Finding zeroes of polynomial functions

So far, when we are looking for zeroes to polynomial functions, we have been lucky enough to "stumble into them" by trial and error.

Sometimes we might not be so lucky!

Consider the following problem:

Find the zeroes and graph the function:

$$p(x) = -3x^3 + 8x^2 + 10x - 8$$

Finding the zeroes requires us to be able to **factor** $p(x)$.

Since $p(x)$ is a cubic polynomial, there is no easy method!

We will go searching for zeroes by trial and error, using synthetic substitution.

This is time-consuming but's okay . . .

. . . we have to **graph** $p(x)$ . . .

. . . so we need to **get the y-values** anyway!

Let's plug x-values into $p(x)$ in an organized way.

We will start with $x = 0$ and keep going in the ***positive direction*** until the y-values seem like they're going to infinity . . . then start over at $x = 0$ and do the same thing in the negative direction.

You are encouraged to try this on your own!

$$p(x) = -3x^3 + 8x^2 + 10x - 8$$

x-values (inputs)

	-3	-8	10	-8
0				*-8*
1	*-3*	*5*	*15*	*7*
2	*-3*	*2*	*14*	*20*
3	*-3*	*-1*	*7*	*13*
4	*-3*	*-4*	*-6*	*-32*
5	*-3*	*-7*	*-25*	*-133*
0				*-8*
-1	*-3*	*11*	*-1*	*-7*
-2	*-3*	*14*	*-18*	*28*
-3	*-3*	*17*	*-41*	*115*

y-values (outputs)

Unfortunately . . .

. . . **we found no zeroes!!**

Or didn't we?

Check again:

	-3	-8	10	-8
0				*-8*
1	*-3*	*5*	*15*	*7*
2	*-3*	*2*	*14*	*20*
3	*-3*	*-1*	*7*	*13*
4	*-3*	*-4*	*-6*	*-32*
5	*-3*	*-7*	*-25*	*-133*
0				*-8*
-1	*-3*	*11*	*-1*	*-7*
-2	*-3*	*14*	*-18*	*28*
-3	*-3*	*17*	*-41*	*115*

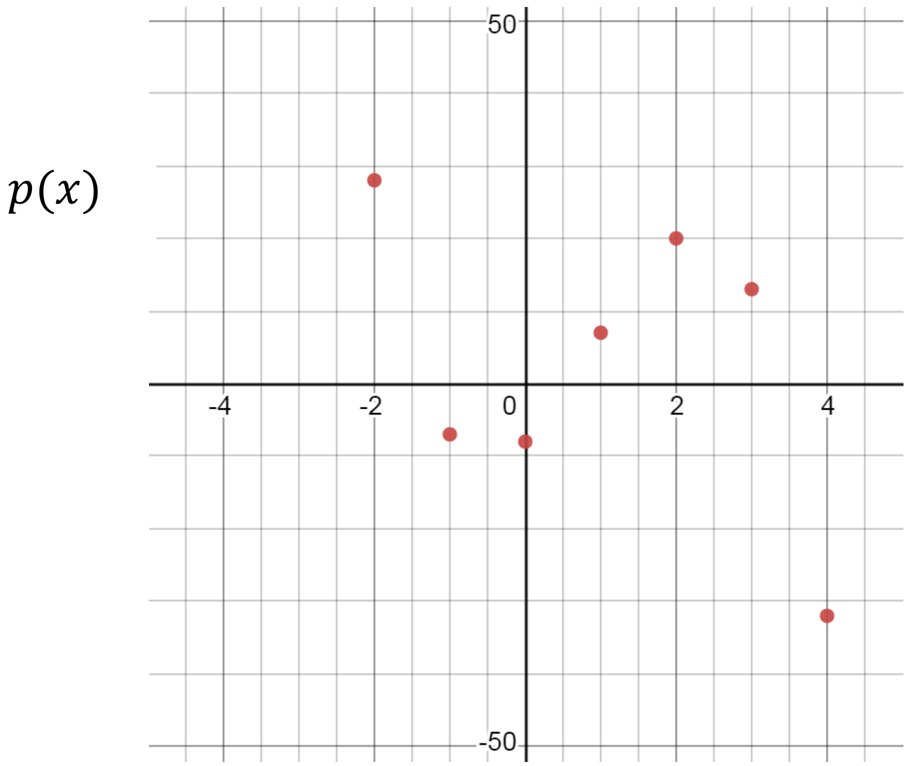
We see three places where the **y-values change sign** . . .

. . . as we go up by one x.

This means the function crossed the x-axis!!

