Logarithms and logarithmic functions

Let's return to our problem involving car depreciation. We had that

$$V(t) = the\ value\ of\ a\ car\ depreciating\ at\ 20\%\ per\ year$$
$$= 25{,}000 * 0.8^t$$

What if we wanted to know how long before it's only worth $2000?

We could easily set up that as an equation:

$$V(t) = 25{,}000 * 0.8^t = 2000$$

Solving, we would first divide by 25,000:

$$25{,}000 * 0.8^t = 2000$$

$$\frac{25{,}000 * 0.8^t}{25000} = \frac{2000}{25000}$$

$$0.8^t = \frac{2}{25}$$

At this point, we might be unsure about what to do next. We haven't yet solved any equations in this class such as this, where the variable is an exponent!

That said, we may remember the operation called a **logarithm** that addresses a very similar situation!

One way to think of logarithms is as the **inverse** to exponentiation.

For example, we know that

$$3 + 7 = 10$$

can be rewritten as

$$7 = 10 - 3$$

because addition and subtraction are inverse operations.

And we know that

$$4 * 9 = 36$$

can be rewritten as

$$4 = 36 \div 9$$

because multiplication and division are inverse operations.

The same can be said for an exponential statement such as

$$10^4 = 10{,}000$$

Which can be rewritten as

$$\log_{10} 10{,}000 = 4$$

In general, logarithms are defined as follows:

If $b^x = a$ then $\log_b a = x$

And we can sometimes figure out a logarithm by reverting to its exponential form:

Ex: find $\log_9 27$

One way to figure this out is to rewrite it in exponential form:

$$\log_9 27 = x$$

this little number is the base

becomes

$$9^x = 27$$

Noting that both sides of the equation can be written in terms of base 3:

$$9^x = 27$$

$$(3^2)^x = 3^3$$

$$3^{2x} = 3^3$$

Which means that

$$2x = 3$$

$$x = \frac{3}{2}$$

So

$$\log_9 27 = \frac{3}{2}$$

Going back to our original problem, we had the equation

$$0.8^t = \frac{2}{25}$$

which could be rewritten using logarithms as

$$\log_{0.8} \frac{2}{25} = t$$

The only problem here is . . .

. . . our calculator does not have $\log_{0.8} x$

The standard scientific calculator only has two logarithmic functions:

$$\log_{10} x = \mathbf{log}\, x$$

which is to say “log base 10”, written simply as $\log x$

and

$$\log_e x = \mathbf{ln}\, x$$

which is to say “log base e”, pronounced simply as “$\ln x$”

The calculator will not calculate

$$\log_{0.8} \frac{2}{25}$$

We need another method!

We can find it in the properties of logarithms:

The Properties of Logarithms

I. $\log_b x * y = \log_b x + \log_b y$

II. $\log_b \frac{x}{y} = \log_b x - \log_b y$

III. $\log_b x^n = n \log_b x$

The third law of logarithms is especially useful for solving exponential equations. Let’s look in more detail how it works:

We have that, for example,

$$\log_2 16^3 = 3\log_2 16$$

and

$$\log 8^2 = 2\log 8$$

and

$$\ln x^5 = 5\ln x$$

All of these examples use the third law of logarithms to move an exponent out in front of the logarithm.

Now let's go back to our original equation that we were trying to solve:

$$0.8^t = \frac{2}{25}$$

Our problem is that we are trying to get the variable t by itself! We can't do this without logarithms!

So we can take the logarithm of both sides:

$$0.8^t = \frac{2}{25}$$

$$\log(0.8^t) = \log\left(\frac{2}{25}\right)$$

and using the third law of logarithms:

$$\log(0.8^t) = \log\left(\frac{2}{25}\right)$$

Becomes

$$t * \log(0.8) = \log\left(\frac{2}{25}\right)$$

Which allows us to solve for t:

$$t = \frac{\log \frac{2}{25}}{\log 0.8}$$

$$t = 11.3$$

It will take 11.3 years until the car's value depreciates to be \$5000.

