Continuously Compounded Interest

Now suppose that another bank, in the suburbs, offers continuously compounded interest?

This means that the interest is compounded every tiny fraction of a second!

In other words, the value of n would be very, very big. It would be infinite.

So to figure out the formula for continuously compounded interest, we would use the general compound interest formula:

$$A(t) = P * \left(1 + \frac{r}{n}\right)^{nt}$$

and let $n \to \infty$!!!!!

*You do not **need** to know how I figure this out.*

If you prefer, skip to the end of this section.

We are trying to find

$$\lim_{n\to\infty} P\left(1 + \frac{r}{n}\right)^{nt}$$

This is a bit complicated!

First we will **simplify the formula** a bit by doing a substitution:

$$Let\ k = \frac{n}{r},$$

Which means that

$$n = kr$$

This variable, k, we are introducing to make the formula "cleaner".

It becomes:

$$\lim_{n \to \infty} P\left(1 + \frac{1}{k}\right)^{krt}$$

And since $n \to \infty$, and k must be greater than n, we can get rid of n entirely, replacing it in the limit with k:

$$\lim_{k \to \infty} P\left(1 + \frac{1}{k}\right)^{krt}$$

Which, using some fancy algebra, becomes

$$P\left(\underbrace{\lim_{k \to \infty}\left(1 + \frac{1}{k}\right)^{k}}\right)^{rt}$$

we will now see that this complicated-looking stuff simplifies!

Now if we look at this strange, complicated-looking expression in the middle, we see that it depends entirely on k.

What happens as $k \to \infty$?

To see, let's look at a table of values, with k getting bigger and bigger:

$\boldsymbol{k}$	$\left(\boldsymbol{1+\frac{1}{k}}\right)^{\boldsymbol{k}}$
$\mathbf{1}$	2
$\mathbf{10}$	2.594
$\mathbf{100}$	2.705
$\mathbf{1000}$	2.720
$\mathbf{10,000}$	2.718

Here, the values coming out of the formula seem to be getting closer together (converging) even though k is becoming exponentially larger.

They are converging on a very special number, called, simply,

"e"

This number is very close to 2.718, but it's irrational, so you can never write it down exactly using decimals.

Suffice to say, $$e \approx 2.718$$

and is sometimes called “the natural base”. You may have to wait until Calculus to discover why, for now, we merely note that

$$\lim_{n\to\infty}\left(1+\frac{1}{k}\right)^{k} \cong 2.718 \approx e$$

which means that our formula for continuously compound interest becomes

$$P\left(\lim_{k\to\infty}\left(1+\frac{1}{k}\right)^{k}\right)^{rt}$$

$$= Pe^{rt}$$

We have derived the following formula for continuously compounded interest:

$$A = Pe^{rt}$$

where

$$P = principal$$

$$r = annual\ interest\ rate$$

$$t = number\ of\ years\ invested$$

$$A = amount\ in\ account$$

In the case of our original example, if our suburban bank offers 4% apr compounded continuously, after 8 years our $1000 investment would become

$$\begin{aligned} A_s(8) &= 1000e^{0.04*8} \\ &= 1000e^{0.32} \\ &= 1000 * 1.37713 \\ &= \$1377.13 \end{aligned}$$

which is about two more dollars than we earned with quarterly compounding, and about nine more dollars than we earned with annual compounding.

Note that

$$f(x) = e^x$$

Has a very similar graph to that of all the others that modeled exponential increase:

$$f(x) = e^x$$