Periodic Functions

In this course we have studied *many kinds of functions*, but there are two in particular that are the most fundamental.

One of these is **linear**.

The other is **exponential**.

Both linear and exponential functions have a **starting point**:

Linear function

$$y = b + mx$$

Exponential function

$$y = A_0 * (1 + r)^t$$

starting point

Which appear on the graph as the $\boldsymbol{y}$**-intercept**.

From there, the linear function values increase (or decrease) by the same (constant) additive quantity, called the slope:

$$y = b \quad + \quad \underbrace{m + m + m + \ldots + m}_{x \text{ times}}$$

Whereas the exponential function values increase (or decrease) by the same multiplicative factor (which is $1 + r$):

$$y = A_0 \; * \; \underbrace{(1+r)(1+r)(1+r) * \ldots * (1+r)}_{t \text{ times}}$$

What is common about both types of functions is that they are either

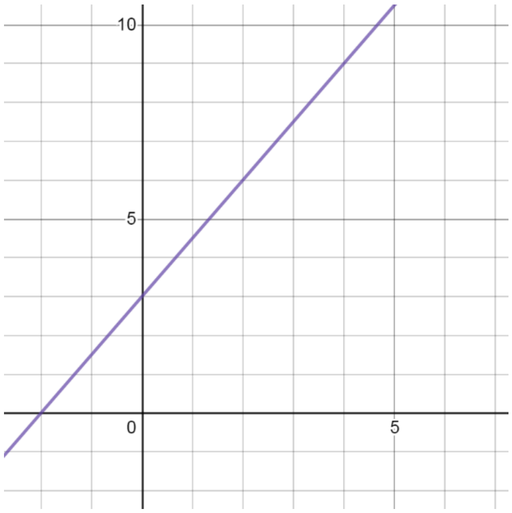
always increasing

or

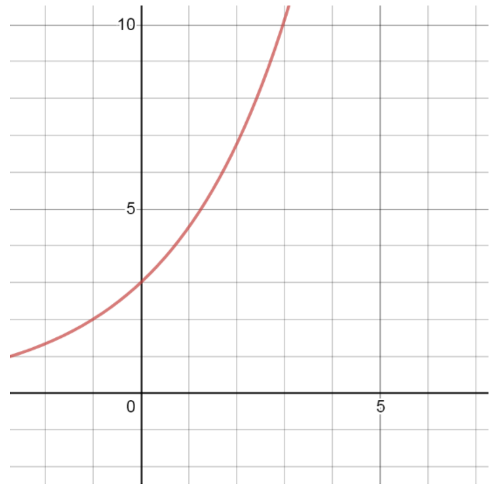
always decreasing

which we can easily see from their graphs:

$y = 3 + 1.5x$

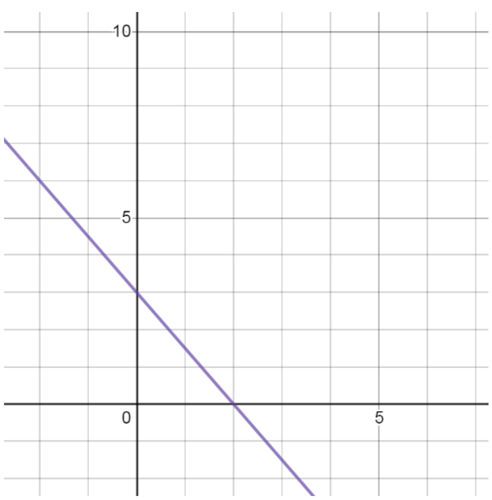


$y = 3 * 1.5^x$

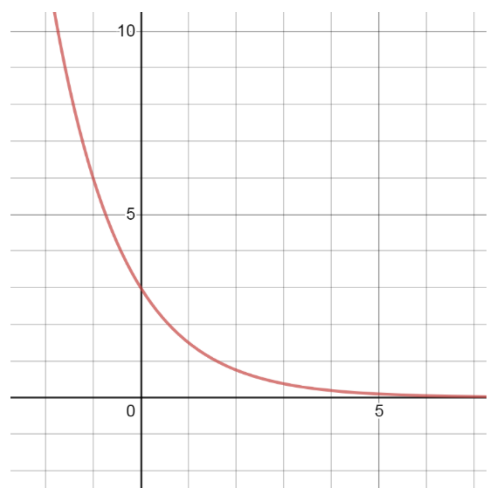


always increasing

$y = 3 - 1.5x$



$y = 3 * 0.5^x$



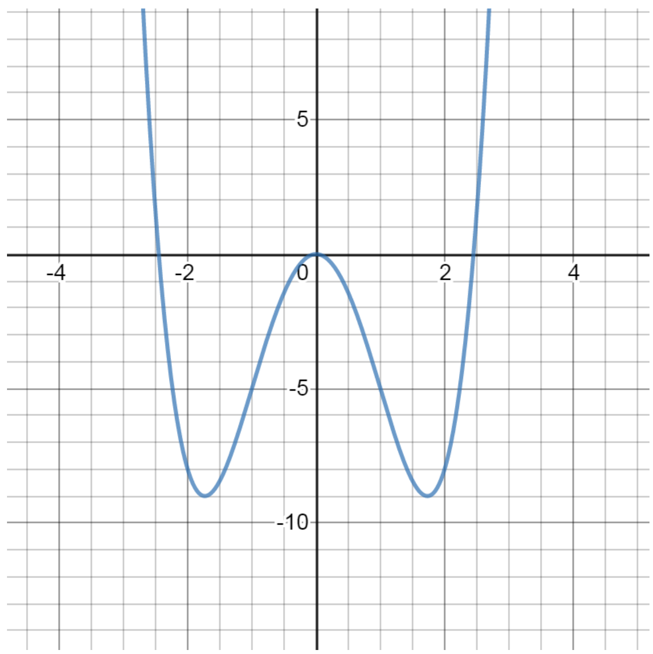
always decreasing

We need a function to describe quantities increasing . . .

. . . and decreasing!

Of course we do have this happen with polynomial functions:

$g(x) = x^4 - 6x^2$



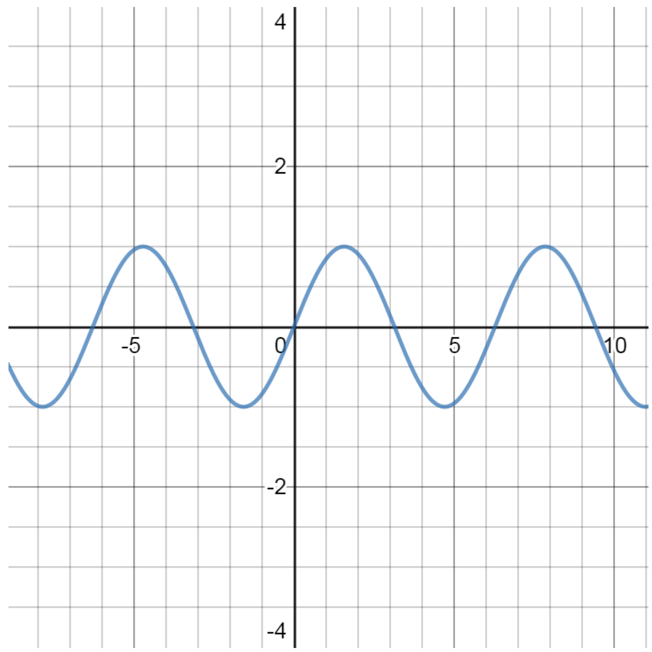
But to make our function truly fundamental . . .

. . . like linear and exponential change . . .

We would want our function to **increase and decrease in regular ways**.

Something like this . . .

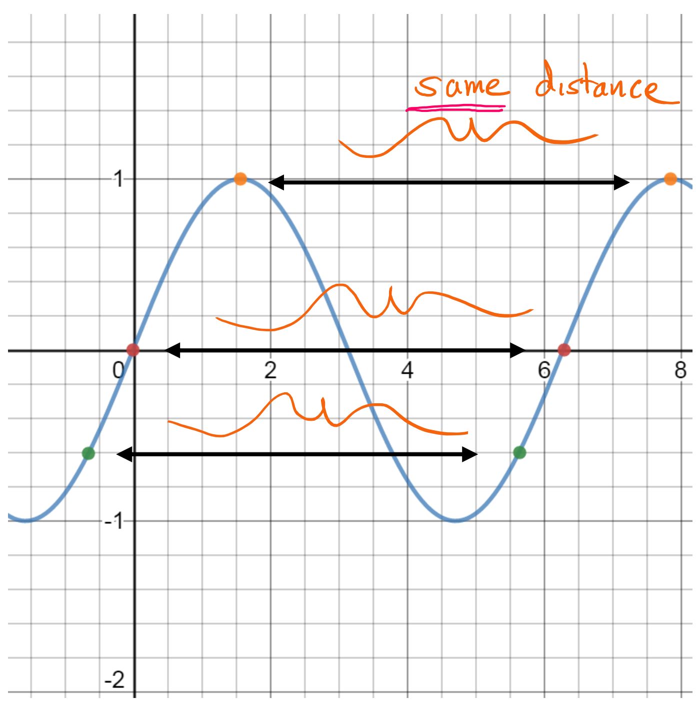
$h(x)$



This type of function is called **periodic**

The **period** is the length of time (distance along the x-axis) . . .

. . . to **complete one full cycle**:



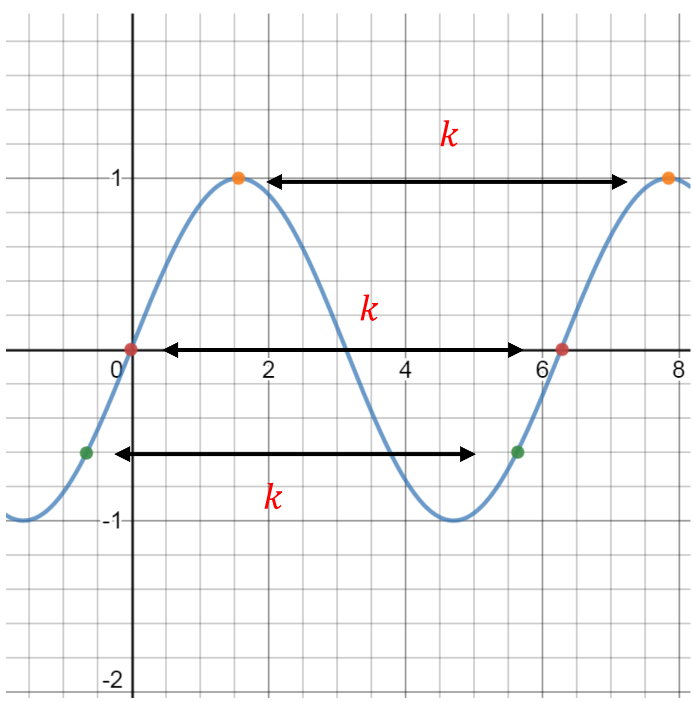
The period of a periodic function is the same everywhere on the function.

The formal way to define this type of function is as follows:

A function $\boldsymbol{f}$ is periodic if $\boldsymbol{f(x) = f(x \pm k)}$ for all $\boldsymbol{x}$ in its domain.

The smallest constant value k for which this is true is the period.

Again, that value k is the horizontal distance between equal y-values for all points on the graph:



The next question is . . .

. . . what kind of formula would produce this type of function?

Another way to think of periodic is that it runs in **cycles** . . .

cycle ⟷ **circle**

bicycle

Because when you go completely around a circle . . .

. . . or make a bicycle wheel turn one full revolution . . .

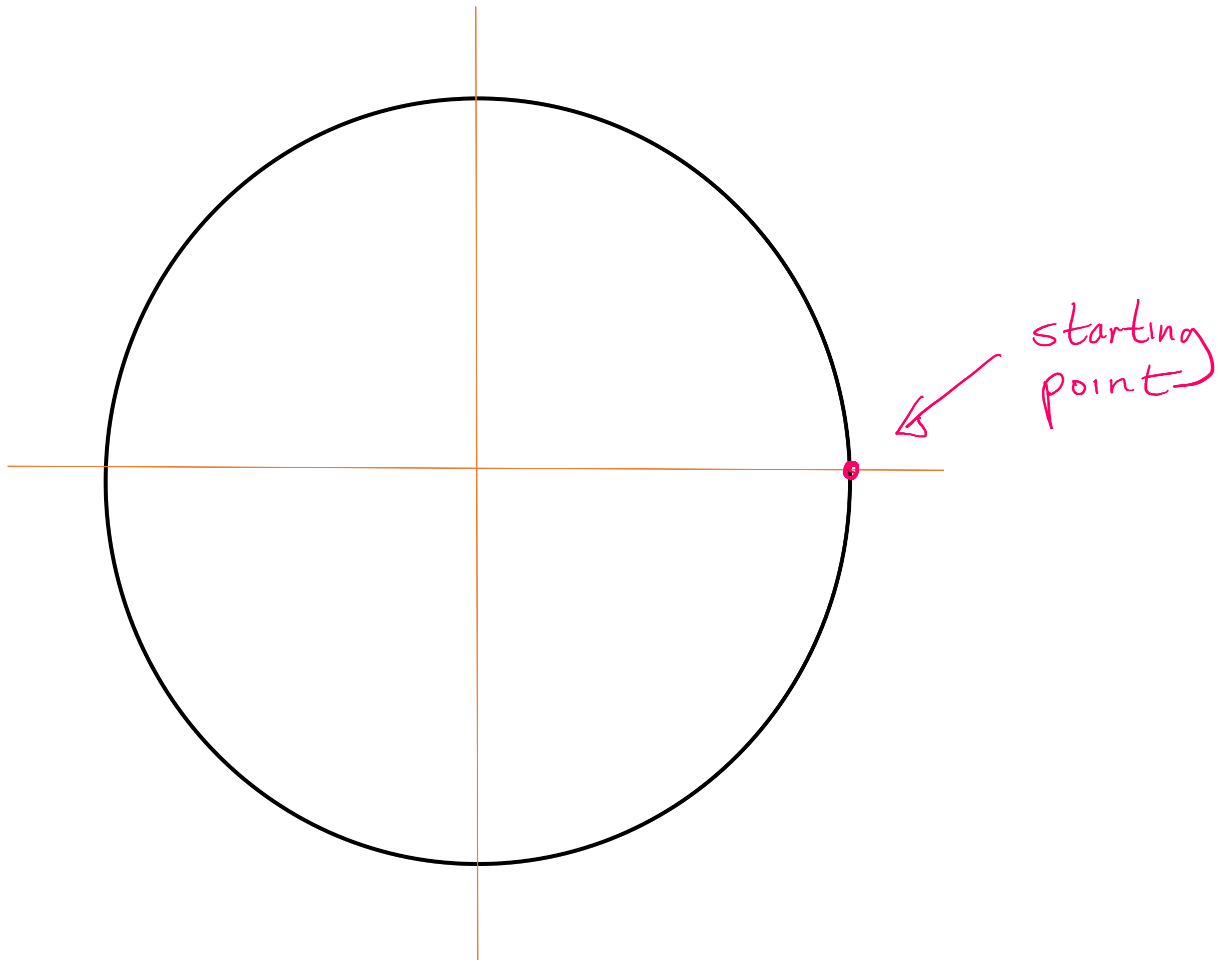
. . . you get back to the same place!

