The Income Fluctuation Problem with Capital Income Risk: Optimality and Stability[1]

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ABSTRACT. This paper studies the income fluctuation problem with capital income risk (i.e., dispersion in the rate of return to wealth). Wealth returns and labor earnings are allowed to be serially correlated and mutually dependent. Rewards can be bounded or unbounded. Under rather general conditions, we develop a set of new results on the existence and uniqueness of solutions, stochastic stability of the model economy, as well as efficient computation of the ergodic wealth distribution. A variety of applications are discussed. Quantitative analysis shows that both stochastic volatility and mean persistence in wealth returns have nontrivial impact on wealth inequality.

Keywords: Income fluctuation, optimality, stochastic stability, wealth distribution.

1. INTRODUCTION

The income fluctuation problem refers to the broad class of decision problems that characterize the optimal consumption-saving behavior for agents facing stochastic income streams. In most cases, agents are subject to idiosyncratic shocks and borrowing constraints. Markets are incomplete so idiosyncratic risks cannot be fully diversified or hedged. The model represents one of the fundamental workhorses of modern macroeconomics, and has been adopted to study a large variety of important topics, ranging

from asset pricing, life-cycle choice, fiscal policy, social security, to income and wealth inequality, among many others. See, for example, Schechtman (1976), Deaton and Laroque (1992), Huggett (1993), Aiyagari (1994), Carroll (1997), Chamberlain and Wilson (2000), Cagetti and De Nardi (2008), De Nardi et al. (2010), Guner et al. (2011), Guvenen (2011), Meghir and Pistaferri (2011), Meyer and Sullivan (2013), Guvenen and Smith (2014) and Heathcote et al. (2014).

In recent years, researchers have come to investigate an important mechanism in the income fluctuation framework—the dispersion in rates of return to wealth, referred to below as the capital income risk. Early studies are provided by Angeletos and Calvet (2005) and Angeletos (2007). These works highlight that the macroeconomic effects of idiosyncratic capital income risk can be both qualitatively distinct from those of idiosyncratic labor income risk and quantitatively significant.

An especially important set of applications concerns wealth inequality. As is well known in the literature, the classic income fluctuation frameworks of Huggett (1993) and Aiyagari (1994), in which returns to wealth are homogeneous across agents, fail to reproduce the high inequality and the fat upper tail of wealth distributions in many economies. Such empirical failure has prompted researchers to investigate models with uninsured capital income risk. Entrepreneurial risk, a representative example of capital income risk, is studied by Quadrini (2000) and Cagetti and De Nardi (2006). By introducing heterogeneity across agents in their work and entrepreneurial ability, these studies successfully generate skewed wealth distributions that are more similar to those observed in the U.S. data.

Moreover, in an OLG economy with intergenerational transmission of wealth, Benhabib et al. (2011) show that capital income risk is the driving force of the heavy-tail properties of the stationary wealth distribution. In a Blanchard-Yaari style economy, Benhabib et al. (2016) show that idiosyncratic investment risk has a big impact on generating a double Pareto stationary wealth distribution. In another important contribution, Gabaix et al. (2016) point out that a positive correlation of returns with wealth ("scale dependence") in addition to persistent heterogeneity in returns ("type dependence") can well explain the speed of changes in the tail inequality observed in

the data. An important work that is highly pertinent to the present paper is Benhabib et al. (2015). In a stylized infinite horizon income fluctuation problem with capital income risk, the authors prove that there exists a unique stationary wealth distribution that displays fat tail.

On the empirical side, using twelve years of population data from Norway's administrative tax records, Fagereng et al. (2016a,b) document that individuals earn markedly different average returns to both their financial assets (a standard deviation of 14%) and net worth (a standard deviation of 8%). Wealth returns are heterogeneous both within and across asset classes. Returns are positively correlated with the wealth level and highly persistent over time. In addition, wealth returns are (mildly) correlated across generations.

Although theoretical, empirical and quantitative studies all reveal the significant economic impact of capital income risk, existing models of capital income risk in the income fluctuation framework are highly stylized. For example, the assumptions of IID labor income process, IID wealth return process and their mutual independence made by Benhabib et al. (2015) are rejected by the empirical data in several economies (see, e.g., Kaplan and Violante (2010), Guvenen and Smith (2010) and Fagereng et al. (2016a,b)). As Benhabib et al. (2015) point out, adding positive correlations in labor earnings and wealth returns enriches model dynamics in that it captures economic environments with limited social mobility.

To our best knowledge, a general theory of capital income risk in the income fluctuation framework has been missing in the literature. This raises concerns about whether or not existing views on the economic impact of capital income risk hold in general, as well as whether or not modeling capital income risk in more generic and realistic settings is technically achievable. To be specific, several important questions are:

- Do correlations in the wealth return process (e.g., those caused by mean persistence or stochastic volatility of wealth returns) enhance or dampen the macroeconomic impact of capital income risk?
- What if, in addition to serial correlation, the wealth return process and the labor earnings process are mutually correlated?

- Does an optimal policy always exist in these generalized settings? If it does, is it unique?
- Does the stochastic law of motion for optimal wealth accumulation yield a stationary distribution of wealth?
- If it does, is the model economy globally stable, in the sense that the stationary distribution is unique and can be approached by the distributional path from any starting point?
- How do we compute the optimal policy and the stationary wealth distribution in practice?

These questions are highly significant, in the sense that a negative answer to any of them will pose a threat to the existing findings concerning capital income risk. However, due to technical limitations, these questions have not been investigated in a general income fluctuation framework. In this paper, we attempt to fill this gap. To this end, we extend the standard income fluctuation problem by characterizing the following essential features.

- Agents face idiosyncratic rate of return to wealth $\{R_t\}$ (capital income risk) and idiosyncratic labor earnings $\{Y_t\}$ (labor income risk), both of which are affected by a generic, exogenous Markov process $\{z_t\}$.
- Supports of $\{R_t\}$ and $\{Y_t\}$ are bounded or unbounded, and, in either case, allowed to contain zero.
- The reward (utility) function is bounded or unbounded, and no specific structure is imposed beyond differentiability, concavity and the usual slope conditions.

As can be seen, general $\{R_t\}$ and $\{Y_t\}$ processes that are serially correlated and mutually dependent are covered by our framework. Moreover, consumption can become either arbitrarily small or arbitrarily large, so that agents are allowed to borrow up to the highest sustainable level of debt, creating rich and substantial model dynamics reflecting agents' borrowing activity.[2]

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[2]See the discussion of Rabault (2002).

We make several tightly connected contributions on optimality, stochastic stability and computation of this generalized income fluctuation problem.

First, we prove that the Coleman operator adapted to this framework is indeed an "n-step" contraction mapping in a complete metric space of candidate consumption policies, even when rewards are unbounded. The unique fixed point is shown to be the optimal policy (also unique in the candidate space), and several important properties (e.g., continuity and monotonicity) are derived. To tackle unboundedness, we draw on and extend Li and Stachurski (2014) by adding capital income risk and constructing a metric that evaluates consumption differences in terms of marginal utility. To obtain contractions under a minimal level of restriction, we focus our key assumption on bounding the *long-run growth rate* of wealth returns.

We show that this assumption is indeed equivalent to bounding the spectral radius of an expected wealth return operator (a bounded linear operator) by $1/\beta$. As a result, it is similar to the assumptions made by recent literature regarding the operator theoretic method, which have been proven both necessary and sufficient for the existence and uniqueness of solutions in a variety of models (see, e.g., Hansen and Scheinkman (2009, 2012), Borovička and Stachurski (2017) and Toda (2018)). Our assumption is easy to verify numerically. For example, when the state space for the exogenous Markov process $\{z_t\}$ is finite, verifying this assumption is as convenient as finding the largest modulus of the set of eigenvalues for a given matrix.

Second, as our most significant contribution, we show that the model economy is globally stable, even in the presence of capital income risk. Specifically, there exists a unique stationary distribution for the state process (including wealth and the exogenous Markov state), and, given any initial state, the distributional path of the state process generated via optimal consumption and wealth accumulation converges to the stationary distribution as time iterates forward. The idea of proof goes as follows. Based on the optimality results established in the previous step, existence of a stationary distribution is guaranteed under some further restrictions on agents' level of patience, plus some mild assumptions on the stochastic properties of the exogenous state and the labor income processes. The key is to show that the wealth process is bounded in probability.

The proof of global stability is more tricky and separated into two scenarios.

(Scenario I) When the exogenous state process $\{z_t\}$ is independent and identically distributed, so are $\{R_t\}$ and $\{Y_t\}$, and wealth is the only state variable remaining. We show that, with some additional concavity structure imposed, the model economy is monotone, allowing us to use some new results in the field of stochastic stability (due to Kamihigashi and Stachurski (2014, 2016)). Based on these results, both global stability and the Law of Large Numbers are established. In this case, convergence of the distributional path to its stationarity is in the form of weak convergence. Moreover, the added concavity assumption holds for standard utilities such as CRRA or the logarithm utility. Notably, even in the current case, our theory extends the stability theory of Benhabib et al. (2015), since we allow $\{R_t\}$ and $\{Y_t\}$ to be dependent on each other (a more detailed comparison is given below).

(Scenario II) When the exogenous state process $\{z_t\}$ is Markovian, $\{R_t\}$ and $\{Y_t\}$ are in general autocorrelated and mutually dependent, and the structure of monotone economy is lost due to the added exogenous state. As a result, the order theoretic approach used in the previous case is no longer applicable. In response to that, we aim to exploit the traditional theory of stochastic stability (see, e.g., Meyn and Tweedie (2009)). Specifically, we provide sufficient conditions for the state process to be ψ-irreducible, strongly aperiodic and a positive Harris chain, which in turn guarantee global stability and the Law of Large Numbers. Convergence here is in total variation norm distance, which is stronger than weak convergence. Our sufficient conditions are easy to verify in applications, and centered around existence of density representations for the exogenous state process and the labor earnings process. We only require that supports of the two densities contain respectively a nontrivial compact subset and a certain "small" interval. Importantly, no further concavity structure is required.

Moreover, we show in this scenario that if we add the same concavity structure as we do in scenario I and some other mild assumptions (e.g., existence of densities for the wealth return process and geometric drift property of the labor earnings process), then the model economy is indeed V-geometrically ergodic. As a result, convergence to the stationary distribution occurs at a geometric speed.

Since an IID process is a special Markov process, as a byproduct, the theory in scenario II serves as an alternative stability theory when the exogenous state process is IID. As can be seen from the discussion above, neither of the two theories is "stronger" than the other in this circumstance. On the one hand, global stability in scenario I is established under an additional concavity assumption, which is not required for global stability in scenario II. On the other hand, we make no assumptions on the density structure of the key stochastic processes in scenario I as we do in scenario II.

Based on the established stability and ergodicity results, the unique stationary distribution can be approximated via tracking a single state process simulated according to the optimal consumption and wealth accumulation rules, which is highly efficient. The real caveat is that, in presence of capital income risk, there can be very large realized values of wealth (and consumption), causing serious problems to numerical computation of the optimal policy. However, this problem is alleviated in our setting. We show that, under our maintained assumptions, the optimal policy is concave and asymptotically linear with respect to the wealth level. Hence, at large levels of wealth, the optimal consumption rule can be well approximated via linear extrapolation.

We provide several important applications. First, we illustrate how our theory can be applied to modeling capital income risk in different scenarios. Then, we provide a numerical example in which we explore the quantitative effect of stochastic volatility and mean persistence of the wealth return process on wealth inequality. In the calibrated economy, our quantitative analysis shows that both these two factors lead to lower tail exponents of the stationary wealth distribution and higher Gini coefficients, and thus a higher level of wealth inequality.

In terms of connections to the existing literature, the most closely related results are those found in the recent paper Benhabib et al. (2015). Like us, the authors study capital income risk in an income fluctuation framework. On the one hand, their paper proves an important theoretical result—the stationary wealth distribution has a fat tail, a topic not treated by the present paper (tail properties are only studied by us numerically).

On the other hand, our theory of optimality and stochastic stability is considerably sharper and covers a much broader range of applications. Specifically, to avoid

technical complication, Benhabib et al. (2015) assume that $\{R_t\}$ and $\{Y_t\}$ are IID, mutually independent, supported on bounded closed intervals with strictly positive lower bounds, and that their distributions are represented by densities. Albeit helpful for simplifying analysis and deriving tail properties, these assumptions rule out important features observed in the real economy (e.g., mean persistence and stochastic volatility in the empirical labor earnings and wealth return processes, as discussed). Moreover, the strictly positive lower bound for $\{Y_t\}$ prevents agents from borrowing up to the highest sustainable level of debt, hiding substantial model dynamics.[3] As described above, all these assumptions are relaxed in our framework.

Regarding earlier literature, specific types of capital income risk are modeled by Quadrini (2000), Angeletos and Calvet (2005), Cagetti and De Nardi (2006) and Angeletos (2007) in general equilibrium frameworks. In comparison, the present paper focuses on constructing a "more general" one-sector framework and deriving sharper theoretical results, which, of course, could potentially benefit "more general" general equilibrium analysis.

Moreover, since we tackle unbounded rewards and the associated technical complication, our paper is also related to Rabault (2002), Carroll (2004), Kuhn (2013) and Li and Stachurski (2014). These works develop different methods to handle the issue of unboundedness in standard income fluctuation problems (ones without capital income risk). While Carroll (2004) constructs a weighted supremum norm contraction and works with the Bellman operator, the other three works focus on the Coleman operator. In particular, Rabault (2002) exploits the monotonicity structure, Kuhn (2013) applies a version of the Tarski's fixed point theorem, while Li and Stachurski (2014) constructs a contraction mapping based on a metric that evaluates consumption differences in marginal values. As discussed above, the present paper draws on and extends Li and Stachurski (2014) by incorporating capital income risk.

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[3]As discussed in Rabault (2002), in this case, agents are guaranteed a strictly positive minimum level of consumption, so the marginal value of consumption is bounded, and the problem can be easily solved by constructing supremum norm contractions. However, relaxing this assumption allows agents to systematically avoid exhausting their borrowing capacity.

The rest of this paper is structured as follows. Section 2 formulates the problem. Section 3 establishes optimality results. Sufficient conditions for the existence and uniqueness of optimal policies are discussed. Section 4 focuses on stochastic stability. Global stability and some further properties are studied. Section 5 provides a set of applications. All proofs are deferred to the appendix.

2. SET UP

This section sets up the income fluctuation problem to be studied. As a first step, we introduce some mathematical techniques and notation used in this paper.

2.1. **Preliminaries.** Let $\mathbb{N}$, $\mathbb{R}$ and $\mathbb{R}_+$ be the natural, real and nonnegative real numbers respectively. Given topological space S, let $\mathscr{B}(\mathsf{S})$ be the Borel σ-algebra and let $\mathscr{P}(\mathsf{S})$ be the set of probability measures on $\mathscr{B}(\mathsf{S})$. A *stochastic kernel* Q on S is a map $Q \colon \mathsf{S} \times \mathscr{B}(\mathsf{S}) \to [0,1]$ such that

- $x \mapsto Q(x, B)$ is $\mathscr{B}(\mathsf{S})$-measurable for each $B \in \mathscr{B}(\mathsf{S})$ and
- $B \mapsto Q(x, B)$ is a probability measure on $\mathscr{B}(\mathsf{S})$ for each $x \in \mathsf{S}$.

Let $bc\mathsf{S}$ be the set of bounded continuous functions on S. A stochastic kernel Q is called *Feller* if $x \mapsto \int h(y)Q(x, \mathrm{d}y)$ is in $bc\mathsf{S}$ whenever $h \in bc\mathsf{S}$.

For all $t \in \mathbb{N}$, we define the *t-th order kernel* as

$$Q^1 := Q, \quad Q^t(x, B) := \int Q^{t-1}(y, B) Q(x, \mathrm{d}y) \quad (x \in \mathsf{S}, B \in \mathscr{B}(\mathsf{S})).$$

The value $Q^t(x, B)$ represents the probability of transitioning from x to B in t steps. Furthermore, for all $\mu \in \mathscr{P}(\mathsf{S})$, we define $\mu Q^t \in \mathscr{P}(\mathsf{S})$ as

$$(\mu Q^t)(B) := \int Q^t(x, B) \mu(\mathrm{d}x) \qquad (B \in \mathscr{B}(\mathsf{S})).$$

A sequence $\{\mu_n\} \subset \mathscr{P}(\mathsf{S})$ is called *tight*, if, for all $\varepsilon > 0$, there exists a compact $K \subset \mathsf{S}$ such that $\mu_n(\mathsf{S} \backslash K) \leq \varepsilon$ for all n. We say that μ_n *converges to μ weakly* and write $\mu_n \xrightarrow{w} \mu$ if $\mu \in \mathscr{P}(\mathsf{S})$ and $\int h \, \mathrm{d}\mu_n \to \int h \, \mathrm{d}\mu$ for all bounded continuous $h \colon \mathsf{S} \to \mathbb{R}$.

A stochastic kernel Q is called *bounded in probability* if the sequence $\{Q^t(x, \cdot)\}_{t \geq 0}$ is tight for all $x \in \mathsf{S}$. We call $\psi \in \mathscr{P}(\mathsf{S})$ *stationary* for Q if $\psi Q = \psi$. We say that

Q is *globally stable* if there exists a unique stationary distribution ψ in $\mathscr{P}(\mathsf{S})$ and $\psi_0 Q^t \overset{w}{\to} \psi$ for all $\psi_0 \in \mathscr{P}(\mathsf{S})$.

Let K be a bounded linear operator from $bc\mathsf{S}$ to itself and $\|\cdot\|$ be the supremum norm on $bc\mathsf{S}$. The *operator norm* and *spectral radius* of K are defined by

$$\|K\| := \sup\{\|Kg\| : g \in bc\mathsf{S}, \ \|g\| \leq 1\} \quad \text{and} \quad r(K) := \lim_{m \to \infty} \|K^m\|^{1/m}.$$

In particular, when S is finite, K becomes a square matrix, and the spectral radius $r(K)$ reduces to $\max_\lambda |\lambda|$, where λ ranges over the set of eigenvalues of K. (See, e.g., page 663 of Aliprantis and Border (2006)).

In what follows, $(\Omega, \mathscr{F}, \mathbb{P})$ is a fixed probability space on which all random variables are defined, while $\mathbb{E}$ is expectations with respect to $\mathbb{P}$.

2.2. **The income fluctuation problem.** We introduce capital income risk and consider a generalized income fluctuation problem as follows

$$\max \mathbb{E} \left\{ \sum_{t \geq 0} \beta^t u(c_t) \right\}$$

$$\text{s.t.} \quad a_{t+1} = R_{t+1}(a_t - c_t) + Y_{t+1}, \tag{1}$$

$$0 \leq c_t \leq a_t, \quad (a_0, z_0) = (a, z) \text{ given,}$$

where $\beta \in [0, 1)$ is a state-independent discount factor, u is the utility function, the control process $\{c_t\}_{t \geq 0}$ is consumption, $\{R_t\}_{t \geq 1}$ is a gross rate of return on wealth and $\{Y_t\}_{t \geq 1}$ is labor income. The return and income processes obey

$$R_t = R(z_t, \zeta_t), \quad \{\zeta_t\}_{t \geq 1} \overset{\text{IID}}{\sim} \nu,$$

$$Y_t = Y(z_t, \eta_t), \quad \{\eta_t\}_{t \geq 1} \overset{\text{IID}}{\sim} \mu, \tag{2}$$

where R and Y are nonnegative real-valued measurable functions, $\{\zeta_t\}$ and $\{\eta_t\}$ are innovations, and $\{z_t\}_{t \geq 0}$ is a time-homogeneous Z-valued Markov process with Feller stochastic kernel P, where Z is a Borel subset of $\mathbb{R}^m$ paired with the usual relative topology.

Throughout we make the following assumption on the agent's utility.

Assumption 2.1. The utility function $u\colon \mathbb{R}_+ \to \{-\infty\} \cup \mathbb{R}$ is twice differentiable on $(0, \infty)$ and satisfies

(1) $u' > 0$ and $u'' < 0$ everywhere on $(0, \infty)$, and
(2) $u'(c) \to \infty$ as $c \to 0$ and $u'(c) \to 0$ as $c \to \infty$.

Example 2.1. A typical example that meets assumption 2.1 is the CRRA utility

$$u(c) = c^{1-\gamma}/(1-\gamma) \quad \text{if } \gamma > 0,\ \gamma \neq 1 \quad \text{and} \quad u(c) = \log c \quad \text{if } \gamma = 1, \tag{3}$$

where $\gamma > 0$ is the coefficient of relative risk aversion.

2.3. **Further notation.** We use x and $\hat{x}$ to denote respectively the current and next period random variables. In addition,

$$\mathbb{E}_{a,z} := \mathbb{E}\left[\,\cdot\,\middle|\,(a_0, z_0) = (a, z)\right] \quad \text{and} \quad \mathbb{E}_z := \mathbb{E}\left[\,\cdot\,\middle|\, z_0 = z\right]. \tag{4}$$

In particular, for any integrable function f,

$$\mathbb{E}_z f(\hat{z}, \hat{R}, \hat{Y}) = \int f\left[\hat{z}, R(\hat{z}, \hat{\zeta}), Y(\hat{z}, \hat{\eta})\right] P(z, \mathrm{d}\hat{z})\nu(\mathrm{d}\hat{\zeta})\mu(\mathrm{d}\hat{\eta}). \tag{5}$$

3. Optimality Results

In this section, we show that, with bounded or unbounded rewards, the Coleman operator adapted to the income fluctuation problem above is an n-step contraction mapping on a complete metric space of candidate policies, and that the unique fixed point is the optimal policy. To that end, we make the following assumptions.

