The Rational Zero Test

We can summarize our method for finding **possible rational zeroes**:

The Rational Zero Test

If $f(x) = a_n x^n + a_{n-1}x^{n-1} + a_{n-2}x^{n-2} + \ldots + a_2 x^2 + a_1 x + a_0$

then every rational zero has the form $\boldsymbol{\frac{p}{q}}$

where

p is a factor of a_0

and

q is a factor of a_n

For example:

with

$$h(x) = 3x^3 - 4x^2 + 6x - 6$$

Any rational zero of this function must be of the form $\frac{p}{q}$

Where p is a factor of 3 and q is a factor of -6

Which gives the following possible zeros:

$$\pm\frac{1}{3},\ \pm\frac{2}{3},\ \pm\frac{3}{3},\ \pm\frac{6}{3}$$

duplicates

$$\pm\frac{1}{1},\ \pm\frac{2}{1},\ \pm\frac{3}{1},\ \pm\frac{6}{1}$$

Eliminating the duplicates, we get:

$$\pm\frac{1}{3},\quad \pm\frac{2}{3},\quad \pm 1,\quad \pm 2,\quad \pm 3,\quad \pm 6$$

Another example: find the possible rational zeroes of

$$p(x) = 4x^4 - 12x^3 - 2x^2 + 5x - 10$$

We are creating possible fractions in which …

- ✓ factors of the leading coefficient (4) go in the denominator
- ✓ factors of the constant term (-10) go in the numerator

Here are the factors of 4:

$$1,\quad 2,\quad 4$$

Here are the factors of 10:

$$1,\quad 2,\quad 5,\quad 10$$

This gives us many possibilities initially:

$$\pm\frac{1}{4},\ \pm\frac{2}{4},\ \pm\frac{5}{4},\ \pm\frac{10}{4}$$

$$\pm\frac{1}{2},\ \pm\frac{2}{2},\ \pm\frac{5}{2},\ \pm\frac{10}{2}$$

$$\pm\frac{1}{1},\ \pm\frac{2}{1},\ \pm\frac{5}{1},\ \pm\frac{10}{1}$$

Which after eliminating duplicates gives us:

$$\pm\frac{1}{4}, \quad \pm\frac{1}{2}, \quad \pm\frac{5}{4}, \quad \pm\frac{5}{2}, \quad \pm 1, \quad \pm 2, \quad \pm 5, \quad \pm 10$$

Let's try a whole problem now:

Find the zeroes and graph:

$$g(x) = 2x^3 + 9x^2 - 6x - 18$$

First, since we need to graph $g(x)$, we will get all the points we need for plotting the graph (which will also help our search for zeroes):

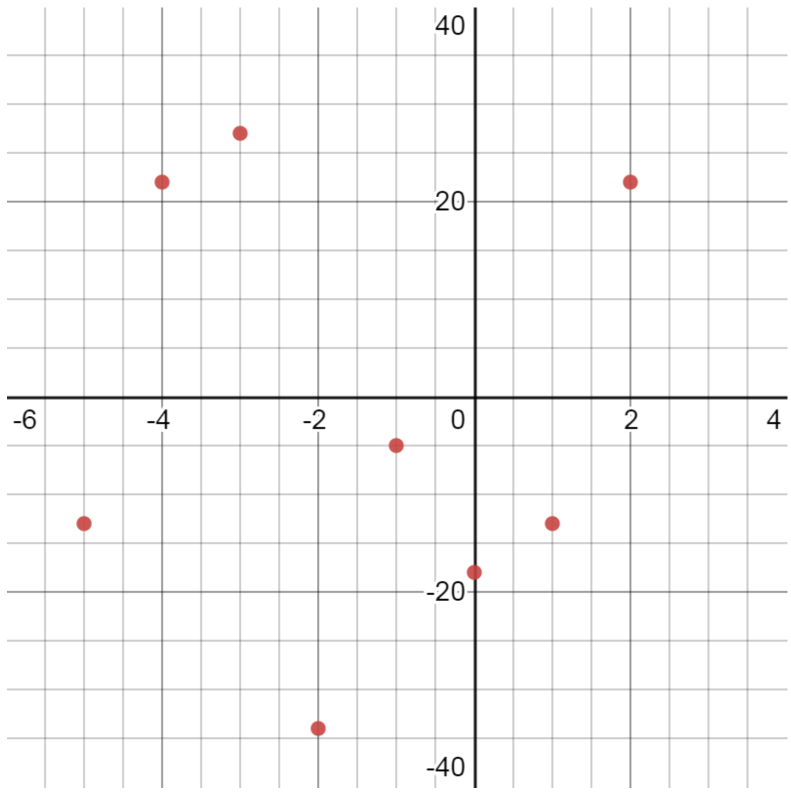
	2	**9**	**−6**	**−18**	
0				−18	
1	2	11	5	−13	zero alert!
2	2	13	20	22	
3	2	15	39	99	
0				−18	
−1	2	7	−13	−5	
−2	2	5	−16	−34	zero alert!
−3	2	3	−15	27	
−4	2	1	−10	22	zero alert!
−5	2	−1	−1	−13	
−6	2	−3	12	−90	

