Complex Zeroes

Consider the following function:

$$f(x) = x^2 - 4$$

Can you figure out in your head what the zeroes are?

Try to do the same for the following functions:

$$g(x) = x^2 - 3$$

$$p(x) = x^2 - 2$$

$$h(x) = x^2 - 1$$

$$q(x) = x^2$$

$$r(x) = x^2 + 1$$

With $f(x) = x^2 - 4$ we simply solve the equation:

$$x^2 - 4 = 0$$

$$x^2 = 4$$

$$x = \pm 2$$

With the following functions, the process is the same:

$$g(x) = x^2 - 3 = 0$$

$$x = \pm\sqrt{3}$$

$$p(x) = x^2 - 2 = 0$$

$$x = \pm\sqrt{2}$$

$$h(x) = x^2 - 1 = 0$$

$$x = \pm 1$$

$$q(x) = x^2$$

$$x = 0$$

Note that some of these functions have **rational** zeros . . .

. . . and some have **irrational** zeroes.

And for one function, $q(x)$, there is only **one** zero.

We can see all of these zeroes graphically as **x-intercepts**:

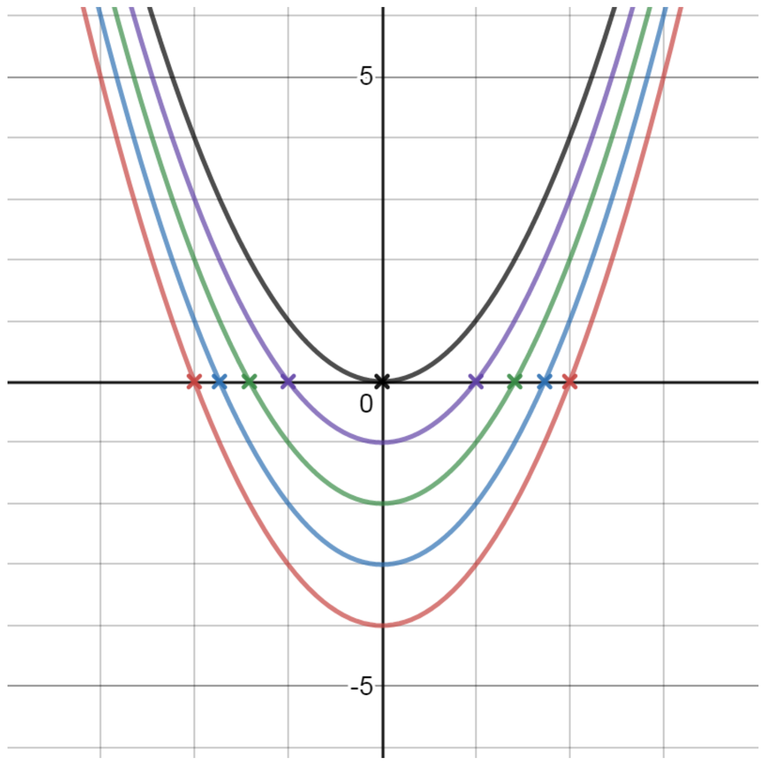
$q(x) = x^2$

$h(x) = x^2 - 1$

$p(x) = x^2 - 2$

$g(x) = x^2 - 3$

$f(x) = x^2 - 4$



But what about the function $r(x) = x^2 + 1$??

If we try to set it equal to zero and solve, here's what happens:

$$r(x) = x^2 + 1 = 0$$

$$x^2 = -1$$

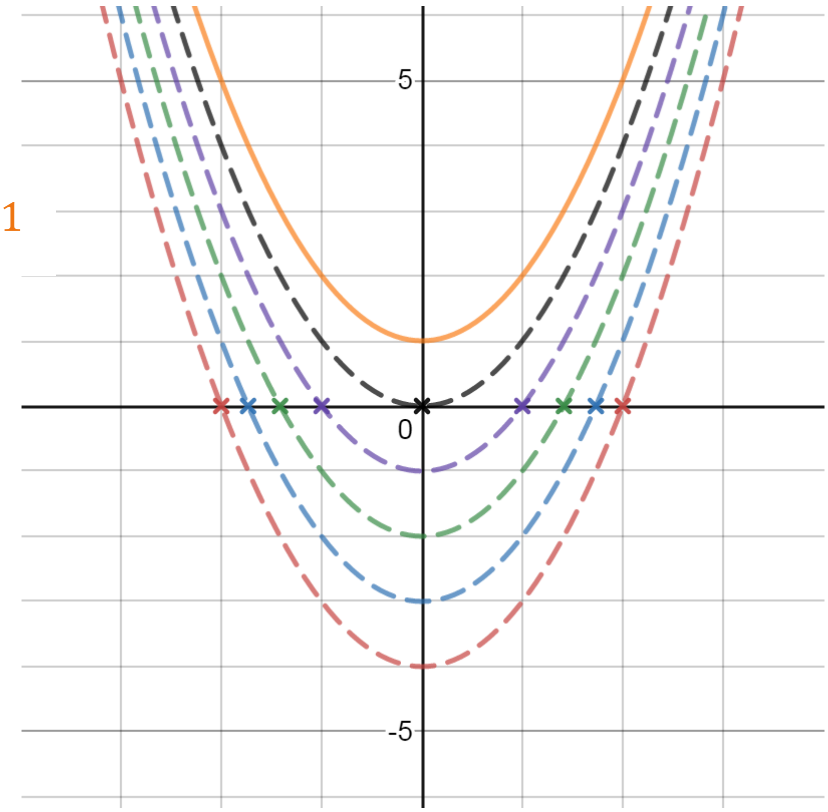
$$x = \pm\sqrt{-1}$$

$\sqrt{-1}$ is not a real number!

What does this mean?

Let's graph the function next to the other graphs to see:

$r(x) = x^2 + 1$



Can you see that $r(x)$ has no x-intercepts?

It exists entirely above the x-axis.

This is what happens when the zeroes to a function are not real numbers.

But they are still numbers, and they are still zeroes!

They just don't make x-intercepts!

In the case of $r(x)$, its zeroes are entirely **imaginary**:

$$\pm\sqrt{-1} = \pm i$$

The imaginary number i is defined in this way:

$$i^2 = -1$$

These numbers cannot be found on the real number line. That's why they will never be x-intercepts.

Not all numbers are real or imaginary!

To see this, let's find the zeroes of the following function:

$$p(x) = x^2 - 4x + 13 = 0$$

$$\begin{aligned}
x^2 - 4x + 13 &= 0 \\
x^2 - 4x &= -13 \\
x^2 - 4x + \mathbf{4} &= -13 + \mathbf{4} \\
(x-2)^2 &= -9 \\
x - 2 &= \pm\sqrt{-9} \\
x - 2 &= \pm\sqrt{9}\sqrt{-1} \\
x - 2 &= \pm 3i \\
x &= 2 \pm 3i
\end{aligned}$$

The zeroes of $p(x)$ turn out to have both a real part and an imaginary part:

$$2 \pm 3i$$

Real part   Imaginary part

So if it's not a real number or an imaginary number . . .

. . . what kind of number is it?

It's a **complex** number.

All numbers in math are complex numbers.

So all zeroes to functions are complex numbers!

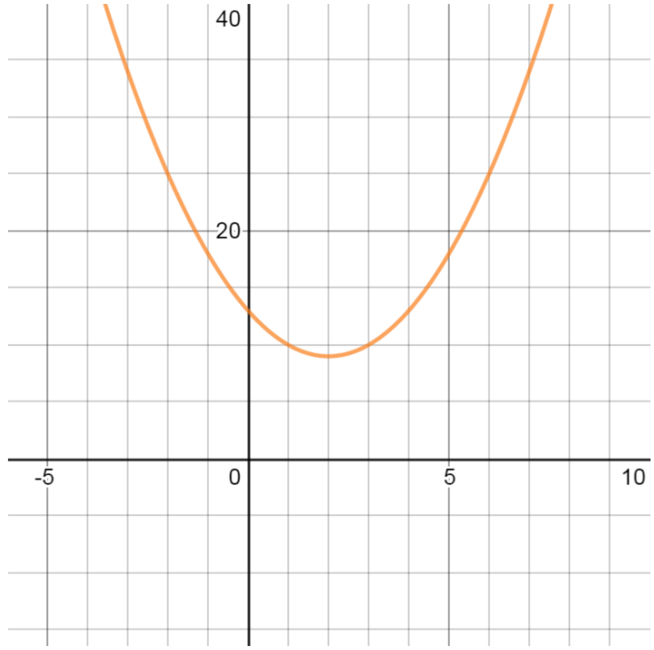
So far, all of the zeroes we have been finding have zero imaginary part.