We can see this by plotting the graph:

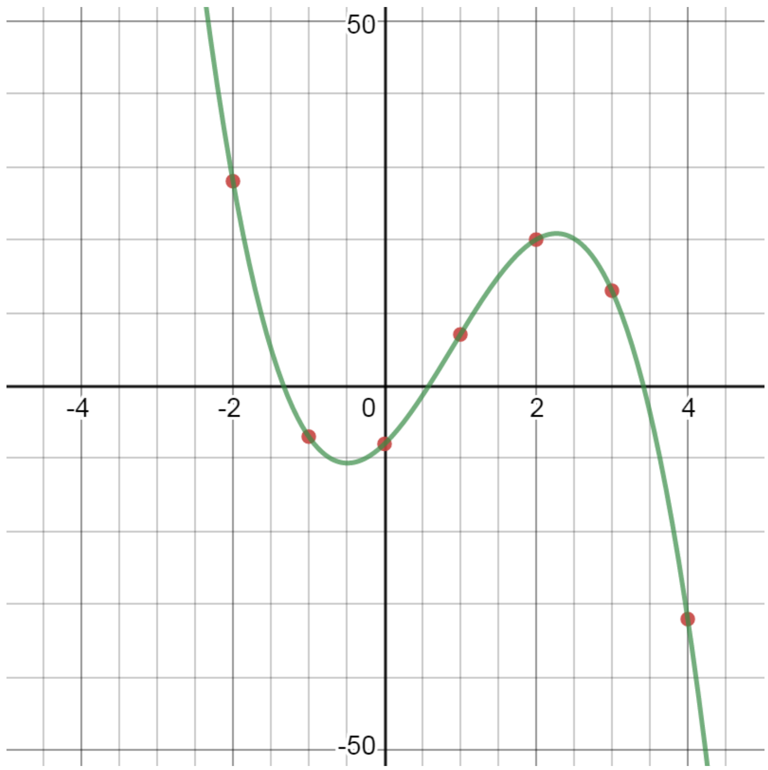
(note that we need to use different scales for x and y)



Can you see the three places where the function has to cross the x-axis?

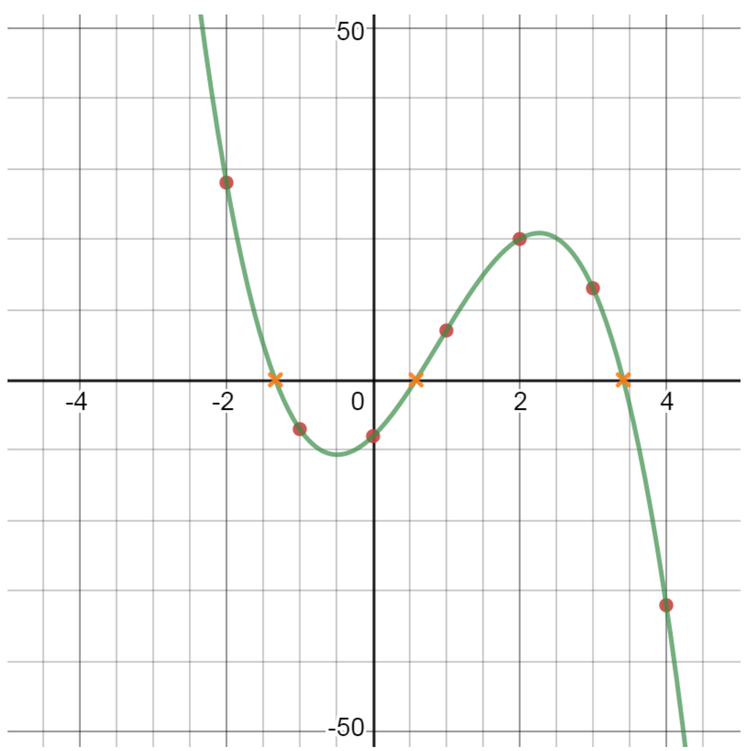
Now let's draw the graph by connecting the points with a smooth curve:

$p(x)$



Yes, we can see that the function must cross the x-axis in three places:

$p(x)$



But here's the problem . . .

. . . we know *about* **where** they are . . .

. . . but we need to know **exactly what** they are!

And all we need is to know one of them . . .

. . . and then use synthetic division!

Here's the good news:

If one of them is a rational number . . .

. . . meaning some sort of fraction . . .

. . . we can figure out what it might be:

Here's how!

this next page or two is technical and you dont need to know it.

We have that

$$p(x) = -3x^3 + 8x^2 + 10x - 8$$

Think about how its factorization would have to work if it had a rational zero:

$$-3x^3 + 8x^2 + 10x - 8$$

$$= (\underline{a}\,x \pm \underline{b})(\underline{\quad}\,x^2 \pm \underline{\quad}\,x \pm \underline{\quad})$$

multiply to $-3x^3$

multiply to -8

We are really only interested in that first factor for now . . .

