Practice Session 3
Nonlinear Fourier Transforms

1 Introduction

In the last two lectures we found that solutions of Lax-integrable system have an especially simple representation in terms of "spectral data" that is obtained by an analysis of the spectrum of a linear operator. The evolution of this data was found to be simpler than in the time domain.

The map from a signal $u(t,z)$, t fixed, to the spectral data is called *nonlinear Fourier transform (NFT)*, while the inverse map from the spectral data to $u(t,z)$ is called *inverse NFT (INFT)*. Hence, the problem

$$u(0,z) \overset{\text{Lax-integrable system}}{\longrightarrow} u(t,z)$$

can be solved as follows using NFTs:

$$u(0,z) \overset{\text{NFT}}{\longrightarrow} \text{spectral data at time } 0 \overset{\text{simple evolution}}{\longrightarrow} \text{spectral data at time } t \overset{\text{INFT}}{\longrightarrow} u(t,z). \tag{1.1}$$

This approach generalizes Fourier's method for solving the heat equation, which led to the name NFT.

2 Computation of the KdV-NFT for Multi-Solitons

In Lecture 5, we discussed the NFT of the Korteweg–de Vries (KdV) equation

$$\dot{u} + 6uu' + u''' = 0, \qquad u(t,z) \to 0 \text{ for } z \to \pm\infty \text{ "fast enough"}, \tag{2.1}$$

for the special case that the continuous spectrum of the Schroedinger operator

$$L(t)x(z) = x''(z) + u(t,z)x(z)$$

is negligible. Solutions $u(z,t)$ of the KdV equation that fulfill this condition are known as *multi-solitons* in the literature. (The term reflectionless potentials is also common.) Their spectral data is given by

$$\lambda_1, \dots, \lambda_N, \quad \delta_1(t), \dots, \delta_N(t),$$

where the λ_n are the eigenvalues of the Schroedinger operator $L(t)$. If the corresponding eigenfunctions $\psi^n(z)$ are normalized such that $(\psi^n, \psi^n) = 1$ and $(\psi^n, \psi^m) = 0$ for $m \neq n$, then the δ_n are given by

$$\psi^n(z \to +\infty) \simeq \delta_n e^{-\sqrt{\lambda_n} z}, \qquad \psi^n(z \to -\infty) \simeq \alpha_n e^{\sqrt{\lambda_n} z}, \quad \alpha_n > 0. \tag{2.2}$$

Note that the δ_n implicitly depend on t because the $\psi^n(z)$ are the eigenfunctions of $L(t)$.

Task 1 Write a script that computes the eigenvalues λ_n and eigenfunctions ψ^n for $u(0,z) = 12\,\text{sech}^2(z)$ with $z \in [-25, 25]$. You should find three eigenvalues, the first two being one and four.

Hints: In the previous practice session, you learned how to compute eigenvalues and eigenfunctions of linear operators using Chebfun's `eigs` command. Prefer eigenvalues with larger real parts, and discard non-positive values (since we know from the lecture that eigenvalues have to be positive).

Task 2 We now estimate $\delta_1(0)$ using your results from Task 1. Plot $e^{\sqrt{\lambda_1} z}\psi^1(z)$. Due to (2.2), you should see a plateau close to the right boundary at approximately 3.464. That is our estimate for $\delta_1(0)$.

Hint: If your estimate of $\delta_1(0)$ has the wrong sign, the eigenfunction likely has to be normalized further. Next to $(\psi^1, \psi^1) = 1$, it should also satisfy $\psi^1(z \to -\infty) \simeq \alpha e^{\sqrt{\lambda} z}$ with $\alpha > 0$. See Slide 8 of Lecture 5.

Note that we have determined (most of) the spectral data of $u(0,z)$. That is, we computed a NFT!

3 Evolution of the Spectral Data

During the lecture, we had seen that the spectral data of $u(t, z)$ changes with t in a simple way. We will now verify numerically that the eigenvalues λ_n are independent of t, while the constants $\delta_n(t)$ satisfy

$$\delta_n(t) = e^{4\sqrt{\lambda_n}\lambda_n t}\delta_n(0). \tag{3.1}$$

Task 3 Compute $u(0.5, z)$ for $u(0, z)$ given in Task 1 by solving the KdV equation (2.1) numerically.