Note that the reason we used the log function was that our scientific calculator allows us to compute it. We could have also used ln:

$$\frac{\ln \frac{2}{25}}{\ln 0.8} = 11.3$$

This process can be summarized in a formula (for those who prefer formulas) . . . called the “change of base formula”:

Change of Base Formula

$$\log_b(a) = \frac{\log b}{\log a}$$

This formula can be used to solve exponential equations; however, I prefer to use the third law of logarithms to solve these problems.

Let’s do another one:

The population of the U.S. is expected to grow by 1% per year for the near future. If the population in 2015 was approximately 321 million, how many years until the U.S. contains half a billion people?

First we need to come up with a function representing the population of the U.S.:

$$P(t) = population\ of\ the\ U.S.(in\ millions)\ t\ years\ after\ 2015$$

Again, note that we have to set up our function so that the variable t represents years **since** a given starting point.

Applying the general formula for percent change, we get

$$P(t) = 321 * (1.01)^t$$

and to find when the population reaches a half-billion people, we solve

$$P(t) = 500$$

resulting in the equation

$$321 * (1.01)^t = 500$$

We will solve it by first dividing away the constant out front:

$$\frac{321 * (1.01)^t}{321} = \frac{500}{321}$$

which gives us

$$1.01^t = \frac{500}{321}$$

Here, you could use the change of base formula. But I will use the process of taking the logarithm of both sides:

$$\ln(1.01^t) = \ln\left(\frac{500}{321}\right)$$

Using the second law of logarithms

$$t * \ln(1.01) = \ln(500) - \ln(321)$$

$$t = \frac{\ln(500) - \ln(321)}{\ln(1.01)}$$

$$t = 44.5$$

We would predict a half-billion people in the U.S. 44.5 years from 2015, or in the year 2060.

Let's do one final example:

$20,000 is invested at 2.7% apr compounded continuously. How long until the investment doubles in value?

This is an easy one. First we set up a function representing the amount of our investment:

$$A(t) = amount\ in\ account\ after\ t\ years$$

And then apply the formula for continuously compounded interest:

$$A(t) = 20000e^{0.027t}$$

Then set this function equal to twice its starting amount:

$$A(t) = 40000$$

which results in the equation

$$20000e^{0.027t} = 40000$$

Dividing both sides by 20000, we get

$$\frac{20000e^{0.027t}}{20000} = \frac{40000}{20000}$$

$$e^{0.027t} = 2$$

This equation can be solved directly with logarithms, because our calculator **has** logarithim base e!

$$e^{0.027t} = 2$$

becomes

$$\log_e 2 = 0.027t$$

or

$$\ln 2 = 0.027t$$

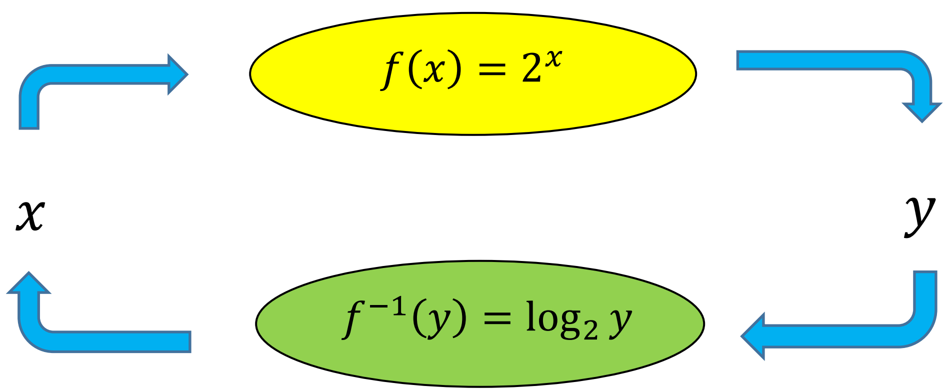
$$\frac{\ln(2)}{0.027} = \frac{0.027t}{0.027}$$