Let's first go ahead and plot these points.

This will give us a visual sense of what's going on.

It will also help us to identify any mistakes in our computations!

$g(x)$



Looking at my points, there's one that definitely seems not right.

Can you see which one?

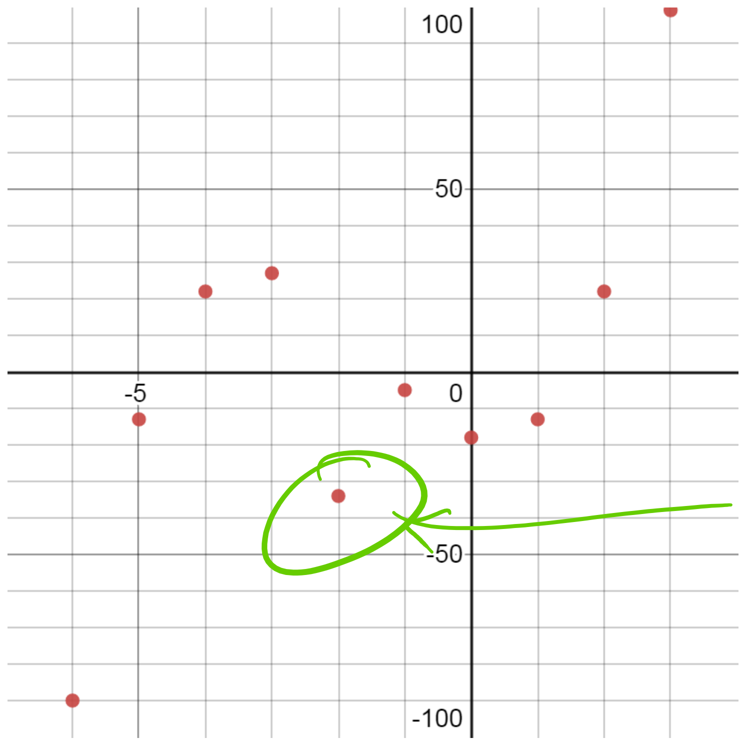
If not, maybe I will go ahead and plot the two points that I was unsure about plotting because they were two big or two small:

$$g(3) = 99$$

$$g(-6) = -90$$

Of course here I must change the scale on my y-axis. I will go from -100 to 100 on the vertical axis.

$g(x)$



this point doesn't look right!

Now the wrong point is more obvious . . .

