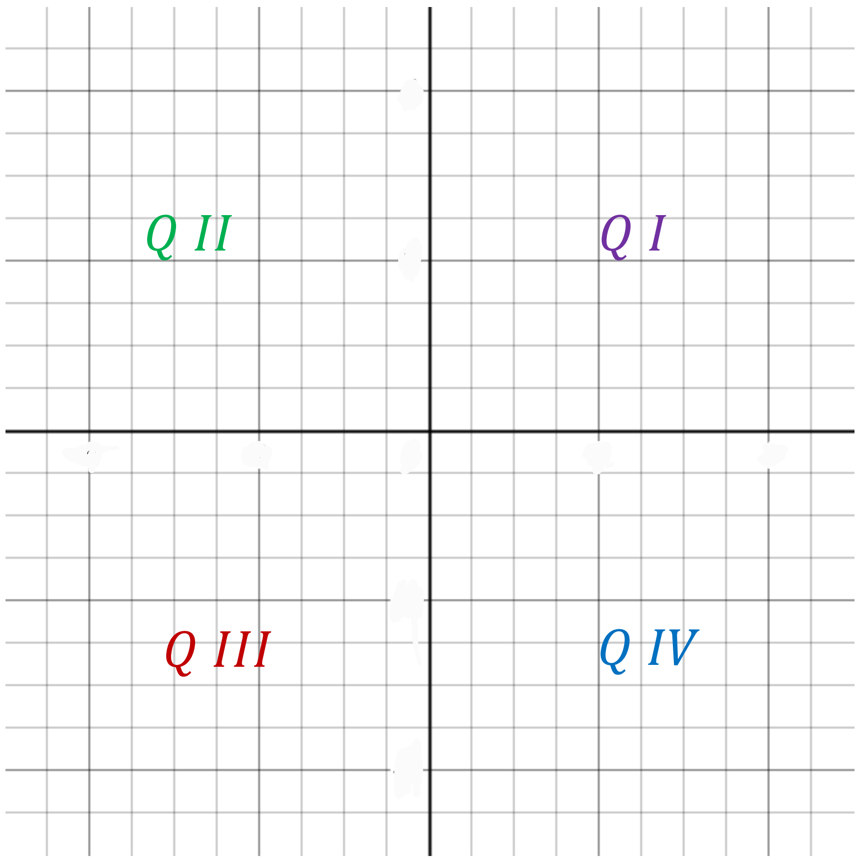
Trigonometric functions of angles from other quadrants

Earlier I mentioned the place called “Quadrant I”. So far, the angles of our special triangles were all from this quadrant of the coordinate axis. We now want to be able to find the sine, cosine, and tangent from angles within other quadrants. First, let’s define where those quadrants are:

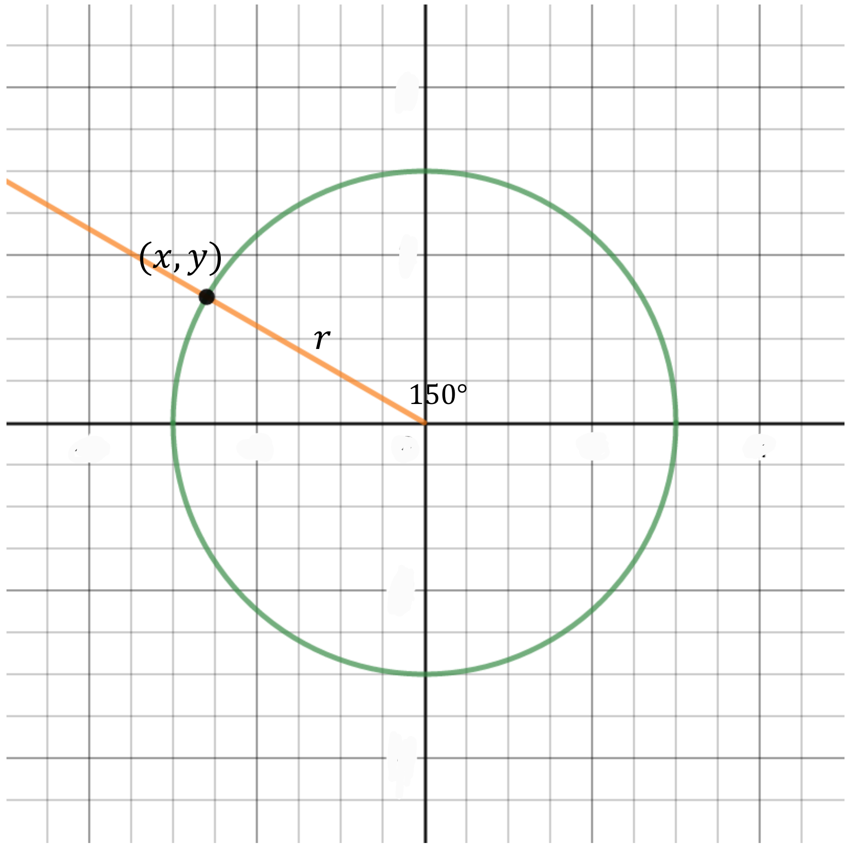


Consider the following problem:

Find the function value

$$\sin(150°)$$

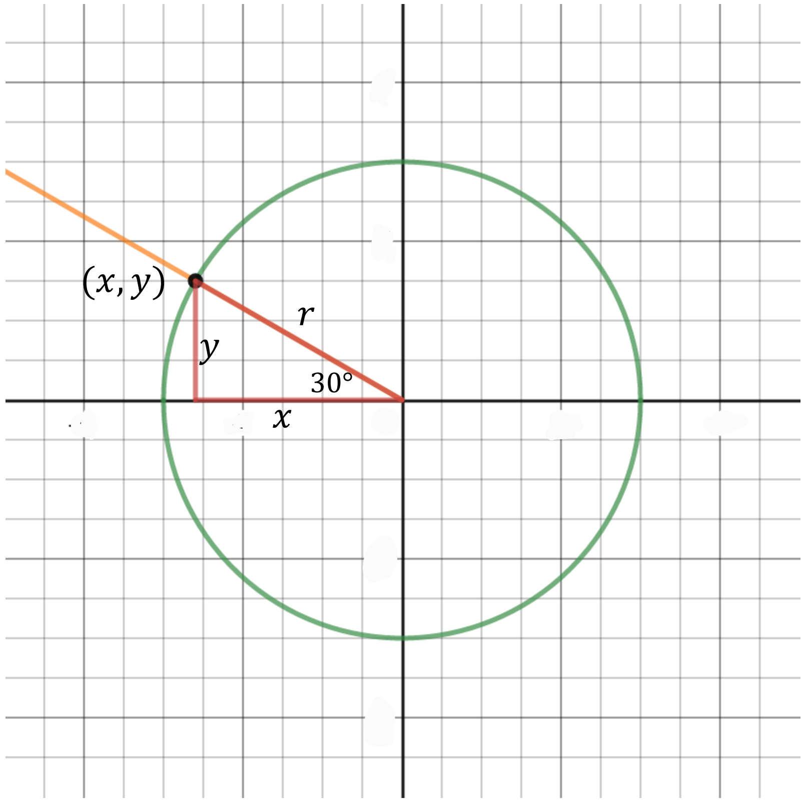
This angle is in Q II:



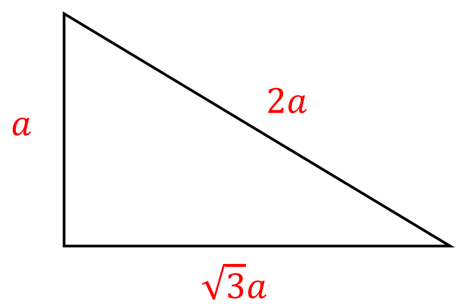
There are **two ways** to do this kind of problem.

The first way is to use the ratios of special triangles. This method is based on *understanding*.

