Radian Angle Measure

You might never have wondered why we use the numbers we use for angles.

For example, why is a right angle 90°, rather than, say, 100°?

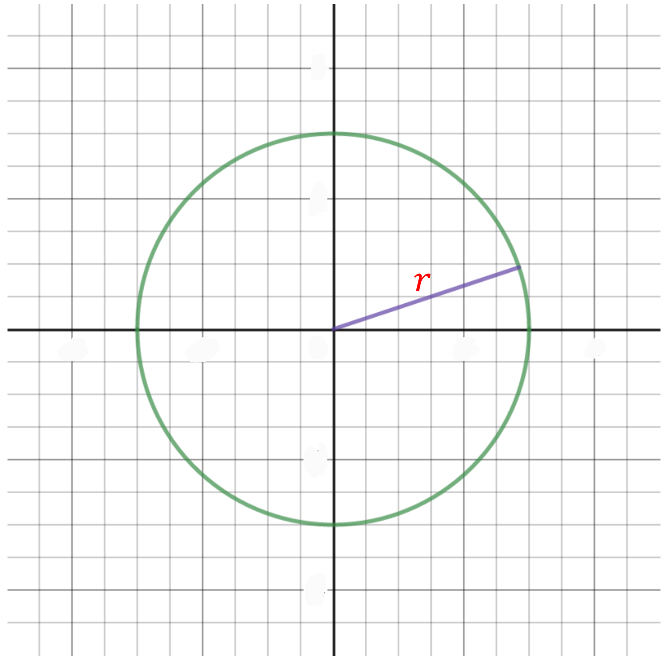
The answer to this question is that the numbers we use for degree measures of angles are ultimately **arbitrary**.

That means there is no particular reason for them, or no "good" mathematical reason.

In advanced math, they use a different kind of angle measurement . . . one that really does have a good reason for it.

Here's how it goes:

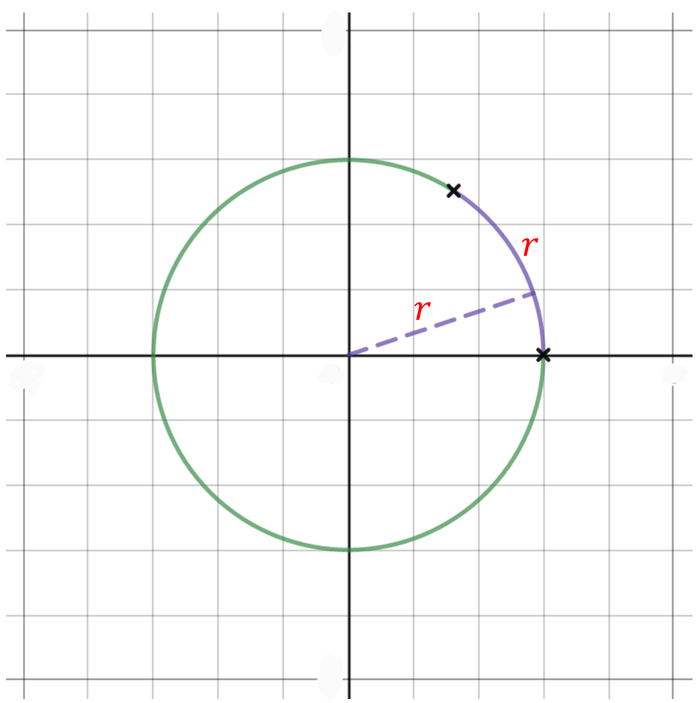
Take a circle and measure the radius of the circle:



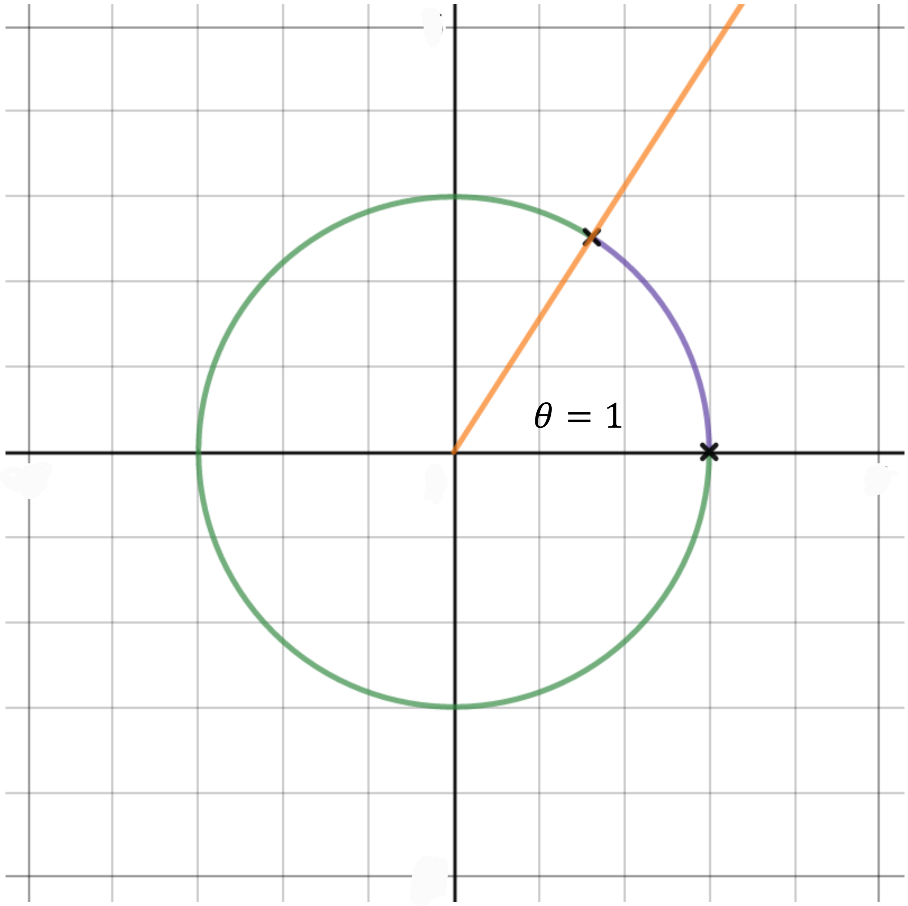
Now, take that *length of radius* and make it so that it can bend . . .

. . . then pick it up and move it so that it lies on the edge of the circle . . .

. . . starting at where angles usually start:



The angle formed by this arc is 1 radian!



That means that the radian angle measure is directly related to the distance around the circle, measured in terms of the radius!

So how many radius's does it take to *get all the way around* (360°)?

The distance around a circle is the circumference.

There's a very important formula involving the circumference of a circle:

$$\frac{c}{d} = \pi$$

where d is the diameter of the circle!

In fact this is the very definition of the number “pi”, π.

We know that

$$\pi \cong 3.1416$$

But π is an irrational number, so we can never fully write it out.

We will just use the symbol π for it in this class!

Now, as I was saying . . . we have that

$$\frac{c}{d} = \pi$$

and since the diameter is twice the radius,

$$\frac{c}{2r} = \pi$$

or

$$c = 2\pi r$$

That means that the distance around a circle is 2π times the radius.

So the angle in radians that takes us all the way around a circle is

$$\theta = 2\pi$$

This gives us a way to “translate” back and forth between degrees and radians:

$$360° \leftrightarrow 2\pi$$

Which combined with the fact that

$$0° \leftrightarrow 0$$

Gives us the following proportional formula for converting between degrees and radians:

$$\frac{degrees}{360} = \frac{radians}{2\pi}$$

Which you could simplify if you want to be

$$\frac{degrees}{180} = \frac{radians}{\pi}$$

For me, this is the easiest conversion method to remember!

If I wanted to convert, let's say, the radian angle $\frac{5\pi}{3}$ into degrees I would just plug it into this proportion and solve for x:

$$\frac{x}{180} = \frac{\frac{5\pi}{3}}{\pi}$$

Cross-multiplying, we get

$$\pi x = \left(\frac{5\pi}{3}\right) * 180$$

$$\pi x = \left(\frac{5\pi}{3}\right) * 180$$
$$\pi x = 5\pi(60)$$
$$x = 300$$

It turns out that $\frac{5\pi}{3}$ radians is equivalent to 300°.