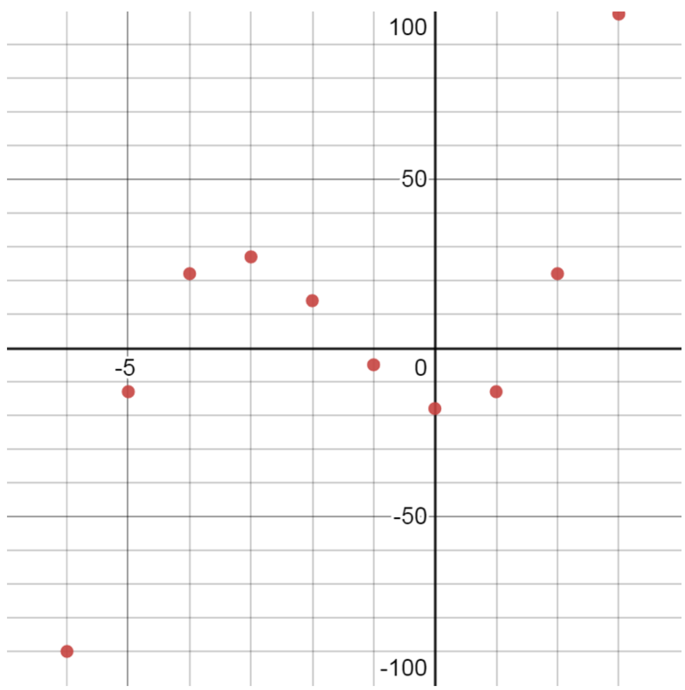
So I go back to my synthetic substitution table and try $x = -2$ again:

	2	**9**	$\mathbf{-6}$	$\mathbf{-18}$
0				-18
1	2	11	5	-13
2	2	13	20	22
3	2	15	39	99
0				-18
$\mathbf{-1}$	2	7	-13	-5
$\mathbf{-2}$	2	5	-16	$\cancel{-34}$ 14
$\mathbf{-3}$	2	3	-15	27
$\mathbf{-4}$	2	1	-10	22
$\mathbf{-5}$	2	-1	-1	-13

$$-\mathbf{6} \;\big|\; 2 \quad -3 \quad 12 \quad -90$$

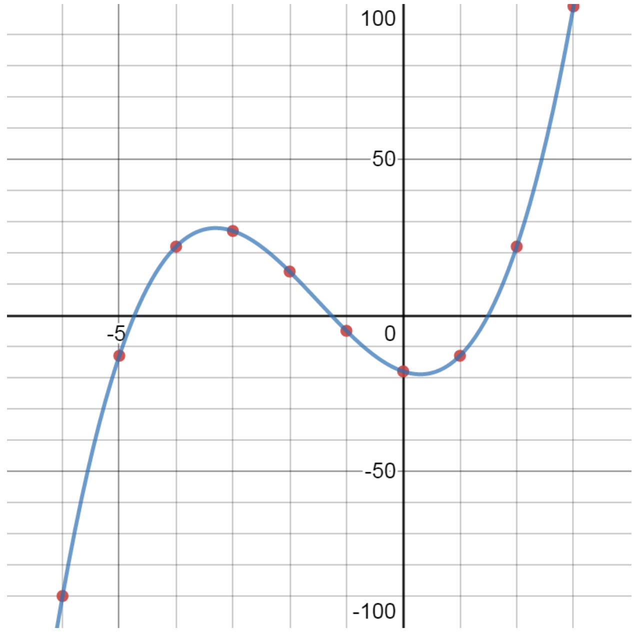
And then fix my graph:

$g(x)$



Now the points can be seen as belonging to a smooth cubic curve, so I will draw the curve over them:

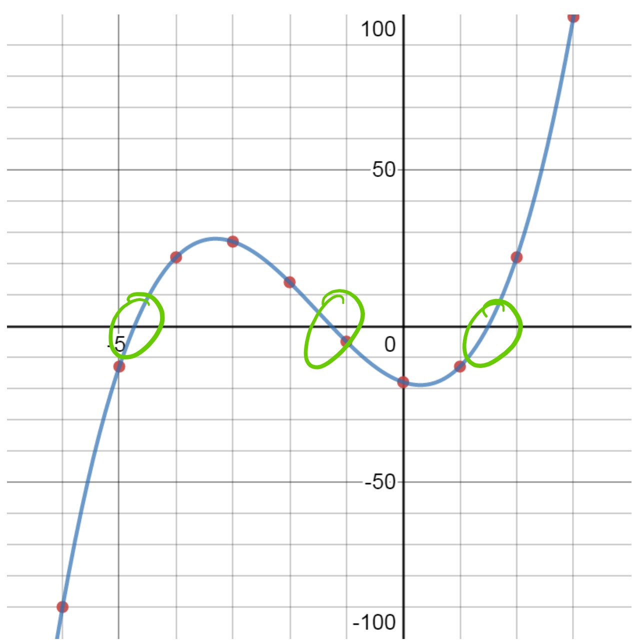
$g(x)$



Now let's find the zeroes!

