Practice Session 4

Data-Driven Koopman Analysis of Nonlinear Systems

1 Introduction

In the last two lectures, we considered *nonlinear* systems of the form

$$\dot{x} = f(x), \quad y = g(x) = \begin{bmatrix} g_1(x) & g_2(x) & \cdots \end{bmatrix}^T, \quad x(0) = x_0 \in \mathbb{C}^N, \tag{1.1}$$

where the $g_1, g_2, \ldots$ are measurement functions. We found that with the right measurement functions (usually infinitely many), the evolution of the measurements becomes *linear*:

$$\dot{y}(t) = A_c y(t), \quad y(0) = g(x_0). \tag{1.2}$$

If the measurements $y(t)$ are rich enough, then we can use them to recover the state $x(t)$ of the nonlinear system at any time. In such cases, we have transformed a nonlinear system into a linear one. We found that the eigenfunctions of the Koopman operator can achieve this. However, we can only use finitely many measurement functions in practice. Thus, our linearization will only be an approximation. The advantage over "normal" linearizations based on Taylor series is that it is global, and not local.

In this practice session, we numerically verify some of the results from Lecture 7 for the system

$$\dot{x}_1 = -x_1, \quad \dot{x}_2 = -x_2 - x_1^2, \quad x(0) = x_0, \tag{1.3}$$

which we already encountered (with a different sign) in Lecture 7.

2 Data-driven approximation of the Koopman operator

Our approach is as follows. First, we fix $\tau > 0$ and discretize the linear system (1.2):

$$y[n] := y(n\tau) \quad \Longrightarrow \quad y[n+1] = Ay[n], \quad A = \exp(\tau A_c). \tag{2.1}$$

Using *dynamic mode decomposition (DMD)*, we will try to find a matrix A such that (2.1) is satisfied when the $y[n]$ contain measurements of the nonlinear system. The eigenfunctions of the Koopman operator would provide ideal measurements, but we do not know them at this point. In such cases, we saw that a basis for the space of eigenfunctions as measurement functions can be used instead in order to linearize the dynamics. In the following, we shall the following basis of monomials:

$$\mathscr{B}_d := \left\{ h : \mathbb{R}^2 \to \mathbb{R} \mid h(x) = x_1^{k_1} x_2^{k_2}, \text{ where } k_1, k_2 \in \{0, 1, \ldots, d\} \text{ and } k_1 + k_2 \leq d \right\}. \tag{2.2}$$

For example, the basis $\mathscr{B}_2$ consists of the measurement functions

$$g_1(x) = x_1^0 x_2^0, \quad g_2(x) = x_1^0 x_2^1, \quad g_3(x) = x_1^0 x_2^2, \quad g_4(x) = x_1^1 x_2^0, \quad g_5(x) = x_1^1 x_2^1, \quad g_6(x) = x_1^2 x_2^0.$$

Note that $x_1^0 = x_2^0 = 1$. The ordering of the functions is not important and could be different.

The larger d is chosen, the better we can approximate the eigenfunctions of the Koopman operator using the basis elements in $\mathscr{B}_d$. We will now discuss how the matrix A can be estimated if the measurement functions are the elements of $\mathscr{B}_d$. In principle, one can recover the Koopman eigenfunctions from A.

Task 1 Write a Matlab function `y=task_1(x,d)`, where $x \in \mathbb{R}^2$ and $d \in \mathbb{N}$, that returns the vector

$$y = g(x) = \begin{bmatrix} g_1(x) & g_2(x) & \ldots & g_{|\mathscr{B}_d|}(x) \end{bmatrix}^T.$$

Here, $|\mathscr{B}_d|$ denotes the number of elements in the basis $\mathscr{B}_d$.

Hint: For $x = [\ \frac{1}{2} \quad \frac{1}{3}\ ]^T$ and $d = 2$, you should get $y = [\ 1 \quad \frac{1}{3} \quad \frac{1}{9} \quad \frac{1}{2} \quad \frac{1}{6} \quad \frac{1}{4}\ ]^T$ (up to a permutation).

Since DMD is a data-driven method, we now will generate some data numerically.

Task 2 Write a Matlab function `Xp=task_2(X,tau)`, where

$$X = \left[\ x_0^{(1)} \quad x_0^{(2)} \quad \dots \quad x_0^{(K)}\ \right]$$

is a matrix of initial conditions and $\tau > 0$. For each $x_0^{(k)}$, the function computes the value of the resulting trajectory $x^{(k)}(t)$ at $t = \tau$ by solving (1.3) numerically. The function returns the vector

$$X' = \left[\ x^{(1)}(\tau) \quad x^{(2)}(\tau) \quad \dots \quad x^{(K)}(\tau)\ \right].$$

Hint: You can use Matlab's `ode45` function to solve (1.3) numerically. Run `help ode45` in Matlab to get more information. For $X = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ and $\tau = 0.1$, your function should return $X' = \begin{bmatrix} 0.9048 & 0 \\ -0.0861 & 0.9048 \end{bmatrix}$.

