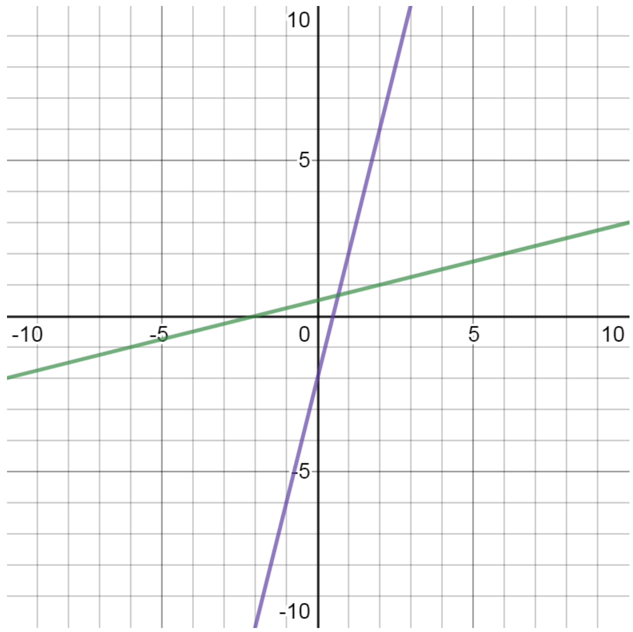
Graphs of Inverse Functions

Let's examine the graphs of both pairs of inverse functions we've recently seen:

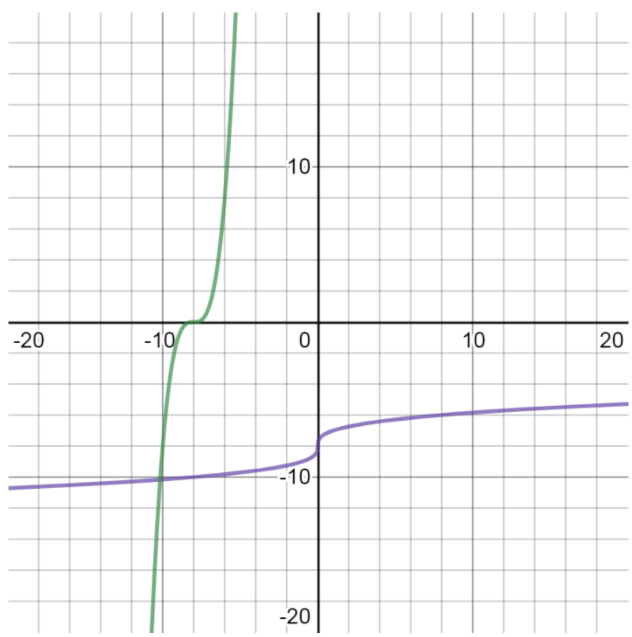
$h(x) = 4x - 2$

$h^{-1}(x) = \frac{x+2}{4}$




$q(x) = \sqrt[3]{x} - 8$

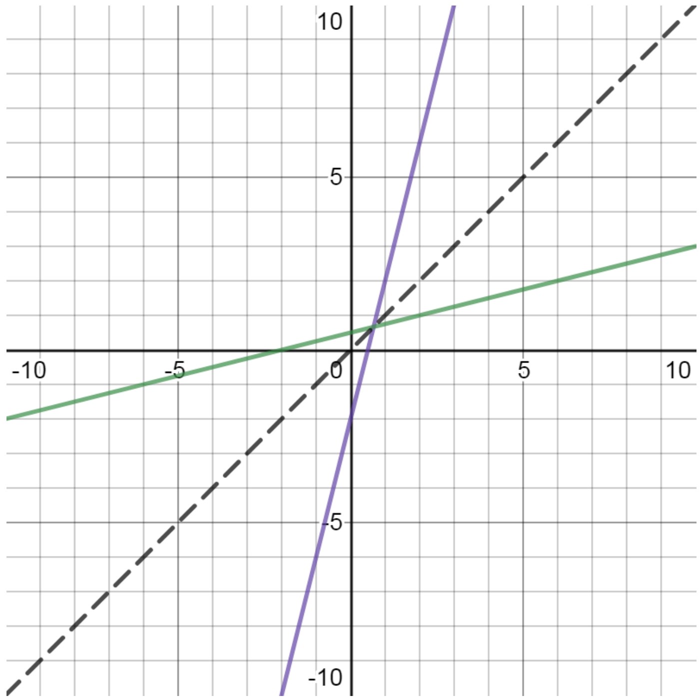
$q^{-1}(x) = (x+8)^3$

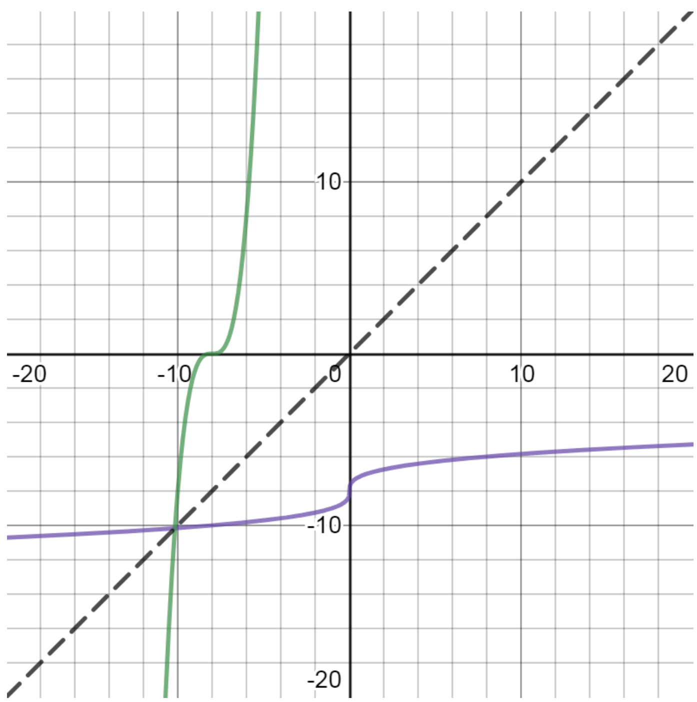



Think for a moment . . .

. . . do you see any commonalities between the two pairs of graphs?

If not, now look at them:

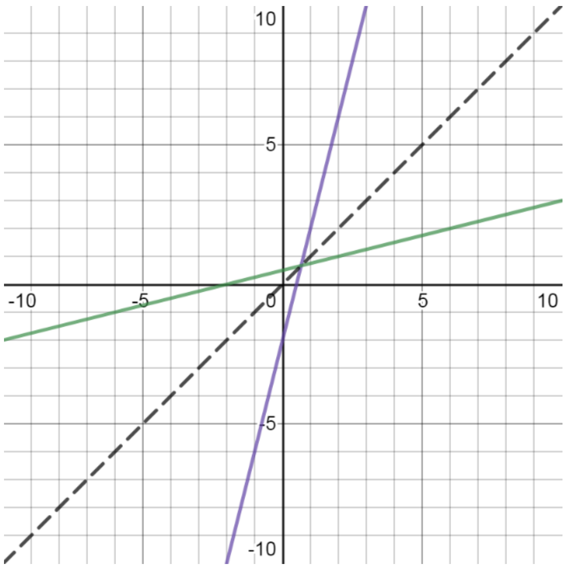


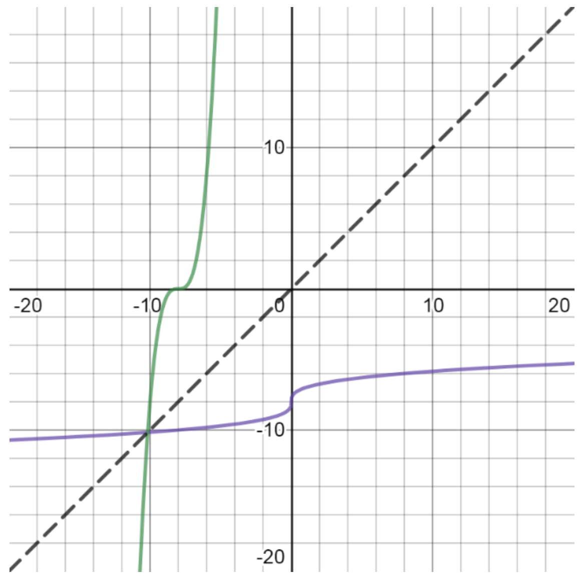



If you think about it, each pair of inverse functions are . . .

. . . ***reflections*** of each other . . .

. . . across the line $y = x$




Note that the line $y = x$ . . .

. . . is another way of saying the identity function, $f(x) = x$!

This is not a coincidence.

It's a rule:

**The graphs of inverse functions are reflections of each other across the line $y = x$.**

Domain and Range of Inverse Functions

Consider the function:

$$f(x) = -\sqrt{x}$$

What is its domain?

We considered a very similar situation a couple weeks ago!

