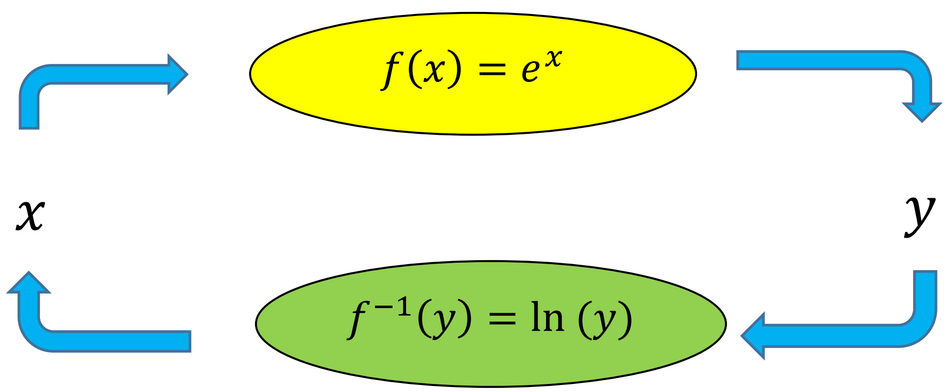
Inverse Trigonometric Functions

We saw in our section on exponential functions that all exponential functions have inverse functions associated with them, known as logarithmic functions:

$$g^{-1}(x) = \ln(x)$$



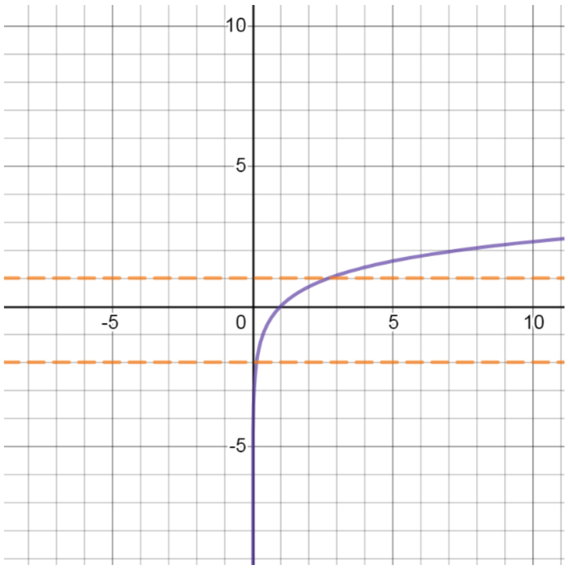
Does the same thing happen with trigonometric functions?

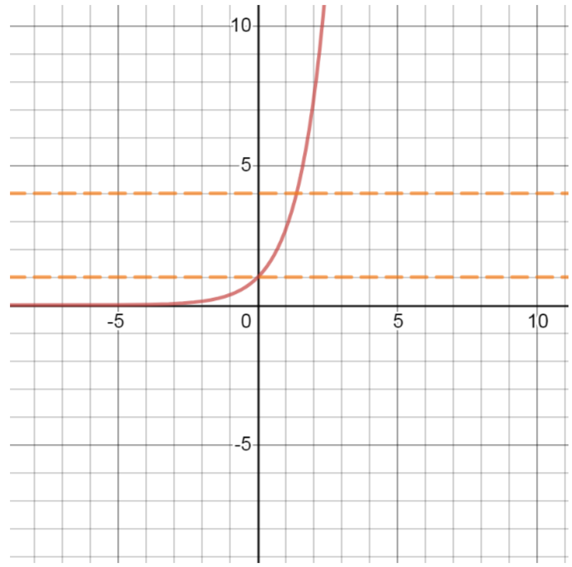
Answer: ***not exactly***.

The difference is that exponential functions are **one-to-one** . . .

. . . meaning that they pass the horizontal line test:

$f(x) = \ln(x)$ $\qquad\qquad$ $g(x) = e^x$





Meaning that any y-value that comes out of the function corresponds with only one x-value that went into the function. Hence these functions

***have* an inverse**

What about

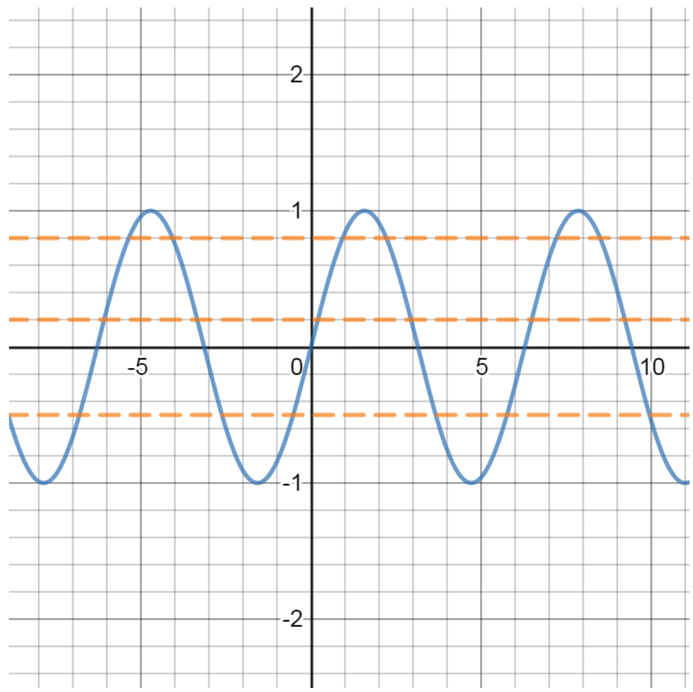
$$f(x) = \sin(x)\ ?$$

Does this function have an inverse?

