Nonlinear Programming Assignment

SC42056 Optimization for Systems and Control 2022/2023

E_1, E_2 and E_3 are parameters changing from 0 to 18 for each group according to the sum of the last three numbers of the student IDs:

$$E_1 = D_{a,1} + D_{b,1}, \qquad E_2 = D_{a,2} + D_{b,2}, \qquad E_3 = D_{a,3} + D_{b,3}$$

with $D_{a,3}$ the right-most digit of the student ID of the first student, $D_{b,3}$ the right-most digit of the student ID of the other student, $D_{a,2}$ the one but last digit of the student ID of the first student, and so on.

Important: Please note that all questions regarding this assignment should be asked via the Brightspace Discussion forum.

In recent years, the number of vehicles on our roads has grown larger and larger, and even though large and soundly designed traffic networks (freeways and urban roads) have already been constructed, traffic congestion still cannot be avoided. Moreover, it is often too time- and money-consuming to build new transportation infrastructure or reconstruct existing infrastructure. Therefore, traffic jams occur frequently and have a severe impact when many people need to use the traffic infrastructure with limited capacity simultaneously, especially during rush hours. Traffic congestion can lead to transportation delays, economic losses, pollution, and other problems. Effective traffic control methods are thus necessary to reduce traffic jams and promote efficiency in traveling. In this context, traffic control strategies are one of the most efficient and effective methods to address traffic congestion problems.

In this assignment, we consider traffic signal control, the main control measure used in cities. The intersection of a traffic network, shown in **Figure 1**, will be used to implement the control strategy.

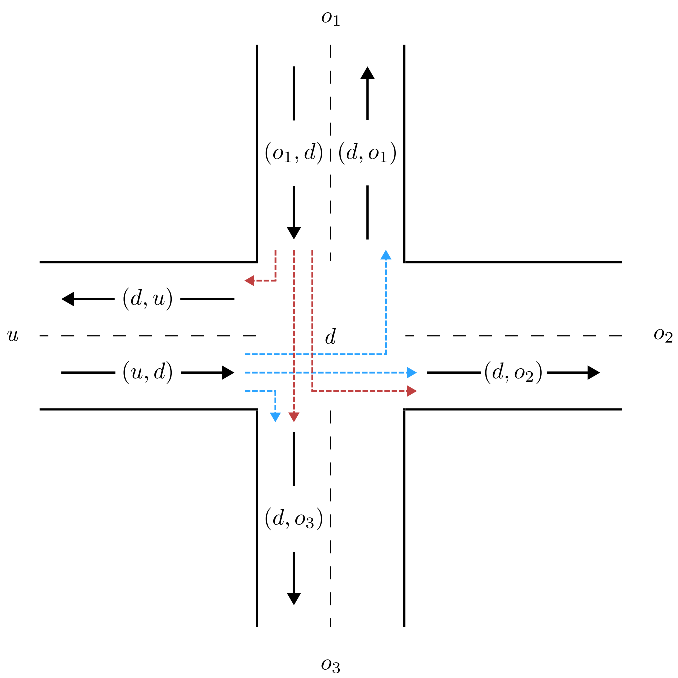

Figure 1: Intersection in an urban traffic network

In order to describe the urban traffic model [1][2] (*S-model*), we define J as the set of nodes (intersections) and L as the set of directed links (roads) in the urban traffic network. In our situation, we have:

$$J = \{u, d, o_1, o_2, o_3\}$$
$$L = \{(u, d), (d, u), (o_1, d), (d, o_1), (d, o_2), (d, o_3)\}$$

Moreover, each link can be marked by its downstream nodes, where traffic flow will go after crossing the intersection. Considering, for example link (u, d), its set of downstream nodes is defined as $O_{u,d} = \{o_1, o_2, o_3\}$. Also, each link has three lanes where vehicles can turn left, go straight, and turn right, respectively, as indicated by the colored arrows in **Figure 1**.

Traffic signal control of node d is considered in this assignment, where traffic flows entering link (u, d) via node u, as well as link (o_1, d) via node o_1, are given a priori. As a consequence, traffic flow dynamics of links (u, d) and (o_1, d) will be taken into account in the following. However, to simplify the presentation of the modeling process, only the equations for link (u, d) will be provided, assuming the same structure for the dynamics of link (o_1, d), with some differences that will be highlighted when necessary.

