Solving Trigonometric Equations

The final topic of this course involves solving a new kind of equation:

$$2\sin\theta - 1 = 0$$

To figure this out, we need to combine our understanding of **algebra** . . .

. . . with our understanding of **trigonometry**!

With these equations, the algebra always comes first.

We want to find θ.

So what would you do first to begin isolating for θ?

First get the $\sin\theta$ by itself:

$$2\sin\theta - 1 = 0$$

$$2\sin\theta = 1$$

$$\sin\theta = \frac{1}{2}$$

At this point, we might try to get θ by canceling the sine, using $\sin^{-1}x$:

$$\sin^{-1}(\sin\theta) = \sin^{-1}(\frac{1}{2})$$

There's only one problem to this.

Which is that the function $\sin^{-1} x$ doesn't actually exist.

Not exactly.

See, we discussed the inverse sine function in week 10, and there we created a function . . .

$$y = \arcsin\,(x)$$

that sort of worked as an inverse sine.

But it was defined to have a specific range:

$$\left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$$

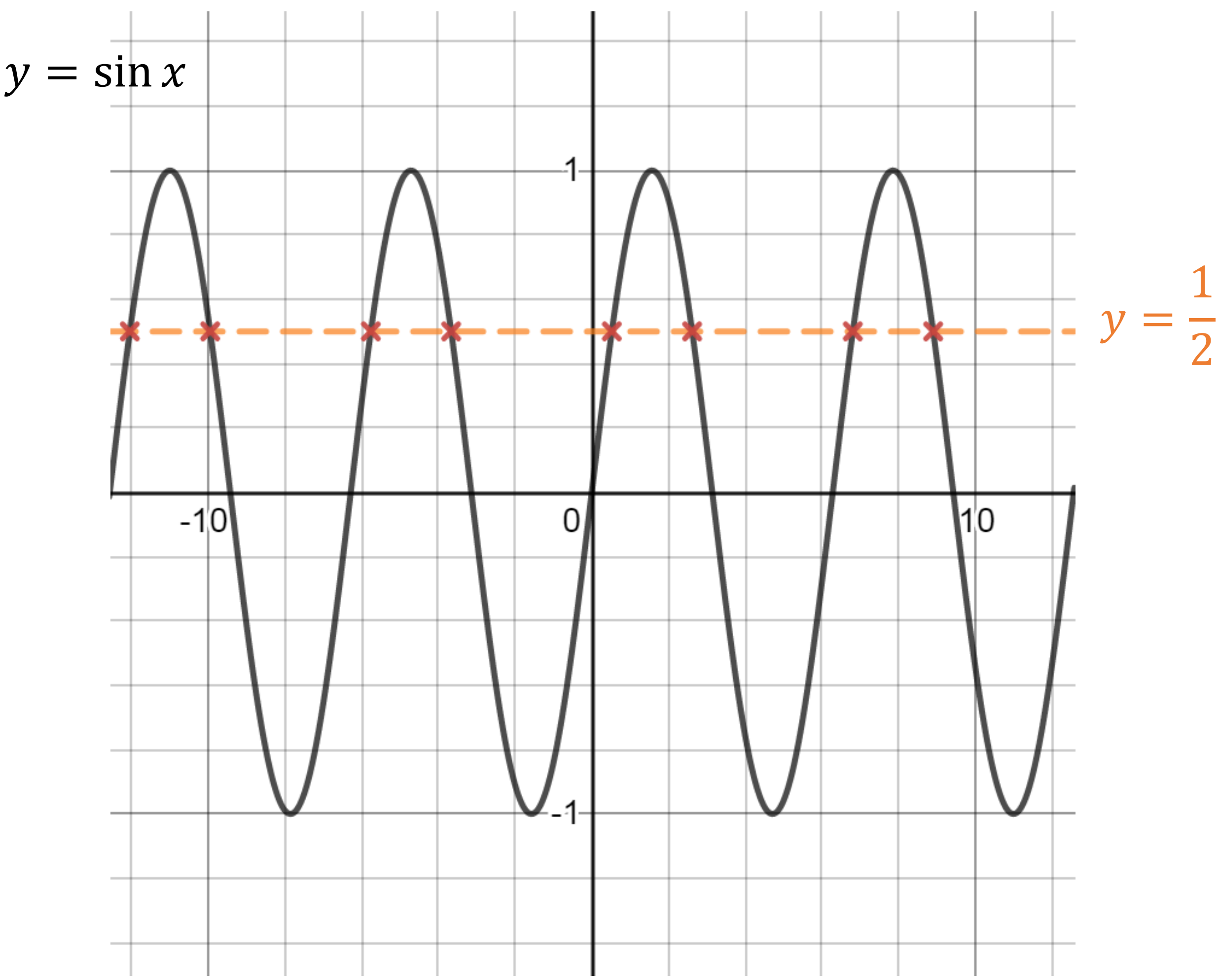
Because there are a lot of angles θ that produce any given sine value.

This is complicated, so let's go to the graph to review this situation, and put it together with our current problem.

We are currently trying to figure out the solution to

$$\sin\theta = \frac{1}{2}$$

By looking at the graph of sine, we can see that there are in fact many solutions to this equation:



Can you see that all of these points represent places where

$$\sin\theta = \frac{1}{2} \quad \textbf{?}$$

You need to understand this first, to understand these problems!

Take a moment to absorb this.

To most trigonometric equations like this . . .

. . . there will be an infinite number of solutions!!!!

