Practice Session 5

In this final practice session, we will reproduce some results from recent literature.

1 Nonlinear Fourier Transform

In Practice Session 3, we investigated the part of the nonlinear Fourier transform for the Korteweg–de Vries equation that corresponds to the discrete spectrum of the Schrödinger operator

$$L(t) = \partial_z^2 + M_u, \quad D(L) \subset \mathscr{L}^2(\mathbb{R}) \text{ dense}, \quad u(t,z) \to 0 \text{ for } z \to \pm\infty \text{ "fast enough".}$$

In this practice session, we will have a look at the continuous spectrum. Remember that the continuous spectrum in this case consists of the λ for which

$$L(t)\psi = \lambda\psi \quad \Longleftrightarrow \quad \frac{\partial^2 \psi}{\partial z^2} = (\lambda - u)\psi \tag{1.1}$$

has a bounded solution, but not a finite energy solution. Since any solution asymptotically behaves as

$$\psi(z \to -\infty) \simeq \alpha e^{\sqrt{\lambda}z} + \beta e^{-\sqrt{\lambda}z}, \qquad \psi(z \to +\infty) \simeq \gamma e^{\sqrt{\lambda}z} + \delta e^{-\sqrt{\lambda}z}, \tag{1.2}$$

we found in Lecture 5 that the continuous spectrum consists of all real $\lambda \leq 0$.
In order to recover $u(t_0, z)$ from the spectrum of the Schrödinger operator $L(t)$, one needs to characterize the corresponding generalized eigenfunctions ψ. In practice, it turns out that if one chooses

$$\alpha = 1, \quad \beta = 0$$

in (1.2), then $\rho(\lambda) = \delta(\lambda)/\gamma(\lambda)$ is sufficient to characterize the continuous spectrum.

> Our goal for Part 1 is to reproduce Figure 23b from the paper T. Trodgon et al., *"Numerical inverse scattering for the Korteweg–de Vries and modified Korteweg–de Vries equations"* Physica D, Vol. 241, No. 11, pp. 1003–1025, Jun. 2012. Link: `https://www-sciencedirect-com.tudelft.idm.oclc.org/science/article/pii/S016727891200053X`.

Task 1 Write a Matlab script that computes $\psi(a)$ and $\psi'(a) := (\partial_z\psi)(a)$ for the initial condition

$$\psi(-a) = e^{\sqrt{\lambda}(-a)}, \quad \psi'(-a) = \sqrt{\lambda}e^{\sqrt{\lambda}(-a)},$$

by solving (1.1) numerically, for any given $a > 0$, $\lambda \leq 0$ and u.
Hint: You can use Chebfun to solve this task. See Section 10.2 of the Chebfun guide[1].
Hint: For $a = 2$, $\lambda = -1$ and $u(z) = 1$, you should find $\psi(a) \approx -0.7140 - 0.5642i$ and $\psi'(a) \approx 0.3917 - 1.0909i$.

In order to recover γ and δ, we assume that a is large enough such that (1.2) can be used:

$$\psi(a) \approx \gamma e^{\sqrt{\lambda}a} + \delta e^{-\sqrt{\lambda}a} = \begin{bmatrix} e^{\sqrt{\lambda}a} & e^{-\sqrt{\lambda}a} \end{bmatrix} \begin{bmatrix} \gamma \\ \delta \end{bmatrix}.$$

By taking the derivative of (1.2), we furthermore get

$$\psi(a) \approx \sqrt{\lambda}\gamma e^{\sqrt{\lambda}a} - \sqrt{\lambda}\delta e^{-\sqrt{\lambda}a} = \begin{bmatrix} \sqrt{\lambda}e^{\sqrt{\lambda}a} & -\sqrt{\lambda}e^{-\sqrt{\lambda}a} \end{bmatrix} \begin{bmatrix} \gamma \\ \delta \end{bmatrix}.$$

[1] `https://www.chebfun.org/docs/guide/guide10.html`

Task 2 Write a Matlab script that recovers γ and δ by solving the linear equation system

$$\begin{bmatrix} \psi(a) \\ \psi'(a) \end{bmatrix} = \begin{bmatrix} e^{\sqrt{\lambda}a} & e^{-\sqrt{\lambda}a} \\ \sqrt{\lambda}e^{\sqrt{\lambda}a} & -\sqrt{\lambda}e^{-\sqrt{\lambda}a} \end{bmatrix} \begin{bmatrix} \gamma \\ \delta \end{bmatrix}$$

for any given $\psi(a)$, $\psi'(a)$ and $\lambda \leq 0$.

