}{\cos(60°)} = \frac{1}{\frac{1}{2}} = 2$$

$$\cot(45°) = \frac{1}{\tan(45°)} = \frac{1}{1} = 1$$

$$\csc(180°) = \frac{1}{\sin(180°)} = \frac{1}{0} = undefined$$