But at least there is a pattern!

We will need to understand that pattern.

What is it?

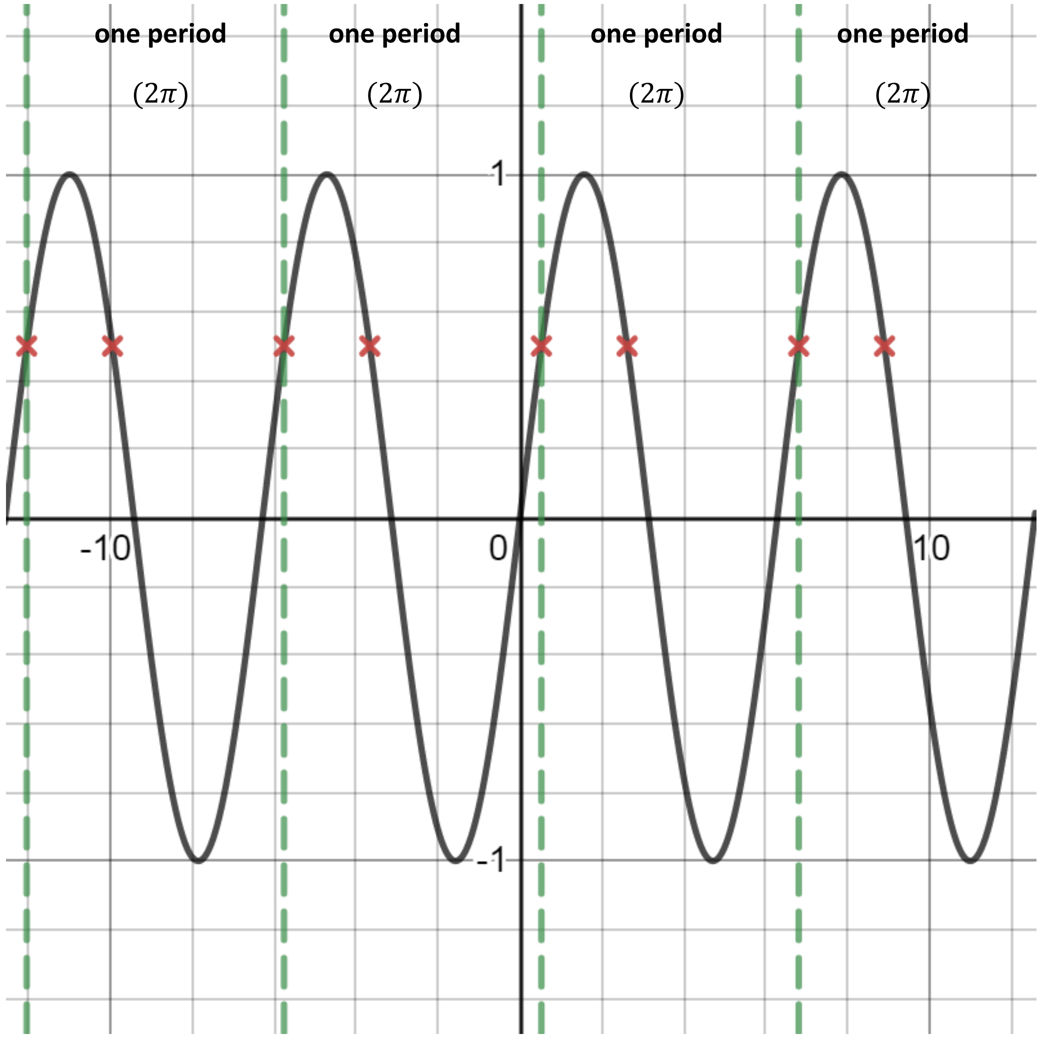
The solutions shown in the graph seem to come in pairs . . .

. . . but it's better to think of them as groups . . .

. . . separated by the **period** of the function!

Let's go to the graph again to show what I mean:

$y = \sin x$



Can you see that all of these vertical lines cross the intercepts . . .

. . . on the ***upward*** part of the sine function?

All of these points are separated from each other by a **full** period . . .

