Polynomial Functions

We have been looking at ***quadratic functions***:

$$f(x) = ax^2 + bx + c$$

Which are are ***degree 2* polynomial functions**.

Here are some **other** examples of **polynomial functions**:

$$g(x) = x^3 - 2x^2 + 4x + 7$$

$$h(x) = 2x^5 - 3x^3 + 6$$

$$p(x) = -\frac{1}{2}x^4 + \frac{2}{3}x^3 - \frac{1}{6}x^2 - 8$$

$$q(x) = x^8 + 1$$

$$r(x) = 3x + 1$$

In general, a polynomial function is a function of the form:

leading term　　degree

$$p(x) = a_n x^n + a_{n-1}x^{n-1} + a_{n-2}x^{n-2} + \ldots. + a_2x^2 + a_1x + a_0$$

leading coefficient

KEY FACTS ABOUT POLYNOMIAL FUNCTIONS:

- ✓ they are sums of algebraic terms
- ✓ the variables in the terms have whole number exponents
- ✓ the greatest exponent is called the degree
- ✓ they are generally written in descending order of power

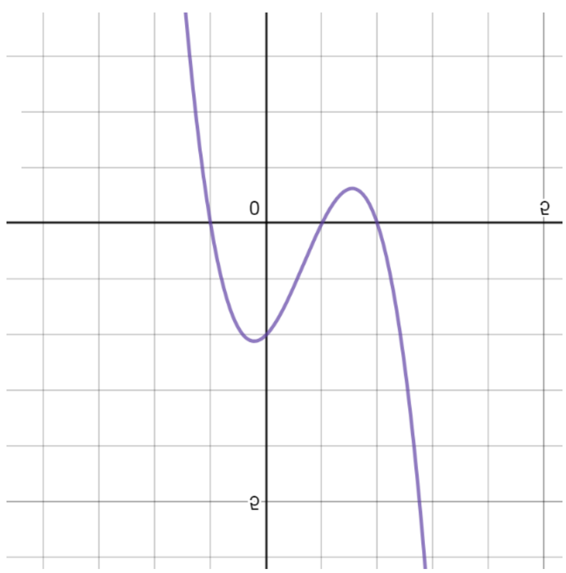
In Calculus, many problems involve polynomial functions.

For a quadratic function, the most important point is the **vertex**.

For polynomial functions of higher degree than 2, there may be more than one vertex-like point, which are sometimes called **turns**.

Consider, for example, the degree 3 polynomial function shown here:

$$p(x) = x^3 - 2x^2 - x + 2$$



There are **two** "turns" in the graph.

It is one of the primary goals of **Calculus I** to find these points where the function "turns." In Calculus, these points are called **relative extrema**.

We will not be finding relative extrema in this class . . .

(except for quadratic functions . . . which we've already learned how to do)

What we will be doing is finding the **zeroes**.

The **zeroes** are the x-values that make the **function equal zero**.

In other words . . .

a is a **zero** of $p(x)$ if

$$p(a) = 0$$

Of course $p(x)$ represents the y-values on the graph . . .

