Synthetic Division

To see this, let's go back . . .

. . . back to the point when we found our first zero . . .

. . . using ***synthetic substitution***:

We decided to try $x = 3$ as a possible zero:

$$\begin{array}{r|rrrr} 3 & 2 & 3 & -23 & -12 \\ & & 6 & 27 & 12 \\ \hline & 2 & 9 & 4 & \color{red}{0} \end{array}$$

Do you notice anything now about the synthetic substitution numbers?

. . . anything ***familiar??***

Check again:

$$\begin{array}{r|rrrr} 3 & 2 & 3 & -23 & -12 \\ & & 6 & 27 & 12 \\ \hline & 2 & 9 & 4 & \color{red}{0} \end{array}$$

$$\begin{array}{rl} & \quad 2x^2 + 9x + 4 \\ x - 3 & \overline{)2x^3 + 3x^2 - 23x - 12} \\ & \underline{-2x^3 \color{red}{+} 6x^2 \qquad\qquad\quad} \\ & \qquad\quad 9x^2 - 23x - 12 \\ & \underline{\qquad \color{red}{-}9x^2 \color{red}{+} 27x \qquad} \\ & \qquad\qquad\qquad 4x - 12 \\ & \underline{\qquad\qquad\quad \color{red}{-}4x \color{red}{+} 12} \\ & \qquad\qquad\qquad\quad \color{red}{0} \end{array}$$

That's right . . . the numbers in the bottom row of the synthetic substitution turn out to be **the coefficients of the other polynomial factor!**

Not convinced? Let's return to an earlier problem, finding the zeroes of

$$f(x) = x^3 - 5x^2 + 2x + 8$$

In our work for that problem, we first found that

$$f(2) = 0$$

then concluded that

$$(x - 2) \text{ is a factor}$$

Then divided $f(x)$ by $(x - 2)$ using polynomial long division:

$$\begin{array}{r|l} & x^2 - 3x - 4 \\ \hline x - 2 & x^3 - 5x^2 + 2x + 8 \\ & -x^3 + 2x^2 \\ \hline & -3x^2 + 2x + 8 \\ & +3x^2 - 6x \\ \hline & -4x + 8 \\ & -4x + 8 \\ \hline & 0 \end{array}$$

To get

$$f(x) = (x-2)(x^2-3x-4)$$

Let’s show the same result using **synthetic substitution/synthetic division!**

We use

$$f(x) = x^3 - 5x^2 + 2x + 8$$

to set up synthetic substitution:

$$\begin{array}{c|cccc} & 1 & -5 & 2 & 8 \\ & & & & \\ \hline & 1 & & & \end{array}$$

And then substitute $x = 2$:

$$\begin{array}{c|cccc} \color{red}{2} & 1 & -5 & 2 & 8 \\ & & \color{red}{2} & & \\ \hline & 1 & & & \end{array}$$

Go ahead and try continuing the process on your own!

Remember, it’s just multiply, then add, multiply, then add . . .

$$\begin{array}{c|cccc} 2 & 1 & -5 & 2 & 8 \\ & & 2 & -6 & -8 \\ \hline & 1 & -3 & -4 & 0 \end{array}$$

Here, we should not surprised that we ended up with zero . . .

. . . because we already knew that $f(2) = 0$.

But what we also can see is that the bottom row of digits . . .

. . . match up with the coefficients of the other factor to $f(x)$!

$$\begin{array}{c|cccc} 2 & 1 & -5 & 2 & 8 \\ & & 2 & -6 & -8 \\ \hline & 1 & -3 & -4 & 0 \end{array}$$

$$\begin{array}{r} x^2 - 3x - 4 \\ x - 2 \,\overline{)\, x^3 - 5x^2 + 2x + 8} \\ -x^3 + 2x^2 \qquad\qquad \\ \hline -3x^2 + 2x + 8 \\ +3x^2 - 6x \qquad \\ \hline -4x + 8 \\ -4x + 8 \\ \hline 0 \end{array}$$

Let's do another problem . . .and thus time we will **draw the graph!!**

Problem: Find the zeroes of the following function, and draw the graph:

$$h(x) = x^3 + 2x^2 - 6x - 4$$

This problem is asking for a graph . . .

. . . so we will start the problem by substituting values into $h(x)$ . . .