Assumption 3.1. There exists $n \in \mathbb{N}$ such that $\theta := \beta^n \sup_{z \in \mathsf{Z}} \mathbb{E}_z R_1 \cdots R_n < 1$.

Assumption 3.2. For all $z \in \mathsf{Z}$, we have $\sum_{t=1}^{\infty} \beta^t \mathbb{E}_z Y_t < \infty$.

Assumption 3.3. $\sup_{z \in \mathsf{Z}} \mathbb{E}_z \hat{R} < \infty$, $\sup_{z \in \mathsf{Z}} \mathbb{E}_z u'(\hat{Y}) < \infty$ and $\sup_{z \in \mathsf{Z}} \mathbb{E}_z \hat{R} u'(\hat{Y}) < \infty$.

Assumption 3.4. The functions $z \mapsto R(z, \zeta)$, $z \mapsto Y(z, \eta)$, $z \mapsto \mathbb{E}_z \hat{R}$ and $z \mapsto \mathbb{E}_z \hat{R}\, u'(\hat{Y})$ are continuous.

Example 3.1. For all bounded continuous function f on Z, define

$$Kf(z) := \mathbb{E}_z \hat{R} f(\hat{z}), \quad z \in \mathsf{Z}.$$

Then K is a bounded linear operator by assumption 3.3. Let $r(K)$ be the spectral radius of K. Then assumption 3.1 holds if and only if $\beta r(K) < 1$. We prove this result in the appendix.

Example 3.2. Let $\{z_t\}$ be a finite-state Markov chain on $\mathsf{Z} := \{i_1, \cdots, i_N\}$ with transition matrix Π (a "discrete" stochastic kernel). Let $\mathrm{diag}(\cdot)$ denote the diagonal matrix generated by elements in the bracket, and, with slight abuse of notation, let

$$\mathbb{E}\, R(z, \zeta) := \int R(z, \zeta) \nu(\mathrm{d}\zeta) \quad \text{and} \quad D := \mathrm{diag}\left(\mathbb{E}\, R(i_1, \zeta), \cdots, \mathbb{E}\, R(i_N, \zeta)\right).$$

In this case, the operator K in example 3.1 reduces to the matrix $K = \Pi D$. Therefore, assumption 3.1 holds if and only if $r(\Pi D) < 1/\beta$. In particular, $r(\Pi D)$ equals the largest modulus of all the eigenvalues of ΠD.

Example 3.3. Based on the Hölder's inequality, to show assumption 3.3, it suffices to find some $p, q \in [1, \infty]$ such that $1/p + 1/q = 1$ and

$$\sup_{z \in \mathsf{Z}} \mathbb{E}_z \hat{R}^p < \infty \quad \text{and} \quad \sup_{z \in \mathsf{Z}} \mathbb{E}_z u'(\hat{Y})^q < \infty.$$

To establish the required results, we (temporarily) assume $a_0 > 0$ and set the asset space as $(0, \infty)$. The state space for the state process $\{(a_t, z_t)\}_{t \geq 0}$ is then[4]

$$\mathsf{S}_0 := (0, \infty) \times \mathsf{Z} \ni (a, z).$$

Consider the maximal asset path $\{\tilde{a}_t\}$ defined by

$$\tilde{a}_{t+1} = R_{t+1}\, \tilde{a}_t + Y_{t+1} \quad \text{and} \quad (\tilde{a}_0, \tilde{z}_0) = (a, z) \text{ given.} \tag{6}$$

Lemma 3.1. *If assumptions 3.1–3.2 hold, then $\sum_{t \geq 0} \beta^t \mathbb{E}_{a,z}\, \tilde{a}_t$ is finite for all $(a, z) \in$* S_0.

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[4]Note that the second condition of assumption 2.1 and assumption 3.3 imply that $\mathbb{P}\{Y_t > 0\} = 1$ for all $t \geq 1$ (although Y_t is allowed to be arbitrarilly close to zero). Hence, $\mathbb{P}\{a_t > 0\} = 1$ for all $t \geq 1$ by the law of motion (1). It thus makes no difference to optimality to exclude zero from the asset space. Doing this simplifies analysis since u and u' are finite away from zero. It actually allows us to propose a useful metric and apply the contraction approach, as to be shown later.

A *feasible policy* is a Borel measurable function $c \colon \mathsf{S}_0 \to \mathbb{R}$ with $0 \le c(a, z) \le a$ for all $(a, z) \in \mathsf{S}_0$. Given any feasible policy c and initial condition $(a, z) \in \mathsf{S}_0$, the *asset path* generated by $(c, (a, z))$ is the sequence $\{a_t\}_{t \ge 0}$ in (1) when $c_t = c(a_t, z_t)$ and $(a_0, z_0) = (a, z)$. The *lifetime value* of any feasible policy c is the function $V_c \colon \mathsf{S}_0 \to \{-\infty\} \cup \mathbb{R}$ defined by

$$V_c(a, z) = \mathbb{E}_{a,z} \left\{ \sum_{t \ge 0} \beta^t u\left[c(a_t, z_t)\right] \right\},$$

where $\{a_t\}$ is the asset path generated by $(c, (a, z))$. Notice that $V_c(a, z) < \infty$ for any feasible c and any $(a, z) \in \mathsf{S}_0$. This is because, by assumption 2.1, there exists a constant L such that $u(c) \le c + L$, and hence

$$V_c(a, z) \le \mathbb{E}_{a,z} \sum_{t \ge 0} \beta^t u(a_t) \le \mathbb{E}_{a,z} \sum_{t \ge 0} \beta^t u(\tilde{a}_t) \le \sum_{t \ge 0} \beta^t \mathbb{E}_{a,z}\, \tilde{a}_t + \frac{L}{1 - \beta}.$$

The last expression is finite by lemma 3.1.

A feasible policy c^* is called *optimal* if $V_c \le V_{c^*}$ on S_0 for any feasible policy c. In the present setting, the finiteness of V_c for each feasible policy, the strict concavity of u, and the convexity of the set of feasible policies from each $(a, z) \in \mathsf{S}_0$ imply that for each given parameterization, at most one optimal policy exists.

A feasible policy is said to satisfy the *first order optimality conditions* if

$$(u' \circ c)(a, z) \ge \beta\, \mathbb{E}_z\, \hat{R}\,(u' \circ c)\left(\hat{R}\left[a - c(a, z)\right] + \hat{Y},\, \hat{z}\right) \tag{7}$$

for all $(a, z) \in \mathsf{S}_0$, and equality holds when $c(a, z) < a$. Moreover, a feasible policy is said to satisfy the *transversality condition* if, for all $(a, z) \in \mathsf{S}_0$,

$$\lim_{t \to \infty} \beta^t \mathbb{E}_{a,z}\left[(u' \circ c)(a_t, z_t)\, a_t\right] = 0. \tag{8}$$

Theorem 3.1. *If assumptions 2.1 and 3.1–3.2 hold, and c is a feasible policy that satisfies both the first order optimality conditions and the transversality condition, then c is an optimal policy.*

When does an optimal policy exist, and how can we compute it? To answer these questions, following Li and Stachurski (2014), we use a contraction argument, where the underlying function space is set to $\mathscr{C}$, the functions $c \colon \mathsf{S}_0 \to \mathbb{R}$ such that

(1) c is continuous,

(2) c is increasing in the first argument,

(3) $0 < c(a, z) \le a$ for all $(a, z) \in \mathsf{S}_0$, and

(4) $\sup_{(a,z)\in\mathsf{S}_0} |(u' \circ c)(a, z) - u'(a)| < \infty$.

To compare two policies, we pair $\mathscr{C}$ with the distance

$$\rho(c, d) := \|u' \circ c - u' \circ d\| := \sup_{(a,z)\in\mathsf{S}_0} |(u' \circ c)(a, z) - (u' \circ d)(a, z)| \qquad (9)$$

that evaluates the maximal difference in terms of marginal utility. Note that

$$c \in \mathscr{C} \implies \exists K \in \mathbb{R}_+ \text{ s.t. } u'(a) \le (u' \circ c)(a, z) \le u'(a) + K, \ \forall (a, z) \in \mathsf{S}_0. \qquad (10)$$

Moreover, while elements of $\mathscr{C}$ are not generally bounded, one can show that ρ is a valid metric on $\mathscr{C}$. In particular, ρ is finite on $\mathscr{C}$ since $\rho(c, d) \le \|u' \circ c - u'\| + \|u' \circ d - u'\|$, and the last two terms are finite by the definition of $\mathscr{C}$.

Proposition 3.1. *$(\mathscr{C}, \rho)$ is a complete metric space.*

Proposition 3.2. *If assumptions 2.1 and 3.1–3.3 hold, $c \in \mathscr{C}$, and, for all $(a, z) \in \mathsf{S}_0$,*

$$(u' \circ c)(a, z) = \max \left\{ \beta\, \mathbb{E}_z\, \hat{R}\, (u' \circ c) \left(\hat{R}\,[a - c(a, z)] + \hat{Y}, \, \hat{z} \right), u'(a) \right\}, \qquad (11)$$

then c satisfies both the first order optimality conditions and the transversality condition. In particular, c is an optimal policy.

Inspired by proposition 3.2, we aim to characterize the optimal policy as the fixed point of the *Coleman operator* T defined as follows: for fixed $c \in \mathscr{C}$ and $(a, z) \in \mathsf{S}_0$, the value of the image Tc at (a, z) is defined as the $\xi \in (0, a]$ that solves

$$u'(\xi) = \psi_c(\xi, a, z), \qquad (12)$$

where ψ_c is the function on

$$G := \{(\xi, a, z) \in \mathbb{R}_+ \times (0, \infty) \times \mathsf{Z} : 0 < \xi \le a\} \qquad (13)$$

defined by

$$\psi_c(\xi, a, z) := \max \left\{ \beta \mathbb{E}_z \hat{R}(u' \circ c)[\hat{R}(a - \xi) + \hat{Y}, \, \hat{z}], \, u'(a) \right\}. \qquad (14)$$

The following propositions show that the Coleman operator T is a well-defined self-map from the candidate space $(\mathscr{C}, \rho)$ into itself.

Proposition 3.3. *If assumptions 2.1 and 3.1–3.3 hold, then for each $c \in \mathscr{C}$ and $(a, z) \in \mathsf{S}_0$, there exists a unique $\xi \in (0, a]$ that solves (12).*

Proposition 3.4. *If assumptions 2.1 and 3.1–3.4 hold, then $Tc \in \mathscr{C}$ for all $c \in \mathscr{C}$.*

Recall n and θ defined in assumption 3.1. We now provide our key optimality result.

Theorem 3.2. *If assumptions 2.1 and 3.1–3.4 hold, then T^n is a contraction mapping on $(\mathscr{C}, \rho)$ with modulus θ. In particular,*

(1) *T has a unique fixed point $c^* \in \mathscr{C}$.*
(2) *The fixed point c^* is the unique optimal policy in $\mathscr{C}$.*
(3) *For all $c \in \mathscr{C}$ and $k \in \mathbb{N}$, we have $\rho(T^{nk}c, c^*) \leq \theta^k \rho(c, c^*)$.*

4. STOCHASTIC STABILITY

This section focuses on stochastic stability of the generalized income fluctuation problem. We first provide sufficient conditions for the existence of a stationary distribution and then explore conditions for uniqueness and ergodicity.

Now we add zero back into the asset space, and consider a larger state space for the state process $\{(a_t, z_t)\}_{t \geq 0}$, denoted by

$$\mathsf{S} := [0, \infty) \times \mathsf{Z} \ni (a, z).$$

We extend c^* to S by setting $c^*(0, z) = 0$ for all $z \in \mathsf{Z}$. Together, c^* and the transition functions for $\{a_t\}$, $\{R_t\}$ and $\{Y_t\}$ determine a Markov process with state vector $s_t := (a_t, z_t)$ taking values in the state space S. Let Q denote the corresponding stochastic kernel. The law of motion of $\{s_t\}$ is

$$a_{t+1} = R\left(z_{t+1}, \zeta_{t+1}\right)\left[a_t - c^*\left(a_t, z_t\right)\right] + Y\left(z_{t+1}, \eta_{t+1}\right),$$
$$z_{t+1} \sim P\left(z_t, \cdot\right) \tag{15}$$

4.1. **Existence of a stationary distribution.** To obtain existence of a stationary distribution, we make the following assumptions.

Assumption 4.1. There exists $\alpha \in (0,1)$ such that

(1) $\beta \mathbb{E}_z \hat{R} u'[\hat{R}(1-\alpha)a] \leq u'(a)$ for all $(a,z) \in \mathsf{S}_0$,[5] and

(2) there exists $n \in \mathbb{N}$ such that $(1-\alpha)^n \sup_{z \in \mathsf{Z}} \mathbb{E}_z R_1 \cdots R_n < 1$.

Assumption 4.2. $\sup_{t \geq 1} \mathbb{E}_z Y_t < \infty$ for all $z \in \mathsf{Z}$.

Assumption 4.3. The stochastic kernel P is bounded in probability.

Example 4.1. For homogeneous utility functions (e.g., CRRA), if the first condition of assumption 4.1 holds for some $a \in (0,\infty)$, then it must hold for all $a \in (0,\infty)$. To see this, let k be the degree of homogeneity. Then we have

$$\beta \mathbb{E}_z \hat{R} u'[\hat{R}(1-\alpha)a]/u'(a) = \beta \mathbb{E}_z \hat{R}^{1+k}(1-\alpha)^k \quad \text{for all } a \in (0,\infty).$$

The right hand side is constant in a.

Example 4.2. Recall example 3.2, where $\{z_t\}$ is a finite-state Markov chain. Consider the CRRA utility defined in (3). Define further the column vector

$$V := \left(\mathbb{E}\, R(i_1,\zeta)^{1-\gamma}, \cdots, \mathbb{E}\, R(i_N,\zeta)^{1-\gamma} \right)'.$$

Then, assumption 4.1 holds whenever

$$\max\{r(\Pi D), 1\} < (\beta\|\Pi V\|)^{-1/\gamma}. \tag{16}$$

To see this, the first condition of assumption 4.1 holds if there exists $\alpha \in (0,1)$ such that $(1-\alpha)^{-\gamma}\beta \mathbb{E}_z \hat{R}^{1-\gamma} \leq 1$ for all $z \in \mathsf{Z}$. Since Z is finite, this is equivalent to the existence of an $\alpha \in (0,1)$ such that $(1-\alpha)^{-\gamma}\beta\|\Pi V\| \leq 1$. Similar to example 3.2, the second condition of assumption 4.1 holds if $r(\Pi D) < 1/(1-\alpha)$ for the same α. Together, these requirements are equivalent to (16).

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[5]Here we adopt the convention that $0 \cdot \infty = 0$ so that assumption 4.1 does not rule out the case $\mathbb{P}\{R_t = 0 \mid z_{t-1} = z\} > 0$. Indeed, as would be shown in proofs, all the conclusions of this paper still hold if we replace this condition by the weaker alternative: $\beta \mathbb{E}_z \hat{R} u'[\hat{R}(1-\alpha)a + \alpha\hat{Y}] \leq u'(a)$ for all $(a,z) \in \mathsf{S}_0$, while maintaining the second part of assumption 4.1.

Example 4.3. Benhabib et al. (2015) consider the CRRA utility and assume that $\{R_t\}$ and $\{Y_t\}$ are IID, mutually independent, supported on bounded closed intervals of strictly positive real numbers with their distributions represented by densities, and that $\beta\mathbb{E}\,R_t^{1-\gamma} < 1$ and $(\beta\mathbb{E}\,R_t^{1-\gamma})^{\frac{1}{\gamma}}\mathbb{E}\,R_t < 1$. Under these conditions, assumptions 4.2–4.3 obviously hold. Assumption 4.1 is satisfied by letting $\alpha := 1 - (\beta\mathbb{E}\,R_t^{1-\gamma})^{\frac{1}{\gamma}}$ and $n := 1$. The first condition of assumption 4.1 holds since $\alpha \in (0,1)$ and

$$\beta\mathbb{E}_z\hat{R}\,u'[\hat{R}(1-\alpha)a]/u'(a) = (1-\alpha)^{-\gamma}\beta\mathbb{E}\,R_t^{1-\gamma} = \left(\beta\mathbb{E}\,R_t^{1-\gamma}\right)^{-1}\beta\mathbb{E}\,R_t^{1-\gamma} = 1,$$

while the second condition holds for $n = 1$ since $(1-\alpha)\mathbb{E}\,R_t = (\beta\mathbb{E}\,R_t^{1-\gamma})^{\frac{1}{\gamma}}\mathbb{E}\,R_t < 1$.

Let c^* be the unique optimal policy obtained from theorem 3.2 and α be defined as in assumption 4.1. The next proposition establishes a strictly positive lower bound on the optimal consumption rate.

Proposition 4.1. *If assumptions 2.1, 3.1–3.4 and 4.1 hold, then $c^*(a,z) \geq \alpha a$ for all $(a,z) \in \mathsf{S}$.*

From this result the existence of a stationary distribution is not difficult to verify.

Theorem 4.1. *If assumptions 2.1, 3.1–3.4 and 4.1–4.3 hold, then Q is bounded in probability and admits at least one stationary distribution.*

4.2. **Further Optimality Properties.** Slightly digressed from our main topics, we show that the optimal policy satisifies several other important properties under the following assumption.

Assumption 4.4. The map $s \mapsto (u')^{-1}\left[\beta\mathbb{E}_z\hat{R}\,(u' \circ c)\,(\hat{R}s + \hat{Y},\,\hat{z})\right]$ is concave on $\mathbb{R}_+$ for each fixed $z \in \mathsf{Z}$ and $c \in \mathscr{C}$ that is concave in its first argument.

Example 4.4. Assumption 4.4 imposes some concavity structure on the utility function. It holds for CRRA and logarithmic utilities, as shown in appendix B.

The next proposition implies that, with this added concavity structure, the optimal policy is concave and asymptotically linear with respect to the wealth level.

Proposition 4.2. *If assumptions 2.1, 3.1–3.4, 4.1 and 4.4 hold, then*

(1) $a \mapsto c^*(a, z)$ is concave for all $z \in \mathsf{Z}$, and

(2) for all $z \in \mathsf{Z}$, there exists $\alpha' \in [\alpha, 1)$ such that $\lim_{a \to \infty}[c^*(a, z)/a] = \alpha'$.[6]

By proposition 4.2, as a gets large, $c^*(a, z) \approx \alpha' a + b(z)$ for some function b, which is helpful for numerical computation. In the presence of capital income risk, there can be large realized values of wealth and consumption. This proposition then provides a justification for the linear extrapolation technology adopted when computing the optimal policy at large wealth levels.

4.3. **Global stability.** We start with the case of IID $\{z_t\}$ process, which allows us to exploit the monotonicity structure of the stochastic kernel Q. We then discuss general Markov $\{z_t\}$ processes. Since Q is not generally monotone in these settings,[7] global stability is established via a different approach.

4.3.1. *Case I:* IID $\{z_t\}_{t \geq 0}$ *process.* In this case, both $\{R_t\}$ and $\{Y_t\}$ are IID processes, though dependence between $\{R_t\}$ and $\{Y_t\}$ are allowed. The optimal policy is then a function of asset only, and the transition function (15) reduces to

$$a_{t+1} = R_{t+1}\left[a_t - c^*(a_t)\right] + Y_{t+1}. \tag{17}$$

In particular, we have a Markov process $\{a_t\}_{t \geq 0}$ taking values in $\mathbb{R}_+$. The next result extends theorem 3 of Benhabib et al. (2015).

Theorem 4.2. *If assumptions 2.1, 3.1–3.4, 4.1–4.2 and 4.4 hold, then Q is globally stable.*[8]

Let ψ^* be the unique stationary distribution of Q, obtained in theorem 4.2. Let $\mathscr{L}$ be the linear span of the set of increasing ψ^*-integrable functions $h \colon \mathbb{R}_+ \to \mathbb{R}$.[9] Recall

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[6] Here we rule out the trivial situation $\mathbb{P}\{R_t = 0 \mid z_{t-1} = z\} = 1$, in which case $\alpha' = 1$.

[7] Since the optimal policy $c^*(a, z)$ is not generally monotone in z, we cannot conclude from (15) that a_{t+1} is monotone in z_t. Hence, (a_{t+1}, z_{t+1}) is not necessarily increasing in (a_t, z_t) and monotonicity might fail.

[8] Since $\{z_t\}$ is IID, conditional expectations reduce to unconditional ones. Hence, to verify assumptions 3.1–3.4 and 4.2, it suffices to show: $\mathbb{E}\,R_t^2 < \infty$, $\beta\mathbb{E}\,R_t < 1$, $\mathbb{E}\,Y_t < \infty$ and $\mathbb{E}\,[u'(Y_t)]^2 < \infty$.