There's a second method, which involves memorizing a formula:

If we were to take this basic proportion from before:

$$\frac{degrees}{180} = \frac{radians}{\pi}$$

And solve for degrees in terms of radians, we would get:

$$(180)\frac{degrees}{180} = (180)\frac{radians}{\pi}$$

$$degrees = \left(\frac{180}{\pi}\right) * (radians)$$

In this case, if we wanted to convert $\frac{7\pi}{6}$ radians to degrees we would just plug it into the formula:

$$degrees = \left(\frac{180}{\pi}\right) * \left(\frac{7\pi}{6}\right)$$

$$= \frac{180}{\pi} * \frac{7\pi}{6}$$

$$= 30 * 7$$

$$= 210$$

So we have that the degree measure equivalent of $\frac{7\pi}{6}$ radians is 210°.

Finally, there is an intuitive way to figure this out, but it involves remembering some basic equivalencies between radian and degrees:

We know that

$$360° \leftrightarrow 2\pi$$

so

$$180° \leftrightarrow \pi$$

which means that

$$90° \leftrightarrow \frac{\pi}{2}$$

and

$$45° \leftrightarrow \frac{\pi}{4}$$

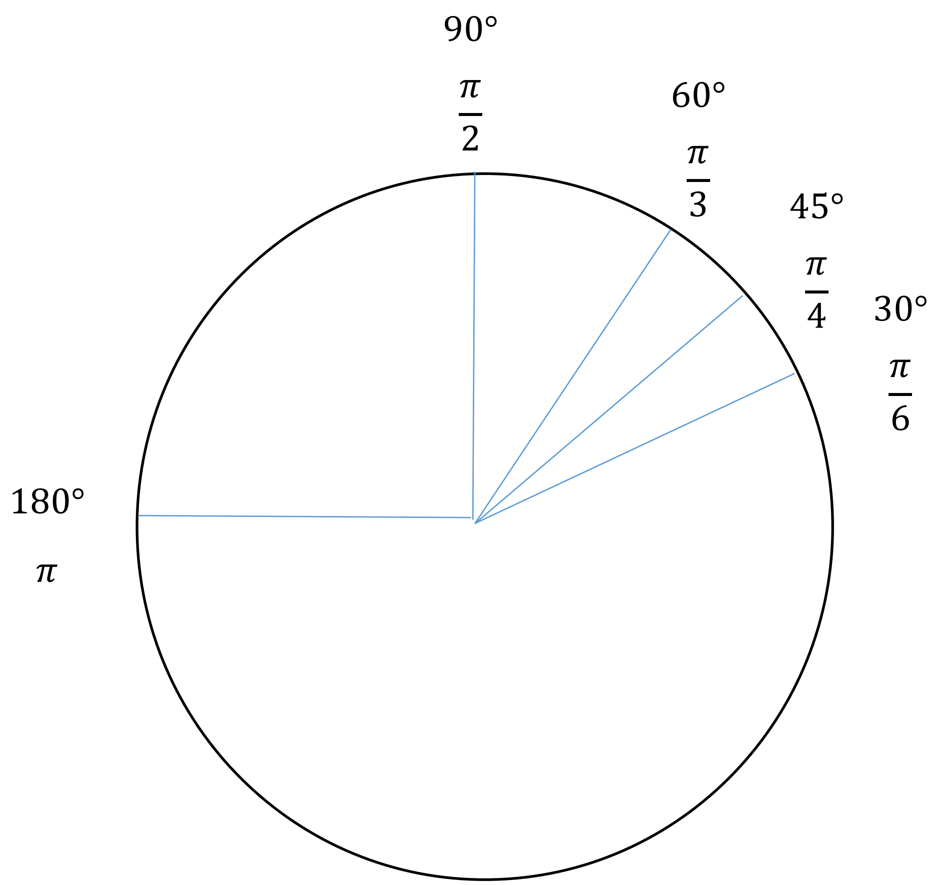
and also, dividing 90° by 3:

$$30° \leftrightarrow \frac{\pi}{6}$$

which if doubled becomes

$$60° \leftrightarrow \frac{\pi}{3}$$

If you do a lot of problems with radians, you quickly remember these relationships, which are sometimes expressed in a diagram:



By knowing these basic relationships, you can figure out all the common conversions intuitively (by simple common sense):

$$\frac{5\pi}{3} = 5\left(\frac{\pi}{3}\right) = 5 * 60° = 300°$$

$$\frac{7\pi}{6} = 7\left(\frac{\pi}{6}\right) = 7 * 30° = 210°$$

$$\frac{3\pi}{4} = 3\left(\frac{\pi}{4}\right) = 3 * 45° = 135°$$

$$\frac{7\pi}{2} = 7\left(\frac{\pi}{2}\right) = 7 * 90° = 630°$$

Now we can do some problems involving trig calculations with radian measure angles!

1. Calculate

$$\cos\left(\frac{2\pi}{3}\right)$$

First, let's convert this radian measure angle to degrees so that we can use our previous methods:

$$\frac{x}{180} = \frac{\frac{2\pi}{3}}{\pi}$$

Cross-multiplying, we have:

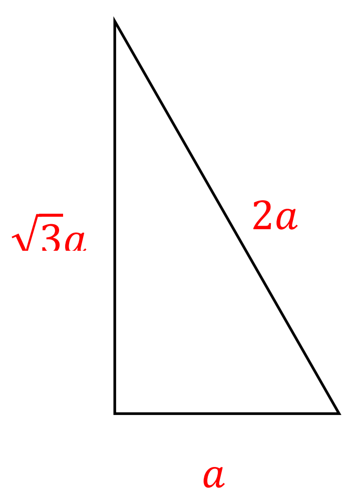
$$x\pi = 180 * \left(\frac{2\pi}{3}\right)$$

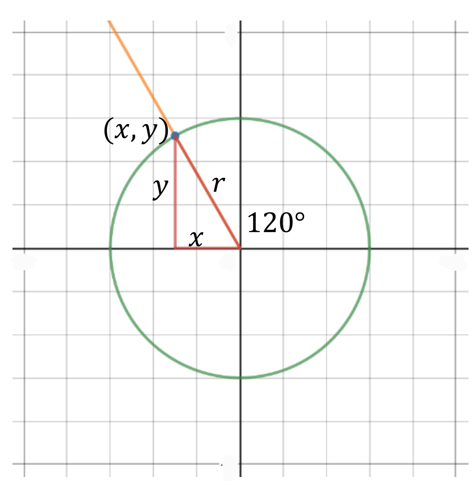
$$x = 60 * 2$$

$$x = 120$$

So we are working with a 120° angle!

Now, to get the cosine of that angle, we might use the diagram method:





So we get that

$$\cos(120°) = \frac{x}{r} = \frac{-a}{2a} = -\frac{1}{2}$$

2. Calculate:

$$\csc\left(-\frac{9\pi}{4}\right)$$

First we convert to degrees. Here I will use the formula:

$$degrees = \left(\frac{180}{\pi}\right) * \left(-\frac{9\pi}{4}\right)$$

$$= -45 * 9$$

$$= -405°$$

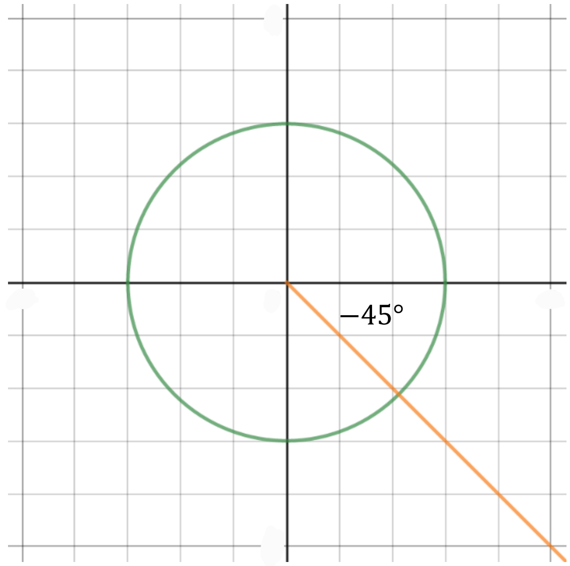
Now since this degree measure is more than 360°, it has completed one full revolution. Its trigonometric function values will all be the same as

$$-405° + 360° = -45°$$

(note that if the angle was +405° we would subtract 360° to get a value θ such that $|\theta| < 360°$)

Let's figure out $csc\ (-45°)$ using the reference angle method.

