Composition of Functions

Consider this real-life example:

> We have a business making hand-made designer dresses. Our business requires that we rent space in a loft for producing our dresses; this rent is \$1200 per month. Also, each dress costs \$60 to make. Write a function expressing our **total monthly costs** in terms of **quantity of dresses produced**.

We define our function as follows:

$$C(x) = monthly\ cost\ (in\ dollars)\ of\ producing\ x\ dresses$$

What goes into our monthly costs?

We have *fixed monthly costs* of \$1200 per month that we owe no matter what.

We also have *variable monthly costs* of \$60 per dress that depend on the number of dresses we make. This part of our function involves x.

So,

$$C(x) = \underbrace{1200}_{\text{fixed cost}} + \underbrace{60x}_{\text{variable cost}}$$

What would be our monthly costs for producing 30 dresses?

$$C(30) = 1200 + 60(30)$$

$$= 3000$$

We will spend \$3000 to make 30 dresses in a given month.

Now, suppose that it takes 3 hours of labor to make one dress.

We can define the following function:

$$x(t) = number\ of\ dresses\ produced\ given\ t\ hours\ of\ work$$

What would the formula be?

Since each hour of work translates into one-third of a dress being made,

$$x(t) = \frac{1}{3}t$$

Now . . . find the function for **our total monthly costs** . . .

. . . in terms of the **number of hours of work** put into dress-making

In other words, find:

$$C(t) = total\ monthly\ costs\ given\ t\ hours\ of\ work$$

This is actually very easy!

Given that we already have $C(x)$ and $x(t)$ . . .

. . . we can simply put them together:

$$C(x) = 1200 + 60x$$

$$x(t) = \frac{1}{3}t$$

Or rather, put $x(t)$ into $C(x)$:

$$C\big(x(t)\big) = 1200 + 60(\frac{1}{3}t)$$

$$C(t) = 1200 + 20t$$

What are our total monthly costs if we work for 24 hours in a given week?

$$\begin{aligned} C(24) &= 1200 + 20(24) \\ &= 1200 + 480 \\ &= 1680 \end{aligned}$$

This example involved something called the **composition of functions**.

To understand the wording, what do you think of when you hear the term,

decompose

or

decomposition?

I think of those words as meaning something like . . . "breaking down"

or

"taking apart"

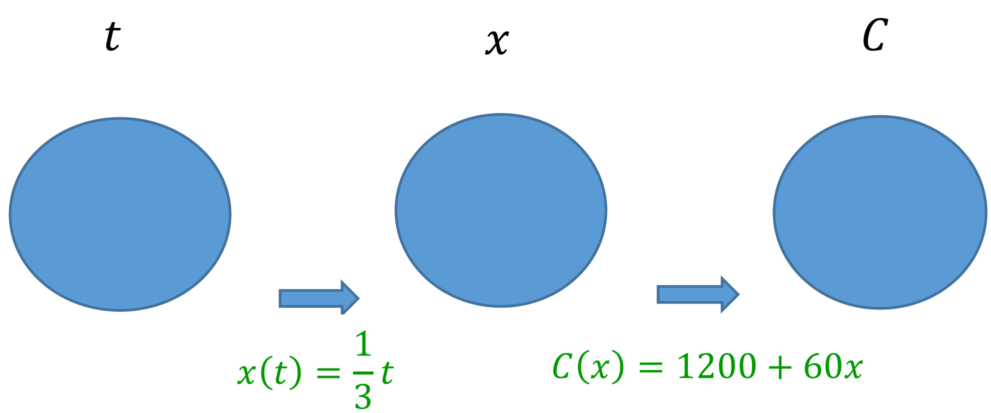
"Composition" is the opposite of that . . .

. . . so it means "putting together"

In the previous example, we **put together** $C(x)$ with $x(t)$ . . .

. . . or put differently, we put $x(t)$ into $C(x)$.

