Compound Interest

Consider the following problem:

You have $1000 to invest at 4% annually compounded interest. How much will you have in the account in 8 years?

Here's one way you might try to do the problem:

$$4\%\ of\ 1000 = 0.04 * 1000 = \$40$$

$$\frac{\$40}{year} * 8\ years = \$320$$

What do you think of this answer?

If you are skeptical, you have good reason to be. This answer is **wrong**.

The reason it's wrong is that this method did not **compound** the interest.

When you invest (or borrow) money, the interest is usually added into your total investment, so that the next time you are awarded interest, that interest is calculated as a percentage of your total accumulated investment amount.

We did not do this here; rather, we just added $40 every year to the account. This is called **simple interest**, when the interest is not compounded.

Let's try again:

After the **first** year, we are awarded 4% of our total investment as interest:

$$0.04 * 1000 = 40$$

And this amount is added to our account, giving us:

$$1000 + 40 = 1040$$

After the **second** year, we are awarded 4% of our total investment:

$$0.04 * 1040 = 41.60$$

And this amount is added to our account, giving us:

$$1040 + 41.60 = 1081.60$$

After the third year, we are awarded 4% of our total investment:

$$0.04 * 1081.60 = 43.26$$

And this amount is added to our account, giving us

$$1081.60 + 43.26 = 1124.86$$

And the process continues . . . doing this 5 more times would get us our answer.

And yet . . . this method ***takes too long!***

There's got to be a better way!

We would like to develop a **formula**.

And to find that formula, we need to see this process a bit more simply.

Let's look at what we have so far:

$years$	$amount\ in\ account$
0	1000
1	$1000 \quad + \quad 0.04 * 1000 \quad = 1040$
2	$1040 \quad + \quad 0.04 * 1040 \quad = 1081.60$
3	$1081.60 + \quad 0.04 * 1081.60 = 1124.86$

Notice that each step of the process, the same calculation happens:

$$A_{n+1} = A_n + 0.04 * A_n$$

Can you see that?

Now what could you do, algebraically, with the expression

$$A_n + 0.04 * A_n$$

To simplify it?

this is just like factoring

If you're not sure, try factoring:

$x + 6x$

$= x(1+6)$

$$A_n + 0.04 * A_n = A_n * (1 + 0.04)$$

Here, I just factored out the greatest common factor. Now I simplify:

$$A_n * (1 + 0.04) = A_n * 1.04$$

because $1+0.04$

$= 1.04$

Check out what this means! Putting this together, we have:

$$A_{n+1} = A_n * 1.04$$

In other words, every time a year goes by and we get our interest . . .

. . . our new investment amount is **1.04 times** our previous amount!

Now we can look at the process in a more simplified way:

$time\ (years)$	$amount\ in\ account$
$\mathbf{0}$	1000
$\mathbf{1}$	$1000 * 1.04$
$\mathbf{2}$	$1000 * 1.04 * 1.04$
$\mathbf{3}$	$1000 * 1.04 * 1.04 * 1.04$

Or, writing the computation even more simply (using exponents):

$time\ (years)$	$amount\ in\ account$
$\mathbf{0}$	1000
$\mathbf{1}$	$1000 * 1.04$
$\mathbf{2}$	$1000 * 1.04^2$
$\mathbf{3}$	$1000 * 1.04^3$
t	$1000 * 1.04^t$

Which can be written as a function:

$$A(t) = amount\ in\ account\ after\ t\ years$$

$$A(t) = 1000 * 1.04^t$$

We can use this function to answer the original problem:

After 8 years, the investment will have grown to be

$$A(8) = 1000 * 1.04^8 = 1368.56905$$

or

$$\$1368.57$$

which is a bit more (though not that much more) than the $1320 we calculated using simple interest!

The compound interest process has produced a totally **new** kind of function!

Because

$$A(t) = 1000 * 1.04^t$$

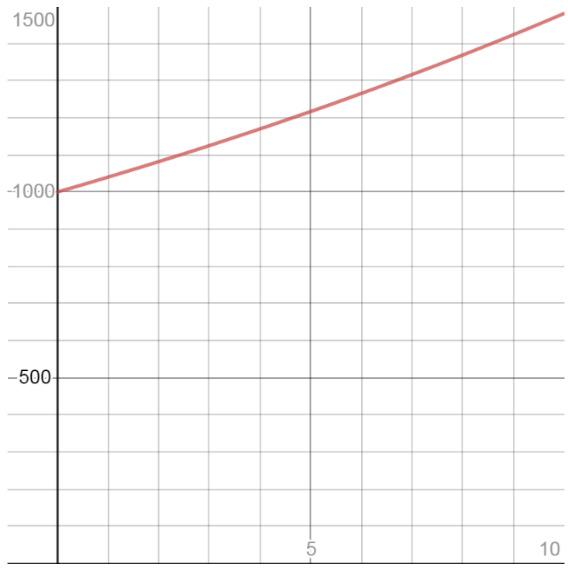
is the first function we have seen where the variable is in an exponent!

