Practice Session 1
Toeplitz Operators

1 Introduction

In this practice session, we investigate the connection between Toeplitz operators and robust control. We investigate the system-theoretic interpretation of the operator norm, and how it is related to the H^∞-norm of the transfer function. We implement numerical methods to approximate them.

Let us consider a discrete-time single-input single-output state-space system

$$\begin{cases} w_{n+1} &= Aw_n + bx_n \\ y_n &= c^T w_n + dx_n \end{cases}, \quad n = 1, 2, 3, \cdots, \tag{1.1}$$

where $A \in \mathbb{C}^{d\times d}$, $b, c \in \mathbb{C}^{d\times 1}$ and $d \in \mathbb{C}$. For the numerical examples, we shall use the values

$$A = \begin{bmatrix} 0.5 & 3 & 2 \\ & -0.5 & -1 \\ & & 0.2 \end{bmatrix}, \quad b = \begin{bmatrix} 1 \\ 0 \\ -2 \end{bmatrix}, \quad c = \begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix}, \quad d = 0.2. \tag{1.2}$$

If the initial state of the system is zero, $w_1 = 0$, then the relation between the input sequence $x = \{x_k\}_{k=1}^\infty$ and the output sequence $y = \{y_k\}_{k=1}^\infty$ is given by the linear operator equation

$$\underbrace{\begin{bmatrix} y_1 \\ y_2 \\ y_3 \\ y_4 \\ \vdots \end{bmatrix}}_{y} = \underbrace{\begin{bmatrix} h_0 & & & & \\ h_1 & h_0 & & & \\ h_2 & h_1 & h_0 & & \\ h_3 & h_2 & h_1 & h_0 & \\ \vdots & \ddots & \ddots & \ddots & \ddots \end{bmatrix}}_{T} \underbrace{\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ \vdots \end{bmatrix}}_{x}, \quad h_n = \begin{cases} 0 & n < 0 \\ d, & n = 0 \\ c^T A^{n-1} b, & n \geq 1 \end{cases}. \tag{1.3}$$

Note that the matrix representation $T = [T_{m,n}]$ satisfies $T_{m,n} = h_{m-n}$. Such operators are known as *Toeplitz operators* in the literature. If all eigenvalues of the matrix A are in the open unit disc $\mathbb{D} := \{\lambda \in \mathbb{C} : |\lambda| < 1\}$, then T is a bounded linear operator on ℓ_2, that is, $T \in \mathscr{L}(\ell_2)$. The operator norm of T,

$$\|T\| = \sup_{\substack{x\in\ell_2 \\ x\neq 0}} \frac{\|Tx\|_2}{\|x\|_2},$$

provides us the maximum gain by which the system (1.1) can amplify the energy of an input. This quantity plays a fundamental role in robust control.

2 A first method to compute the operator norm $\|T\|$

In this section, we utilize a finite-dimensional approximation of T to estimate $\|T\|$. The operator

$$P_N \in \mathscr{L}(\ell_2), \quad \begin{bmatrix} x_1 & \dots & x_N & x_{N+1} & x_{N+2} & \cdots \end{bmatrix}^T \mapsto \begin{bmatrix} x_1 & \dots & x_N & 0 & 0 & \cdots \end{bmatrix}^T$$

is a unitary projection (i.e., $P^2 = P = P^*$). The norm of $T^* P_N$ provides us a lower bound for $\|T\|$:

$$\|T^* P_N\| = \sup_{\substack{x\in\ell_2 \\ x\neq 0}} \frac{\|T^* P_N x\|_2}{\|x\|_2} = \sup_{\substack{x\in P_N\ell_2 \\ x\neq 0}} \frac{\|T^* x\|_2}{\|x\|_2} \leq \sup_{\substack{x\in\ell_2 \\ x\neq 0}} \frac{\|T^* x\|_2}{\|x\|_2} = \|T^*\| = \|T\|.$$

Furthermore, one can show that this lower bound becomes tight as N grows large:

$$\lim_{N\to\infty} \|T^* P_N\| = \|T\|.$$

The operator $T^* P_N$ turns out to be finite-dimensional and can thus be represented by a finite matrix.

Task 1 Determine the matrix representation of $T^* P_N$ and write a Matlab program that, for any given (stable) state-space system (1.1) and N, computes $\|T^* P_N\|$. Use your program to estimate $\|T\|$ for the state-space system (1.2) (increase N until $\|T^* P_N\|$ converges).

Hints: Use the Matlab commands **toeplitz** and **norm**.

