Percentage Decrease and Exponential Decay

Here's a question:

Does exponential change always mean that the numbers increase?

Since exponential change comes from percent change, the same question could be put this way:

Can numbers ever decrease by a percent?

The answer is YES.

Here's an example:

The value of a car was $25000 brand new in 2015. One year later, that car's value had depreciated to $20000. If the car continues to depreciate at the same rate, predict the car's value in 2025.

Here, we have a situation in which a car's value is depreciating over time because the car is becoming older and more used. The problem suggest that the car will continue to depreciate **at the same rate**. But what does that mean?

The reason this question is important, is that we have seen two different kinds of rates in this course.

One is **rate of change** in a ***linear*** formula.

Here the **change** is **constant**:

$$m = slope = \frac{change\ in\ value}{change\ in\ time} = \frac{20000 - 25000}{1} = -5000\ per\ year$$

If we let

$$V = value\ of\ the\ car$$

$$t = \#\ of\ years\ since\ 2015$$

And using our basic linear formula:

$$y = mx + b$$

Which with our variables becomes

$$V = mt + b$$

Given that b is our starting value, we get the following formula (function):

$$V(t) = -5000t + 25000$$

linear rate of change
(slope)

This works with our data, since $V(1) = -5000(1) + 25000 = \$20,000$

Which was what was supposed to be the value in 2016.

However, we have a problem here!!!!

Look what happens when we try to predict the value of the car in 2025:

This is 10 years after 2015, so $t = 10$.

We get:

$$\begin{aligned} V(10) &= -5000(10) + 25000 \\ &= -50000 + 25000 \\ &= -\$25{,}000 \end{aligned}$$

???

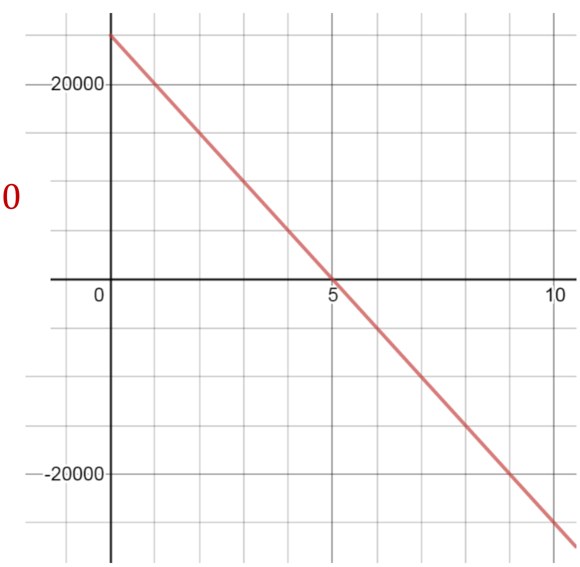
Obviously this doesn't make any sense!

The value of a car can't be negative!

If we look at the graph, we see the problem even better:

$V(t) = -5000t + 25000$

value of car



After 5 years, the car is worth \$0, and after that, the value goes into the negatives.

We need to reconceive this function.

It seems that **linear change** does not work for depreciation!

Instead, we will use **percent change** to explain the same data!

If the value went from \$25000 to \$20000 in 1 year, it decreased by \$5000.

We will measure this change as a ***percent of the original amount***:

$$percent\ change = -\frac{\$5000}{\$25{,}000} = 0.2 = 20\%$$

Instead of thinking of the car's value decreasing by \$5000 per year . . .

. . . we will think of the car's value decreasing by 20% per year . . .

So we can use our general formula for percentage change:

$$A(t) = A_0(1+r)^t$$

Where

$$V_0 = 25{,}000$$

$$r = -0.2$$

−20% as a decimal

Giving us the following function:

$$V(t) = 25{,}000 * (1 - 0.2)^t$$

$$V(t) = 25{,}000 * 0.8^t$$

Notice that the value in the formula of 0.8 makes sense because . . .

$$0.8 \leftrightarrow 80\%$$

and after each year of a 20% value decrease . . .

. . . the new value is 80% of the previous value.

Notice that this exponential formula for depreciation matches our data:

Which explains the same numbers:

linear version:

$V(t) = -5000t + 25000$

$\boldsymbol{t}$	$\boldsymbol{V(t)}$
0	25000
1	20000
2	15000
3	10000
4	5000
5	0
6	-5000