Notice that the interior angle of the triangle formed by the sides x and y is 30°:



And we know the ratios of the sides of a 30-60-90 triangle:



So we use this set of ratios, along with the formula for sine, to get

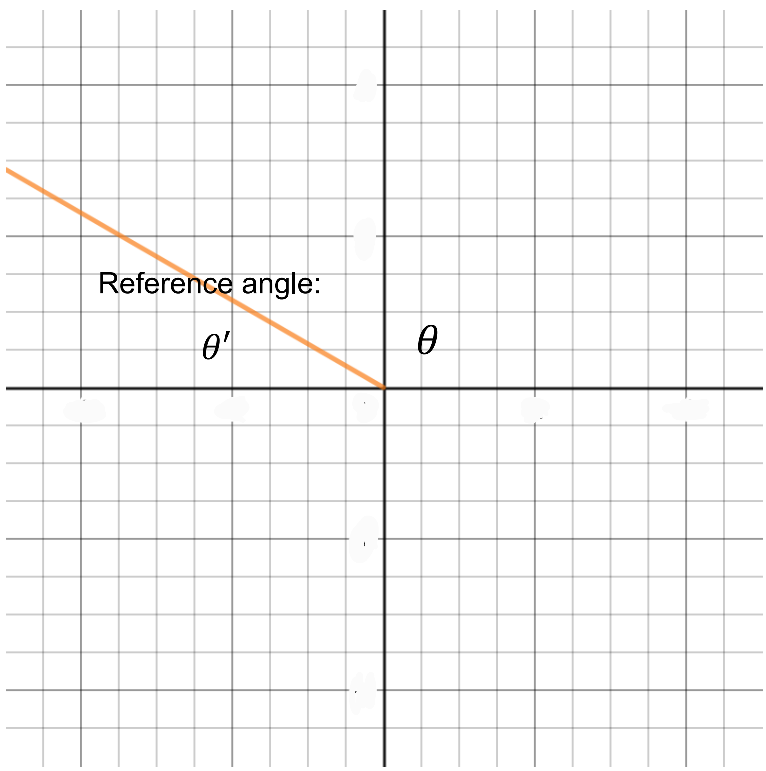
$$\sin(150°) = \frac{y}{r} = \frac{a}{2a} = \frac{1}{2}$$

Method II involves the use of something called ***reference angles***.

Here is the official definition:

Let θ be an angle in the standard position. Its **reference angle** is the acute angle θ' formed by the terminal side of θ and the horizontal axis.

Now isn't that what we just did in the first method? Pretty much! The reference angle for 150° is 30°:



Now, once you know the reference angle, you know what the trigonometric function value is from our table of values:

θ	$0°$	$30°$	$45°$	$60°$	$90°$	$180°$	$270°$
$\sin(\theta)$	0	$\boldsymbol{\frac{1}{2}}$	$\frac{1}{\sqrt{2}}$	$\frac{\sqrt{3}}{2}$	1	0	-1
$\cos(\theta)$	1	$\frac{\sqrt{3}}{2}$	$\frac{1}{\sqrt{2}}$	$\frac{1}{2}$	0	-1	0
$\tan(\theta)$	0	$\frac{\sqrt{3}}{3}$	1	$\sqrt{3}$	$undef.$	0	$undef.$

So the answer is

$$\sin(150°) = \sin(30°) = \frac{1}{2}$$

So the Reference Angle method is based more on *memorization* than *understanding.* It works pretty well . . .

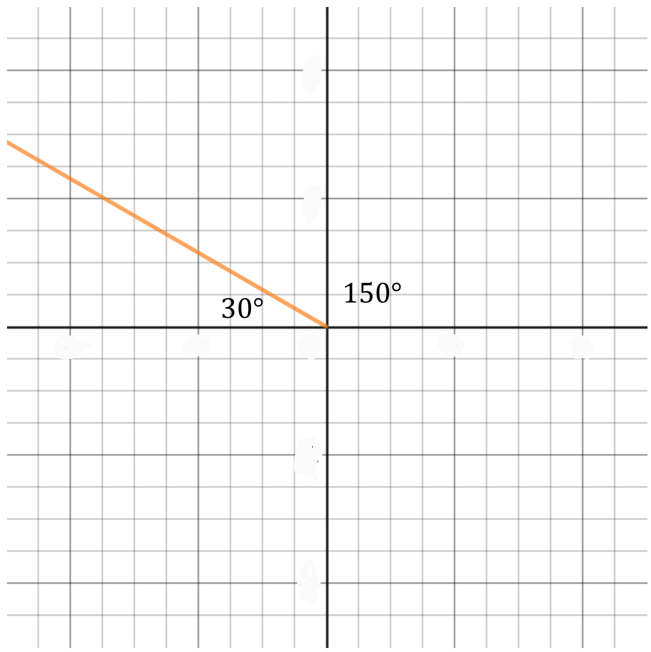
. . . there's just one catch . . .