First, let's find the reference angle:



The reference angle, remember, is the angle formed between the terminal side of the angle and the x-axis. In this case, that angle is

$$45°$$

We check the table (in our memory):

θ	$0°$	$30°$	$45°$	$60°$	$90°$	$180°$	$270°$
$\sin(\theta)$	0	$\frac{1}{2}$	$\mathbf{\frac{1}{\sqrt{2}}}$	$\frac{\sqrt{3}}{2}$	1	0	-1
$\cos(\theta)$	1	$\frac{\sqrt{3}}{2}$	$\frac{1}{\sqrt{2}}$	$\frac{1}{2}$	0	-1	0
$\tan(\theta)$	0	$\frac{\sqrt{3}}{3}$	1	$\sqrt{3}$	$undef.$	0	$undef.$

Note that we are interested in $\sin(45°)$ because $\csc(\theta) = \frac{1}{\sin(\theta)}$.

Also note that because we are in Q IV, where $y < 0$, we get that

$$\sin(-45°) = -\frac{1}{\sqrt{2}}$$

so

$$\csc\left(-\frac{9\pi}{4}\right)$$

$$= \csc(-405°)$$

$$= \csc(-45°)$$

$$= \frac{1}{\sin(-45°)}$$

$$= \frac{1}{-\sin(45°)}$$

$$= \frac{1}{-\frac{1}{\sqrt{2}}}$$

$$= -\sqrt{2}$$

3. Calculate

$$\tan\left(\frac{11\pi}{2}\right)$$

First we convert to degrees . . .
. . . this time I will use the “intuitive method”:

$$\frac{11\pi}{2} = 11\left(\frac{\pi}{2}\right) = 11 * 90° = 990°$$

Which will have all the same trigonometric function values as

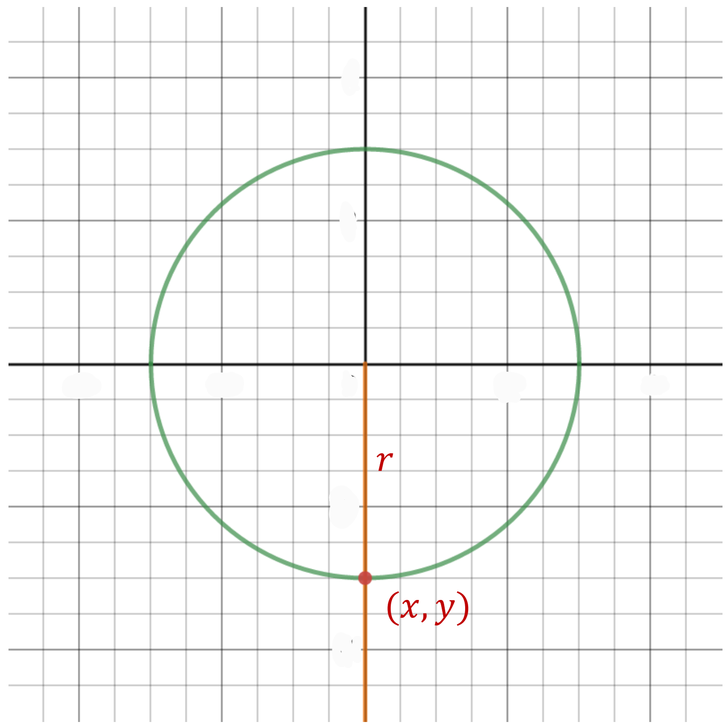
$$990° - 360° = 630° = 630° - 360° = 270°$$

(here I had to subtract 360° twice because it had completed more than two revolutions).

