exponential functions and the compound interest formula

All of the examples we have seen in this section are real-life situations. It's true that many things in the real world change exponentially . . .

. . . because many quantities in the real world change in a way that . . .

. . . depends on how much is there:

Real-life contexts for exponential **growth** (percent increase):

- Compound interest
- Population change
- Disease (epidemic)

Real-life contexts for exponential **decay** (percent decrease):

- Depreciation
- Radioactive decay
- Drug levels in bloodstream

Now we will be looking at exponential change from a purely mathematical perspective.

Which is to say, we will be looking at exponential functions.

One difference here, is that while in the real world, exponential change usually describes a quantity changing over time . . .

. . . and both the quantity changing and time must be positive . . .

But in the mathematical world, negative numbers can still exist.

So let's consider the exponential function:

$$f(x) = 10 * 2^x$$

What is its domain and range?

To decide on its domain, we need to think about the following question:

## Can an exponent be negative?

What would happen if we tried to plug in the value $x = -1$?

$$f(-1) = 10 * 2^{-1}$$

In fact, negative exponents are okay!!!!

This is the rule about them:

$$a^{-n} = \frac{1}{a^n}$$

So

$$
\begin{aligned}
f(-1) &= 10 * 2^{-1} \\
&= 10 * \left(\frac{1}{2^1}\right) \\
&= 10 * \left(\frac{1}{2}\right) \\
&= \quad 5
\end{aligned}
$$

In fact, any real number, positive or negative, can be an exponent.

So the domain of

$$f(x) = 10 * 2^x$$

is $(-\infty, \infty)$.

How about the range?

Here things are a bit different.

The relevant point is this:

$b^n > 0$ , for any $b > 0$

This is to say that as long as the "base" of an exponent is positive . . .

(and we will always make the base positive!)

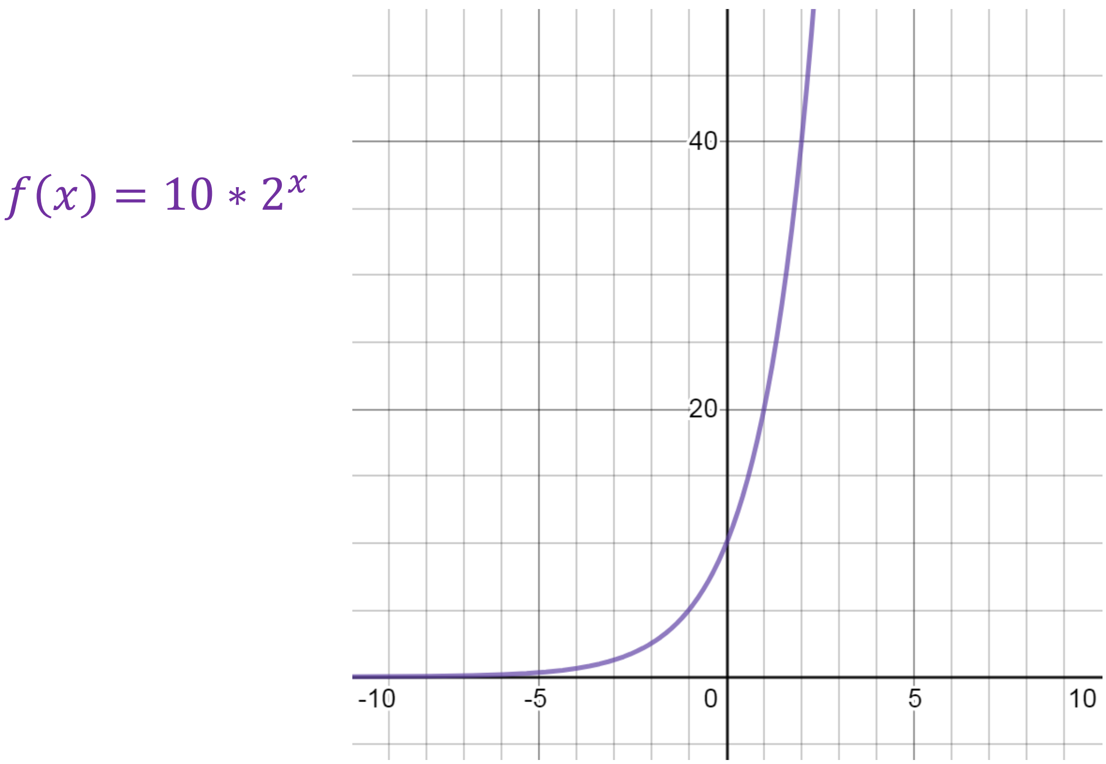
Then taking that number to **any** power will always produce a positive result!