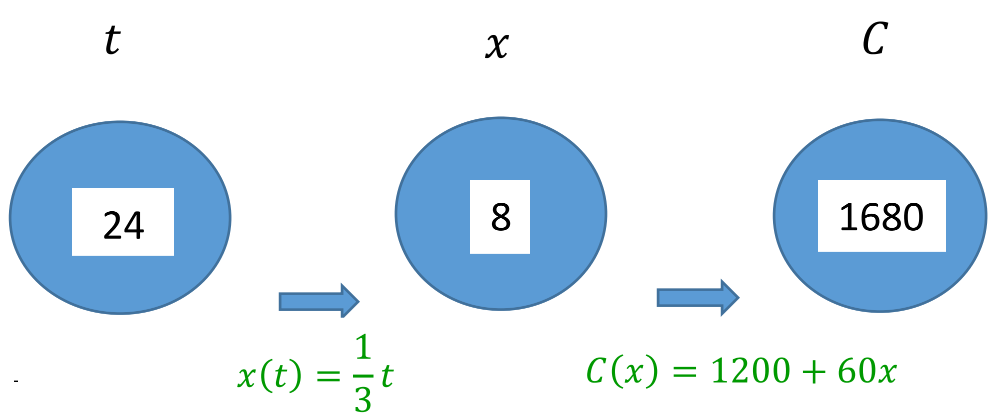
Here's a diagram showing the process:



Here we see that 24 hours of work . . .

. . . results in 8 dresses

. . . for $1680 in total monthly costs



Maybe the simplest way to describe the compositions of functions is . . .

. . . a function of a function.

Let's look at a different, more abstract example:

Consider the following pair of functions:

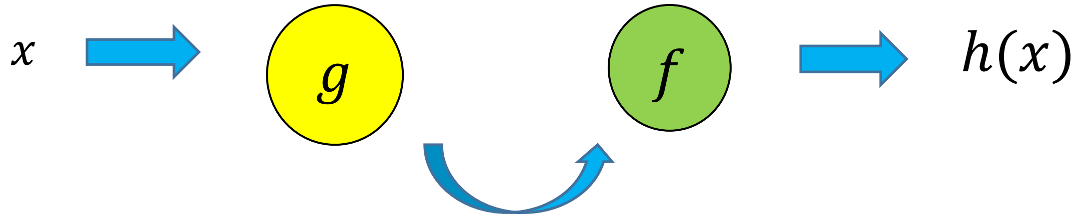
$$f(x) = x^2 \quad \text{and} \quad g(x) = 2x + 1$$

We will **compose** these two functions in the following way:

We will create a new function, $h(x)$, where

$$h(x) = f((g(x))$$

A diagram describing this might look like:



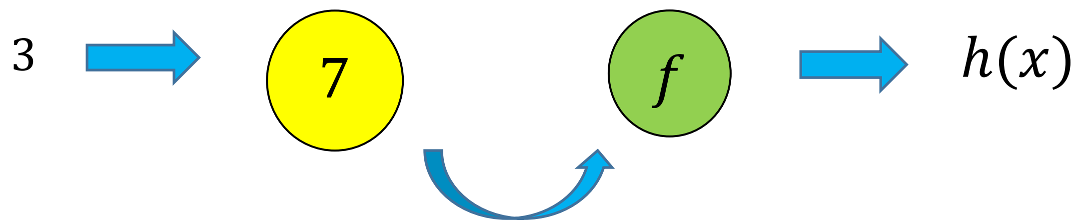
Where in the specific case of $f(x) = x^2$ and $g(x) = 2x + 1$