. . . a very important one . . .

To see this, let's look at a similar problem:

$$\cos(150°)$$

Here, the reference angle is also 30° . . .



So we would look to our memorized trigonometric value to get the answer:

θ	$0°$	$30°$	$45°$	$60°$	$90°$	$180°$	$270°$
$\sin(\theta)$	0	$\frac{1}{2}$	$\frac{1}{\sqrt{2}}$	$\frac{\sqrt{3}}{2}$	1	0	-1
$\cos(\theta)$	1	$\boldsymbol{\frac{\sqrt{3}}{2}}$	$\frac{1}{\sqrt{2}}$	$\frac{1}{2}$	0	-1	0
$\tan(\theta)$	0	$\frac{\sqrt{3}}{3}$	1	$\sqrt{3}$	$undef.$	0	$undef.$

To get

$$\cos(150°) = \cos(30°) = \frac{\sqrt{3}}{2}$$

And yet this answer is **WRONG**!

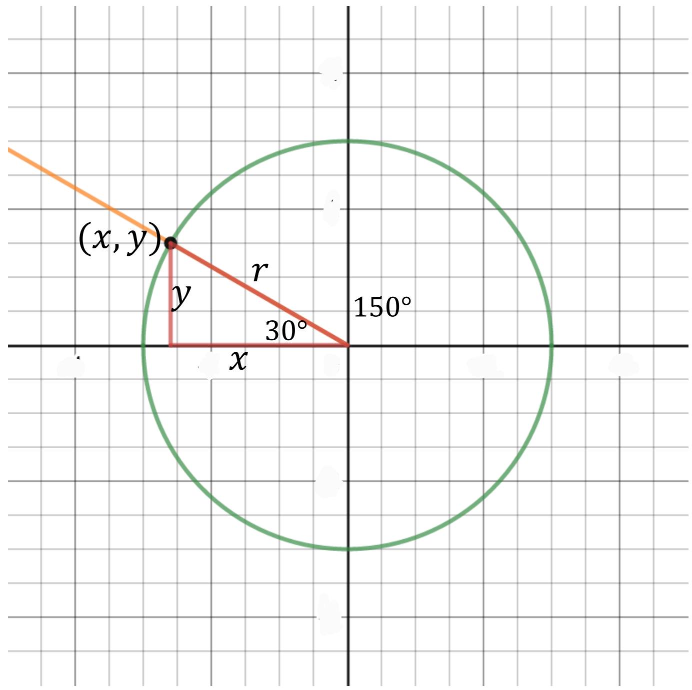
The correct answer is

$$-\frac{\sqrt{3}}{2}$$

Why?

To understand why, let’s go back to the original method:

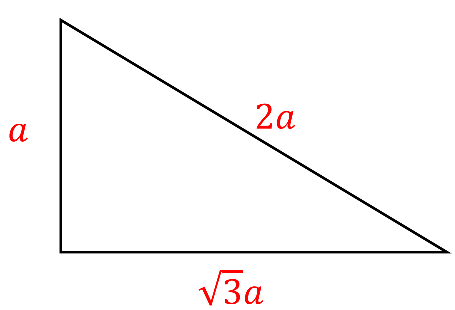
Here we actually look at what’s happening with the sides of the triangle:



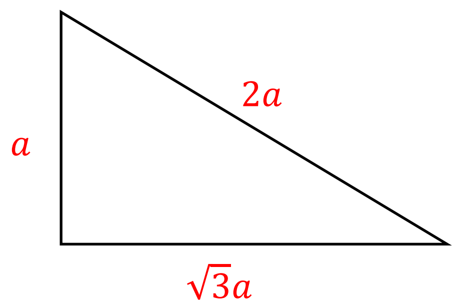
We know that

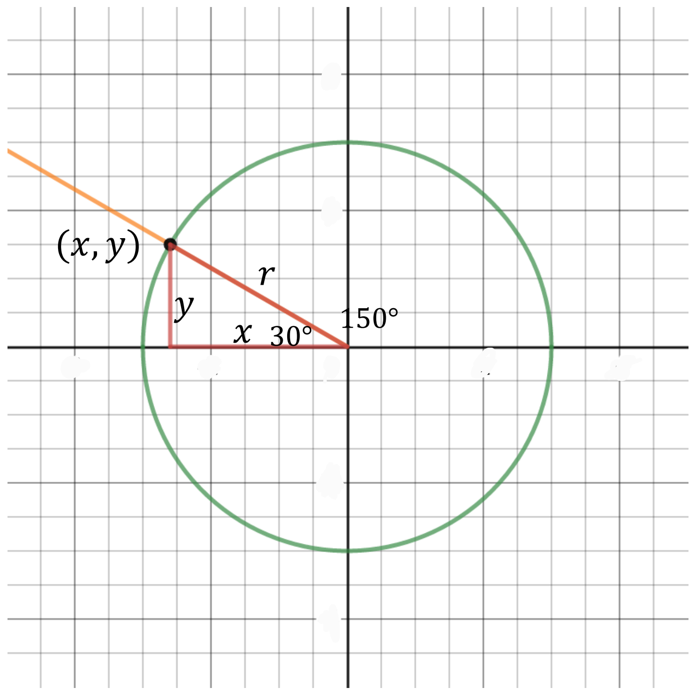
$$\cos(150°) = \frac{x}{r}$$

And we know that the ratio of sides in this triangle is



But because we are looking at the actual point in the coordinate plane, we see that the x-coordinate is **negative**!





So when we use the formula for cosine, we note that the x-value has to be **negative**.

Note that r stands for the **radius**, which is *always positive*.

So we get that

$$\cos(150°) = \frac{x}{r} = \frac{-\sqrt{3}a}{2a} = -\frac{\sqrt{3}}{2}$$

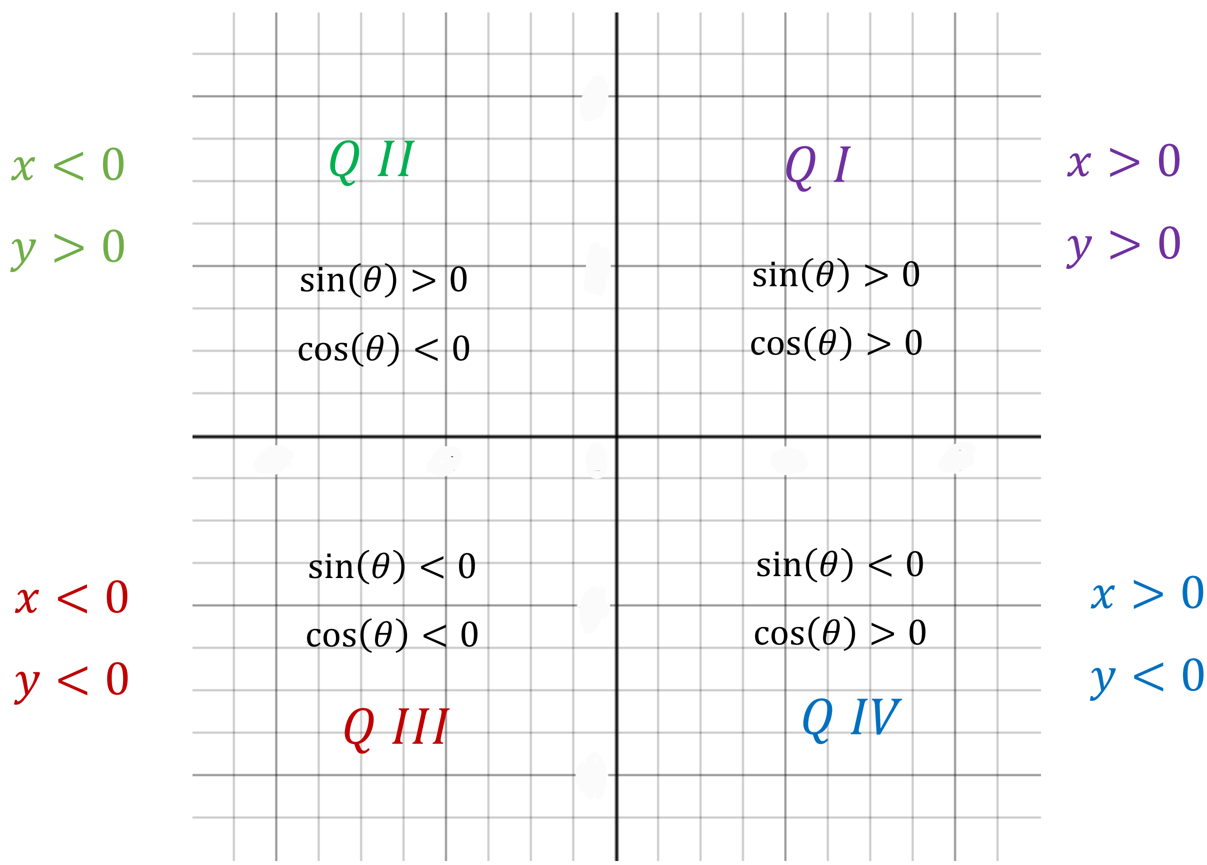
We did this problem using the original method of looking at the triangle inside the coordinate axis.