. . . because once we set it equal to zero . . .

. . . it will give us the zero we are looking for:

$$(ax \pm b) = 0$$

$$ax \pm b = 0$$

$$ax = \pm b$$

$$x = \pm \frac{b}{a}$$

So we are looking for $\boldsymbol{a}$ and b in the below scenario:

$$-3x^3 + 8x^2 + 10x - 8$$

$$= \quad (\underline{\underline{a\,x}} \quad \pm \quad b \;) \;\; (\quad \underline{\underline{x^2}} \quad \pm \quad x \quad \pm \quad)$$

multiplies to $-3x^3$

First think about $\boldsymbol{a}$ . . .

. . . can you see that it has to divide in evenly into -3?

It must be a **factor** of the leading coefficient to $p(x)$.

Now, since the zero we are looking for is x where

$$x = \pm \frac{b}{a}$$

We can conclude that the denominator of x must be a factor of -3!

There are only two possibilities, given that 3 is a prime number:

$$1 \quad or \quad 3$$

So our zero has to look like

$$\pm\frac{\quad}{1} \quad or \quad \pm\frac{\quad}{3}$$

Now let's think about what the numerator must be:

$$-3x^3 + 8x^2 + 10x - 8$$

$$= \quad (ax \pm b)(x^2 \pm x \pm \quad)$$

multiplies to -8

$$x = \pm\frac{b}{a}$$

Our numerator comes from ***b***, and ***b*** multiplies some number to be -8!

So ***b*** is a **factor** of -8!

There are four different factors of 8:

$$1 * 8 = 8$$

$$2 * 4 = 8$$

So the numerator of our zero must be either

$$1 \quad or \quad 2 \quad or \quad 4 \quad or \quad 8$$

Combining this with what we learned about the denominator . . .

. . . (either 1 or 3) . . .

We have the following possibilities for our zero:

$$\pm\frac{1}{1} \quad \pm\frac{2}{1} \quad \pm\frac{4}{1} \quad \pm\frac{8}{1} \quad \pm\frac{1}{3} \quad \pm\frac{2}{3} \quad \pm\frac{4}{3} \quad \pm\frac{8}{3}$$

If we simplify, this becomes:

$$\pm 1 \quad \pm 2 \quad \pm 4 \quad \pm 8 \quad \pm\frac{1}{3} \quad \pm\frac{2}{3} \quad \pm\frac{4}{3} \quad \pm\frac{8}{3}$$

That's a total of sixteen possibilities! That's a lot to try!

But we don't need to try out all of them!

Check out our synthetic substitution table again:

	-3	-8	10	-8
0				-8
1	-3	5	15	7
2	-3	2	14	20
3	-3	-1	7	13
4	-3	-4	-6	-32
5	-3	-7	-25	-133
0				-8
-1	-3	11	-1	-7
-2	-3	14	-18	28
-3	-3	17	-41	115

our possible zeroes happen here

(because the graph is crossing from negative to positive or back)

Our only possible zeroes are . . .

- between 0 and 1
- between 3 and 4
- between -1 and -2

How many of the below zeroes qualify?

$$+1 \quad +2 \quad +4 \quad +8 \quad +\frac{1}{3} \quad +\frac{2}{3} \quad +\frac{4}{3} \quad +\frac{8}{3}$$

$$-1 \quad -2 \quad -4 \quad -8 \quad -\frac{1}{3} \quad -\frac{2}{3} \quad -\frac{4}{3} \quad -\frac{8}{3}$$

None of the integers!!!

$$\cancel{+1} \quad \cancel{+2} \quad \cancel{+4} \quad \cancel{+8} \quad +\frac{1}{3} \quad +\frac{2}{3} \quad +\frac{4}{3} \quad +\frac{8}{3}$$

$$\cancel{-1} \quad \cancel{-2} \quad \cancel{-4} \quad \cancel{-8} \quad -\frac{1}{3} \quad -\frac{2}{3} \quad -\frac{4}{3} \quad -\frac{8}{3}$$

Between 0 and 1 we have $\frac{1}{3}$ and $\frac{2}{3}$, but there are none between 3 and 4:

$$\cancel{+1} \quad \cancel{+2} \quad \cancel{+4} \quad \cancel{+8} \quad +\frac{1}{3} \quad +\frac{2}{3} \quad \cancel{+\frac{4}{3}} \quad \cancel{+\frac{8}{3}}$$

$$\cancel{-1} \quad \cancel{-2} \quad \cancel{-4} \quad \cancel{-8} \quad -\frac{1}{3} \quad -\frac{2}{3} \quad -\frac{4}{3} \quad -\frac{8}{3}$$