**No negative numbers** are allowed into the function!

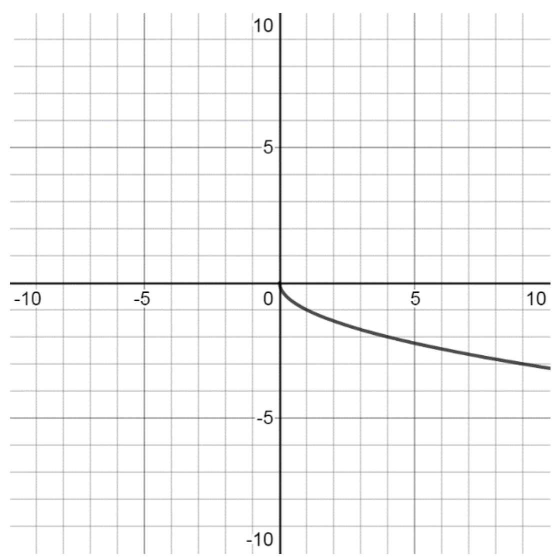
So the domain is $[0, \infty)$.

How about the range?

We can see from the formula that *only negative numbers* (and zero) come out of this function.

$$f(x) = -\sqrt{x}$$

So the range is $(-\infty, 0]$.



Now let's think about what $f^{-1}(x)$ would be! This is going to be a bit tricky!!

Why? Because we need to consider the domain and range!

Suppose we tried to find $f^{-1}(x)$ . . . the "formal" way:

$$y = -\sqrt{x}$$

$$x = -\sqrt{y}$$

$$(x)^2 = \left(-\sqrt{y}\right)^2$$

$$x^2 = y$$

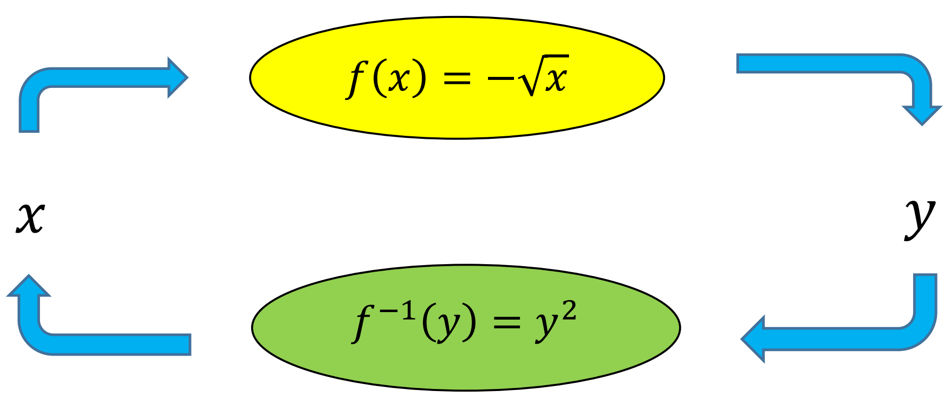
$y = x^2$

Giving us

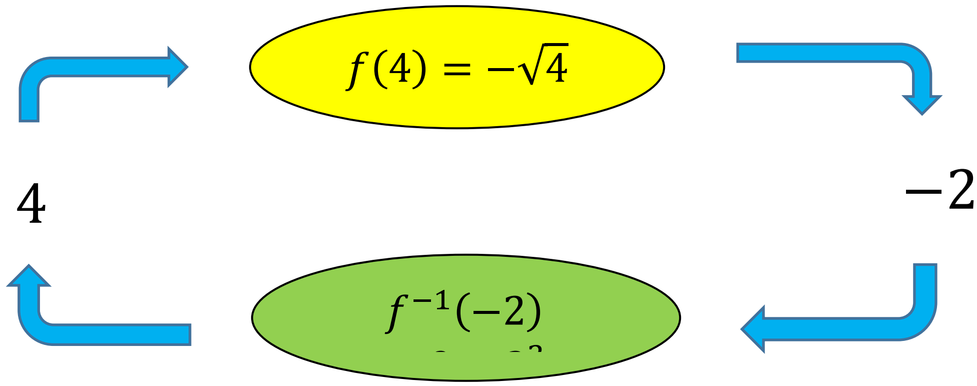
$$f^{-1}(x) = x^2$$

Everything *seems* right . . . but there's a problem!