Does it pass the horizontal line test?

Well let's see:

$g(\theta) = \sin(\theta)$



Clearly, the $sine$ function **fails** the horizontal line test!!!

To see this in an example, we have that

$$\sin\left(\frac{\pi}{2}\right) = 1$$

and

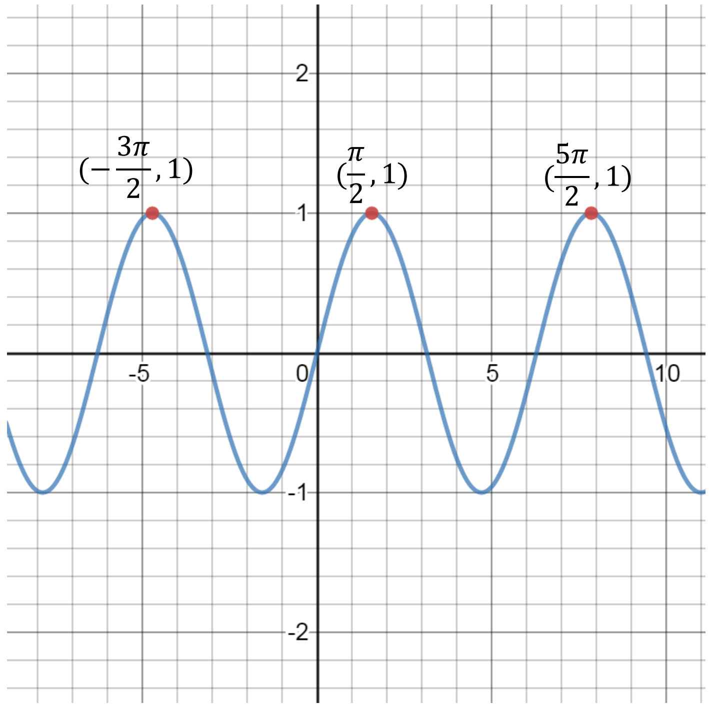
$$\sin\left(-\frac{3\pi}{2}\right) = 1$$

and

$$\sin\left(\frac{5\pi}{2}\right) = 1$$

this means you can't "go back" from 1

Which can be illustrated in the following graph:



So suppose you tried to find

$$\sin^{-1}(1)$$

What would it be?

There are an infinite number of angles that have a sine of 1.

Therefore,

$$f(x) = \sin(x)$$

has no inverse!

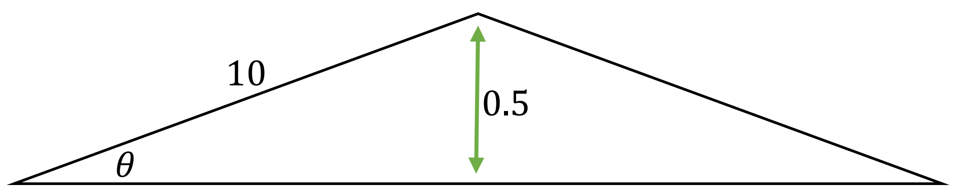
But wait!!!

We kind of need an inverse for sine!

For example, consider the following real-life problem:

A road leads uphill. The top of the hill is 0.5 miles higher in elevation from the bottom of the hill, and the road is 10 miles long. What is the angle that a vehicle must ascend to go up the hill?

This is a problem from what is called “right-triangle” trigonometry. We can illustrate the situation with the following diagram:



We are trying to figure out the angle θ!

To do this, we can set up an equation, using the following formula from right-triangle trigonometry:

$$\sin(\theta) = \frac{opposite\ side}{hypotnuse}$$

Which of course is the right triangle version of

$$\sin(\theta) = \frac{y}{r}$$

The main difference between right-triangle trigonometry, and trigonometric functions, is that in right triangle trigonometry the angles are all between $0°$ and $90°$.

Going back to our problem, we have the following equation:

$$\sin(\theta) = \frac{0.5}{10}$$

or

$$\sin(\theta) = \frac{1}{20}$$

To solve for θ, we would like to be able to find the inverse of sine:

$$\sin^{-1}(\sin(\theta)) = \sin^{-1}\left(\frac{1}{20}\right)$$

Which would cancel the sine on the left side to isolate the angle:

$$\theta = \sin^{-1}\left(\frac{1}{20}\right)$$

Now if you actually look at your scientific calculator, you might see that there is a button that looks like

$$\sin^{-1}$$

And if you use it on the value of $\frac{1}{20}$, you actually get the answer:

$$2.9°$$

How could this be? We just said that the sine function **doesn't have an inverse!!!**

The answer is a bit complicated. We **need** an inverse sine function . . .