Hint: With the parameters from the hint for Task 1, you should find $\gamma \approx -0.0590+1.0195i$ and $\delta \approx 0.2072i$.

Task 3 Write a Matlab script that plots $\mathscr{R}e\{\rho(k)\}$ and $Im\{\rho(k)\}$ over k for

$$u(z) = 2.4\,\text{sech}^2(z), \quad k = -\sqrt{-\lambda} = -4, -3.99, \cdots, -0.01, \quad a = 15.$$

Hint: Remember that $\rho = \delta/\gamma$. Your result should match the $k < 0$ part of Figure 23b in the paper.

Part 2: Koopman Stability Analysis

In Practice Session 4, we considered nonlinear systems of the form

$$\dot{x} = f(x), \quad y = g(x) = \begin{bmatrix} g_1(x) & g_2(x) & \cdots \end{bmatrix}^T, \quad x(0) = x_0 \in \mathbb{C}^N,$$

and developed software that would provide us with an estimate for a matrix A such that

$$y(n\tau + \tau) \approx Ay(n\tau), \quad n = 0, 1, 2, \cdots, \quad \tau > 0,$$

if the measurement functions $g_n(x)$ are chosen from

$$\mathscr{B}_d := \left\{ h : \mathbb{R}^2 \to \mathbb{R} \mid h(x) = x_1^{k_1} x_2^{k_2}, \text{ where } k_1, k_2 \in \{0, 1, \ldots, d\} \text{ and } k_1 + k_2 \leq d \right\}.$$

We had verified that the matrix A is a finite-dimensional approximation of the Koopman operator.

Our goal for Part 2 is to reproduce Figure 2 from the paper A. Mauroy and I. Mezic, *"Global Stability Analysis Using the Eigenfunctions of the Koopman Operator,"* IEEE Transactions on Automatic Control, Vol. 61, No. 11, Nov. 2016. Link: `https://doi-org.tudelft.idm.oclc.org/10.1109/TAC.2016.2518918`.

Task 4 Use your scripts from Practice Session 4 to compute the matrix A for the system

$$\dot{x}_1 = -\frac{3}{4}x_1 - \frac{1}{8}x_2 + \frac{1}{4}x_1x_2 - \frac{1}{4}x_2^2 - \frac{1}{2}x_1^3, \quad \dot{x}_2 = -\frac{1}{8}x_1 - x_2,$$

and parameters $K = 1000$, $d = 25$ and $\tau = 0.001$.

Hint: You should find Koopman eigenvalues at $\lambda \approx -0.698$ and $\lambda \approx -1.052$.

In Lecture 7, we found that if c is an eigenvector of A^T with respect to the eigenvalue $e^{\tau\lambda}$,

$$A^T c = e^{\tau\lambda} c,$$

then under the right circumstances $\psi(x) = c^T g(x)$ is an eigenfunction of the Koopman operator:

$$[U(\tau)\psi](x) = e^{\tau\lambda}\psi(x).$$

Task 5 Compute the eigenvectors c_1 and c_2 of the matrix A^T that correspond to $\lambda \approx -0.698$ and $\lambda \approx -1.052$, respectively. Compute the functions $\psi_1(x) = c_1^T g(x)$ and $\psi_2(x) = c_2^T g(x)$ on the grid $x_1, x_2 \in \{-1, -0.98, -0.96, \ldots, 1\}$. Then, use this data and make a contour plot for each of the functions.

Hint: Use Matlab's `contour` command with activated `'showtext'` option to create the plots. Run `help contour` to get more information.

Hint: Your plots should match Figure 2 (left and right) in the paper for $-1 \leq x_1, x_2 \leq 1$.

Remember from Lecture 7 that the zero-level sets $\psi(x) = 0$ of continuous Koopman eigenfunctions with $\mathscr{R}e\{\lambda\} < 0$ are positively invariant[2] and globally asymptotically stable. Note that this result does not apply directly in our case since we approximated the nonlinear system only in the region

$$M = \{x \in \mathbb{R}^2 \,:\, -1 \leq x_1, x_2 \leq 1\}.$$

However, the result could still be applied assuming that M is positively invariant. In that case, any trajectory in M would have to approach both the zero-level sets in both of our two plots. Since they only cross at the origin, this would imply that the origin is globally asymptotically stable in M.

[2]That is, any trajectory that starts on the level sets stays on the level set.