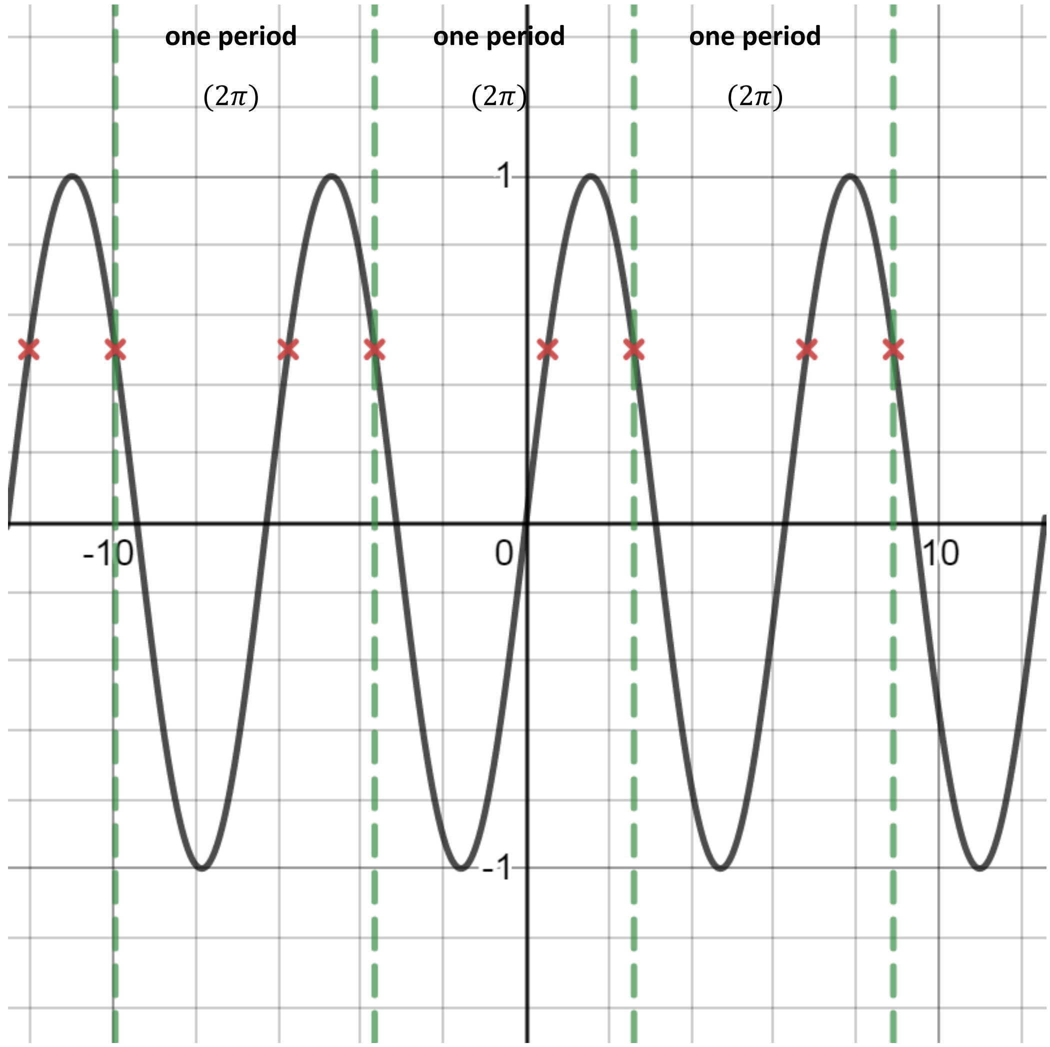
. . . which for this function is 2π.

If we could find **one** of these solutions . . .

. . . we would **know all the others**

Now let's look at the other group of solutions:

$y = \sin x$



Notice that these intercepts all happen on the . . .

. . . ***downward*** part of the graph.

They are all separated from each other by a full period of 2π . . .

. . . so if were to find one of them, we would know them all!!

Here is how we would write these two (infinite) groups of solutions:

$$\theta_1 + 2\pi * k$$

$$\theta_2 + 2\pi * k$$

Where k is any integer

See if you can understand how I came up with the above expressions.

Do you get it?   If not, let me explain . . .

Remember how we had two groups of solutions . . .

. . . and if you knew one . . .

. . . you knew them all?

$$\theta_1 \quad \text{and} \quad \theta_2$$

represent any angle in each of the categories. When you add

$$2\pi * k$$

you are including all the other angles separated by one period . . .

. . . two periods . . .

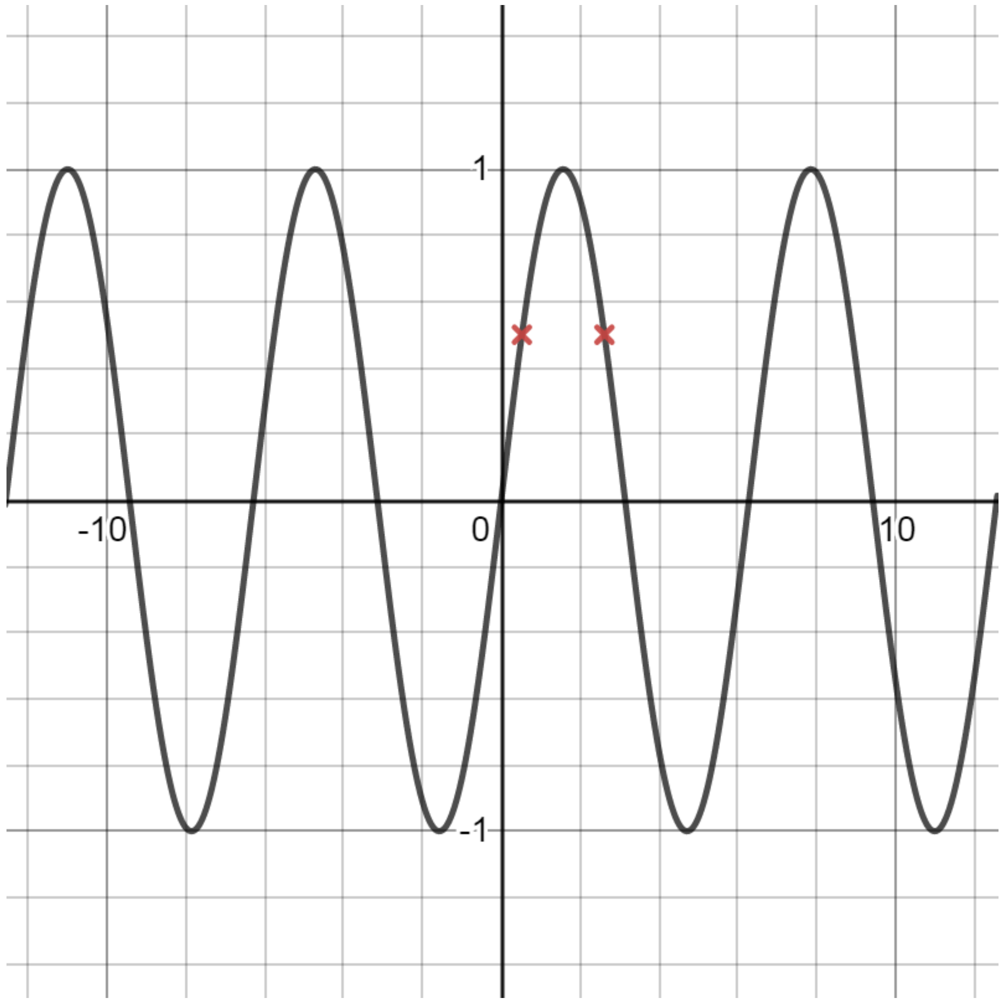
. . . three periods . .