So we create one by **restricting the range**!

Here's how we do it . . .

The sine function fails the horizontal line test . . . but if we just look at a very specific part of it . . .

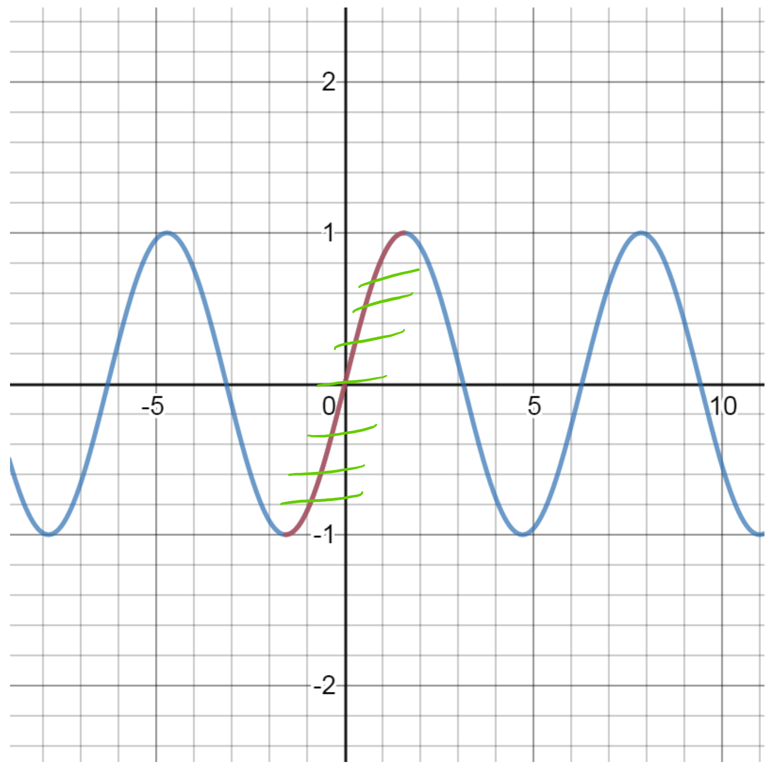
. . . **it passes!**

Check it out:

$f(\theta) = \sin(\theta)$

$g(\theta) = \sin(\theta)$

$-\frac{\pi}{2} < \theta < \frac{\pi}{2}$



passes the horizontal line test

As you can see, as long as we restrict the **domain** of sine to

$$\left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$$

the sine function passes the horizontal line test (is one-to-one).

This means that the **range** of

$$\sin^{-1}$$

must be restricted to the same set of values.

To make clear that we are making a special case for this inverse sine function, we usually use a special name for it:

arcsine

We define the inverse sine function to be

$$h(x) = \arcsin(x) = \theta$$

where

$$\sin(\theta) = x$$

and

$$-\frac{\pi}{2} \leq \theta \leq \frac{\pi}{2}$$

Let’s look at a couple of examples!

Find

$$\arcsin(1)$$

As we saw earlier, there are many angles θ such that $\sin(\theta) = 1$.

However, there is only one such angle θ such that

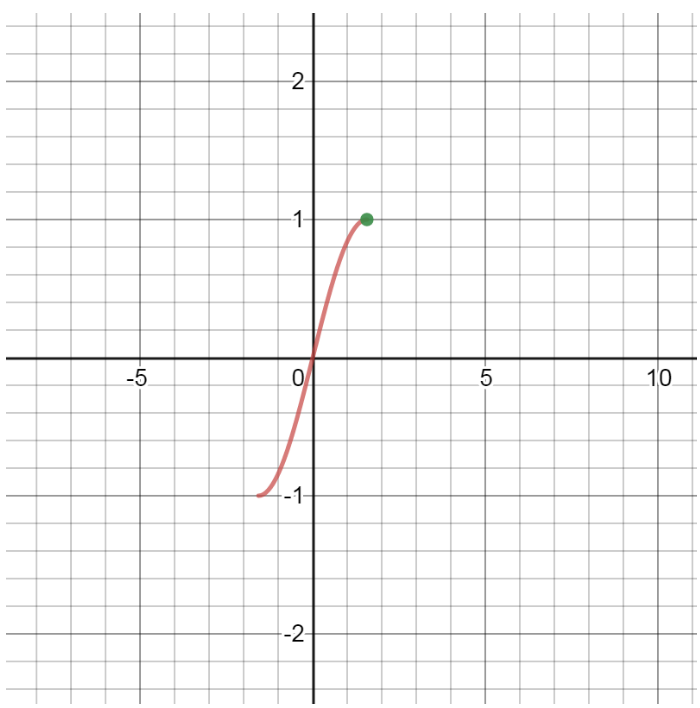
$$\sin(\theta) = 1$$

and

$$-\frac{\pi}{2} \leq \theta \leq \frac{\pi}{2}$$

That angle is

$$\frac{\pi}{2}$$



Here’s another example:

Find

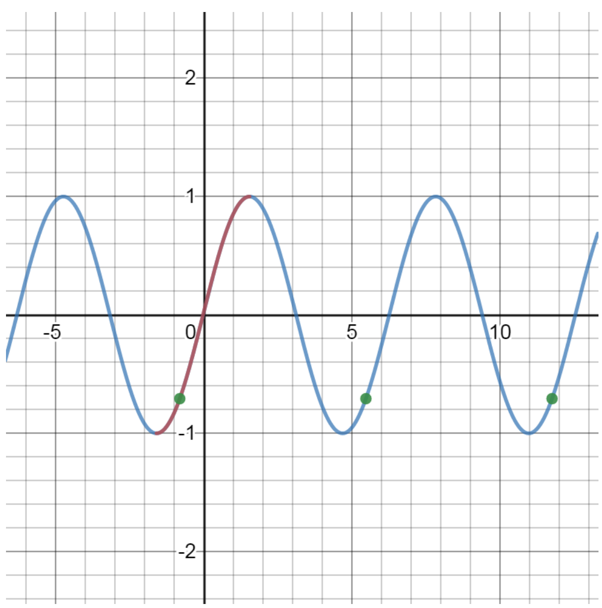
$$\arcsin\left(-\frac{1}{\sqrt{2}}\right)$$

We have many angles that have a sine of $-\frac{1}{\sqrt{2}}$:

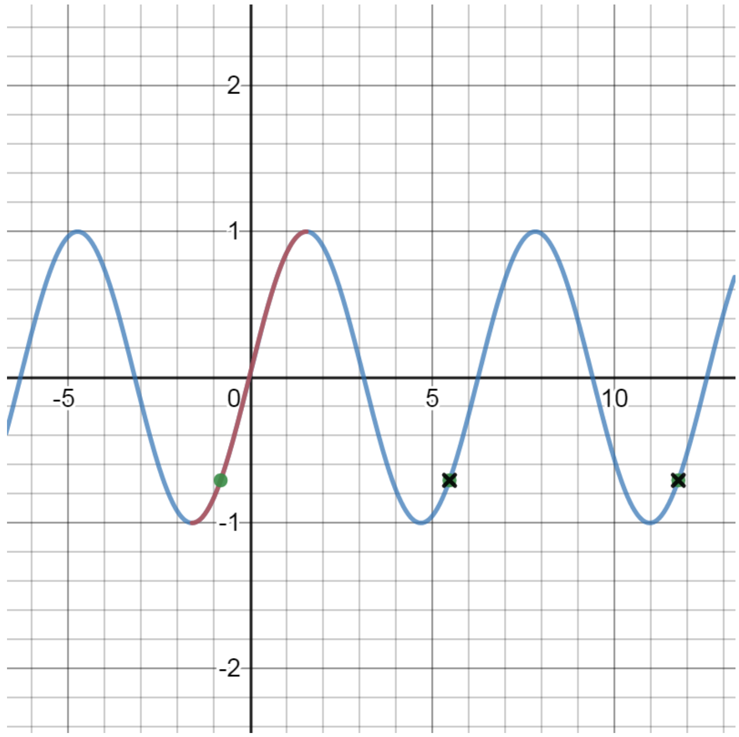
$$\sin\left(-\frac{\pi}{4}\right) = -\frac{1}{\sqrt{2}}$$

$$\sin\left(\frac{7\pi}{4}\right) = -\frac{1}{\sqrt{2}}$$

$$\sin\left(\frac{15\pi}{4}\right) = -\frac{1}{\sqrt{2}}$$



But only one of those angles is between $-\frac{\pi}{2}$ and $\frac{\pi}{2}$:



And that is the angle

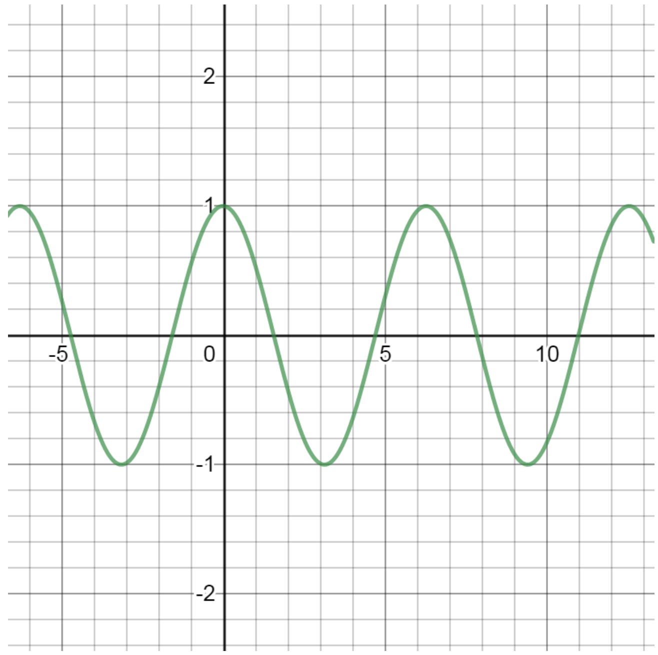
$$\theta = -\frac{\pi}{4}$$

(otherwise known in degree language as -45°).

