Inverse functions

The **simplest** function that is known to exist is

$$f(x) = x$$

This function is called the **identity function**.

Why? The concept of "identity" in math means "producing the same result".

The **additive** identity is **0** because $A + 0 = A$

The **multiplicative** identity is 1 because $A * 1 = A$

So the identity **function** is $f(x) = x$ because $f(A) = A$

It's not a very interesting function!

But it's useful to understand an important **concept** for functions!

Suppose that

$$f(x) = x$$

And

$$f(x) = g(h(x))$$

What would this mean?

Well, we would have

$$g(h(x)) = x$$

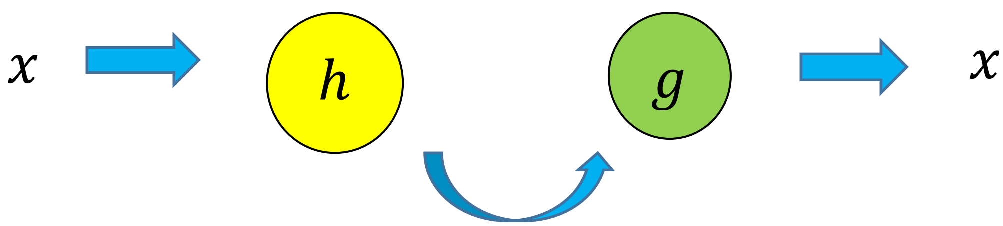
This means that if we start with some (*any*) x . . .

. . . put that x into h . . .

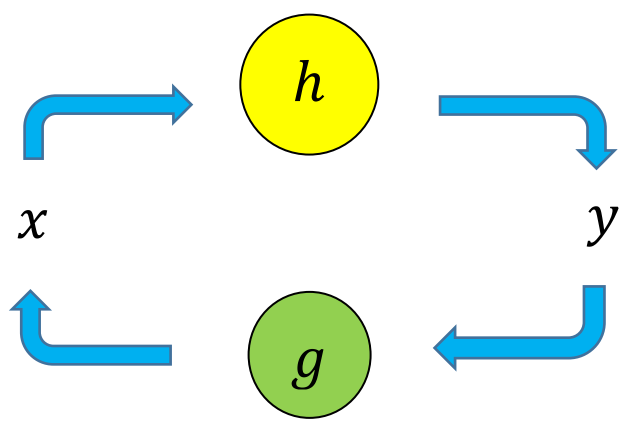
. . . then put $h(x)$ into g . . .

What would come out would be the original input x!

Our functional diagram for this process might look like this:



But an even better diagram might look like this:



What this diagram captures is that the function g . . .

. . . seems to **take us back where we started**.

Put differently, g **reverses** h

In function language, we say that g is the inverse of h.

Let's see an example.

Let

$$h(x) = x + 4$$

What would be the inverse of $h(x)$?

It would **reverse the process** of h.

Since the process of h was about **adding 4** . . .

reversing that process would be about **subtracting 4**.

So the inverse of $h(x)$ would be the function

$$g(x) = x - 4$$

To check, find the composition of these functions:

$$g(h(x)) \quad \textit{and} \quad h(g(x))$$

For g and h to be ***inverse functions***, both of the following must be true:

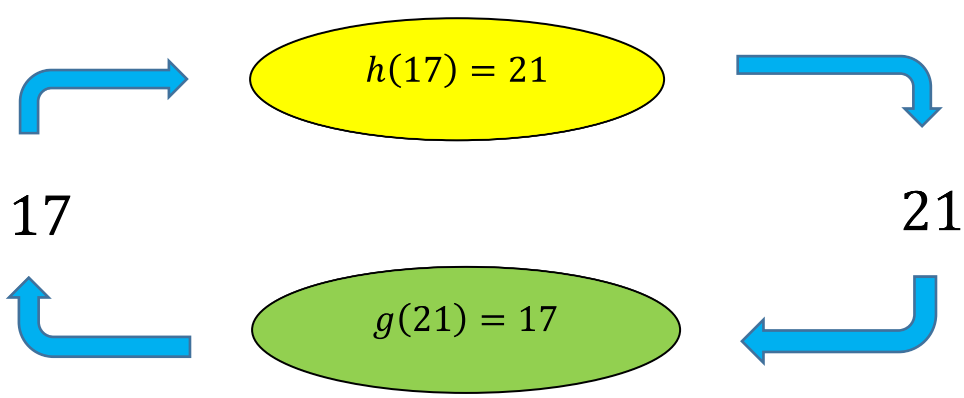
$$g\big(h(x)\big) = x \quad \textit{and} \quad h\big(g(x)\big) = x$$

