The Horizontal Line Test

In the previous example, we started with the function $f(x) = -\sqrt{x}$ . . . .

. . . and tried to find its inverse f^{-1}

. . . *it turned out to be* $f^{-1}(x) = x^2$ *but only on* $(-\infty, 0]$ . . .

This time, let's start with

$$g(x) = x^2$$

And try to find its inverse!

Doing it the "formal" way . . .

$$y = x^2$$

$$x = y^2$$

$$\pm\sqrt{x} = \sqrt{y^2}$$

$$\pm\sqrt{x} = y$$

so

$$g^{-1}(x) = \pm\sqrt{x}$$

But wait a minute!!

THIS ISN'T A FUNCTION!

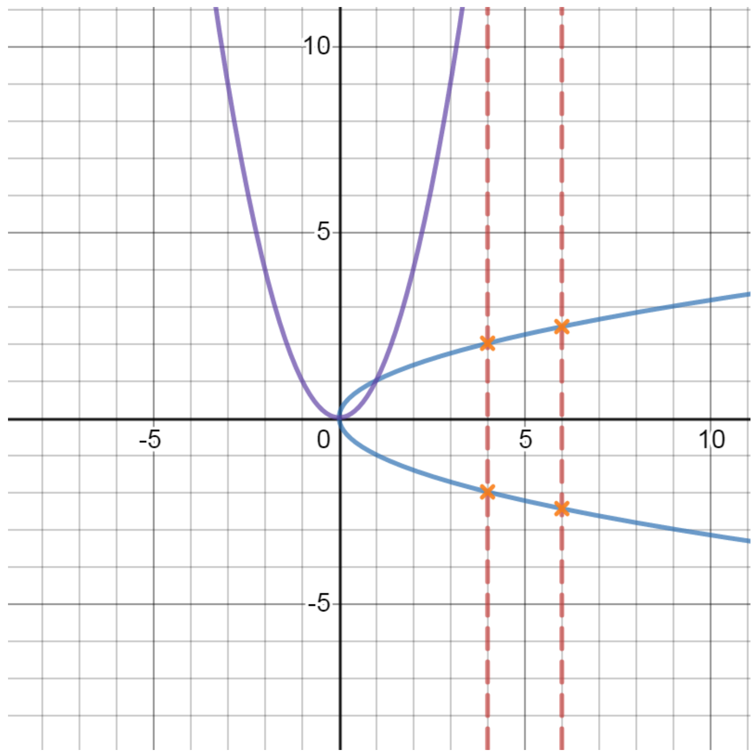
Remember the rule of functions:

There can only be one output for any input

We can see this problem visually if we try to graph g and g^{-1}:

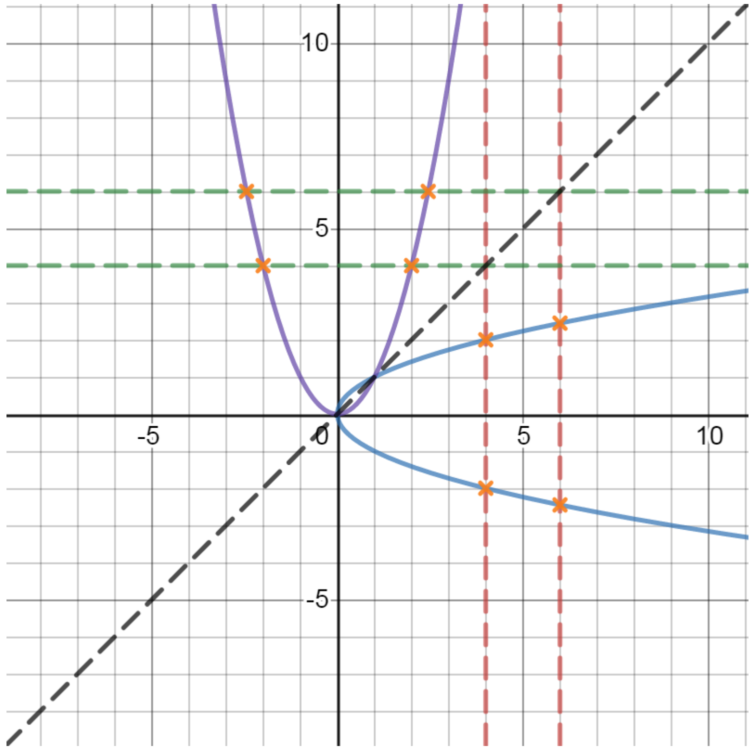
$g(x) = x^2$

$g^{-1}(x) = \pm\sqrt{x}$



Is there a way we could have seen this problem in advance?

YES!



