Graphing Rational Functions

Let's put all these ideas together, and do a few problems from start to finish:

Find all asymptotes and graph the function:

$$h(x) = \frac{2x - 1}{x + 2}$$

First, find any vertical asymptotes:

The vertical asymptotes happen where the function is undefined . . .

. . . which happens with the denominator is zero.

$$x + 2 = 0$$
$$x = -2$$

Next, find any horizontal asymptotes:

The horizontal asymptotes happen at

$$y = \lim_{x\to\infty} \frac{2x - 1}{x + 2} = \lim_{x\to\infty} \frac{2x + 0}{x + 0} = \lim_{x\to\infty} \frac{2x}{x} = \lim_{x\to\infty} 2 = 2$$

So we have a horizontal asymptote at $y = 2$.

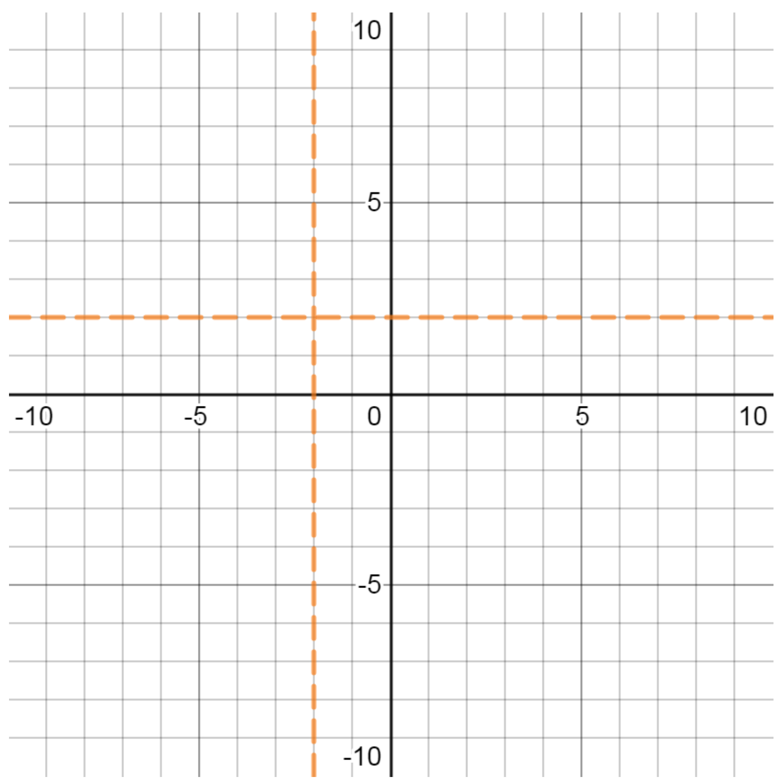
Let’s graph these asymptotes (while choosing an appropriate scale for our graph). Choosing an appropriate scale means we should think about how many points we need to plot.

To graph a rational function you must plot at least two points on either side of all vertical asymptotes.

And we should ***leave some more room*** on either side . . .

. . . to draw our graph!