To see it, let's consider the circular inverse function diagram:



Now suppose we plug 4 into $f(x)$:

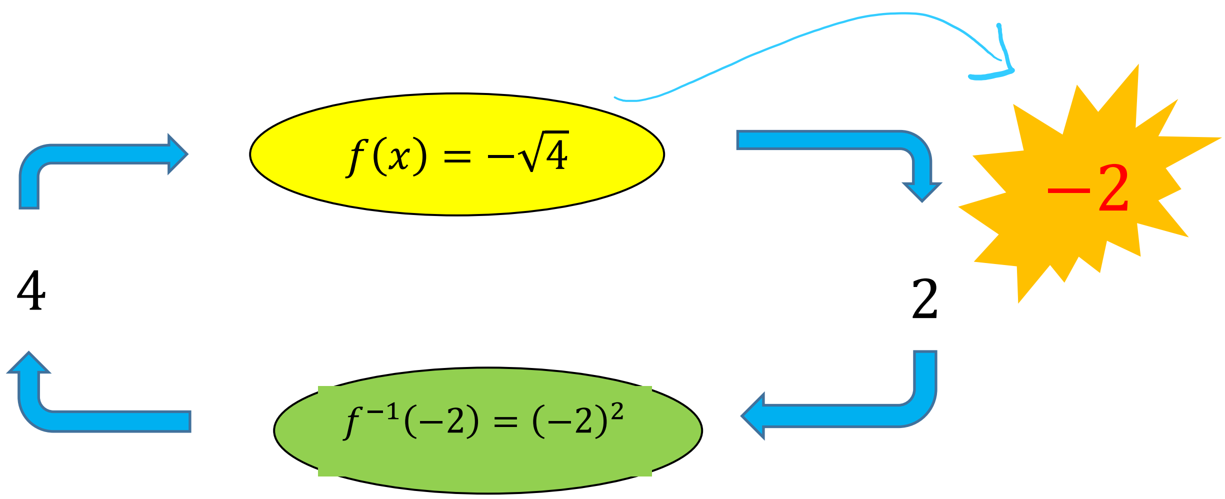



It works great!!! $f(4) = 2$ and $f^{-1}(2) = 4$.

This is the way it's supposed to work!

***However . . .***

Look what happens when you plug 2 into $f^{-1}(x)$:




Do you see the problem? We start with +2 and end up with -2!!!

*This is not how inverse functions are supposed to work!!*

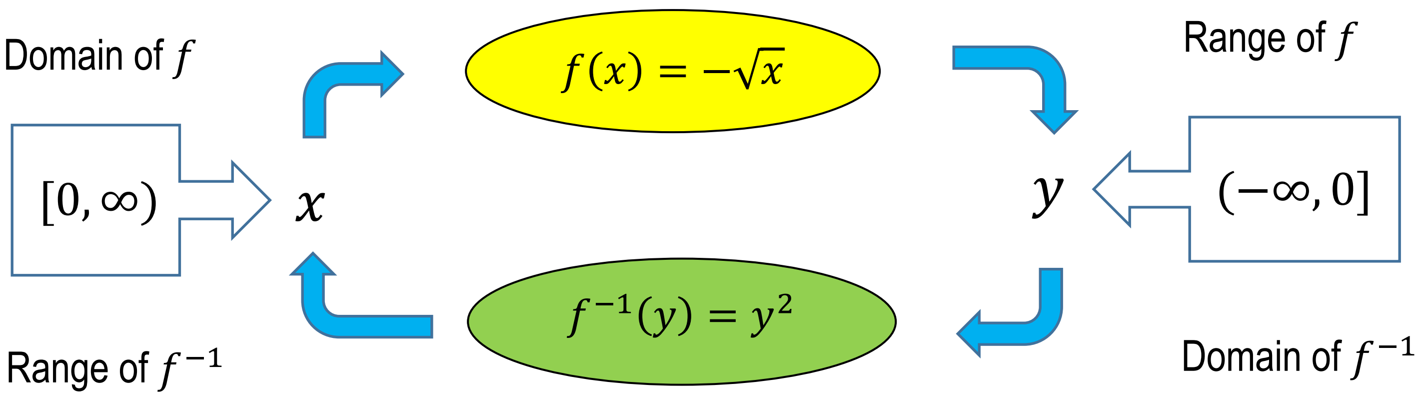
*WHAT WENT WRONG???*

To understand this, we need to look at the domain and range!

Remember that the domain for $f(x) = -\sqrt{x}$ was $[0, \infty)$.

Remember that the range for $f(x) = -\sqrt{x}$ was $(-\infty, 0]$.

