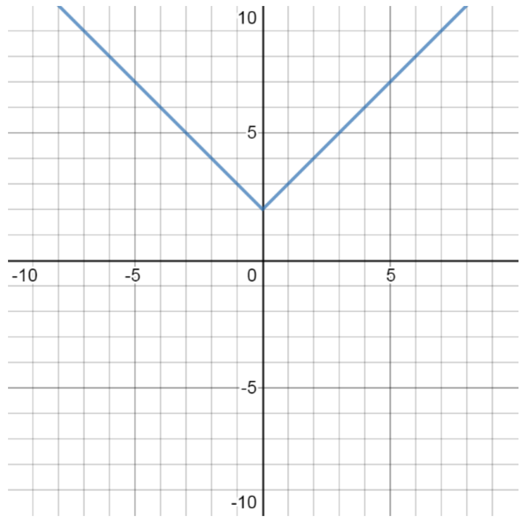
Transformations of Functions

Consider the following graph:

$q(x)$



Does it look familiar?

It should!

It's very similar to the graph we just studied: $h(x) = |x|$

But *not exactly* the same. Go back to the previous graph and compare.

Do you see the difference?

challenge   Can you guess what the formula for it is?

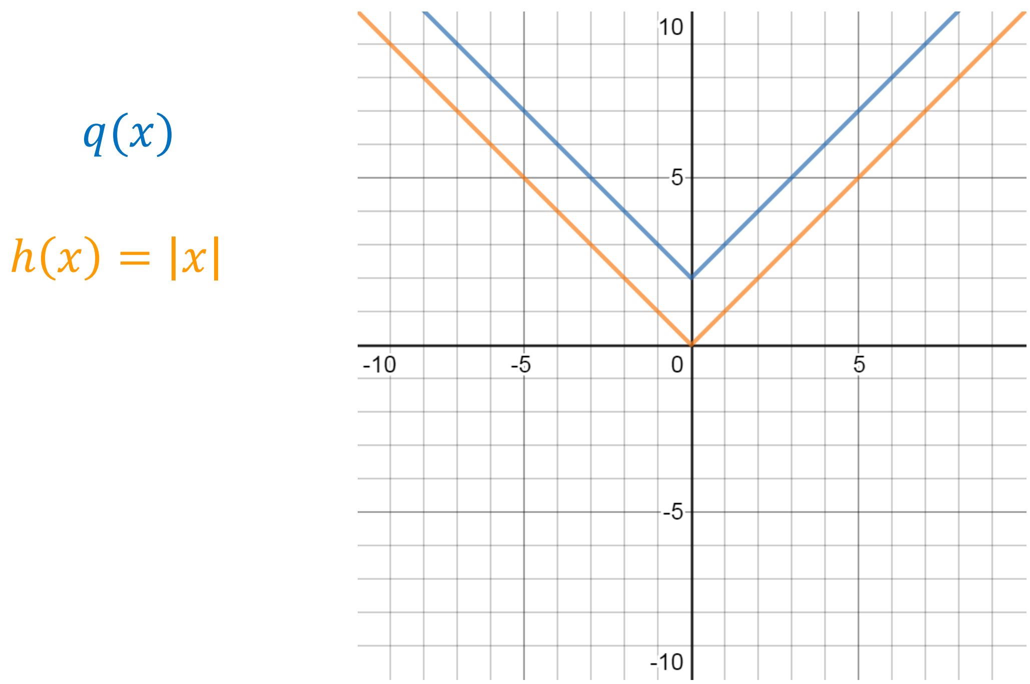
To understand the answer, let's graph both

$$h(x) = |x|$$

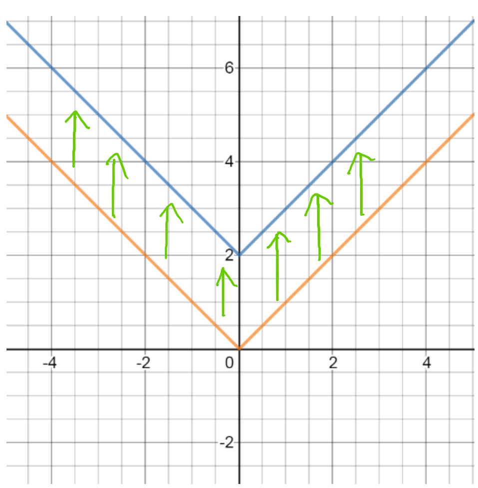
and

$$q(x)$$

on the same coordinate axis:



We can see that $q(x)$ has the same graph as $h(x)$ but **shifted up**:



The question is . . . what does this mean for its formula?

Since all of the y-values have increased by **2** …

We have basically taken $h(x) = |x|$

and *ADDED* 2!

So the formula for $q(x)$ is . . .

$$q(x) = |x| + 2$$

We have just introduced an important concept for this course.

It's called a

transformation of a function.

Anytime you take a given function,

and change it,

you are doing a ***transformation***

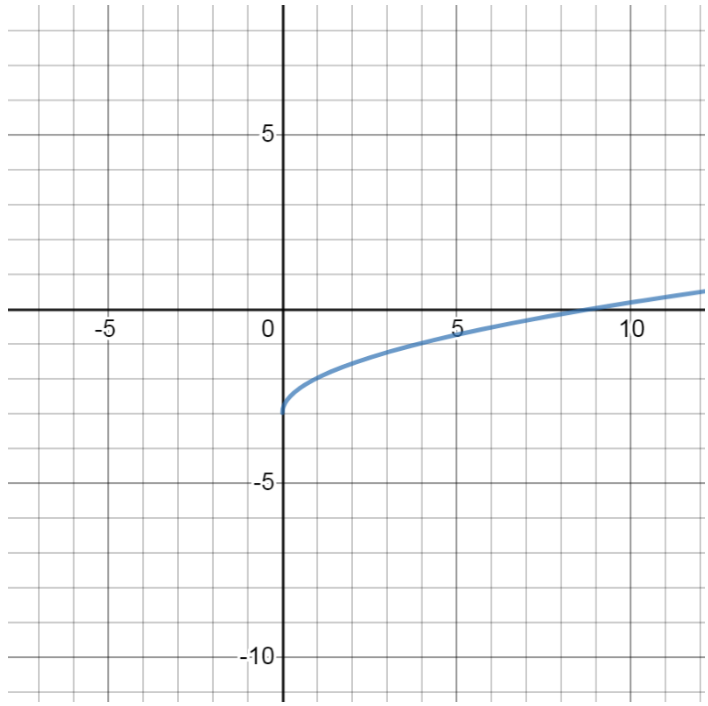
We will be looking at five different kinds. The kind of transformation we saw here was a

vertical shift

Let's see another.

Try to guess the formula for the following graph:

$f(x)$



First, try to see which **type of basic (parent) function** it resembles.

Then try to figure out *what transformation* was done to that basic function.