Both of these are easily verified:

$$\begin{aligned} g\big(h(x)\big) &= g(x+4) \\ &= (x+4) - 4 \\ &= x \\ h\big(g(x)\big) &= h(x-4) \\ &= (x-4) + 4 \\ &= x \end{aligned}$$

We can see this using our previous diagram but plugging in any value:

$$h(x) = x + 4$$



$$g(x) = x - 4$$

There's a special notation we use for inverse functions:

The inverse of $f(x)$ is called $f^{-1}(x)$.

This is probably because the multiplicative inverse of any number x is x^{-1}.

This means that

$$f\left(f^{-1}(x)\right) = x$$

And

$$f^{-1}\left(f(x)\right) = x$$

Let's look at another example.

Let

$$g(x) = x^3$$

Find the inverse function to g, $g^{-1}(x)$

Intuitively, we are looking to **reverse the cubing process**.

What is the reverse of cubing a number?

Finding its cube root!

Therefore,

$$g^{-1}(x) = \sqrt[3]{x}$$

Now consider the function

$$h(x) = 4x - 2$$

Find $h^{-1}(x)$.

Here, we must recognize $h(x)$ as a two-stage process.

First, h takes the input x and . . . **multiplies by 4**.

Then it takes that result and . . . **subtracts 2**.

How would we **reverse the process**?

The answer is that we would first want to **reverse the last step** . . .

(so we would **add 2**)

. . . **and then reverse the first step**.

(so we would **divide by 4**)

The formula executing these two steps would be written as

$$h^{-1}(x) = \frac{x + 2}{4}$$

To check our answer . . .

$$h\left(h^{-1}(x)\right) = h(\frac{x+2}{4})$$
$$= 4\left(\frac{x+2}{4}\right) - 2$$
$$= x + 2 - 2$$
$$= x$$

And,

$$h^{-1}\left(h(x)\right) = h^{-1}(4x-2)$$
$$= \frac{(4x-2)+2}{4}$$
$$= \frac{4x}{4}$$
$$= x$$

Again, we called that method of ***finding the inverse function*** . . .

. . . an “intuitive” method.

which means using your common sense to figure something out.

There’s also a **formal method** of doing the same problem.

This means working step-by-step through a procedure.

Let’s look at how this is done:

How to find the inverse function step-by-step:

Find the inverse function to $h(x) = 4x - 2$

Step 1: write the function using x and y: $y = 4x - 2$

Step 2: reverse the x and y: $x = 4y - 2$

Step 3: solve for y in terms of x: $x = 4y - 2$

adding 2 to both sides

$$x + 2 = 4y$$

dividing both sides by 4

$$\frac{x+2}{4} = y$$

Step 4: Let $y = h^{-1}(x)$: $h^{-1}(x) = \frac{x+2}{4}$

Using the same method, find the inverse function $q^{-1}(x)$ for

$$q(x) = \sqrt[5]{x} - 8$$

Let

$$y = \sqrt[5]{x} - 8$$

Now reverse the variables:

$$x = \sqrt[5]{y} - 8$$

And solve for y:

$$x = \sqrt[5]{y} - 8$$

$$x + 8 = \sqrt[5]{y}$$

$$(x + 8)^5 = y$$

So if

$$q(x) = \sqrt[5]{x} - 8$$

then,

$$q^{-1}(x) = (x + 8)^5$$

Take a moment to confirm (by thinking about it) that these two-stage functions are indeed the inverse or reverse of each other.