This happens with the hands of a clock.

This happens with the spinning of the earth.

This happens with the earth’s circuit around the sun.

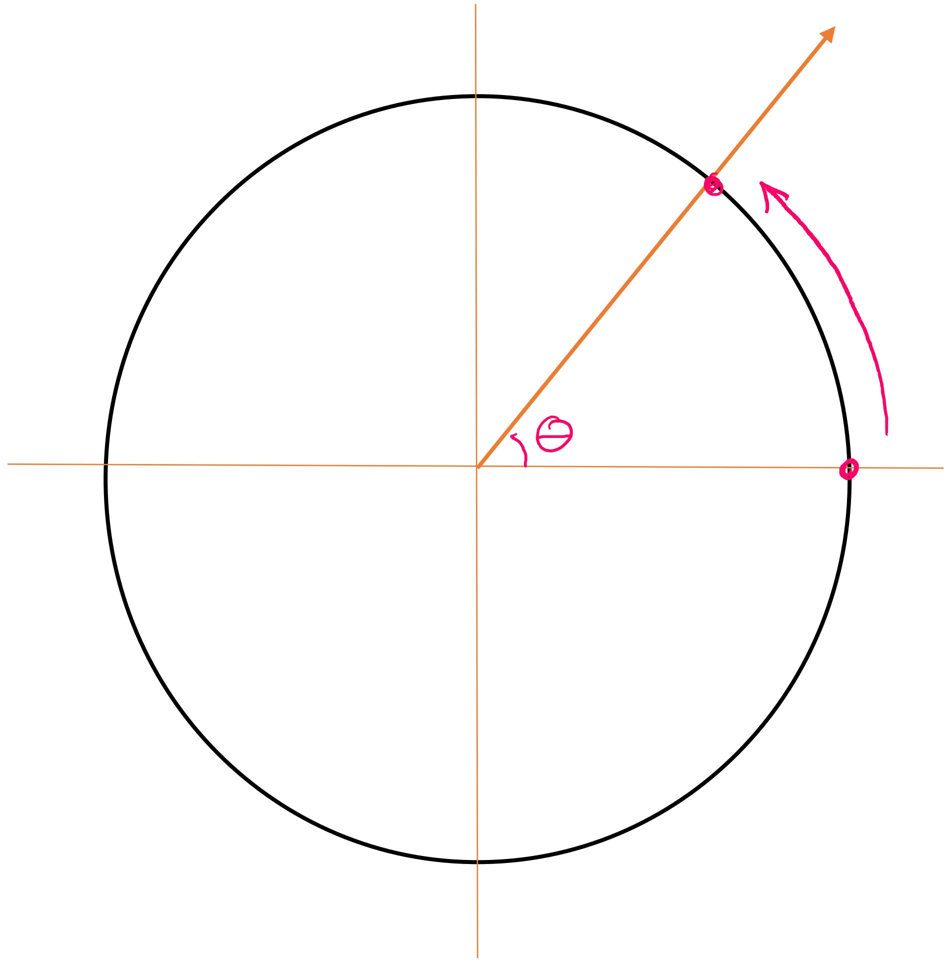
To conceptualize this, we will draw a circle, centered at the origin of a coordinate axis, and designate the starting position as a point on the circle on the far right edge:



To complete one full cycle (period), this point must travel all the way around the circle, counter-clockwise.

How will we measure its progress?

We can do this with an **angle**:



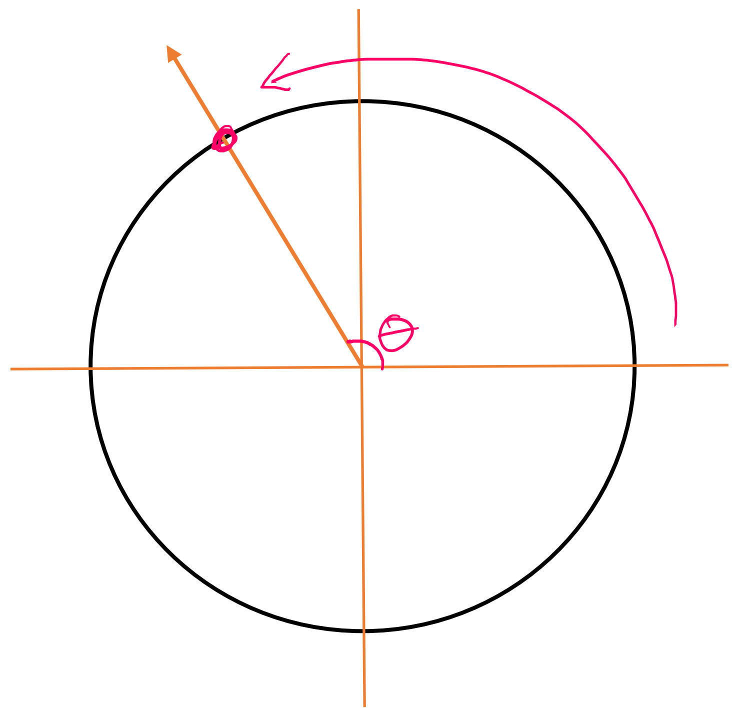
The Greek alphabetic letter “theta” is often used to designate angles in math.