It turns out that $f(x)$ is the square root function $p(x) = \sqrt{x}$

shifted ***down***

by 3

So

$$f(x) = \sqrt{x} - 3$$

To sum up the concept of the **vertical shift** in function language:

The ***transformed*** function $f(x)$ has its graph shifted k vertical units by adding or subtracting k to or from the original function $g(x)$:

$$f(x) = g(x) \pm k$$

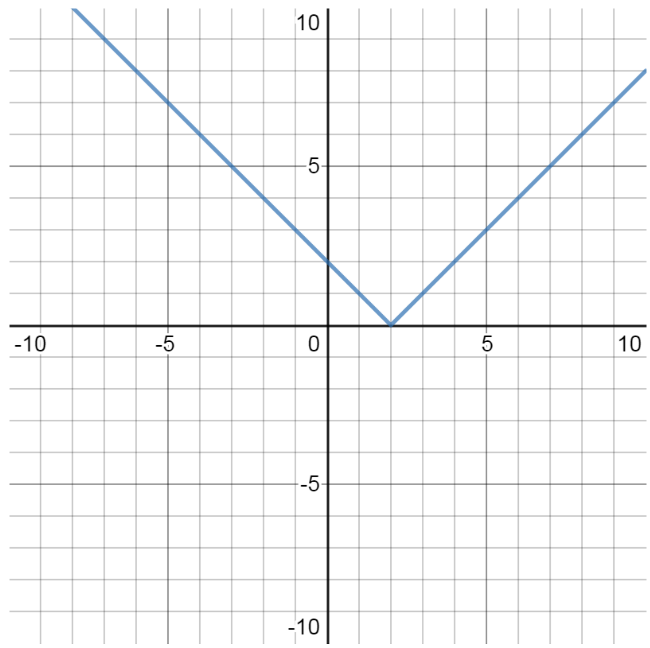
this number is the vertical shift

Now consider the function:

$$h(x) = |x - 2|$$

The graph of the function is shown below:

$$h(x) = |x - 2|$$



What do you notice about this graph?

Do you recognize it as a *transformation* of a function?

If so, what kind of transformation is it?

And what is the original (parent) function?

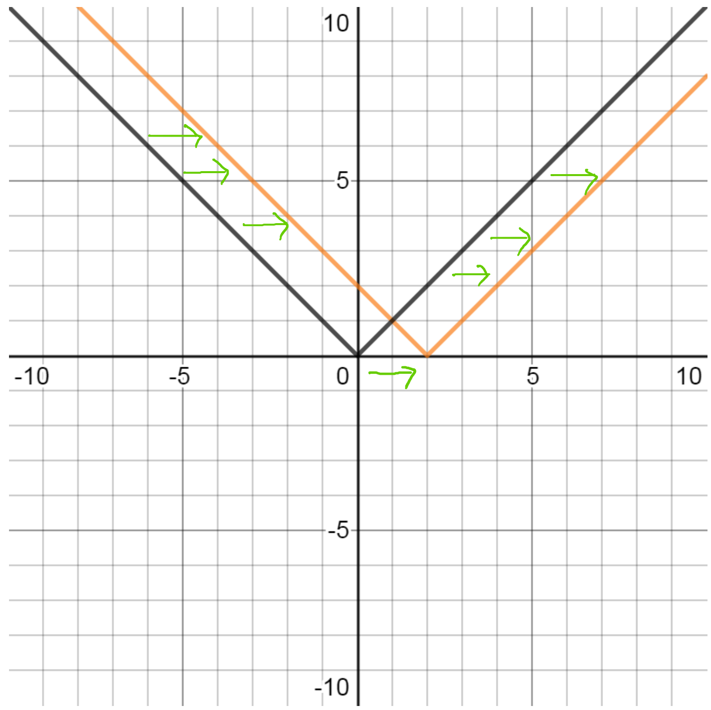
Those are the two things you have to know to get these problems right!

The original function is $f(x) = |x|$

And its graph has been **shifted** to the right:

$$f(x) = |x|$$

$$h(x) = |x - 2|$$



This is an example of a

horizontal **shift.**

How would we know this from the formula?

In other words, how does a *horizontal shift* "show up" in the formula?

The key to see the difference between a horizontal and vertical shift is to understand the difference between the input and the output of the function:

Input "affects" the x (horizontal)

Output "affects" the y (vertical)

So for $h(x) = |x - 2|$, because the 2 is subtracted from the x,
we think of it as affecting the input, which shows up on the horizontal axis.

To sum up the concept of the **horizontal shift** in function language:

The ***transformed*** function $f(x)$ has its graph shifted k horizontal units by subtracting k from the x in original function $g(x)$:

subtraction means shift right

$$f(x) = g(x - k)$$

this number is the horizontal shift

Let's now look at a function with both horizontal and vertical shifts:

$$p(x) = \sqrt{x+3} - 4$$

Can you see that this function is our square root (parent) function $y = \sqrt{x}$

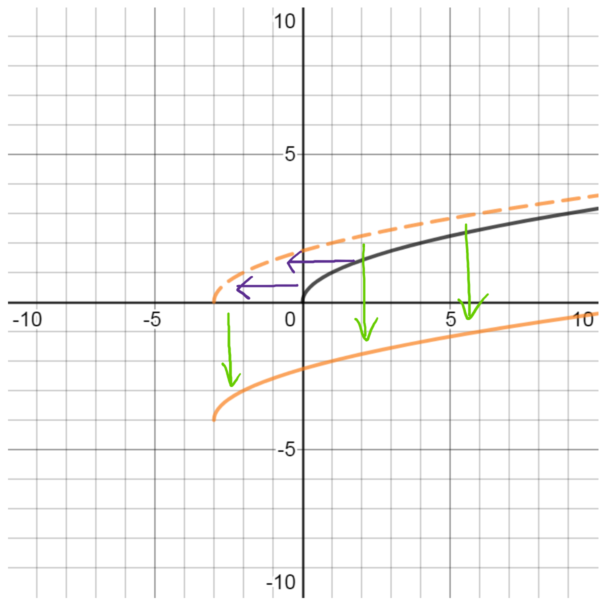
shifted vertically *and* horizontally?

And can you see which is which?

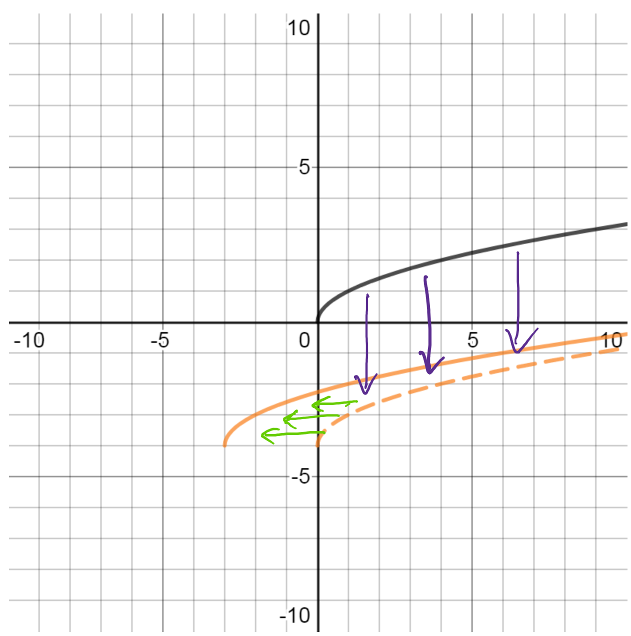
$$p(x) = \sqrt{x+3} - 4$$

If you were to graph $p(x)$, you would start with the original $y = \sqrt{x}$ and then do the two shifts. It wouldn't matter what order you do them in!

