Quadratic Functions

The function we just obtained:

$$P(p) = -2p^2 + 400p - 14000$$

Tells us how much profit we can expect to earn for each price that we choose.

Let's plug in some possible prices to see what happens:

At $40 per dress our profit is:

$$P(40) = -2(40)^2 + 400(40) - 14000 = \mathbf{-\$1200}$$

This makes sense because at $40 we are selling them at cost, so we lose our $1200 per month fixed costs (rent)

At $60 per dress our profit is:

$$P(60) = -2(60)^2 + 400(60) - 14000 = \$\mathbf{2800}$$

At $80 per dress our profit is:

$$P(80) = -2(80)^2 + 400(80) - 14000 = \$\mathbf{5200}$$

At $100 per dress our profit is:

$$P(100) = -2(100)^2 + 400(100) - 14000 = \$\mathbf{6000}$$

At $120 per dress our profit is:

$$P(120) = -2(120)^2 + 400(120) - 14000 = \$\mathbf{5200}$$

For a while, increasing our price brings higher profits, because we're making more off each dress.

But then there comes a point where increasing our price seems to bring our profit down! Why do you think this is?

Our goal is to maximize our profit!!

The problem, then is what price will maximize our profit?

This is the same as finding the **maximum value** of $P(p)$.

How do we know for sure that this function even has a maximum value?

Have we seen any functions in this course that are similar to $P(p)$?

One function you might remember is

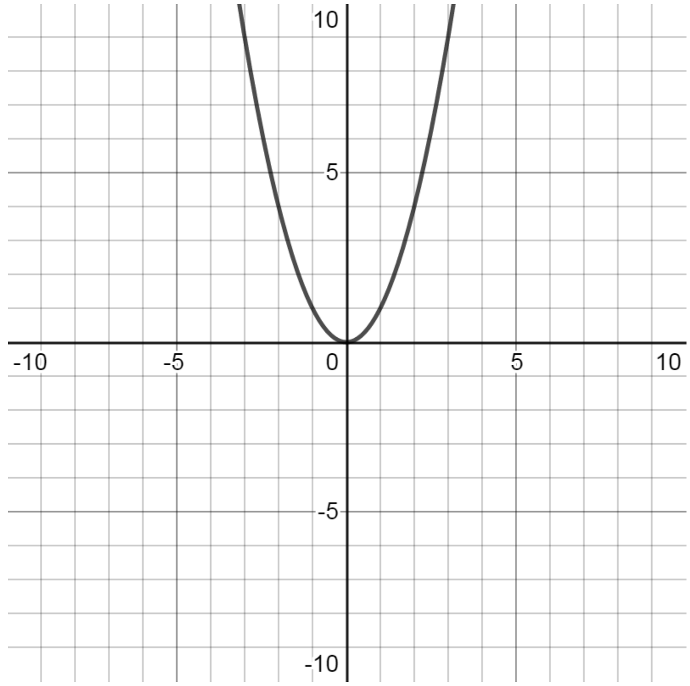
$$f(x) = x^2$$

This function, like $P(p)$, is a polynomial function of ***degree 2***.

Otherwise known as a **quadratic function**.

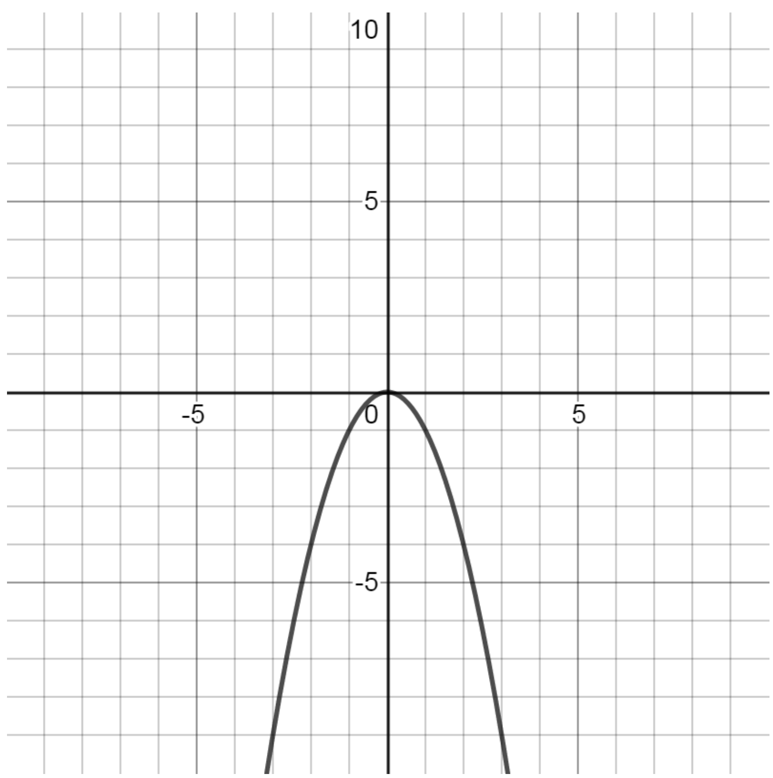
And it does not have a maximum, but it has a minimum:

$f(x) = x^2$



Of course if it were **vertically reflected**, ***then*** it would have a maximum:

$f(x) = -x^2$



Still, this function seems pretty simple compared to

$$P(p) = -2p^2 + 400p - 14000$$

But what if we were to perform additional transformations on $f(x) = -x^2$?

In particular, what if we were to apply a **vertical stretch** (times 2):

$$f(x) = -2x^2$$

And then a **horizontal** and **vertical** **shift**:

$$f(x) = -2(x-3)^2 + 5$$

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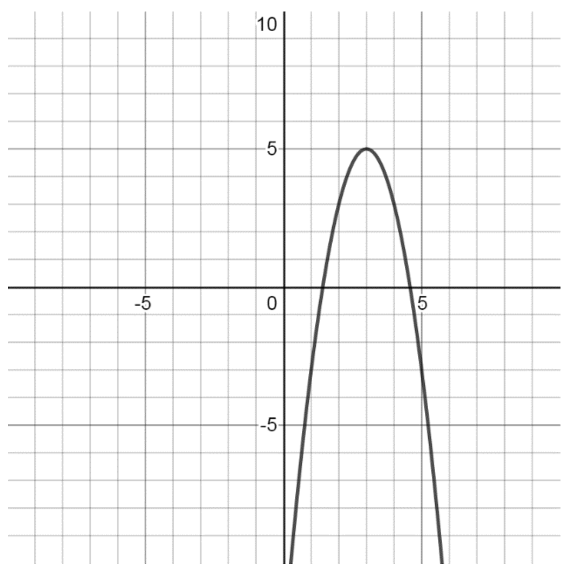


Note that if we were to multiply out the square, the formula would be

$$f(x) = -2(x^2 - 6x + 9) + 5$$
$$= -2x^2 + 12x - 14$$

Which is beginning to look very similar to our function $P(p)$!

WHAT IS THE POINT OF ALL THIS??

The point is that

$$P(p) = -2p^2 + 400p - 14000$$

is a transformation of

$$P(p) = p^2$$

And since $P(p) = p^2$ has a **vertex at** $(\mathbf{0}, \mathbf{0})$ . . .

To find the vertex of

$$P(p) = -2p^2 + 400p - 14000$$

Which will be the point of maximum profit . . .

We just need to see the **horizontal and vertical shift** from $P(p) = p^2$!

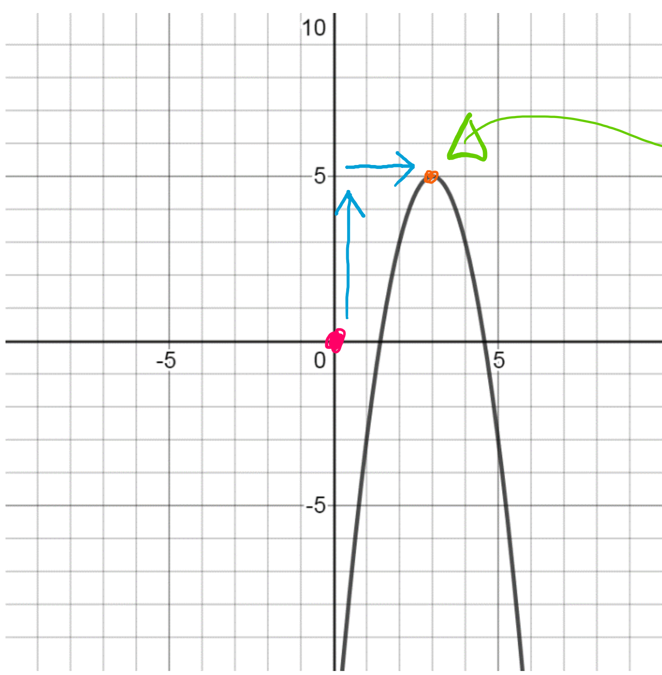
Which means we need to rewrite

$$P(p) = -2p^2 + 400p - 14000$$

As

$$P(p) = -2(p - h)^2 + k$$

Let's look at this graphically again to better understand what's going on:



the vertex is shifted 5 up and 3 to the right

so finding the maximum point of any quadratic function amounts
h ertical

to finding t e v
and horizontal shift
horizontal x^2 shift from $(0,0)$
vertical shift

And now let's do a very simple example.

Find the maximum point of:

$$g(x) = -(x+3)^2 + 2$$

We recognize $g(x)$ as a transformation of $y = x^2$.

Three transformations have been applied:

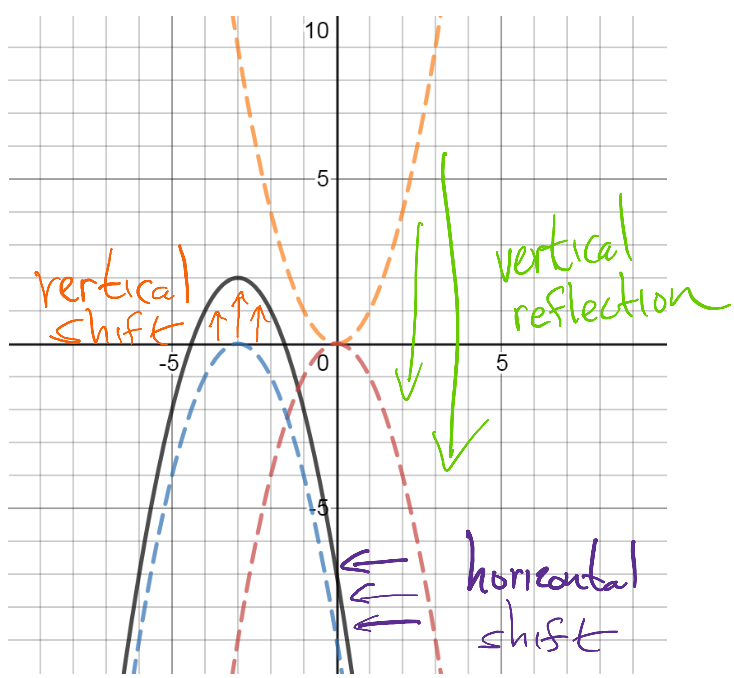
- ✓ vertical reflection
- ✓ horizontal shift (3 left)
- ✓ vertical shift (up 2)

$y = x^2$

$y = -x^2$

$y = -(x+3)^2$

$$g(x) = -(x+3)^2 + 2$$



The vertex of $g(x)$ starts from $(0, 0)$ and is shifted 3 left and 2 up . . .

. . . to $(-3, 2)$.

The maximum point is $(-3, 2)$ and the maximum value of $g(x)$ is 2.

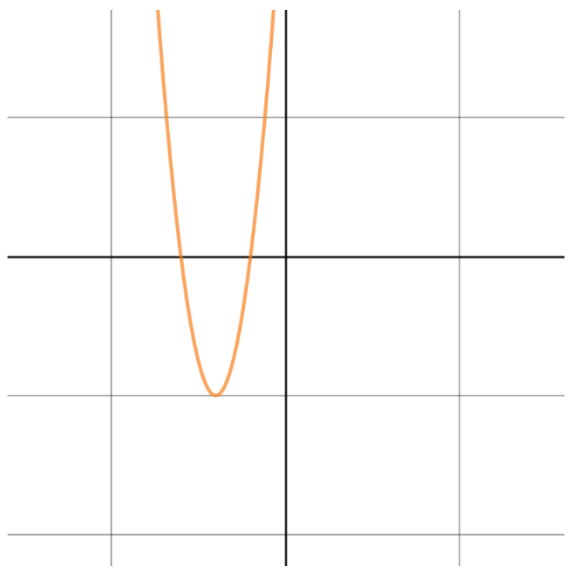
All of the examples we are looking at are functions of the GENERAL form:

$$f(x) = ax^2 + bx + c$$

These are called **quadratic functions** . . .

. . . and the graphs of these functions . . .