$$t = \frac{\ln(2)}{0.027} = 25.67\ years = 25\ years, 8\ months$$

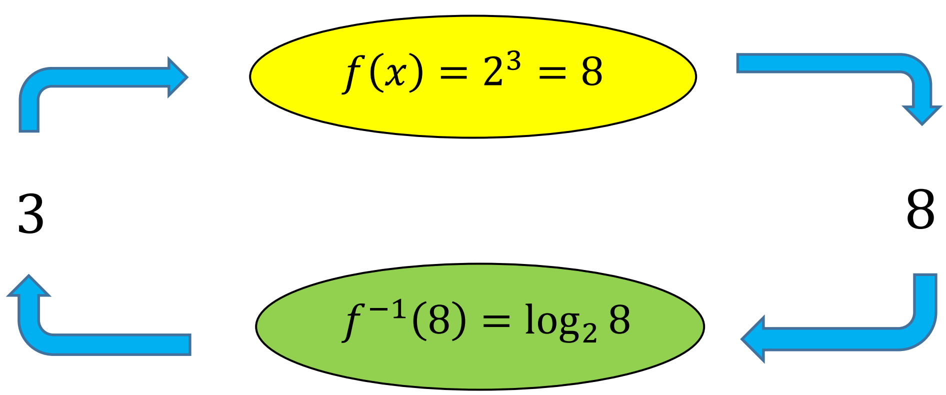
Logarithmic Functions

In the previous section we found that exponential functions have inverse functions associated with the same base.

Let’s see this with a diagram:



Where if we were to plug in the value of $x = 3$, we would get:

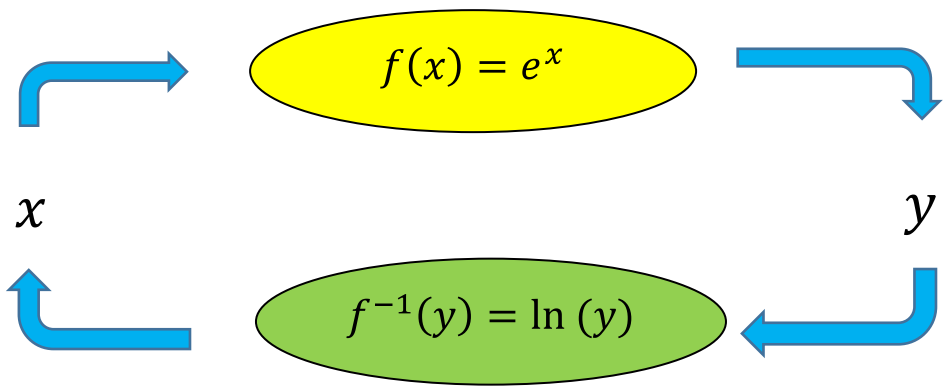


Similarly, we have that for

$$g(x) = e^x$$

we have that

$$g^{-1}(x) = \ln(x)$$



We might ask, what does the graph of

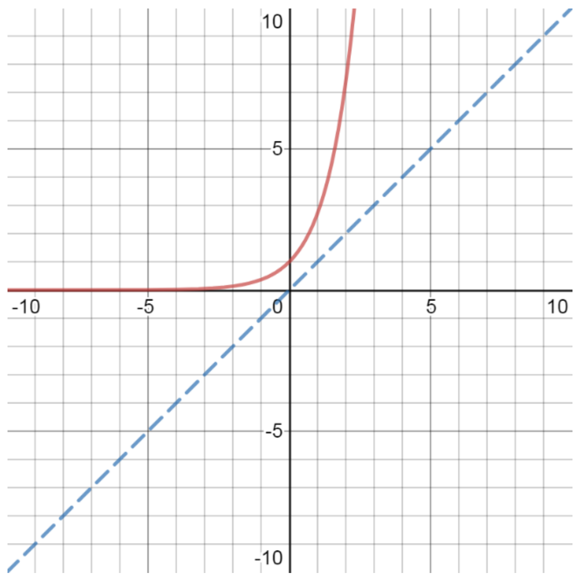
$$h(x) = \ln(x)$$

look like?