Here I do the horizontal shift first:



And here I do the vertical shift first:



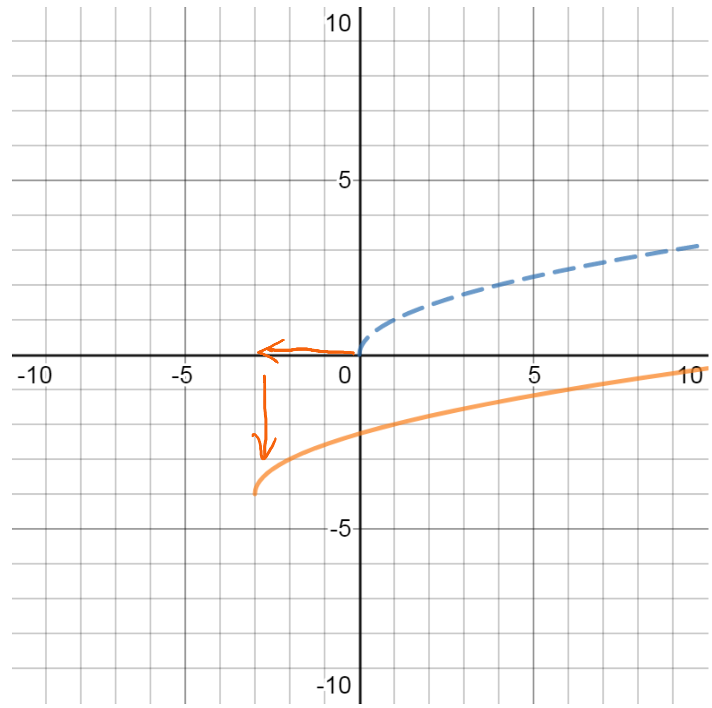
As you note, the end result is the same either way:

$f(x) = \sqrt{x}$

$p(x) = \sqrt{x+3} - 4$

addition means horizontal shift

left



Decide if the following transformations indicate a vertical or horizontal shift:

$$h(x) = (x-2)^3$$

$$q(x) = |x+5|$$

$$r(x) = \sqrt{x} + 7$$

$$t(x) = x^2 - 4$$

$$f(x) = h(x+6)$$

$$g(x) = p(x) - 3$$

Answers:

$h(x) = (x-2)^3$ horizontal shift 2 (to the right)

$q(x) = |x+5|$ horizontal shift 5 (to the left)

$r(x) = \sqrt{x} + 7$ vertical shift 7 (up)

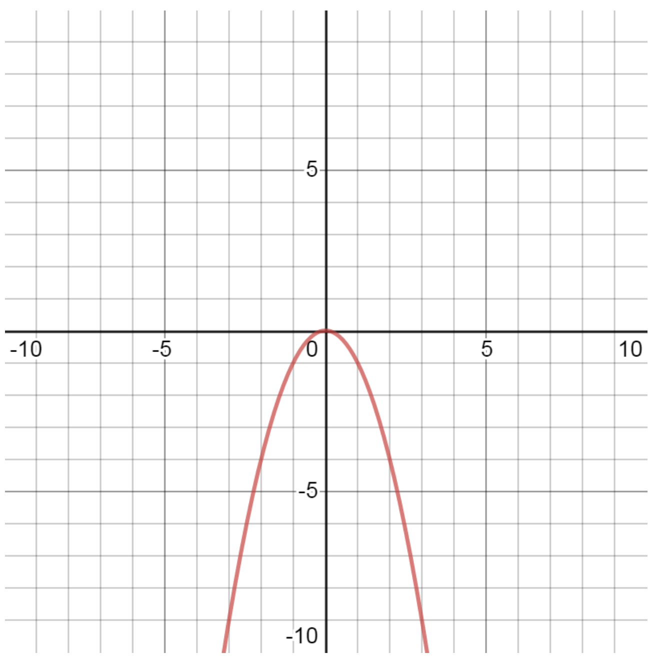
$t(x) = x^2 - 4$ vertical shift 4 (down)

$f(x) = h(x+6)$ horizontal shift 6 (to the left)

$g(x) = p(x) - 3$ vertical shift 3 (down)

Now consider the following graph:

$g(x)$



Does this graph look at all familiar? If so, try to identify:

the original (parent) function

the type of transformation

You might have recognized the graph as a **parabola** . . .

which suggests that the original (parent) function is $f(x) = x^2$.

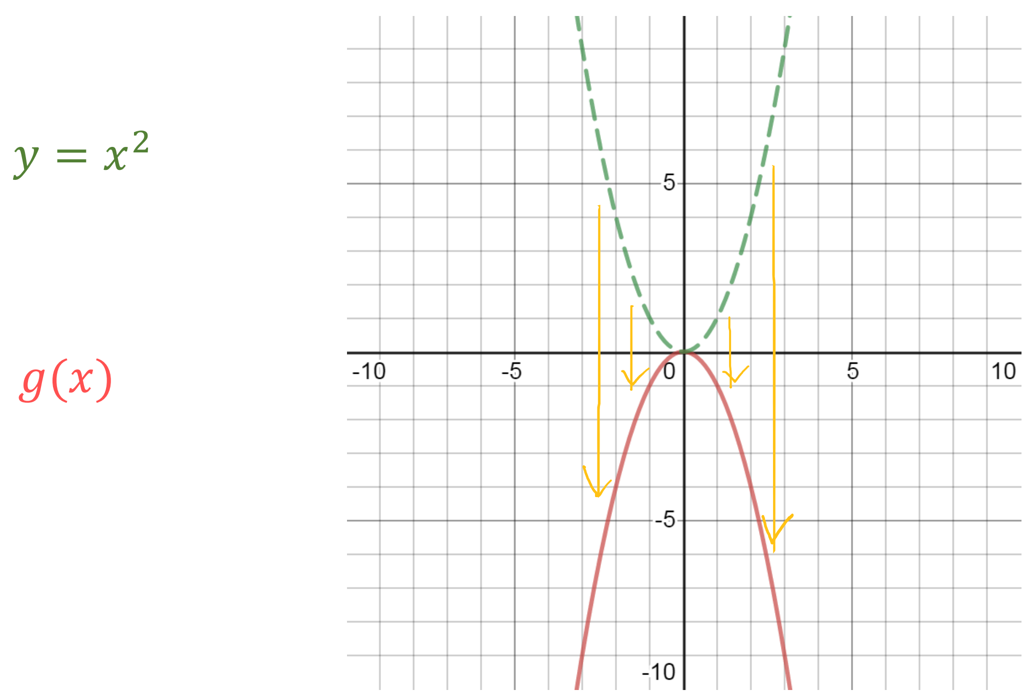
What type of transformation is happening might be difficult to see at first.

It's *not* a **shift**.

The orientation of the graph has changed.

It's been **reflected** . . .