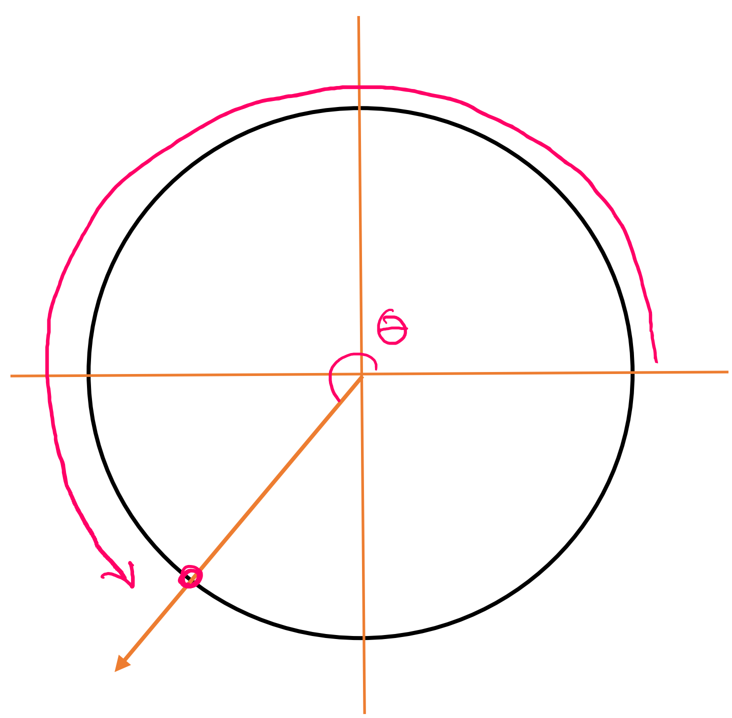
Key point:

As this angle increases . . .

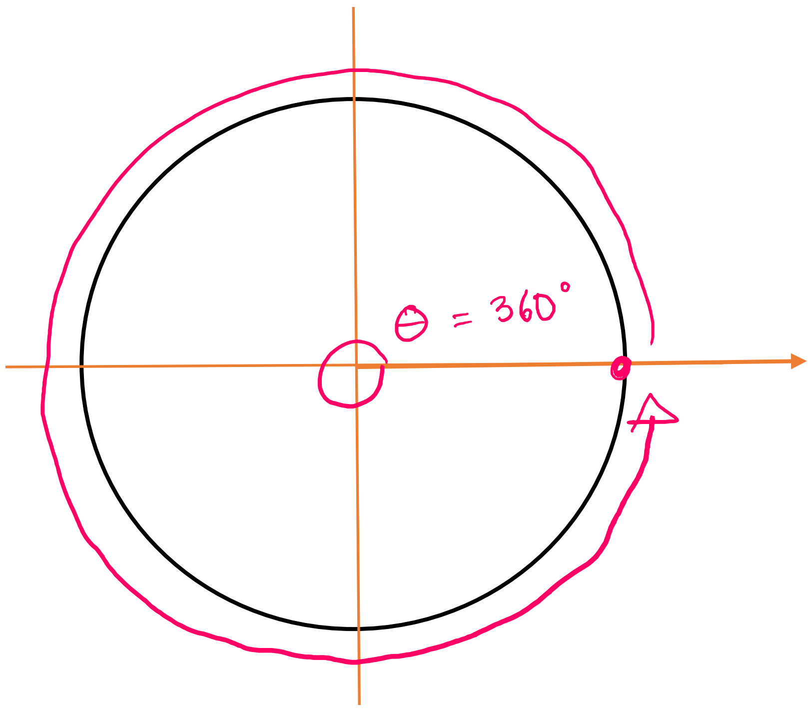
. . . the point on its terminal side moves counter-clockwise . . .

. . . around the circle!


θ


θ

Until θ eventually reaches $360°$ . . .



And we are back where we started, after completing a full period!!!

So the **independent variable** of our function . . .

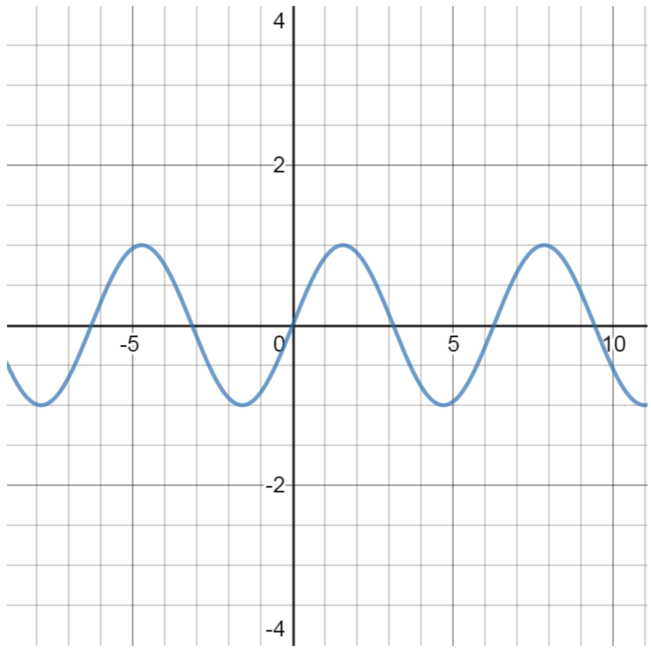
. . . the number that will be on the **horizontal axis** . . .

is the angle θ

Our function will be $f(\theta)$

But what will the dependent variable be??

