Homework exercise IV

Filtering and Identification (SC42025)

Hand in pictures / scans of your hand-written solutions as a PDF for theoretical exercises. For the MATLAB exercise, please export your live script as a PDF (instructions in template). Then, **merge all files as a single PDF and upload them through Brightspace** on **11-01-2023** before **18:00**. You are allowed and encouraged to discuss the exercises together but you need to hand in individual solutions.

Please highlight your final answer!

Exercise 1 [35 points]

Consider the unknown minimal LTI system:

$$\begin{aligned} x_{k+1} &= Ax_k + Ke_k, \qquad x_k \in \mathbb{R}^n, \quad e_k \in \mathbb{R}, \\ y_k &= Cx_k + e_k, \qquad y_k \in \mathbb{R}, \\ e_k &\sim \mathcal{N}(0, \sigma_e^2 \neq 0). \end{aligned} \tag{1}$$

The system is assumed to be observable and reachable. You may further assume that $\max \operatorname{eig}(A) < 1$ and that all signals are ergodic.
To solve this exercise, assume that the block size is $s > n$ and that you have access to a number of measurements $N \gg s$, organized in a vector $\{y_k\}_{k=1}^N$.

(a) **[5 points]** Show that the data equation for the outputs of model (1) can be written as:

$$\begin{bmatrix} Y_p \\ Y_f \end{bmatrix} = \mathcal{O}_s \begin{bmatrix} X_p \\ X_f \end{bmatrix} + \mathcal{S}_s \begin{bmatrix} E_p \\ E_f \end{bmatrix}, \tag{2}$$

where the past signals are $Y_p = Y_{1,s,N-2s+1}$, $X_p = X_{1,N-2s+1}$ and $E_p = E_{1,s,N-2s+1}$, and the future signals are $Y_f = Y_{s+1,s,N-2s+1}$, $X_f = X_{s+1,N-2s+1}$ and $E_f = E_{s+1,s,N-2s+1}$. Make sure you write out at least the first and last column of the matrices in terms of their components.

(b) **[5 points]** Show that the future states X_f can be written in terms of past noise such that

$$X_f = \Omega X_p + \Gamma E_p. \tag{3}$$

Clearly indicate what matrices Ω and Γ are.

(c) **[5 points]** We now aim to design a set of instrumental variables to eliminate the influence of the noise matrix from the data equation above. What properties need to be verified such that $Z = Y_p := Y_{1,s,N-2s+1}$ can be used an instrumental-variable matrix?

(d) **[5 points]** Show that, if the properties in (c) apply, then $\operatorname{rank}\left(Y_f Y_p^\intercal\right) = n$, and therefore $\operatorname{range}\left(Y_f Y_p^\intercal\right) = \operatorname{range}(O_s)$.

In the following questions, consider the following LQ factorization:

$$\begin{bmatrix} Y_p \\ Y_f \end{bmatrix} = \begin{bmatrix} L_{11} & 0 \\ L_{21} & L_{22} \end{bmatrix} \begin{bmatrix} Q_1^\intercal \\ Q_2^\intercal \end{bmatrix} \tag{4}$$

(e) **[5 points]** Show that, if $\left(\lim_{N\to\infty} \frac{1}{N} Y_p Y_p^\intercal\right)$ has full rank, then $\operatorname{rank}(L_{11}^\intercal) = s$.
Hint: Take the SVD of L_{11} to show that $\operatorname{rank}(L_{11}) = \operatorname{rank}(L_{11}L_{11}^\intercal)$

(f) **[5 points]** Prove that $\operatorname{range}\left(Y_f Y_p^\intercal\right) = \operatorname{range}(L_{21}L_{11}^\intercal)$.

(g) **[5 points]** Show that the result obtained in (f) can be simplified such that $\operatorname{range}(L_{21}) = \operatorname{range}(\mathcal{O}_s)$, and that $\operatorname{rank}(L_{21}) = n$. What does this result imply when trying to identify the system matrices?

Exercise 2 [25 points]

Consider the closed-loop configuration depicted in Figure 1:

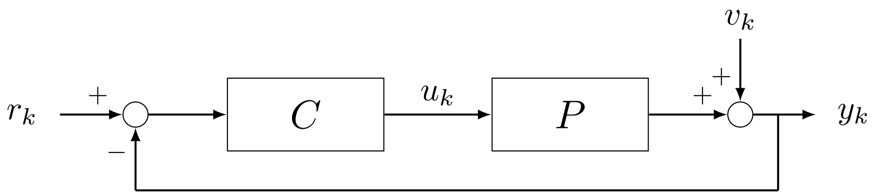

Figure 1: Closed-loop configuration of a known controller C with an unknown LTI plant P of unknown order n.

with known controller C given by the state-space model:

$$\begin{aligned} z(k+1) &= Fz(k) + G(r(k) - y(k)) \\ u(k) &= Hz(k) + J(r(k) - y(k)), \end{aligned}$$

where $z(k)$ is the state-space variable of the controller. Consider additionally the unknown LTI system of unkown order n that is assumed to have the following state-space model:

$$\begin{aligned} x(k+1) &= Ax(k) + Bu(k) \\ y(k) &= Cx(k) + v(k) \end{aligned} \tag{5}$$

and data equation for future measurements:

$$Y_{s,s,N} = \mathcal{O}_s X_{s,N} + \mathcal{T}_s U_{s,s,N} + V_{s,s,N}. \tag{6}$$

The input sequence $r(k)$ is assumed statistically independent from $v(\ell)$ for all k, ℓ and the initial conditions of the system and the controller also statistically independent from $v(\ell)$ for all ℓ. It may also be assumed that the feedback system is stable. Noise $v(k)$ is assumed zero-mean and all stochastic processes may assumed to be ergodic.