. . . **across the x-axis**

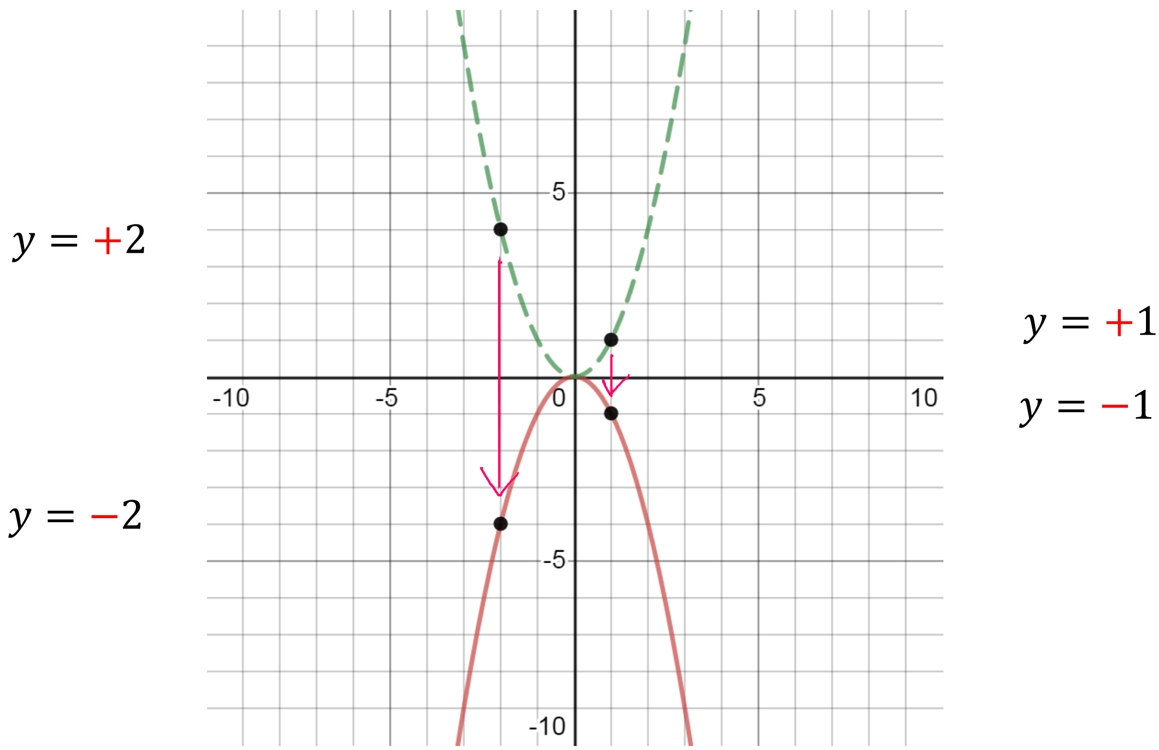


This is called a ***vertical* reflection**.

How does it “show up” in the formula?

If you look at the graph of $g(x)$, you might see that all the y-values

Are *exactly the negative* of the corresponding y-values in $y = x^2$.



So the **entire output** of the original function, $y = x^2$, has been ***negated***!

This means that

$$g(x) = -x^2$$

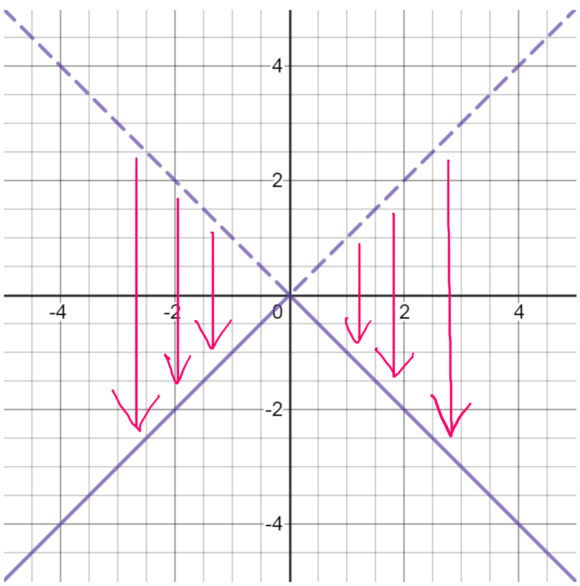
Thus a ***vertical reflection*** happens when

the **output of a function** is ***negated***:

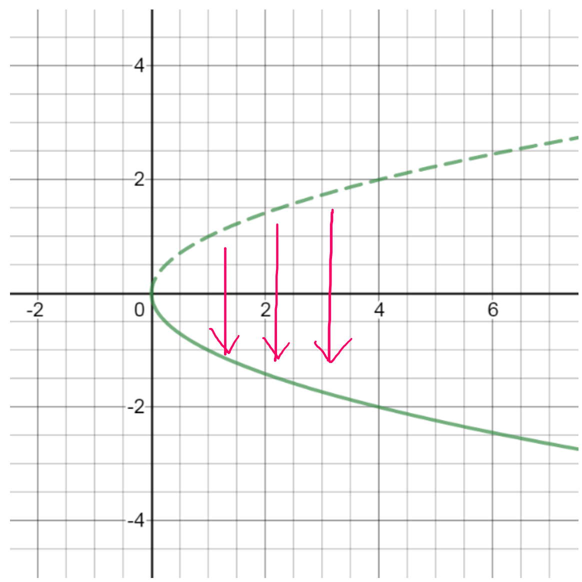
$$g(x) = -f(x)$$

The axis of reflection (think of it as the mirror) is the x-axis.

Here are two more examples of vertical reflections:



$f(x) = -|x|$



$g(x) = -\sqrt{x}$

Question: Why were these reflections **vertical** . . .

. . . rather than **horizontal**?

Answer: Just as a **vertical** shift results from adding to the **output**,

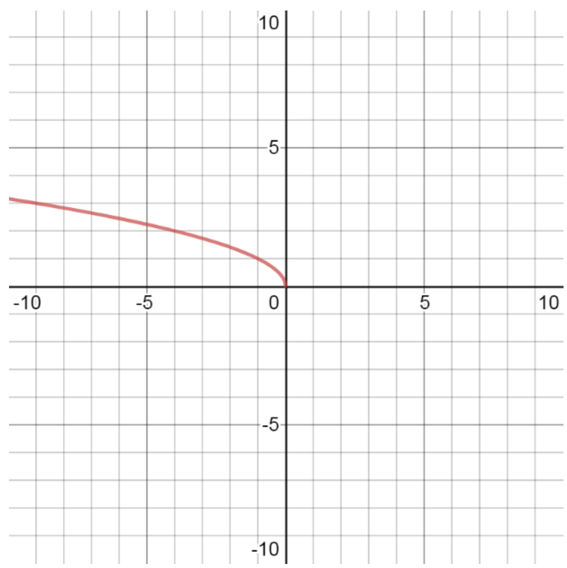
A **vertical reflection** results from negating the **output**.

Conversely, just as a **horizontal** shift results from adding to the **input**,

A **horizontal reflection** results from negating the **input**.

Consider the following graph:

$h(x)$



Can you see what is the original (parent) function?

. . . and the type of transformation?

The original function is of course $y = \sqrt{x}$.

And the type of transformation is a **horizontal reflection**.

Again, since the transformation type is horizontal,

it comes from affecting the input . . .

