Horizontal Asymptotes

The horizontal asymptote of a rational function $f(x) = \frac{p(x)}{q(x)}$

is obtained by finding

$$\lim_{x \to \pm\infty} f(x)$$

Again, for our purposes we can simplify that to

$$\lim_{x \to \infty} f(x)$$

To figure this out in the previous problem, we constructed a table of values.

But this takes too long!!

There's a faster way!

Here's the concept:

Watch, more closely, *what is happening to the formula* . . .

. . . as $x \to \infty$

x	$g(x)$
$\mathbf{10}$	$\dfrac{\mathbf{10}+1}{\mathbf{10}-1} = 1.22$
$\mathbf{100}$	$\dfrac{\mathbf{100}+1}{\mathbf{100}-1} = 1.02$
$\mathbf{1000}$	$\dfrac{\mathbf{1000}+1}{\mathbf{1000}-1} = 1.002$
$\mathbf{10000}$	$\dfrac{\mathbf{10000}+1}{\mathbf{10000}-1} = 1.0002$

these numbers matter less and less

(it's like they're overshadowed by the other ones)

$\downarrow$ ∞ $\qquad$ $\downarrow$ 1

Notice that in the formula, some of the numbers get big (the x). . .

. . . and some stay the same (the constant value of 1).

The constants become **relatively insignificant**

. . . which means they matter less and less!

Here's the key concept:

The **constant** parts of the formula (that matter less and less) . . .

. . . *don't matter in the long run!*

Remember: the **limit** is about ***where the function values are going***.

Which is why we are allowed to say that

$$\lim_{x\to\infty} \frac{x+1}{x-1} \quad = \quad \lim_{x\to\infty} \frac{x+0}{x-0} \quad = \quad \lim_{x\to\infty} \frac{x}{x}$$

and

$$\lim_{x\to\infty} \frac{x}{x} \quad = \quad \lim_{x\to\infty} 1$$

and because the number known as 1 doesn't care what happens with x . . .

(because it's just "$\mathbf{1}$")

$$\lim_{x\to\infty} 1 \quad = \quad 1$$

Let's do another example:

Find the horizontal asymptote of

$$h(x) = \frac{1-5x}{2x+3}$$

We need to find

$$\lim_{x\to\infty} \frac{1-5x}{2x+3}$$

Here goes:

$$\lim_{x\to\infty} \frac{1-5x}{2x+3}$$

$$= \lim_{x\to\infty} \frac{0-5x}{2x+0}$$

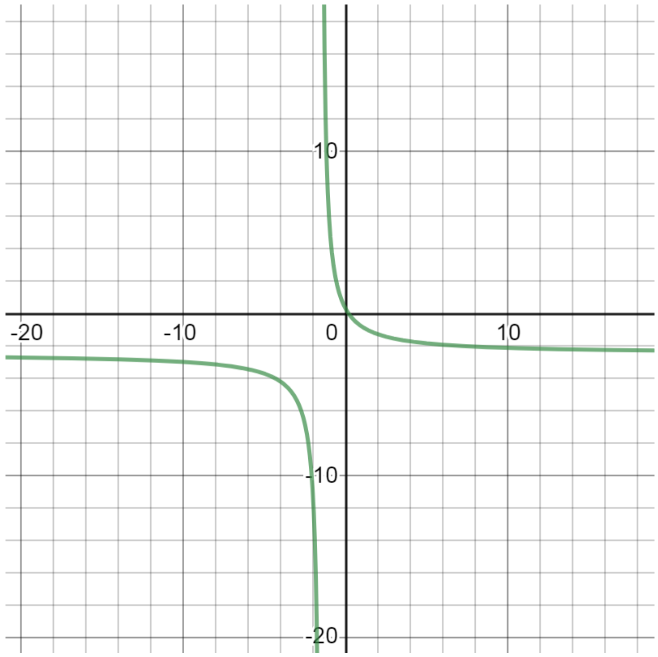
$$= \lim_{x\to\infty} \frac{-5x}{2x}$$

$$= \lim_{x\to\infty} \frac{-5}{2}$$

$$= -\frac{5}{2}$$

Which in decimal form is -2.5. Let's look at the graph to see that the horizontal asymptote is indeed -2.5:

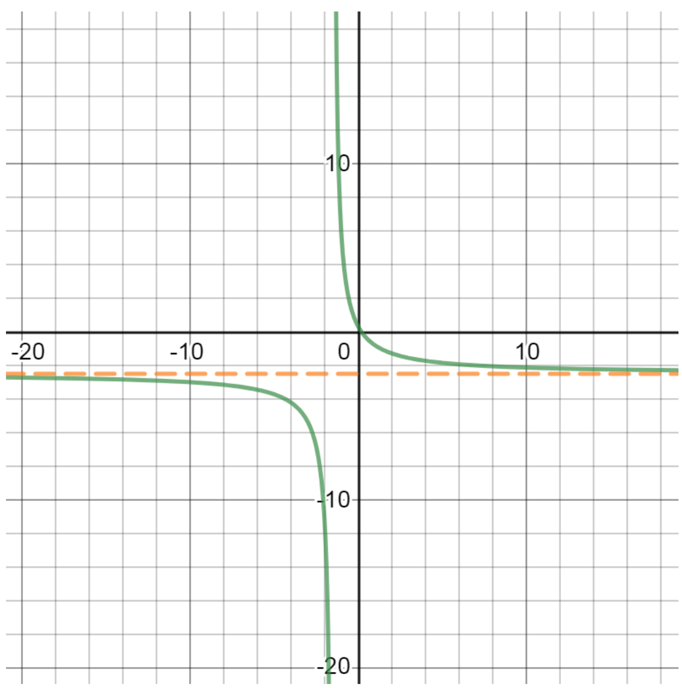
$$h(x) = \frac{1 - 5x}{2x + 3}$$



Maybe that's not clear enough . . . so let's graph the line $y = -2.5$ on the same axis to show that this is the horizontal asymptote:

$$h(x) = \frac{1 - 5x}{2x + 3}$$

$$y = -2.5$$



We need to some more examples!

Find the horizontal asymptote for

$$h(x) = \frac{2x^2 - 1}{x^2 - 3x}$$

We need to find

$$\lim_{x \to \infty} \frac{2x^2 - 1}{x^2 - 3x}$$

Here goes:

At first it seems that we can only eliminate the constant 1:

$$\lim_{x \to \infty} \frac{2x^2 - 1}{x^2 - 3x} = \lim_{x \to \infty} \frac{2x^2 - 0}{x^2 - 3x} = \lim_{x \to \infty} \frac{2x^2}{x^2 - 3x}$$

But there's more we can do . . .

Looking at the denominator, will both terms be equally significant . . .

. . . *in the long run*?

No! The first term, x^2, will become much "bigger" for large values of x:

x	x^2	$3x$
10	100	30
100	10000	300
1000	1000000	3000

getting smaller
by comparison

And since the limit only cares about where the function values are going . . .

. . . we can **eliminate the $3x$** as relatively insignificant:

$$\lim_{x\to\infty} \frac{2x^2}{x^2 - 3x} = \lim_{x\to\infty} \frac{2x^2}{x^2 - 0} = \lim_{x\to\infty} \frac{2x^2}{x^2} = \lim_{x\to\infty} 2 = 2$$

Here's a slightly more formally correct way of doing the same problem:

$$\lim_{x\to\infty} \frac{2x^2+1}{x^2+3x} = \lim_{x\to\infty} \frac{2x^2+1}{x(x+3)} = \lim_{x\to\infty} \frac{2x^2+0}{x(x+0)} = \lim_{x\to\infty} \frac{2x^2}{x^2} = \lim_{x\to\infty} 2 = 2$$

Here we only eliminate constant terms by factoring the denominator.