On the negative side, there is one fraction, $\frac{-4}{3}$, between -1 and -2:

$$\cancel{+1} \quad \cancel{+2} \quad \cancel{+4} \quad \cancel{+8} \quad +\frac{1}{3} \quad +\frac{2}{3} \quad \cancel{+\frac{4}{3}} \quad \cancel{+\frac{8}{3}}$$

$$\cancel{-1} \quad \cancel{-2} \quad \cancel{-4} \quad \cancel{-8} \quad \cancel{-\frac{1}{3}} \quad \cancel{-\frac{2}{3}} \quad -\frac{4}{3} \quad \cancel{-\frac{8}{3}}$$

So we end up with only three possibilities: $\{\frac{1}{3}.\frac{2}{3},-\frac{4}{3}\}$

We try them out using synthetic substitution:

	$\mathbf{-3}$	$\mathbf{8}$	$\mathbf{10}$	$\mathbf{-8}$
$\mathbf{\frac{1}{3}}$	-3	7	$\frac{37}{3}$	$-\frac{35}{9}$
$\mathbf{\frac{2}{3}}$	-3	6	14	$\frac{4}{3}$
$-\mathbf{\frac{4}{3}}$	-3	12	-6	$\mathbf{0}$

We found it!

$$\boldsymbol{x = -\frac{4}{3}}$$

is our zero!!!!

And it's our ONLY rational zero!

We can use synthetic division to find the other two:

	$\mathbf{-3}$	$\mathbf{8}$	$\mathbf{10}$	$\mathbf{-8}$
$\mathbf{\frac{1}{3}}$	-3	7	$\frac{37}{3}$	$-\frac{35}{9}$
$\mathbf{\frac{2}{3}}$	-3	6	14	$\frac{4}{3}$
$-\mathbf{\frac{4}{3}}$	-3	12	-6	$\mathbf{0}$

$$p(x) = \left(x - \frac{4}{3}\right)(-3x^2 + 12x - 6) = 0$$

$$-3x^2 + 12x - 6 = 0$$

$$-3(x^2 - 4x + 2) = 0$$

Dividing both sides by -3, we get:

$$x^2 - 4x + 2 = 0$$

I'll solve this by completing the square:

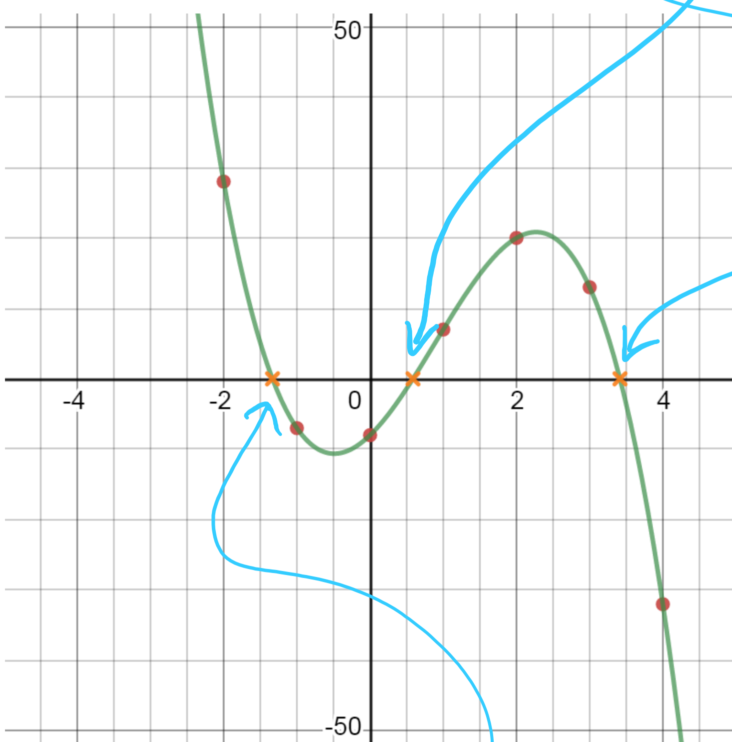
$$\begin{aligned} x^2 - 4x &= -2 \\ x^2 - 4x + \mathbf{4} &= -2 + \mathbf{4} \\ (x-2)(x-2) &= 2 \\ (x-2)^2 &= 2 \\ (x-2) &= \pm\sqrt{2} \\ x &= 2 \pm \sqrt{2} \end{aligned}$$

We can see that these zeroes check out with our graph:

$$x_1 = 2 - \sqrt{2} \cong 0.6$$

$$x_2 = 2 + \sqrt{2} \cong 3.4$$

$p(x)$



As does of course our rational zero, $x_3 = -\frac{4}{3} \cong -1.33$