.

And so on! Adding $2\pi * k$ gives us the **general solution**.

Here would be the best (easiest to find) versions of θ_1 and θ_2:

$y = \sin x$



They are the easiest to find because they are angles between

$$0 \text{ and } 2\pi$$

Which is to say between

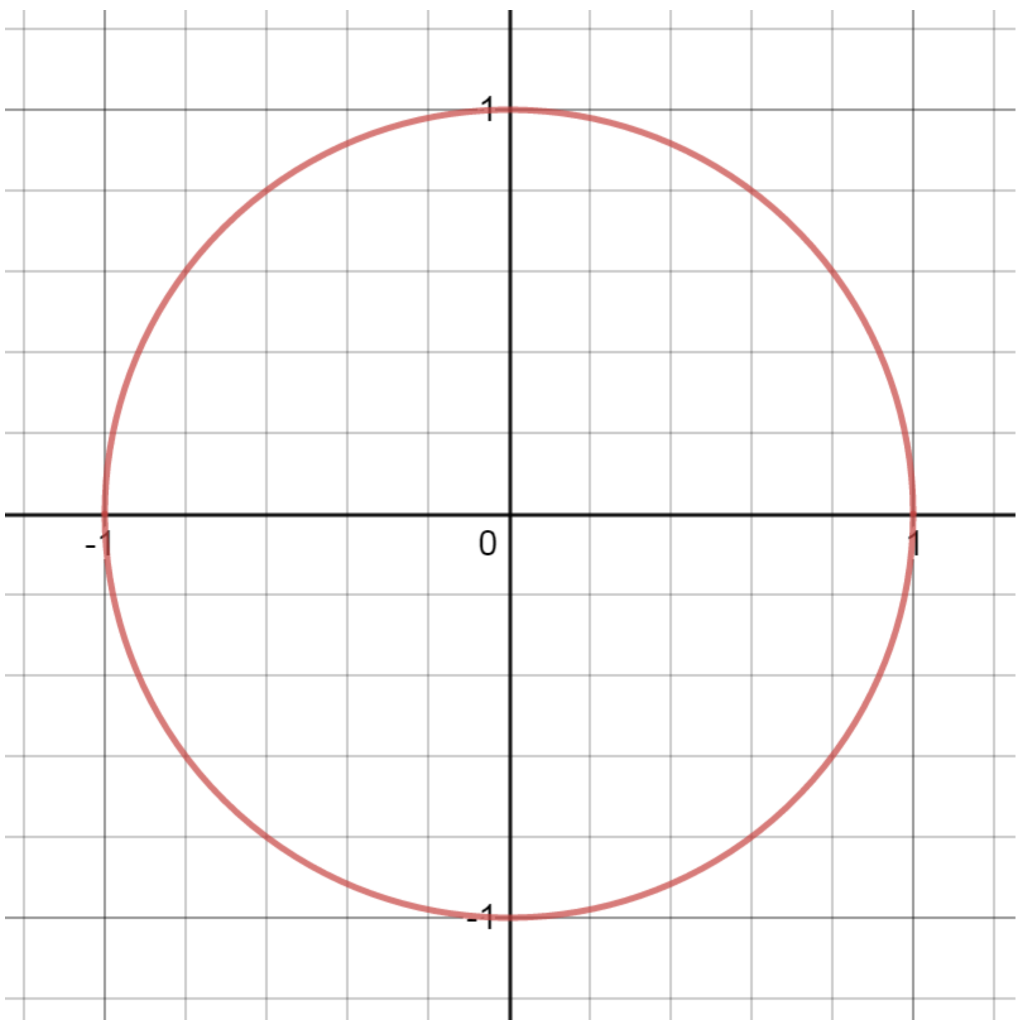
$$0° \text{ and } 360°$$

We just need to figure out what those two angles are!!!