What would the graph of such a function look like?

If we try t-values from $t = 1$ to $t = 10$ we get:

$A(t) = 1000 * 1.04^t$

(amount of investment)

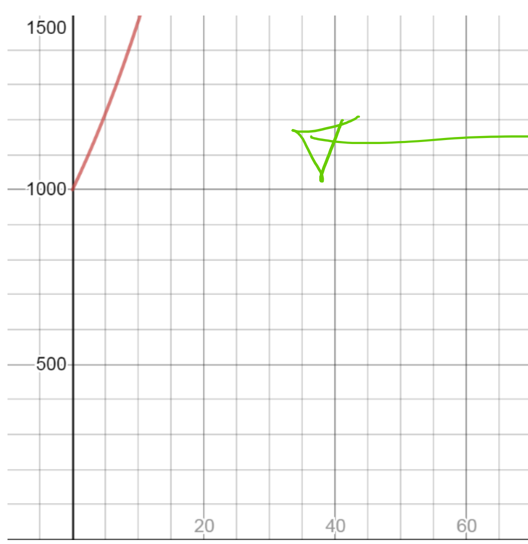


t

(years invested)

This graph seems to be linear (maybe).

Now let's choose a larger scale, looking at the investment over 70 years:

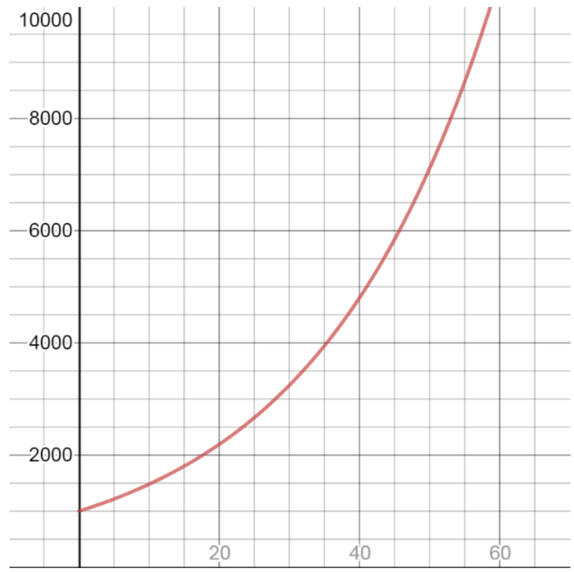


terrible graph!

because the scaling is bad

Clearly I need to change the scale in the y-direction as well! After expanding the y-scale to include investment amounts up to \$10,000:

the graph gets cut off here



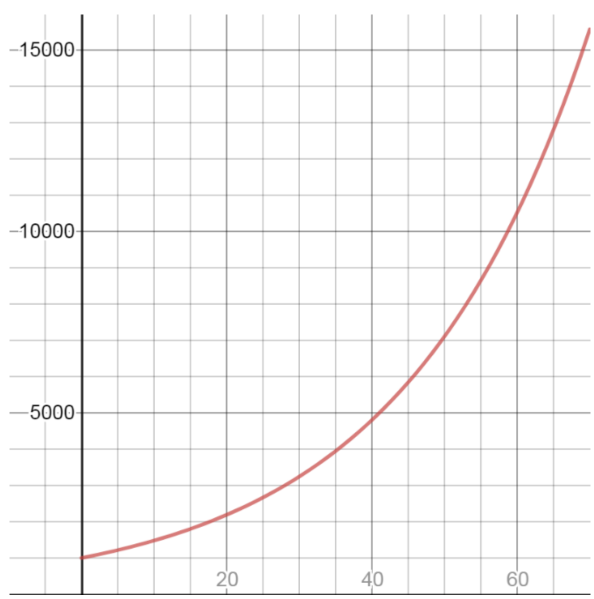
This is still not good enough! To find the maximum y-value I need to plot, I evaluate the function at $t = 70$ years:

$$A(70) = 1000 * 1.04^{70} = 15571.62$$

So I will scale my graph to go from $y = 0$ to $y = 16000$:

$$A(t) = 1000 * 1.04^{t}$$

(amount of investment)



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