Practice Session 2

Infinite-dimensional linear state-space systems

1 Introduction

The main goal of this practice session is to point out several useful applications of the functional calculus for linear operators that we discussed briefly during the last lecture.

We will concentrate ourselves on linear partial differential equations (PDEs) of the form

$$\boxed{\frac{\partial u}{\partial t} = a_0 u + a_1 \frac{\partial u}{\partial \Omega} + a_2 \frac{\partial^2 u}{\partial \Omega^2} + \cdots + a_{N-1} \frac{\partial^{N-1} u}{\partial \Omega^{N-1}}, \quad u(0, \Omega) = u_0(\Omega)}$$

where $u = u(t, \Omega)$ is a scalar function of $t \geq 0$ and $\Omega \in \mathbb{R}$, and the a_n are scalar coefficients. Using the polynomial functional calculus, we can rewrite the PDE as an infinite-dimensional state-space system for the state $x(t) := u(t, \cdot)$ and the initial condition $x(0) = u_0$:

$$\boxed{\frac{dx}{dt} = Tx, \quad T := p\left(\frac{d}{d\Omega}\right),}$$

where $\frac{d}{d\Omega}$ is the derivative operator and $p(z) = a_0 + a_1 z + \cdots + a_{N-1} z^{N-1}$ is a scalar polynomial.

2 The spectrum of the differential operator $T = p\left(\frac{d}{d\Omega}\right)$

We first have to choose spaces and domains. Our Hilbert space will be

$$L_2^{\text{per}} := \{f : \mathbb{R} \to \mathbb{C} \,:\, f(\Omega) = f(\Omega + 2\pi) \text{ periodic}, \, \|f\|_2 < \infty\}, \; (f, g) = \frac{1}{2\pi} \int_{-\pi}^{\pi} \overline{f(\Omega)} g(\Omega) d\Omega.$$

A function f is in L_2^{per} if and only if it has a Fourier series representation

$$f(\Omega) = \sum_{n=-\infty}^{\infty} c_n e^{-j\Omega n}, \qquad \{c_n\}_{n=-\infty}^{\infty} \in \ell_2^{\mathbb{Z}},$$

where $\ell_2^{\mathbb{Z}}$ is the Hilbert space of bi-infinite square-summable sequences from the second lecture.

The derivative operator is well-defined on the following domain, which is dense in L_2^{per}:

$$D\left(\frac{d}{d\Omega}\right) = \left\{ f \in L_2^{\text{per}} \,:\, f(\Omega) = \sum_{n=-N}^{N} c_n e^{-j\Omega n} \text{ for suitable coefficients } c_{-N}, \cdots, c_N \text{ with } N < \infty \right\}.$$

(It would be possible to use a larger domain here, but this one is already sufficient to show that the operator is densely defined.) For any $f \in D\left(\frac{d}{d\Omega}\right)$, we have

$$\frac{d}{d\Omega} f(\Omega) = \frac{d}{d\Omega} \sum_{n=-N}^{N} c_n e^{-j\Omega n} = -j \sum_{n=-N}^{N} n c_n e^{-j\Omega n}.$$

Task 1 Write a Matlab program that given the polynomial coefficients a_n plots the spectrum of the operator $T : D(T) \to L_2^{\text{per}}$, $D(T) = D\left(\frac{d}{d\Omega}\right)$, in a user-specified box $\{\lambda \in \mathbb{C} \,:\, \mathscr{Re}(\lambda), \mathscr{Im}(\lambda) \in [-L, L]\}$. Plot the spectrum for the operator that arises from the PDE $\frac{\partial u}{\partial t} = -2u + 3\frac{\partial u}{\partial \Omega} + \frac{\partial^2 u}{\partial \Omega^2} + 2\frac{\partial^3 u}{\partial \Omega^3}$ with $L = 15$.