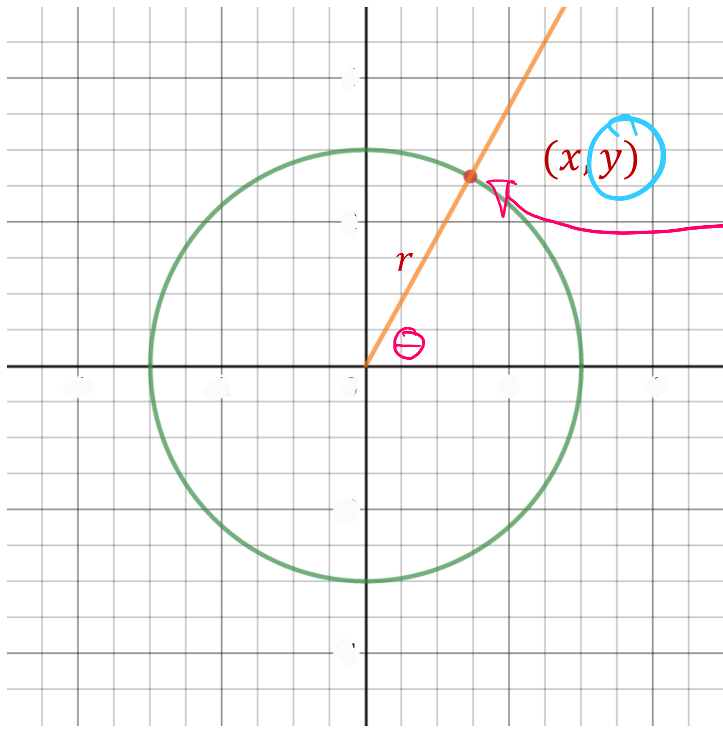
Remember, we want our graph to look like . . .



Which is to say that we want the y-values to go up and down.

To get y-values into our periodic function model, let's look at the coordinates of the point that is revolving around the circle as the angle θ changes.

Now we need to situate our circle in an actual coordinate axis:



as θ increases, the y-value of (x, y) will go up and down

You may notice that this circle has a radius of 3. That's not important. The circles we are looking at can have any radius . . . but must be centered at the origin.

See if you can visualize this . . .

As the angle θ increases . . .

and the point (x, y) moves around the circle . . .

what happens to the value of y?

Can you see that it goes up and down??????

This is the key to understanding our function!!!

Both the y and the x-coordinates of the point go up and down!

We will define our periodic functions in terms of these two numbers!!!

We define: $$f(\theta) = \sin(\theta) = \frac{y}{r}$$

and

$$g(\theta) = \cos(\theta) = \frac{x}{r}$$

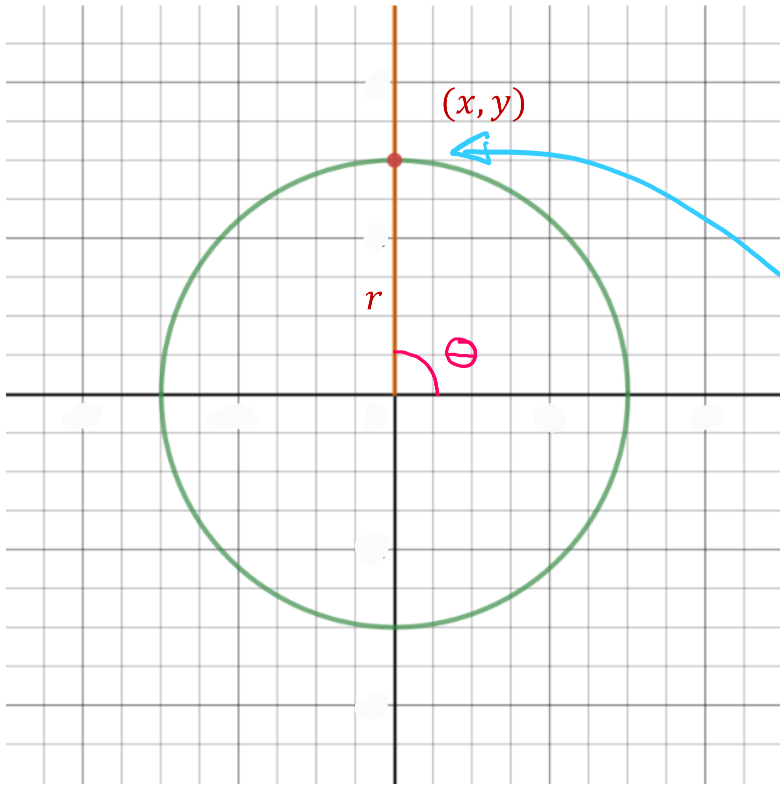
where r represents the radius of the circle.