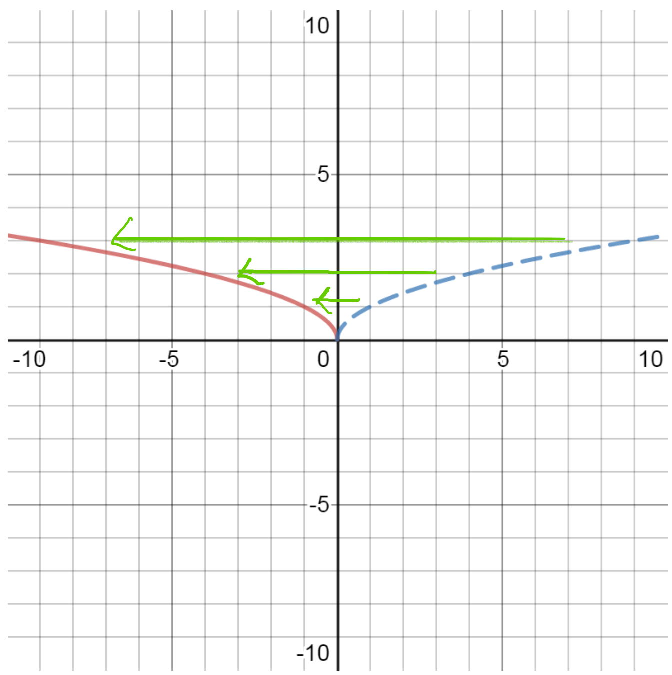
. . . or in this case, **negating** the input . . .

. . . which is the $\boldsymbol{x}$.

So the function is $h(x) = \sqrt{-x}$.

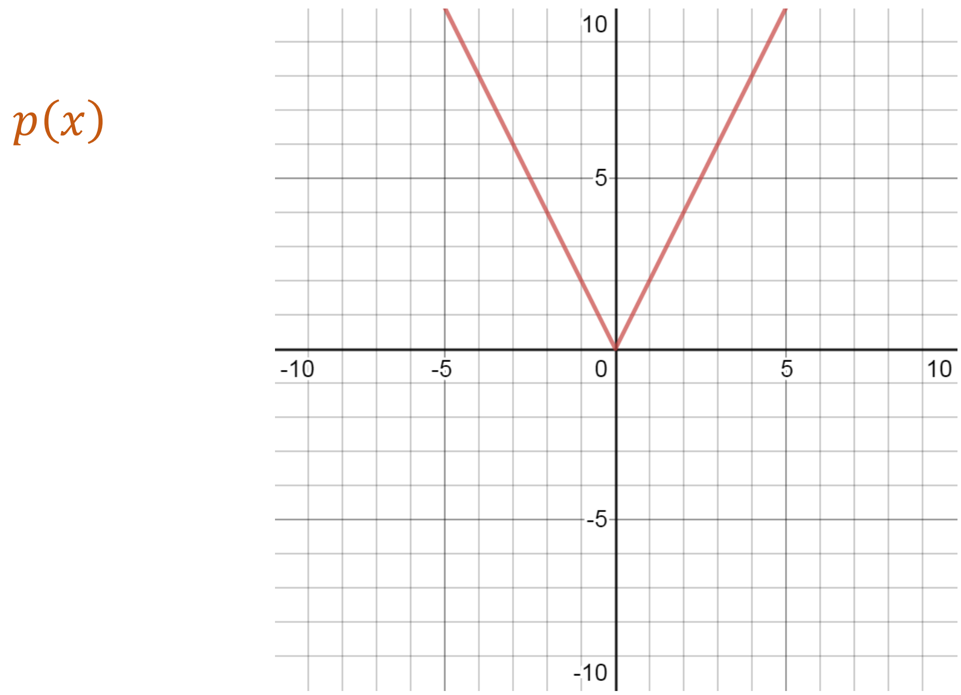
$y = \sqrt{x}$

$h(x) = \sqrt{-x}$



We have one final type of transformation to look at.

Consider the following graph:



As always, the questions are . . .

What is the original (parent) function?

What is the type of transformation?

Do you recognize the “V”-shape of the graph?

It should remind you of the absolute value function, $y = |x|$.

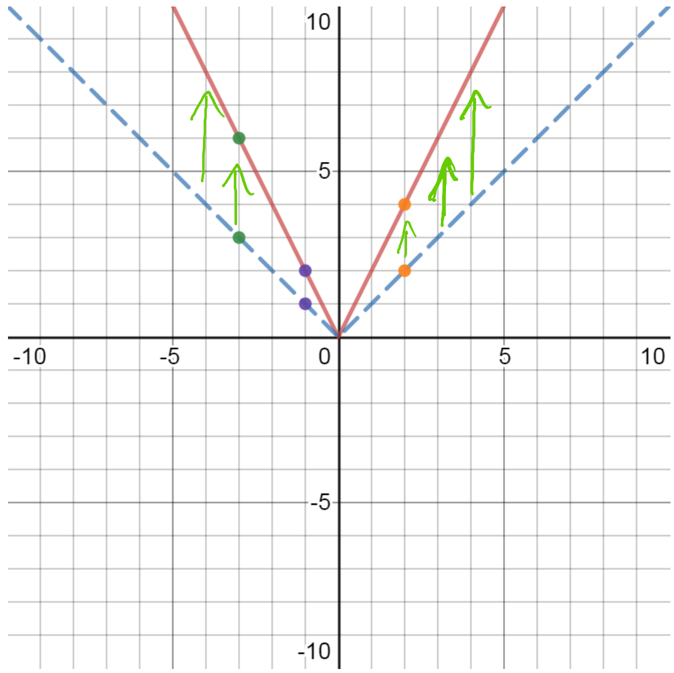
What has *been done* to it??

Answer: It has been pulled away (up) from the x-axis.

Put differently, it has been stretched (up) from the x-axis.

$y = |x|$

$p(x)$



What is the formula for the transformed function?

Note that for each point on the original graph, the point corresponding to the transformed graph . . .

has its y-value **doubled**.

Hence this type of transformation must involve **multiplying by 2**.

And since it's happening **vertically**, this multiplication is happening to the

output.

So the function is

$$p(x) = 2|x|$$

Thus a ***vertical stretch*** happens when

the **output of a function** is ***multiplied***:

$$g(x) = kf(x)$$

The stretch pulls each point on the graph **away** from the x-axis . . .

. . . by k times.

NOTE: it's really only a stretch if $k > 1$!

If $k < 1$ then the points on the graph are being ***pushed toward*** the x-axis.

In other words, it's a ***vertical shrink***!

For example, consider the function

$$g(x) = \frac{1}{2}x^2 - 2$$

What is the original (parent) function?

And what transformation(s) are happening to it?

Answer:

The original function is $y = x^2$.

It's undergoing two transformations:

vertical shrink

vertical shift

1. vertical shrink (by a factor of ½)
2. vertical shift (down 2).