From the graph we can see that the $g(x)$ crosses the x-axis three times . . .

. . . giving us three zeroes for the function:



zeroes!

However, since none of these were integers . . .

. . . we can only hope that at least one is **rational** . . .

. . . so we can find it using the

Rational Zero Test

Our function is

$$g(x) = 2x^3 + 9x^2 - 6x - 18$$

So any rational zero $x = \frac{p}{q}$ must be such that

p is a factor of 18: $\{1, 2, 3, 6, 9, 18\}$

and

q is a factor of 2: $\{1, 2\}$

We have the following possibilities:

$$\pm\frac{1}{2}, \quad \pm\frac{2}{2}, \quad \pm\frac{3}{2}, \quad \pm\frac{6}{2}, \quad \pm\frac{9}{2}, \quad \pm\frac{18}{2}$$

$$\pm\frac{1}{1}, \quad \pm\frac{2}{1}, \quad \pm\frac{3}{1}, \quad \pm\frac{6}{1}, \quad \pm\frac{9}{1}, \quad \pm\frac{18}{1}$$

Which after simplifying and eliminating duplicates becomes:

$$\pm\frac{1}{2}, \quad \pm\frac{3}{2}, \quad \pm\frac{9}{2}, \quad \pm 1, \quad \pm 2, \quad \pm 3, \quad \pm 6, \quad \pm 9, \quad \pm 18$$

But we know from both our table and graph that . . .

. . . our zeroes are **not integers**!

So those can be eliminated:

$$\pm\frac{1}{2}, \quad \pm\frac{3}{2}, \quad \pm\frac{9}{2}, \quad \cancel{\pm 1}, \quad \cancel{\pm 2}, \quad \cancel{\pm 3}, \quad \cancel{\pm 6}, \quad \cancel{\pm 9}, \quad \cancel{\pm 18}$$

Leaving:

$$\pm\frac{1}{2}, \quad \pm\frac{3}{2}, \quad \pm\frac{9}{2}$$

Now let's look closely at our table to see which of the above could work:

	2	**9**	**−6**	**−18**	
0				−18	
1	2	11	5	−13	zero
2	2	13	20	22	
3	2	15	39	99	
0				−18	
−1	2	7	−13	−5	zero
−2	2	5	−16	14	
−3	2	3	−15	27	
−4	2	1	−10	22	zero
−5	2	−1	−1	−13	
−6	2	−3	12	−90	

The table shows three zeroes:

- between 1 and 2
- between -1 and -2
- between -4 and -5

these are the x-values

Thus we can now eliminate all possible rational zeroes except:

$$\frac{3}{2} \ and \ -\frac{5}{2}$$

We need to try them: Let's use synthetic substitution:

	$\mathbf{2}$	$\mathbf{9}$	$\mathbf{-6}$	$\mathbf{-18}$
$\mathbf{\frac{3}{2}}$	2	12	12	$\mathbf{0}$
$-\mathbf{\frac{5}{2}}$				

We find a zero at $x = \frac{3}{2}$. So we don't need to try $-\frac{5}{2}$!

Now we have that our function can be factored:

$$g(x) = (x - \frac{3}{2})(2x^2 + 12x + 12)$$

So to find our remaining zeroes we must solve:

$$2x^2 + 12x + 12 = 0$$

$$x^2 + 6x + 6 = 0$$

Which I will solve using the quadratic formula:

$$x = \frac{-6 \pm \sqrt{6^2 - 4(1)(6)}}{2(1)}$$

$$x = \frac{-6 \pm \sqrt{36 - 24}}{2}$$

$$x = \frac{-6 \pm \sqrt{12}}{2}$$

$$x = \frac{-6 \pm 2\sqrt{3}}{2}$$

$$x = \frac{\cancel{2}(-3 \pm \sqrt{3})}{\cancel{2}}$$

Factor the GCF from the numerator and cancel

$$x = -3 \pm \sqrt{3}$$

So our zeroes to the function are:

$$\left\{ \frac{3}{2}, \quad -3 + \sqrt{3}, \; -3 - \sqrt{3} \right\}$$