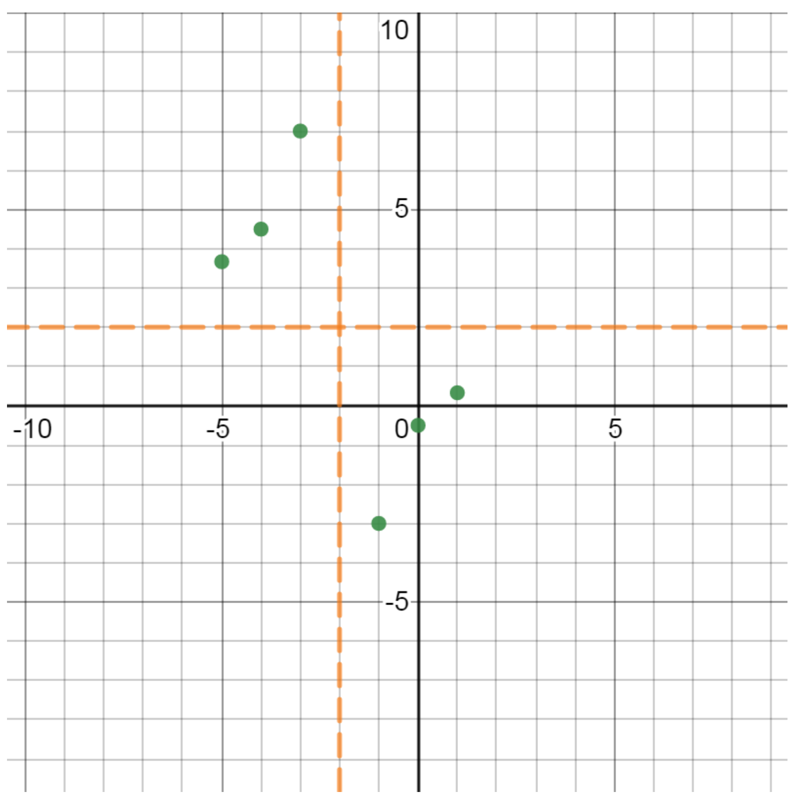
So let’s make sure that our graph goes all the way from $x = -10$ to $x = 10$:



Now we need to plot some points. I will do three x-values on either side of the vertical asymptote:

$\boldsymbol{x}$	$\boldsymbol{h(x)}$
$\mathbf{-3}$	7
$\mathbf{-4}$	4.5
$\mathbf{-5}$	2.67
$\mathbf{-1}$	-3
$\mathbf{0}$	-0.5
$\mathbf{1}$	-0.33

And then plot these points:

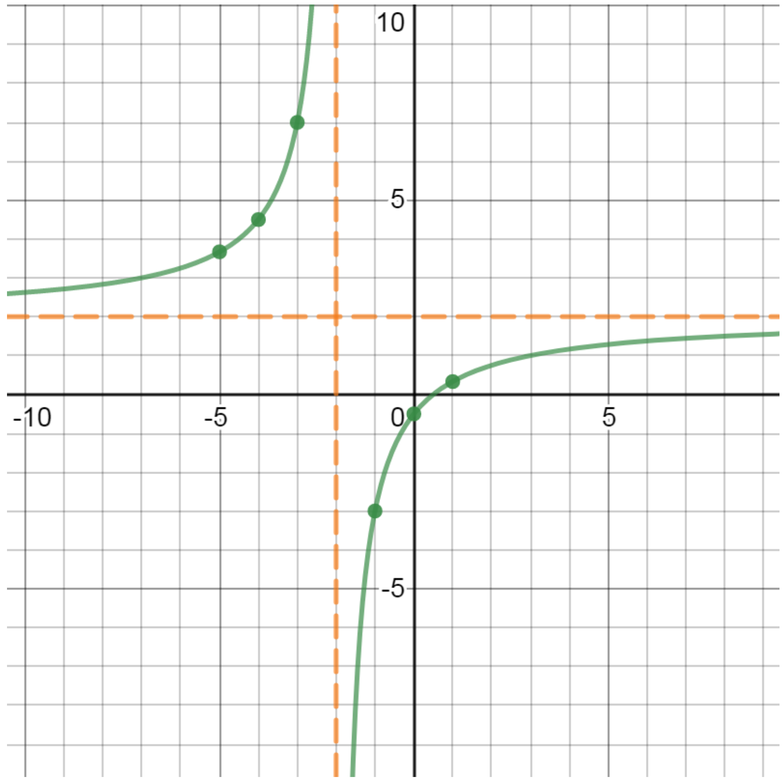


Now, with the help of the asymptotes, we can draw the graph:

$$h(x) = \frac{2x-1}{x+2}$$

$$x = -2$$

$$y = 2$$



Let's do another problem:

Find all asymptotes and graph:

$$q(x) = \frac{x}{x^2 - 9}$$

<u>Vertical asymptotes:</u>

The vertical asymptotes to $q(x)$ occur where $q(x)$ is undefined:

$$x^2 - 9 = 0$$
$$x^2 = 9$$
$$x = \pm\sqrt{9}$$
$$x = \pm 3$$

<u>Horizontal asymptote:</u>

The horizontal asymptote can be obtained by finding

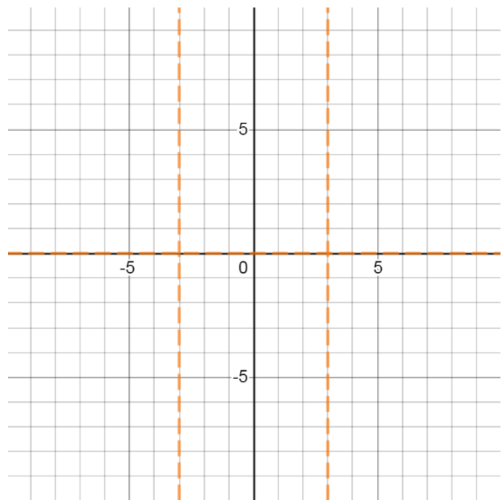
$$y = \lim_{x\to\infty} q(x)$$

$$y = \lim_{x\to\infty} \frac{x}{x^2 - 9} = \lim_{x\to\infty} \frac{x}{x^2 - 0} = \lim_{x\to\infty} \frac{x}{x^2} = \lim_{x\to\infty} \frac{1}{x} = 0$$

There is a horizontal asymptote at $y = 0$

We need to plot at least two points to either side of each vertical asymptote.

Adding some more room, let's graph $q(x)$ between $x = -10$ and $x = 10$:

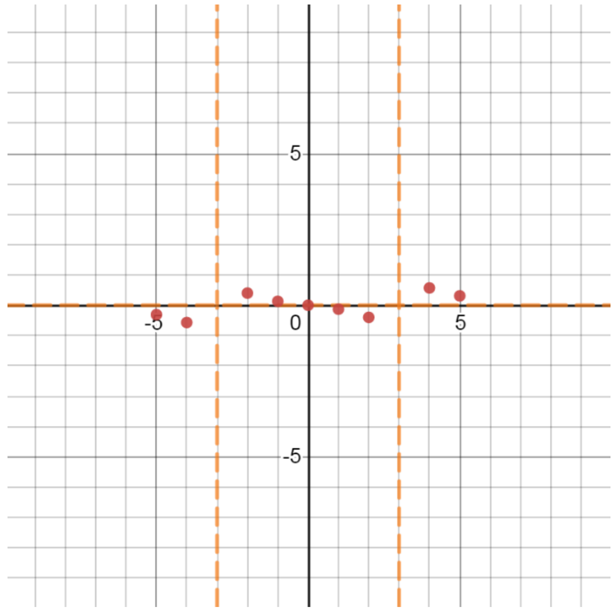


Now we will generate a table of values:

$\boldsymbol{x}$	$\boldsymbol{q(x)}$
$\mathbf{-5}$	-0.31
$\mathbf{-4}$	-0.57
$\mathbf{-2}$	0.4
$\mathbf{-1}$	0.13
$\mathbf{0}$	0
$\mathbf{1}$	-0.13
$\mathbf{2}$	-0.4
$\mathbf{4}$	0.57
$\mathbf{5}$	0.31

And plot them:

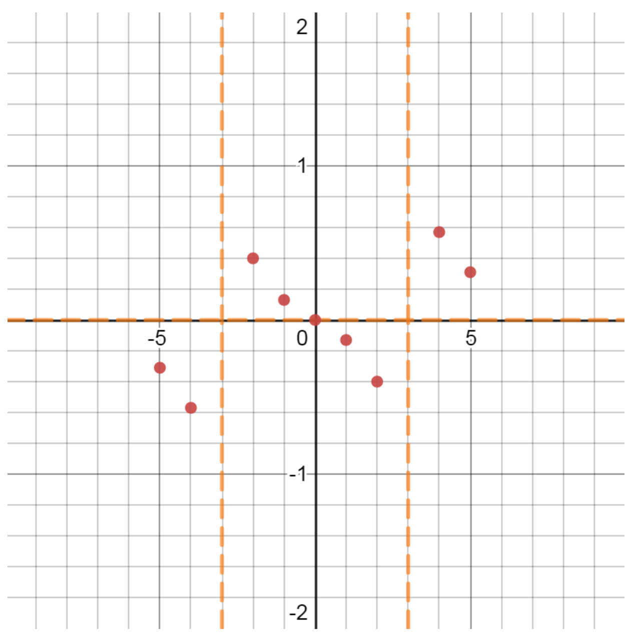
$$q(x) = \frac{x}{x^2 - 9}$$



This graph is not well-scaled! All of the points are very close to the x-axis.

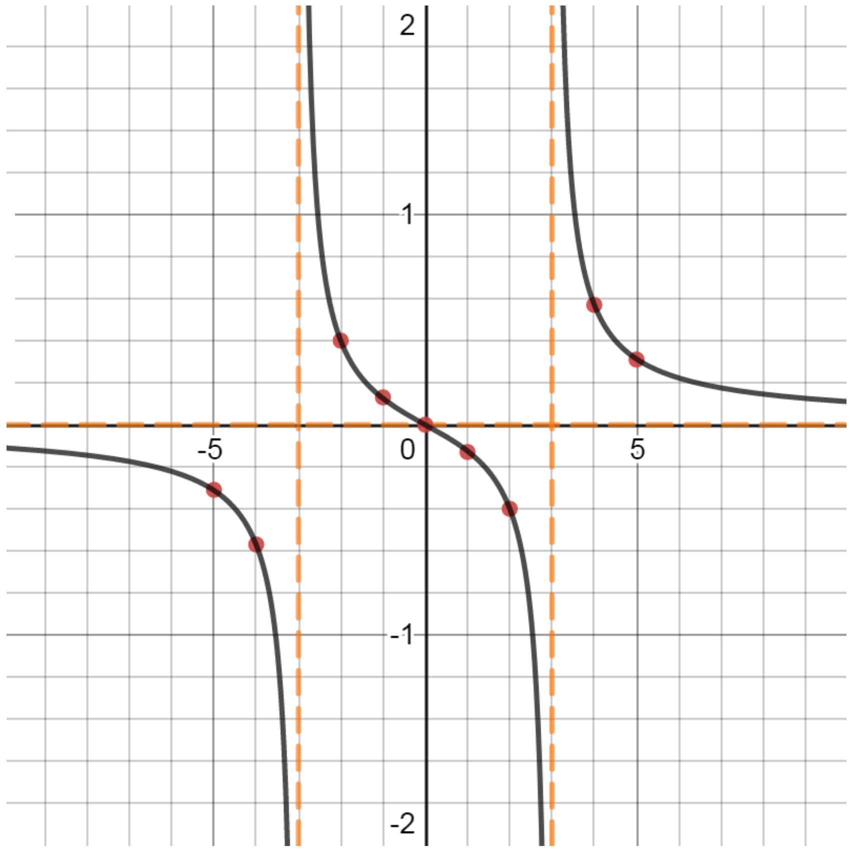
We need to shrink the scale for the y-values. After all, all of those y-values that we found are between $y = -1$ and $y = 1$. So that we have some room to graph to the asymptotes, let's choose a scale between $y = -2$ and $y = 2$

$$q(x) = \frac{x}{x^2 - 9}$$



Now we can connect the points with a smooth curve:

$$q(x) = \frac{x}{x^2 - 9}$$



Now that's a good-looking graph!