. . . using **synthetic substitution**!

$$\begin{array}{r|rrrr} 1 & 1 & 2 & -6 & -4 \\ & & 1 & 3 & -3 \\ \hline & 1 & 3 & -3 & -7 \end{array}$$

$$\begin{array}{r|rrrr} 2 & 1 & 2 & -6 & -4 \\ & & 2 & 8 & 4 \\ \hline & 1 & 4 & 2 & 0 \end{array}$$

zero alert!

$$\begin{array}{r|rrrr} 3 & 1 & 2 & -6 & -4 \\ & & 3 & 15 & 27 \\ \hline & 1 & 5 & 9 & 23 \end{array}$$

$$\begin{array}{r|rrrr} 4 & 1 & 2 & -6 & -4 \\ & & 4 & 24 & 72 \\ \hline & 1 & 6 & 18 & 68 \end{array}$$

getting pretty big

Note that we can **stop plugging in *positive* x −values** here!

The graph of $h(x)$ ***seems*** to be **leaving for infinity** . . .

. . . so the subsequent y-values will not fit nicely on our graph . . .

(in fact, $h(4) = 68$ should not be plotted, either)

. . . and we will find no more zeroes in this direction!!!

So now let's start with $x = -1$ and proceed further in the negative direction:

$$\begin{array}{r|rrrr} -1 & 1 & 2 & -6 & -4 \\ & & -1 & -1 & 7 \\ \hline & 1 & 1 & -7 & 3 \end{array}$$

$$\begin{array}{r|rrrr} -2 & 1 & 2 & -6 & -4 \\ & & -2 & 0 & 12 \\ \hline & 1 & 0 & -6 & 8 \end{array}$$

$$\begin{array}{r|rrrr} -3 & 1 & 2 & -6 & -4 \\ & & -3 & 3 & 9 \\ \hline & 1 & -1 & -3 & 5 \end{array}$$

$$
\begin{array}{r|rrrr}
-4 & 1 & 2 & -6 & -4 \\
 & & -4 & 8 & -8 \\
\hline
 & 1 & -2 & 2 & -13
\end{array}
$$

$$
\begin{array}{r|rrrr}
-5 & 1 & 2 & -6 & -4 \\
 & & -5 & 15 & -45 \\
\hline
 & 1 & -3 & 9 & -49
\end{array}
$$

getting pretty big negative

We can **stop plugging in *negative* x −values** here!

The graph of $h(x)$ ***seems*** to be **leaving for negative infinity** . . .

. . . so the subsequent y-values will not fit nicely on our graph . . .

(in fact, $h(-5) = -49$ should not be plotted, either)

. . . and we will find no more zeroes in this direction!!!

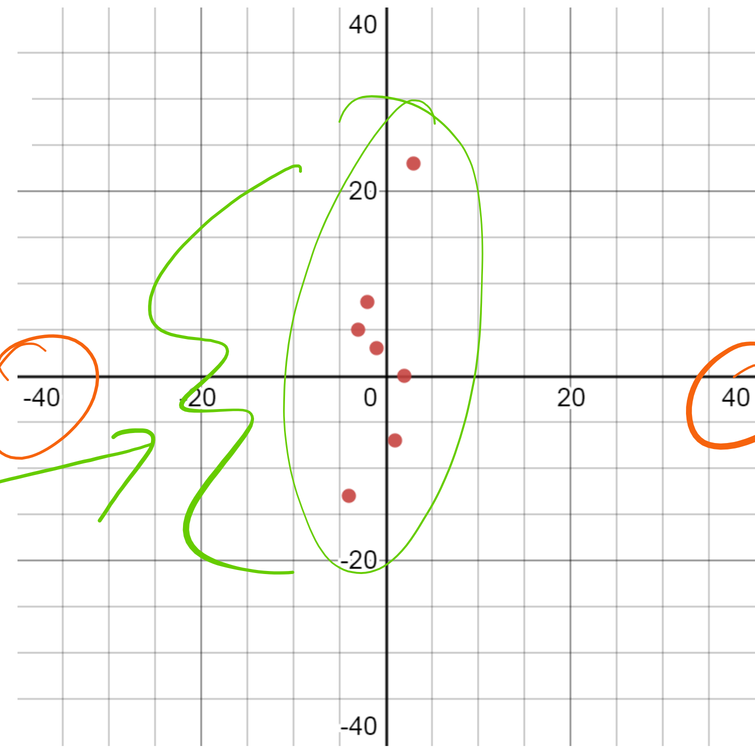
Note finally, that for our graph, we also need to plug in $\boldsymbol{x = 0}$**!!**

Which we don't need synthetic substitution for:

$$\mathrm{h}(0) = (0)^3 + 2(0) - 6(0) - 4 = \mathbf{-4}$$

Now, let's go ahead and plot the points we've obtained:

$h(x) = x^3 + 2x^2 - 6x - 4$



these are too big!

confusing graph!