3 A second method to compute the operator norm $\|T\|$

In the previous section, we have computed the operator norm of T in the time-domain. Now, we derive a frequency domain method. Equipped with the inner product $(f, g) = \frac{1}{2\pi}\int_0^{2\pi} \overline{f(\Omega)} g(\Omega) d\Omega$, the following space of 2π-periodic square-integrable functions

$$H^2 = \left\{ X : [0, 2\pi) \to \mathbb{C} \,:\, X(\Omega) = \sum_{n=1}^{\infty} x_n e^{-j\Omega n},\ \|X\|_2 := \sqrt{(X, X)} < \infty \right\}$$

forms a Hilbert space. The functions $\phi^n(\Omega) = e^{-j\Omega n}$, $n = 1, 2, \cdots$, provide a countable infinite basis for H^2. We know from the lecture that the operator $U : H^2 \to \ell_2$, $X \mapsto \{(\phi^n, X)\}_n$ is unitary,[1] and H^2 hence isometric to ℓ_2. Since U is unitary, the operator norm of T satisfies

$$\|T\| = \|U^* T U\|$$

and we can consider the operator $U^* T U$ instead. Note that the infinite matrix equation (1.3) describes a convolution and $U^{-1} = U^*$ the discrete-time Fourier transform. One finds that

$$U^* T U \,:\, X(\Omega) \mapsto Y(\Omega) = H(\Omega) X(\Omega), \quad \text{where } H(\Omega) = \sum_{n=0}^{\infty} h_n e^{-j\Omega n}.$$

Finally, we obtain the following upper bound on the operator norm of T:

$$\|T\| = \|U^* T U\| = \sup_{\substack{X \in H^2 \\ X \neq 0}} \frac{\|HX\|_2}{\|X\|_2} = \sup_{\substack{X \in H^2 \\ X \neq 0}} \frac{\sqrt{\frac{1}{2\pi}\int_0^{2\pi} |H(\Omega) X(\Omega)|^2 d\Omega}}{\|X\|_2} \leq \sup_{\Omega \in [0, 2\pi)} |H(\Omega)| \cancel{\sup_{\substack{X \in L^2_{p+} \\ X \neq 0}} \frac{\|X\|_2}{\|X\|_2}}$$

One can also show that this upper bound is tight, but we will skip this step. The quantity $\sup_{\Omega \in [0, 2\pi)} |H(\Omega)|$ is known as the *H^∞-norm* of the system (1.1) in the literature.[2]

Task 2 Write a Matlab program that uses the chebfun and **max** functions from the Chebfun software package (`www.chebfun.org`) to compute $\|T\| = \sup_{\Omega \in [0, 2\pi)} |H(\Omega)|$ for any given (stable) state-space system (1.1). Use your program to compute $\|T\|$ for the state-space system (1.2)

Hint: Since the state-space system is stable, the Neumann series $(I - T)^{-1} = \sum_{k=0}^{\infty} T^k$ implies that

$$d + e^{-j\Omega} c^T (I - e^{-j\Omega} A)^{-1} b = d + e^{-j\Omega} c^T \left(I + A e^{-j\Omega} + A^2 e^{-2j\Omega} + \cdots \right) b = H(\Omega).$$

Task 3 Use Matlabs **norm** command (for linear systems[3]) to verify your result from Task 2.

4 Final remarks

The idea for the first approach for computing $\|T\|$ is adapted from [4]. The relation between the operator norm of T and the H_∞-norm $\sup_\Omega |H(\Omega)|$ is a well-known standard result. In practice, however more sophistic algorithms are used. The Matlab command **norm** e.g. utilizes the algorithm described in [5].

[1] Note that $(\phi^n, X) = \frac{1}{2\pi}\int_0^{2\pi} e^{j\Omega n} X(\Omega) d\Omega$ is nothing but the inverse discrete-time Fourier transform.

[2] Basically, one has to construct a sequence of functions $X^1, X^2, \cdots$ in H^2 such that $|X^n(\Omega)|^2$ converges towards a Dirac δ-pulse centered at a frequency at which the supremum is (close-to) being achieved.

[3] See `https://de.mathworks.com/help/control/ref/norm.html`.

[4] Wahls, Boche and Pohl: *"Zero-forcing precoding for frequency selective MIMO channels with H-infinity criterion and causality constraint,"* Signal Processing, vol. 89, no. 9, pp. 1754–1761, Sep. 2009.

[5] Bruisma and Steinbuch: *"A Fast Algorithm to Compute the H_∞-Norm of a Transfer Function Matrix,"* System Control Letters, 14 (1990), pp. 287-293.