Okay, now we need to try a different type of rational function:

Find the horizontal asymptotes of:

$$p(x) = \frac{x}{x^2 - 9}$$

We need to find:

$$\lim_{x \to \infty} \frac{x}{x^2 - 9}$$

Here goes:

$$\lim_{x \to \infty} \frac{x}{x^2 - 9} = \lim_{x \to \infty} \frac{x}{x^2 - 0} = \lim_{x \to \infty} \frac{x}{x^2} = \lim_{x \to \infty} \frac{1}{x}$$

And this is as far as we can simplify the limit!!!

At this point, we just have to figure out what $\lim_{x \to \infty} \frac{1}{x}$ actually is!

You can do it! Think about what would happen to

$$\frac{1}{x}$$

As the denominator gets bigger . . . and bigger . . . and bigger . . .

$$\frac{1}{10} \to \frac{1}{100} \to \frac{1}{1000} \to \frac{1}{10000} \to \frac{1}{100000} \to . . .$$

Those same numbers, seen as decimals, are

$$0.1 \;\rightarrow 0.01 \;\rightarrow 0.001 \;\rightarrow 0.0001 \;\rightarrow 0.00001 \;\rightarrow . . .$$

Where are these numbers going?

Or, put a better way, what number are they getting closer to?

That's right . . . zero!

We have that

$$\lim_{x\to\infty} \frac{1}{x} = 0$$

Therefore the horizontal asymptote to the function

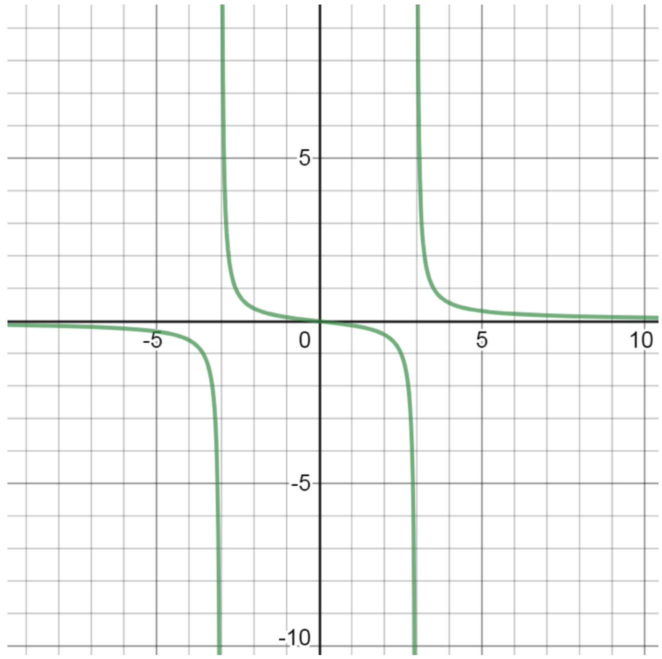
$$p(x) = \frac{x}{x^2 - 9}$$

Is the line

$$y = 0$$

Which is of course the x-axis. We can see this is true from the graph:

$$p(x) = \frac{x}{x^2 - 9}$$



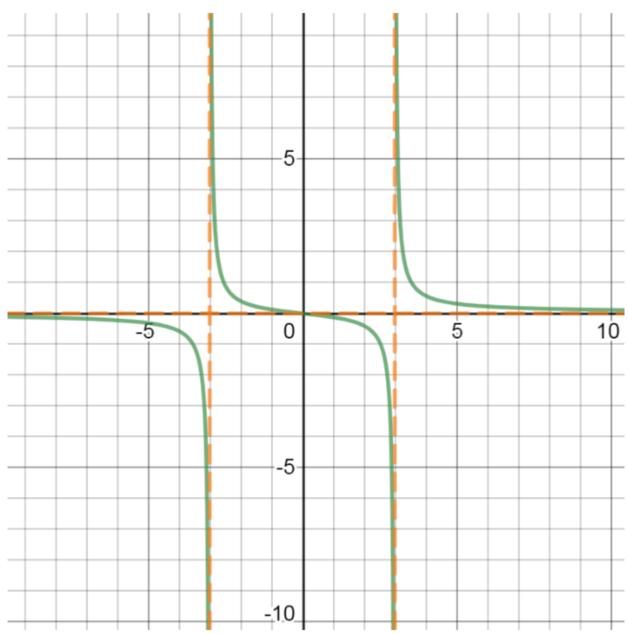
Note that the denominator of $p(x)$ becomes zero at $x = \pm 3$ . . .

Which is why we have vertical asymptotes at $x = 3$ and $x = -3$:

$$p(x) = \frac{x}{x^2 - 9}$$

$x = \pm 3$

$y = 0$



But we can see one important difference between horizontal asymptotes …

… and vertical asymptotes.

Can you see what it is?

The graph of $p(x)$ crosses the line $y = 0$!

It crosses the horizontal asymptote!

This can happen with horizontal asymptotes!

Because the horizontal asymptote is only to show what happens at the ends of the graph!!

The graph can never cross the vertical asymptote!

One final note on the horizontal asymptote:

We have been figuring out the horizontal asymptote using the limit as $x \to \infty$.

There are really only three ways this can work out.

Let's look at them, and see if we can note a pattern, in case you prefer to memorize a formula for this type of problem.

we're about to consider a "formula" approach to the horizontal asymptote

We are trying to find

$$\lim_{x \to \infty} \frac{p(x)}{q(x)}$$

Where p and q are polynomial functions. Here, what ends up being the most important is the **degree of the function**, because the **leading term** (the term with the highest degree) will always be the *most significant term*:

most important term

for example, in the below function:

$$p(x) = x^3 - 2x^2 + 12x - 5$$

The leading term, x^3 will be so big (for large x-values) compared to the other terms, it's the only one that's going to matter as $x \to \infty$.

Thus we can come up with a formula for finding the horizontal asymptotes that is based on the degrees of the polynomials in the numerator and denominator of the function, and the coefficients of their leading terms.

Here, we will use the notation:

$$\deg(f(x))$$

to represent the degree of a function.

Consider any rational function

$$f(x) = \frac{p(x)}{q(x)}$$

suppose this has a lesser degree

this becomes dominant

Where p and q are polynomials:

(I) Suppose $\deg(p(x)) < \deg(q(x))$

Then $q(x)$ will be the dominant part of the fraction, getting bigger and bigger relative to the numerator.

Hence,

$$\frac{p(x)}{q(x)} \to 0$$

and

$$\lim_{x \to \infty} \frac{p(x)}{q(x)} = 0$$

(II) Suppose deg($p(x)$) = deg($q(x)$)

Then the leading term of both the top and the bottom will end up being the only significant terms, and

$$\frac{p(x)}{q(x)} \to \frac{leading\ term\ of\ p(x)}{leading\ term\ of\ q(x)} = \frac{a_n x^n}{b_n x^n}$$

But the x^n will cancel, hence we will have

$$\lim_{x\to\infty} \frac{p(x)}{q(x)} = \frac{a_n}{b_n}$$

Where a_n and b_n are the leading coefficients, respectively, of $p(x)$ and $q(x)$

(III) Suppose deg($p(x)$) $>$ deg($q(x)$)

Here the degree of the numerator is greater than the degree of the denominator. We did not look at any examples of this type of rational function. Here, the dominant part of the fraction is the numerator, which means that the function itself will just get bigger and bigger as $x \to \infty$.

Hence

$$\lim_{x\to\infty} \frac{p(x)}{q(x)} = \infty$$

and there is **no horizontal asymptote**