Vertical Asymptotes

“Asymptote” means something that you approach but never reach.

It’s a “vertical” asymptote because in this case, the line is vertical.

Why is it vertical?

It’s there because the function is undefined at $x = 1$.

There can be no points where it’s undefined . . .

. . . so the graph can never cross!

This will (almost) always be true when the function’s **denominator** $= \mathbf{0}$.

(it’s only **not** true when the numerator is zero at the same time).

So we have . . .

If $f(x) = \frac{p(x)}{q(x)}$, then the graph of $f(x)$ has a vertical asymptote at $x = a$

wherever $q(a) = 0$

the denominator equals zero → undefined

Put less formally . . .

To find the vertical asymptote of a rational function . . .

. . . find where the denominator equals zero.

For example, consider the problem of graphing the function

$$r(x) = \frac{2}{x-2}$$

Since this is a rational function, we would first check for vertical asymptotes to help us draw our graph.

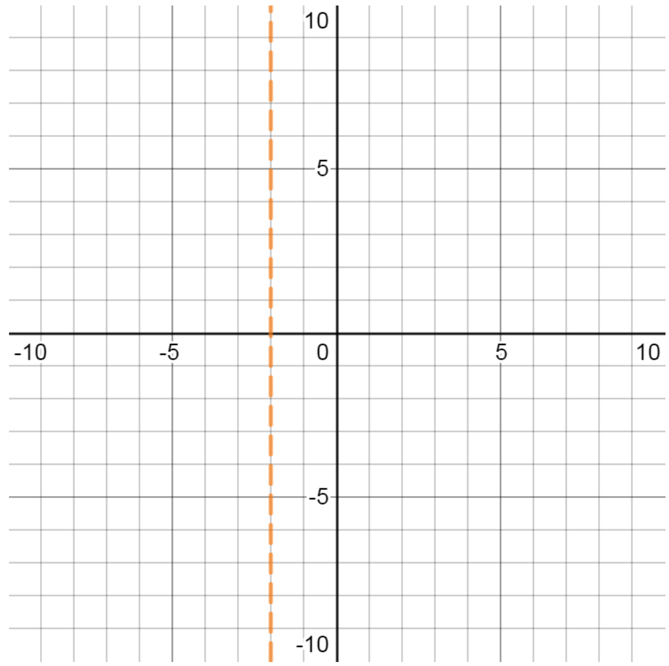
To find the vertical asymptotes, find where $r(x)$ is undefined.

To find where $r(x)$ is undefined, find where its denominator equals zero:

$$x - 2 = 0$$

$$x = 2$$

$r(x)$ has a vertical asymptote at $x = 2$:



Let's do another example.

Consider the function

$$h(x) = \frac{x^2}{x^2 - 9}$$

Does the function have vertical asymptotes, and if so, where?

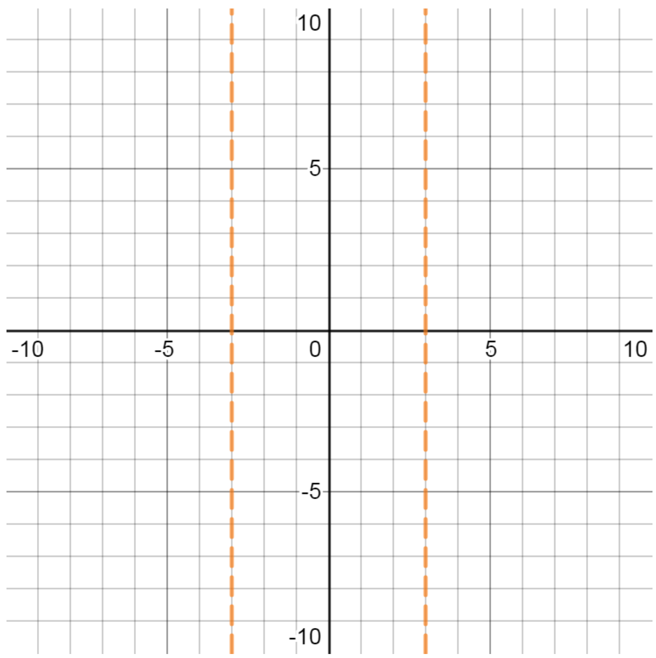
$h(x)$ will have vertical asymptotes wherever it is undefined . . .