The cycle of the traffic light in node d is divided into 2 phases: the green-light phase when the vehicles in link (u, d) can cross the intersection, and the red-light phase when the vehicles in link (u, d) are not allowed to cross the intersection. In this particular case, the vehicles that turn right are not influenced by the traffic signal, as shown in **Figure 2**. The same consideration applies to link (o_1, d).

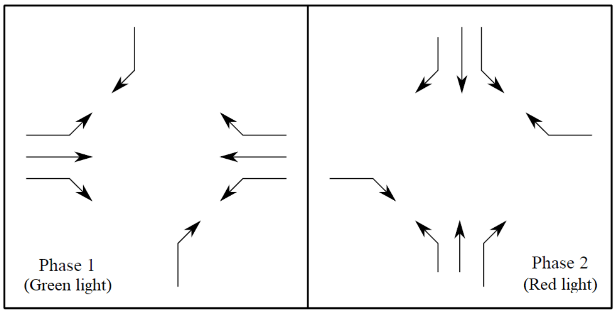

Figure 2: Signal phases of the traffic light control at the intersection.

Taking the cycle time c as the length of the simulation time interval and k as the corresponding cycle counter, the number of the vehicles in link (u, d) is updated according to the input and output average flow rate over the entire cycle of length c by:

$$n_{u,d}(k+1) = n_{u,d}(k) + \left(\alpha_{u,d}^{\text{enter}}(k) - \alpha_{u,d}^{\text{leave}}(k)\right) \cdot c \tag{1}$$

The leaving average flow rate $\alpha_{u,d}^{\text{leave}}(k)$ is the sum of the leaving flow rates turning to each output link:

$$\alpha_{u,d}^{\text{leave}}(k) = \sum_{i \in O_{u,d}} \alpha_{u,d,i}^{\text{leave}}(k) \quad . \tag{2}$$

The leaving average flow rate over the cycle time c is determined by the capacity of the intersection, the number of cars waiting and arriving, and the available space in the output link. In particular, $\alpha^{\text{leave}}_{u,d,i}(k)$ is expressed as:

$$\alpha^{\text{leave}}_{u,d,i}(k) = \min\left(\tfrac{\beta_{u,d,i}\cdot\mu_{u,d}\cdot g_{u,d,i}(k)}{c},\ \tfrac{q_{u,d,i}(k)}{c} + \alpha^{\text{arrive}}_{u,d,i}(k),\ \tfrac{C_{d,i}(k)}{c}\right) \quad \text{for} \quad i \in O_{u,d} \tag{3}$$

where $\mu_{u,d}$ is the saturated flow rate leaving link (u,d), $\beta_{u,d,i}$ the the fraction of traffic in link (u,d) that will turn to link (d,i), $g_{u,d,i}(k)$ the green-light time length during the time cycle k for the traffic flow in link (u,d) directed towards link (d,i), $q_{u,d,i}(k)$ is the length of the queue at time step k in link (u,d) turing to link (d,i), $\alpha^{\text{arrive}}_{u,d,i}(k)$ is the avarage flow rate arriving at the tail of the queue $q_{u,d,i}(k)$ at k, and $C_{d,i}(k)$ is the (possibly time-varying) available space in the corresponding downstream link (d,i). Note that the lanes turning right are not influenced by the traffic light, e.g. $g_{u,d,o_3}(k) = c\ \forall k$, and the green-light time duration for lanes going straight and turning left are assumed to be the same, e.g. $g_{u,d,o_1}(k) = g_{u,d,o_2}(k)$.

Then, the number of vehicles waiting in the queue of link (u,d) and directed to the downstream node $i \in O_{u,d}$ is updated as:

$$q_{u,d,i}(k+1) = q_{u,d,i}(k) + \left(\alpha^{\text{arrive}}_{u,d,i}(k) - \alpha^{\text{leave}}_{u,d,i}(k)\right)\cdot c \quad \text{for} \quad i \in O_{u,d} \tag{4}$$

and the total number of vehicles in the queue of link (u,d) at k is:

$$q_{u,d}(k) = \sum_{i\in O_{u,d}} q_{u,d,i}(k) \tag{5}$$

The vehicles that entered link (u,d) during cycle k will reach the tail of the queue after a certain time delay. Therefore, for the average traffic flow arriving at the queue, the following expression can be used:

$$\alpha^{\text{arrive}}_{u,d}(k) = (1-\gamma_{u,d}(k))\cdot\alpha^{\text{enter}}_{u,d}(k-\tau_{u,d}(k)) + \gamma_{u,d}(k)\cdot\alpha^{\text{enter}}_{u,d}(k-\tau_{u,d}(k)-1) \tag{6}$$