[9] In other words, $\mathscr{L}$ is the set of all $h \colon \mathbb{R}_+ \to \mathbb{R}$ such that $h = \alpha_1 h_1 + \cdots + \alpha_k h_k$ for some scalars $\{\alpha_i\}_{i=1}^k$ and increasing measurable $\{h_i\}_{i=1}^k$ with $\int |h_i|\,\mathrm{d}\psi^* < \infty$.

that $bc\mathbb{R}_+$ is the set of continuous bounded functions $h\colon \mathbb{R}_+ \to \mathbb{R}$. The following theorem shows that the Law of Large Numbers holds in this framework.

Theorem 4.3. *If the assumptions of theorem 4.2 hold, then the following statements hold:*

(1) *For all $\mu \in \mathscr{P}(\mathbb{R}_+)$ and $h \in \mathscr{L}$, we have*

$$\mathbb{P}_\mu \left\{ \lim_{T \to \infty} \frac{1}{T} \sum_{t=1}^{T} h(a_t) = \int h \, \mathrm{d}\psi^* \right\} = 1.$$

(2) *For all $\mu \in \mathscr{P}(\mathbb{R}_+)$, we have*

$$\mathbb{P}_\mu \left\{ \lim_{T \to \infty} \frac{1}{T} \sum_{t=1}^{T} h(a_t) = \int h \, \mathrm{d}\psi^* \ \text{ for all } h \in cb\mathbb{R}_+ \right\} = 1.$$

4.3.2. *Case II: Markovian $\{z_t\}_{t \geq 0}$ process.* In this case, $\{R_t\}$ and $\{Y_t\}$ are in general non-IID and mutually dependent processes.[10]

We assume that the stochastic processes $\{z_t\}$ and $\{Y_t\}$ admit density representations denoted respectively by $p(z' \mid z)$ and $f_L(Y \mid z)$. Specifically, there exists a nontrivial measure ϑ on $\mathscr{B}(\mathsf{Z})$ such that

$$P(z, A) = \int_A p(z' \mid z)\vartheta(\mathrm{d}z'), \qquad (A \in \mathscr{B}(\mathsf{Z}), z \in \mathsf{Z}),$$

and for $\mathrm{d}Y := \lambda(\mathrm{d}Y)$, where λ is the Lebesgue measure,

$$\mathbb{P}\{Y_t \in A \mid z_t = z\} = \int_A f_L(Y \mid z) \, \mathrm{d}Y, \qquad (A \in \mathscr{B}(\mathbb{R}_+), z \in \mathsf{Z}).$$

Assumption 4.5. The following conditions hold:

(1) the support of ϑ contains a compact subset C that has nonempty interior,[11]
(2) $p(z' \mid z)$ is strictly positive on $\mathsf{C} \times \mathsf{Z}$ and continuous in z, and
(3) there exists $\delta_Y > 0$ such that $f_L(Y \mid z)$ is strictly positive on $(0, \delta_Y) \times \mathsf{C}$.

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[10]Since this framework encorporates the IID $\{z_t\}$ structure as a special case, this section provides an alternative ergodic theory for the IID framework as a byproduct. By comparing the assumptions of theorem 4.2 and those of theorem 4.4 below, we see that the latter holds without assumption 4.4, so neither of the two theories is more powerful than the other.

[11]The *support* of the measure ϑ is defined as the set of points $z \in \mathsf{Z}$ for which every open neighborhood of z has positive ϑ measure.

Assumption 4.5 is easy to verify in applications. The following examples are some simple illustrations, while more complicated applications are treated in section 5.

Example 4.5. If Z is a countable subset of $\mathbb{R}^m$, then $\{z_t\}$ is a countable state Markov chain, in which case ϑ is the counting measure and $p(z' \mid z)$ reduces to a transition matrix Π. In particular, each single point in Z is a compact subset in the support of ϑ that has nonempty interior (itself), and p is continuous in z by definition. Hence, conditions (1)–(2) of assumption 4.5 hold as long as at least one column of Π is strictly positive (i.e., each element of that column is positive).

Example 4.6. Since Z is a Borel subset of $\mathbb{R}^m$, if ϑ can be chosen as the Lebesgue measure, then condition (1) of assumption 4.5 holds trivially. Indeed, since $P(z, \mathsf{Z}) = 1$, the support of ϑ must contain a nonempty open box (i.e., sets of the form $\Pi_{i=1}^m (a_i, b_i)$ with $a_i < b_i$, $i = 1, \cdots, m$), inside which a compact subset with nonempty interior can be found.

For all measurable map $f \colon \mathsf{S} \to [1, \infty)$ and $\mu \in \mathscr{P}(\mathsf{S})$, we define

$$\|\mu\|_f := \sup_{g \colon |g| \leq f} \left| \int g \, \mathrm{d}\mu \right|.$$

We say that the stochastic kernel Q corresponding to $\{(a_t, z_t)\}_{t \geq 0}$ is f-*ergodic* if

(a) there exists a unique stationary distribution $\psi^* \in \mathscr{P}(\mathsf{S})$ such that $\psi^* Q = \psi^*$,
(b) $f \geq 1$, $\int f \, \mathrm{d}\psi^* < \infty$, and, for all $(a, z) \in \mathsf{S}$,

$$\lim_{t \to \infty} \|Q^t((a, z), \cdot) - \psi^*\|_f = 0.$$

We say that Q is f-*geometrically ergodic* if, in addition, there exist constants $r > 1$ and $M \in \mathbb{R}_+$ such that, for all $(a, z) \in \mathsf{S}$,

$$\sum_{t \geq 0} r^t \|Q^t((a, z), \cdot) - \psi^*\|_f \leq M f(a, z).$$

In particular, if $f \equiv 1$, then Q is called *ergodic/geometrically ergodic*.

The following theorem establishes ergodicity and the Law of Large Numbers. Notably, assumption 4.4 is NOT required for these results.

Theorem 4.4. *If assumptions 2.1, 3.1–3.4, 4.1–4.3 and 4.5 hold, then*

(1) Q *is ergodic, in particular,*

$$\sup_{A \in \mathscr{B}(\mathsf{S})} \left| Q^t\left((a, z), A\right) - \psi^*(A) \right| \to 0 \quad as \ t \to \infty.$$

(2) *For all* $\mu \in \mathscr{P}(\mathsf{S})$ *and map* $h : \mathsf{S} \to \mathbb{R}$ *with* $\int |h| \, \mathrm{d}\psi^* < \infty$,

$$\mathbb{P}_\mu \left\{ \lim_{T \to \infty} \sum_{t=1}^{T} h(a_t, z_t) = \int h \, \mathrm{d}\psi^* \right\} = 1.$$

We next show that geometric ergodicity is guaranteed under some further assumptions. Suppose $\{R_t\}$ admits a density representation $f_C(R \mid z)$, in other words,

$$\mathbb{P}\{R_t \in A \mid z_t = z\} = \int_A f_C(R \mid z) \, \mathrm{d}R, \qquad (A \in \mathscr{B}(\mathbb{R}), \ z \in \mathsf{Z}),$$

where $\mathrm{d}R := \lambda(\mathrm{d}R)$. Recall the IID innovations $\{\zeta_t\}$ and $\{\eta_t\}$ defined by (2) and the compact subset $\mathsf{C} \subset \mathsf{Z}$ defined by assumption 4.5.

Assumption 4.6. The following conditions hold:

(1) there exists $\delta_R > 0$ such that $f_C(R \mid z)$ is strictly positive on $(0, \delta_R) \times \mathsf{C}$,
(2) there exist $q \in [0, 1)$ and $q' \in \mathbb{R}_+$ such that $\mathbb{E}_z Y_2 \leq q \mathbb{E}_z Y_1 + q'$ for all $z \in \mathsf{Z}$,
(3) the innovations $\{\zeta_t\}$ and $\{\eta_t\}$ are mutually independent.

Example 4.7. If either $\{Y_t\}$ is a bounded process or Z is a finite set, then the second condition of assumption 4.6 holds trivially. In particular, if Z is finite, then we can let q be an arbitrary number in $[0, 1)$ and let $q' := \sup_{z \in \mathsf{Z}} \mathbb{E}_z Y_2$, which is finite by assumption 3.2. More general examples are discussed in the next section.

Let the measurable map $V : \mathsf{S} \to [1, \infty)$ be defined by

$$V(a, z) := a + m \, \mathbb{E}_z \hat{Y} + 1, \tag{18}$$

where m is a sufficiently large constant defined in the proof of theorem 4.5 below.

Theorem 4.5. *If assumptions 2.1, 3.1–3.4 and 4.1–4.6 hold, then* Q *is* V-*geometrically ergodic.*

5. Applications

We now turn to several substantial applications of the theory described above. We first illustrate how our theory can be applied to modeling capital income risk in different situations. We then provide a numerical example and study the quantitative effect of stochastic volatility and mean persistence of the wealth return process on wealth inequality.

Throughout this section, we work with the CRRA utility function defined by (3). Recall that $\gamma > 0$ is the coefficient of relative risk aversion.

5.1. **Modeling Capital Income Risk.** Suppose the income process contains both persistent and transient components (see, e.g., Blundell et al. (2008), Browning et al. (2010), Heathcote et al. (2010), Kaplan and Violante (2010), Kaplan (2012), DeBacker et al. (2013), and Carroll et al. (2017)). In particular, we consider

$$\log Y_t = \chi_t + \eta_t,$$

where the persistent component $\{\chi_t\}_{t \geq 0}$ is a finite-state Markov chain with transition matrix Π_χ, and the transient component $\{\eta_t\}_{t \geq 1}$ is an IID sequence with $\mathbb{E}\, e^{\eta_t} < \infty$ and $\mathbb{E}\, e^{-2\gamma\eta_t} < \infty$. Moreover, $\{\chi_t\}$ and $\{\eta_t\}$ are mutually independent.

As a natural extension of the IID financial return process assumed by Benhabib et al. (2015), we consider $\{R_t\}_{t \geq 1}$ taking form of

$$\log R_t = \mu_t + \sigma_t \zeta_t,$$

where $\{\zeta_t\}_{t \geq 1} \stackrel{\text{IID}}{\sim} N(0, 1)$, $\{\mu_t\}_{t \geq 0}$ and $\{\sigma_t\}_{t \geq 0}$ are respectively finite-state Markov chains with transition matrices Π_μ and Π_σ, $\{\sigma_t\}$ is positive, and $\{\mu_t\}$, $\{\sigma_t\}$ and $\{\zeta_t\}$ are mutually independent.[12] Such a setup, as it appears, allows us to capture both mean persistence and stochastic volatility.

The state spaces of $\{\chi_t\}$, $\{\mu_t\}$ and $\{\sigma_t\}$ are respectively (sorted in increasing order)

$$\mathsf{Z}_\chi := \{\ell_1, \cdots, \ell_K\}, \quad \mathsf{Z}_\mu := \{i_1, \cdots, i_M\} \quad \text{and} \quad \mathsf{Z}_\sigma := \{j_1, \cdots, j_N\}.$$

---

[12]Note that $\{Y_t\}$ and $\{R_t\}$ are allowed to be dependent on each other since, for example, we allow $\{\chi_t\}$ and $\{\mu_t\}$ to be mutually dependent, as we do for $\{\eta_t\}$ and $\{\sigma_t\}$, etc.

Let diag $(\cdot)$ be the diagonal matrix created by elements in the bracket, and let

$$D_\mu := \mathrm{diag}\left(\mathrm{e}^{i_1}, \cdots, \mathrm{e}^{i_M}\right) \quad \text{and} \quad D_\sigma := \mathrm{diag}\left(\mathrm{e}^{j_1^2/2}, \cdots, \mathrm{e}^{j_N^2/2}\right).$$

Furthermore, we define the column vectors

$$V_\mu := \left(\mathrm{e}^{(1-\gamma)i_1}, \cdots, \mathrm{e}^{(1-\gamma)i_M}\right)' \quad \text{and} \quad V_\sigma := \left(\mathrm{e}^{(1-\gamma)^2 j_1^2/2}, \cdots, \mathrm{e}^{(1-\gamma)^2 j_N^2/2}\right)'.$$

For any square matrix A, let $r(A)$ be its spectral radius. We assume that

$$r(\Pi_\mu D_\mu) \cdot r(\Pi_\sigma D_\sigma) < 1/\beta \quad \text{and} \tag{19}$$

$$\max\left\{r(\Pi_\mu D_\mu) \cdot r(\Pi_\sigma D_\sigma),\ 1\right\} < \left(\beta \|\Pi_\mu V_\mu\| \cdot \|\Pi_\sigma V_\sigma\|\right)^{-1/\gamma}. \tag{20}$$

This problem can be placed in our framework by setting

$$z_t := (\chi_t, \mu_t, \sigma_t) \quad \text{and} \quad \mathsf{Z} := \mathsf{Z}_\chi \times \mathsf{Z}_\mu \times \mathsf{Z}_\sigma.$$

To simplify notation, we denote $z := z_0$ and $(\chi, \mu, \sigma) := (\chi_0, \mu_0, \sigma_0)$.

5.1.1. *Optimality Results.* Since $\{\zeta_t\} \overset{\text{IID}}{\sim} N(0,1)$, by the Fubini theorem,

$$\beta^n \mathbb{E}_z R_1 \cdots R_n = \beta^n \mathbb{E}_z \mathrm{e}^{\mu_1 + \sigma_1 \zeta_1} \cdots \mathrm{e}^{\mu_n + \sigma_n \zeta_n} = \beta^n (\mathbb{E}_\mu \mathrm{e}^{\mu_1} \cdots \mathrm{e}^{\mu_n})(\mathbb{E}_\sigma \mathrm{e}^{\sigma_1^2/2} \cdots \mathrm{e}^{\sigma_n^2/2}).$$

For all bounded functions f on Z_μ and h on Z_σ, we define

$$K_1 f(\mu) := \mathbb{E}_\mu \mathrm{e}^{\mu_1} f(\mu_1) \quad \text{and} \quad K_2 h(\sigma) := \mathbb{E}_\sigma \mathrm{e}^{\sigma_1^2/2} h(\sigma_1).$$

Similar to example 3.1, $\beta^n \sup_z \mathbb{E}_z R_1 \cdots R_n < 1$ for some $n \in \mathbb{N}$ if and only if $\beta r(K_1) r(K_2) < 1$.[13] The latter obviously holds since (19) holds, and, similar to example 3.2, $r(K_1) = r(\Pi_\mu D_\mu)$ and $r(K_2) = r(\Pi_\sigma D_\sigma)$. Assumption 3.1 is verified.

Using the fact that Z is a finite space, we have

$$\sup_{t \geq 0} \sup_z \mathbb{E}_z \mathrm{e}^{\chi_t} = \sup_{t \geq 0} \sup_\chi \mathbb{E}_\chi \mathrm{e}^{\chi_t} \leq \sup_{t \geq 0} \sup_\chi \mathbb{E}_\chi \mathrm{e}^{\ell_K} = \mathrm{e}^{\ell_K} < \infty. \tag{21}$$

Since in addition $\{\eta_t\}$ is IID with $\mathbb{E}\,\mathrm{e}^{\eta_t} < \infty$, we have

$$\sup_{t \geq 0} \sup_z \mathbb{E}_z Y_t = \sup_{t \geq 0} \sup_z \mathbb{E}_z \mathrm{e}^{\chi_t + \eta_t} = \left(\sup_{t \geq 0} \sup_z \mathbb{E}_z \mathrm{e}^{\chi_t}\right) \mathbb{E}\,\mathrm{e}^{\eta_1} < \infty.$$

---

[13]As in example 3.1, we have $\|K_1^n\| = \sup_\mu \mathbb{E}_\mu \mathrm{e}^{\mu_1} \cdots \mathrm{e}^{\mu_n}$ and $\|K_2^n\| = \sup_\sigma \mathbb{E}_\sigma \mathrm{e}^{\sigma_1^2/2} \cdots \mathrm{e}^{\sigma_n^2/2}$. Then $\beta r(K_1) r(K_2) < 1$ iff $\beta \|K_1^n\|^{1/n} \|K_2^n\|^{1/n} < 1$ for some $n \in \mathbb{N}$ iff $\beta^n \|K_1^n\| \|K_2^n\| < 1$ for some $n \in \mathbb{N}$ iff $\sup_z \beta^n \mathbb{E}_z R_1 \cdots R_n < 1$ for some $n \in \mathbb{N}$.

Hence, assumption 3.2 holds. As a byproduct, we have also verified assumptions 4.2 and 4.6-(2) (recall example 4.7). Similarly, since $\sup_z \mathbb{E}_z e^{-2\gamma\chi_1} \leq e^{-2\gamma\ell_1} < \infty$ and $\mathbb{E} e^{-2\gamma\eta_t} < \infty$, we have

$$\sup_z \mathbb{E}_z \left[u'(Y_1)\right]^2 = \sup_z \mathbb{E}_z e^{-2\gamma(\chi_1+\eta_1)} = \left(\sup_z \mathbb{E}_z e^{-2\gamma\chi_1}\right) \mathbb{E} e^{-2\gamma\eta_1} < \infty. \qquad (22)$$

Moreover, for all $z \in \mathsf{Z}$, based on the Fubini theorem,

$$\mathbb{E}_z \hat{R}^2 = \mathbb{E}_z e^{2\mu_1 + 2\sigma_1\zeta_1} = \mathbb{E}_\mu e^{2\mu_1} \mathbb{E}_\sigma e^{2\sigma_1\zeta_1} = \mathbb{E}_\mu e^{2\mu_1} \mathbb{E}_\sigma e^{2\sigma_1^2} \leq e^{2i_M + 2j_N^2} < \infty.$$

Hence, assumption 3.3 holds (see example 3.3). Since Z is a finite space, this in turn implies that $z \mapsto \mathbb{E}_z \hat{R} u'(\hat{Y})$ must be continuous, so assumption 3.4 holds.

In summary, we have verified all the assumptions of section 3. All the related optimality results have been established.

5.1.2. *Existence of Stationary Distributions.* Similar to examples 4.1–4.2, assumption 4.1-(1) holds if $(1-\alpha)^{-\gamma}\beta\mathbb{E}_z\hat{R}^{1-\gamma} \leq 1$ for all z. Since

$$\mathbb{E}_z \hat{R}^{1-\gamma} = \mathbb{E}_\sigma e^{(1-\gamma)(\mu_1+\sigma_1\zeta_1)} = \mathbb{E}_\mu e^{(1-\gamma)\mu_1} \mathbb{E}_\sigma e^{(1-\gamma)\sigma_1\zeta_1}$$
$$= \mathbb{E}_\mu e^{(1-\gamma)\mu_1} \mathbb{E}_\sigma e^{(1-\gamma)^2\sigma_1^2/2} \leq \|\Pi_\mu V_\mu\| \cdot \|\Pi_\sigma V_\sigma\|,$$

it suffices to show that $\beta\|\Pi_\mu V_\mu\| \cdot \|\Pi_\sigma V_\sigma\| \leq (1-\alpha)^\gamma$. Moreover, similar to verifying assumption 3.1, assumption 4.1-(2) holds as long as $(1-\alpha)r(\Pi_\mu D_\mu)r(\Pi_\sigma D_\sigma) < 1$. In summary, assumption 4.1 holds whenever there exists $\alpha \in (0,1)$ that satisfies

$$r(\Pi_\mu D_\mu) \cdot r(\Pi_\sigma D_\sigma) < 1/(1-\alpha) \leq (\beta\|\Pi_\mu V_\mu\| \cdot \|\Pi_\sigma V_\sigma\|)^{-1/\gamma}.$$

This is guaranteed by (20). Moreover, assumption 4.2 has been verified in the previous section, assumption 4.3 is trivial since Z is finite, and assumption 4.4 has been verified in example 4.4.

In summary, all the assumptions up to section 4.1 have been verified. As a result, all the conclusions of propositions 4.1–4.2 and theorem 4.1 hold.

5.1.3. *Global Stability.* Regarding ergodicity and the Law of Large Numbers (theorem 4.4), it remains to verify assumption 4.5. This is true if we assume further

- there are strictly positive columns in each of the matrices Π_χ, Π_μ and Π_σ (recall example 4.5), and
- $\{\eta_t\}$ has a density that is strictly positive on $(-\infty, \delta)$ for some $\delta \in \mathbb{R}$.

Regarding geometric ergodicity (theorem 4.5), it remains to verify assumption 4.6. Condition (1) is trivial since $\{\zeta_t\} \overset{\text{IID}}{\sim} N(0,1)$. Condition (2) has been verified in previous sections. Hence, the model is V-geometrically ergodic as long as the innovations $\{\eta_t\}$ and $\{\zeta_t\}$ are mutually independent.

5.2. **Modeling Generic Stochastic Returns.** Indeed, our theory works for more general setups. To illustrate, consider the following labor income process[14]

$$Y_t = \chi_t \, \varphi_t + \nu_t \quad \text{and} \quad \ln \chi_{t+1} = \rho \ln \chi_t + \varepsilon_{t+1}, \tag{23}$$

where $\chi_0 \in (0, \infty)$ and $\rho \in (0, 1)$ are given, $\{\varepsilon_t\}_{t \geq 1} \overset{\text{IID}}{\sim} N(0, \delta^2)$, $\{\nu_t\}_{t \geq 1}$ and $\{\varphi_t\}_{t \geq 1}$ are positive IID sequences with finite second moments, and $\mathbb{E}\, \nu_t^{-2\gamma} < \infty$. Moreover, $\{\chi_t\}$, $\{\varphi_t\}$ and $\{\nu_t\}$ are mutually independent. Similar setups appear in a lot of applied literature. See, for example, Heathcote et al. (2010), Kaplan and Violante (2010), Huggett et al. (2011), Kaplan (2012) and DeBacker et al. (2013).