(a) **[5 points]** Express the matrices $\mathcal{O}_s$ and $\mathcal{T}_s$ in terms of the unknown system matrices.

(b) **[10 points]** Show that in the data equation (6) the noise matrix $V_{s,s,N}$ can be removed asymptotically by performing the following correlation,

$$\lim_{N\to\infty} \frac{1}{N} Y_{s,s,N} \begin{bmatrix} U_{0,s,N} \\ Y_{0,s,N} \end{bmatrix}^{\mathsf{T}}.$$

(c) **[10 points]** Let R be the Hankel matrix constructed from the signal r_k and let $\mathcal{T}_c$ be the Toeplitz matrix similar to $\mathcal{T}_s$ of Part (a), but determined from the system matrices of the controller C. If we parametrize the data equation of the closed-loop system via the matrices F_1 and F_2 as follows,

$$Y = F_1 \mathcal{O}_s X + F_1 \mathcal{T}_s (F_2 + \mathcal{T}_c R) + F_1 V$$

such that the matrix F_1 does not depend on Y. Then first give an expression for the matrices F_1 and F_2, with that of F_2 only given in terms of known data. Second, show why the matrix F_1 is full rank.

Exercise 3 [40 points]

Use the live-script template `Homework4_template.mlx` for this exercise. Make sure to read the instructions in the live-script carefully.
Consider the following Output Error system given in Homework 3:

$$y(k) = \frac{b_1 q^{-1} + b_2 q^{-2} + b_3 q^{-3}}{1 + a_1 q^{-1} + 2_2 q^{-2} + a_3 q^{-3} + a_4 q^{-4} + a_5 q^{-5}} u(k) + e(k). \tag{7}$$

In the next questions you will use MATLAB to identify the parameters and initial conditions of the given system using the input-output data from `iodata.mat`.

(a) **[5 points]** Split the data set in training and validation data by adapting the code below and motivate your choice.

(b) We are now going to implement the PO-MOESP algorithm for the Output Error System and apply it to the training set (no validation step). To facilitate the implementation, we split it in several subproblems. To speed up the runtime of the implementation and the testing, we consider only a small subset of the given data. The block size is selected as $s = 100$. In the identification part in section c), we will consider the full data set.

- **[3 points]** Compute the Hankel matrices `U_hankel` and `Y_hankel` in `Homework4_template.mlx`.
- **[3 points]** Compute the matrix L_32 with $\begin{bmatrix} U_f \\ Z \\ Y_f \end{bmatrix} = \begin{bmatrix} L_{11} & 0 & 0 \\ L_{21} & L_{22} & 0 \\ L_{21} & L_{32} & L_{33} \end{bmatrix} \begin{bmatrix} Q_1^T \\ Q_2^T \\ Q_3^T \end{bmatrix}$, $Z = \begin{bmatrix} U_p \\ Y_p \end{bmatrix}$ in `Homework4_template.mlx`.
- **[4 points]** Plot the singular values of L_32 by adapting in `Homework4_template.mlx`. Motivate, which model order you would choose. Also, make sure to visualize the graph in a meaningufull manner, for example, by using a logarithmic representation and markers. Please note that a full graph includes titles and axis labels.
- **[3 points]** Compute the system matrices A and C by adapting the code below. For this exercise, we choose $n = 3$.
- **[3 points]** Compute the x_0 and the system matrices B and D by adapting the code below.
- **[4 points]** Combine the above segments in the function `pomoesp`, such that it computes the matrices `A_po_moesp`, `B_po_moesp`, `C_po_moesp`, `D_po_moesp` and the initial condition `x0_po_moesp`, given the inputs `u_small`, the measured outputs `y_small` and the block size s. You can use the file `pomoesp.m` provided and then copy the whole function to the bottom of the live-script or write it in the live-script directly. Make sure to plot the singular values within the function and to provide the user with the option to input the system order based on the plot of the singular values. For this exercise, please choose $n = 3$.

(c) Identification with PO-MOESP

- **[3 points]** Apply the PO-MOESP algorithm on the training set with a training validation split of $\frac{2}{3}$ - $\frac{1}{3}$ to compute the matrices `A_po_moesp`, `B_po_moesp`, `C_po_moesp`, `D_po_moesp` and the initial condition `x0_po_moesp`. Do not perform a validation step yet. Do not change the given training-validation split and block size $s = 100$. Please select $n = 4$ during runtime.
- **[5 points]** Write a function `simsystem` that simulates a dynamic system given any input vector u and matrices `A_po_moesp`, `B_po_moesp`, `C_po_moesp`, `D_po_moesp` and the initial condition `x0_po_moesp`.You can use the file `simsystem.m` provided and then copy the whole function to the bottom of the live-script or write it in the live-script directly. Simulate the system to compute measured and estimated outputs `y_train` and `yhat_po_moesp_train`, respectively.
- **[5 points]** Write a function `VAF` and `RMSE`, that computes the Variance Acocunted For (VAF) and root mean squared error (RMSE), respectively, given the measured and estimated outputs`y_train` and `yhat_po_moesp_train`, respectively. You can use the file `VAF.m` and `RMSE.m` provided and then copy the whole function to the bottom of the live-script or write it in the live-script directly.
- **[2 points]** Test the PO-MOESP model on the validation data and compute the VAF and RMSE for the validation data. Plot your results with the code provided below.