If the zero has **zero imaginary part**, it's a **real** zero, and will show up on the graph as an x-intercept.

If the zero has a **non-zero imaginary part**, it will have ***no x-intercept***.

For example, let's look at the graph of our previous function:

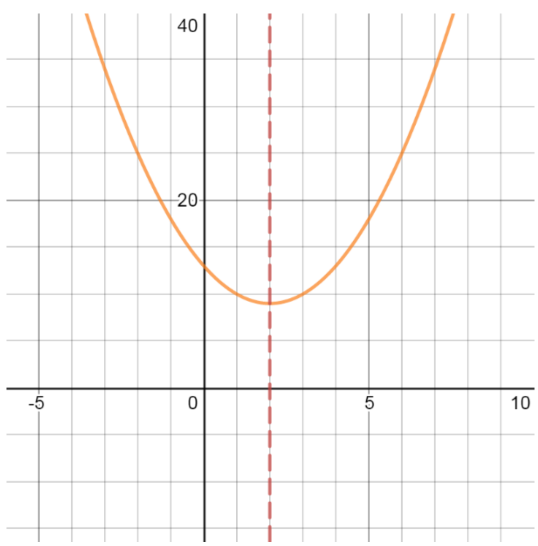
$p(x) = x^2 - 4x + 13$



As you can see, $p(x)$ has no x-intercepts!

Note that its vertical axis is at $x = 2$ . . .

. . . which was the **real part** of its zeroes, $2 \pm 3i$



Let’s do an example.

Find the zeroes of

$$g(x) = x^3 - x^2 - 4x - 6$$

and graph the function.

	$\mathbf{1}$	$\mathbf{-1}$	$\mathbf{-4}$	$\mathbf{-6}$	
$\mathbf{0}$				-6	
$\mathbf{1}$	1	0	-4	-10	
$\mathbf{2}$	1	1	-2	-10	
$\mathbf{3}$	1	2	2	0	zero!
$\mathbf{4}$	1	3	8	26	
$\mathbf{0}$				-6	
$\mathbf{-1}$	1	-2	-2	-4	
$\mathbf{-2}$	1	-3	2	-10	
$\mathbf{-3}$	1	-4	8	-30	

We found only one zero: $x = 3$.

This was the only place the function crossed the x-axis!

To find the other zeroes, we will switch to synthetic division:

	$\mathbf{1}$	$\mathbf{-1}$	$\mathbf{-4}$	$\mathbf{-6}$
$\mathbf{0}$				-6
$\mathbf{1}$	1	0	-4	-10
$\mathbf{2}$	1	1	-2	-10
$\mathbf{3}$	1	2	2	0
$\mathbf{4}$	1	3	8	26
$\mathbf{0}$				-6
$\mathbf{-1}$	1	-2	-2	-4
$\mathbf{-2}$	1	-3	2	-10
$\mathbf{-3}$	1	-4	8	-30

We have that

$$g(x) = (x-3)(x^2+2x+2)$$

And so we must now solve $x^2+2x+2=0$

Using the complete the square method:

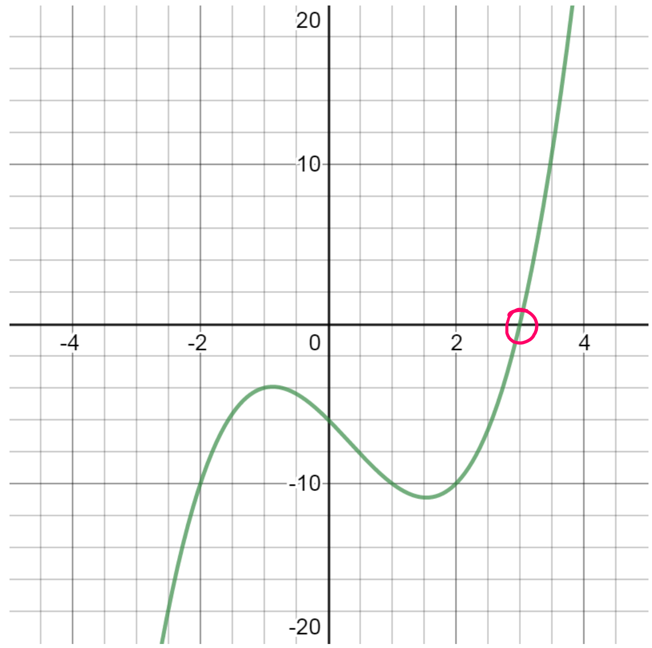
$$\begin{aligned}
x^2+2x &= -2 \\
x^2+2x\color{red}{+\mathbf{1}} &= -2\color{red}{+\mathbf{1}} \\
(x+1)^2 &= -1 \\
x+1 &= \pm\sqrt{-1} \\
x+1 &= \pm i
\end{aligned}$$

The solution is $x = -1 \pm i$

So the zeroes of the function are: $\{3, -1+i, -1-i\}$

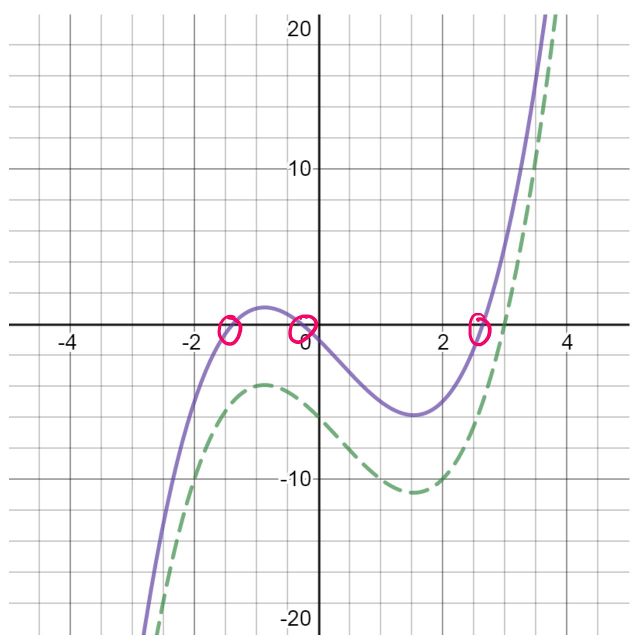
We can see from the graph that there is only one x-intercept, $x = 3$.

$$g(x) = x^3 - x^2 - 4x - 6$$



Notice also that if we were to shift up the function by +4, we could get 2 more x-intercepts:

$$q(x) = g(x) + 5$$



Let's do another example:

$$h(x) = 2x^4 - 13x^3 + 25x^2 + 7x - 33$$

Find the zeroes.

This problem is not telling us to graph. Therefore we don't need to find all of the y-values if we don't want to. I will go searching and see what happens. First, let's plug in positive x-values:

	2	**−13**	**25**	**7**	**−33**
0					−33
1	2	−11	14	21	−12
2	2	−9	14	21	7
3	2	−7	4	19	24
4	2	−5	5	29	75

Zero alert!

getting big! stop here!

I will stop there with the positive x-values because I "sense" that $h(x)$ is going to continue on to infinity.

Notice that we locate one real zero, between $x = 1$ and $x = 2$.

We could try to use the Rational Zero Test to find it, but that would take much time, and we don't even know whether it's rational!