Can you see what's **wrong** with this graph??

All the points are bunched in the middle!

This means the scale for the x-values is too broad!

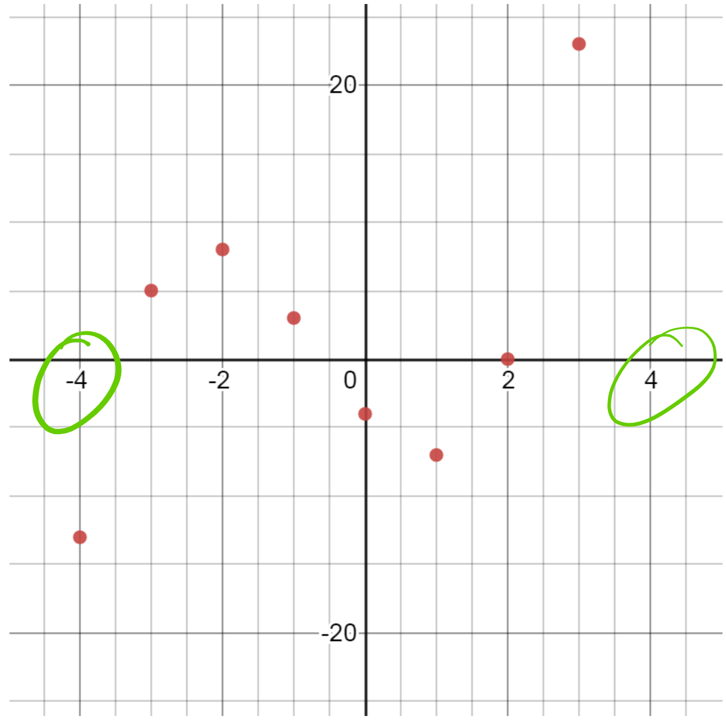
We only need the x-values to go from about -5 to $+5$.

Whereas the y-values we are plotting are bigger numbers!

So the scale for the y-values seems to be okay!!!

Let's revise the scale of our graph, restricting the x-axis to $[-5, 5]$:

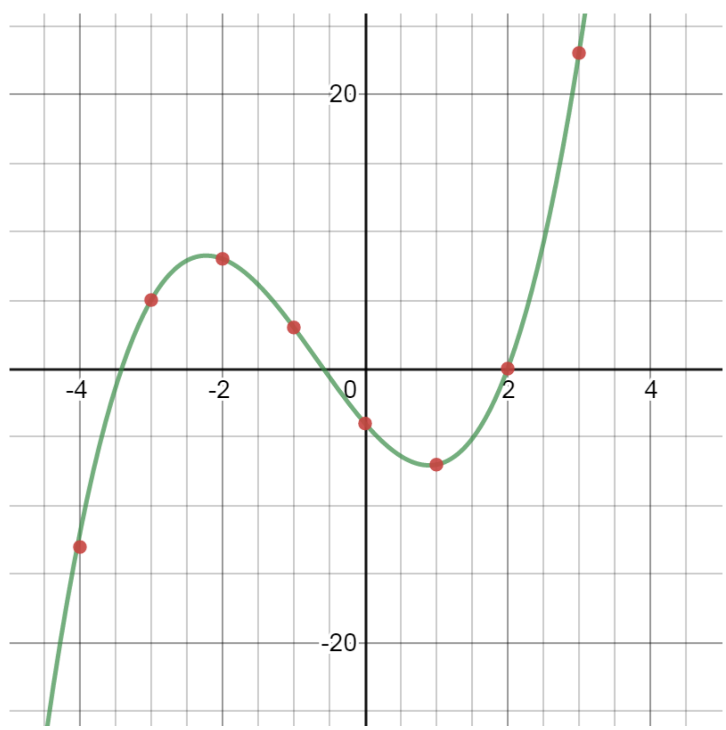
$h(x) = x^3 + 2x^2 - 6x - 4$



we are graphing x from -4 to 4

This looks much better! Now we will connect the points with a smooth curve:

$h(x)$



That's a good graph! It's harder to do by hand . . .

. . . but you will be asked to do it on the test!!!

That said, most of the points on this problem will be for finding the zeroes.