where $\alpha^{\text{enter}}_{u,d}(k)$ is the average traffic flow entering link (u,d) at k, and it is split in two components arriving at the queue with a delay of $\tau_{u,d}(k)$ and $\tau_{u,d}(k)+1$ time steps respectively. The proportion of the flow affected by the two distinct delays is provided by the factor $\gamma_{u,d}(k) \in [0,1]$. Here, $\tau_{u,d}(k)$ and $\gamma_{u,d}(k)$ are expressed as:

$$\tau_{u,d}(k) = \text{floor}\left\{\frac{(C_{u,d}(k) - q_{u,d}(k))\cdot l_{\text{veh}}}{N^{\text{lane}}_{u,d}\cdot v^{\text{free}}_{u,d}\cdot c}\right\} \tag{7}$$

$$\gamma_{u,d}(k) = \text{frac}\left\{(C_{u,d}(k) - q_{u,d}(k))\cdot l_{\text{veh}},\ N^{\text{lane}}_{u,d}\cdot v^{\text{free}}_{u,d}\cdot c\right\} \tag{8}$$

where l_{veh} is the average space occupied by a vehicle, $N^{\text{lane}}_{u,d}$ is the number of lanes in link (u,d), and $v^{\text{free}}_{u,d}$ the free-flow vehicle speed in (u,d). The functions floor$\{x\}$, frac$\{x,y\}$ refer respectively to the largest integer smaller or equal to x, and to the fraction of the division[1] of x by y.

For links (u,d), (o_1,d) maximum capacities are assumed to be constant in this assignment, and they are computed using link paramenters, e.g. for $C_{u,d}(k)$ holds:

$$C_{u,d}(k) = \frac{N^{\text{lane}}_{u,d}\cdot l_{u,d}}{l_{\text{veh}}} \qquad \forall k \tag{9}$$

[1]Please note that when using the Matlab function 'rem(a,n)' in connection with the division $a/n = q + (r/n)$, the Matlab function 'rem' returns the term r. What is required for the assignment is instead $\gamma_{i,j}(\cdot) = r/n$.

For downstream links, time-varying capacities will be considered depending on the traffic conditions, e.g. traffic incidents that may temporarily block part of the road, which will be provided later.

Before reaching the tail of the waiting queues in link (u, d), the flow rate of arriving vehicles needs to be divided according to the turning rates:

$$\alpha_{u,d,i}^{\text{arrive}}(k) = \beta_{u,d,i} \cdot \alpha_{u,d}^{\text{arrive}}(k) \qquad \text{for} \qquad i \in O_{u,d} \tag{10}$$

Equations (1) – (10) were formulated for link (u, d). Therefore, a similar system should be derived to model traffic flow dynamics in link (o_1, d). Then, the total time spent (TTS) by the drivers on links (u, d) and (o_1, d) is taken as the output of the system:

$$y\,(k) = \left(n_{u,d}\,(k) + n_{o_1,d}\,(k)\right) \cdot c \tag{11}$$

The meaning of the symbols appearing in the equations is reported below:

Symbol	Unit	Meaning
c	[h]	Cycle time
k		Cycle counter
$n_{u,d}\,(k)$	[veh]	Number of vehicles in link (u, d) at the beginning of cycle k
$N_{u,d}^{\text{lane}}$		Number of lanes in link (u, d)
$v_{u,d}^{\text{free}}$	[km/h]	Free flow vehicle speed in link (u, d)
l_{veh}	[m]	Average space occupied by a vehicle
$l_{u,d}$	[m]	Lenght of link (u, d)
$q_{u,d}(k)$	[veh]	Queue length at the beginning of cycle k in link (u, d) expressed in number of vehicles
$q_{u,d,i}(k)$	[veh]	Queue length of the sub-stream turning to link (d, i) at the beginning of cycle k
$\alpha_{u,d}^{\text{leave}}(k)$	[veh/h]	Average flow rate leaving link (u, d) during cycle k
$\alpha_{u,d,i}^{\text{leave}}(k)$	[veh/h]	Average flow rate of the sub-stream leaving link (u, d) during cycle k and going toward link (d, i)
$\alpha_{u,d}^{\text{arrive}}(k)$	[veh/h]	Average flow rate arriving at the queue in (u, d) during cycle k
$\alpha_{u,d,i}^{\text{arrive}}(k)$	[veh/h]	Average flow rate arriving at the queue in (u, d) during cycle k and directed to link i
$\alpha_{u,d}^{\text{enter}}(k)$	[veh/h]	Average flow rate entering link (u, d) during cycle k
$\beta_{u,d,i}$		Fraction of the traffic in link (u, d) anticipating to turn to link (d, i)
$\mu_{u,d}$	[veh/h]	Saturation flow rate leaving link (u, d)
$C_{u,d}(k)$	[veh]	Storage capacity of link (u, d) during cycle k
$g_{u,d,i}(k)$	[h]	Green-time length during cycle k for the traffic stream towards link (d, i) in link (u, d)

Equivalent symbols are required to model the dynamics of link (o_1, d).