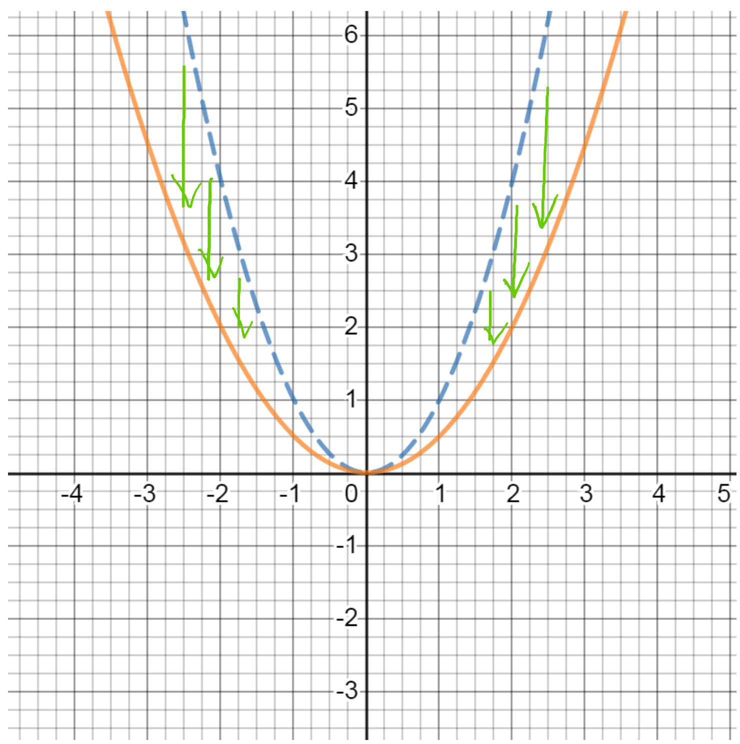
$$g(x) = \frac{1}{2}x^2 - 2$$

Note: when multiple transformations occur, they should be done according to the order of operations. Since addition and subtraction are the last two operations in that list, the shifts should be done last.

To graph $g(x)$, we first apply the ***vertical shrink***:

$y = x^2$

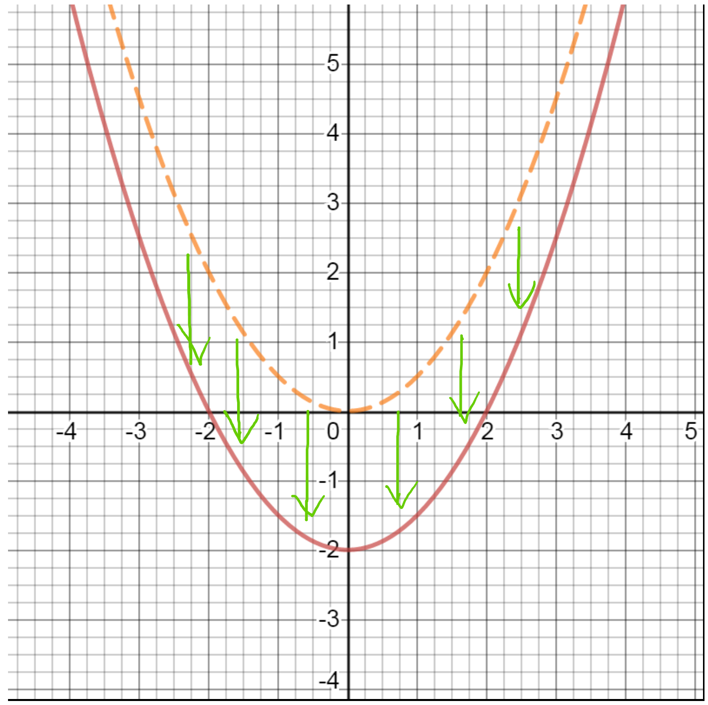
$y = \frac{1}{2}x^2$



And then apply the vertical shift:

$$y = \frac{1}{2}x^2$$

$$g(x) = \frac{1}{2}x^2 - 2$$



The previous example involved applying more than one transformation to a function. Let's look at a problem involving **three** different transformations.

Consider

$$f(x) = -|x - 2| + 5$$

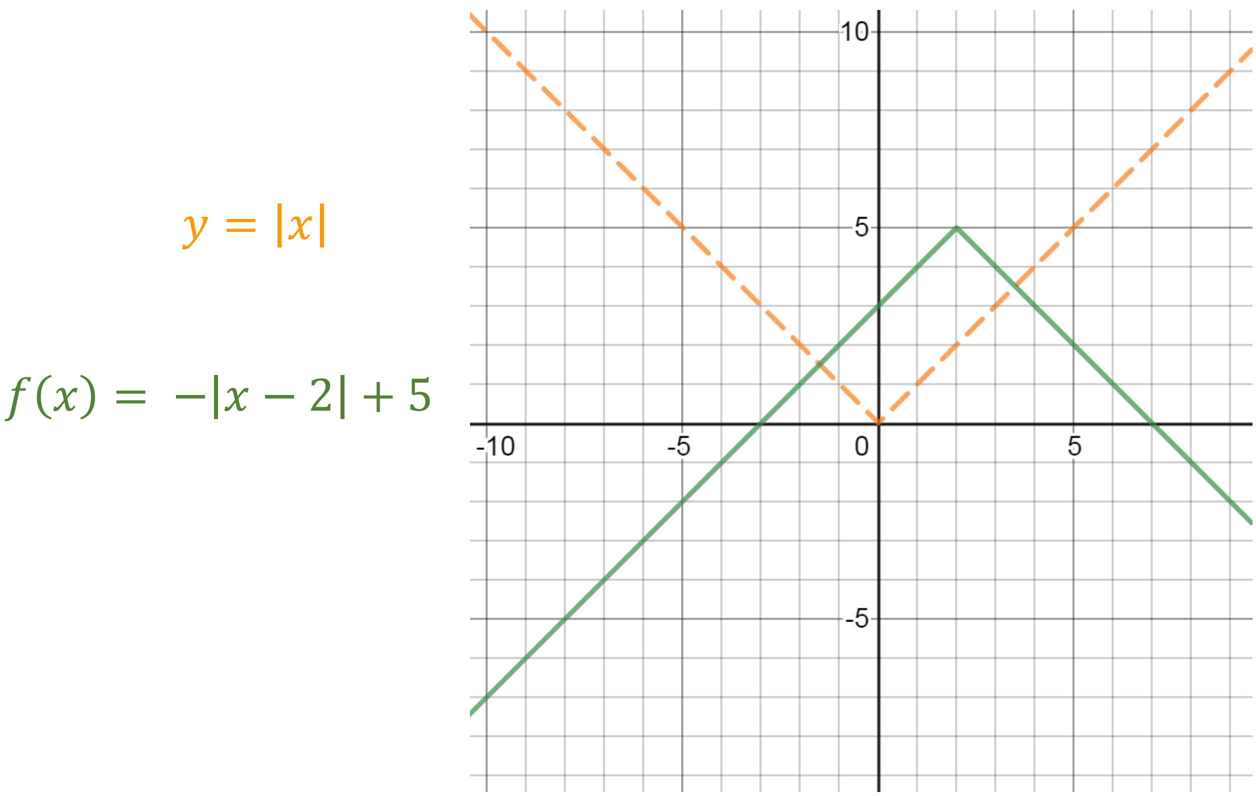
Draw the graph of $f(x)$.

Answer: here we know that our original function is $y = |x|$.

