Finding zeroes of polynomial functions by trial and error

Problem: find the zeroes of

$$f(x) = x^3 - 5x^2 + 2x + 8$$

How to do this?

We can't factor $f(x)$ using any of the methods we've developed before!

Maybe we will just have to try out different possibilities!!!!

Let's guess $x = 1$!

$$\begin{aligned} f(1) &= (1)^3 - 5(1)^2 + 2(1) + 8 \\ &= 1 - 5(1) + 2 + 8 \\ &= 1 - 5 + 2 + 8 \\ &= 6 \end{aligned}$$

Nope! $f(1) = 6 \neq \mathbf{0}$!

Next let's try $x = 2$:

$$\begin{aligned} f(2) &= (2)^3 - 5(2)^2 + 2(2) + 8 \\ &= 8 - 5(4) + 4 + 8 \\ &= 8 - 20 + 4 + 8 \\ &= \mathbf{0} \end{aligned}$$

We got lucky and found one!!

Not only that, we can now ***find the other zeroes*** much more easily!!!

How?

Because if $x = 2$ is a zero of $f(x)$ . . .

. . . then $(x - 2)$ is a factor of $f(x)$!

Do you see why this is true? Think of a simpler example:

$$g(x) = x^2 - 7x + 12$$

To find the zeroes, we would factor:

$$\begin{aligned} g(x) = x^2 - 7x + 12 &= 0 \\ (x - 3)(x - 4) &= 0 \end{aligned}$$

$$x = 3 \quad x = 4$$

x = 3 is a zero

so ...

(x - 3) is a factor

This is ALWAYS true!! Here's the official rule:

The Factor Theorem:

A polynomial $f(x)$ has a factor $x - k$ if and only if $f(k) = 0$

What does this mean for our original problem?

It means **this**:

Since

$$f(x) = x^3 - 5x^2 + 2x + 8$$

and

$$f(2) = 0$$

so $(x-2)$ is a factor

(by the Factor Theorem)

we can conclude that

$$f(x) = (x - 2) * g(x)$$

where $g(x)$ is some polynomial that we can find by DIVISION!!

Why division? Let me explain . . .

Polynomials represent numbers, right?

Suppose we were trying to **factor** a big number like $\mathbf{676}$ . . .

. . . and we knew that $\mathbf{13}$ was a factor.

How would we proceed?

We would divide **13** into **676**:

$$676 \div 13 = 52$$

So,

$$676 = (13)(52)$$

We can do the same with our polynomial $f(x)$:

$$(x^3 - 5x^2 + 2x + 8) \div (x - 2)$$

Do you remember how to do polynomial division?

We start with the long-division set-up the same as with numbers:

$$x - 2 \;\overline{)x^3 - 5x^2 + 2x + 8}$$

Then we divide the first term into the first term:

$$\begin{array}{r|l} & x^2 \\ \hline x-2 & x^3-5x^2+2x+8 \end{array}$$

$x \cdot (x^2) = x^3$

And then multiply this quotient by the divisor:

$$\begin{array}{r|l} & x^2 \\ \hline x-2 & x^3-5x^2+2x+8 \\ & x^3-2x^2 \end{array}$$

Finally, we subtract the lower line from the upper line. Be carefull!! To subtract a negative, we are actually adding!!

we are subtracting by changing the signs and adding

$$\begin{array}{r|l} & x^2 \\ \hline x-2 & x^3-5x^2+2x+8 \\ & -x^3+2x^2 \\ \hline & \quad -3x^2+2x+8 \end{array}$$

And now, we repeat the process another round, dividing the leading terms:

$$\begin{array}{r|l} & x^2-3x \\ \hline x-2 & x^3-5x^2+2x+8 \\ & -x^3+2x^2 \\ \hline & \quad -3x^2+2x+8 \end{array}$$

$x \cdot (-3x) = -3x^2$

Multiplying:

$$\begin{array}{r l}
 & x^2 - 3x \\
x - 2 & \overline{)x^3 - 5x^2 + 2x + 8} \\
 & \underline{-x^3 + 2x^2} \\
 & -3x^2 + 2x + 8 \\
 & -3x^2 + 6x
\end{array}$$

And subtracting the bottom line from the top line:

$$\begin{array}{r l}
 & x^2 - 3x \\
x - 2 & \overline{)x^3 - 5x^2 + 2x + 8} \\
 & \underline{-x^3 + 2x^2} \\
 & -3x^2 + 2x + 8 \\
 & \underline{+3x^2 - 6x} \\
 & -4x + 8
\end{array}$$

Then, finally, one last round of dividing leading terms:

$$\begin{array}{r l}
 & x^2 - 3x - 4 \\
x - 2 & \overline{)x^3 - 5x^2 + 2x + 8} \\
 & \underline{-x^3 + 2x^2} \\
 & -3x^2 + 2x + 8 \\
 & \underline{+3x^2 - 6x} \\
 & -4x + 8
\end{array}$$

Multiplying:

$$
\begin{array}{r l}
 & x^2 - 3x - 4 \\ \cline{2-2}
x - 2 & \big)\, x^3 - 5x^2 + 2x + 8 \\
 & -x^3 + 2x^2 \\ \cline{2-2}
 & \qquad -3x^2 + 2x + 8 \\
 & \qquad +3x^2 - 6x \\ \cline{2-2}
 & \qquad\qquad -4x + 8 \\
 & \qquad\qquad {\color{red}-4x + 8}
\end{array}
$$

And subtracting . . .

$$
\begin{array}{r l}
 & x^2 - 3x - 4 \\ \cline{2-2}
x - 2 & \big)\, x^3 - 5x^2 + 2x + 8 \\
 & -x^3 + 2x^2 \\ \cline{2-2}
 & \qquad -3x^2 + 2x + 8 \\
 & \qquad +3x^2 - 6x \\ \cline{2-2}
 & \qquad\qquad -4x + 8 \\
 & \qquad\qquad -4x + 8 \\ \cline{2-2}
 & \qquad\qquad\quad {\color{red}0}
\end{array}
$$

Getting a **zero remainder**!!!

But we knew we would . . .

. . . because we knew that $x - 2$ was a **factor**!

meaning it divides in evenly

So what we have is that

$$f(x) = x^3 - 5x^2 + 2x + 8$$
$$= (x-2)(x^2 - 3x - 4)$$

And our remaining factor is a quadratic . . .

. . . so we can try to factor it!!!

$$f(x) = x^3 - 5x^2 + 2x + 8$$
$$= (x-2)(x^2 - 3x - 4)$$
$$= (x-2)(x-4)(x+1)$$

And now we have all our zeroes:

$$x = 2 \quad x = 4 \quad x = -1$$

MORAL OF THE STORY:

Finding *one zero* (by chance) can lead to finding ***more zeroes*** . . .

. . . using **division**!

ONE DRAWBACK:

Polynomial long division takes a long time!

yeah it does!

ANOTHER DRAWBACK:

Finding a zero **by chance** could take a long time!!

for sure!

GOOD NEWS:::::

There is a way to make both of these problems MUCH easier!!!!!

It's called . . .

Synthetic Substitution

Also known as . . .

Synthetic Division

you're gonna like this

It goes by these two different names for a good reason . . .

. . . (that we will discover soon)