Notice that the points that show **Vertical Line Test** *fail*

are reflections across $y = x$

of what could be called a **Horizontal Line Test** *fail* ***!!!!!***

From this example, we get

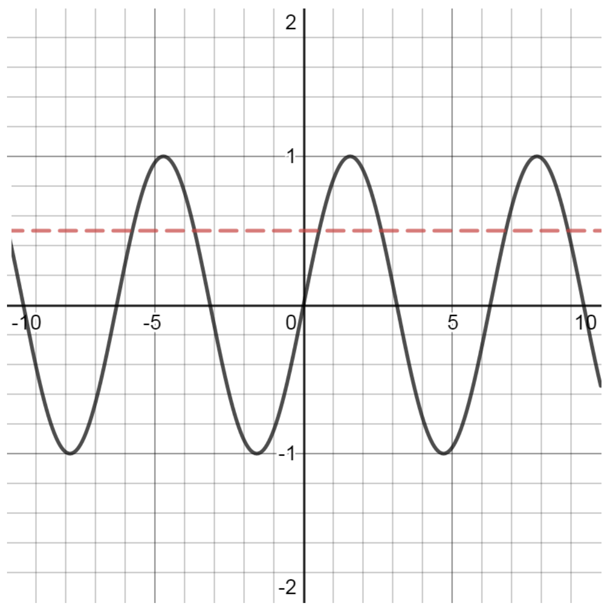
the Horizontal Line Test:

If a horizontal line crosses a function graph in more than one point . . .

. . . that function does not have an inverse.

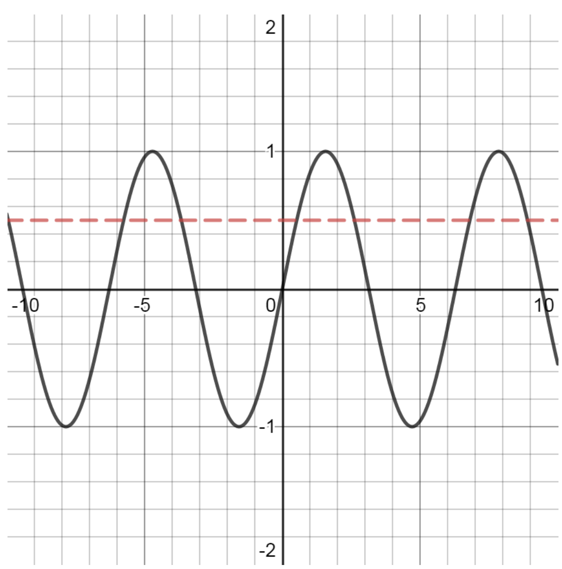
NOTE: It’s **okay** for a horizontal line to cross a function in many places:

$h(x) = \sin(x)$



It just means that the function *HAS NO INVERSE*.

Why is that again?



Because if we tried to get $h^{-1}(0.5)$, we would find many possible outputs!

So $h^{-1}(x)$ would not be a function!

And if $h^{-1}(x)$ is not a function, then $h(x)$ has no inverse!

Unless . . .

. . . we restrict its domain.

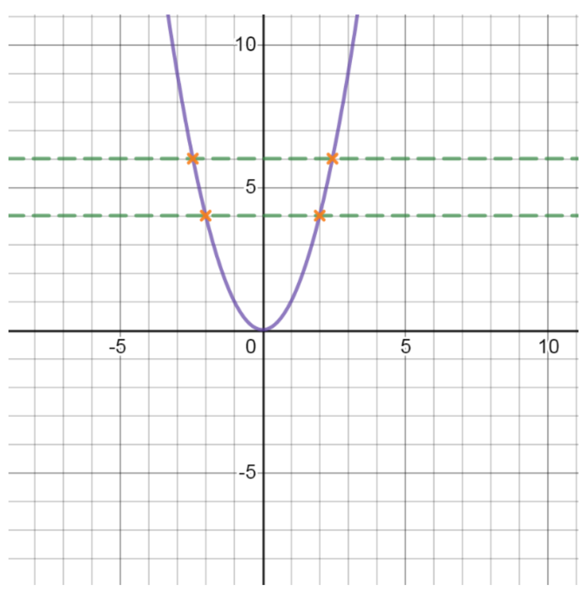
Consider again the function:

$$g(x) = x^2$$

We found that this function **has no inverse** . . .

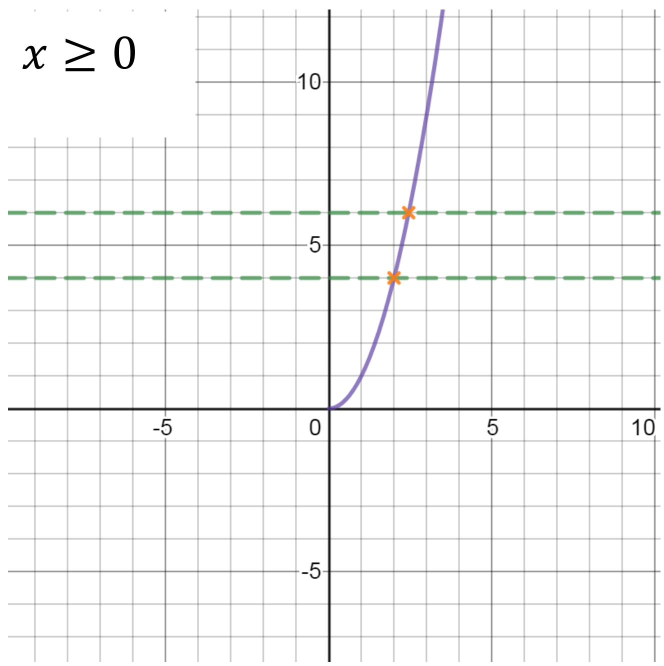
. . . because it *failed* the Horizontal Line Test:

$$g(x) = x^2$$



However, suppose we were to restrict its domain to be $[0, \infty)$:

$$g(x) = x^2 \quad x \geq 0$$



Now the function **DOES NOT FAIL** the Horizontal Line Test!

It's inverse would be:

$$g^{-1}(x) = \sqrt{x}$$

And we can see that here,

the domain of $g(x) \equiv$ the range of $g^{-1}(x)$

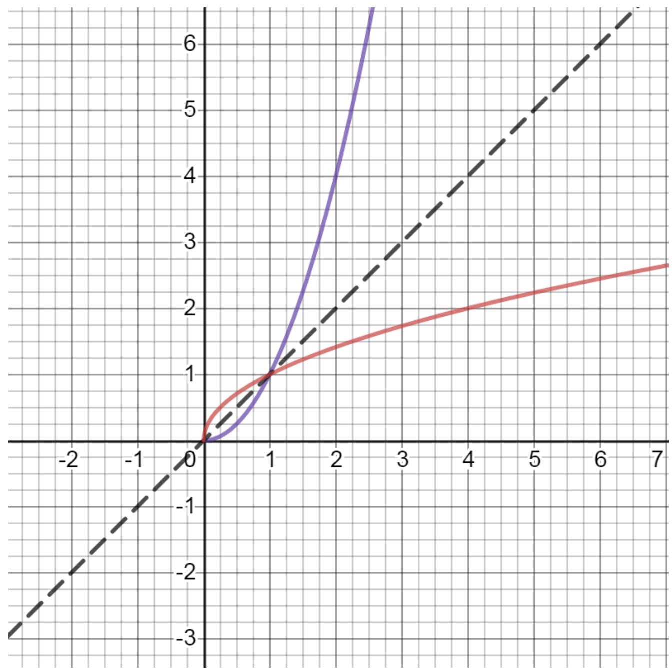
and

the range of $g(x) \equiv$ the domain of $g^{-1}(x)$

Because both (all) of these are $[0, \infty)$:

$g(x) = x^2 \quad x \geq 0$

$g^{-1}(x) = \sqrt{x}$



We should ask a more general question . . .

What kinds of functions will always **pass the Horizontal Line Test**??

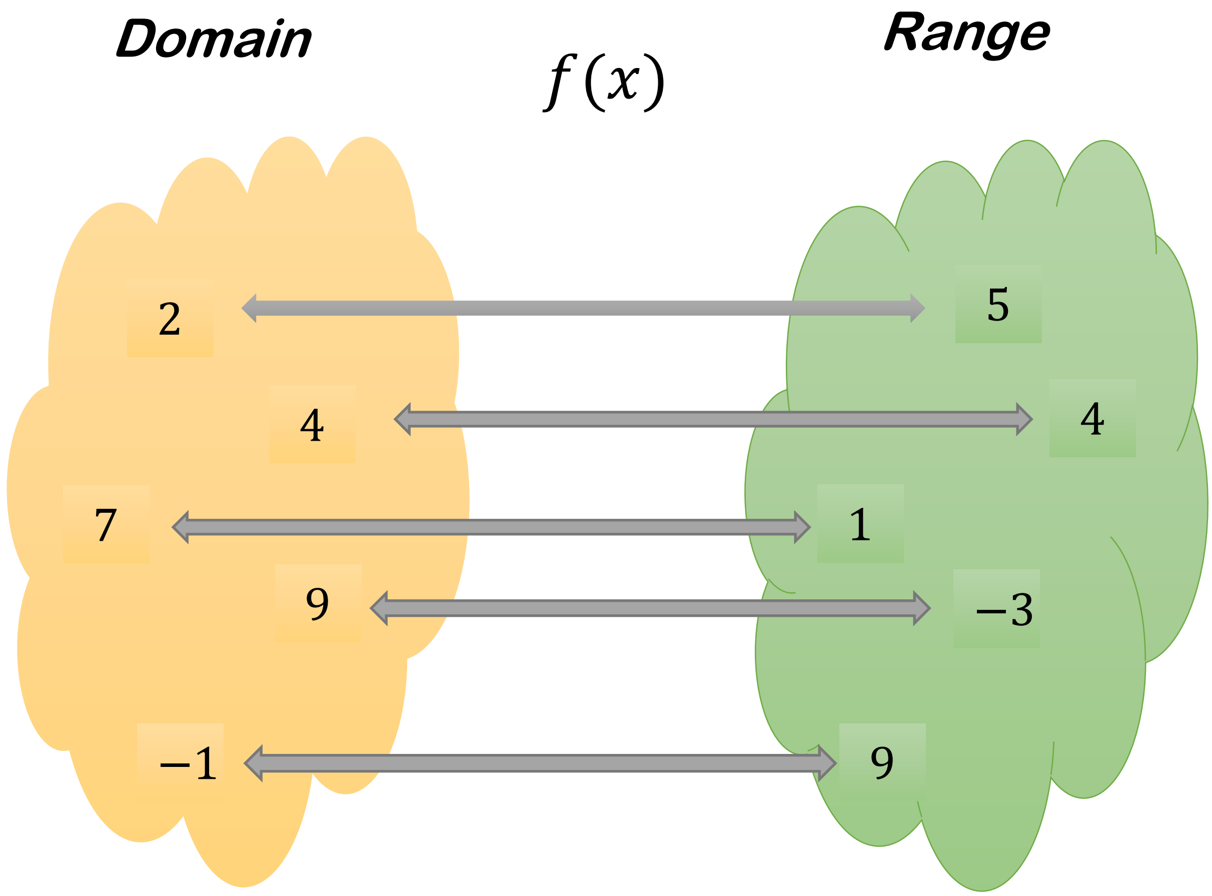
And therefore HAVE AN INVERSE???