Now we also need an inverse function for cosine!

Check out the graph of the cosine function if you don't believe me:

$h(\theta) = \cos(\theta)$



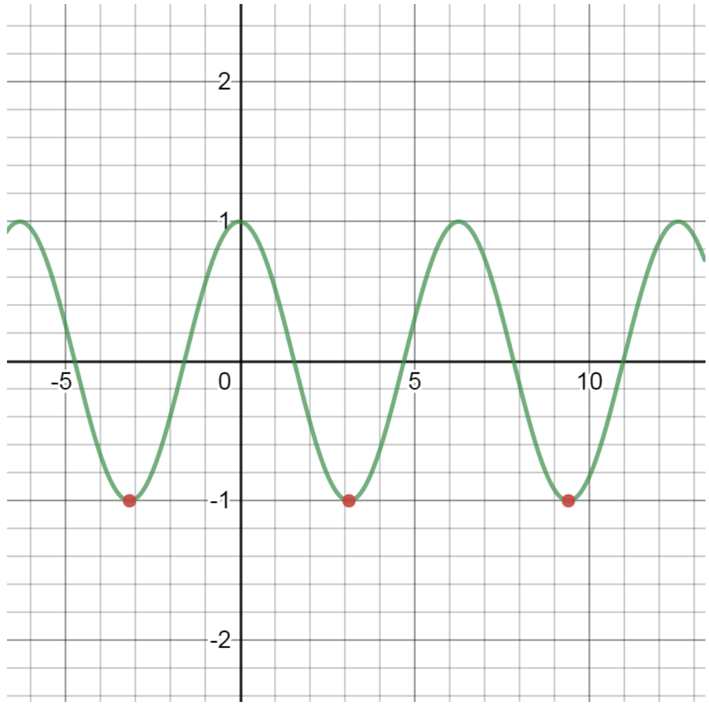
Clearly this function fails the Horizontal Line Test!

So if were trying to find the angle θ such that

$$\cos(\theta) = -1$$

We would not be able to find just one:

$h(\theta) = \cos(\theta)$



Again, there are many angles θ that have $\cos(\theta) = -1$:

$$\cos(-\pi) = -1$$
$$\cos(\pi) = -1$$
$$\cos(3\pi) = -1$$

And many more!

And again, since

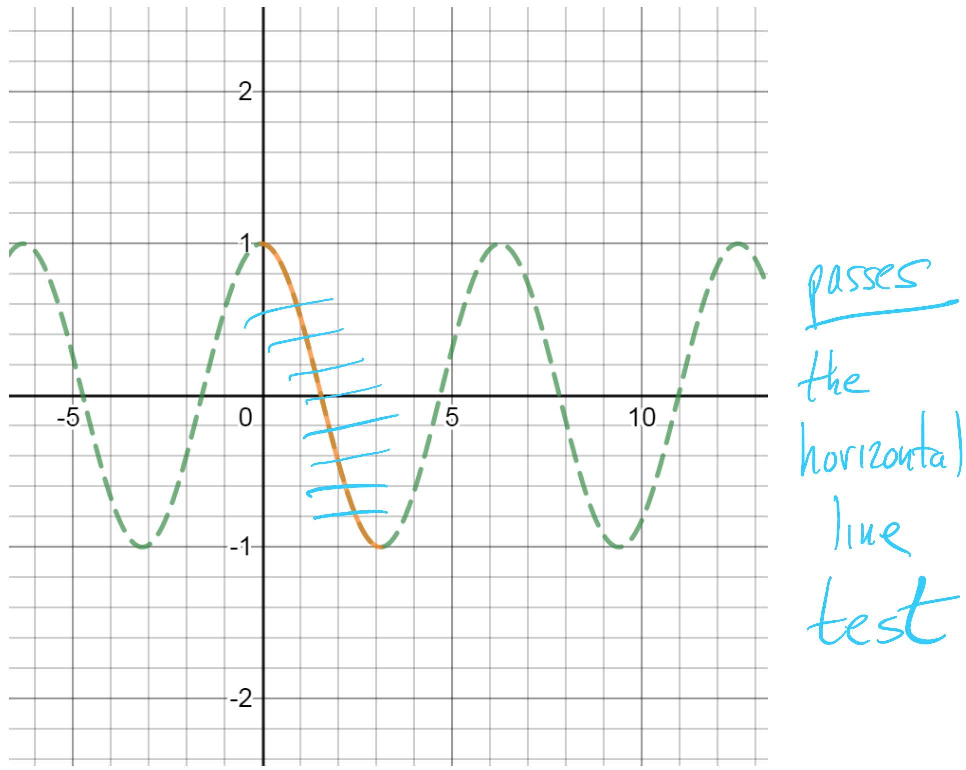
$$\cos^{-1}(-1)$$

does not have a unique output, it is **not a function**.