We need to include r in our formula so that the size of the circle is irrelevant . . . only the angle θ will matter.

Note that $\sin(\theta)$ is written formally and pronounced as the “sine” of theta

And that $\cos(\theta)$ is written formally and pronounced as the “cosine” of theta.

Notice that the largest that $\sin(\theta)$ can ever get is 1:

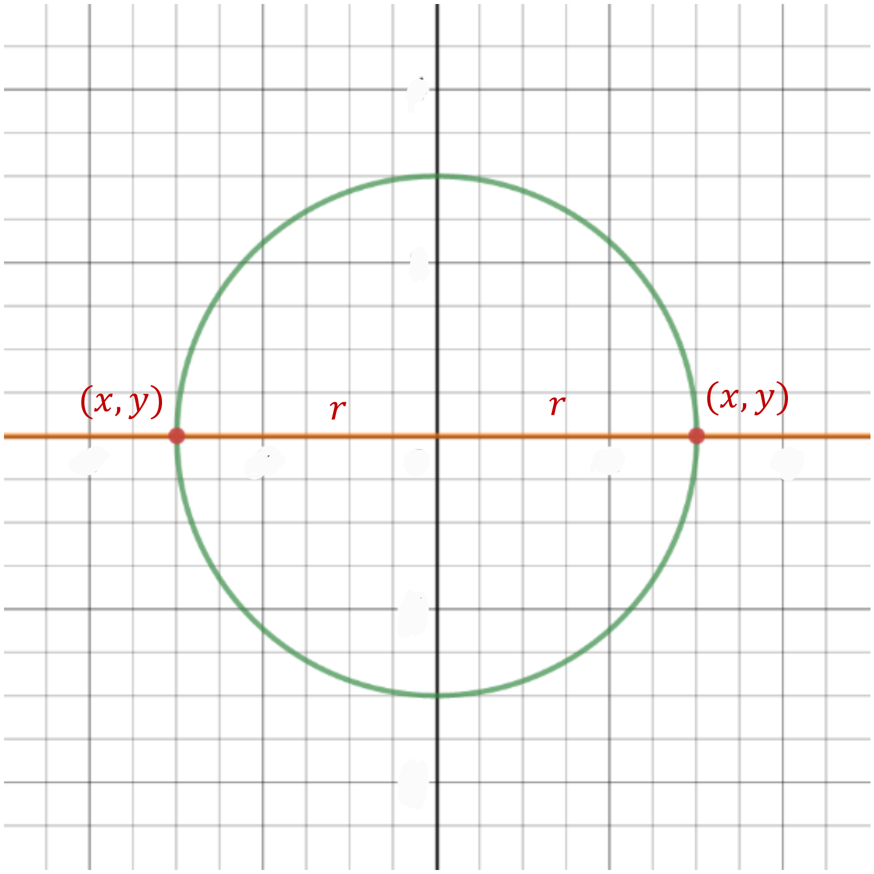


at $\theta = 90°$, the value of y is the biggest it can get.

Because when $\theta = 90°$, the value of y becomes equal to the radius r . . .

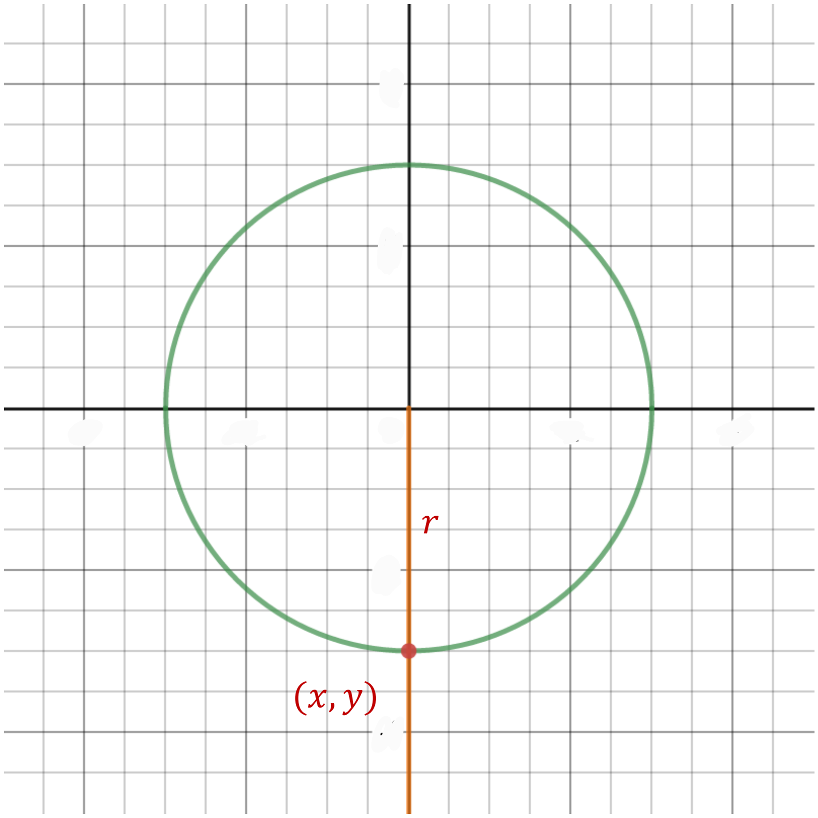
$$\sin(90°) = \frac{y}{r} = \frac{r}{r} = 1$$