These functions are called

one-to-one

Let's look at a diagram to understand this important **concept**:

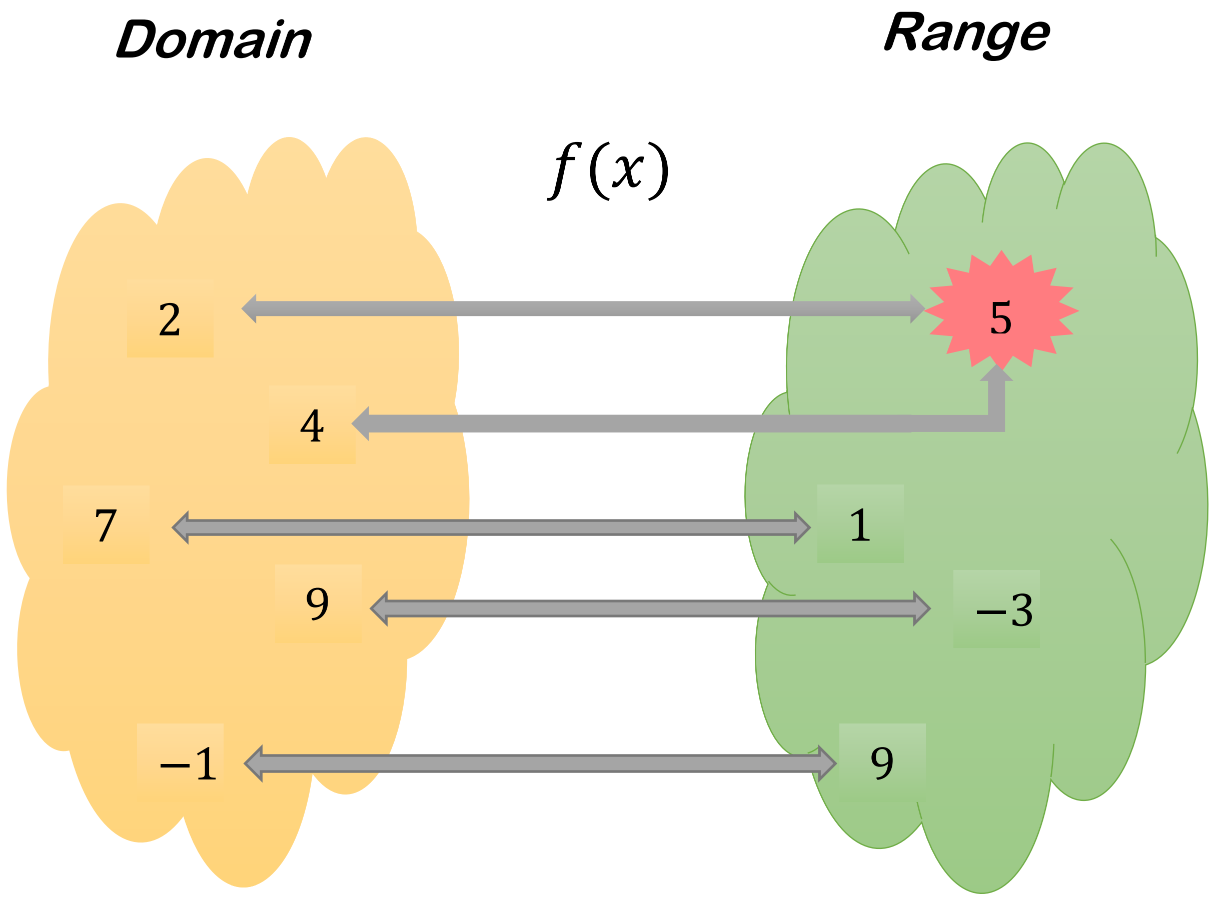
A visual representation of a one-to-one function:



Key point:

Every value in the domain corresponds to a ***unique*** value in the range

A visual representation of a not one-to-one function:



Key point:

Here, two values in the domain go to the same value in the range

It's **still a function** . . .