Hints: First, determine the spectrum of $\frac{d}{d\Omega}$ analytically. (When considered as an operator on $\ell_2^{\mathbb{Z}}$, $\frac{d}{d\Omega}$ is very similar to an example discussed during the third lecture.) Then, exploit that $\sigma(T) = p\left(\sigma(\frac{d}{d\Omega})\right)$.

In Chebfun, we can directly define differential operators using the `chebop` command. Run the command `help chebop` in Matlab to learn more about it. Also, it might be helpful to study the Chebfun example "A periodic ODE system"[1]. Chebfun's `eigs` command furthermore lets us compute eigenvalues of such operators. Run the command `help chebop/eigs` to learn more about it.

Task 2 Use Chebfuns `eigs` command to compute a few eigenvalues of the operator from Task 1. Check if the results match with what you found in Task 1.

3 Solving infinite-dimensional linear state-space systems

We will now investigate how the infinite-dimensional system

$$\frac{dx}{dt} = Tx, \quad x(0) = x_0,$$

can be solved. Similar to the finite-dimensional case, it turns out that

$$\boxed{x(t) = \exp(tT)x(0),}$$

where the operator $\exp(tT)$ is again defined using functional calculus with $\exp(z) := e^z$. The Chebfun command `expm` lets us solve the equation above. Run `help chebop/expm` for more information.

Task 3 Write a Matlab program that computes and plots $x(t)$ at $t = 0.1, 0.5, 2$ using Chebfuns `expm` command with T from Task 1 and the initial condition $x(0) = x_0$ given by $x_0(\Omega) = \sin^2(\Omega) + 0.5\cos(\Omega)$.

Task 4 Repeat Task 3 for $\frac{\partial u}{\partial t} = 2u + 3\frac{\partial u}{\partial \Omega} + \frac{\partial^2 u}{\partial \Omega^2} + 2\frac{\partial^3 u}{\partial \Omega^3}$. You should see a qualitatively different behavior. Also repeat Task 2 and try to explain the difference in behavior by comparing the spectra for both cases.

4 Final Remarks

There are other interesting problem classes beyond linear PDEs that can be represented using infinite-dimensional state-space systems. Delayed state-space systems of the form

$$\dot{x}(t) = A_0 x(t) + \sum_{i=1}^{p} A_i x(t - it_0), \quad x(0) = x_0 \in \mathbb{C}^N,$$

can for example be written as $\dot{x} = Tx$, $T = p(\delta)$, where $\delta : x(t) \mapsto x(t - t_0)$ is a delay operator and $p(z) = A_0 + A_1 z + \cdots + A_p z^p$ is a (matrix-valued) polynomial. The book of Curtain and Zwart[2] is a rich source on the topic. You should have found in Task 4 that the spectrum of the two systems is connected to the stability as in the linear case. However, be aware that is true only under certain conditions:

> *"Unfortunately, for infinite-dimensional systems stability is no longer determined by the position of the spectrum and one may have operators with 'poles in the left half-plane' which are not stable. A simple example is given in Chapter 5 using the shift operator. Fortunately, in the case of Riesz spectral operators and retarded delay systems, the stability is determined by the spectrum"* – S. P. Banks, Automatica 37(4), 2001. (This article is a book review of [2].)

Extra Task for Interested Students Similarly to Task 1, functional calculus can be used to compute the spectra of upper triangular Toeplitz operators as discussed in the previous practice session (replace $\frac{d}{d\Omega}$ with a shift operator). Write a Matlab program based on this idea that plots the spectrum of the Toeplitz operator from the previous practice session. You can check your result using EigTool[3].

[1] `http://www.chebfun.org/examples/ode-linear/PeriodicSystem.html`
[2] R. F. Curtain and H. J. Zwart, *"An Introduction to Infinite-Dimensional Linear Systems Theory,"* Springer, 1995.
[3] `http://www.cs.ox.ac.uk/projects/pseudospectra/eigtool/`