To figure out $\tan(270°)$ we can either look at the diagram:

$$\tan(270°) = \frac{y}{x} = \frac{y}{0}$$

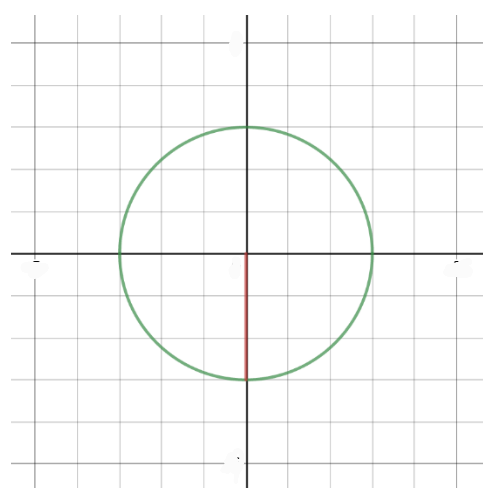
$$= undefined$$



Or realize that the reference angle is 90°:

Then know that $\tan(90°)$ is undefined . . .

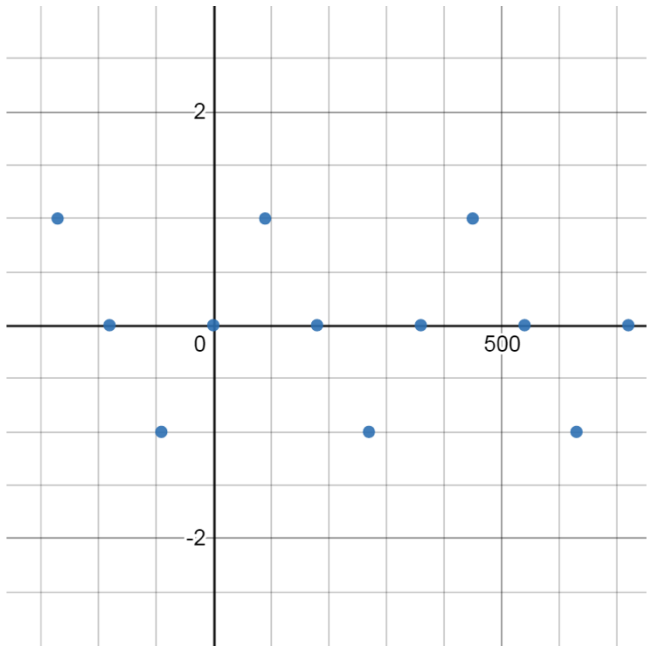
And realize that putting a negative on "undefined" (because $y < 0$) makes no sense.



Now that we have a way of getting the function values for $f(x) = \sin(x)$ we can finish the graph we started earlier.

Here is what we had:

$g(\theta) = \sin(\theta)$

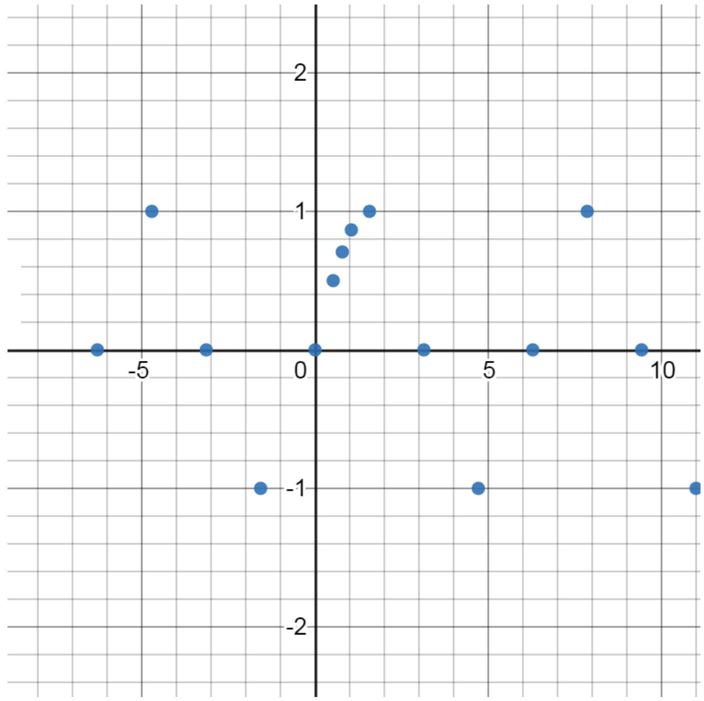


Now let's add the points we get from our basic table of values from the Quadrant I:

θ	$0°$	$30°$	$45°$	$60°$	$90°$	$180°$	$270°$
$\sin(\theta)$	0	$\mathbf{\frac{1}{2}}$	$\mathbf{\frac{1}{\sqrt{2}}}$	$\mathbf{\frac{\sqrt{3}}{2}}$	1	0	-1
$\cos(\theta)$	1	$\frac{\sqrt{3}}{2}$	$\frac{1}{\sqrt{2}}$	$\frac{1}{2}$	0	-1	0
$\tan(\theta)$	0	$\frac{\sqrt{3}}{3}$	1	$\sqrt{3}$	$undef.$	0	$undef.$

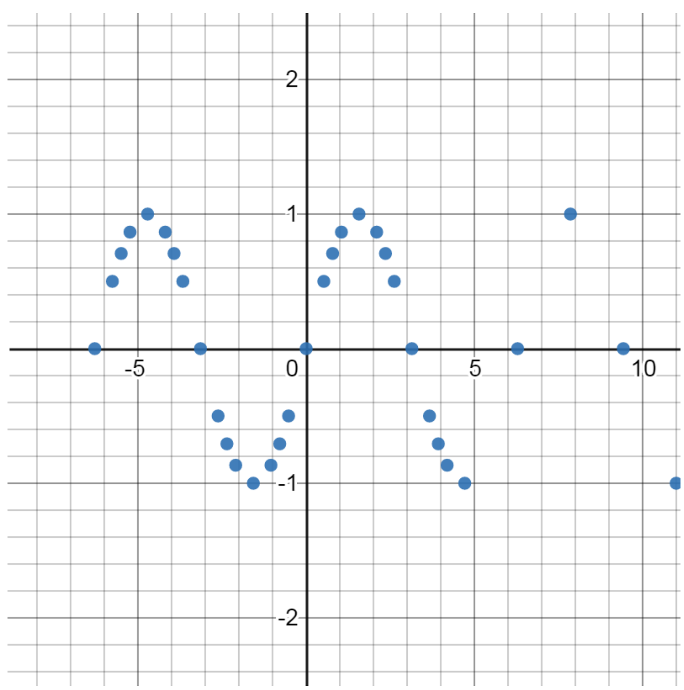
Which gives us

$g(\theta) = \sin(\theta)$



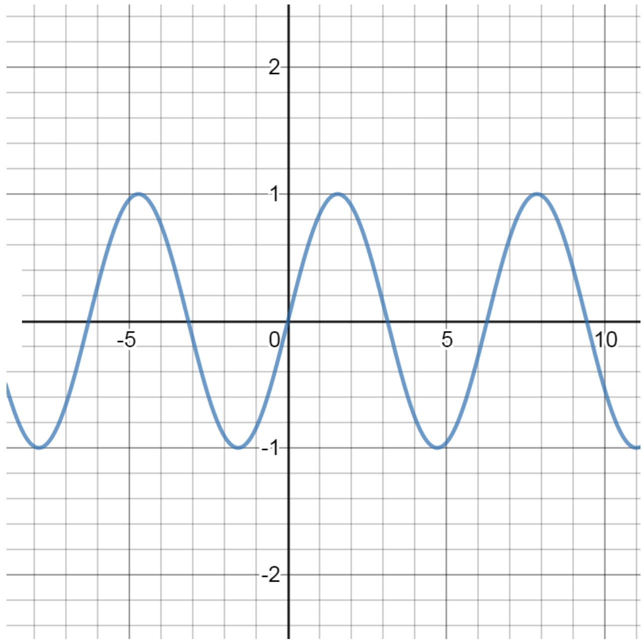
Now, if we create function values for the angles between -2π and 4π that are multiples of 30° and 45° ($\frac{\pi}{6}$ and $\frac{\pi}{4}$ in radians), we get:

$g(\theta) = \sin(\theta)$



Which becomes

$g(\theta) = \sin(\theta)$



Notice some things about this graph:

Its range is $[-1, 1]$

Its domain is $(-\infty, \infty)$

Its period is 2π.