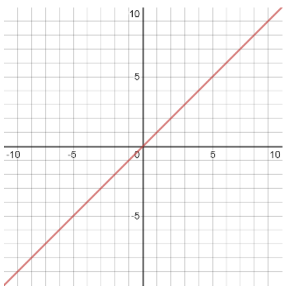
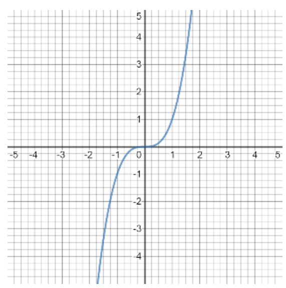
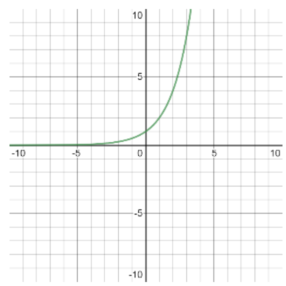
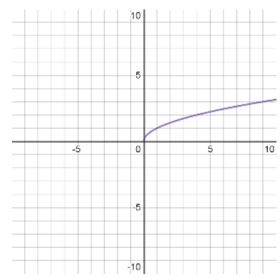
. . . but ***not one-to-one***

Definition: A **one-to-one** function is a function in which . . .

. . . every value in the domain corresponds to a unique value in the range

Examples of one-to-one functions:

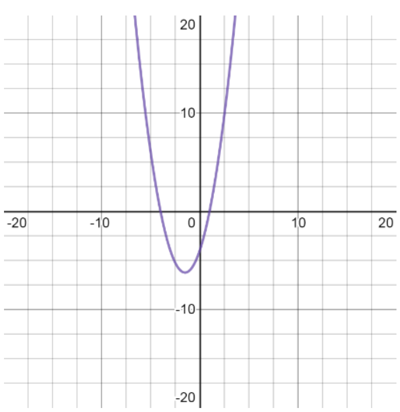
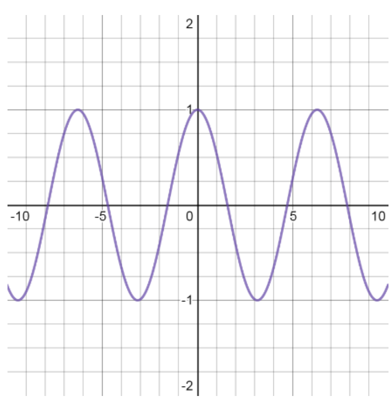
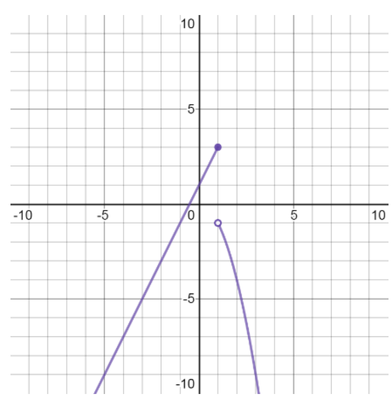
$f(x) = x$ $\quad$ $q(x) = x^3$ $\quad$ $g(x) = 2^x$ $\quad$ $h(x) = \sqrt{x}$

All of these graphs **PASS** the Horizontal Line Test

Examples of NOT one-to-one functions

$g(x) = x^2 - 3x + 4$ $\quad$ $p(x) = \cos(x)$ $\quad$ $r(x) = \begin{cases} 2x+1 & x \leq 1 \\ -x^2 & x > 1 \end{cases}$

All of these graphs ***FAIL*** the Horizontal Line Test

* this next example would be only for extra credit on a test

*

Let's do a hard(er) example!!!

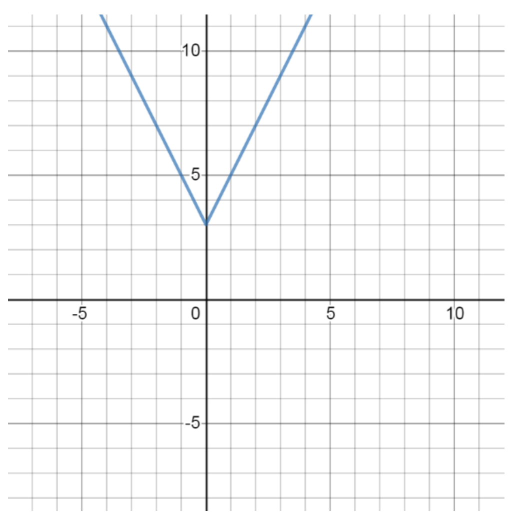
Consider the function:

$$p(x) = 2|x| + 3$$

Does this function have an inverse?