these numbers don't make sense

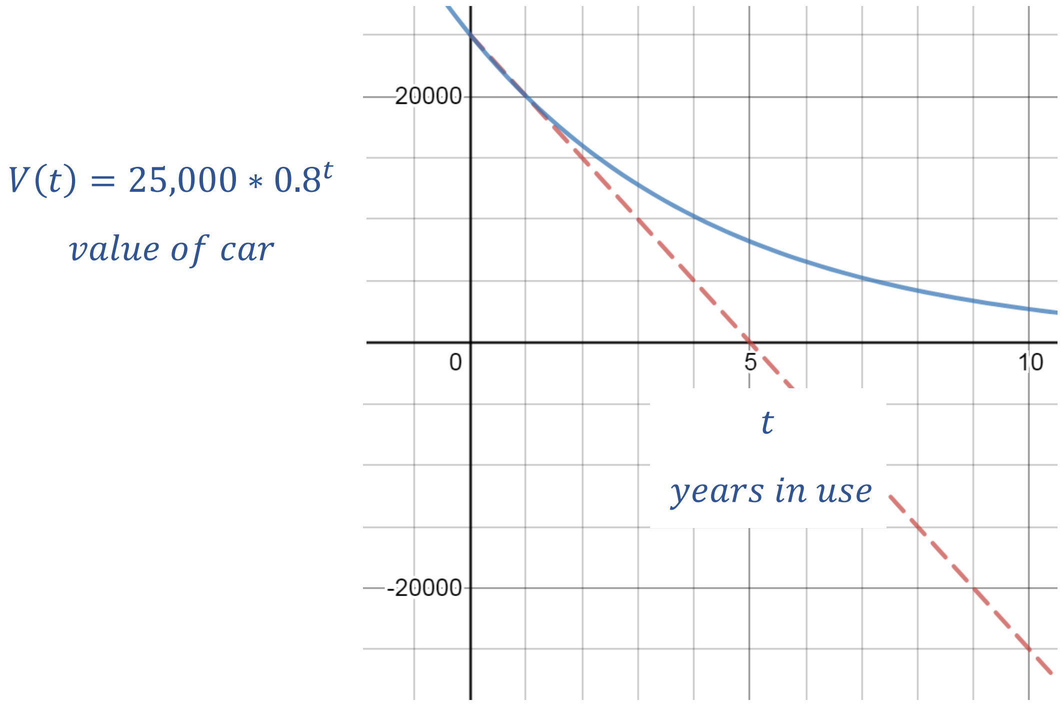
exponential version:

$V(t) = 25000 * 0.8^t$

$\boldsymbol{t}$	$\boldsymbol{V(t)}$
0	25000
1	20000
2	16000
3	12800
4	10240
5	8192
6	6554

these numbers make sense

But the exponential version of the model does not produce negative values for the car:



Just as exponential increase is often called ***"exponential growth,"***

Exponential decrease is often called ***"exponential decay."***

This is literally happening in the case of radioactive materials.

Radioactive materials have what's called a "half-life."

This is the time it takes for the radiation to decay to be half of what it was.

It's exponential change (and percent change) because half is 50%.

Many drugs are described in terms of their half-life, because most of the drug is eliminated from the body very fast, but some lingers on.

For example, the half-life of caffeine in the body is 6 hours.

If you drink a cup of coffee at 8am, how much of that caffeine is still in your system at 11pm when you go to bed?

We will define an exponential function representing the amount of that cup of coffee in your body over time. We will measure time in hours!

The problem here is that the amount of caffeine is going down by 50% every six hours, instead of every one hour. We will find a way to deal with that problem. But first, let's define the function:

$$C(t) = amount\ of\ coffee\ caffeine\ in\ the\ body\ after\ t\ hours$$

We might immediately try to apply the general formula for percentage change:

$$A(t) = A_0(1+r)^t$$

Where in this case,

$$C_0 = 1$$

$$r = -0.5$$

minus 50% as a decimal

However, this won't work, because the percentage change of 50% happens every six hours, instead of every hour.

We will have to be a bit creative!

Let's go back to basics, and think of the table of values that would result from reducing the caffeine by 50% every six hours:

$\boldsymbol{t}$ $(\boldsymbol{hours})$	$\boldsymbol{C}$ $(\boldsymbol{cups\ worth\ of\ caffeine\ left\ in\ the\ body})$
0	$1 = 1 * 0.5^0$
6	$1 * 0.5 = 1 * 0.5^1$
12	$1 * 0.5 * 0.5 = 1 * 0.5^2$
18	$1 * 0.5 * 0.5 * 0.5 = 1 * 0.5^3$

Here, in this example, the variable and the exponent are not the same.

However, if you look closely, you will see a pattern:

$t\ (hours)$	$C\ (cups\ worth\ of\ caffeine\ left\ in\ the\ body)$
0	$1 = 1 * 0.5^0$
6	$1 * 0.5 = 1 * 0.5^1$
12	$1 * 0.5 * 0.5 = 1 * 0.5^2$
18	$1 * 0.5 * 0.5 * 0.5 = 1 * 0.5^3$

this exponent is $12 \div 6 = 2$

The exponents are always one-sixth of the number of hours!

Which gives us the formula:

$t\ (hours)$	$C\ (cups\ worth\ of\ caffeine\ left\ in\ the\ body)$
0	$1 = 1 * 0.5^0$
6	$1 * 0.5 = 1 * 0.5^1$
12	$1 * 0.5 * 0.5 = 1 * 0.5^2$
18	$1 * 0.5 * 0.5 * 0.5 = 1 * 0.5^3$
t	$1 * 0.5^{\frac{t}{6}}$

And in turn, the function: $C(t) = 1 * 0.5^{\frac{t}{6}}$

At 11pm, 15 hours have elapsed from the time you drank the cup of coffee.

$$C(15) = 1 * 0.5^{\frac{15}{6}} = 0.18$$

So about 18% of the caffeine from that original cup is still in your body!