Sorry if I am belaboring this point, but I find this concept kind of hard!

In order to make an inverse cosine function work, we need to restrict its range. Let's see how:

$$g(\theta) = \cos(\theta)$$
$$0 \leq \theta \leq \pi$$



As you can see, as long as we restrict the **domain** of sine to

$$[0, \pi]$$

the sine function passes the horizontal line test (is one-to-one).

This means that the **range** of

$$\cos^{-1}$$

must be restricted to the same set of values.

To make clear that we are making a special case for this inverse sine function, we usually use a special name for it:

arccosine

We define the inverse cosine function to be

$$h(x) = \arccos(x) = \theta$$

where

$$\cos(\theta) = x$$

and

$$0 \le \theta \le \pi$$

Based on this definition, then, the only angle θ such that $0 \leq \theta \leq \pi$

and

$$\cos(\theta) = -1$$

is the angle

$$\pi$$

(otherwise known as 180°)

and therefore

$$\arccos(-1) = \pi$$

Let's do another example!

Find

$$\arccos\left(-\frac{\sqrt{3}}{2}\right)$$

Here we are looking for the one (*and only one!*) angle θ that lies . . .

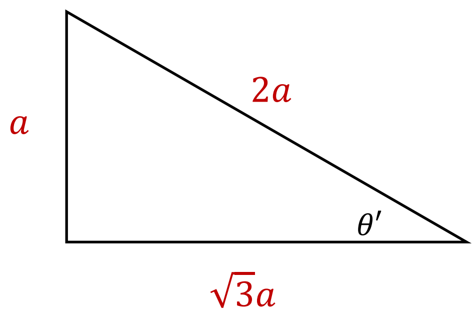
. . . between 0 and π . . .

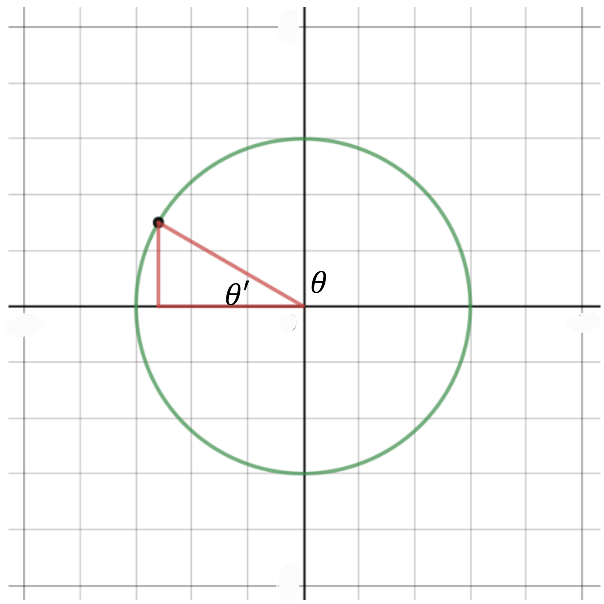
(otherwise known, in degree-world, as between 0° and 180°)

Such that

$$\cos(\theta) = -\frac{\sqrt{3}}{2}$$

I might examine my circle diagram to figure this out:





We know that the angle that makes the ratio of sides such that

$$\cos(\theta) = \frac{x}{r} = -\frac{\sqrt{3}}{2} = \left(-\frac{\sqrt{3}a}{2a}\right)$$