Notice also that when $\theta = 0°$ or $180°$ or $360°$ . . . (etc), the sine is zero:



$$\sin(0°) = \sin(180°) = \sin(360°) = \frac{y}{r} = \frac{0}{r} = 0$$

And finally, notice that when θ becomes $270°$, the value of sine is -1:



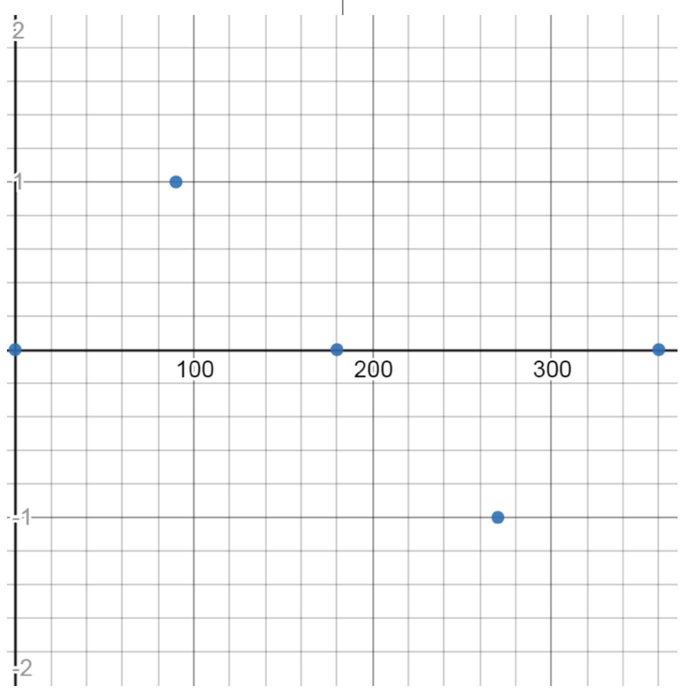
Because when $\theta = 90°$, the value of y becomes equal to the **negative** of the radius r . . .

$$\sin(270°) = \frac{y}{r} = \frac{-r}{r} = -1$$

Putting these values into a table, we begin to get the sense of the graph:

θ	$sin\ (\theta)$
$0°$	0
$90°$	1
$180°$	0
$270°$	-1
$360°$	0

$g(\theta) = \sin\ (\theta)$



But remember, the angle θ doesn't need to stop at $360°$!

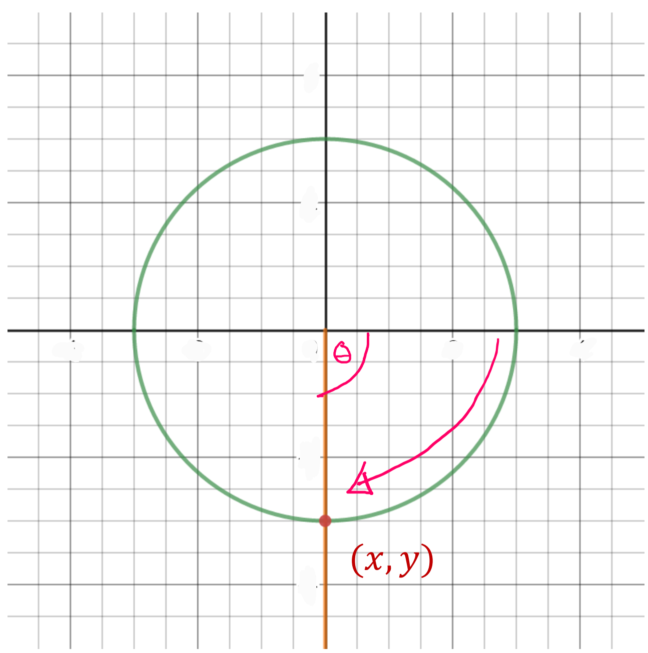
We can continue to increase θ for more cycles . . . which will repeat the process we have been doing:

θ	$sin\ (\theta)$
$0°$	0
$90°$	1
$180°$	0
$270°$	-1
$360°$	0
$450°$	1
$540°$	0
$630°$	-1
$720°$	0

And we can also move the angle θ in the negative direction!

This will mean moving the terminal side clockwise:

$$\sin(-90°) = \frac{y}{r} = \frac{-r}{r} = -1$$



θ = negative $90°$

Which in turn gives us more points:

θ	$sin\ (\theta)$
$0°$	0
$90°$	1
$180°$	0
$270°$	-1
$360°$	0
$450°$	1
$540°$	0
$630°$	-1
$720°$	0
$-90°$	-1
$-180°$	0
$-270°$	1

And giving us a graph that shows the **periodicity** even more clearly:

$g(\theta) = \sin(\theta)$