We are trying to find out the **two** angles, $0 \leq \theta < 2\pi$, such that

$$\sin\theta = \frac{1}{2}$$

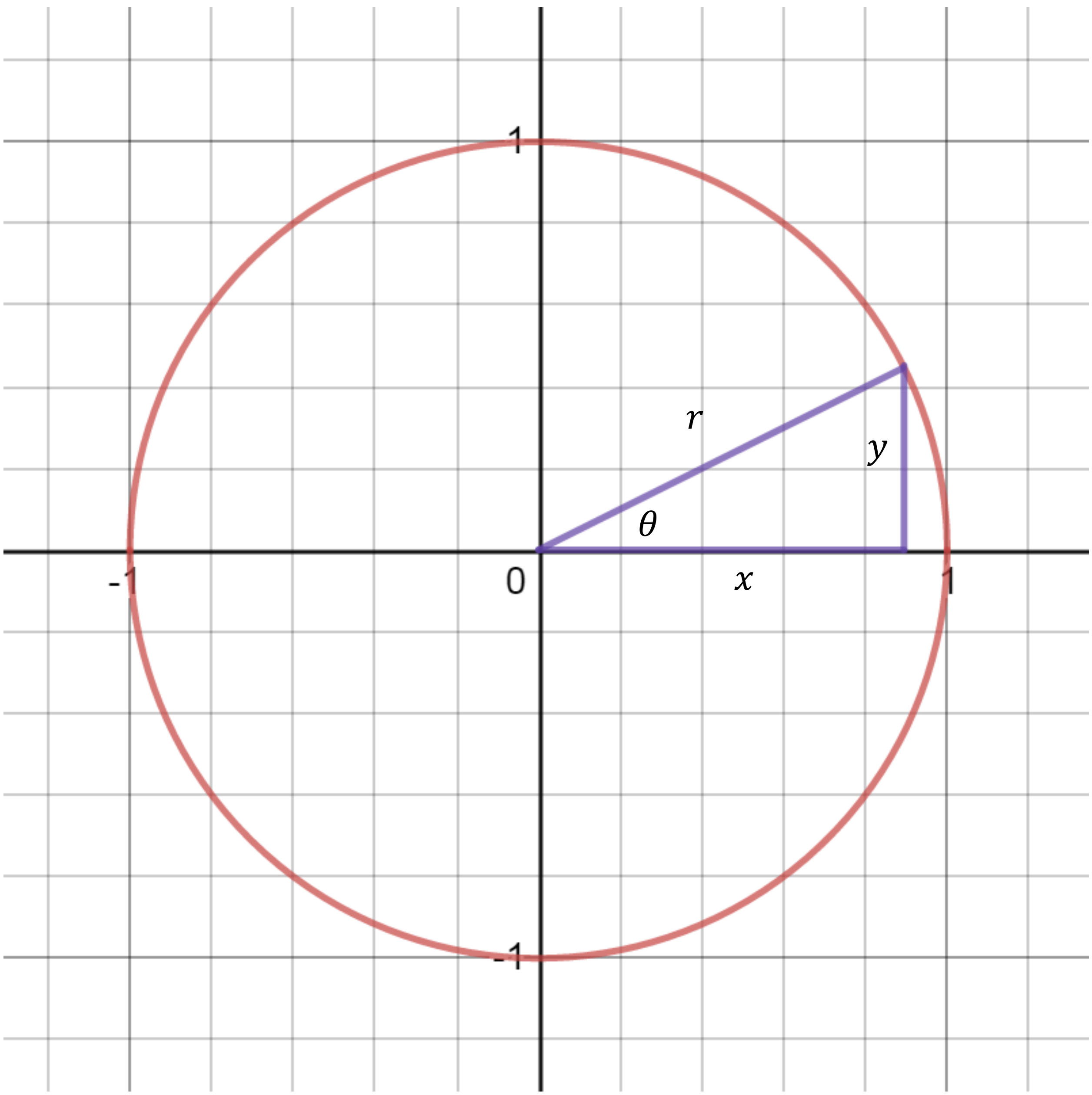
Let's try to find them by looking back at the unit circle:



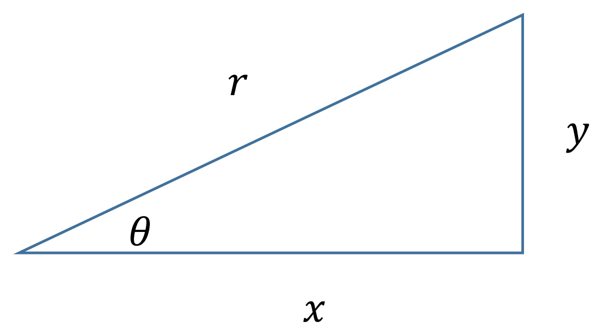
First, we remember what sine means:

$$\sin\theta = \frac{y}{r}$$

So let's take a random angle and incorporate these variables into our circle:



As you can see, the sine comes down to two sides of that right triangle:

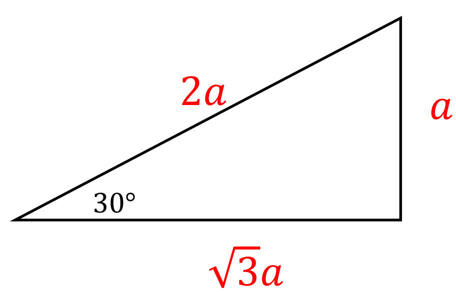


We want to find the angle θ that makes

$$\frac{y}{r} = \frac{1}{2}$$

That ratio might be familiar!!!

We saw it first when we were looking at a particular special triangle:



This was the special triangle we called the 30-60-90 triangle.

We have that

$$\sin 30° = \frac{y}{r} = \frac{a}{2a} = \frac{1}{2}$$

So 30° is one of our angles!