is

$$150°$$

or

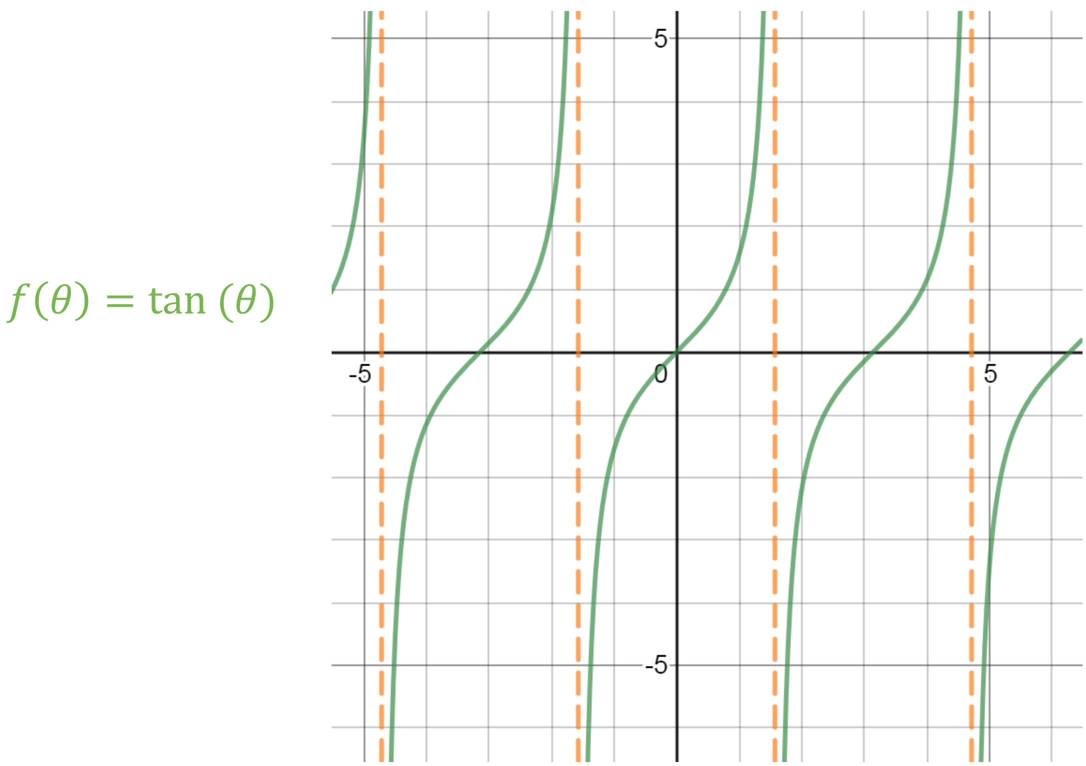
$$\frac{5\pi}{6}$$

Thus

$$\arccos\left(\frac{\sqrt{3}}{2}\right) = \frac{5\pi}{6}$$

Finally we must also find an inverse function for tangent!

We haven't looked at the graph of the tangent function yet, but here goes:



We will look at this graph more closely in the next section . . .

. . . but we certainly notice here that it fails the Horizontal Line Test!

So to define a working inverse tangent function we will also need to restrict the range of the function in the same kind of way that we did with sine and cosine.

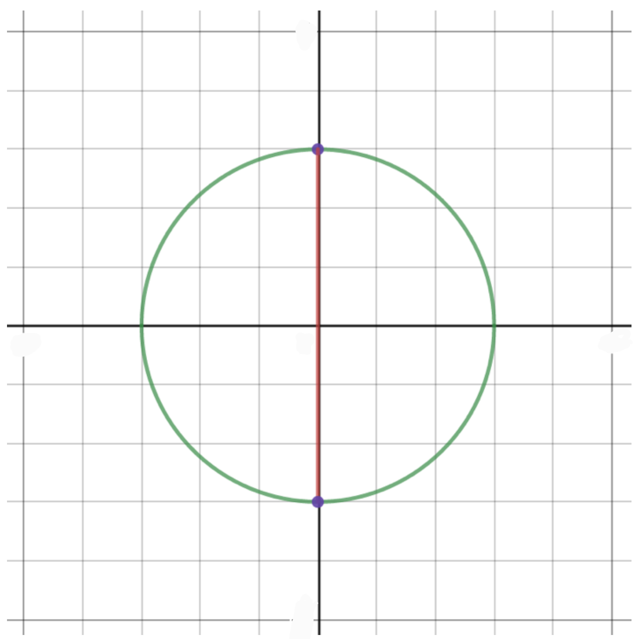
This graph has vertical asymptotes whenever tangent becomes undefined.

Since

$$\tan(\theta) = \frac{y}{x}$$

This happens whenever $x = 0$.

And we know that $x = 0$ many many times:



including such angles as

$$90° \; or \; \frac{\pi}{2}$$

$$270° \; or \; \frac{3\pi}{2}$$

$$-90° \; or -\frac{\pi}{2}$$

and so on.

This is why there are vertical asymptotes on the graph!

But notice that **between** the vertical asymptotes . . .

. . . the function **passes the horizontal line test** (is one-to-one)!

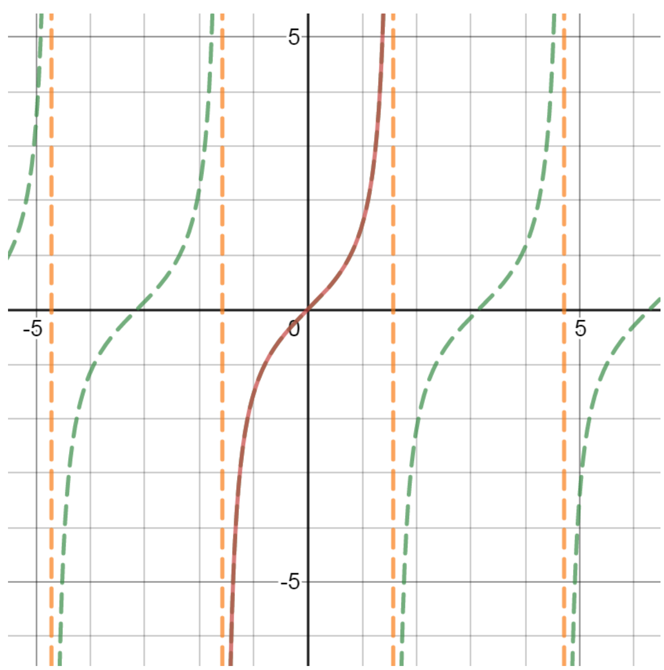
We will therefore choose one of these intervals . . .