To decide, consider its graph:

$p(x) = 2|x| + 3$



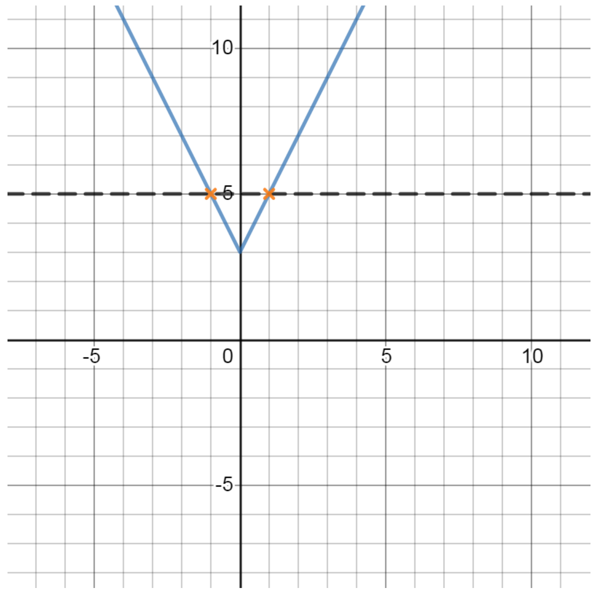
Clearly this function FAILS the horizontal line test:

$$p(-1) = 5$$

$$p(1) = 5$$

Can't go backwards to x

from $p(x) = 5$!



Problem: Restrict the domain of $p(x)$ so that $p(x)$ has an inverse . . .

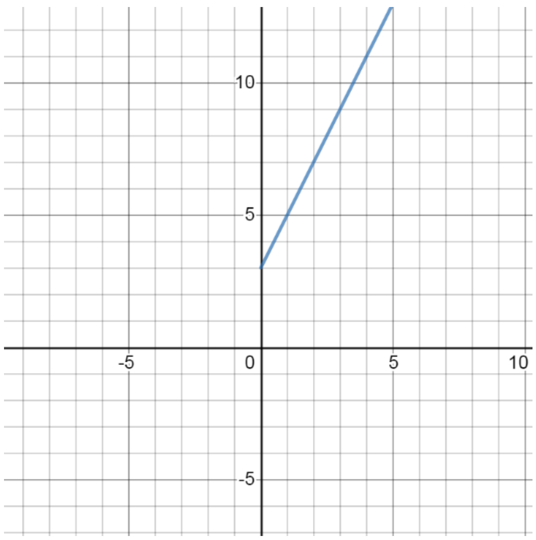
. . . then find $p^{-1}(x)$ . . .

. . . and state the domain and range of both functions

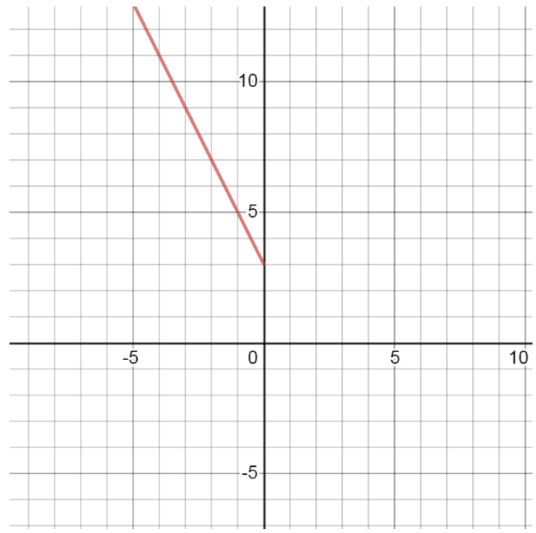
There are two ways to do this!

To restrict the domain of $p(x)$ so that it passes the Horizontal Line Test . . .

We must either restrict the domain to $(-\infty, 0]$ OR $[0, \infty)$:



$p(x) = 2|x| + 3 \qquad x \geq 0$



$p(x) = 2|x| + 3 \qquad x \leq 0$

Let's choose the second one . . . it's a little trickier!!!

Now to find $p^{-1}(x)$, we have to something that's going to seem complicated . . . . because it is!

We can't use the "formal" method of finding $p^{-1}(x)$ . . .

. . . until we get rid of the absolute value (| |) sign . . .

. . . because we can't solve the equation for y:

$$x = 2|y| + 3$$

We need to **rewrite $p(x) = 2|x| + 3$** without the absolute value sign!!

Here's how we do it:

Since our function

$$p(x) = 2|x| + 3$$

Is only defined for $x \leq 0$ . . .