Of course

$$30° \equiv \frac{\pi}{6} radians$$

So we have that

$$\theta_1 = \frac{\pi}{6}$$

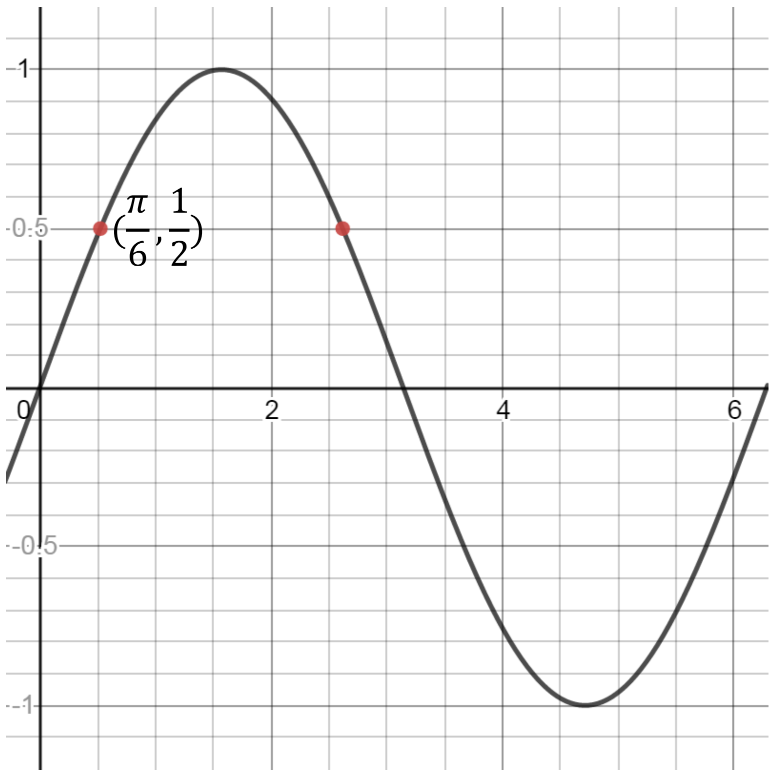
What about θ_2?

Here, we go back to our unit circle diagram and think if there are any other angles that have a sine of $\frac{1}{2}$.

Once again, that means that we want

$$\frac{y}{r} = \frac{1}{2}$$

And we know it’s not that far away from the first angle:

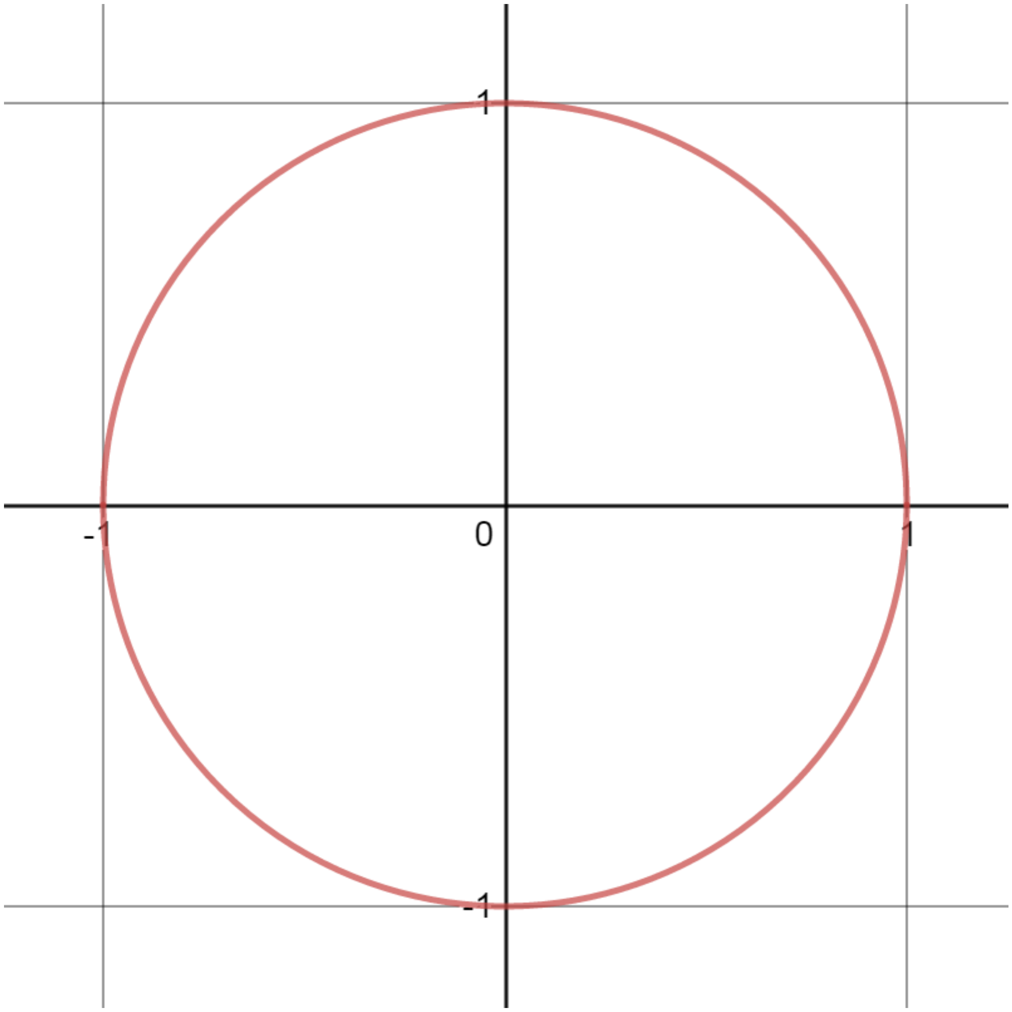


This is a zoomed-in version of the same graph we saw earlier.