. . . are always **parabolas.**



The most important point on a parabola is the **vertex** . . .

. . . in part because in a real-life situation . . .

. . . it represents the **maximum** or **minimum**.

Therefore, the most important goal of this section is finding the vertex.

How to find the vertex of a parabola:

* One method is to find the **horizontal** and **vertical** **shift** of $y = x^2$:

* there is another method ... so this part is optional

Which can be seen if the function is put into STANDARD form:

$$g(x) = a(x - h)^2 + k$$

In which the vertex of the parabola is

$$(h, k)$$

Now, all quadratic functions of the GENERAL form:

$$g(x) = ax^2 + bx + c$$

can be put into STANDARD form:

$$g(x) = a(x - h)^2 + k$$

By an algebraic process called . . . *completing the square*.

Watch as I do it to the below function:

$$h(x) = x^2 - 6x + 2$$

The basic approach to *completing the square* involves working backwards from

$$(x - h)^2$$

This is complicated!

There is another method, involving a formula, that's easier!

Feel free to skip to that method!

Finding the vertex by completing the square (optional!)

We have that

$$h(x) = x^2 - 6x + 2$$

We need to turn this into

$$h(x) = (x - h)^2 + k$$

to see the vertex.

To do this, let's move the 2 to the side in the formula:

$$h(x) = x^2 - 6x \qquad\qquad + 2$$

We do this because this number is not part of our complete the square process, and we need to get it out of the way (for now).

Now, let's add and subtract 9 within the formula:

$$h(x) = x^2 - 6x {\color{red}{}+ 9} {\color{blue}{}- 9} + 2$$

I chose the number 9 for a reason that I will discuss later!

Look what happens to the function now:

$$h(x) = \underbrace{x^2 - 6x + 9}_{\text{factored}} \qquad -9 + 2$$

$$h(x) = (x-3)(x-3) \qquad -9 + 2$$

$$h(x) = (x-3)^2 - 7$$

Now, we have the function in the form we were looking for:

vertical shift

$$h(x) = a(x-h)^2 + k$$

horizontal shift

The vertex of the function is the point (h, k), or in this case, $(3, -7)$.

How did we do this?

The key was choosing the correct number to complete the square.

In this case, *that number was 9!*

In general, that number is

$$\left(-\frac{b}{2a}\right)^2$$

Where

$$f(x) = ax^2 + bx + c$$

NOTE: In order for this number to work, we need to factor a from the first two terms in the function:

so this coefficient is no longer b

$$f(x) = a(x^2 + \_\_x) \quad +c$$

To complete the square, we **add and subtract** $\left(\frac{b}{2a}\right)^2$ . . .

. . . within the parenthesis, then **factor**.

Let's see this with another example:

$$p(x) = x^2 + 8x + 11$$

Here, $a = 1$ so there's no need to factor it out!

What number completes the square?? Remember, it's

$$\left(\frac{b}{2a}\right)^2$$

And since for $p(x)$, $a = 1$ and $b = 8$, we have that

$$\left(\frac{b}{2a}\right)^2 = \left(\frac{4}{1}\right)^2 = 16$$

Which we will then **add and subtract** within the formula:

$$p(x) = x^2 + 8x \qquad\qquad + 11$$

$$p(x) = x^2 + 8x \mathbf{+ 16 - 16} + 11$$

Then factor:

$$p(x) = (x + 4)(x + 4) - 16 + 11$$

$$p(x) = (x + 4)^2 - 5$$

So the vertex is at $(-4, 5)$.

Let's do a slightly harder one:

$$q(x) = -4x^2 + 8x - 3$$

First, we need to factor out a from the first two terms:

$$q(x) = -4(x^2 - 2x \qquad\qquad) - 3$$

Next find the number that completes the square. It is

$$\left(\frac{b}{2a}\right)^2 = \left(\frac{8}{2(-4)}\right)^2 = (-1)^2 = 1$$

Let’s add and subtract this number within the parenthesis:

$$q(x) = -4(x^2 - 2x \qquad\qquad) - 3$$

$$q(x) = -4(x^2 - 2x + \mathbf{1} - \mathbf{1}) - 3$$

Now, the first three terms within the parenthesis factor:

$$x^2 - 2x + 1 = (x + 1)^2$$

But don’t forget the -1 that’s left over from the factoring:

$$q(x) = -4(x^2 - 2x + 1 - \mathbf{1}) - 3$$

Which must be multiplied by the -4 on the outside of the parenthesis!