We can replace $|x|$ with $-(x)$!

Think about it . . .

. . . since x is negative (or zero) . . . the | | just changes the sign!

We can do that just as well by putting a minus sign in front of x!

So our function

$$p(x) = 2|x| + 3 \qquad x \leq 0$$

becomes

$$p(x) = 2(-x) + 3 \qquad x \leq 0$$

or

$$p(x) = -2x + 3 \qquad x \leq 0$$

And now we can find $p^{-1}(x)$ in the formal way:

$$y = -2x + 3$$

$$x = -2y + 3$$

$$x - 3 = -2y$$

$$\frac{x-3}{-2} = y$$

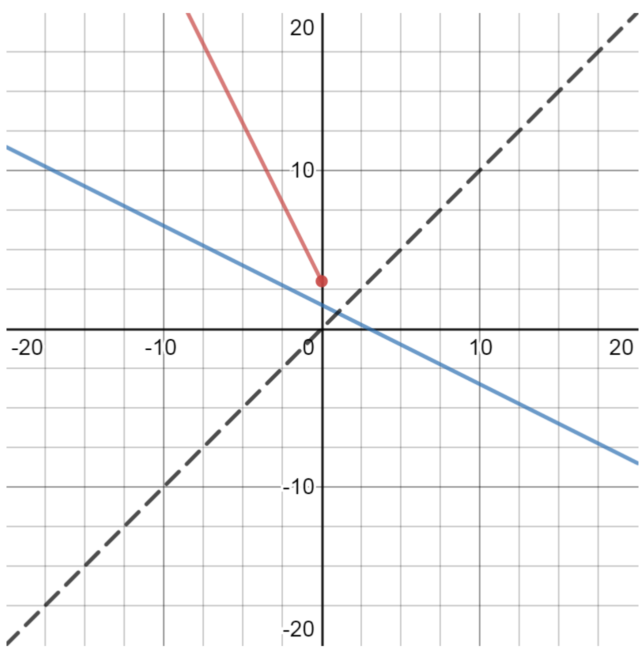
$$p^{-1}(x) = -\frac{1}{2}x + \frac{3}{2}$$

Now suppose we were to graph this whole function (for all x) . . .

. . . alongside $p(x)$:

$p(x) = -2x + 3 \quad x \leq 0$

$p^{-1}(x) = -\frac{1}{2}x + \frac{3}{2}$



Can you see what's wrong?

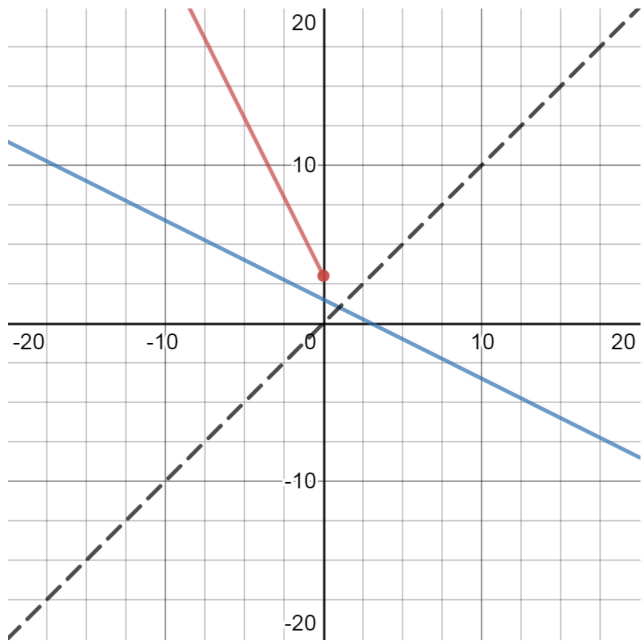
Well, you might have noticed two things wrong, which are the same thing:

First, something is funny about the graphs . . .

. . . they don't quite look like reflections across $y = x$ yet:

$p(x) = -2x + 3 \quad x \leq 0$

$p^{-1}(x) = -\frac{1}{2}x + \frac{3}{2}$



And then there's something funny about the formulas themselves . . .

We see that $p(x)$ has a **restricted domain** . . .

. . . should $p^{-1}(x)$ have one too?

Yes!

Remember:

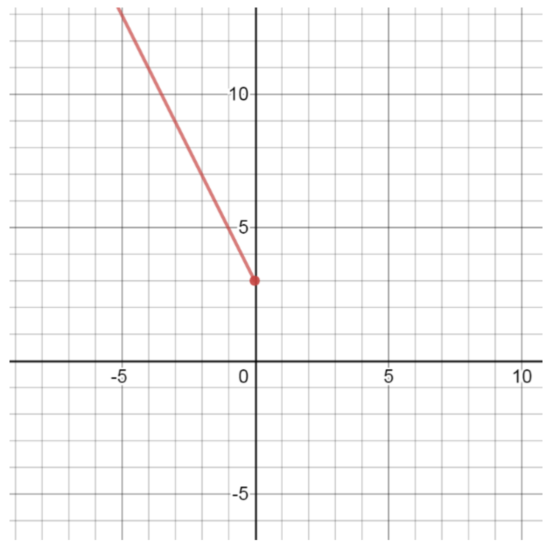
The domain of any function must be the range of its inverse

This means that the **domain of** $p^{-1}(x)$ . . .