We have that plugging in $x = 3$:

$$h(3) = f(g(3))$$

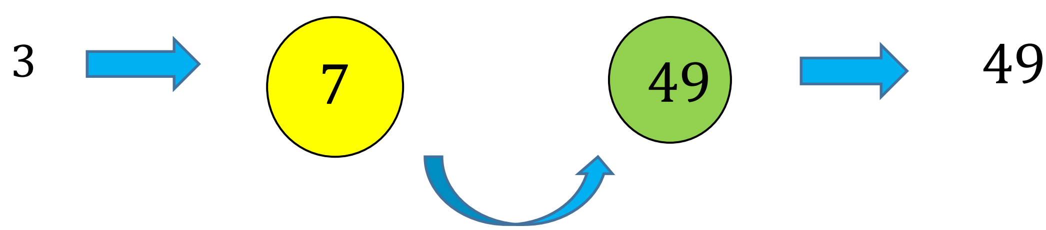
Would first send the value of 3 into $g(x)$:

$$g(3) = 2(3) + 1 = 7$$



and then, send the value of 7 into $f(x)$:

$$f(7) = 7^2 = 49$$



Alternatively, we can find the formula for $h(x) = f(g(x))$:

$$h(x) = f(g(x))$$
$$= f(2x + 1)$$
$$= (2x + 1)^2$$

$$h(3) = (2(3) + 1)^2$$
$$= (7)^2$$
$$= 49$$

Note that the “official” way to right $f(g(x))$ using conventional notation is

$$f \circ g(x)$$

If we wanted to, we could make the function compose the other way:

$$g \circ f(x)$$

Which would work out like this:

$$g\big(f(x)\big) = g(x^2) = 2(x^2) + 1 = 2x^2 + 1$$

Now, we want to try to decompose a **compound** function . . .

. . . into its component parts.

Consider the following function:

$$h(x) = \sqrt{3x - 5}$$

This function can be thought of as having two ***processes*** or ***phases***:

What happens to the input **first**?

It goes through the process:

$$u = 3x - 5$$

What happens **second**?

The result of the first process goes through the process:

$$y = \sqrt{u}$$

Therefore we can make two separate functions:

$$f(x) = 3x - 5 \quad \text{and} \quad g(x) = \sqrt{x}$$

And say,

$$h(x) = g(f(x))$$

Notice that f is the ***inside*** function, because x goes into f first!

Problem: Decompose the function

$$p(x) = \frac{1}{(2x-5)^3}$$

into three functions, $f(x)$, $g(x)$, and $h(x)$, such that

$$p(x) = f \circ g \circ h(x)$$

Or, put less formally,

$$p(x) = f(g(h(x)))$$

Answer:

The innermost function, $h(x)$, is the function that "happens to x" ***first***.

We can see that this would be the process: $y = 2x - 5$.

So,

$$h(x) = 2x - 5$$

Once this output is calculated, it is cubed. Since the output of h goes into g,

$$g(x) = x^3$$

The final stage of $p(x)$ is taking the output of g and dividing it into 1.

So,

$$f(x) = \frac{1}{x}$$

Note that for each of these functions, $h, g,$ and f, the x-value is different.

The x that goes into the innermost function h will come out of h a different number, y, which then turns into the x that goes into g.

This could be confusing!

The key to understanding this, I think, is to think of functions as **processes** rather than variables.

Problem: Decompose the function

$$g(x) = \left(\sqrt{x} - 12\right)^3$$

Into three component functions, and then write $g(x)$ as the composition of those functions.

Answer: First let's decide what should be the innermost function.

What's the mathematical "process" that "happens to" x first?

The first thing to happen to x . . .

. . . is that **its radical** (positive square root) is calculated.

So let's call our innermost function $r(x)$ and say $r(x) = \sqrt{x}$.

What happens next? That's easy: **12 is subtracted** from the output of $r(x)$.

So let's call our next function $s(x) = x - 12$.

The last thing to happen is that this result is **cubed**.

So let's let our final function be $c(x) = x^3$

Putting it all together, we have

$$g(x) = c(s(r(x)))$$

Or

$$g(x) = c \circ s \circ r(x)$$

Note: either of the above answers is correct notation.

Also note: Using r, s, and c as the "names" of our function is entirely optional. We could have used f, h, and q. This problem did not specify the names, but rather simply asked us to come up with three component functions.