Tasks

1. Formulate the discrete-time state space model that predicts the number of vehicles on link (u,d) and link (o_1,d) for the next simulation cycle $k+1$ as follows:

$$x(k+1) = f(x(k), u(k))$$
$$y(k) = g(x(k))$$

where f and g are vector-valued, nonlinear functions.

Directions: The parameters of the link (u,d) are given in **Table 1**.

c	$N_{u,d}^{\text{lane}}$	$v_{u,d}^{\text{free}}$	l_{veh}	$l_{u,d}$	β_{u,d,o_1}	β_{u,d,o_2}	β_{u,d,o_3}	$\mu_{u,d}$
60 [s]	3	60 $\left[\frac{\text{km}}{\text{h}}\right]$	7 [m]	4000 [m]	0.4	0.3	0.3	1800 $\left[\frac{\text{veh}}{\text{h}}\right]$

Table 1: Parameters for link (u,d)

The traffic flow entering link (u,d) is given as follows:

$$\alpha_{u,d}^{\text{enter}}(k) = \begin{cases} 2300 + 10 \cdot E_1 & [\text{veh/h}] \quad \text{if} \quad k \leq 20 \\ 1800 + 10 \cdot E_2 & [\text{veh/h}] \quad \text{if} \quad 20 < k \leq 40 \\ 2100 + 10 \cdot E_3 & [\text{veh/h}] \quad \text{if} \quad k > 40 \end{cases}$$

We assume that the available space in downstream links is known. In particular, the capacity of downstream links of link (u,d) is given as follows:

$$C_{d,o_3}(k) = \begin{cases} 10 - \frac{E_3}{2} & [\text{veh}] \quad \text{if} \quad k \leq 20 \\ 10 + \frac{E_3}{2} & [\text{veh}] \quad \text{if} \quad k > 20 \end{cases}$$

$$C_{d,o_2}(k) = \begin{cases} 10 + \frac{E_2}{2} & [\text{veh}] \quad \text{if} \quad k \leq 20 \\ 10 + \frac{E_2}{2} - \frac{2 \cdot k}{E_1+1} & [\text{veh}] \quad \text{if} \quad 20 < k \leq 35 \\ 20 + \frac{E_2}{2} & [\text{veh}] \quad \text{if} \quad 35 < k \leq 45 \\ 20 + \frac{E_3}{2} + \frac{2 \cdot k}{E_1+1} & [\text{veh}] \quad \text{if} \quad k > 45 \end{cases}$$

$$C_{d,o_1}(k) = C_{d,o_2}(k) - \frac{2 \cdot k}{E_1 + 1} \quad [\text{veh}] \quad \forall k$$

Note that all capacities $C_{i,j}(\cdot)$ must have nonnegative value. Therefore, assume $C_{i,j}(\cdot)$ equal to zero if the exspression given above returns a negative value for your combination of parameters E_1, E_2, E_3.

The parameters for link (o_1, d) are given in **Table 2**.

c	$N^{\text{lane}}_{o_1,d}$	$v^{\text{free}}_{o_1,d}$	l_{veh}	$l_{o_1,d}$	$\beta_{o_1,d,u}$	β_{o_1,d,o_2}	β_{o_1,d,o_3}	$\mu_{o_1,d}$
60 [s]	3	50 $\left[\frac{\text{km}}{\text{h}}\right]$	7 [m]	4000 [m]	0.3	0.4	0.3	1700 $\left[\frac{\text{veh}}{\text{h}}\right]$

Table 2: Parameters for link (o_1, d)

The traffic flow entering link (o_1, d) is assumed to be:

$$\alpha^{\text{enter}}_{o_1,d}(k) = 1800 + 10 \cdot E_1 \qquad [\text{veh/h}] \qquad \forall k$$

The available space for downstream links of link (o_1, d) is defined as follows: $C_{d,o_2}(k)$ and $C_{d,o_3}(k)$ have been already specified; $C_{d,u}(k)$ is given by:

$$C_{d,u}(k) = \begin{cases} 20 - \frac{E_3}{2} & [\text{veh}] \quad \text{if} \quad k \leq 40 \\ 20 + \frac{E_3}{2} & [\text{veh}] \quad \text{if} \quad k > 40 \end{cases}$$

Hints: Choose the states properly so that the output can be expressed as a function of the states; the inputs are green-time lengths. All parameters must be expressed with the same units: h, km, veh.