. . . and $h(x)$ is undefined wherever $x^2 - 9 = 0$

$$\begin{aligned} x^2 - 9 &= 0 \\ (x+3)(x-3) &= 0 \end{aligned}$$

$$x = -3 \qquad x = 3$$

$h(x)$ has vertical asymptotes at $x = -3$ and $x = 3$.



Now wait a minute . . .

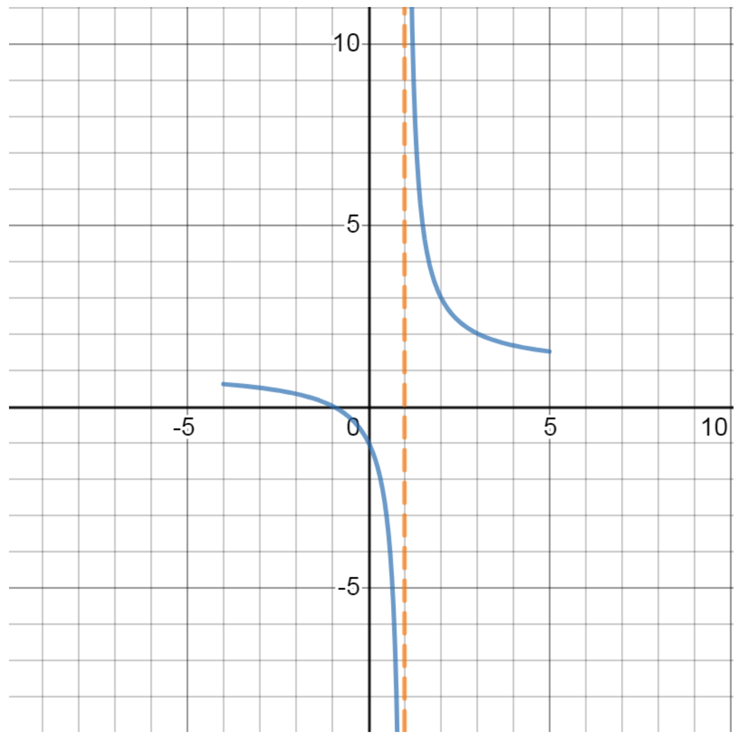
We **never finished** our original graph of

$$g(x) = \frac{x+1}{x-1}$$

Here’s what we had so far:

$$g(x) = \frac{x+1}{x-1}$$

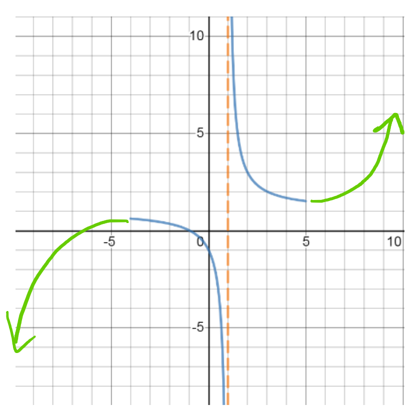
$$-4 \le x \le 5$$

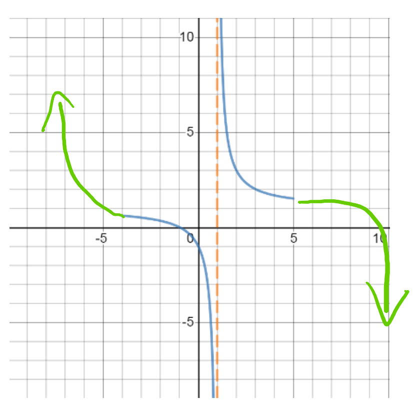


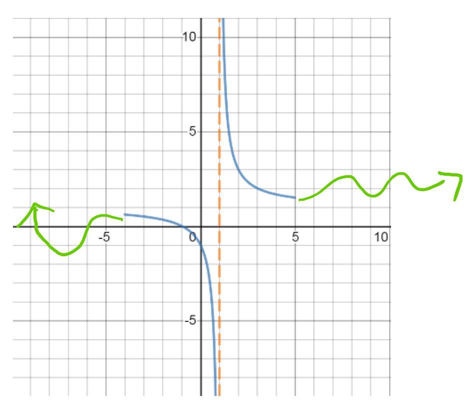
We didn’t finish this graph because we didn’t quite indicate what happened on the ends of the graph.

What do you think happens as we go out, away from the middle?

I can imagine several possibilities:



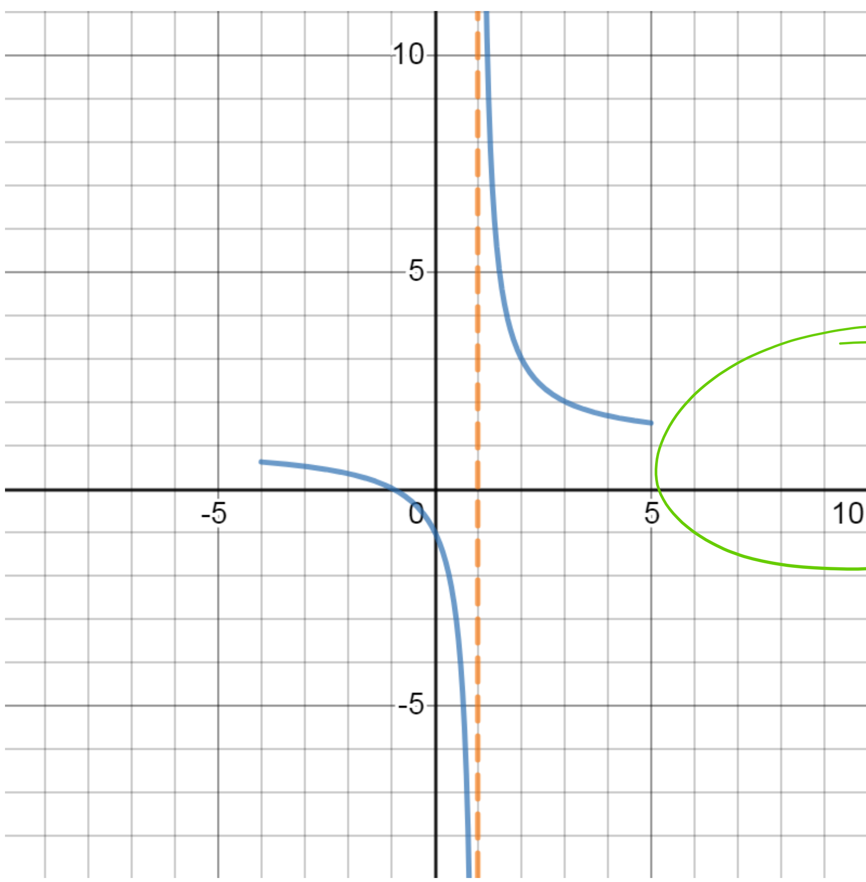




To truly figure this out, we need to conceptualize what happens as

$$x \to \pm\infty$$

This is the only way to know for sure what happens at the "ends" of the graph!



what's happening here?

see where it's going by letting $x \to \infty$

To do this, we will preview a very important concept from Calculus . .

. . . which is finding the **limit** of a function!

In math, the "limit" refers to **where the y-values are going**.

For $g(x)$, we want to know where the y-values are going as $x \to \pm\infty$.

So we want to find

$$\lim_{x\to\infty} g(x) \quad \text{and} \quad \lim_{x\to-\infty} g(x)$$

For all the problems we will be doing here, both of these limits are the same,

So we will be just looking at

$$\lim_{x\to\infty} g(x)$$

The simplest way to see this is with a table . . .

