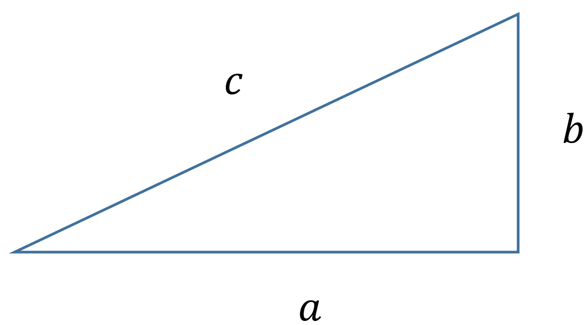
Simplifying Trigonometric Identities

One of the most fundamental rules of mathematics is known as the . . .

Pythagorean Theorem

It goes like this:

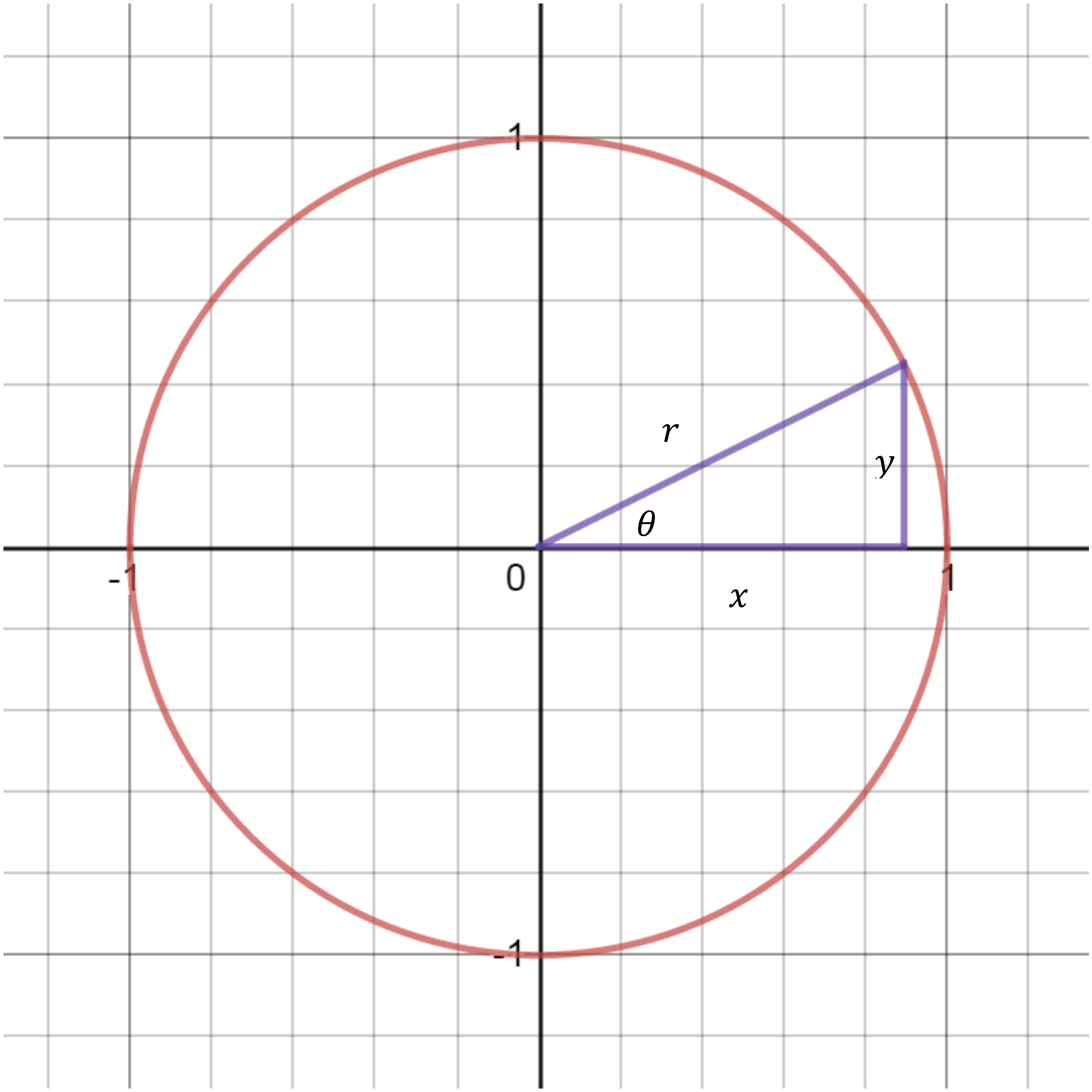
For any right triangle:



We have that

$$a^2 + b^2 = c^2$$

Now, trigonometry and right triangles have a lot in common:



Which is to say that for any point on the terminal side of an angle, (x, y),

$$x^2 + y^2 = r^2$$

Now if we divide both sides of this equation by r^2, we get

$$\frac{x^2}{r^2} + \frac{y^2}{r^2} = \frac{r^2}{r^2} = 1$$

Now remember that

$$\sin(\theta) = \frac{y}{r}, \;\; \text{so} \sin^2\theta = \frac{y^2}{r^2}$$

and

$$\cos(\theta) = \frac{x}{r}, \;\; \text{so} \cos^2\theta = \frac{x^2}{r^2}$$

Thus,

$$\sin^2\theta + \cos^2\theta = 1$$

The above formula is called the

Pythagorean Identity

It is the most important "identity" in trigonometry.

Identities can be thought of as simple formulas. The trigonometric ones become especially important in Integral Calculus (Calculus II).

Here are some other basic trigonometric identities that you should know:

$$\sin(\theta) = \frac{1}{\csc(\theta)} \quad and \quad \csc(\theta) = \frac{1}{\sin(\theta)}$$

$$\cos(\theta) = \frac{1}{\sec(\theta)} \quad and \quad \sec(\theta) = \frac{1}{\cos(\theta)}$$

$$\tan(\theta) = \frac{1}{\cot(\theta)} \quad and \quad \cot(\theta) = \frac{1}{\tan(\theta)}$$

And here are a couple of others that go with the Pythagorean Identity:

$$\sin^2\theta + \cos^2\theta = 1$$

If you divide both sides of this equation by $\sin^2\theta$, you get

$$\frac{\sin^2\theta}{\sin^2\theta} + \frac{\cos^2\theta}{\sin^2\theta} = \frac{1}{\sin^2\theta}$$

which becomes

$$1 + \cot^2\theta = \csc^2\theta$$

Also, if you take the standard Pythagorean Identity

$$\sin^2\theta + \cos^2\theta = 1$$

and divide both sides of the equation by $\cos^2\theta$, you get

$$\frac{\sin^2\theta}{\cos^2\theta} + \frac{\cos^2\theta}{\cos^2\theta} = \frac{1}{\cos^2\theta}$$

which becomes

$$\tan^2\theta + 1 = \sec^2\theta$$

So taken together, we really have three Pythagorean identities:

$$\sin^2\theta + \cos^2\theta = 1$$

$$1 + \cot^2\theta = \csc^2\theta$$

$$\tan^2\theta + 1 = \sec^2\theta$$

You may need to use any of these three for the upcoming problems, so you might want to memorize all of them. Personally, I don't like to memorize lots of formulas. So I just remember the first one, and how to get the other two!

Here are some other identities (though I will not test you on these):

$$\sin(-\mu) = -\sin(\mu)$$

$$\cos(-\mu) = \cos(\mu)$$

$$\tan(-\mu) = -\tan(\mu)$$

$$\sin\left(\frac{\pi}{2}-\mu\right)=\cos(\mu)$$

$$\cos\left(\frac{\pi}{2}-\mu\right)=\sin(\mu)$$

The first type of problem we will do to help learn these identities goes like this:

Simplify the given expression:

$$\frac{\csc(\theta)}{\sec(\theta)}$$

Here, the problem is telling you that there's a simpler way to write this!