So we get

$$q(x) = -4(x - 1)^2 + 4 - 3$$

$$q(x) = -4(x - 1)^2 + 1$$

So the vertex of the parabola is $(1,1)$.

Returning to our original example of profit as a function of price, we had:

$$P(p) = -2p^2 + 400p - 14000$$

Let's complete the square, and thereby convert $P(p)$ into the general form:

$$P(p) = -a(p-h)^2 + k$$

We first must factor out the leading coefficient a, which here is -2:

$$P(p) = \ -2(p^2 - 200p \qquad\qquad\qquad) - 14000$$

Now we complete the square . . .

. . . by adding $\left(-\frac{b}{2a}\right)^2 = \left(-\frac{400}{2(-2)}\right)^2 = (-100)^2 = 10000$

WITHIN the parenthesis:

$$P(p) = \ -2(p^2 - 200p \qquad\qquad\qquad\qquad\qquad) - 14000$$
$$P(p) = \ -2(p^2 - 200p + 10000 - 10000\ ) - 14000$$

and then factoring:

$$P(p) = -2((p-100)^2 - 10000) - 14000$$

$$= -2(p-100)^2 - 2(-10000) - 14000$$

$$= -2(p-100)^2 + 20000 - 14000$$

$$= -2(p-100)^2 + 6000$$

The vertex of the parabola is at $(100, 6000)$

Which means that the price that maximizes profit is $100

And that maximum profit will be $6000.

Whew!!! This is the optional method, for extra credit on the exam!

Finding the vertex using the formula

Using an algebraic method called completing the square . . .

. . . we get that the value

$$\bar{x} = -\frac{b}{2a}$$

will always be the x-coordinate of the vertex, for any quadratic function . . .

. . . in GENERAL FORM:

$$f(x) = ax^2 + bx + c$$

So for example, consider the quadratic function

$$p(x) = x^2 + 6x + 2$$

The x-coordinate of the parabola vertex is

$$\bar{x} = -\frac{b}{2a} = -\frac{6}{2(1)} = -3$$

To find the y-coordinate, just plug the x-coordinate into the function:

$$
\begin{aligned}
p(-3) &= (-3)^2 + 6(-3) + 2 \\
&= 9 - 18 + 2 \\
&= -7
\end{aligned}
$$

So the vertex of $p(x) = x^2 - 6x + 2$ is $(-3, -7)$.

Now let’s do the function:

$$q(x) = -4x^2 + 8x - 3$$

The vertex of $q(x)$ is at

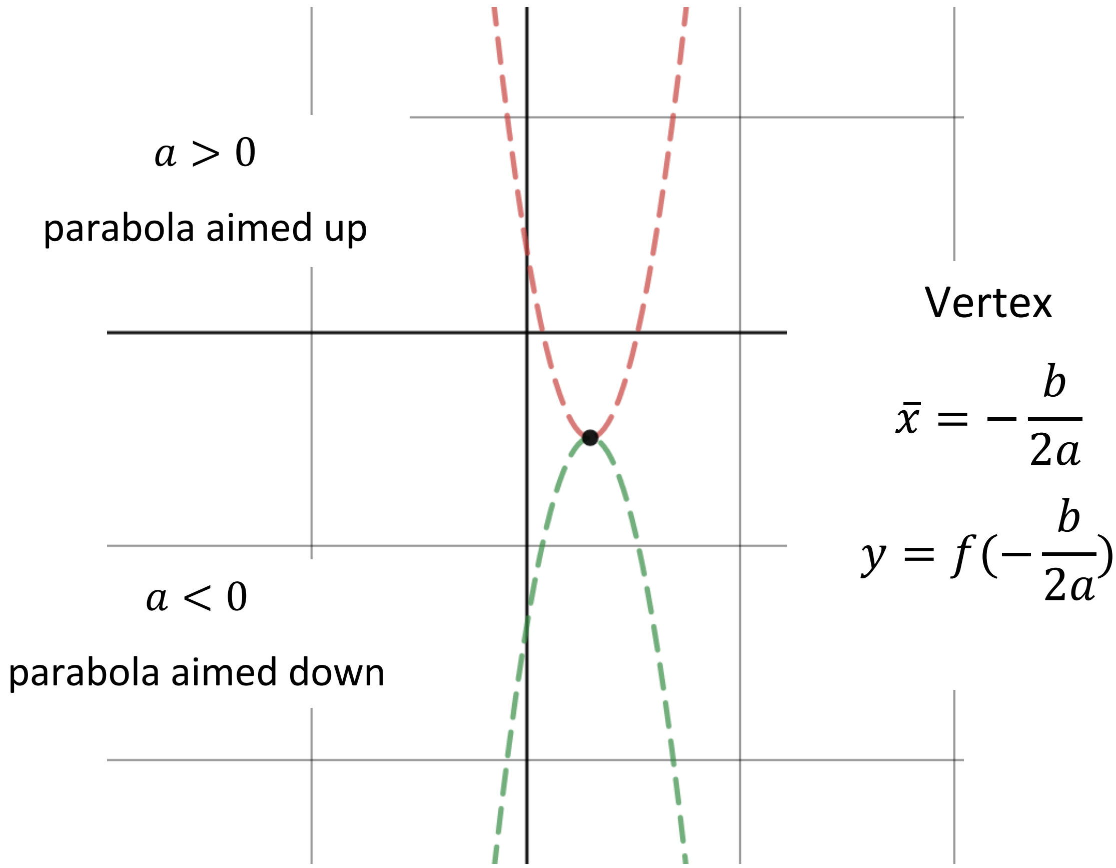
$$x = -\frac{b}{2a} = -\frac{8}{2(-4)} = 1$$