Further, we can see that three transformations have been applied to it:

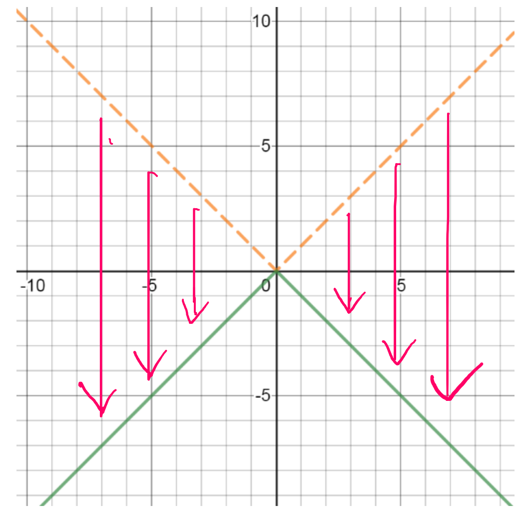
$$f(x) = -|x-2| + 5$$

To draw the graph, we will do the vertical reflection first, then follow with the two shifts in either order:

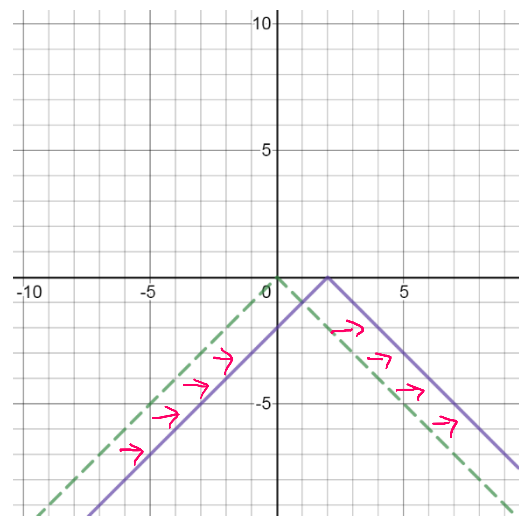


To see how we get from the original function graph to the final graph of $f(x)$ step by step . . .

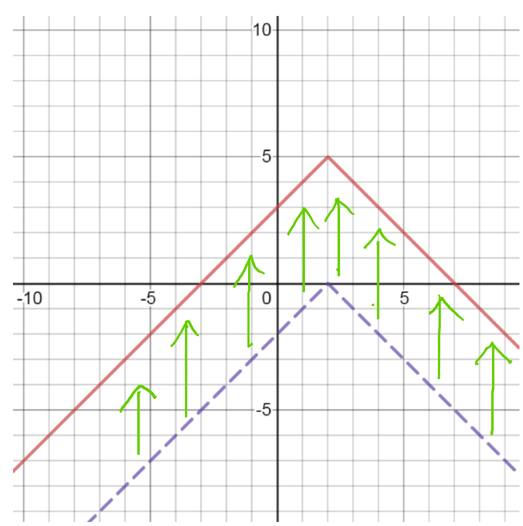
First the vertical reflection:



Then the horizontal shift:

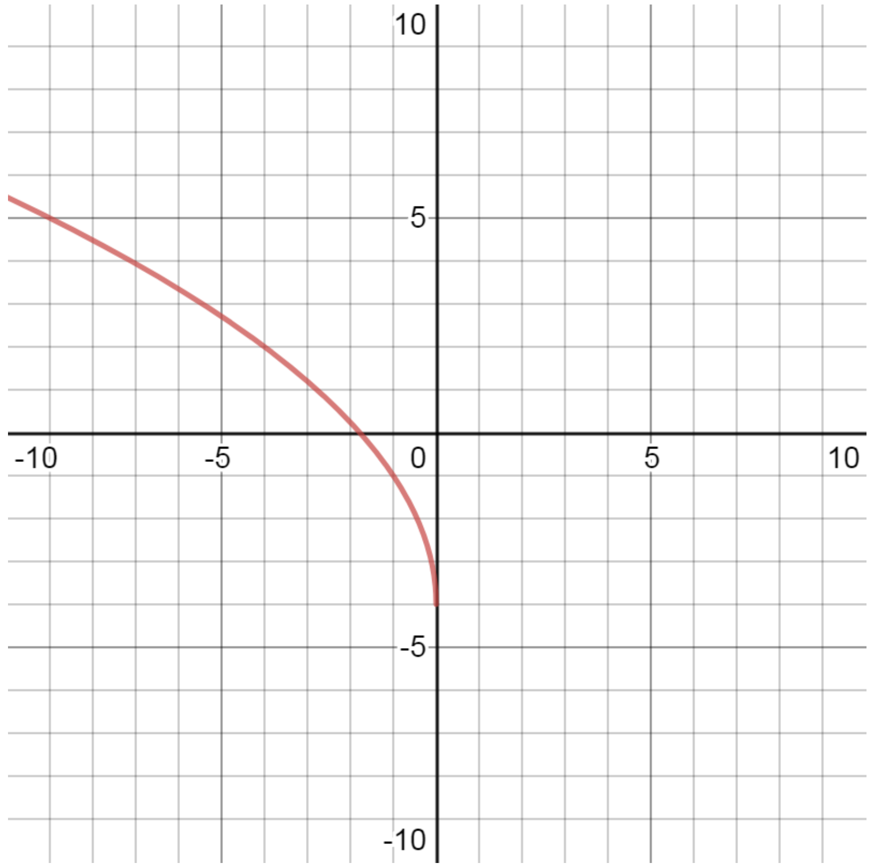


then the vertical shift:



Consider the following graph:

$g(x)$



Write the formula for $g(x)$.

To find the formula, we need to first identify the original (parent) function.