. . . should be the **range of** $p(x)$!

What is the range of $p(x)$?

$$p(x) = -2x + 3 \quad x \leq 0$$



range of $p(x)$

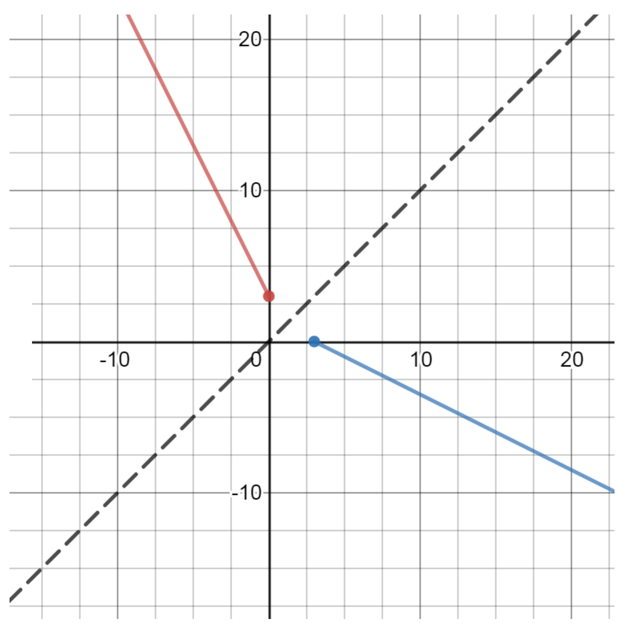
$[3, \infty)$

We can see from the graph that the range of $p(x)$ is $[3, \infty)$.

This, therefore, must be the domain of $p^{-1}(x)$:

$$p(x) = -2x + 3 \quad x \leq 0$$

$$p^{-1}(x) = -\frac{1}{2}x + \frac{3}{2} \quad x \geq 3$$



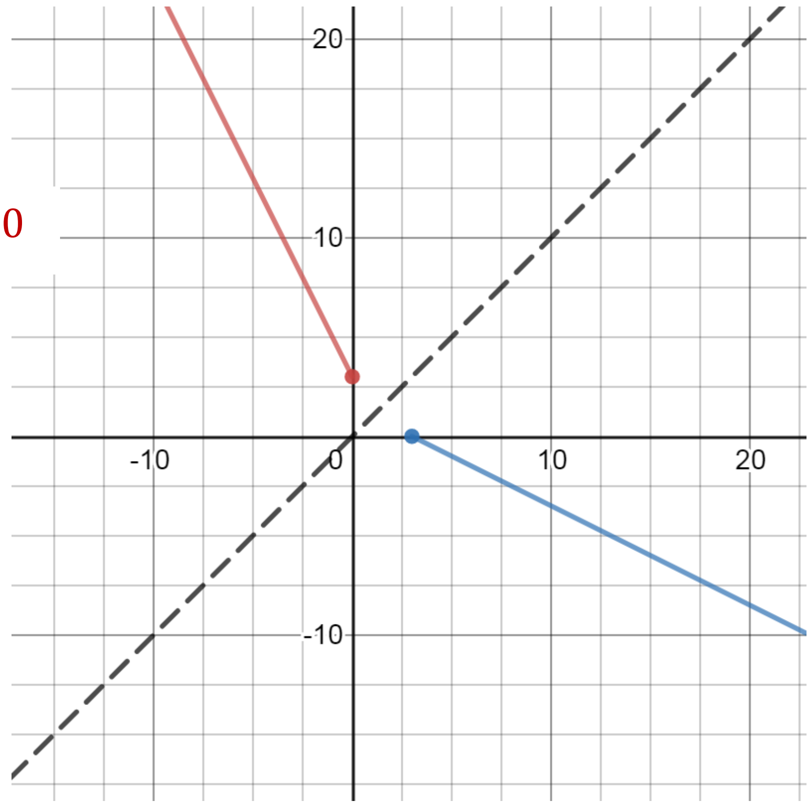
domain of $p^{-1}(x)$

$[3, \infty)$

Notice, also, that the range of $p^{-1}(x)$ . . .

. . . which is $(-\infty, 0]$:

$p(x) = -2x + 3 \quad x \leq 0$



. . . is also the domain of $p(x)$ . . .

. . . which we wrote in the formula as $x \leq 0$

This follows the rule . . .

The domain of any function must be the range of its inverse

And this(very tricky!) example concludes our first section of the course!!!!

(whew!)