This setup can be placed in our framework by setting $\eta_t := (\varphi_t, \nu_t)$. Next, we aim to verify all the assumptions related to $\{Y_t\}$.

Based on (23), for all $t \geq 0$, the distribution of χ_t given χ_0 follows

$$(\chi_t \mid \chi_0) \sim LN\left(\rho^t \ln \chi_0, \ \delta^2 \sum_{k=0}^{t-1} \varphi^{2k}\right).$$

We denote $\chi := \chi_0$ for simplicity. Then for all $t \geq 0$ and $s \in \mathbb{R}$, we have[15]

$$\mathbb{E}_\chi \chi_t^s = \exp\left[s\rho^t \ln \chi + \frac{s^2 \delta^2 (1 - \rho^{2t})}{2(1 - \rho^2)}\right].$$

---

[14]Similar extensions can be made to the $\{R_t\}$ process.

[15]Recall that for $X \sim LN(\mu, \sigma^2)$ and $s \in \mathbb{R}$, we have $\mathbb{E}(X^s) = \exp\left(s\mu + s^2\sigma^2/2\right)$.

In particular, since $\rho \in (0,1)$, this implies that $\sup_{t \geq 0} \mathbb{E}_{\chi} \chi_t^s < \infty$ for all $s \in \mathbb{R}$ and $\chi \in (0, \infty)$. Hence,

$$\sup_{t \geq 0} \mathbb{E}_{\chi} Y_t = \sup_{t \geq 0} \mathbb{E}_{\chi} \chi_t \varphi_t + \mathbb{E}\, \nu_t \leq \left(\sup_{t \geq 0} \mathbb{E}_{\chi} \chi_t \right) \mathbb{E}\, \varphi_t + \mathbb{E}\, \nu_t < \infty$$

for all $\chi \in (0, \infty)$, and assumptions 3.2 and 4.2 hold. Moreover, since $Y_t \geq \nu_t$,

$$\sup_{\chi} \mathbb{E}_{\chi} \left[u'(Y_t) \right]^2 \leq \mathbb{E}\, \left[u'(\nu_t) \right]^2 = \mathbb{E}\, \nu_t^{-2\gamma} < \infty,$$

and the second part of assumption 3.3 holds. Regarding assumption 4.6-(2), since $\rho \in (0,1)$, we can choose $\bar{\chi} > 0$ such that

$$q := e^{\delta^2 \rho^2 / 2} \bar{\chi}^{\rho(\rho-1)} < 1.$$

Then for $\chi \leq \bar{\chi}$, we have $\mathbb{E}_{\chi} \chi_2 \leq e^{\rho^2 \ln \bar{\chi} + \delta^2 (1+\rho^2)/2} =: d$, and for $\chi > \bar{\chi}$, we have

$$\mathbb{E}_{\chi} \chi_2 = e^{\delta^2 (1+\rho^2)/2} \chi^{\rho^2} = \frac{e^{\delta^2 (1+\rho^2)/2} \chi^{\rho^2}}{e^{\delta^2/2} \chi^{\rho}} \cdot e^{\delta^2/2} \chi^{\rho}$$

$$= e^{\delta^2 \rho^2 / 2} \chi^{\rho(\rho-1)} \cdot \mathbb{E}_{\chi} \chi_1 \leq e^{\delta^2 \rho^2 / 2} \bar{\chi}^{\rho(\rho-1)} \cdot \mathbb{E}_{\chi} \chi_1 = q\, \mathbb{E}_z \chi_1.$$

Hence, $\mathbb{E}_{\chi} \chi_2 \leq q\, \mathbb{E}_{\chi} \chi_1 + d$ for all χ. Since in addition $\mathbb{E}\, \varphi_t < \infty$, $\mathbb{E}\, \nu_t < \infty$ and

$$\mathbb{E}_{\chi} Y_2 = \mathbb{E}_{\chi} \chi_2 \, \mathbb{E}\, \varphi_2 + \mathbb{E}\, \nu_2,$$

assumption 4.6-(2) follows immediately.

Finally, assumption 4.5-(3) holds as long as the distributions of $\{\varphi_t\}$ and $\{\nu_t\}$ have densities that are strictly positive on $(0, \bar{\delta})$ for some $\bar{\delta} > 0$.

5.3. **Numerical Example.** What are the "wealth inequality effects" of mean persistence and stochastic volatility in the rate of return to wealth? This is an important question that is rarely explored by the existing literature. In what follows we attempt to provide an answer via simulation. In doing this, we will also explore the generality of our theory by testing the stability properties of the economy for a broad range of parameters. Our study is based on the model of section 5.1.

Regarding the finite-state Markov chains $\{\chi_t\}$, $\{\mu_t\}$ and $\{\eta_t\}$, we use the method of Tauchen and Hussey (1991) and discretize the following AR(1) processes

$$\chi_t = \rho_\chi \chi_{t-1} + \varepsilon_t^\chi, \qquad \{\varepsilon_t^\chi\} \overset{\text{IID}}{\sim} N(0, \delta_\chi^2),$$

$$\mu_t = (1 - \rho_\mu)\bar{\mu} + \rho_\mu \mu_{t-1} + \varepsilon_t^\mu, \qquad \{\varepsilon_t^\mu\} \overset{\text{IID}}{\sim} N(0, \delta_\mu^2),$$

$$\log \sigma_t = (1 - \rho_\sigma)\bar{\sigma} + \rho_\sigma \log \sigma_{t-1} + \varepsilon_t^\sigma, \qquad \{\varepsilon_t^\sigma\} \overset{\text{IID}}{\sim} N(0, \delta_\sigma^2),$$

into N_χ, N_μ and N_σ states, respectively.

Regarding the parameters of the $\{Y_t\}$ process, we set $\{\eta_t\}$ to be a normal distribution with mean 0 and variance $\delta_\eta^2 = 0.075$. In addition, we set $\rho_\chi = 0.9770$ and $\delta_\chi^2 = 0.02$. These values are chosen broadly in line with the existing literature. See, for example, Heathcote et al. (2010), Kaplan and Violante (2010), and DeBacker et al. (2013).

Our calibration of the $\{R_t\}$ process is based on Fagereng et al. (2016a), in which the authors report the average and standard deviation of the financial return process of Norway from 1993–2013.[16] We transform the two series to match our model and run first-order autoregressions, which yield $\bar{\mu} = 0.0281$, $\rho_\mu = 0.5722$, $\delta_\mu = 0.0067$, $\bar{\sigma} = -3.2556$, $\rho_\sigma = 0.2895$ and $\delta_\sigma = 0.1896$. Based on this parameterization, the stationary mean and standard deviation of the $\{R_t\}$ process are approximately 1.03 and 4%, respectively.

However, to distinguish the different effect of stochastic volatility and mean persistence, as well as to mitigate the computational burden caused by high state dimensionality, we consider two subsidiary model economies. The first model reduces $\{\mu_t\}$ to its stationary mean $\bar{\mu}$, while the second model reduces $\{\sigma_t\}$ to its stationary mean $\hat{\sigma} := \exp(\bar{\sigma} + \delta_\sigma^2/2(1 - \rho_\sigma^2))$. In summary, $\{R_t\}$ satisfies

$$\log R_t = \bar{\mu} + \sigma_t \zeta_t \qquad \text{(Model I)}$$

$$\log R_t = \mu_t + \hat{\sigma} \zeta_t \qquad \text{(Model II)}$$

---

[16]This is the only data source we can find that has a full record of financial returns. Although our calibration is based on this dataset, we have conducted sensitivity analysis for different groups of parameters. The results show that their qualitative effects are broadly the same, although their quantitative effects vary, as one would expect.

To test the stability properties of the economy, we set $\beta = 0.95$, $N_\chi = 5$ and consider respectively $\gamma = 1$ and $\gamma = 2$. Furthermore, in model I, we set $N_\sigma = 5$ and consider a broad neighborhood of the calibrated $(\rho_\sigma, \delta_\sigma)$ pairs, and in model II, we set $N_\mu = 5$ and consider a large neighborhood around the calibrated (ρ_μ, δ_μ) values. Each scenario, we hold the rest of the parameters as in the benchmark. The results are shown in figure 1 and figure 2.

Since the dot points (calibrated parameter values) lie in the stable range in all cases, both the two calibrated models are globally stable, and stationary wealth distributions can be computed by the established ergodic theorems (theorem 4.4 and theorem 4.5). Moreover, the broad stability range indicates that our theory can handle a wide range of parameter setups, including highly persistent and volatile $\{R_t\}$ processes.

Our next goal is to explore the quantitative impact of capital income risk on wealth inequality. As a first step, we compute the optimal policy. This can be realized by iterating the Coleman opeartor and evaluating the distance between loops via the designed metric ρ. The algorithm is guaranteed to converge based on theorem 3.2. Specifically, we assign 100 grid points to wealth equally spaced in $[10^{-4}, 50]$. Expectations with respect to the IID innovations are evaluated via Monte Carlo with 1000 draws. Moreover, in all cases, we use piecewise linear interpolation to approximate policies. Policy function evaluation outside of the grid range is via linear extrapolation, as is justified by proposition 4.2.

Once the optimal policy is obtained, we then simulate a single time series of 5×10^7 agents in each case and compute the stationary distribution based on our ergodic theorems 4.4–4.5. As a final step, we compare the key properties of the stationary wealth distributions in different economies. In particular, we estimate the tail exponent based on the wealth level of the top 5% and top 10% of the simulated agents.[17]

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[17]Recall that a random variable X is said to have a *heavy upper tail* if there exist constants $A, \alpha > 0$ such that $\mathbb{P}\{X > x\} \geq Ax^{-\alpha}$ for large enough x, where α is refered to as the *tail exponent*. The smaller the tail exponent is, the fatter the distribution tail is, and thus a higher level of inequality exists. It is common in the literature to estimate the tail exponent via linearly regressing the log-ranks over the log-wealth levels of the top 5% and top 10% most wealthy agents.
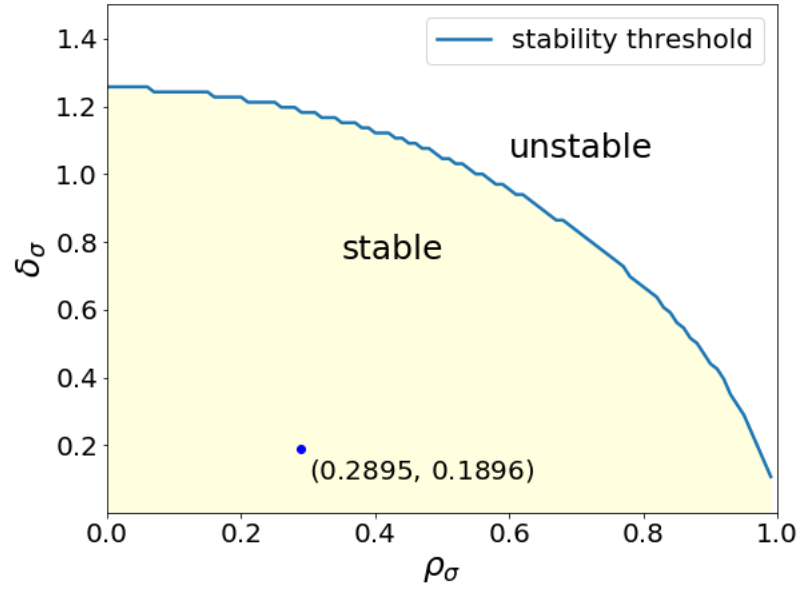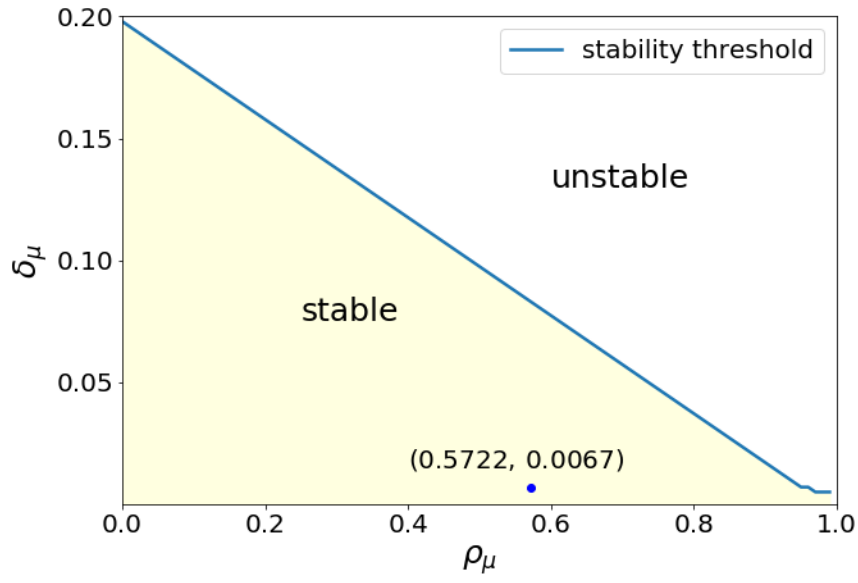
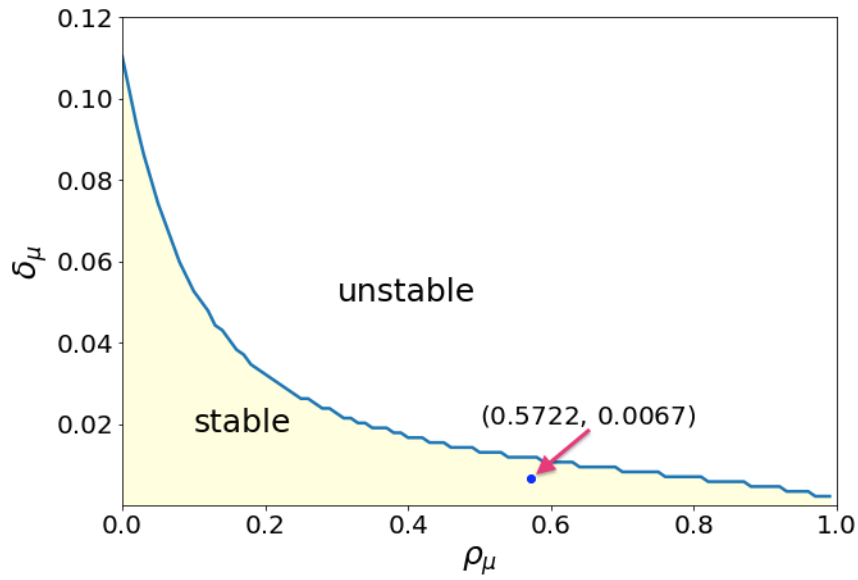(a) Model I : $\beta = 0.95,\quad \gamma = 1,\quad \bar{\mu} = 0.0281$
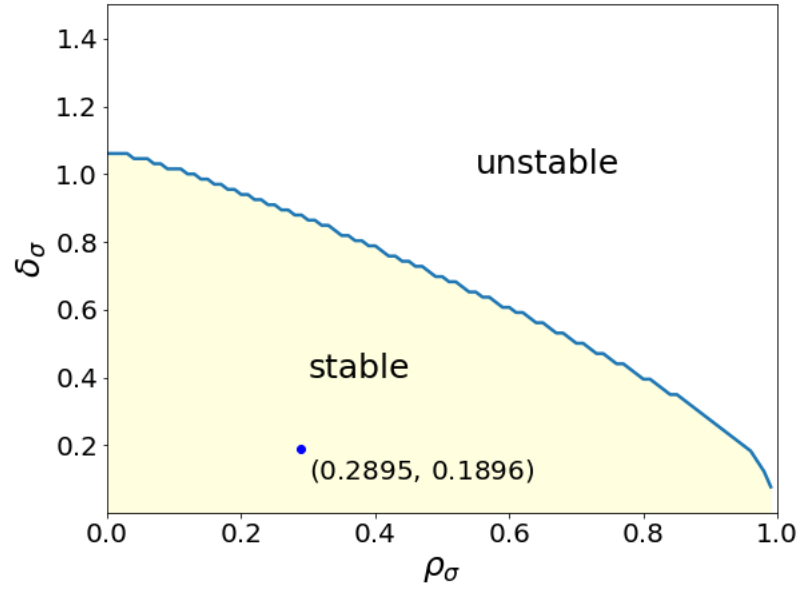
(b) Model I : $\beta = 0.95,\quad \gamma = 2,\quad \bar{\mu} = 0.0281$

FIGURE 1. Stability Range and Threshold of Model I

(a) Model II : $\beta = 0.95, \quad \gamma = 1, \quad \hat{\sigma} = 0.0393$

(b) Model II : $\beta = 0.95, \quad \gamma = 2, \quad \hat{\sigma} = 0.0393$

FIGURE 2. Stability Range and Threshold of Model II

Moreover, we estimate the Gini coefficient and provide a detailed analysis of the wealth share in each case.

All simulations are processed in a standard Julia environment on a laptop with a 2.9 GHz Intel Core i7 and 32GB RAM.

TABLE 1. Tail Exponent and Gini Coefficient

Model Economy		Model I	Model II	IID $\{R_t\}$	Constant $\{R_t\}$
Tail Exponent	Top 5%	3.0	2.9	4.4	4.4
	Top 10%	2.6	2.5	3.7	3.7
Gini Coefficient		0.47	0.45	0.34	0.33

Parameters: $\beta = 0.95$, $\gamma = 2$, $\bar{\mu} = 0.0281$, $\bar{\sigma} = -3.2556$, $\rho_\sigma = 0.2895$, $\delta_\sigma = 0.1896$, $\rho_\mu = 0.5722$ and $\delta_\mu = 0.0067$.

TABLE 2. Wealth Share (in percentage)

Poorest agents (%)	5%	10%	15%	20%	25%	30%	35%	40%	45%	50%
Model I	0.8	1.8	3.1	4.6	6.2	8.2	10.4	12.9	15.7	18.7
Model II	1.1	2.4	3.9	5.7	7.6	9.7	12.1	14.7	17.5	20.6
IID $\{R_t\}$	1.5	3.4	5.6	8.0	10.6	13.4	16.5	19.8	23.4	27.3
Constant $\{R_t\}$	1.6	3.5	5.6	8.0	10.7	13.5	16.6	20.0	23.6	27.5
Poorest agents (%)	55%	60%	65%	70%	75%	80%	85%	90%	95%	100%
Model I	22.1	25.9	30.0	34.7	40.3	47.0	55.1	64.8	77.0	100
Model II	24.1	27.8	31.9	36.6	42.0	48.5	56.3	65.7	77.5	100
IID $\{R_t\}$	31.4	35.9	40.7	46.0	51.8	58.4	65.7	74.2	84.3	100
Constant $\{R_t\}$	31.6	36.1	41.0	46.3	52.0	58.5	65.9	74.3	84.4	100

Parameters: same as table 1. In the first and sixth rows, $N\%$ denotes the $N\%$ of agents with lowest levels of wealth.

We compare our models with two other models, in which $\{R_t\}$ is respectively an IID process and a constant.[18] The difference between the results of model I and model II and the results of the other two models reflects the role of stochastic volatility

---

[18]In the former case, we set $N_\sigma = 1$ in model I (so that σ_t reduces to its stationary mean) or $N_\mu = 1$ in model II (so that μ_t reduces to its stationary mean). In the latter case, we reduce $\{R_t\}$ to its stationary mean.
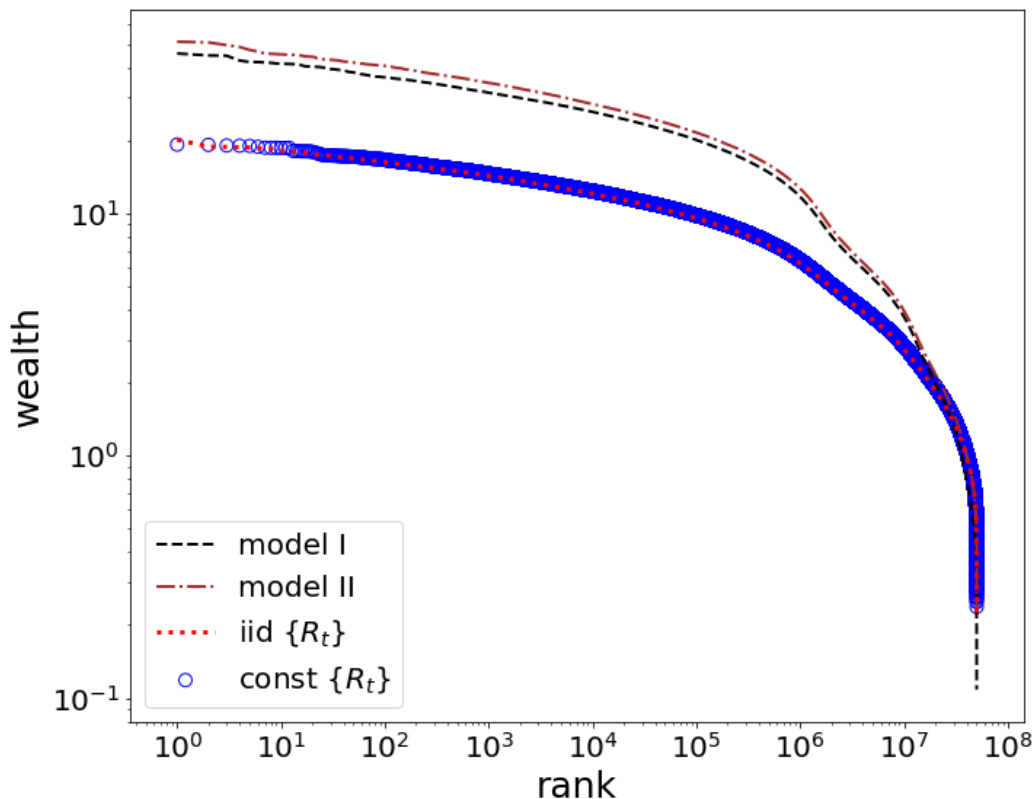
FIGURE 3. The Zipf Plot

and mean persistence of the wealth return process. Parameter setups and results are reported in tables 1–2.[19]

As can be seen in table 1, the tail exponents of model I and model II are smaller than the tail exponents when $\{R_t\}$ is IID or constant. In other words, both stochastic volatility and mean persistence in wealth returns lead to a higher degree of wealth inequality. Moreover, mean persistence results in slightly lower tail exponents than stochastic volatility does.