Instead, I suggest we continue our search by trying some negative x-values:

	2	**−13**	**25**	**7**	**−33**	
0					−33	
1	2	−11	14	21	−12	
2	2	−9	14	21	7	
3	2	−7	4	19	24	
4	2	−5	5	29	75	
0					−33	
−1	2	−15	40	−33	**0**	← Zero!

Good news: we found a zero: $x = -1$

Now, since we don't have to graph the function, we can continue our search for zeroes by dividing $h(x)$ by $(x + 1)$ using synthetic division:

	2	**−13**	**25**	**7**	**−33**
0					−33
1	2	−11	14	21	−12
2	2	−9	14	21	7
3	2	−7	4	19	24
4	2	−5	5	29	75
0					−33
−1	2	−15	40	−33	0

$$h(x) = (x + 1)\underbrace{(2x^3 - 15x^2 + 40x - 33)}$$

So now we are solving the equation

$$2x^3 - 15x^2 + 40x - 33 = 0$$

What should we do? We have several options at this point. One is to continue to plug in negative x-values to expand our synthetic substitution table:

	2	**−13**	**25**	**7**	**−33**
0					-33
1	2	-11	14	21	-12
2	2	-9	14	21	7
3	2	-7	4	19	24
4	2	-5	5	29	75
0					-33
−1	2	-15	40	-33	0
−2	2	-17	59	-111	189

getting very big!

But when we try to do so, the function values begin to get very large!

So let's go back to the zero we found, between $x = 1$ and $x = 2$.

Hopefully this is a rational zero!

To see what possible rational numbers it could be, we apply the Rational Zero Test.

We could use the coefficients of the equation

$$2x^3 - 15x^2 + 40x - 33 = 0$$

Because we know that the zeroes must be solutions to this equation, but it turns out, in this case, that they would be the same coefficients of the original equation:

$$2x^4 - 13x^3 + 25x^2 + 7x - 33 = 0$$

Sometimes the reduced equation will have simpler numbers to work with!

There are four factors of -33: $1,\quad 3,\quad 11,\quad 33$

And two factors of 2: $1,\quad 2$

So our possible rational zeroes are

$$\pm\frac{1}{1}, \quad \pm\frac{3}{1}, \quad \pm\frac{11}{1}, \quad \pm\frac{33}{1}$$
$$\pm\frac{1}{2}, \quad \pm\frac{3}{2}, \quad \pm\frac{11}{2}, \quad \pm\frac{33}{2}$$

But only one of these is between $x = 1$ and $x = 2$!!!!

Can you see which one it is?

That’s right, only $x = \frac{3}{2}$ can work!

Let’s try it. We might as well use the coefficients of the reduced equation:

$$2x^3 - 15x^2 + 40x - 33 = 0$$

$$\begin{array}{c|cccc}
 & \mathbf{2} & \mathbf{-15} & \mathbf{40} & \mathbf{-33} \\
\hline
\mathbf{\frac{3}{2}} & 2 & -12 & 22 & \color{red}{\mathbf{0}}
\end{array}$$

It works!!!!

And the remaining zeroes must be solutions to the equation

$$2x^2 - 12x + 22 = 0$$

Let’s first divide both sides by 2 to simplify it a bit:

$$x^2 - 6x + 11 = 0$$

It’s not factorable, so let’s solve by the quadratic formula this time:

$$x = \frac{-(-6) \pm \sqrt{(-6)^2 - 4(1)(11)}}{2(1)}$$

$$x = \frac{6 \pm \sqrt{36 - 44}}{2}$$

$$x = \frac{6 \pm \sqrt{-8}}{2}$$

$$x = \frac{6 \pm \sqrt{4 * 2 * -1}}{2}$$

$$x = \frac{6 \pm 2\sqrt{2}i}{2}$$

$$x = \frac{\cancel{2}(3 \pm \sqrt{2}i)}{\cancel{2}}$$

factor the numerator, then cancel to simplify

$$x = 3 \pm \sqrt{2}i$$

So the zeroes to the function are

$$\left\{ -1, \quad \frac{3}{2}, \quad 3 + \sqrt{2}i, \quad 3 - \sqrt{2}i \right\}$$