Let's build this into our inverse function diagram:




The key idea here is the following:

**The domain of any function must be the range of its inverse**

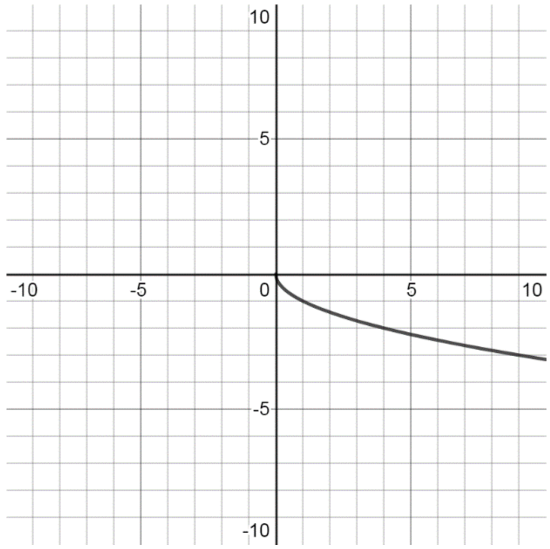
So we were not allowed to plug $x = 2$ into $f^{-1}$ . . .

. . . because 2 is not in the range of $f$!

Let's look at the graphs to understand this better!

We saw that the graph of $f(x) = -\sqrt{x}$ looked like this:

$f(x) = -\sqrt{x}$



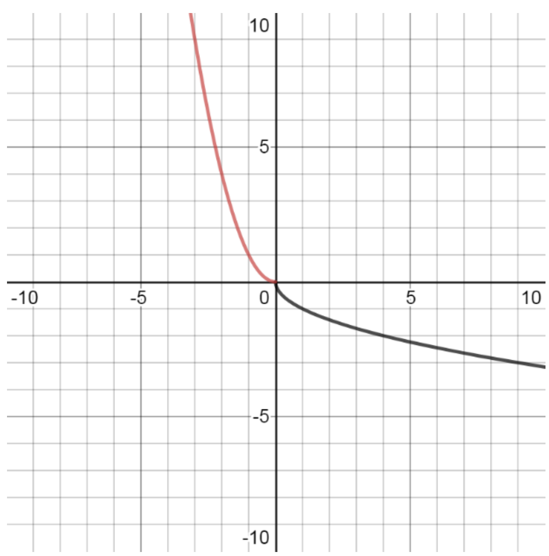

Since the range of $f(x)$ is $(-\infty, 0]$ . . .

. . . we must **restrict** the domain of $f^{-1}(x)$ to $(-\infty, 0]$

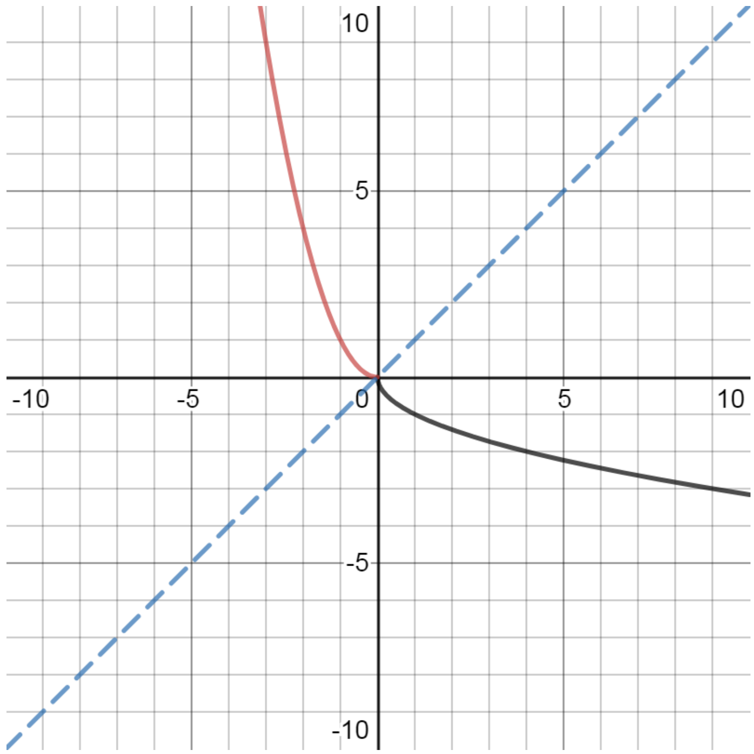
So the graph of $f^{-1}(x) = x^2$ will only exist for $x \leq 0$:

$f(x) = -\sqrt{x}$

$f^{-1}(x) = x^2$



Notice how both graphs are **reflections across the line $y = x$**:




This brings us to a final important concept of inverse functions!!!