And we already found one:

$$\begin{array}{c|cccc} 2 & 1 & 2 & -6 & -4 \\ & & 2 & 8 & 4 \\ \hline & 1 & 4 & 2 & 0 \end{array}$$

$x = 2$ is a zero so $(x-2)$ is a factor

We can find the remaining zeroes by using **synthetic division**:

$$\begin{array}{c|cccc} 2 & 1 & 2 & -6 & -4 \\ & & 2 & 8 & 4 \\ \hline & 1 & 4 & 2 & 0 \end{array}$$

$$h(x) = (x-2)(x^2+4x+2)$$

so we need to solve:

$$x^2 + 4x + 2 = 0$$

This does not factor! Last time we used completing the square to solve the non-factorable quadratic equation. This time let's use the quadratic formula!

$$1x^2 + 4x + 2 = 0$$

$a \quad b \quad c$

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

$$x = \frac{-4 \pm \sqrt{4^2 - 4(1)(2)}}{2(1)}$$

$$x = \frac{-4 \pm \sqrt{16 - 8}}{2}$$

$$x = \frac{-4 \pm \sqrt{8}}{2}$$

$$x = \frac{-4 \pm 2\sqrt{2}}{2}$$

separate into two fractions

$$x = -\frac{4}{2} \pm \frac{2\sqrt{2}}{2}$$

$$x = -2 \pm \sqrt{2}$$

then simplify

So our zeroes for $h(x)$ are

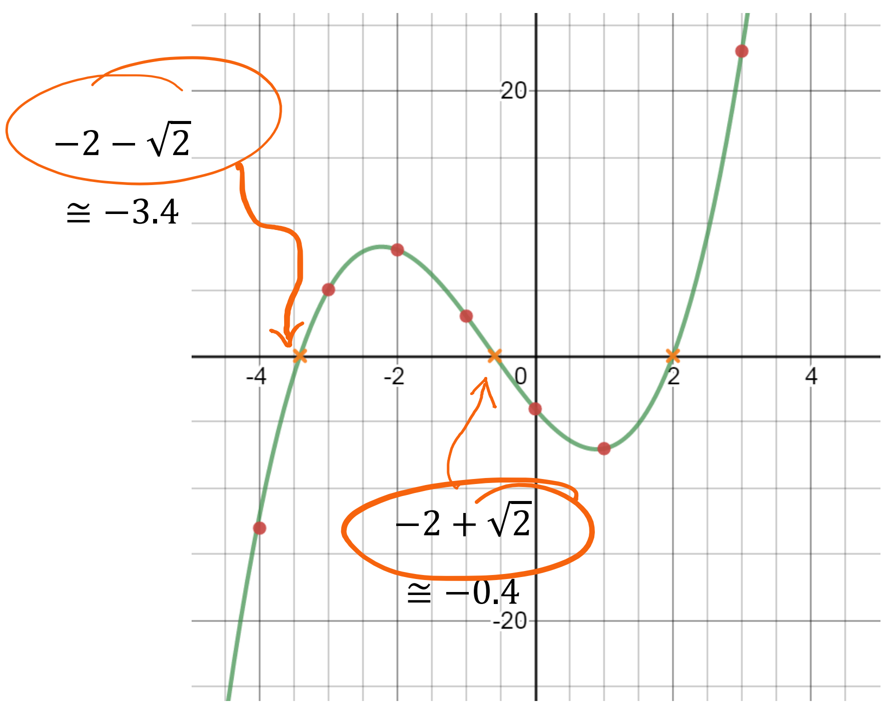
$$\{2, -2+\sqrt{2}, -2-\sqrt{2}\}$$

Note that the approximate decimal form for the irrational zeroes is:

$$-2+\sqrt{2} \approx -0.4$$

$$-2-\sqrt{2} \approx -3.4$$

Which checks out with our graph:



Finally, let's discuss how to do synthetic substitution more efficiently:

Instead of setting up a **different table** every time we plug a value in:

do this part in your head

1	1	2	-6	-4
		1	3	-3
	1	3	-3	-7

2	1	2	-6	-4
		2	8	4
	1	4	2	0

3	1	2	-6	-4
		3	15	27
	1	5	9	23

too many numbers

If we want, we can do the middle row of numbers in our heads…

… and put the entire set of inputs into one big table:

	1	2	-6	-4
0				-4
1	1	3	-3	-7
2	1	4	2	0
3	1	5	9	23
0				-4
-1	1	1	-7	3
-2	1	-1	-6	8
-3	1	-1	-3	5

these are the important numbers

This will help us to be organized as well!

I suggest trying this “streamlined approach” as you do the homework!