Notice that the angle has to be less than π, because after θ increases past π, the sine values turn negative.

And we know that $\theta > \frac{\pi}{2}$, because the function is decreasing beyond that point (beyond 90°).

That means that this angle must be somewhere in Quadrant II:

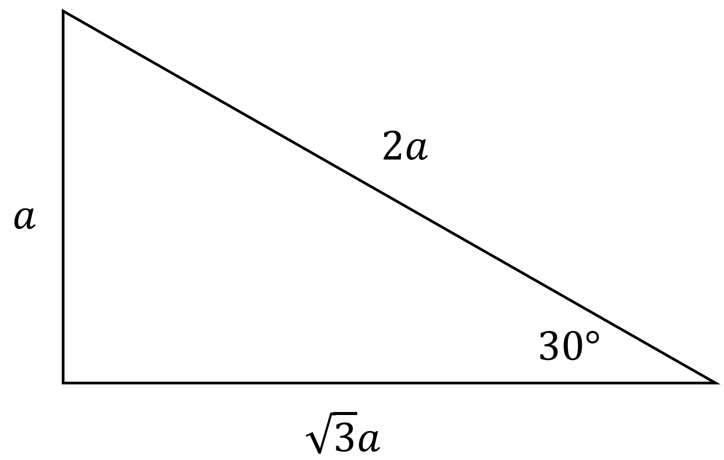


Can you put a triangle in this quadrant in which

$$\frac{y}{r} = \frac{1}{2} \quad ?$$

HINT: it's the ***same ratio between sides*** of a triangle as in θ_1!

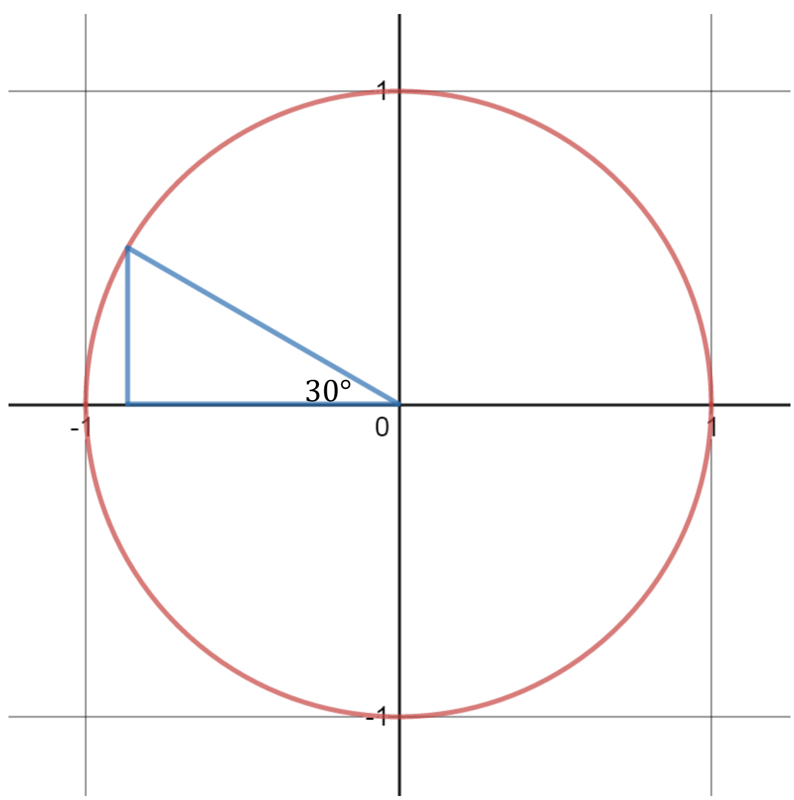
It has to be another 30-60-90 triangle!



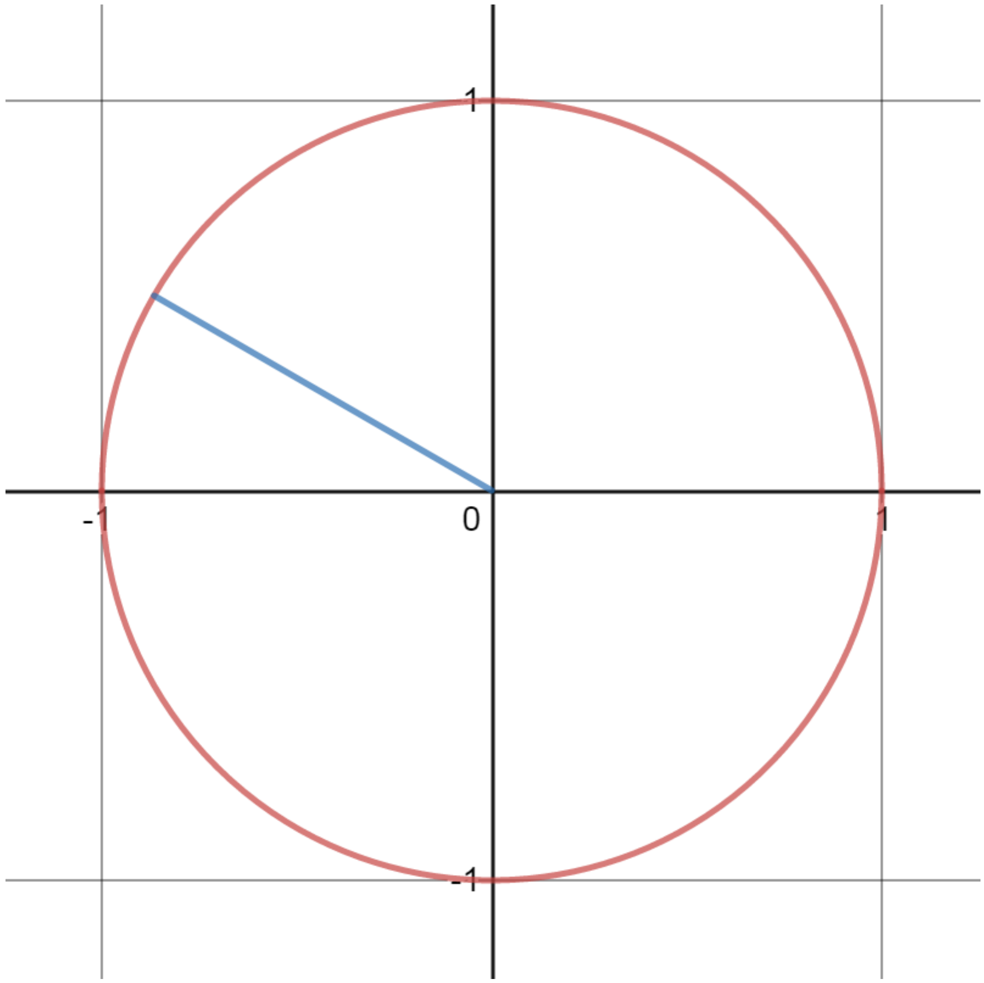
These are the only triangles where the shortest side . . . .