. . . the interval from $-\frac{\pi}{2}$ to $\frac{\pi}{2}$ . . .

As our range for arctangent!

$$f(\theta) = \tan(\theta)$$

$$-\frac{\pi}{2} < \theta < \frac{\pi}{2}$$



Note that this is the same range as the arcsine function!

We define the inverse tangent function to be

$$h(x) = \arctan(x) = \theta$$

where

$$\tan(\theta) = x$$

and

$$-\frac{\pi}{2} \leq \theta \leq \frac{\pi}{2}$$

Let's do an example!

Find

$$\arctan(-1)$$

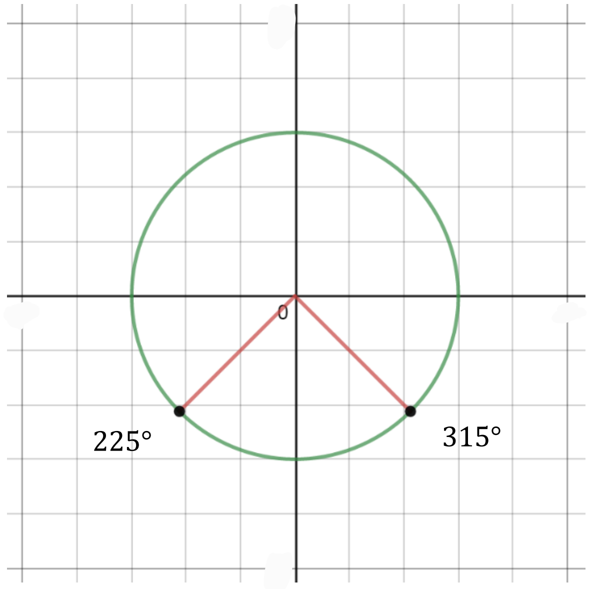
Here, if I have chosen to memorize my table of basic function values I would look for an angle giving me a tangent equal to 1:

θ	$0°$	$30°$	$45°$	$60°$	$90°$	$180°$	$270°$
$\sin(\theta)$	0	$\frac{1}{2}$	$\frac{1}{\sqrt{2}}$	$\frac{\sqrt{3}}{2}$	1	0	-1
$\cos(\theta)$	1	$\frac{\sqrt{3}}{2}$	$\frac{1}{\sqrt{2}}$	$\frac{1}{2}$	0	-1	0
$\tan(\theta)$	0	$\frac{\sqrt{3}}{3}$	**1**	$\sqrt{3}$	$undef.$	0	$undef.$

And see that the reference angle is 45°.

However, I am looking for a tangent of **-**1!

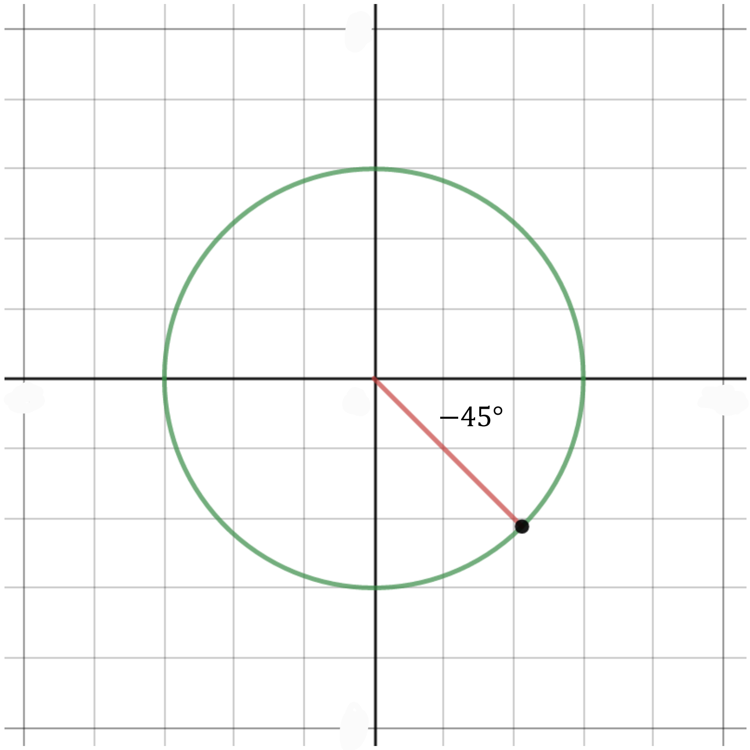
There are two angles between 0° and 360° that have a reference angle of 45° and a negative tangent:



Those angles are 225° and 315°.

However, neither of these angles are in the correct range!

We realize that -45° is the correct angle that will give us a tangent of -1.



Which in radians is

$$-\frac{\pi}{4}$$

Therefore,

$$\arctan(-1) = -\frac{\pi}{4}$$