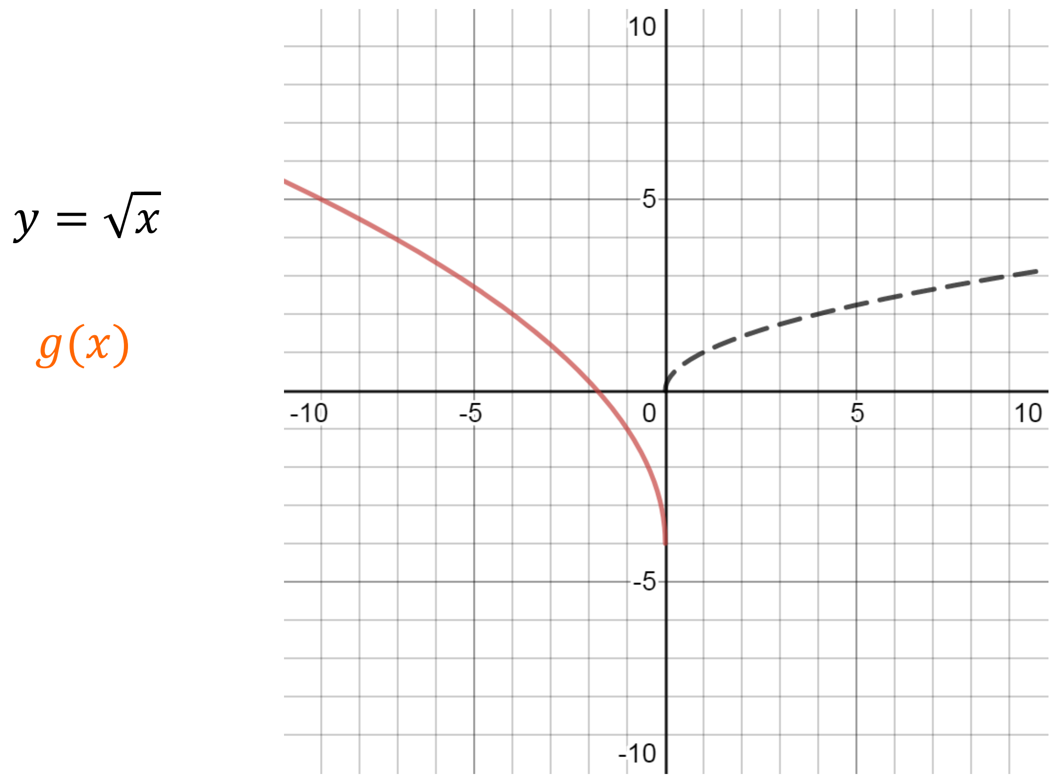
This is of course,

$$y = \sqrt{x}$$

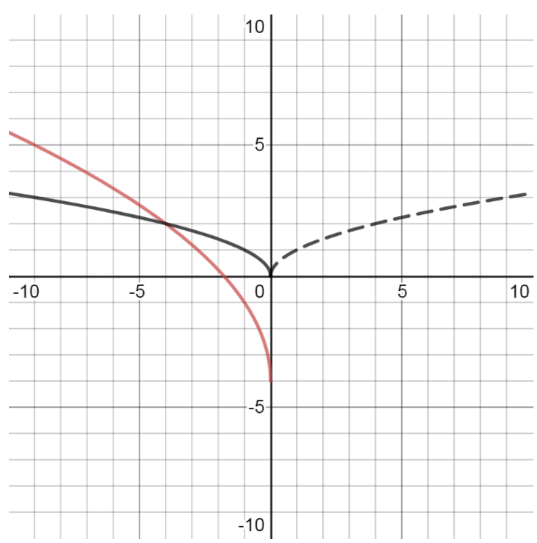
Next, what transformations have been applied to it?

Answer:

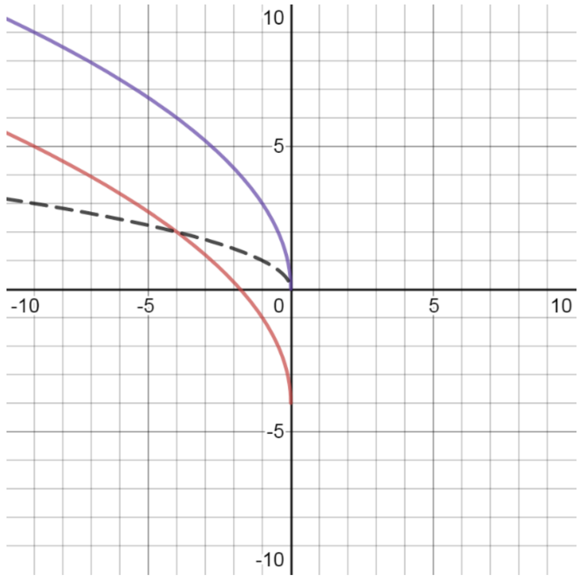
Consider the transformed function $g(x)$ on the same graph with $y = \sqrt{x}$:



Maybe the first transformation that you notice is a **horizontal reflection**:



Then notice that the (reflected) graph has been **vertically stretched**:



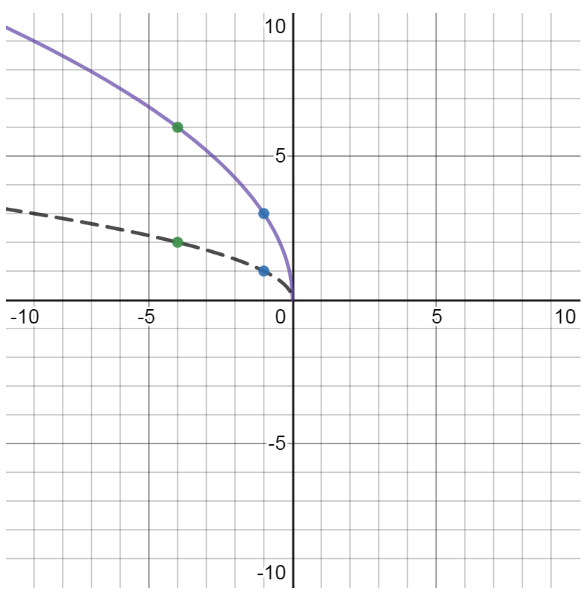
Remember, all vertical stretches (or shrinks) are determined by a . . .

. . . **multiplicative factor.**

To see what it is, take any y-value from the stretched graph . . .

. . . and divide it by the corresponding y-value from the pre-stretched graph.

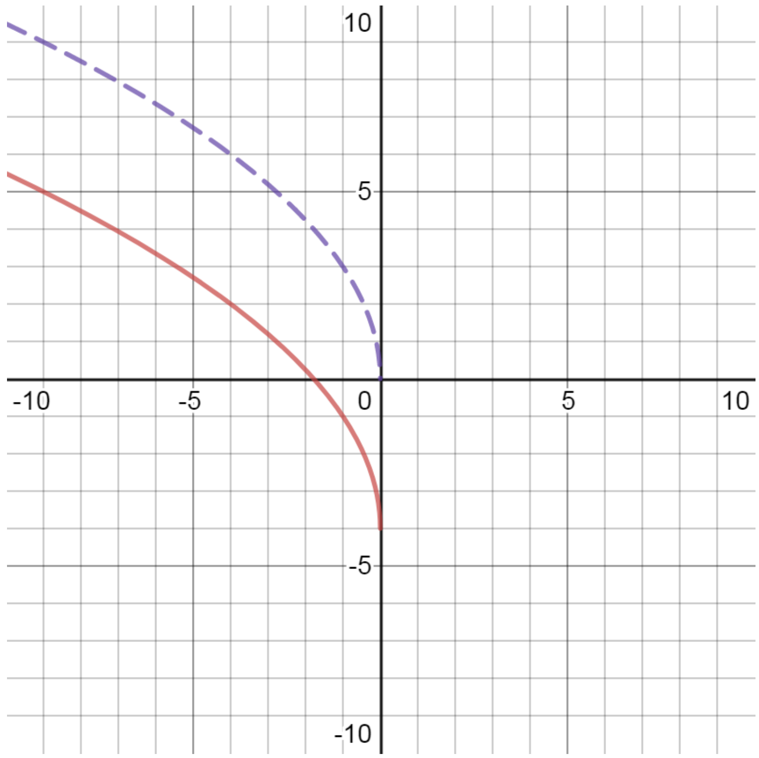
Or, simply, figure out **how many times** the y-values were multiplied by.



Finally, we have a vertical shift (down) 4:

$y = 3\sqrt{-x}$

$g(x)$



So we have our formula for $g(x)$ must incorporate

- ✓ Horizontal reflection
- ✓ Vertical stretch (times 3)
- ✓ Vertical shift (minus 4)

We get:

$$g(x) = 3\sqrt{-x} - 4$$