Therefore the range of

$$f(x) = 10 * 2^x$$

is $(0, \infty)$.

We can see all of this most easily from the graph:




The graph gets very close to the $x$-axis, but actually never touches it!

That's right, the graph of $f(x)$ has a horizontal asymptote at $y = 0$.

In this case, it's because $\lim_{x \to -\infty} f(x) = 0$

A basic exponential function has the form:

$$f(x) = a * b^x$$

Depending on how we choose $a$ and $b$, this function has some different possible graphs . . .

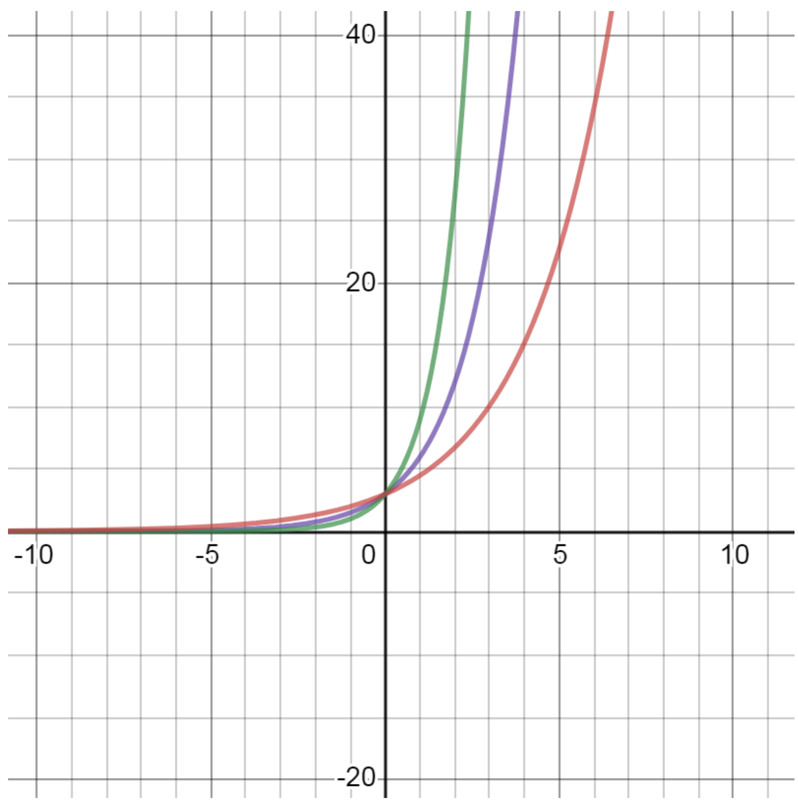
Here we have $f(x) = 3 * b^x$ for different bases $b$:

$$f(x) = 3 * b^x$$

$\boldsymbol{b = 3}$

$\boldsymbol{b = 2}$

$\boldsymbol{b = 1.5}$




The important key here is that for all bases, $b$

$$b > 1$$

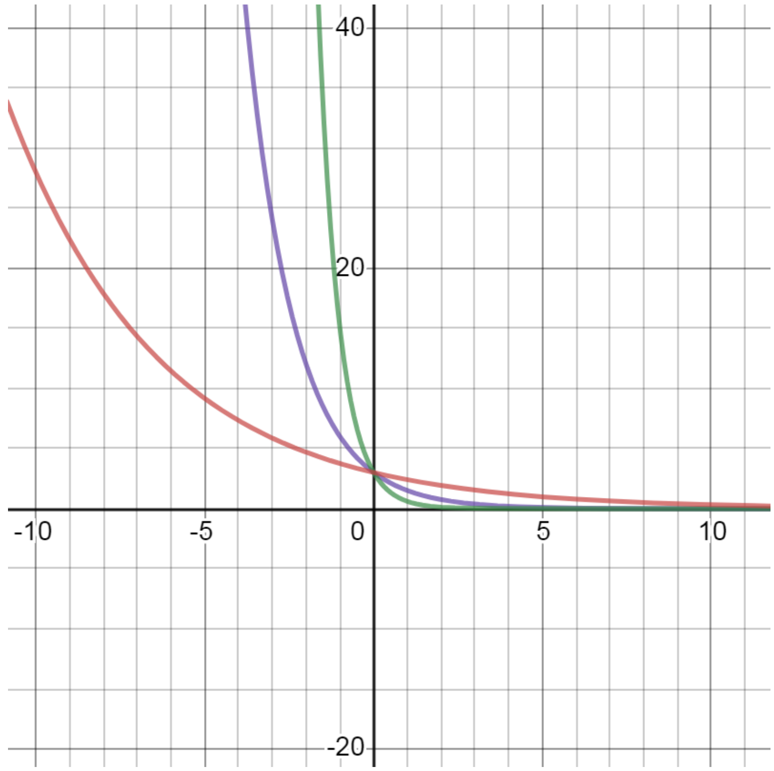
Now look at some other possibilities for the same basic function:

$f(x) = 3 * b^x$

$\boldsymbol{b = 0.2}$

$\boldsymbol{b = 0.5}$

$\boldsymbol{b = 0.8}$




Here, the takeaway is that the exponential function decreases when

$$0 < b < 1$$

Now, what all of these examples have in common is the coefficient 3.

Now let’s vary this value and see what happens with the graph:

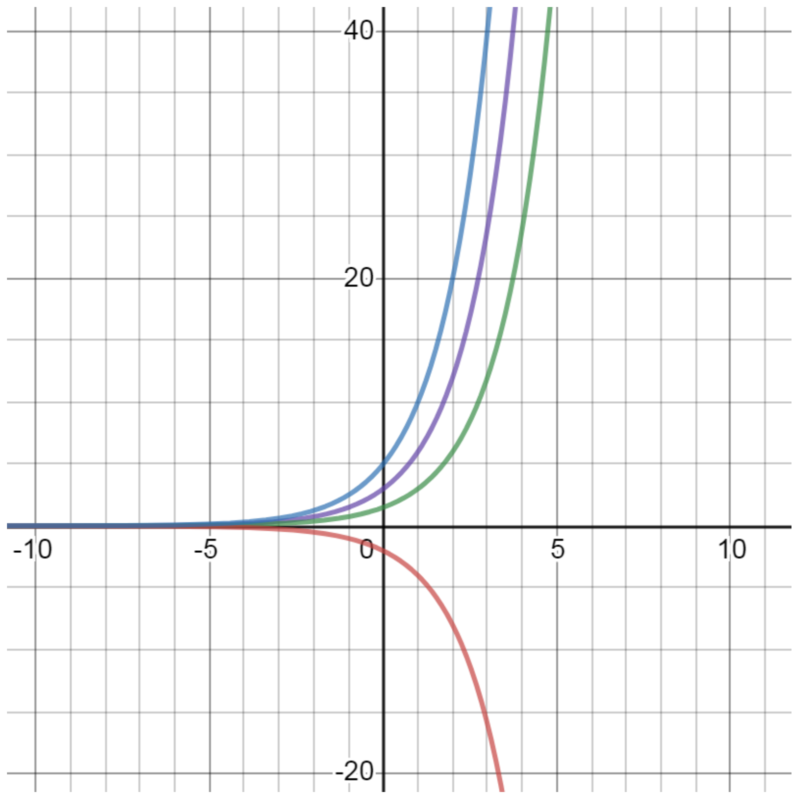
$$f(x) = a * 2^x$$

$\boldsymbol{a = 5}$

$\boldsymbol{a = 3}$

$\boldsymbol{a = 1.5}$

$\boldsymbol{a = -2}$




What is different with these graphs is that they all cross the $y$-axis at a different point.

In the case of $a = -2$, think of the graph as a **vertical reflection** of the graph

$$y = 2 * 2^x$$

While all of these functions have a domain of $(-\infty, \infty)$

The last function, $f(x) = -2 * 2^x$ has a different range: $(-\infty, 0)$

Now we must return to the real-life example of compound interest.

In our original problem, $1000 was invested at 4% annually compounded interest.

Now, in this new problem, we suppose that a different bank, located uptown, offers a competing opportunity: instead of annual compounding, this bank offers **semiannual compounding**.

$1000 is invested at 4% annual interest, compounded semiannually. How much will be in the account in 8 years?

To figure this out, we have to first realize that semiannual compounding means that the bank will award interest every six months.

But 4% is the **annual** interest rate! (**apr**)

So when the bank awards interest, will it be awarding 4%?

No!

Since only six months have elapsed since any interest payment, and six months is half a year, the bank will award half of the annual interest rate:

2% interest (every six months)

Now to increase any number by 2% is done most simply by multiplying by

$$1.02$$

So but since this will be happening twice every year, our function will be

$$A_u(t) = Amount\ in\ the\ account\ at\ Uptown\ bank\ after\ t\ years$$

$$A_u(t) = 1000 * 1.02^{2t}$$

because interest is awarded every 6 months – twice per year

To see why the exponent is $2t$, just note that . . .

$when\ t = 1,\quad A_u = 1000 * 1.02^2$ (because interest has been awarded 2 times)

$when\ t = 2,\quad A_u = 1000 * 1.02^4$ (because interest has been awarded 4 times)

$when\ t = 3,\quad A_u = 1000 * 1.02^6$ (because interest has been awarded 6 times)

So the exponent of the function must be 2 times the number of years!

After 8 years, we have . . .

$$A_u(8) = 1000 * 1.02^{2*8}$$

$$= 1000 * 1.02^{16}$$

$$= \$1372.79$$

Notice that this is a bit more than *four dollars more* than the amount we would have if we invested at the original bank offering annual compounding.

Now what if a downtown bank offers **quarterly compounding** at the same annual interest rate?

Here, our interest is awarded 4 times a year, but we will only receive

$$\frac{4\%}{4} = 1\%$$

interest each time our interest is compounded.

Our function will consequently be

$$A_d(t) = amount\ in\ account\ at\ Downtown\ bank\ after\ t\ years$$

$$A_d(t) = 1000 * 1.01^{4t}$$

After 8 years, we end up with

$$A_d(8) = 1000 * 1.01^{4*8}$$
$$= 1000 * 1.01^{32}$$
$$= \$1374.92$$

8 years has 32 quarters!

Which is bit more than *two dollars more* than the amount we would have at semiannual compounding.

The above process can be done for any compounding interval. This leads to the **general compound interest formula**:

$$A(t) = P * \left(1 + \frac{r}{n}\right)^{nt}$$

where

$$P = principal\ (amount\ invested)$$

$$t = number\ of\ years\ invested$$

$$r = annual\ interest\ rate\ (in\ decimal\ form)$$

$$n = number\ of\ times\ interest\ is\ compounded\ per\ year$$