One way to speculate is to remember that the graphs of inverse functions are ***reflections*** of each other **across the line $y = x$**

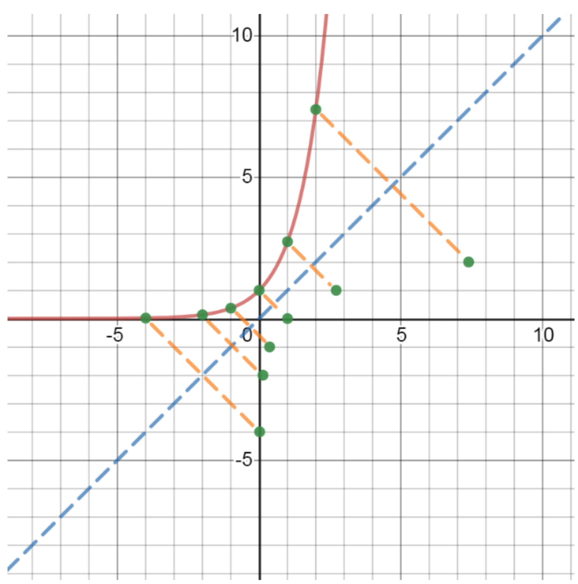
So to see the graph of $g(x) = \ln(x)$, let's try to get the reflection across $y = x$ of our original graph $h(x) = e^x$:

$h(x) = e^x$



We might try to plot a few individual points and find their reflections:

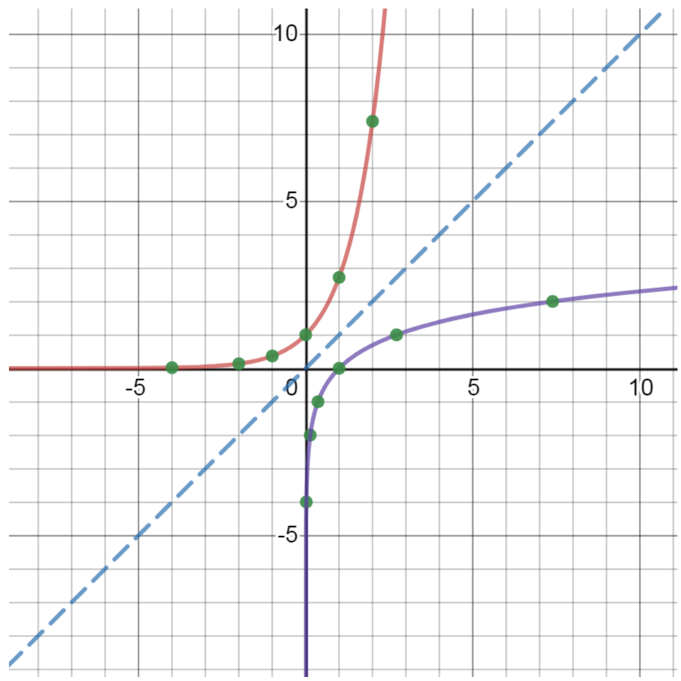
$h(x) = e^x$



Which gives us

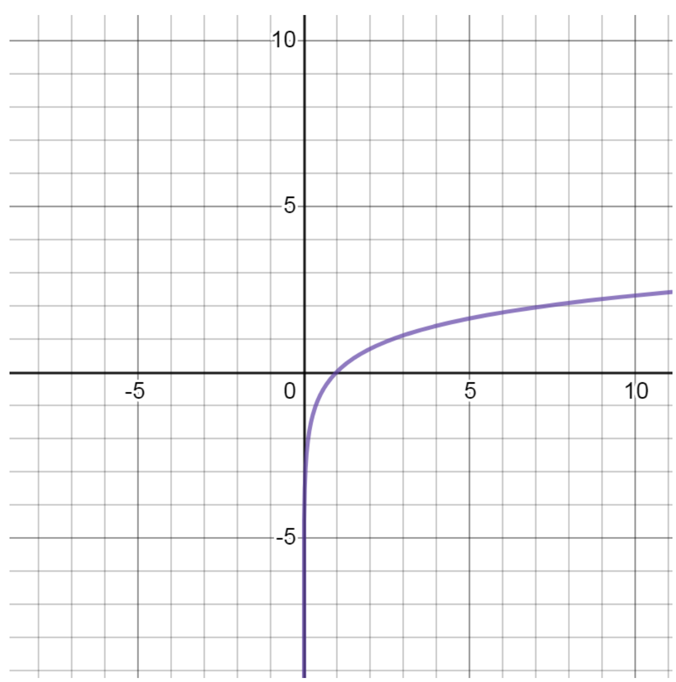
$h(x) = e^x$

$g(x) = \ln(x)$

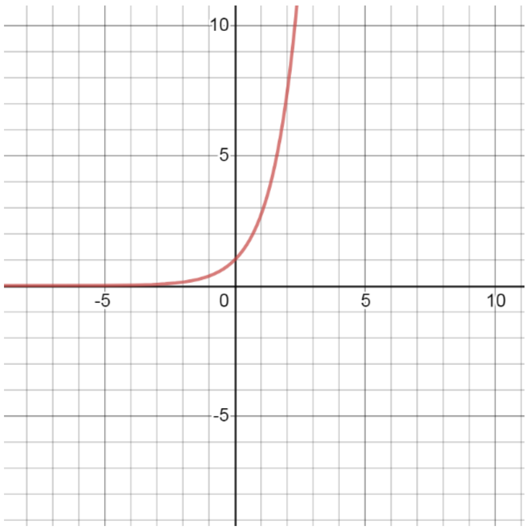


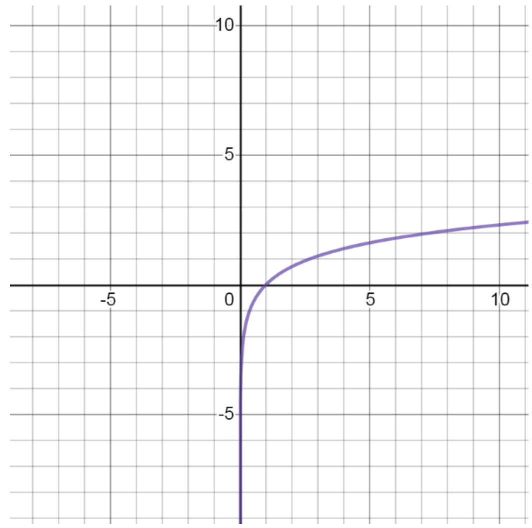
Or, seen by itself:

$g(x) = \ln(x)$



We notice that the domain and range of this function are the opposite of what they are for their inverse, $h(x) = e^x$:





$h(x) = e^x$

domain:

$(-\infty,\ \infty)$

range:

$(0, \infty)$

$g(x) = \ln(x)$

domain:

$(-\infty,\ \infty)$

range:

$(-\infty, \infty)$

We might ask ourselves . . . what is the “purpose” of a logarithmic function, outside of the examples we have seen in which they can help to solve exponential equations?

Is there any real-life situation that is “modeled” with logarithmic functions?

The answer is yes: **logarithmic scales**

Sometimes, the raw numbers that measure things are so different in magnitude . . .

. . . meaning some of the numbers are very small . . .