These problems have ***different ways to figure them out*** . . .

. . . but one way is to go straight to one of the main identities . . .

. . . and substitute it directly in!

We have that

$$\csc(\theta)=\frac{1}{\sin(\theta)} \quad \text{and} \quad \sec(\theta)=\frac{1}{\cos(\theta)}$$

so,

$$\frac{\csc(\theta)}{\sec(\theta)}$$

$$= \frac{\frac{1}{\sin(\theta)}}{\frac{1}{\cos(\theta)}}$$

from here we might try to cancel the denominator to the big fraction:

$$= \frac{\frac{1}{\sin(\theta)} * (\cos(\theta))}{\frac{1}{\cos(\theta)} * (\cos(\theta))}$$

which gives us

$$= \frac{\cos(\theta)}{\sin(\theta)}$$

$$= \cot(\theta)$$

This is the answer to the problem—it is a more simplified expression than what we started with!

We didn't know how we would get here . . .

. . . we just tried substituting some things . . .

. . . and it worked!!!!

These are kind of like puzzles . . . you might have to try different things.

Let's do some more:

Simplify the trigonometric expression:

$$\sin(x) + \cot(x)\cos(x)$$

Last time, converting everything into sine and cosine worked well . . .

. . . so here I will leave those alone and just change the $\cot(x)$.

We know that

$$\cot(x) = \frac{\cos(x)}{\sin(x)}$$

So

$$\sin(x) + \cot(x)\cos(x)$$

$$= \sin(x) + \frac{\sin(x)}{\cos(x)} * \cos(x)$$

Canceling the $\cos(x)$ we get

$$= \sin(x) + \frac{\sin(x)}{\cos(x)} * \cos(x)$$

$$= \sin(x) + \sin(x)$$

$$= 2\sin(x)$$

Now let's do some harder ones:

Simplify the trigonometric expression:

$$\sin\ x(\cos^2 x - 1)$$

This expression is different from the previous ones in that it has a squared term.

Whenever you see one of the squared trigonometric functions, you might want to use one of the Pythagorean identities.

Here we have $\cos^2 x$ involved in the expression. Remember our primary Pythagorean Identity:

$$\sin^2 \theta + \cos^2 \theta = 1$$

To get $\cos^2 \theta$ by itself, we can subtract $\sin^2 \theta$ from both sides:

$$\sin^2 \theta + \cos^2 \theta = 1$$

to get

$$\cos^2 \theta = 1 - \sin^2 \theta$$

Substituting this into our original expression, we get

$$\sin\ x(\cos^2 x - 1)$$

$$= \sin\ x(1 - \sin^2 x - 1)$$

$$= \sin\ x(-\sin^2 x)$$

$$= -\sin^3 x$$

And we are done.

Here's another:

Simplify the given trigonometric expression:

$$\cos^2 x + \cos^2 x * \tan^2 x$$

There are a number of ways to start!

But you might first recognize that some algebra can be used . . .

. . . in particular, factoring:

$$\cos^2 x + \cos^2 x \tan^2 x$$

$$= \cos^2 x(1 + \tan^2 x)$$

Here, I just recognized that the terms had a greatest common factor.

Now, since we have some squared trigonometric terms, we might try using one of the Pythagorean Identities. Let’s use the one involving tangent:

$$\tan^2\theta + 1 = \sec^2\theta$$

Substituting this into our expression, we get

$$= \cos^2 x(1 + \tan^2 x)$$

$$= \cos^2 x\,(\sec^2 x)$$

And now let’s use one of our basic reciprocal identities:

$$= \cos^2 x\,(\sec^2 x)$$

$$= \cos^2 x\,(\frac{1}{\cos^2 x})$$

$$= \quad 1$$