DMD now tries to estimate A in (2.1) as follows. Let $y^{(1)}[n], \dots, y^{(K)}[n]$ denote the solutions of (2.1) for K different random initial conditions x_0. We build data matrices

$$Y = \left[\ y^{(1)}[0] \quad \cdots \quad y^{(K)}[0]\ \right], \quad Y' = \left[\ y^{(1)}[1] \quad \cdots \quad y^{(K)}[1]\ \right] \tag{2.3}$$

and approximate the matrix A by solving

$$\|A\|_F = \min \text{ subject to } \|Y' - AY\|_F^2 = \min_{\tilde{A}} \|Y' - \tilde{A}Y\|_F^2. \tag{2.4}$$

Task 3 Write a Matlab function `[A,Y,Yp]=task_3(K,d,tau)`, where $K, d \in \mathbb{N}$ and $\tau > 0$. The function builds the data matrices (2.3) for initial conditions $x_0^{(k)}$ that are chosen randomly from the box $[-1,1]^2$ with a uniform distribution, and then solves (2.4) for A.

Hint: Use your functions from Task 1 and 2. Use Matlab's `lsqminnorm` function to solve (2.4). For $K = 10000$, $d = 1$ and $\tau = 1$, you should find

$$A \approx \begin{bmatrix} 1 & 0 & 0 \\ -0.08 & 0.37 & 0 \\ 0 & 0 & 0.37 \end{bmatrix}.$$

The matrix A thus describes the action of the Koopman operator for a specific set of measurement functions. Since our measurement functions from a basis, we can use A to find Koopman eigenfunctions.

3 Relation to Koopman Eigenvalues and Eigenfunctions

In Lecture 6, we found that if the Koopman semigroup generator U_0 has an eigenfunction

$$\psi(x) = c^T g(x) = \sum_{n=1}^{|\mathscr{B}_d|} c_n g_n(x) \text{ such that } U_o\psi = \lambda\psi$$

and (2.1) is fulfilled exactly, then the coefficient vector c is an eigenvector of A^T with eigenvalue $e^{\lambda\tau}$:

$$A^T c = e^{\lambda\tau} c.$$

We now want to verify this relation numerically. Therefore, we first verifying the accuracy of the linearization and then compute the numerical approximations of the Koopman eigenvalues.

Task 4 Run your function from Task 3 for $K = 1000$, $d = 10$ and $\tau = 0.001$, and verify that the linearization error

$$\frac{1}{\tau} \frac{\|Y' - AY\|_F}{\|Y'\|_F}$$

is small. Compute the eigenvalues $e^{\lambda\tau}$ and eigenvectors c of A^T and plot the Koopman eigenvalues λ.

Hint: The Koopman eigenvalues in the range $\mathscr{R}e(\lambda) \geq -5.5$ should lie at -5,-4, $\ldots$, -1, 0.

In Lecture 7, we found that the functions

$$\psi^{(1)}(x) = x_1 \text{ and } \psi^{(2)}(x) = x_1^2 - x_2 \tag{3.1}$$

are eigenfunctions of the Koopman generator U_0 with eigenvalue $\lambda = -1$.

Task 5 Find the coefficient vectors c_1 and c_2 that correspond to $\psi^{(1)}$ and, respectively, $\psi^{(2)}$. Verify that $A^T c_1 = e^{\lambda\tau} c_1$ and $A^T c_2 = e^{\lambda\tau} c_2$.

4 Invariant Sets from Koopman Eigenfunctions

The Koopman generator U_0 has the two eigenfunctions (3.1) for the eigenvalue $\lambda = -1$. Since

$$\lim_{t\to\infty} |\psi^{(k)}(x(t))| = \lim_{t\to\infty} |e^{\lambda t}\psi^{(k)}(x(0))| = 0, \qquad k = 1, 2,$$

every trajectory $x(t)$ of the system must approach the corresponding zero-level sets

$$Z_k := \left\{ x \in \mathbb{R}^2 \,:\, \psi^{(k)}(x) = 0 \right\}.$$

(Note that this only works for continuous eigenfunctions.)

Task 6 Plot the zero-level sets Z_1 and Z_2 in the range $-4 \leq x_1, x_2 \leq 4$.

You should find that the zero-level sets only intersect at the origin, $Z_1 \cap Z_2 = \{0\}$. Since every solution must approach both of them, we find that every solution must converge towards the origin!

The zero-level are furthermore positively invariant since

$$|\psi^{(k)}(x(0))| = 0 \quad \Longrightarrow \quad |\psi^{(k)}(x(t))| = |e^{-\lambda t}\psi^{(k)}(x(0))| = 0, \quad \forall t \geq 0.$$

That is, any trajectory starting on Z_1 or Z_2 stays on Z_1 or Z_2, respectively, at all times.

Task 7 Compute the trajectories $x(t)$ for the initial conditions $x_0 = [\ \sqrt{2} \quad 2\ ]^T$ and $x_0 = [\ 0 \quad -2\ ]^T$, and add them to your plot from Task 6.

Hint: The initial conditions lie on the invariant sets. Hence, the trajectories should not leave them.