And since

$$
\begin{aligned}
q(1) &= -4(1)^2 + 8(1) - 3 \\
&= -4 + 8 - 3 \\
&= 1
\end{aligned}
$$

The vertex of $q(x)$ is the point $(1, 1)$.

Graphically, we can summarize this as shown:



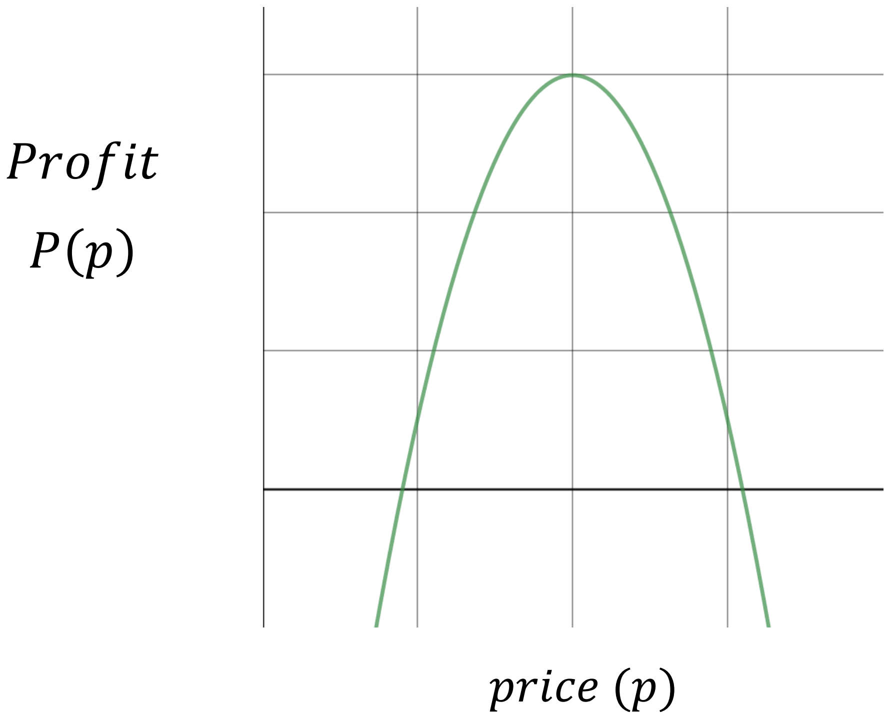
$$f(x) = ax^2 + bx + c$$

Let’s go back to our function for profit:

$$P(p) = -2p^2 + 400p - 14000$$

We see that for this quadratic function, $a = -2$, so $a < 0$.

That shows that the parabola will be **aimed down** . . .



To find its maximum point of $P(p) = -2p^2 + 400p - 14000$ . . .

. . . (which will tell us the maximum profit) . . .

We find

$$\bar{p} = -\frac{b}{2a} = -\frac{400}{2(-2)} = -100$$

So, the price that maximizes profit is \$100!

This price will give us a profit of $P(100) = \$6000$.