. . . is $\frac{1}{2}$ of the longest side!

Here is where it would appear in our unit circle:

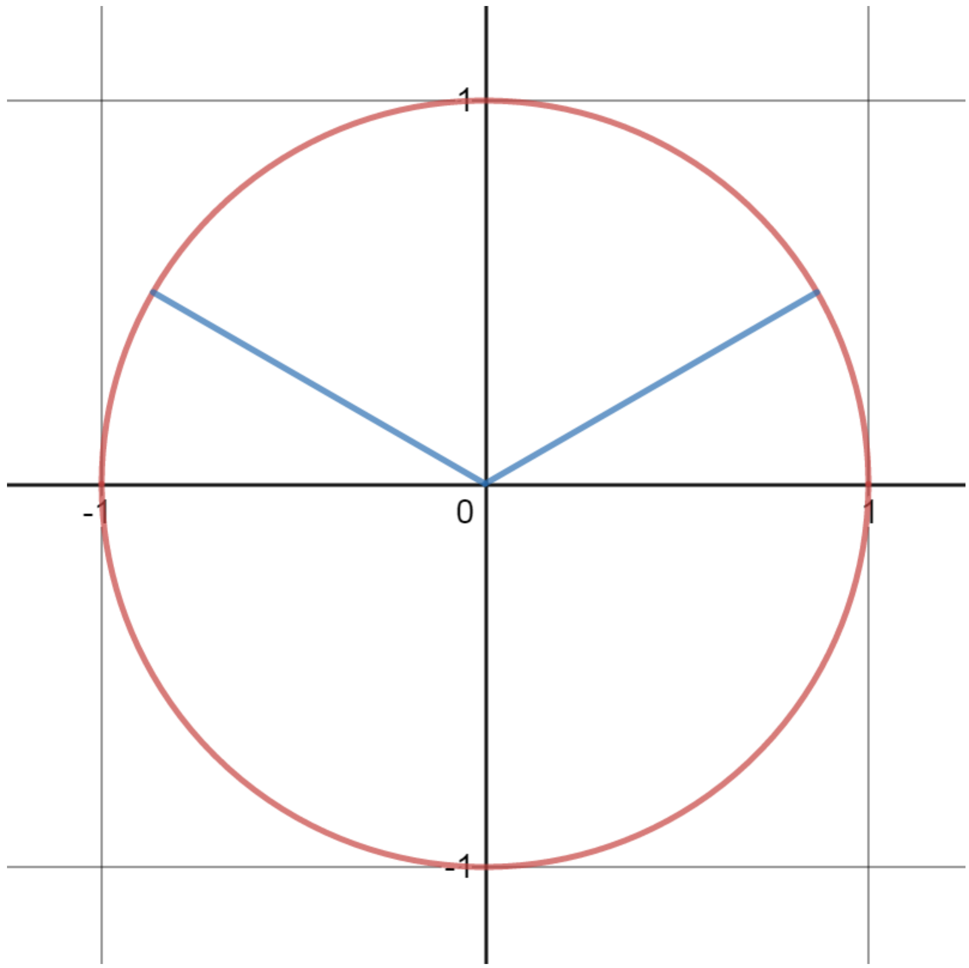


But of course we need to find the actual angle of that terminal side:



Which is 150°, otherwise known as $\frac{5\pi}{6}$ radians.

Here are θ_1 and θ_2 as they appear in our circle:



Once again, these are the only angles between 0 and 2π such that

$$\sin\theta = \frac{y}{r} = \frac{1}{2}$$

And the way we figured them out was by knowing that the ratio of the small side to the diagonal of a 30-60-90 triangle is $\frac{1}{2}$.

So our answer to the problem is

$$\frac{\pi}{6} + 2\pi * k$$

$$\frac{5\pi}{6} + 2\pi * k$$

WHEW!!!

That took a while!!!