. . . so the **zeroes** make $y = 0$

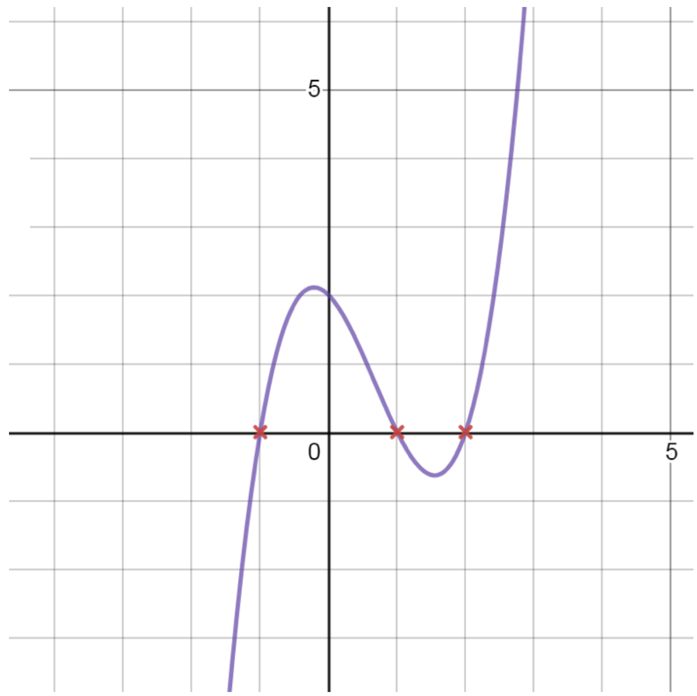
And given that $y = 0$ is the equation of the x-axis . . .

The **zeroes** of a function are the x-intercepts!

Let's see this on the graph of

$$p(x) = x^3 - 2x^2 - x + 2$$

$p(x)$



From the graph itself, we can see that the x-intercepts are:

$$x = -1,\ x = 1,\ x = 2$$

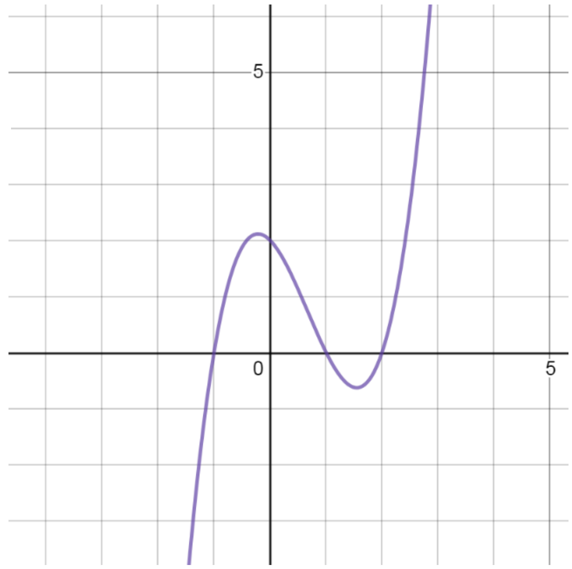
Which is to say that the set of zeroes of $p(x)$ is

$$\{-1, 1, 2\}$$

Notice also about $p(x)$ that the graph falls down on the left . . .

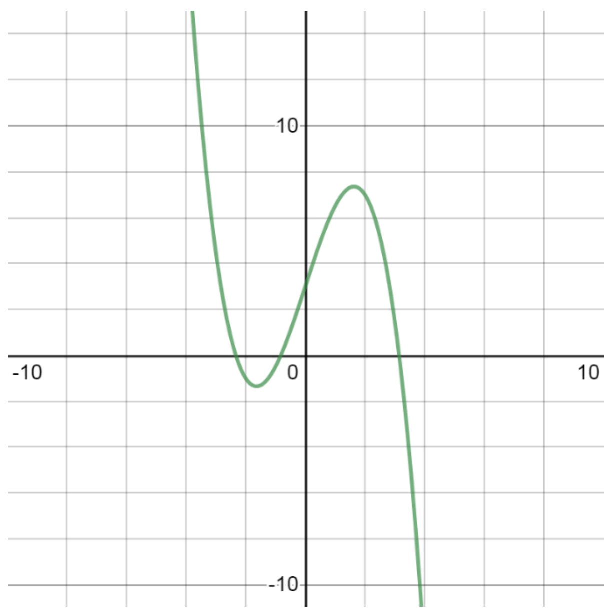
. . . and rises up on the right:

$$p(x) = x^3 - 2x^2 - x + 2$$



This is not always the case. Consider another degree 3 polynomial function:

$$h(x) = -\frac{1}{2}x^3 + 4x + 3$$



This graph **rises to the left** and **falls to the right**.

