Solving Quadratic Equations that can't be factored

Consider the quadratic equation:

$$x^2 + 4x + 3 = 0$$

This equation can easily be solved by factoring:

$$x^2 + 4x + 3 = 0$$

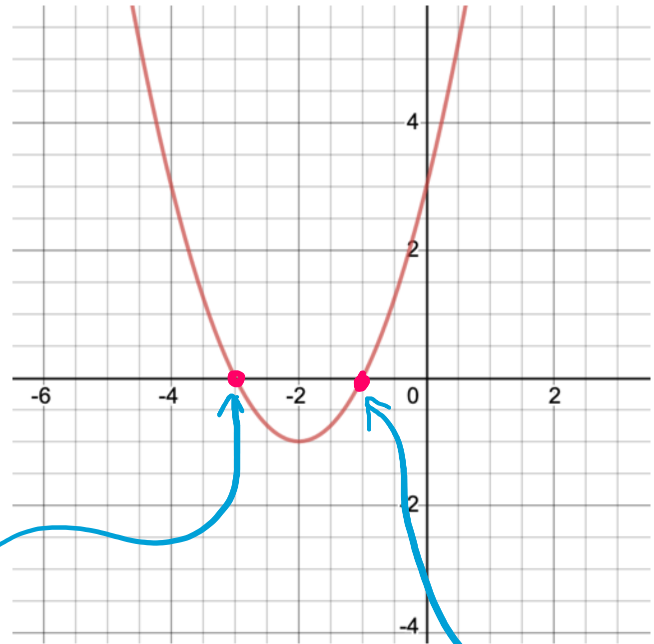
$$(x + 1)(x + 3)$$

$$x + 1 = 0 \;\; or \;\; x + 3 = 0$$

$$x = -1 \;\; or \; x = -3$$

Note that we can see these solutions on the graph of $y = x^2 + 4x + 3$:

$y = x^2 + 4x + 3$



the solutions to the equation are the x-values that make $y = 0$

$x = -3$

$x = -1$

Now, let's do a similar equation:

$$x^2 + 4x + \cancel{3} = 0$$

$$x^2 + 4x + 2 = 0$$

What happens when we try to factor?

$$x^2 + 4x + 2 = 0$$

$$(x + \quad)(x + \quad)$$

multiplies to 2

adds to 4

?

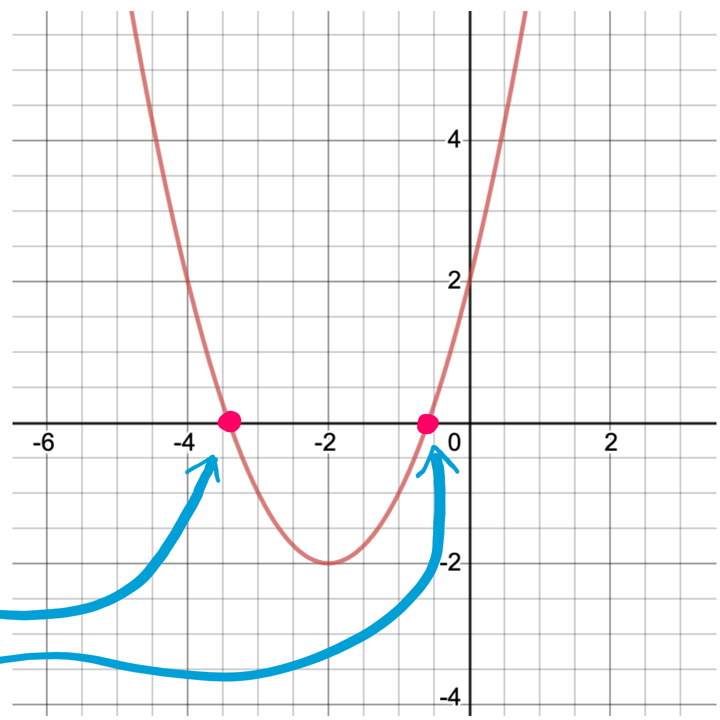
It **can't** be factored!

So how are we going to solve the equation?

Does this equation even have a solution?

Let's check the graph of $y = x^2 + 4x + 2$:

$$y = x^2 + 4x + 2$$



these x-values make y equal zero, so they solve the equation!

The graph crosses the $x - axis$, which means that there are solutions!

How to find them?

We will use a technique called **completing the square**:

To use the complete the square method . . .

. . . we begin with the equation in standard form:

$$x^2 + 4x + 2 = 0$$

If there were a coefficient to x^2 other than 1 . . .

. . . we would **divide both sides** by this number.

In this case we don't need to.

Next, we move the constant term to the other side . . .

. . . because we need to make room for a different number:

$$x^2 + 4x + 2 = 0$$

$$x^2 + 4x \qquad = \; -2$$

We will now complete the square . . .

