Verifying Trigonometric Identities

We are now going to do a different version of the problems about simplifying trigonometric identities. These problems are very similar!

They also come down to substituting trigonometric identities . . .

. . . combined with a bit of algebra.

Here's the first one:

Verify the given identity:

$$\frac{\sec^2\theta - 1}{\sec^2\theta} = \sin^2\theta$$

Here, we are being asked to ***show*** that the left side equals the right side.

It's much easier to show this by starting with the more complex side . . .

. . . and simplifying it down.

(again, these problems are very similar to the previous ones we did)

So let's begin with the left side and see how we can simplify it.

$$\frac{\sec^2\theta - 1}{\sec^2\theta} = \sin^2\theta$$

Note: the numerator contains a squared trigonometric expression, $\sec^2\theta$.

So does the denominator! **But let’s just do the numerator** . . .

. . . because of the -1!

We have that one of our Pythagorean Identities is

$$\tan^2\theta + 1 = \sec^2\theta$$

Subtracting 1 from both sides, we get

$$\tan^2\theta = \sec^2\theta - 1$$

So we can substitute this into the original left side as follows:

$$\frac{\sec^2\theta - 1}{\sec^2\theta}$$

$$= \frac{\tan^2\theta}{\sec^2\theta}$$

Now, let’s reduce this to sines and cosines:

$$\frac{\tan^2\theta}{\sec^2\theta}$$

$$= \frac{\frac{\sin^2\theta}{\cos^2\theta}}{\frac{1}{\cos^2\theta}}$$

which, once we multiply top and bottom by $\cos^2\theta$:

$$\frac{\frac{\sin^2\theta}{\cos^2\theta} * \cos^2\theta}{\frac{1}{\cos^2\theta} * \cos^2\theta}$$

gives us

$$\sin^2\theta$$

which was the right-hand side of the original identity!

In other words, we got to the place we wanted to go just by doing what seemed like “natural” steps of simplifying!

Let’s do another one.

Verify the given identity:

$$\sec x - \cos x = \sin x \tan x$$

Here, both sides seem relatively simplified. However, let's start with the right-hand side, if for no other reason than the terms are further apart.

$$\sec x - \cos x$$

The only thing that occurs to me to do is to reduce the secant term to its cosine element:

$$\sec x - \cos x$$

$$= \frac{1}{\cos x} - \cos x$$

The only thing I can think to do next is subtract the terms. For this we need to convert to the LCD, which is $\cos x$:

$$\frac{1}{\cos x} - \cos x$$

$$= \frac{1}{\cos x} - \cos x * \left(\frac{\cos x}{\cos x}\right)$$

$$= \frac{1}{\cos x} - \frac{\cos^2 x}{\cos x}$$

$$= \frac{1 - \cos^2 x}{\cos x}$$

This gives a squared trigonometric term! We might remember seeing this expression in the numerator before; if not, we can go back to our basic Pythagorian Identity and subtract the $\cos^2 x$ from both sides:

$$\sin^2 x + \cos^2 x = 1$$

$$\sin^2 x = 1 - \cos^2 x$$

Now we can substitute $\sin^2 x$ directly into the numerator of our expression:

$$= \frac{1 - \cos^2 x}{\cos x}$$

$$= \frac{\sin^2 x}{\cos x}$$

At this point we might need to check back to our goal. We wanted to get to this expression:

$$\sin x \tan x$$

We can make this final transition by factoring the numerator:

$$\frac{\sin^2 x}{\cos x} = \frac{\sin x * \sin x}{\cos x} = \sin x * \left(\frac{\sin x}{\cos x}\right) = \sin x \tan x$$

And we are done!

Let's do one more (harder) problem:

Verify the given identity:

$$2\sec^2\theta = \frac{1}{1-\sin\theta} + \frac{1}{1+\sin\theta}$$

For this one, it would be better to start with the left-hand side . . .

. . . and try to somehow simplify it.

$$\frac{1}{1-\sin\theta} + \frac{1}{1+\sin\theta}$$

As with the last problem, the only thing that may seem doable is adding these fractions together. We need to convert them to the same LCD:

$$\frac{1}{1-\sin\theta} * \frac{1+\sin\theta}{1+\sin\theta} + \frac{1}{1+\sin\theta} * \frac{1-\sin\theta}{1-\sin\theta}$$

$$= \frac{1+\sin\theta}{(1-\sin\theta)(1+\sin\theta)} + \frac{1-\sin\theta}{(1+\sin\theta)(1-\sin\theta)}$$

Multiplying out the denominators we get

$$= \frac{1+\sin\theta}{1-\sin^2\theta} + \frac{1-\sin\theta}{1-\sin^2\theta}$$

and combining the numerators we get

$$= \frac{2}{1 - \sin^2 \theta}$$

Now on the previous problem we used the basic Pythagorean Identity to show that

$$\sin^2 x = 1 - \cos^2 x$$

We could use the same reasoning to show that

$$\cos^2 x = 1 - \sin^2 x$$

Substituting this identity into our current expression, we get that

$$= \frac{2}{1 - \sin^2 \theta}$$

$$= \frac{2}{\cos^2 \theta}$$

And to finish the problem, let's use the fact that

$$\cos^2 \theta = \frac{1}{\sec^2 \theta}$$

to get

$$\frac{2}{\cos^2 \theta} = \frac{2}{\frac{1}{\sec^2 \theta}} = 2 \sec^2 \theta$$