---

[19]Since the standard Bewley-Ayagari-Hugget model does not generate fat-tailed wealth distribution (see, e.g., Stachurski and Toda (2018)), calculating the tail exponent of the stationary wealth distribution when $\{R_t\}$ is a constant is relatively less standard. However, doing this allows us to reveal the effect of capital income risk on the tail thickness of the stationary wealth distribution.
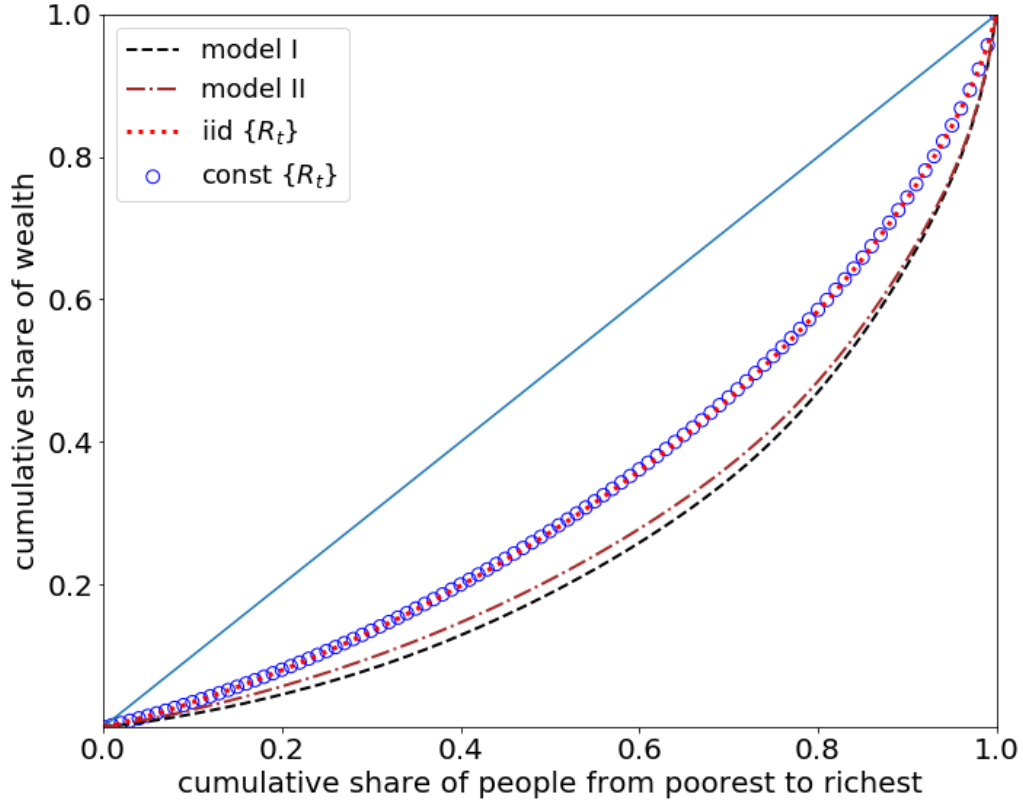
FIGURE 4. The Lorenz Curve

Similarly, the Gini coefficients generated by model I and model II are much higher than those generated by the other two models, illustrating from another perspective that stochastic volatility and mean persistence of wealth returns cause more inequality in wealth. However, different from the previous case, compared with mean persistence, which generates a Gini index 0.45, stochastic volatility has a higher impact on wealth inequality, creating a Gini index 0.47.

Moreover, at least in the current models, IID wealth returns do not have obvious effect on wealth inequality, both in terms of their impact on the tail exponent and in terms of their impact on the Gini coefficient.

The above descriptions are further illustrated in table 2 and figures 3–4. In particular, in table 2 we calculate the wealth share of a given fraction of poorest agents. Notably, the top 10% richest agents hold respectively 35.2%, 34.3%, 25.8% and 25.7% of the total wealth, while the poorest 10% agents hold respectively 1.8%, 2.4%, 3.4%, 3.5% of the total wealth in the four model economies. In figure 3 we create the Zipf plot (i.e., plotting log wealth v.s. log rank). It is clearly indicated that model I and model II generate stationary wealth distributions with fatter upper tails than the other models do, and that the stationary wealth distribution of model II has the fattest upper tail. In figure 4 we plot the Lorenz curve, which can be viewed as a generalized graphical representation of table 2.

Finally, sensitivity analysis with respect to model parameters and a more detailed quantitative analysis can be found in the online appendix of this paper.

6. Appendix A: Proof of Section 3 Results

In proofs we let $\{\mathscr{F}_t\}_{t\geq0}$ be the natural filtration, where $\mathscr{F}_t := \sigma(s_0, \cdots, s_t)$ with $s_t := (a_t, z_t)$ for all t. We start by proving the results of section 3.

Proof of example 3.1. Note that for fixed $n \in \mathbb{N}$,

$$\|K^n\| = \sup_{\|f\|\leq1} \|K^n f\| = \sup_{\|f\|\leq1} \sup_{z\in\mathsf{Z}} |\mathbb{E}_z R_1 \cdots R_n f(z_n)| = \sup_{z\in\mathsf{Z}} \mathbb{E}_z R_1 \cdots R_n.$$

Suppose assumption 3.1 holds. Note that every $t \in \mathbb{N}$ can be written as $t = kn + \ell$ where $k \in \mathbb{N} \cup \{0\}$ and $\ell \in \{0, \cdots, n-1\}$. Since $\|K^t\| = \|K^{kn+\ell}\| \leq \|K^n\|^k \|K^\ell\|$,

$$\|K^t\|^{1/t} = \|K^{kn+\ell}\|^{1/t} \leq \|K^n\|^{k/t} \|K^\ell\|^{1/t} = \|K^n\|^{\frac{1}{n+\ell/k}} \|K^\ell\|^{1/t}.$$

Since $\beta\|K^n\| < 1$ by assumption 3.1 and $\|K^\ell\| \leq \|K\|^\ell < \infty$, letting $t \to \infty$ (and thus $k \to \infty$) yields

$$\beta r(K) = \beta \lim_{t\to\infty} \|K^t\|^{1/t} \leq \beta \lim_{k\to\infty} \|K^n\|^{\frac{1}{n}\frac{n}{n+\ell/k}} \|K^\ell\|^{\frac{1}{kn+\ell}} = \beta\|K^n\|^{1/n} < 1.$$

On the other hand, suppose $\beta r(K) < 1$. Then by the definition of r there exists $n \in \mathbb{N}$ such that $\beta\|K^n\|^{1/n} < 1$. Thus $\beta^n\|K^n\| < 1$ and assumption 3.1 is verified. $\qquad\square$

For the rest of this section, we let n and θ be defined as in assumption 3.1.

Proof of lemma 3.1. Iterating backward on the maximal path (6), we can show that

$$\tilde{a}_t = \left(\prod_{i=1}^t R_i\right) a + \sum_{j=1}^t \left(Y_j \prod_{i=j+1}^t R_i\right).$$

Taking discounted expectation yields

$$\beta^t \mathbb{E}_{a,z} \tilde{a}_t = \left[\mathbb{E}_z\left(\beta^t \prod_{i=1}^t R_i\right)\right] a + \sum_{j=1}^t \mathbb{E}_z\left[\left(\beta^{t-j} \prod_{i=j+1}^t R_i\right)(\beta^j Y_j)\right].$$

Let $M(a, z) := \sum_{t \geq 0} \beta^t \mathbb{E}_{a,z} \tilde{a}_t$. Then the monotone convergence theorem and the Markov property imply that

$$
\begin{aligned}
M(a, z) &= \sum_{t=0}^{\infty} \mathbb{E}_z \left(\beta^t \prod_{i=1}^{t} R_i \right) a + \sum_{t=0}^{\infty} \sum_{j=1}^{t} \mathbb{E}_z \left[\left(\beta^{t-j} \prod_{i=j+1}^{t} R_i \right) (\beta^j Y_j) \right] \\
&= \mathbb{E}_z \left(\sum_{t=0}^{\infty} \beta^t \prod_{i=1}^{t} R_i \right) a + \sum_{j=1}^{\infty} \mathbb{E}_z \mathbb{E}_z \left[(\beta^j Y_j) \left(\sum_{i=0}^{\infty} \beta^i \prod_{k=1}^{i} R_{j+k} \right) \Big| \mathscr{F}_j \right] \\
&= \mathbb{E}_z \left(\sum_{t=0}^{\infty} \beta^t \prod_{i=1}^{t} R_i \right) a + \sum_{j=1}^{\infty} \mathbb{E}_z \left[(\beta^j Y_j) \mathbb{E}_{z_j} \left(\sum_{i=0}^{\infty} \beta^i \prod_{k=1}^{i} R_k \right) \right].
\end{aligned}
$$

By the Markov property and assumption 3.1, for all $k \in \mathbb{N}$ and $z \in \mathsf{Z}$, we have

$$
\begin{aligned}
\mathbb{E}_z \beta^{kn} R_1 \cdots R_{kn} &= \mathbb{E}_z \mathbb{E}_z [\beta^{(k-1)n} R_1 \cdots R_{(k-1)n} \beta^n R_{(k-1)n+1} \cdots R_{kn} \mid \mathscr{F}_{(k-1)n}] \\
&= \mathbb{E}_z \beta^{(k-1)n} R_1 \cdots R_{(k-1)n} \mathbb{E}_{z_{(k-1)n}} (\beta^n R_1 \cdots R_n) \\
&\leq \theta \mathbb{E}_z \beta^{(k-1)n} R_1 \cdots R_{(k-1)n} \leq \cdots \leq \theta^k.
\end{aligned}
$$

Taking supremum on both sides yields

$$
\beta^{kn} \sup_{z \in \mathsf{Z}} \mathbb{E}_z R_1 \cdots R_{kn} \leq \theta^k. \tag{24}
$$

Moreover, assumption 3.3 implies that $K_0 := \sup_{z \in \mathsf{Z}} \mathbb{E}_z \hat{R} < \infty$. Hence,

$$
\begin{aligned}
\mathbb{E}_z \left(\sum_{i=0}^{n-1} \beta^i R_1 \cdots R_i \right) &= \sum_{i=0}^{n-1} \beta^i \mathbb{E}_z R_1 \cdots R_i = \sum_{i=0}^{n-1} \beta^i \mathbb{E}_z R_1 \cdots R_{i-1} \mathbb{E}_{z_{i-1}} R_1 \\
&\leq \sum_{i=0}^{n-1} \beta^i \mathbb{E}_z R_1 \cdots R_{i-1} K_0 \leq \cdots \leq \sum_{i=0}^{n-1} \beta^i K_0^i =: K_1 < \infty
\end{aligned}
$$

for all $z \in \mathsf{Z}$. Taking supremum on both sides yields

$$
\sup_{z \in \mathsf{Z}} \mathbb{E}_z \left(\sum_{i=0}^{n-1} \beta^i R_1 \cdots R_i \right) \leq K_1 < \infty. \tag{25}
$$

Based on (24) and (25), we have

$$\mathbb{E}_z \left(\sum_{i=0}^{\infty} \beta^i \prod_{k=1}^{i} R_k \right) = \sum_{k=0}^{\infty} \mathbb{E}_z \left(\sum_{i=0}^{n-1} \beta^{kn+i} R_1 \cdots R_{kn+i} \right)$$

$$= \sum_{k=0}^{\infty} \mathbb{E}_z \left[\beta^{kn} R_1 \cdots R_{kn} \left(\sum_{i=0}^{n-1} \beta^i R_{kn+1} \cdots R_{kn+i} \right) \right]$$

$$= \sum_{k=0}^{\infty} \mathbb{E}_z \left[\beta^{kn} R_1 \cdots R_{kn} \mathbb{E}_{z_{kn}} \left(\sum_{i=0}^{n-1} \beta^i R_1 \cdots R_i \right) \right]$$

$$\leq \sum_{k=0}^{\infty} \mathbb{E}_z \beta^{kn} R_1 \cdots R_{kn} K_1 \leq \sum_{k=0}^{\infty} \theta^k K_1 := K_2 < \infty$$

for all $z \in \mathsf{Z}$. Hence,

$$\sup_{z \in \mathsf{Z}} \mathbb{E}_z \left(\sum_{i=0}^{\infty} \beta^i \prod_{k=1}^{i} R_k \right) \leq K_2 < \infty.$$

Finally, assumption 3.2 implies that

$$M(a, z) \leq K_2 a + K_2 \sum_{t=1}^{\infty} \beta^t \mathbb{E}_z Y_t < \infty$$

for all $(a, z) \in \mathsf{S}_0$. This concludes the proof. $\qquad \square$

Proof of theorem 3.1. This result extends theorem 1 of Benhabib et al. (2015) and theorem 3.1 of Li and Stachurski (2014). While the assumptions are weaker in our setting, the proof is similar and hence omitted. $\qquad \square$

In the next, we aim to prove proposition 3.1. To that end, we define $\mathscr{H}$ to be the set of functions $h \colon \mathsf{S}_0 \to \mathbb{R}$ that satisfies

(1) h is continuous,

(2) h is decreasing in the first argument, and

(3) $\exists K \in \mathbb{R}$ such that $u'(a) \leq h(a, z) \leq u'(a) + K$ for all $(a, z) \in \mathsf{S}_0$.

On $\mathscr{H}$ we impose the distance

$$d_{\infty}(h, g) := \|h - g\| := \sup_{(a,z) \in \mathsf{S}_0} |h(a, z) - g(a, z)|. \tag{26}$$

While the elements of $\mathscr{H}$ are not bounded, the function d_∞ is a valid metric. Moreover, standard argument shows that $(\mathscr{H}, d_\infty)$ is a complete metric space.

Proof of proposition 3.1. Standard argument shows that ρ is a valid metric. To show completeness of $(\mathscr{C}, \rho)$, it suffices to show that $(\mathscr{C}, \rho)$ and $(\mathscr{H}, d_\infty)$ are isometrically isomorphic.

To see that this is so, let H be the map on $\mathscr{C}$ defined by $Hc = u' \circ c$. It is easy to show that $H : \mathscr{C} \to \mathscr{H}$ and that it is a bijection. Moreover, for all $c, d \in \mathscr{C}$,

$$d_\infty(Hc, Hd) = \|Hc - Hd\| = \|u' \circ c - u' \circ d\| = \rho(c, d).$$

Hence, H is an isometry. The space $(\mathscr{C}, \rho)$ is then complete, as claimed. $\qquad\square$

Proof of proposition 3.2. Let c be a policy in $\mathscr{C}$ satisfying (11). That c satisfies the first order optimality conditions is immediate by definition. It remains to show that any asset path generated by c satisfies the transversality condition (8). To see that this is so, observe that, by (10),

$$\mathbb{E}_{a,z}\beta^t(u' \circ c)(a_t, z_t)a_t \leq \beta^t\mathbb{E}_{a,z}u'(a_t)a_t + \beta^t K\mathbb{E}_{a,z}a_t. \tag{27}$$

Regarding the first term on the right hand side of (27), fix $L > 0$ and observe that

$$\mathbb{E}_{a,z}u'(a_t)a_t = \mathbb{E}_{a,z}u'(a_t)a_t\mathbb{1}\{a_t \leq L\} + \mathbb{E}_{a,z}u'(a_t)a_t\mathbb{1}\{a_t > L\}$$
$$\leq L\mathbb{E}_{a,z}u'(a_t) + u'(L)\mathbb{E}_{a,z}a_t \leq L\mathbb{E}_z u'(Y_t) + u'(L)\mathbb{E}_{a,z}\tilde{a}_t,$$

where $\tilde{a}_t$ is the maximal path defined in (6). We then have

$$\beta^t\mathbb{E}_{a,z}u'(a_t)a_t \leq L\beta^t\mathbb{E}_z u'(Y_t) + u'(L)\beta^t\mathbb{E}_{a,z}\tilde{a}_t. \tag{28}$$

Since $M := \sup_{z \in \mathsf{Z}} \mathbb{E}_z u'(\hat{Y}) < \infty$ by assumption 3.3, the Markov property then implies that for all $z \in \mathsf{Z}$ and $t \geq 1$,

$$\mathbb{E}_z u'(Y_t) = \mathbb{E}_z\mathbb{E}_z\left[u'(Y_t)\,\big|\,\mathscr{F}_{t-1}\right] = \mathbb{E}_z\mathbb{E}_{z_{t-1}}u'(\hat{Y}) \leq \mathbb{E}_z M = M.$$

Hence, $\lim_{t \to \infty} \beta^t\mathbb{E}_z u'(Y_t) = 0$. Since in addition $\lim_{t \to \infty} \beta^t\mathbb{E}_{a,z}\tilde{a}_t = 0$ by lemma 3.1, (28) then implies that $\lim_{t \to \infty} \beta^t\mathbb{E}_{a,z}u'(a_t)a_t = 0$.

Moreover, the second term on the right hand side of (27) is dominated by $\beta^t K \mathbb{E}_{a,z} \tilde{a}_t$, and converges to zero by lemma 3.1. We have thus shown that the term on the right hand side of (27) converges to zero. Hence, the transversality condition holds. $\qquad \square$

Proof of proposition 3.3. Fix $c \in \mathscr{C}$ and $(a, z) \in \mathsf{S}_0$. Because $c \in \mathscr{C}$, the map $\xi \mapsto \psi_c(\xi, a, z)$ is increasing. Since $\xi \mapsto u'(\xi)$ is strictly decreasing, the equation (12) can have at most one solution. Hence uniqueness holds.

Existence follows from the intermediate value theorem provided we can show that

(a) $\xi \mapsto \psi_c(\xi, a, z)$ is a continuous function,

(b) $\exists \xi \in (0, a]$ such that $u'(\xi) \geq \psi_c(\xi, a, z)$, and

(c) $\exists \xi \in (0, a]$ such that $u'(\xi) \leq \psi_c(\xi, a, z)$.

For part (a), it suffices to show that $g(\xi) := \mathbb{E}_z \hat{R}(u' \circ c)\left[\hat{R}(a - \xi) + \hat{Y}, \hat{z}\right]$ is continuous on $(0, a]$. To this end, fix $\xi \in (0, a]$ and $\xi_n \to \xi$. By (10) we have

$$\hat{R}(u' \circ c)\left[\hat{R}(a - \xi) + \hat{Y}, \hat{z}\right] \leq \hat{R}(u' \circ c)(\hat{Y}, \hat{z}) \leq \hat{R}u'(\hat{Y}) + \hat{R}K. \qquad (29)$$

The last term is integrable by assumption 3.3. Hence the dominated convergence theorem applies. From this fact and the continuity of c, we obtain $g(\xi_n) \to g(\xi)$. Hence, $\xi \mapsto \psi_c(\xi, a, z)$ is continuous.

Part (b) clearly holds, since $u'(\xi) \to \infty$ as $\xi \to 0$ and $\xi \mapsto \psi_c(\xi, a, z)$ is increasing and always finite (since it is continuous as shown in the previous paragraph). Part (c) is also trivial (just set $\xi = a$). $\qquad \square$

Proof of proposition 3.4. Fix $c \in \mathscr{C}$. With slight abuse of notation, we denote

$$g(\xi, a, z) := \mathbb{E}_z \hat{R}(u' \circ c)\left[\hat{R}(a - \xi) + \hat{Y}, \hat{z}\right].$$

Step 1. We show that Tc is continuous. To apply a standard fixed point parametric continuity result such as theorem B.1.4 of Stachurski (2009), we first show that ψ_c is jointly continuous on the set G defined in (13). This will be true if g is jointly continuous on G. For any $\{(\xi_n, a_n, z_n)\}$ and (ξ, a, z) in G with $(\xi_n, a_n, z_n) \to (\xi, a, z)$, we need to show that $g(\xi_n, a_n, z_n) \to g(\xi, a, z)$. To that end, we define

$$h_1(\xi, a, \hat{z}, \hat{\zeta}, \hat{\eta}),\, h_2(\xi, a, \hat{z}, \hat{\zeta}, \hat{\eta}) := \hat{R}[u'(\hat{Y}) + K] \pm \hat{R}(u' \circ c)[\hat{R}(a - \xi) + \hat{Y}, \hat{z}],$$

where $\hat{R} := R(\hat{z}, \hat{\zeta})$ and $\hat{Y} := Y(\hat{z}, \hat{\eta})$ as defined in (2). Then h_1 and h_2 are continuous in $(\xi, a, \hat{z})$ by the continuity of c and assumption 3.4, and they are nonnegative since (29) implies that $0 \leq \hat{R}\,(u' \circ c)\,[\hat{R}\,(a - \xi) + \hat{Y}, \hat{z}] \leq \hat{R}[u'(\hat{Y}) + K]$.