To use reference angles to do this problem, we just need to remember that there is an **extra step**:

Once you calculate the (positive) value of the angle from the memorized tables of trig values . . .

. . . you need to decide the **sign**.

You can memorize this part too if you like:

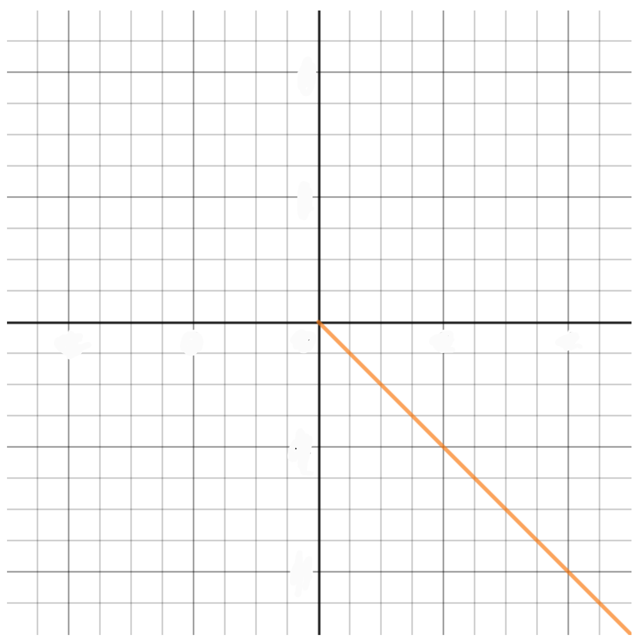


Let's do another example using both methods:

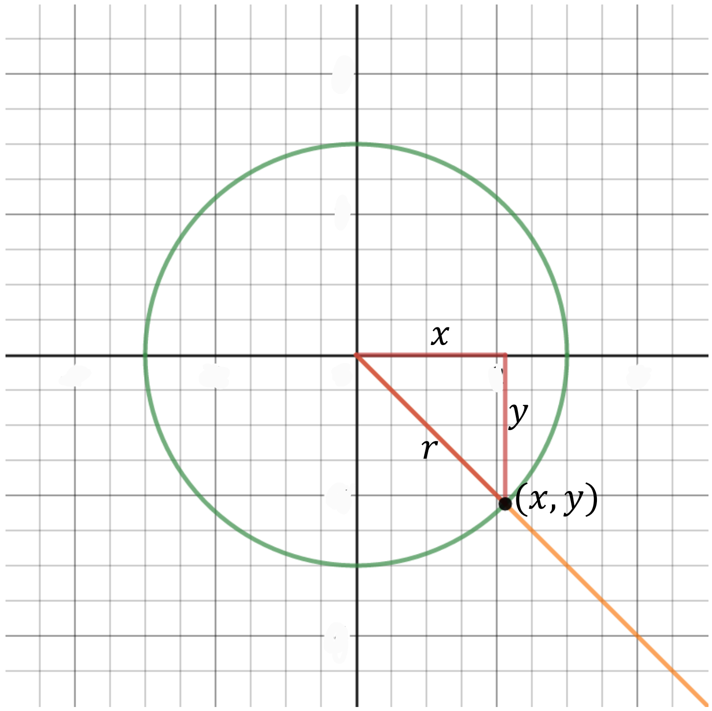
Find

$$\tan(315°)$$

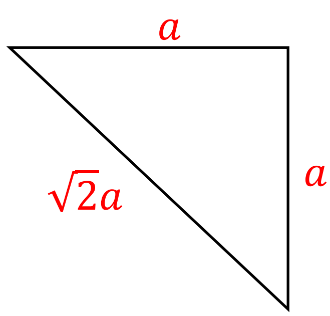
If we want to figure this out using the "understanding" method, we draw a diagram showing an angle at 315°:



Next we put a point somewhere on this line, and then draw a triangle showing the x, y, and r values as sides of that triangle:



Here we see that we are looking at a 45-45-90 right triangle, for which we have memorized the ratio of the sides:

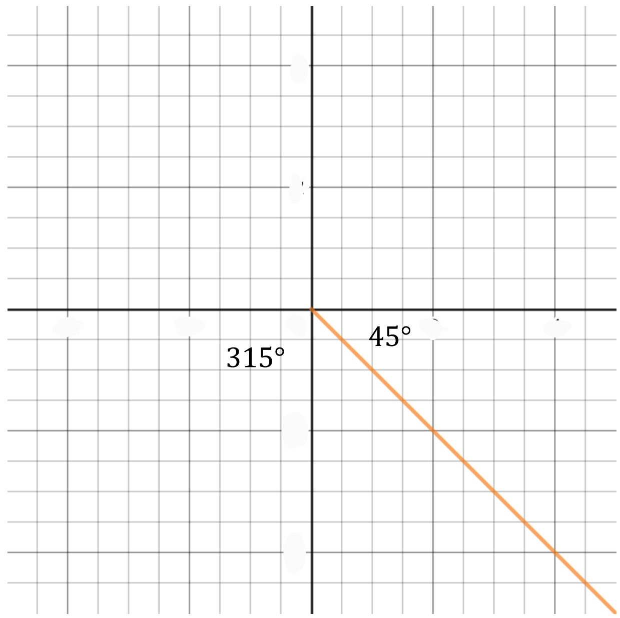


We also note that the y-coordinate of this point (but not the x) is negative!

So we get

$$\tan(315°) = \frac{y}{x} = \frac{-a}{a} = -1$$

If we want to use the reference angle method, we just find the reference angle for 315°:



Which we see is 45°, because that's the angle formed between the terminal side of 315° and the x-axis.

Checking our table of values . . .

θ	$0°$	$30°$	$45°$	$60°$	$90°$	$180°$	$270°$
$\sin(\theta)$	0	$\frac{1}{2}$	$\frac{1}{\sqrt{2}}$	$\frac{\sqrt{3}}{2}$	1	0	-1
$\cos(\theta)$	1	$\frac{\sqrt{3}}{2}$	$\frac{1}{\sqrt{2}}$	$\frac{1}{2}$	0	-1	0
$\tan(\theta)$	0	$\frac{\sqrt{3}}{3}$	$\mathbf{1}$	$\sqrt{3}$	$undef.$	0	$undef.$

We remember that

$$\tan(45°) = 1$$

But we must also ***decide the sign!***

The angle 315° is in Quadrant IV. In this quadrant, we have that

$$x > 0 \; and \; y < 0$$

Since the function of $\tan(\theta) = \frac{y}{x}$, and these are of opposite signs, we conclude that the value formed by this fraction is negative.

So

$$\tan(315°) = -1$$