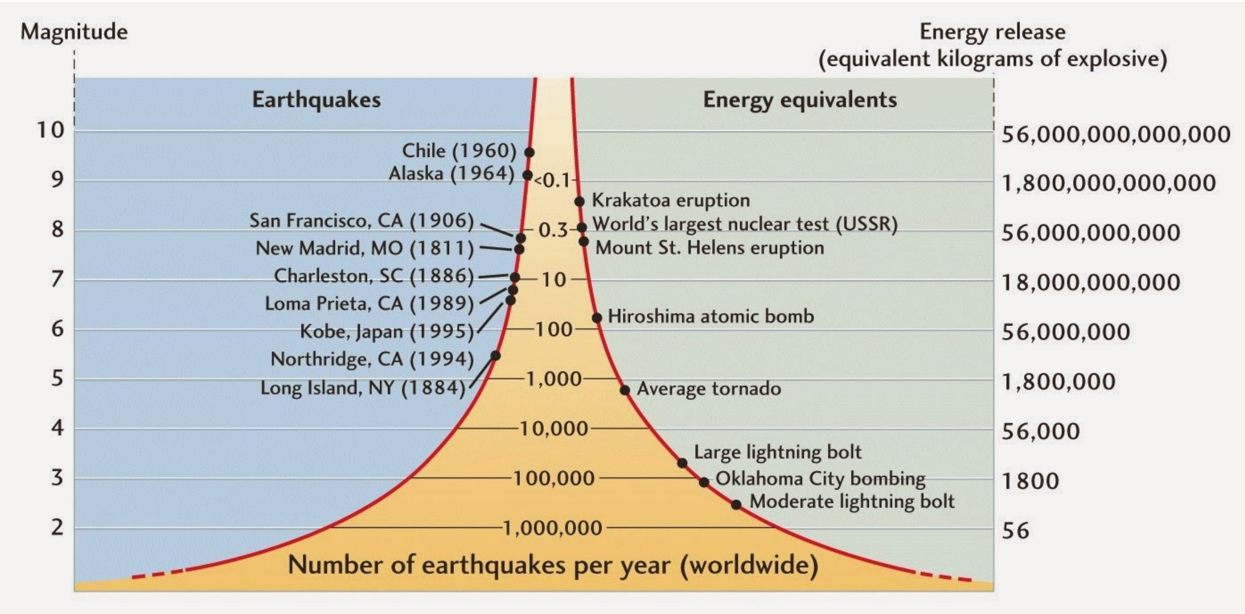
. . . and other of the numbers are very big . . .

. . . that you need to compress the numbers to be more alike.

One of the most commonly seen logarithmic scales is the **Richter Scale**.

The Richter Scale measures the strength of an earthquake.

The raw numbers from an earthquake are measures of energy, which vary very widely from one earthquake to the next. The diagram below illustrates this:



See how different the numbers on the right are?

They wouldn't help people understand how to measure the power of an earthquake because some of them are **too big** (compared to the others)

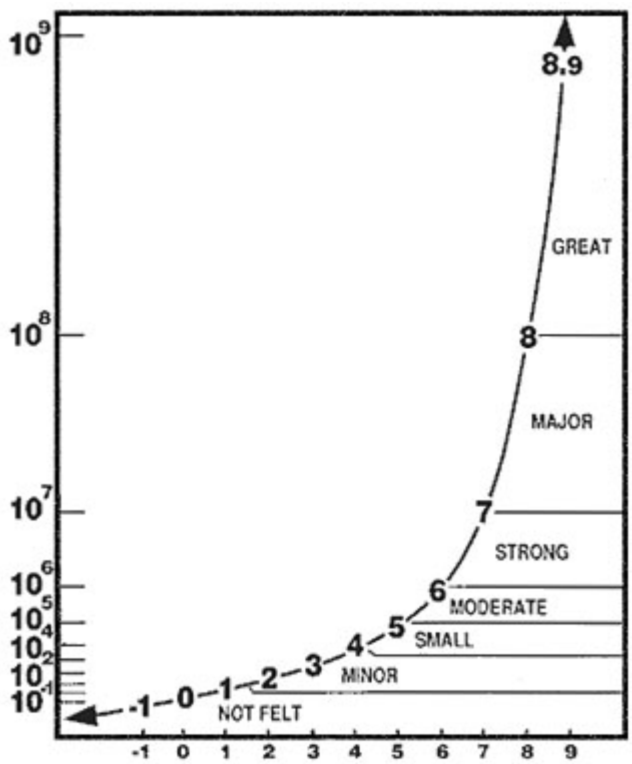
So the raw numbers for energy release are plugged into a logarithmic function (simplified a bit):