Moreover, since the stochastic kernel P is Feller, the product measure satisfies[20]

$$P(z_n, \cdot) \otimes \nu \otimes \mu \xrightarrow{w} P(z, \cdot) \otimes \nu \otimes \mu.$$

Based on the generalized Fatou's lemma of Feinberg et al. (2014) (theorem 1.1),

$$\liminf_{n \to \infty} \int h_i(\xi_n, a_n, \hat{z}, \hat{\zeta}, \hat{\eta}) P(z_n, \mathrm{d}\hat{z}) \nu(\mathrm{d}\hat{\zeta}) \mu(\mathrm{d}\hat{\eta})$$

$$\geq \int h_i(\xi, a, \hat{z}, \hat{\zeta}, \hat{\eta}) P(z, \mathrm{d}\hat{z}) \nu(\mathrm{d}\hat{\zeta}) \mu(\mathrm{d}\hat{\eta}).$$

Since $z \mapsto \mathbb{E}\,_z \hat{R}\,[u'(\hat{Y}) + K]$ is continuous by assumption 3.4, this implies that

$$\liminf_{n \to \infty} \left(\pm \mathbb{E}\,_{z_n} \hat{R}\,(u' \circ c) \left[\hat{R}\,(a_n - \xi_n) + \hat{Y}, \hat{z} \right] \right) \geq \left(\pm \mathbb{E}\,_z \hat{R}\,(u' \circ c) \left[\hat{R}\,(a - \xi) + \hat{Y}, \hat{z} \right] \right).$$

The function g is then continuous since the above inequality is equivalent to

$$\liminf_{n \to \infty} g(\xi_n, a_n, z_n) \geq g(\xi, a, z) \geq \limsup_{n \to \infty} g(\xi_n, a_n, z_n).$$

Hence, ψ_c is continuous on G, as was to be shown. Moreover, since $\xi \mapsto \psi_c(\xi, a, z)$ takes values in the closed interval

$$I(a, z) := \left[u'(a), u'(a) + \mathbb{E}\,_z \hat{R} \left(u'(\hat{Y}) + K \right) \right],$$

the correspondence $(a, z) \mapsto I(a, z)$ is nonempty, compact-valued and continuous. By theorem B.1.4 of Stachurski (2009), $(a, z) \mapsto [u' \circ (Tc)](a, z)$ is continuous. Tc is then continuous on S_0 since u' is continuous.

Step 2. We show that Tc is increasing in a. Suppose that for some $z \in \mathsf{Z}$ and $a_1, a_2 \in (0, \infty)$ with $a_1 < a_2$, we have $\xi_1 := Tc(a_1, z) > Tc(a_2, z) =: \xi_2$. Since c is increasing in a by assumption, ψ_c is increasing in ξ and decreasing in a. Then $u'(\xi_1) < u'(\xi_2) = \psi_c(\xi_2, a_2, z) \leq \psi_c(\xi_1, a_1, z) = u'(\xi_1)$. This is a contradiction.

---

[20]Here $\xrightarrow{w}$ denotes weak convergence, i.e., for all bounded continuous function f, we have

$$\int f(\hat{z}, \hat{\zeta}, \hat{\eta}) P(z_n, \mathrm{d}\hat{z}) \nu(\mathrm{d}\hat{\zeta}) \mu(\mathrm{d}\hat{\eta}) \to \int f(\hat{z}, \hat{\zeta}, \hat{\eta}) P(z, \mathrm{d}\hat{z}) \nu(\mathrm{d}\hat{\zeta}) \mu(\mathrm{d}\hat{\eta}).$$

The formal definition of weak convergence is provided in section 4.3.1.

Step 3. We have shown in proposition 3.3 that $Tc(a, z) \in (0, a]$ for all $(a, z) \in \mathsf{S}_0$.

Step 4. We show that $\|u' \circ (Tc) - u'\| < \infty$. Since $u'[Tc(a, z)] \geq u'(a)$, we have

$$|u'[Tc(a, z)] - u'(a)| = u'[Tc(a, z)] - u'(a)$$

$$\leq \mathbb{E}_z \hat{R} (u' \circ c) \left(\hat{R}[a - Tc(a, z)] + \hat{Y}, \hat{z} \right) \leq \mathbb{E}_z \hat{R} \left[u'(\hat{Y}) + K \right].$$

for all $(a, z) \in \mathsf{S}_0$. Assumption 3.3 then implies that

$$\|u' \circ (Tc) - u'\| \leq \sup_{z \in \mathsf{Z}} \mathbb{E}_z \hat{R} u'(\hat{Y}) + K \left(\sup_{z \in \mathsf{Z}} \mathbb{E}_z \hat{R} \right) < \infty.$$

This concludes the proof. $\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\square$

In the rest of this section, we aim to prove theorem 3.2. Recall $\mathscr{H}$ defined above. Given $h \in \mathscr{H}$, let $\tilde{T}h$ be the function mapping $(a, z) \in \mathsf{S}_0$ into the κ that solves

$$\kappa = \max \left\{ \beta \mathbb{E}_z \hat{R} h \left(\hat{R} \left[a - (u')^{-1} (\kappa) \right] + \hat{Y}, \hat{z} \right), u'(a) \right\}. \tag{30}$$

The next lemma implies that $\tilde{T}$ is a well-defined self-map on $\mathscr{H}$, as well as topologically conjugate to T under the bijection $H : \mathscr{C} \to \mathscr{H}$ defined by $Hc := u' \circ c$.

Lemma 6.1. *The operator* $\tilde{T} : \mathscr{H} \to \mathscr{H}$ *and satisfies* $\tilde{T}H = HT$ *on* $\mathscr{C}$.

Proof of lemma 6.1. Pick any $c \in \mathscr{C}$ and $(a, z) \in \mathsf{S}_0$. Let $\xi := Tc(a, z)$, then ξ solves

$$u'(\xi) = \max \left\{ \beta \mathbb{E}_z \hat{R} (u' \circ c) \left[\hat{R}(a - \xi) + \hat{Y}, \hat{z} \right], u'(a) \right\}. \tag{31}$$

We need to show that HTc and $\tilde{T}Hc$ evaluate to the same number at (a, z). In other words, we need to show that $u'(\xi)$ is the solution to

$$\kappa = \max \left\{ \beta \mathbb{E}_z \hat{R} (u' \circ c) \left(\hat{R} \left[a - (u')^{-1} (\kappa) \right] + \hat{Y}, \hat{z} \right), u'(a) \right\}.$$

But this is immediate from (31). Hence, we have shown that $\tilde{T}H = HT$ on $\mathscr{C}$. Since $H : \mathscr{C} \to \mathscr{H}$ is a bijection, we have $\tilde{T} = HTH^{-1}$. Since in addition $T : \mathscr{C} \to \mathscr{C}$ by proposition 3.4, we have $\tilde{T} : \mathscr{H} \to \mathscr{H}$. This concludes the proof. $\qquad\square$

Lemma 6.2. $\tilde{T}$ *is order preserving on* $\mathscr{H}$. *That is,* $\tilde{T}h_1 \leq \tilde{T}h_2$ *for all* $h_1, h_2 \in \mathscr{H}$ *with* $h_1 \leq h_2$.

Proof of lemma 6.2. Let h_1, h_2 be functions in $\mathscr{H}$ with $h_1 \leq h_2$. Suppose to the contrary that there exists $(a, z) \in \mathsf{S}_0$ such that $\kappa_1 := \tilde{T}h_1(a, z) > \tilde{T}h_2(a, z) =: \kappa_2$. Since functions in $\mathscr{H}$ are decreasing in the first argument, we have

$$
\begin{aligned}
\kappa_1 &= \max\left\{\beta\mathbb{E}_z \hat{R}\, h_1 \left(\hat{R}\left[a - (u')^{-1}(\kappa_1)\right] + \hat{Y},\, \hat{z}\right),\, u'(a)\right\} \\
&\leq \max\left\{\beta\mathbb{E}_z \hat{R}\, h_2 \left(\hat{R}\left[a - (u')^{-1}(\kappa_1)\right] + \hat{Y},\, \hat{z}\right),\, u'(a)\right\} \\
&\leq \max\left\{\beta\mathbb{E}_z \hat{R}\, h_2 \left(\hat{R}\left[a - (u')^{-1}(\kappa_2)\right] + \hat{Y},\, \hat{z}\right),\, u'(a)\right\} = \kappa_2.
\end{aligned}
$$

This is a contradiction. Hence, $\tilde{T}$ is order preserving. $\qquad\square$

Lemma 6.3. $\tilde{T}^n$ *is a contraction mapping on* $(\mathscr{H}, d_\infty)$ *with modulus* θ.

Proof of lemma 6.3. Since $\tilde{T}$ is order preserving and $\mathscr{H}$ is closed under the addition of nonnegative constants, based on Blackwell (1965), it remains to verify: for n and θ given by assumption 3.1,

$$
\tilde{T}^n(h + \gamma) \leq \tilde{T}^n h + \theta\gamma \quad \text{for all } h \in \mathscr{H} \text{ and } \gamma \geq 0.
$$

To that end, by assumption 3.1, it suffices to show that for all $k \in \mathbb{N}$ and $(a, z) \in \mathsf{S}_0$,

$$
\tilde{T}^k(h + \gamma)(a, z) \leq \tilde{T}^k h(a, z) + \gamma\beta^k \mathbb{E}_z R_1 \cdots R_k. \tag{32}
$$

Fix $h \in \mathscr{H}$, $\gamma \geq 0$, and let $h_\gamma(a, z) := h(a, z) + \gamma$. By the definition of $\tilde{T}$, we have

$$
\begin{aligned}
\tilde{T}h_\gamma(a, z) &= \max\left\{\beta\mathbb{E}_z \hat{R}\, h_\gamma \left(\hat{R}\left[a - (u')^{-1}(\tilde{T}h_\gamma)(a, z)\right] + \hat{Y},\, \hat{z}\right), u'(a)\right\} \\
&\leq \max\left\{\beta\mathbb{E}_z \hat{R}\, h \left(\hat{R}\left[a - (u')^{-1}(\tilde{T}h_\gamma)(a, z)\right] + \hat{Y},\, \hat{z}\right), u'(a)\right\} + \gamma\beta\mathbb{E}_z R_1 \\
&\leq \max\left\{\beta\mathbb{E}_z \hat{R}\, h \left(\hat{R}\left[a - (u')^{-1}(\tilde{T}h)(a, z)\right] + \hat{Y},\, \hat{z}\right), u'(a)\right\} + \gamma\beta\mathbb{E}_z R_1.
\end{aligned}
$$

Here, the first inequality is elementary and the second is due to the fact that $h \leq h_\gamma$ and $\tilde{T}$ is order preserving. Hence, $\tilde{T}(h + \gamma)(a, z) \leq \tilde{T}h(a, z) + \gamma\beta\mathbb{E}_z R_1$ and (32) holds for $k = 1$. Suppose that (32) holds for arbitrary k. It remains to show that (32) holds for $k + 1$. Define

$$
f(z) := \gamma\beta^k \mathbb{E}_z R_1 \cdots R_k.
$$

By the induction hypothesis, the monotonicity of $\tilde{T}$ and the Markov property,

$$
\begin{aligned}
\tilde{T}^{k+1}h_\gamma(a,z) &= \max\left\{\beta\mathbb{E}_z\hat{R}\left(\tilde{T}^kh_\gamma\right)\left(\hat{R}\left[a-(u')^{-1}(\tilde{T}^{k+1}h_\gamma)(a,z)\right]+\hat{Y},\hat{z}\right),u'(a)\right\} \\
&\leq \max\left\{\beta\mathbb{E}_z\hat{R}\left(\tilde{T}^kh+f\right)\left(\hat{R}\left[a-(u')^{-1}(\tilde{T}^{k+1}h_\gamma)(a,z)\right]+\hat{Y},\hat{z}\right),u'(a)\right\} \\
&\leq \max\left\{\beta\mathbb{E}_z\hat{R}(\tilde{T}^kh)\left(\hat{R}\left[a-(u')^{-1}(\tilde{T}^{k+1}h_\gamma)(a,z)\right]+\hat{Y},\hat{z}\right),u'(a)\right\} \\
&\quad + \beta\mathbb{E}_zR_1f(z_1) \\
&\leq \max\left\{\beta\mathbb{E}_z\hat{R}(\tilde{T}^kh)\left(\hat{R}\left[a-(u')^{-1}(\tilde{T}^{k+1}h)(a,z)\right]+\hat{Y},\hat{z}\right),u'(a)\right\} \\
&\quad + \gamma\beta^{k+1}\mathbb{E}_zR_1\mathbb{E}_{z_1}R_1\cdots R_k \\
&= \tilde{T}^{k+1}h(a,z) + \gamma\beta^{k+1}\mathbb{E}_zR_1\cdots R_{k+1}.
\end{aligned}
$$

Hence, (32) is verified by induction. This concludes the proof. $\qquad\square$

With the results established above, we are now ready to prove theorem 3.2.

Proof of theorem 3.2. In view of propositions 3.1 and 3.2, to establish all the claims in theorem 3.2, we need only show that

$$
\rho(T^nc,T^nd) \leq \theta\rho(c,d) \quad \text{for all } c,d\in\mathscr{C}.
$$

To this end, pick any $c,d\in\mathscr{C}$. Note that the topological conjugacy result established in lemma 6.1 implies that $\tilde{T} = HTH^{-1}$. Hence,

$$
\tilde{T}^n = (HTH^{-1})\cdots(HTH^{-1}) = HT^nH^{-1} \quad \text{and} \quad \tilde{T}^nH = HT^n.
$$

By the definition of ρ and the contraction property established in lemma 6.3,

$$
\rho(T^nc,T^nd) = d_\infty(HT^nc,HT^nd) = d_\infty(\tilde{T}^nHc,\tilde{T}^nHd) \leq \theta d_\infty(Hc,Hd).
$$

The right hand side is just $\theta\rho(c,d)$, which completes the proof. $\qquad\square$

7. APPENDIX B: PROOF OF SECTION 4 RESULTS

Before working into the results of each subsection, we prove a general lemma that is frequently used in later sections. Recall that, for all $c\in\mathscr{C}$, the value $\xi(a,z):=Tc(a,z)$ solves

$$
(u'\circ\xi)(a,z) = \max\left\{\beta\mathbb{E}_z\hat{R}(u'\circ c)\left(\hat{R}\left[a-\xi(a,z)\right]+\hat{Y},\hat{z}\right),u'(a)\right\}. \tag{33}
$$

Let $c^* \in \mathscr{C}$ denote the optimal policy. For each $z \in \mathsf{Z}$ and $c \in \mathscr{C}$, define

$$\bar{a}_c(z) := (u')^{-1} \left[\beta \mathbb{E}_z \hat{R} \left(u' \circ c \right) (\hat{Y}, \hat{z}) \right] \quad \text{and} \quad \bar{a}(z) := \bar{a}_{c^*}(z). \tag{34}$$

The next result implies that the borrowing constraint binds if and only if wealth is below a certain threshold level.

Lemma 7.1. *For all $c \in \mathscr{C}$, $Tc(a, z) = a$ if and only if $a \leq \bar{a}_c(z)$. In particular, $c^*(a, z) = a$ if and only if $a \leq \bar{a}(z)$.*

Proof of lemma 7.1. Let $a \leq \bar{a}_c(z)$. We claim that $\xi(a, z) = a$. Suppose to the contrary that $\xi(a, z) < a$. Then $(u' \circ \xi)(a, z) > u'(a)$. In view of (33), we have

$$u'(a) < \beta \mathbb{E}_z \hat{R} \left(u' \circ c \right) \left(\hat{R} \left[a - \xi(a, z) \right] + \hat{Y}, \hat{z} \right) \leq \beta \mathbb{E}_z \hat{R} \left(u' \circ c \right) (\hat{Y}, \hat{z}) = u'[\bar{a}_c(z)].$$

From this we get $a > \bar{a}_c(z)$, which is a contradiction. Hence, $\xi(a, z) = a$.

On the other hand, if $\xi(a, z) = a$, then $(u' \circ \xi)(a, z) = u'(a)$. By (33), we have

$$u'(a) \geq \beta \mathbb{E}_z \hat{R} \left(u' \circ c \right) (\hat{Y}, \hat{z}) = u'[\bar{a}_c(z)].$$

Hence, $a \leq \bar{a}_c(z)$. The first claim is verified. The second claim follows immediately from the first claim and the fact that c^* is the unique fixed point of T in $\mathscr{C}$. □

Given $c \in \mathscr{C}$, lemma 7.1 implies that $\xi(a, z) := Tc(a, z) = a$ for $a \leq \bar{a}_c(z)$, and that for $a > \bar{a}_c(z)$, $\xi(a, z)$ solves

$$(u' \circ \xi)(a, z) = \beta \mathbb{E}_z \hat{R} \left(u' \circ c \right) \left(\hat{R} \left[a - \xi(a, z) \right] + \hat{Y}, \hat{z} \right).$$

7.1. **Proof of section 4.1 results.** Our first goal is to prove proposition 4.1. To that end, recall α given by assumption 4.1, and define the subspace $\mathscr{C}_1$ as

$$\mathscr{C}_1 := \left\{ c \in \mathscr{C} : \frac{c(a, z)}{a} \geq \alpha \quad \text{for all } (a, z) \in \mathsf{S}_0 \right\}. \tag{35}$$

Lemma 7.2. *$\mathscr{C}_1$ is a closed subset of $\mathscr{C}$, and $Tc \in \mathscr{C}_1$ for all $c \in \mathscr{C}_1$.*

Proof of lemma 7.2. To see that $\mathscr{C}_1$ is closed, for a given sequence $\{c_n\}$ in $\mathscr{C}_1$ and $c \in \mathscr{C}$ with $\rho(c_n, c) \to 0$, we need to verify that $c \in \mathscr{C}_1$. This obviously holds since $c_n(a, z)/a \geq \alpha$ for all n and $(a, z) \in \mathsf{S}_0$, and, on the other hand, $\rho(c_n, c) \to 0$ implies that $c_n(a, z) \to c(a, z)$ for all $(a, z) \in \mathsf{S}_0$.

We next show that T is a self-map on $\mathscr{C}_1$. Fix $c \in \mathscr{C}_1$. We have $Tc \in \mathscr{C}$ since T is a self-map on $\mathscr{C}$. It remains to show that $\xi := Tc$ satisfies $\xi(a, z) \geq \alpha a$ for all $(a, z) \in \mathsf{S}_0$. Suppose to the contrary that $\xi(a, z) < \alpha a$ for some $(a, z) \in \mathsf{S}_0$. Then

$$u'(\alpha a) < (u' \circ \xi)(a, z) = \max \left\{ \beta \mathbb{E}_z \hat{R} (u' \circ c) \left(\hat{R} [a - \xi(a, z)] + \hat{Y}, \hat{z} \right), u'(a) \right\}.$$

Since $u'(\alpha a) > u'(a)$ and $c \in \mathscr{C}_1$, this implies that

$$
\begin{aligned}
u'(\alpha a) &< \beta \mathbb{E}_z \hat{R} (u' \circ c) \left(\hat{R} [a - \xi(a, z)] + \hat{Y}, \hat{z} \right) \\
&\leq \beta \mathbb{E}_z \hat{R} u' \left(\alpha \hat{R} [a - \xi(a, z)] + \alpha \hat{Y} \right) \\
&\leq \beta \mathbb{E}_z \hat{R} u' \left[\alpha \hat{R}(1 - \alpha)a + \alpha \hat{Y} \right] \leq \beta \mathbb{E}_z \hat{R} u' \left[\hat{R}(1 - \alpha)(\alpha a) \right].
\end{aligned}
$$

This is a contradicted with condition (1) of assumption 4.1 since $(\alpha a, z) \in \mathsf{S}_0$. Hence, $\xi(a, z)/a \geq \alpha$ for all $(a, z) \in \mathsf{S}_0$ and we conclude that $Tc \in \mathscr{C}_1$. $\qquad \square$

With this result, we are now ready to prove proposition 4.1.

Proof of proposition 4.1. Since the claim obviously holds when $a = 0$, it remains to verify that this claim holds on S_0. We have shown in theorem 3.2 that T is a contraction mapping on the complete metric space $(\mathscr{C}, \rho)$, with unique fixed point c^*. Since in addition $\mathscr{C}_1$ is a closed subset of $\mathscr{C}$ and $T\mathscr{C}_1 \subset \mathscr{C}_1$ by lemma 7.2, we know that $c^* \in \mathscr{C}_1$. In summary, we have $c^*(a, z) \geq \alpha a$ for all $(a, z) \in \mathsf{S}$. $\qquad \square$

Our next goal is to prove theorem 4.1. To that end, recall the integer n given by the second condition of assumption 4.1.

Lemma 7.3. $\sup_{t \geq 0} \mathbb{E}_{a,z} a_t < \infty$ *for all* $(a, z) \in \mathsf{S}$.