. . . by adding a number to both sides

$$x^2 + 4x + \mathbf{4} \; = \; -2 + \mathbf{4}$$

Here's how you get that number:

Take the coefficient of x . . .

. . . divide by 2 . . .

. . . and square it:

$$x^2 + 4x \qquad = -2$$

$$\left(\frac{4}{2}\right)^2 = 4$$

Here we end up with the same number by coincidence.

So we have

$$x^2 + 4x + \mathbf{4} = -2 + \mathbf{4}$$

We factor the left side:

$$(x+2)(x+2) = 2$$

Which can be written

$$(x+2)^2 = 2$$

Now we take the square root of both sides:

$$\sqrt{(x+2)^2} = \pm\sqrt{2}$$

(remember . . . every number has two square roots so we need a $\pm$!)

Which gives us

$$x + 2 = \pm\sqrt{2}$$

And then

$$x = -2 \pm \sqrt{2}$$

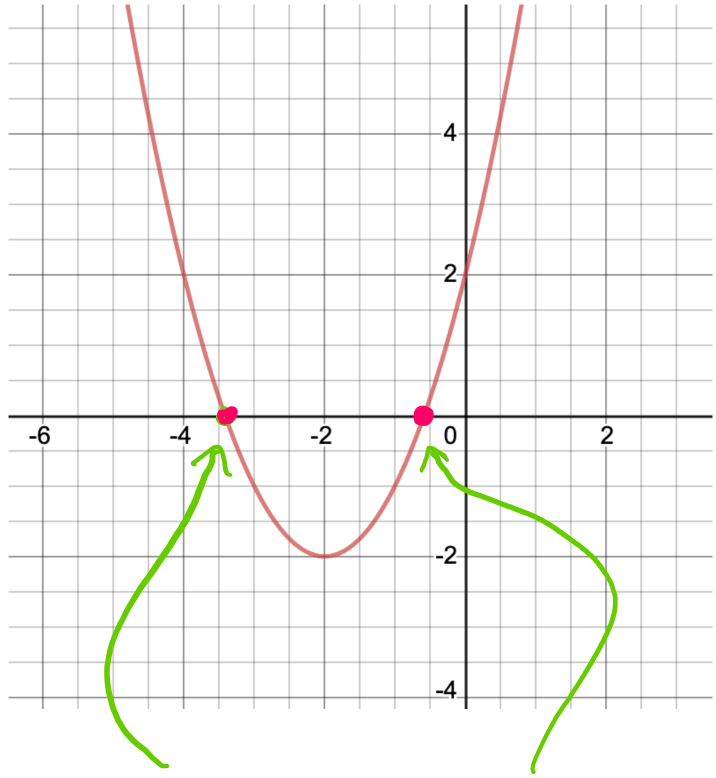
These are the two solutions:

$$x = -2 + \sqrt{2}$$

$$x = -2 - \sqrt{2}$$

We can see these on the graph:

$$y = x^2 + 4x + 2$$



$x = -2 - \sqrt{2}$ $\qquad$ $x = -2 + \sqrt{2}$

$\cong -3.41$ $\qquad$ $\cong -0.39$

Complete the square solves all quadratic equations!

That said, it can get a bit more complicated when the leading coefficient of the equation is not equal to one. Let's do one of these.

Solve:

$$3x^2 - 2x - 4 = 0$$

We might first try to **factor** the quadratic . . . but it **can't** be factored.

We will solve by completing the square.

First, divide both sides by 3:

$$3x^2 - 2x - 4 = 0$$

$$x^2 - \frac{2}{3}x - \frac{4}{3} = 0$$

Next move the constant term to the other side:

$$x^2 - \frac{2}{3}x - \frac{4}{3} = 0$$

$$x^2 - \frac{2}{3}x \qquad = \frac{4}{3}$$

Now find the number that completes the square:

take half of the coefficient of x: $$-\frac{2}{3} \cdot \frac{1}{2} = -\frac{1}{3}$$

then square it! $$\left(-\frac{1}{3}\right)^2 = \frac{1}{9}$$

And we add this number to both sides:

$$x^2 - \frac{2}{3}x + \frac{1}{9} = \frac{4}{3} + \frac{1}{9}$$

The left side factors . . . and the right side must be added:

$$\left(x - \frac{1}{3}\right)^2 = \frac{13}{9}$$

And we take the square root of both sides and solve:

$$\sqrt{\left(x-\frac{1}{3}\right)^2} = \pm\sqrt{\frac{13}{9}}$$

$$x-\frac{1}{3} = \pm\sqrt{\frac{13}{9}}$$

$$x = \frac{1}{3} \pm \frac{\sqrt{13}}{3}$$

$$x = \frac{1 \pm \sqrt{13}}{3}$$

If you don't want to use the complete the square method . . .

. . . you can use a formula instead.

The Quadratic Formula

The solutions to the quadratic equation

$$ax^2 + bx + c = 0$$

are given by

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

To see where the formula comes from . . .

. . . watch the video version of this lesson!