. . . letting the x-values get very large

. . . and seeing what happens to the y-values:

$\boldsymbol{x}$	$\boldsymbol{g(x)}$
10	1.22
100	1.02
1000	1.002
10000	1.0002
$\Downarrow$	$\Downarrow$
∞	?

Can you see a pattern??? Can you see where the y-values are going???

Yes you can!!!

$\boldsymbol{x}$	$\boldsymbol{g(x)}$
10	1.22
100	1.02
1000	1.002
10000	1.0002
$\Downarrow$	$\Downarrow$
∞	1

The numbers coming out of the function are getting closer and closer to 1!

And since they are getting closer and closer . . .

. . . but never getting there . . .

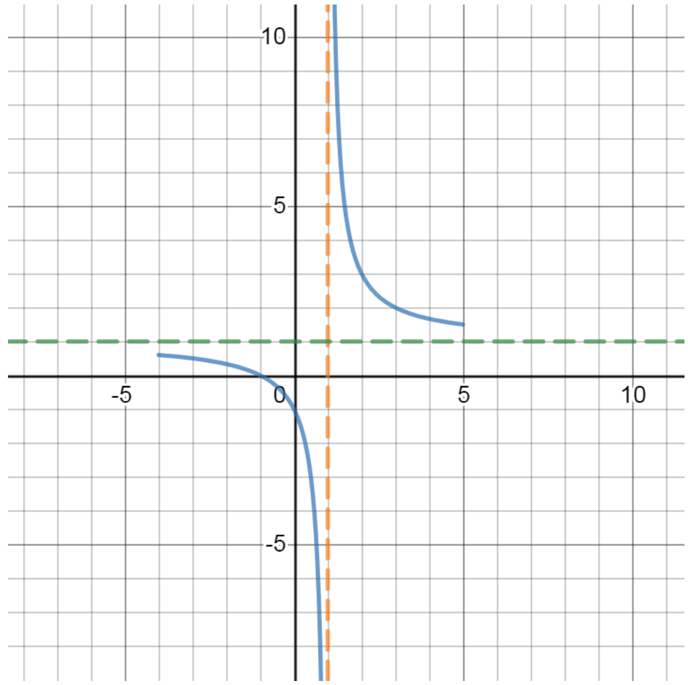
We have a way to draw the graph!

It's an asymptote!

And since it concerns the y-values, it's a

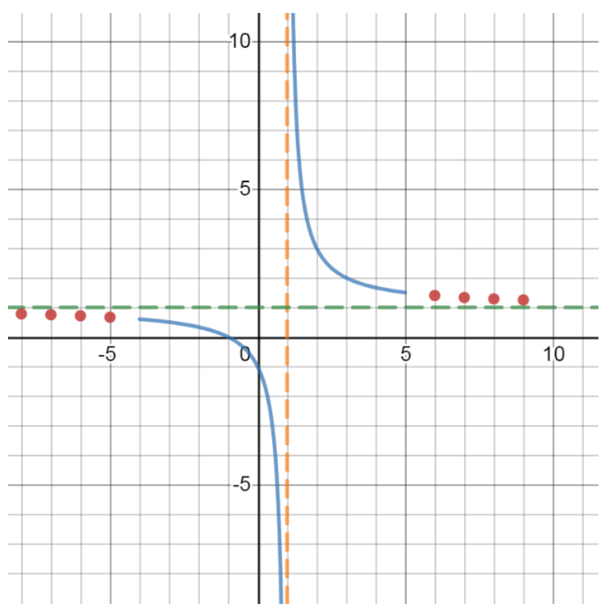
horizontal asymptote

We can see it on the graph:



horizontal asymtote

And while we can’t plot all the points we just came up with (they’re too big), we can plot a few more points on the graph and see how the y-coordinates of those points are getting closer and closer to 1:



So we can finish the graph:

$$g(x) = \frac{x+1}{x-1}$$