Proof of lemma 7.3. Since $c^*(0, z) = 0$, proposition 4.1 implies that $c^*(a, z) \geq \alpha a$ for all $(a, z) \in \mathsf{S}$. For all $t \geq 1$, we have $t = kn + j$ in general, where $k \in \{0\} \cup \mathbb{N}$ and

$j \in \{0, 1, \cdots, n-1\}$. Using these facts and (1), we have:

$$a_t = R_t(a_{t-1} - c_{t-1}) + Y_t \le (1-\alpha)R_t a_{t-1} + Y_t \le \cdots$$

$$\le (1-\alpha)^t R_t \cdots R_1 a + (1-\alpha)^{t-1} R_t \cdots R_2 Y_1 + \cdots + (1-\alpha)R_t Y_{t-1} + Y_t$$

$$= (1-\alpha)^{kn+j} R_{kn+j} \cdots R_1 a + \sum_{\ell=1}^{j} (1-\alpha)^{kn+j-\ell} R_{kn+j} \cdots R_{\ell+1} Y_\ell$$

$$+ \sum_{m=1}^{k} \sum_{\ell=1}^{n} (1-\alpha)^{mn-\ell} R_{kn+j} \cdots R_{(k-m)n+j+\ell+1} Y_{(k-m)n+j+\ell}$$

with probability one. Hence,

$$\mathbb{E}_{a,z} a_t \le (1-\alpha)^t \mathbb{E}_z R_t \cdots R_1 a + \sum_{\ell=1}^{t} (1-\alpha)^{t-\ell} \mathbb{E}_z R_t \cdots R_{\ell+1} Y_\ell$$

$$= (1-\alpha)^{kn+j} \mathbb{E}_z R_{kn+j} \cdots R_1 a + \sum_{\ell=1}^{j} (1-\alpha)^{kn+j-\ell} \mathbb{E}_z R_{kn+j} \cdots R_{\ell+1} Y_\ell$$

$$+ \sum_{m=1}^{k} \sum_{\ell=1}^{n} (1-\alpha)^{mn-\ell} \mathbb{E}_z R_{kn+j} \cdots R_{(k-m)n+j+\ell+1} Y_{(k-m)n+j+\ell}$$

for all $(a, z) \in \mathsf{S}$. Define

$$\gamma := (1-\alpha)^n \sup_{z \in \mathsf{Z}} \mathbb{E}_z R_1 \cdots R_n \quad \text{and} \quad M := \max_{1 \le \ell \le n} \left[(1-\alpha)^\ell \sup_{z \in \mathsf{Z}} \mathbb{E}_z R_\ell \cdots R_1 \right].$$

Note that $\gamma < 1$ by assumption 4.1-(2) and $M < \infty$ by assumption 3.3 and the Markov property. Moreover, $M' := \sup_{t \ge 1} \mathbb{E}_z Y_t < \infty$ by assumption 4.2. The Markov property then implies that for all $(a, z) \in \mathsf{S}$ and $t \ge 0$,

$$\mathbb{E}_{a,z} a_t \le \gamma^k (1-\alpha)^j \mathbb{E}_z R_j \cdots R_1 a + \gamma^k \sum_{\ell=1}^{j} (1-\alpha)^{j-\ell} \mathbb{E}_z R_j \cdots R_{\ell+1} Y_\ell$$

$$+ \sum_{m=0}^{k-1} \gamma^m \sum_{\ell=1}^{n} (1-\alpha)^{n-\ell} \mathbb{E}_z R_{(k-m)n+j} \cdots R_{(k-m-1)n+j+\ell+1} Y_{(k-m)n+j+\ell}$$

$$\le \gamma^k M a + \gamma^k M \sum_{\ell=1}^{j} \mathbb{E}_z Y_\ell + \sum_{m=0}^{k-1} \gamma^m M \sum_{\ell=1}^{n} \mathbb{E}_z Y_{(k-m-1)n+j+\ell}$$

$$\le M a + M M' n + \sum_{m=0}^{\infty} \gamma^m M M' n < \infty.$$

Hence, $\sup_{t \ge 0} \mathbb{E}_{a,z} a_t < \infty$ for all $(a, z) \in \mathsf{S}$, as was claimed. $\qquad \square$

A function $w^*\colon \mathsf{S} \to \mathbb{R}_+$ is called *norm-like* if all its sublevel sets (i.e., sets of the form $\{s \in \mathsf{S}\colon w(s) \leq b\}, b \in \mathbb{R}_+$) are precompact in S (i.e., any sequence in a given sublevel set has a subsequence that converges to a point of S).

Proof of theorem 4.1. Based on lemma D.5.3 of Meyn and Tweedie (2009), a stochastic kernel Q is bounded in probability if and only if for all $s \in \mathsf{S}$, there exists a norm-like function $w_s^*\colon \mathsf{S} \to \mathbb{R}_+$ such that the (Q,s)-Markov process $\{s_t\}_{t \geq 0}$ satisfies $\limsup_{t \to \infty} \mathbb{E}_s [w_s^*(s_t)] < \infty$.

Fix $(a,z) \in \mathsf{S}$. Since P is bounded in probability by assumption 4.3, there exists a norm-like function $w\colon \mathsf{Z} \to \mathbb{R}_+$ such that $\limsup_{t \to \infty} \mathbb{E}_z w(z_t) < \infty$. Then $w^*\colon \mathsf{S} \to \mathbb{R}_+$ defined by $w^*(a_0, z_0) := a_0 + w(z_0)$ is a norm-like function on S. The stochastic kernel Q is then bounded in probability since lemma 7.3 implies that

$$\limsup_{t \to \infty} \mathbb{E}_{a,z}\, w^*(a_t, z_t) \leq \sup_{t \geq 0} \mathbb{E}_{a,z}\, a_t + \limsup_{t \to \infty} \mathbb{E}_z\, w(z_t) < \infty.$$

Regarding existence of stationary distribution, since c^* is continuous and assumption 3.4 holds, and we have shown in the proof of proposition 3.4 that

$$P(z_n, \cdot) \otimes \nu \otimes \mu \xrightarrow{w} P(z, \cdot) \otimes \nu \otimes \mu$$

whenever $z_n \to z$, a simple application of the generalized Fatou's lemma of Feinberg et al. (2014) (theorem 1.1) as in the proof of proposition 3.4 shows that the stochastic kernel Q is Feller. Since in addition Q is bounded in probability, based on the Krylov-Bogolubov theorem (see, e.g., Meyn and Tweedie (2009), proposition 12.1.3 and lemma D.5.3), Q admits at least one stationary distribution. $\qquad \square$

7.2. **Proof of section 4.2 results.** We start by proving example 4.4.

Proof of example 4.4. For each c in $\mathscr{C}$ concave in the first argument, let $h_c(x, \hat\omega) := c(\hat{R}x + \hat{Y}, \hat{z})$, where $\hat\omega := (\hat{R}, \hat{Y}, \hat{z})$. Then $x \mapsto h_c(x, \hat\omega)$ is concave. Since $u'(c) = c^{-\gamma}$,

we have

$$\left[\beta \mathbb{E}_z \hat{R}\, h_c(\alpha x_1 + (1-\alpha)x_2, \hat{\omega})^{-\gamma}\right]^{-\frac{1}{\gamma}} \geq \left[\beta \mathbb{E}_z \hat{R}\left[\alpha h_c(x_1, \hat{\omega}) + (1-\alpha)h_c(x_2, \hat{\omega})\right]^{-\gamma}\right]^{-\frac{1}{\gamma}}$$

$$= \beta^{-\frac{1}{\gamma}} \left(\mathbb{E}_z\left[\alpha \hat{R}^{-\frac{1}{\gamma}} h_c(x_1, \hat{\omega}) + (1-\alpha)\hat{R}^{-\frac{1}{\gamma}} h_c(x_2, \hat{\omega})\right]^{-\gamma}\right)^{-\frac{1}{\gamma}}$$

$$\geq \beta^{-\frac{1}{\gamma}} \left[\left(\mathbb{E}_z\left[\alpha \hat{R}^{-\frac{1}{\gamma}} h_c(x_1, \hat{\omega})\right]^{-\gamma}\right)^{-\frac{1}{\gamma}} + \left(\mathbb{E}_z\left[(1-\alpha)\hat{R}^{-\frac{1}{\gamma}} h_c(x_2, \hat{\omega})\right]^{-\gamma}\right)^{-\frac{1}{\gamma}}\right]$$

$$= \alpha \left[\beta \mathbb{E}_z \hat{R}\, h_c(x_1, \hat{\omega})^{-\gamma}\right]^{-\frac{1}{\gamma}} + (1-\alpha)\left[\beta \mathbb{E}_z \hat{R}\, h_c(x_2, \hat{\omega})^{-\gamma}\right]^{-\frac{1}{\gamma}},$$

where the second inequality is due to the generalized Minkowski's inequality (see, e.g., Hardy et al. (1952), page 146, theorem 198). Hence, assumption 4.4 holds. $\qquad\square$

Next, we aim to prove proposition 4.2. Recall $\mathscr{C}_1$ given by (35). Consider a further subspace $\mathscr{C}_2$ defined by

$$\mathscr{C}_2 := \{c \in \mathscr{C}_1 \colon a \mapsto c(a, z) \text{ is concave for all } z \in \mathsf{Z}\}. \tag{36}$$

Lemma 7.4. $\mathscr{C}_2$ *is a closed subset of the metric space* $(\mathscr{C}, \rho)$, *and* $Tc \in \mathscr{C}_2$ *for all* $c \in \mathscr{C}_2$.

Proof of lemma 7.4. The proof of the first claim is straightforward and thus omitted. We now prove the second claim. Fix $c \in \mathscr{C}_2$. By lemma 7.2 we have $Tc \in \mathscr{C}_1$. It remains to show that $a \mapsto \xi(a, z) := Tc(a, z)$ is concave for all $z \in \mathsf{Z}$. Given $z \in \mathsf{Z}$, lemma 7.1 implies that $\xi(a, z) = a$ for $a \leq \bar{a}_c(z)$ and that $\xi(a, z) < a$ for $a > \bar{a}_c(z)$. Since in addition $a \mapsto \xi(a, z)$ is continuous and increasing, to show the concavity of ξ with respect to a, it suffices to show that $a \mapsto \xi(a, z)$ is concave on $(\bar{a}_c(z), \infty)$.

Suppose to the contrary that there exist some $z \in \mathsf{Z}$, $\alpha \in [0, 1]$, and $a_1, a_2 \in (\bar{a}_c(z), \infty)$ such that

$$\xi\left(\alpha a_1 + (1-\alpha)a_2, z\right) < \alpha \xi(a_1, z) + (1-\alpha)\xi(a_2, z). \tag{37}$$

Let $h(a, z, \hat{\omega}) := \hat{R}\left[a - \xi(a, z)\right] + \hat{Y}$, where $\hat{\omega} := (\hat{R}, \hat{Y})$. Then by lemma 7.1 (and the analysis that follows immediately after that lemma), we have

$$(u' \circ \xi)\left(\alpha a_1 + (1-\alpha)a_2, z\right) = \beta \mathbb{E}_z \hat{R}\, (u' \circ c)\left\{h[\alpha a_1 + (1-\alpha)a_2, z, \hat{\omega}], \hat{z}\right\}$$

$$\leq \beta \mathbb{E}_z \hat{R}\, (u' \circ c)\left[\alpha h(a_1, z, \hat{\omega}) + (1-\alpha)h(a_2, z, \hat{\omega}), \hat{z}\right].$$

Using assumption 4.4 then yields

$$\xi(\alpha a_1 + (1-\alpha)a_2, z) \geq (u')^{-1}\left\{\beta\mathbb{E}_{\,z}\hat{R}\left(u'\circ c\right)\left[\alpha h(a_1, z, \hat{\omega}) + (1-\alpha)h(a_2, z, \hat{\omega}),\, \hat{z}\right]\right\}$$

$$\geq \alpha\,(u')^{-1}\left\{\beta\mathbb{E}_{\,z}\hat{R}\left(u'\circ c\right)\left[h(a_1, z, \hat{\omega}),\, \hat{z}\right]\right\} +$$

$$(1-\alpha)\,(u')^{-1}\left\{\beta\mathbb{E}_{\,z}\hat{R}\left(u'\circ c\right)\left[h(a_2, z, \hat{\omega}),\, \hat{z}\right]\right\}$$

$$= \alpha\,(u')^{-1}\left\{(u'\circ\xi)\,(a_1, z)\right\} + (1-\alpha)\,(u')^{-1}\left\{(u'\circ\xi)\,(a_2, z)\right\}$$

$$= \alpha\,\xi(a_1, z) + (1-\alpha)\,\xi(a_2, z).$$

This contradicts our assumption in (37). Hence, $a \mapsto \xi(a, z)$ is concave for all $z \in \mathsf{Z}$. This concludes the proof. $\qquad\square$

Now we are ready to prove proposition 4.2.

Proof of proposition 4.2. By theorem 3.2, we know that $T\colon \mathscr{C} \to \mathscr{C}$ is a contraction mapping with unique fixed point c^*. Since $\mathscr{C}_2$ is a closed subset of $\mathscr{C}$ and $T : \mathscr{C}_2 \to \mathscr{C}_2$ by lemma 7.4, we know that $c^* \in \mathscr{C}_2$. The first claim is verified.

Regarding the second claim, note that $c^* \in \mathscr{C}_2$ implies that $a \mapsto c^*(a, z)$ is increasing and concave for all $z \in \mathsf{Z}$. Hence, $a \mapsto \frac{c^*(a,z)}{a}$ is a decreasing function for all $z \in \mathsf{Z}$. Since in addition $c^*(a, z) \geq \alpha a$ for all $(a, z) \in \mathsf{S}_0$ by proposition 4.1, we know that $\alpha' := \lim_{a\to\infty}\frac{c^*(a,z)}{a}$ is well-defined and $\alpha' \geq \alpha$. Finally, $\alpha' < 1$ by lemma 7.1 and the fact that $\bar{a}(z) < \infty$ (see footnote 6). Hence, the second claim holds. $\qquad\square$

7.3. **Proof of section 4.3 results.** We first prove the general result that the borrowing constraint binds in finite time with positive probability.

Lemma 7.5. *For all $(a, z) \in \mathsf{S}$, we have $\mathbb{P}_{a,z}\left(\cup_{t\geq 0}\{c_t = a_t\}\right) > 0$.*

Proof of lemma 7.5. The claim holds trivially when $a = 0$. Suppose the claim does not hold on S_0 (recall that $\mathsf{S}_0 = \mathsf{S}\backslash\{0\}$), then $\mathbb{P}_{a,z}\left(\cap_{t\geq 0}\{c_t < a_t\}\right) = 1$ for some $(a, z) \in \mathsf{S}_0$, i.e., the borrowing constraint never binds with probability one. Hence,

$$\mathbb{P}_{a,z}\left\{(u'\circ c)(a_t, z_t) = \beta\mathbb{E}\left[R_{t+1}(u'\circ c)(a_{t+1}, z_{t+1})\big|\mathscr{F}_t\right]\right\} = 1$$

for all $t \geq 0$, where $\mathscr{F}_t := \sigma(s_0, \cdots, s_t)$ with $s_t := (a_t, z_t)$. Then we have

$$
\begin{aligned}
(u' \circ c)\,(a, z) &= \beta^t \mathbb{E}_{a,z}\, R_1 \cdots R_t\,(u' \circ c)\,(a_t, z_t) \\
&\leq \beta^t \mathbb{E}_{a,z}\, R_1 \cdots R_t\,[u'(a_t) + K] \\
&\leq \beta^t \mathbb{E}_z\, R_1 \cdots R_t\,[u'(Y_t) + K]
\end{aligned}
\tag{38}
$$

for all $t \geq 1$. Let $t = kn + 1$, where n is the integer defined by assumption 3.1. Based on assumption 3.3 and the Markov property,

$$
\beta^t \mathbb{E}_z R_1 \cdots R_t = \beta^t \mathbb{E}_z R_1 \cdots R_{t-1} \mathbb{E}_z(R_t \mid \mathscr{F}_{t-1}) = \beta^{t-1} \mathbb{E}_z R_1 \cdots R_{t-1} \beta \mathbb{E}_{z_{t-1}} R_1
$$

$$
\leq \left(\beta \sup_{z \in \mathsf{Z}} \mathbb{E}_z R_1 \right) (\beta^{nk} \mathbb{E}_z R_1 \cdots R_{nk}) \leq \left(\beta \sup_{z \in \mathsf{Z}} \mathbb{E}_z R_1 \right) \theta^k \to 0
$$

as $t \to \infty$, where $\theta \in [0, 1)$ is given by assumption 3.1. Similarly,

$$
\begin{aligned}
\beta^t \mathbb{E}_z R_1 \cdots R_t u'(Y_t) &= \beta^t \mathbb{E}_z R_1 \cdots R_{t-1} \mathbb{E}_z\,[R_t u'(Y_t) \mid \mathscr{F}_{t-1}] \\
&\leq \beta^t \mathbb{E}_z R_1 \cdots R_{t-1} \mathbb{E}_{z_{t-1}}\,[R_1 u'(Y_1)] \\
&\leq \left(\beta \sup_{z \in \mathsf{Z}} \mathbb{E}_z[\hat{R} u'(\hat{Y})] \right) \beta^{nk} \mathbb{E}_z R_1 \cdots R_{nk} \\
&\leq \left(\beta \sup_{z \in \mathsf{Z}} \mathbb{E}_z[\hat{R} u'(\hat{Y})] \right) \theta^k \to 0
\end{aligned}
$$

as $t \to \infty$. Letting $t \to \infty$. (38) implies that $(u' \circ c)\,(a, z) \leq 0$, contradicted with the fact that $u' > 0$. Thus, we must have $\mathbb{P}_{a,z}\,(\cup_{t \geq 0}\{c_t = a_t\}) > 0$ for all $(a, z) \in \mathsf{S}$. □

7.3.1. *Proof of section 4.3.1 results.* The next few results establish global stability and the law of large numbers for the case of IID $\{z_t\}$ process.

We say that a stochastic kernel Q *increasing* if $s \mapsto \int h(s')Q(s, \mathrm{d}s')$ is bounded and increasing whenever $h \colon \mathsf{S} \to \mathbb{R}$ is.

Proof of theorem 4.2. Obviously, assumptions 2.1, 3.1–3.4 and 4.1–4.4 hold under the stated assumptions of theorem 4.2. Based on proposition 4.2, we have $c^* \in \mathscr{C}_2$. In particular, $a \mapsto c^*(a)$ is continuous, and $a \mapsto \frac{c^*(a)}{a}$ is decreasing on $(0, \infty)$. Hence, a_{t+1} is continuous and increasing in a_t (see equation (17)). The stochastic kernel Q is then Feller and increasing. Moreover, Q is bounded in probability by lemma 7.3.

Fix a_0 and a_0' in $\mathbb{R}_+$ with $a_0' \leq a_0$. Let $\{a_t\}$ and $\{a_t'\}$ be two independent Markov processes generated by (17), starting at a_0 and a_0' respectively. Let $\{c_t\}$ and $\{c_t'\}$ be the corresponding optimal consumption paths. By lemma 7.5, $\mathbb{P}_{a_0}(\cup_{t \geq 0}\{c_t = a_t\}) > 0$, i.e., the borrowing constraint binds in finite time with positive probability. Hence, with positive probability, $a_{t+1} = Y_{t+1} \leq R_{t+1}(a_t' - c_t') + Y_{t+1} = a_{t+1}'$. In other words, $\mathbb{P}\{a_{t+1} \leq a_{t+1}'\} > 0$ and Q is order reversing.

Since Q is increasing, Feller, order reversing, and bounded in probability, based on theorem 3.2 of Kamihigashi and Stachurski (2014), Q is globally stable. $\qquad\square$

Proof of theorem 4.3. We have shown in the proof of theorem 4.2 that the stochastic kernel Q is increasing, bounded in probability, and order reversing. Hence, Q is monotone ergodic by proposition 4.1 of Kamihigashi and Stachurski (2016). The two claims of theorem 4.3 then follow from theorem 4.2 (of this paper), and corollary 3.1 and theorem 3.2 of Kamihigashi and Stachurski (2016). In particular, if we pair S with its usual pointwise order $\leq$, then assumption 3.1 of Kamihigashi and Stachurski (2016) obviously holds. $\qquad\square$

7.3.2. *Proof of Section 4.3.2 Results.* Our next goal is to prove theorems 4.4–4.5. In proofs we apply the theory of Meyn and Tweedie (2009). Important definitions (their locations in Meyn and Tweedie (2009)) include: ψ-irreducibility (section 4.2), small set (page 102), strong aperiodicity (page 114), petite set (page 117), Harris chain (page 199), and positivity (page 230).

Note that since $\mathbb{R}^m$ paired with its Euclidean topology is a second countable topological space (i.e., its topology has a countable base), while $\mathbb{R}_+$ and Z are respectively Borel subsets of $\mathbb{R}$ and $\mathbb{R}^m$ paired with the relative topologies, $\mathbb{R}_+$ and Z are also second countable. As a result, for $\mathsf{S} := \mathbb{R}_+ \times \mathsf{Z}$, it always holds that (see, e.g., page 149, theorem 4.44 of Aliprantis and Border (2006))

$$\mathscr{B}(\mathsf{S}) = \mathscr{B}(\mathbb{R}_+) \otimes \mathscr{B}(\mathsf{Z}).$$

Recall the Lebesgue measure λ on $\mathscr{B}(\mathbb{R}_+)$ and the measure ϑ on $\mathscr{B}(\mathsf{Z})$ defined in section 4.3.2. Let $\lambda \times \vartheta$ be the product measure on $\mathscr{B}(\mathsf{S})$.