2. Use the model that you built in Task 1, and formulate the optimization problem to find the time length of the green light at node d for every cycle that minimizes the Total Time Spent (TTS) by the drivers on links (u, d) and (o_1, d) for the following hour (i.e., for the period $[0, 60c]$ $\forall k$).

   Directions: Assume that drivers can always turn right at the intersection, that the green-light duration for going straight and turning left is the same, and that the green-light durations of the two phases reported in **Figure 2** are complementary. Those assumptions are equivalent to the following equalities:

$$\begin{aligned} g_{u,d,o_3}(k) &= g_{o_1,d,u}(k) = c \\ g_{u,d,o_1}(k) &= g_{u,d,o_2}(k) \\ g_{o_1,d,o_2}(k) &= g_{o_1,d,o_3}(k) \\ g_{o_1,d,o_2}(k) &= c - g_{u,d,o_2}(k) \end{aligned}$$

   Moreover, assume that at $k = 0$ there is no traffic, i.e., the numbers of vehicles in links (u, d) and (d, o) are equal to 0, and the numbers of vehicles waiting in the queues are 0 as well. For simplicity, we consider the green time as continuous variables constrained in the interval [15s, 45s]. Select an appropriate optimization algorithm, and explain your choice.

   In addition, we assume that we perfectly know the future inputs and capacities of the traffic network, as they are specified above.

3. Use the chosen optimization algorithm to optimize the traffic flows by choosing two different starting points for the green time lengths of link (u, d): $g_{u,d,i}(k) = 15\text{s}$ $\forall k$ and $g_{u,d,i}(k) = 45\text{s}$ $\forall k$, for $i \in O_{u,d}$. Is there a substantial difference between the obtained solutions? Why? How can the solution be improved (if possible)? Can you prove that the solution obtained is the global optimum?

4. Plot the states and inputs of your simulation in Task 3 and compare them with the ones obtained for the no-control case ($g_{u,d,i}(k) = g_{o_1,d,j}(k) = 30\text{s}\ \forall k$, for $i \in O_{u,d}$, $j \in O_{o_1,d}$). Analyze the traffic situation for the considered simulation period of one hour, including the number of vehicles on the links and the queue length of each lane. Moreover, find the TTS values over the entire simulation period for the different cases (including the no-control case) and compare them. Explain the results obtained.

5. Now consider that the green light times are limited to the following discrete time set: {15s, 20s, 25s, 30s, 35s, 40s, 45s}. Formulate the problem of Task 2 and solve it using an optimization approach that can directly deal with integer optimization variables. Repeat the analysis of Task 4 and compare the results. Also, add a discussion to show/plot/analyze/compare the different algorithms/approaches.

Practical Information

The solutions of the assignment should be uploaded to Brightspace before Monday, October 24, 2022, at 17:00 as <u>one .pdf file</u> (no other formats allowed) containing:

1. A written report on the practical exercise, addressing the required tasks.

2. An appendix to the report containing the Matlab code you used[2]. Please, make sure your code is error free.

After uploading, please verify the file to ensure it is correct and not broken.

Please also note that you will lose 0.5 point from your grade for this assignment for each (started) day of delay in case you exceed the deadline.

References

[1] S. Lin, B. De Schutter, S.K. Zegeye, H. Hellendoorn, and Y. Xi. "Integrated Urban Traffic Control for the Reduction of Travel Delays and Emissions". In: *13th International IEEE Conference on Intelligent Transportation Systems*. 2010, pp. 677–682.

[2] A. Jamshidnejad, S. Lin, Y. Xi, and B. De Schutter. "Corrections to "Integrated Urban Traffic Control for the Reduction of Travel Delays and Emissions"". In: *IEEE Transactions on Intelligent Transportation Systems* 20 (2019), pp. 1978–1983.

[2]In LaTeX, you can use the package `listings` or the environment `verbatim` to create the appendix with the Matlab code.