Hint: You can use the approach described in `https://www.chebfun.org/examples/pde/KdV.html`, but take into account that the coefficients of our KdV (2.1) are slightly different.

Task 4 Apply your scripts for the Tasks 1 and Task 2 to $u(0.5, z)$. You should find that the eigenvalues stay the same, while $\delta_1(0.5)$ satisfies (3.1) (up to a small numerical deviation).

4 The Inverse KdV-NFT for Multi-Solitons

We found that multi-solitons can be reconstructed from their eigenfunctions using the formula

$$u(t_0, z) = \sum_{n=1}^{\infty} 4\sqrt{\lambda_n}[\psi^n(z)]^2. \tag{4.1}$$

Task 5 Verify the reconstruction formula (4.1) using the eigenvalues and -functions from Task 1. Your script should plot $u(0, z)$ directly using the formula from Task 1, and overlay it with the reconstruction from (4.1). Both curves should match since $u(t, z)$ is a multi-soliton. Repeat for $u(0.5, z)$.

Furthermore, we found that the eigenfunctions are completely determined by the spectral data since they could be recovered by solving the following system of differential equations,

$$\psi^{n\prime\prime} + \sum_{m=1}^{N} 4\sqrt{\lambda_m}[\psi^m]^2\psi^n - \lambda_n\psi^n = 0, \qquad \psi^n(z \to +\infty) \simeq \delta_n(t)e^{-\sqrt{\lambda_n}z}, \qquad n = 1, \ldots, N.$$

Our goal is to verify this for the case of one eigenvalue (that is, $N = 1$). The system reduces to

$$\psi'' + 4\sqrt{\lambda}\psi^3 - \lambda\psi = 0, \qquad \psi(z \to +\infty) \simeq \delta e^{-\sqrt{\lambda}z},$$

where $\psi(z) := \psi^1(z)$, $\lambda := \lambda_1$ and $\delta := \delta_1$. For the numerical solution of this differential equation, we replace "$+\infty$" with a large finite value z_1 and approximate $\psi(z_1)$, $\psi'(z_1)$ using the asymptotic behavior:

$$\psi'' + 4\sqrt{\lambda}\psi^3 - \lambda\psi = 0, \qquad \psi(z_1) = \delta e^{-\sqrt{\lambda}z_1}, \quad \psi'(z_1) = -\sqrt{\lambda}\delta e^{-\sqrt{\lambda}z_1}. \tag{4.2}$$

Task 6 Solve the differential equation (4.2) on the domain $z \in [-z_1, z_1]$ for $z_1 = 10$, $\lambda = 1$, $\delta = \sqrt{2}$, and construct $u(t_0, z)$ using (4.1). Plot your result. It should match $u(t_0, z) = 2\,\mathrm{sech}^2(z)$.

Hint: You can adapt the code for the van der Pol oscillator that is given in Section 10.2 of `https://www.chebfun.org/docs/guide/guide10.html` to solve (4.2) numerically. Note that you have to change `N.lbc` to `N.rbc` since our initial condition is given at the right boundary.

Note that we just reconstructed $u(t_0, z)$ from it's spectral data. That is, we computed an inverse NFT!

Task 7 Repeat Task 6 for $\delta = \frac{1}{2}\sqrt{2}$ and $\delta = 2\sqrt{2}$ and plot the resulting $u(t_0, z)$. What do you observe?

5 Final remarks

The computation of the constants δ_n in the spectral data is a numerically challenging problem. The numerical computation of the inverse NFT is also not trivial. See [12] for recent work from our group.

[1] P.J. Prins and S. Wahls, IEEE Access 7(1), Dec. 2019. `https://doi.org/10.1109/ACCESS.2019.2932256`
[2] P.J. Prins and S. Wahls, Comm Nonlin Sci Numer Simul, Nov. 2021. `https://doi.org/10.1016/j.cnsns.2021.105782`