Lemma 7.6. *Let the function $\bar{a}$ be defined as in (34). Then $\inf_{z \in \mathsf{Z}} \bar{a}(z) > 0$.*

Proof of lemma 7.6. Since $c^* \in \mathscr{C}$, there exists a constant $K > 0$ such that

$$0 < (u' \circ c^*)(a, z) \leq u'(a) + K \quad \text{for all } (a, z) \in \mathsf{S}_0.$$

Assumption 3.3 then implies that

$$\sup_{z \in \mathsf{Z}} \mathbb{E}\,_z\hat{R}(u' \circ c^*)(\hat{Y}, \hat{z}) \leq \sup_{z \in \mathsf{Z}} \mathbb{E}\,_z\hat{R}u'(\hat{Y}) + K \sup_{z \in \mathsf{Z}} \mathbb{E}\,_z\hat{R} < \infty.$$

Then, by the definition of $\bar{a}$ and the properties of u,

$$\inf_{z \in \mathsf{Z}} \bar{a}(z) = (u')^{-1}\left[\beta \sup_{z \in \mathsf{Z}} \mathbb{E}\,_z\hat{R}(u' \circ c^*)(\hat{Y}, \hat{z})\right] > 0,$$

as claimed. □

Recall the compact subset $\mathsf{C} \subset \mathsf{Z}$ and $\delta_Y > 0$ given by assumption 4.5. Let

$$\mathsf{C}' := \left[0, \min\left\{\delta_Y, \inf_{z \in \mathsf{Z}} \bar{a}(z)\right\}\right] \quad \text{and} \quad \mathsf{D} := \mathsf{C}' \times \mathsf{C} \in \mathscr{B}(\mathsf{S}). \tag{39}$$

Lemma 7.7. *The Markov process $\{(a_t, z_t)\}_{t \geq 0}$ is ψ-irreducible.*

Proof of lemma 7.7. We define the measure φ on $\mathscr{B}(\mathsf{S})$ by

$$\varphi(A) := (\lambda \times \vartheta)(A \cap \mathsf{D}) \quad \text{for } A \in \mathscr{B}(\mathsf{S}).$$

Then φ is a nontrivial measure. In particular, $\varphi(\mathsf{S}) = (\lambda \times \vartheta)(\mathsf{D}) = \lambda(\mathsf{C}')\vartheta(\mathsf{C}) > 0$ since $\lambda(\mathsf{C}') > 0$ by lemma 7.6 and $\vartheta(\mathsf{C}) > 0$ by assumption 4.5.

For fixed $(a, z) \in \mathsf{S}$ and $A \in \mathscr{B}(\mathsf{S})$ with $\varphi(A) > 0$, by lemma 7.1,

$$\begin{aligned}
\mathbb{P}_{(a,z)}\{(a_{t+1}, z_{t+1}) \in A\} &\geq \mathbb{P}_{(a,z)}\{(a_{t+1}, z_{t+1}) \in A, \, a_t \leq \bar{a}(z_t)\} \\
&= \mathbb{P}_{(a,z)}\{(a_{t+1}, z_{t+1}) \in A \mid c_t = a_t\} \, \mathbb{P}_{(a,z)}\{c_t = a_t\} \\
&= \mathbb{P}_{(a,z)}\{(Y_{t+1}, z_{t+1}) \in A \mid c_t = a_t\} \, \mathbb{P}_{(a,z)}\{c_t = a_t\} \\
&= \mathbb{P}_{(a,z)}\{(Y_{t+1}, z_{t+1}) \in A, \, a_t \leq \bar{a}(z_t)\}. \tag{40}
\end{aligned}$$

Note that for all $z' \in \mathsf{Z}$, by assumption 4.5, $f_L(Y'' \mid z'')p(z'' \mid z') > 0$ whenever $(Y'', z'') \in \mathsf{D}$. Since in addition $\varphi(A) = (\lambda \times \vartheta)(A \cap \mathsf{D}) > 0$, we have

$$\int_A f_L(Y'' \mid z'')p(z'' \mid z')(\lambda \times \vartheta)[\mathrm{d}(Y'', z'')] > 0 \quad \text{for all } z' \in \mathsf{Z}.$$

Let $\triangle := \mathbb{P}_{(a,z)}\{(a_{t+1}, z_{t+1}) \in A\}$ and $E := \{(a', z') \in \mathsf{S} \colon a' \leq \bar{a}(z')\}$. Notice that by lemma 7.1 and lemma 7.5, there exists $t \in \mathbb{N}$ such that

$$Q^t\left((a, z), E\right) = \mathbb{P}_{(a,z)}\{a_t \leq \bar{a}(z_t)\} > 0.$$

Hence, (40) implies that

$$\triangle \geq \int_E \left\{ \int_A f_L(Y'' \mid z'')p(z'' \mid z')(\lambda \times \vartheta)[\mathrm{d}(Y'', z'')] \right\} Q^t\left((a, z), \mathrm{d}(a', z')\right) > 0.$$

Therefore, we have shown that any measurable subset with positive φ measure can be reached in finite time with positive probability, i.e., $\{(a_t, z_t)\}$ is φ-irreducible. Based on proposition 4.2.2 of Meyn and Tweedie (2009), there exists a maximal (in the sense of absolute continuity) probability measure ψ on $\mathscr{B}(\mathsf{S})$ such that $\{(a_t, z_t)\}$ is ψ-irreducible. $\qquad\square$

Lemma 7.8. *The Markov process $\{(a_t, z_t)\}_{t \geq 0}$ is strongly aperiodic.*

Proof of lemma 7.8. By the definition of strong aperiodicity, we need to show that there exists a v_1-small set D with $v_1(D) > 0$, i.e., there exists a nontrivial measure v_1 on $\mathscr{B}(\mathsf{S})$ and a subset $D \in \mathscr{B}(\mathsf{S})$ such that $v_1(D) > 0$ and

$$\inf_{(a,z) \in D} Q\left((a, z), A\right) \geq v_1(A) \quad \text{for all } A \in \mathscr{B}(\mathsf{S}). \tag{41}$$

Let D be defined as in (39). We show that D satisfies the above conditions. Let

$$r(a', z') := f_L(a' \mid z') \inf_{z \in \mathsf{C}} p(z' \mid z), \qquad (a', z') \in \mathsf{S}.$$

Since by assumption 4.5, $p(z' \mid z)$ is strictly positive on $\mathsf{C} \times \mathsf{Z}$ and continuous in z, and $f_L(Y' \mid z')$ is strictly positive on $(0, \delta_Y) \times \mathsf{C}$, the definition of D implies that $r(a', z')$ is strictly positive whenever $(a', z') \in \mathsf{D}$. Define the measure v_1 on $\mathscr{B}(\mathsf{S})$ by

$$v_1(A) := \int_A r(a', z')(\lambda \times \vartheta)[\mathrm{d}(a', z')] \quad \text{for } A \in \mathscr{B}(\mathsf{S}).$$

Since $(\lambda \times \vartheta)(\mathsf{D}) > 0$ as shown in the proof of lemma 7.7 and $r(a', z') > 0$ on D, we have $v_1(\mathsf{D}) > 0$, which also implies that v_1 is a nontrivial measure.

Let $g[(a', z') \mid (a, z)]$ denote the density representation of the stochastic kernel Q when $(a, z) \in \mathsf{D}$. Lemma 7.1 implies that

$$g[(a', z') \mid (a, z)] = f_L(a' \mid z')p(z' \mid z), \qquad (a, z) \in \mathsf{D}.$$

Hence, for all $(a, z) \in \mathsf{D}$ and $A \in \mathscr{B}(S)$,

$$
\begin{aligned}
Q\left((a, z), A\right) &= \int_A g[(a', z') \mid (a, z)](\lambda \times \vartheta)[\mathrm{d}(a', z')] \\
&\geq \int_A r(a', z')(\lambda \times \vartheta)[\mathrm{d}(a', z')] = v_1(A).
\end{aligned}
$$

This implies that condition (41) holds. Hence, $\{(a_t, z_t)\}_{t \geq 0}$ is strongly aperiodic. $\square$

Proof of theorem 4.4. We first show that $\{(a_t, z_t)\}$ is a positive Harris chain. Positivity has been established in theorem 4.1. To show Harris recurrence, by lemma 6.1.4, theorem 6.2.9 and theorem 18.3.2 of Meyn and Tweedie (2009), it suffices to verify

(a) Q is Feller and bounded in probability, and

(b) $\{(a_t, z_t)\}$ is ψ-irreducible, and the support of ψ has non-empty interior.

Claim (a) is already proved in theorem 4.1. Regarding claim (b), in lemma 7.7 we have shown that $\{(a_t, z_t)\}$ is φ-irreducible and thus ψ-irreducible, where ψ is maximal in the sense that $\psi(A) = 0$ implies $\varphi(A) = 0$ for all $A \in \mathscr{B}(\mathsf{S})$. This also implies that $\psi(A) > 0$ whenever $\varphi(A) > 0$. Recall that $\varphi(A) := (\lambda \times \vartheta)(A \cap \mathsf{D})$, where $\mathsf{D} := \mathsf{C}' \times \mathsf{C}$ is defined by (39). Since by assumption 4.5, the support of ϑ contains C that has nonempty interior and the support of λ (the Lebesgue measure) contains the interval C' (of positive λ measure), the support of φ contains $\mathsf{D} = \mathsf{C}' \times \mathsf{C}$ that has nonempty interior. As a result, the support of ψ contains D and thus has nonempty interior. Claim (b) is verified. Therefore, $\{(a_t, z_t)\}$ is a positive Harris chain.

Since in addition we have shown in lemmas 7.7–7.8 that $\{(a_t, z_t)\}$ is ψ-irreducible and strongly aperiodic, based on theorem 13.0.1 and theorem 17.1.7 of Meyn and Tweedie (2009), the stated claims of our theorem hold. This concludes the proof. $\square$

Our next goal is to prove theorem 4.5. We start by proving several lemmas.

Lemma 7.9. *The set* $B := [0, d] \times \{z\}$ *is a petite set for all* $d \in (0, \infty)$ *and* $z \in \mathsf{Z}$.

Proof of lemma 7.9. Since any small set is petite, it suffices to show that B is a v_2-small set, i.e., there exists a nontrivial measure v_2 on $\mathscr{B}(\mathsf{S})$ such that

$$
\inf_{(a,z) \in B} Q^2((a, z), A) \geq v_2(A) \quad \text{for all } A \in \mathscr{B}(\mathsf{S}). \tag{42}
$$

Without loss of generality, we assume that d is large enough. For $a \neq c^*(a, z)$, let

$$f(a' \mid a, z, z') := \frac{1}{a - c^*(a, z)} \int_{[0, a']} f_C\left(\frac{a' - Y'}{a - c^*(a, z)} \,\Big|\, z'\right) f_L\left(Y' \mid z'\right) \mathrm{d}Y', \qquad (43)$$

while $f(\cdot \mid a, z, z') := f_L(\cdot \mid z')$ for $a = c^*(a, z)$. Let $g[(a', z') \mid (a, z)]$ be the density corresponding to the stochastic kernel Q. Since $\{\zeta_t\}$ and $\{\eta_t\}$ are mutually independent by assumption 4.6, g satisfies

$$g[(a', z') \mid (a, z)] = f(a' \mid a, z, z')\, p(z' \mid z).$$

Recall that we have shown in the proof of proposition 4.2 that $a \mapsto c^*(a, z)/a$ is decreasing for all $z \in \mathsf{Z}$. This implies that, for the dynamical system (15), a_{t+1} is increasing in a_t with probability one. Since in addition $c^*(a, z) = a$ if and only if $a \leq \bar{a}(z)$ by lemma 7.1, we have

$$
\begin{aligned}
Q^2((a, z), A) = \mathbb{P}_{a,z}\{(a_2, z_2) \in A\} &\geq \mathbb{P}_{a,z}\{(a_2, z_2) \in A,\ a_1 \leq \bar{a}(z_1)\} \\
&= \mathbb{P}_{a,z}\{(a_2, z_2) \in A \mid a_1 \leq \bar{a}(z_1)\}\, \mathbb{P}_{a,z}\{a_1 \leq \bar{a}(z_1)\} \\
&= \mathbb{P}_{a,z}\{(Y_2, z_2) \in A \mid a_1 \leq \bar{a}(z_1)\}\, \mathbb{P}_{a,z}\{a_1 \leq \bar{a}(z_1)\} \\
&= \mathbb{P}\{(Y_2, z_2) \in A,\ a_1 \leq \bar{a}(z_1) \mid (a_0, z_0) = (a, z)\} \\
&\geq \mathbb{P}\{(Y_2, z_2) \in A,\ a_1 \leq \bar{a}(z_1) \mid (a_0, z_0) = (d, z)\} =: v_2(A)
\end{aligned}
$$

for all $(a, z) \in B$, where the last inequality follows from the fact that a_{t+1} is increasing in a_t (shown above), which indicates that for all fixed $(a, z) \in B$ and $z_1 \in \mathsf{Z}$,

$$\int f(a_1 \mid a, z, z_1) \mathbb{1}\{a_1 \leq \bar{a}(z_1)\}\, \mathrm{d}a_1 \geq \int f(a_1 \mid d, z, z_1) \mathbb{1}\{a_1 \leq \bar{a}(z_1)\}\, \mathrm{d}a_1 > 0.$$

We now show that v_2 defined this way is a nontrivial measure on $\mathscr{B}(\mathsf{S})$. Obviously, v_2 is a measure. Moreover, for fixed $z \in \mathsf{Z}$, $c^*(a, z)/a$ is decreasing in a, strictly less than one as a gets large, and bounded below by $\alpha \in (0, 1)$. Hence, there exists $\alpha' \in (0, 1)$ such that $c^*(a, z)/a \leq \alpha'$ as a gets large. Hence, $a - c^*(a, z) \geq (1 - \alpha')a$, which implies that $a - c^*(a, z) \to \infty$ as $a \to \infty$. Using lemma 7.1 again shows that $f(a' \mid a, z, z')$ satisfies (43) as a gets large. Let $\underline{a} := \inf_{z \in \mathsf{Z}} \bar{a}(z)$. Then $\underline{a} > 0$ by lemma 7.6. Recall $\delta_R > 0$, $\delta_Y > 0$ and the compact subset $\mathsf{C} \subset \mathsf{Z}$ defined by assumption 4.5. Then

$$0 < \frac{\underline{a}}{d - c^*(d, z)} < \delta_R \quad \text{as } d \text{ gets large.}$$

Since in addition $f_L(Y \mid z)$ is strictly positive on $(0, \delta_Y) \times \mathsf{C}$ and $f_C(R \mid z)$ is strictly positive on $(0, \delta_R) \times \mathsf{C}$ by assumptions 4.5–4.6, for d that is large enough, $f(a' \mid d, z, z')$ is defined by (43) and it is strictly positive for all $(a', z') \in (0, \underline{a}) \times \mathsf{C}$. Moreover, since $p(z' \mid z)$ is strictly positive on $\mathsf{C} \times \mathsf{Z}$ and $\vartheta(\mathsf{C}) > 0$ by assumption 4.5,

$$v_2(\mathsf{S}) = \mathbb{P}_{(d,z)}\{a_1 \leq \bar{a}(z_1)\} \geq \mathbb{P}_{(d,z)}\{a_1 \leq \underline{a}\}$$

$$= \int_{\mathsf{Z}} \left[\int_{[0,\underline{a}]} f(a' \mid d, z, z') \, \mathrm{d}a' \right] p(z' \mid z)\vartheta(\mathrm{d}z') > 0.$$

Hence, v_2 is a nontrivial measure on $\mathscr{B}(\mathsf{S})$. Since in addition z is the only element of Z that appears in the analytical form of B, (42) holds and thus B is petite. $\qquad \square$

In the following, we let $\alpha \in [0, 1)$ and $n \in \mathbb{N}$ be defined as in assumption 4.1.

Lemma 7.10. *There exist a petite set B, constants $b < \infty$, $\rho > 0$ and a measurable map $V \colon \mathsf{S} \to [1, \infty)$ such that, for all $(a, z) \in \mathsf{S}$,*

$$\mathbb{E}_{a,z}V(a_n, z_n) - V(a, z) \leq -\rho V(a, z) + b\mathbb{1}\{(a, z) \in B\}.$$

Proof of lemma 7.10. By assumption 4.6, there exists $q'' \in \mathbb{R}_+$ such that

$$\mathbb{E}_z Y_t \leq q^{t-1}\mathbb{E}_z Y_1 + q'' \quad \text{for all } t \in \mathbb{N} \text{ and } z \in \mathsf{Z}.$$

Since $c^*(a, z) \geq \alpha a$ for all $(a, z) \in \mathsf{S}$ by proposition 4.1, $M := \sup_{z \in \mathsf{Z}} \mathbb{E}_z \hat{R} < \infty$ by assumption 3.3, and $\gamma := (1-\alpha)^n \sup_{z \in \mathsf{Z}} \mathbb{E}_z R_n \cdots R_1 < 1$ by assumption 4.1, we have

$$\mathbb{E}_{a,z}a_n \leq (1-\alpha)^n\mathbb{E}_z R_n \cdots R_1 a + \sum_{t=1}^{n}(1-\alpha)^{n-t}\mathbb{E}_z R_n \cdots R_{t+1}Y_t$$

$$\leq \gamma a + \sum_{t=1}^{n}(1-\alpha)^{n-t}M^{n-t}\mathbb{E}_z Y_t \leq \gamma a + \sum_{t=1}^{n}(1-\alpha)^{n-t}M^{n-t}(q^{t-1}\mathbb{E}_z Y_1 + q'').$$

Define $L := \sum_{t=1}^{n}(1 - \alpha)^{n-t}M^{n-t}$ and $\tilde{L} := q''L$. Then $L, \tilde{L} \in \mathbb{R}_+$ and the above inequality implies that

$$\mathbb{E}_{a,z}a_n \leq \gamma a + L\mathbb{E}_z Y_1 + \tilde{L} \quad \text{for all } (a, z) \in \mathsf{S}.$$

Choose $m \in \mathbb{R}_+$ such that $1 - q^n - L/m > 0$ (such an m is available since $q \in [0, 1)$ by assumption 4.6). Let V be defined as in (18), i.e., $V(a, z) = a + m\mathbb{E}_z Y_1 + 1$.

Then the above results imply that

$$\mathbb{E}_{a,z}V(a_n, z_n) = \mathbb{E}_{a,z}a_n + m\,\mathbb{E}_z\mathbb{E}_{z_n}Y_1 + 1 = \mathbb{E}_{a,z}a_n + m\,\mathbb{E}_zY_{n+1} + 1$$

$$\leq \gamma a + L\mathbb{E}_zY_1 + \tilde{L} + m(q^n\mathbb{E}_zY_1 + q'') + 1$$

$$= \gamma a + (L/m + q^n)m\mathbb{E}_zY_1 + \tilde{L} + mq'' + 1.$$

Let $\tilde{\rho} := \min\{1 - \gamma,\ 1 - q^n - L/m\}$. Then $\tilde{\rho} > 0$ by assumption 4.1 and the construction of m. Thus,

$$\mathbb{E}_{a,z}V(a_n, z_n) - V(a, z)$$

$$\leq -(1-\gamma)a - (1 - q^n - L/m)\,m\,\mathbb{E}_zY_1 + \tilde{L} + mq''$$

$$\leq -\tilde{\rho}\left(a + m\,\mathbb{E}_zY_1\right) + \tilde{L} + mq'' = -\tilde{\rho}V(a, z) + \tilde{\rho} + \tilde{L} + mq''. \tag{44}$$

Choose $\rho \in (0, \tilde{\rho})$ and $d \in \mathbb{R}_+$ such that $(\tilde{\rho} - \rho)d > \tilde{\rho} + \tilde{L} + mq''$. Fix $z_0 \in \mathsf{Z}$ and let $B := [0, d] \times z_0$. Lemma 7.9 implies that B is a petite set. Notice that

$$V(a, z) = a + m\,\mathbb{E}_zY_1 + 1 > d \quad \text{for all } (a, z) \notin B.$$

Hence, (44) implies that for all $(a, z) \notin B$, we have

$$\mathbb{E}_{a,z}V(a_1, z_1) - V(a, z) \leq -\tilde{\rho}V(a, z) + \tilde{\rho} + \tilde{L} + mq''$$

$$= -\rho V(a, z) - (\tilde{\rho} - \rho)V(a, z) + \tilde{\rho} + \tilde{L} + mq''$$

$$< -\rho V(a, z) - (\tilde{\rho} - \rho)d + \tilde{\rho} + \tilde{L} + mq'' < -\rho V(a, z). \tag{45}$$

Let $b := \tilde{\rho} + \tilde{L} + mq''$. Then by (44)–(45), we have

$$\mathbb{E}_{a,z}V(a_n, z_n) - V(a, z) \leq -\rho V(a, z) + b\mathbb{1}\{(a, z) \in B\}$$

for all $(a, z) \in \mathsf{S}$. This concludes the proof. $\qquad\square$

Proof of theorem 4.5. That Q is V-geometrically ergodic can be proved by applying theorem 19.1.3 (or proposition 5.4.5 and theorem 15.0.1) of Meyn and Tweedie (2009). All the required conditions in those theorems have been established in our lemmas 7.7–7.10 above. $\qquad\square$

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