Can you see what in the formula may cause this difference?

It turns out that the ***leading coefficient*** of a polynomial function . . .

. . . is the most important term!

It's the most important term . . .

. . . because it multiplies the number . . .

. . . that gets the biggest on either end of the graph!

We can see this by plugging $x = 50$ into both polynomial functions:

makes 'em big

makes the big number negative

$$p(50) = 50^3 - 2(50)^2 - (50) + 2 \qquad h(50) = -\frac{1}{2}(50)^3 + 4(50) + 3$$

$$= 125{,}000 - 5000 - 50 + 2 \qquad = -62500 - 200 + 3$$

biggest numbers

So checking the **sign** of the leading coefficient tells you whether it gets big negative or big positive.

But we also need to know the **degree** of the polynomial.

In this case, the negative x-values are cubed, so they stay negative.

The positive x-values are cubed so they stay positive.

So the graph goes in *different directions on either end!*

This will be the same for any polynomial of ODD DEGREE.

We can sum this up in what's called

The Leading Coefficient Test (ODD DEGREE)

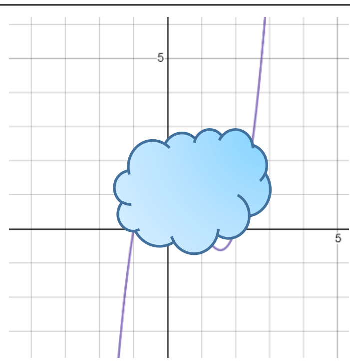
like $g(x) = x^3 - 2x + 6$ odd

If $p(x)$ is a polynomial function of odd degree, then . . .

Leading coefficient positive:

Graph falls to the left . . .

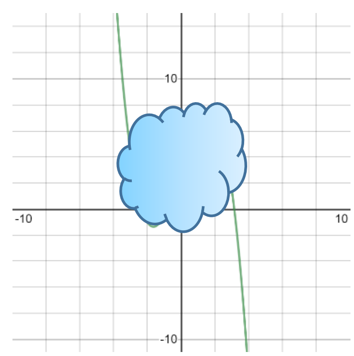
. . . rises to the right



Leading coefficient negative:

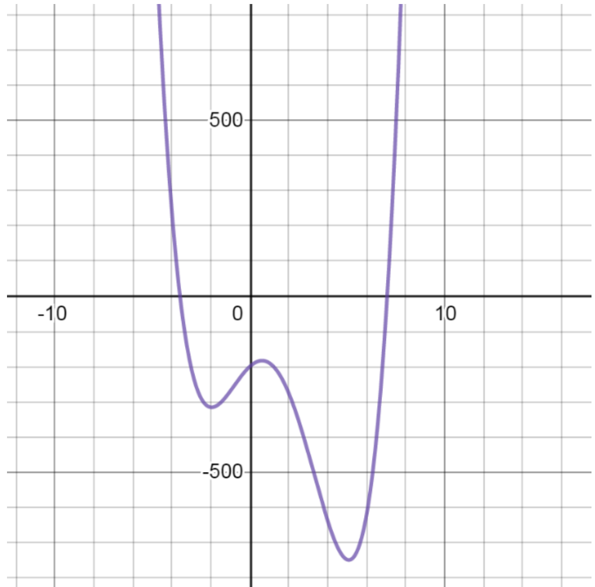
Graph rises to the left . . .

