Zeroes of Polynomial Functions: Factor Method

Some zeroes of polynomial functions can be obtained by factoring.

Consider

$$f(x) = -x^2 + 7x + 30$$

To find the zeroes of $f(x)$, we must solve the equation:

$$f(x) = 0$$

We can do this by factoring. First, multiply both sides by -1 so that the leading coefficient is positive. This creates a standard model for all polynomial equations:

$$-x^2 + 7x + 30 = 0$$

$$-1(-x^2 + 7x + 30) = -1(0)$$

$$x^2 - 7x - 30 = 0$$

making the leading coefficient even

Now factor the polynomial:

$$x^2 - 7x - 30 = 0$$

$$(x - 10)(x + 3) = 0$$

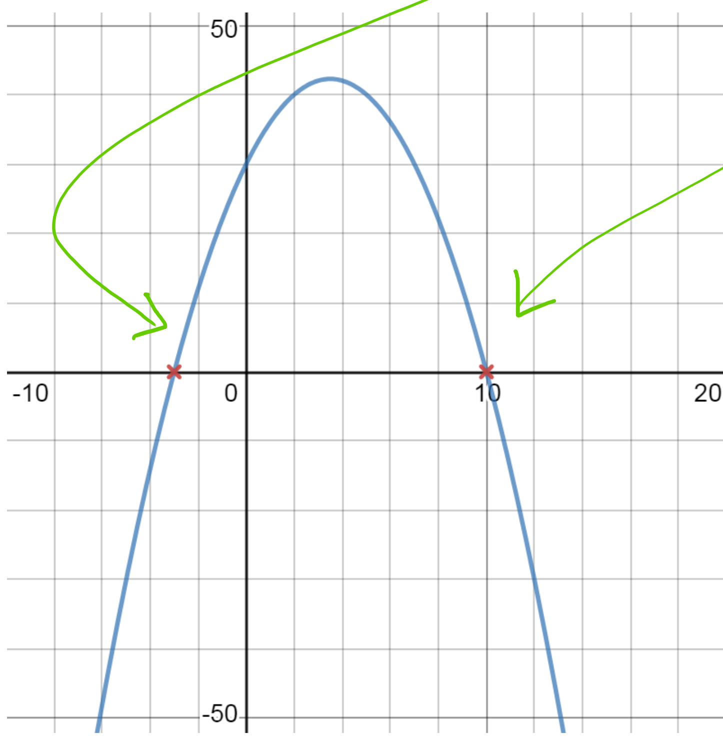
And use the ***zero-product rule*** to find the solutions for x.

$$x - 10 = 0 \qquad x + 3 = 0$$

$$x = 10 \qquad x = -3$$

Looking at the graph, we see that we got it right:

$f(x)$



Here's a slightly more elaborate one:

$$h(x) = x^4 - 9x^2$$

Here, we are trying to solve:

$$x^4 - 9x^2 = 0$$

The first step in factoring is always to factor out the greatest common factor:

$$x^4 - 9x^2 = 0$$

$$x^2(x^2 - 9) = 0$$

And now we factor the binomial to get:

$$x^2(x + 3)(x - 3) = 0$$

$$x^2 = 0 \qquad x + 3 = 0 \qquad x - 3 = 0$$

$$x = 0 \qquad x = -3 \qquad x = 3$$

The set of zeroes is:

$$\{0_2, -3, 3\}$$

Notice the way I wrote the 0_2. The subscript (2) next to the zero stands for

multiplicity

The zero $x = 0$ has multiplicity 2 (sometimes called a double-zero) because the equation