$$\text{Richter Scale Magnitude} = \log(Energy\ Release)$$

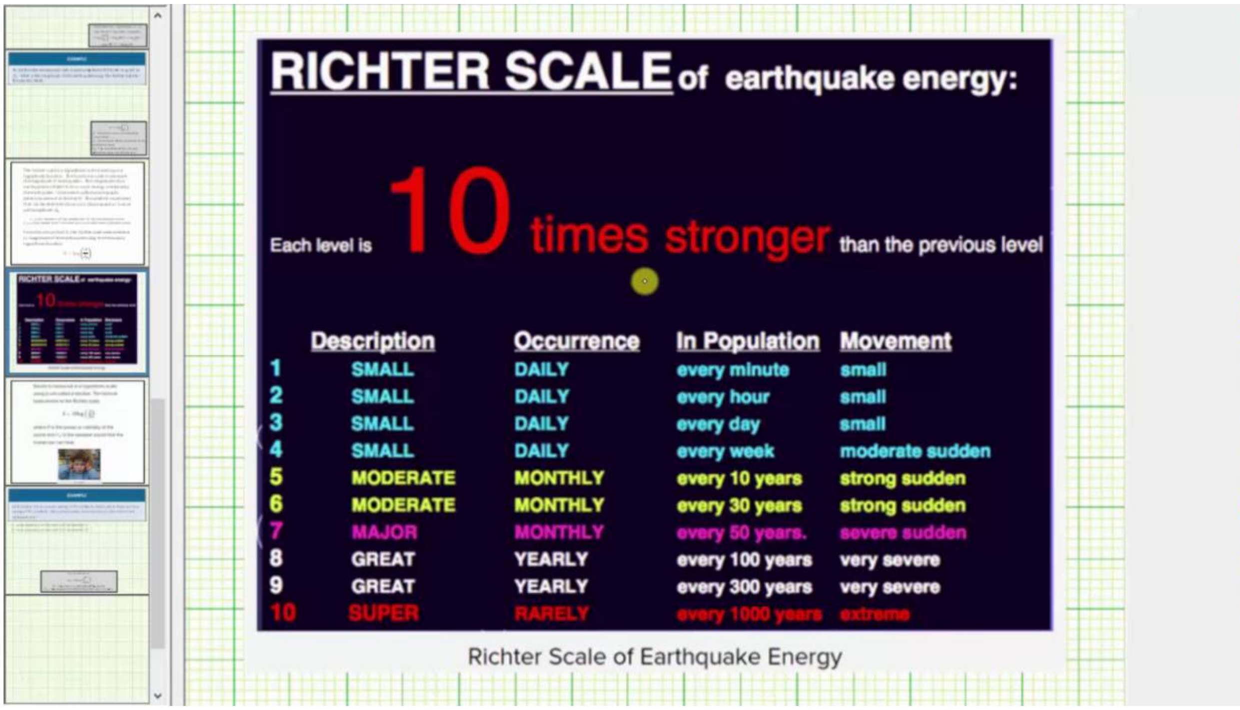
Now remember, logarithms refer to the exponents:

$$\log_{10} 1000 = 3 \text{ because } 10^3 = 1000$$

So the Richter Scale measurements go up by 1 according to an exponent going up by one:



Which can also be seen in this picture:



Put even more simply:

A 6.2 earthquake happens to a given population every 30 years or so . . .

A 7.2 earthquake is **10 times stronger**!!!! And happens every 50 years to the same group of people . . .

An 8.2 earthquake is **10 times stronger than the 7.2**!! And 100 times stronger than the 6.2!! You might experience one in a lifetime at most.