. . . falls to the right



Now let's look at a polynomial function of even degree:

$g(x)$



even

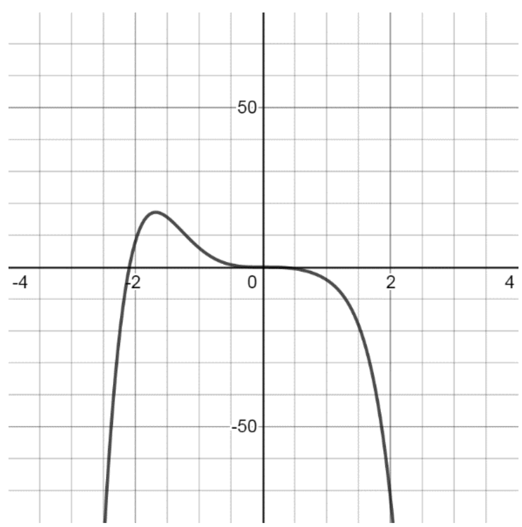
$$g(x) = 2x^4 - 10x^3 - 32x^2 + 50x - 200$$

What do you notice about the graph of $g(x)$ that's different from the previous two polynomial function graphs?

The graph of $g(x)$ **rises on the left** and **rises on the right**.

Conversely, consider the polynomial function $q(x) = -x^6 + 2x^4 - 5x^3$:

$q(x)$



The graph of $q(x)$ **falls to the left** and **falls to the right**

These even degree polynomial functions go in the same direction . . .

on both ends of the graph

Why? Because their leading term becomes positive . . .

. . . after being taken to the even degree:

$$(\pm x)^n \geq 0 \text{ if } n \text{ is even.}$$

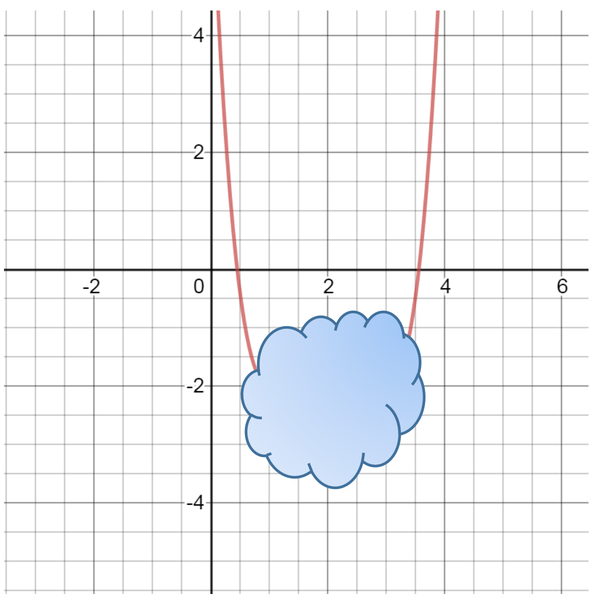
Then the sign of the leading coefficient makes this positive number . . .

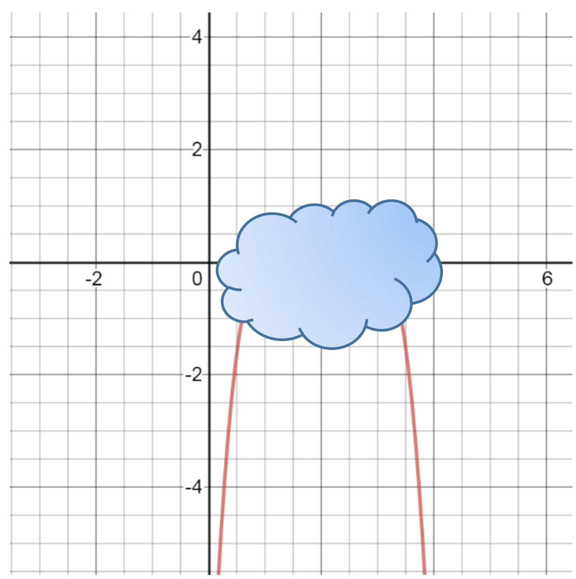
. . . either negative or positive:

The Leading Coefficient Test (EVEN DEGREE)

If $p(x)$ is a polynomial function of **even** degree, then . . .

Leading coefficient positive:	**Leading coefficient negative:**
Graph rises to the left . . .	Graph falls to the left . . .
. . . rises to the right	. . . falls to the right





(note: we are not talking about even or odd functions. Just whether the degree of the function is even or odd)