$$x^2 = 0$$

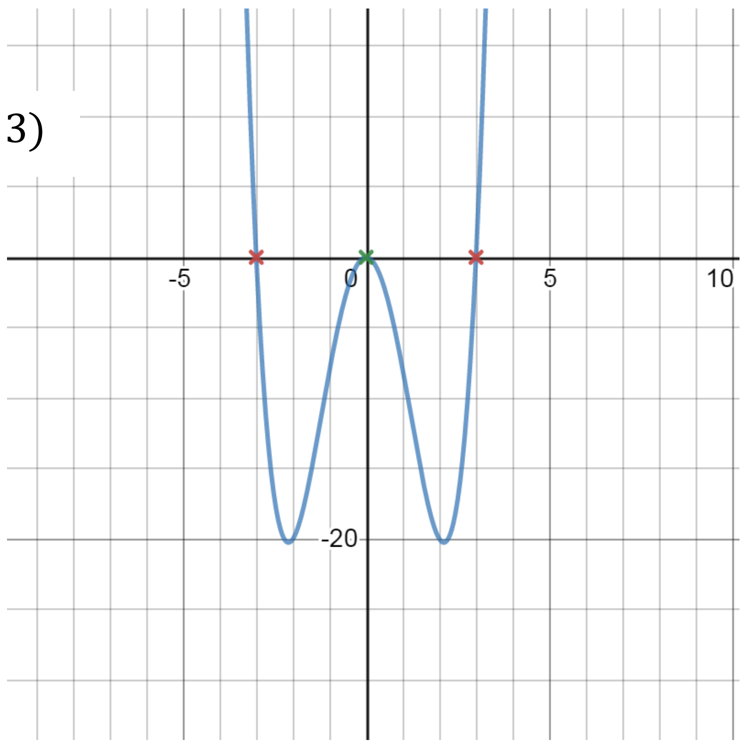
technically could factor into two separate equations:

$$x = 0 \quad \text{and} \quad x = 0$$

What is the *significance* of a zero of multiplicity 2?

Let's look at the graph:

$$h(x) = x^2(x+3)(x-3)$$



Note that at $x = \pm 3$, the graph ***crosses through*** the x-axis.

This is what happens at x-intercepts when the **multiplicity is *one***.

But at $x = 0$, the graph just ***touches*** the x-axis before turning back.

This is what happens at x-intercepts when the **multiplicity is *two***.

Now consider the function

$$q(x) = x^3 - 2x^2 - x$$

To find the zeroes by factoring, we first factor out the GCF:

$$x^3 - 2x^2 - x = 0$$
$$x(x^2 - 2x - 1) = 0$$

But here our factoring efforts get stuck!

$x^2 - 2x - 1$ is *not factorable!!*

What are we going to do??

It's okay . . . all quadratic equations can be solved . . .

. . . by **completing the square** . . .

. . . or **the quadratic formula**

Let's use **completing the square**!

First, we separate the two factors we have and use the zero-product rule:

$$x(x^2 - 2x - 1) = 0$$
$$x = 0 \qquad x^2 - 2x - 1 = 0$$

Now we solve the second equation by moving the constant to the other side:

$$x^2 - 2x - 1 = 0$$

$$\qquad\qquad +1 \qquad +1$$

$$x^2 \underset{\underset{b}{\uparrow}}{- 2x} \qquad = 1$$

Then adding $\left(\frac{b}{2}\right)^2$ to both sides:

$$x^2 - 2x + \left(-\frac{2}{2}\right)^2 = 1 + \left(-\frac{2}{2}\right)^2$$

$$x^2 - 2x + 1 = 1 + 1$$

And then we factor the left side and find it a perfect square:

$$(x-1)(x-1) = 2$$

$$(x-1)^2 = 2$$

Finally we take the square root of both sides:

$$\sqrt{(x-1)^2} = \pm\sqrt{2}$$

because all positive numbers have <u>two</u> square roots

$$x - 1 = \pm\sqrt{2}$$

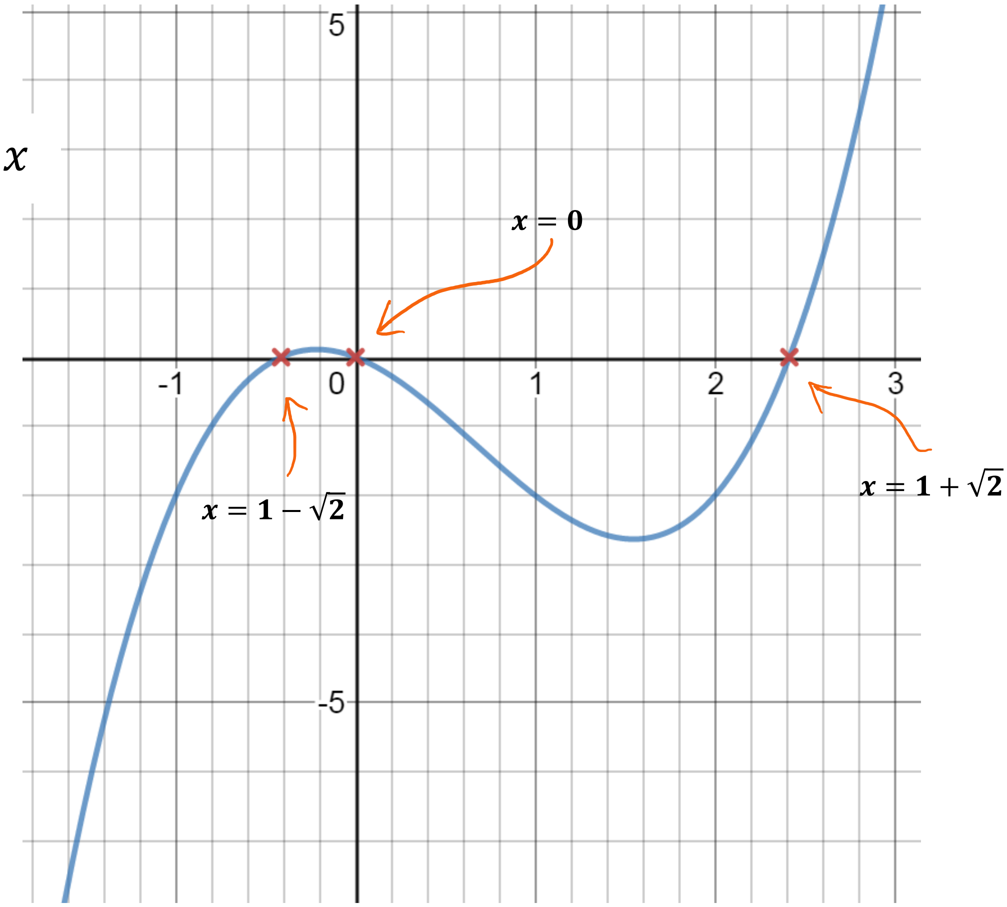
$$x = 1 \pm \sqrt{2}$$

So the zeroes of this function are

$$\{0, 1+\sqrt{2}, 1-\sqrt{2}\}$$

Now let's look at the graph:

$$q(x) = x^3 - 2x^2 - x$$



Notice for example that the largest zero,

$$x = 1 + \sqrt{2}$$

Can be approximated by

$$x = 1 + 1.42 = 2.42$$

Finally, notice that even though $q(x)$ cannot be factored completely . . .

. . . using only **rational numbers**:

$$q(x) = x(x^2 - 2x - 1)$$

It **can** be factored completely if we use **irrational numbers**:

$$x = 0 \qquad x = 1 + \sqrt{2} \qquad x = 1 - \sqrt{2}$$

$$x = 0 \qquad x - (1 - \sqrt{2}) = 0 \qquad x - (1 + \sqrt{2}) = 0$$

$$x(x - (1 - \sqrt{2}))(x - (1 + \sqrt{2}) = 0$$

If you don't believe me, multiply this out:

$$(x - (1 - \sqrt{2}))(x - (1 + \sqrt{2})$$

$$(x - (1 - \sqrt{2}))(x - (1 + \sqrt{2}))$$

$$= x^2 - x(1 + \sqrt{2}) - x(1 - \sqrt{2}) + (1 - \sqrt{2})(1 + \sqrt{2})$$

$$= x^2 - x - \cancel{\sqrt{2}x} - x + \cancel{\sqrt{2}x} + 1 - \cancel{\sqrt{2}} + \cancel{\sqrt{2}} - 2$$

You'll find that it equals

$$x^2 - 2x - 1$$

$$= x^2 - 2x - 1$$