Dynamic optimal choice when rewards are unbounded below[1]

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Abstract. We propose a new approach to solving dynamic decision problems with rewards that are unbounded below. The approach involves transforming the Bellman equation in order to convert an unbounded problem into a bounded one. The major advantage is that, when the conditions stated below are satisfied, the transformed problem can be solved by iterating with a contraction mapping. While the method is not universal, we show by example that many common decision problems do satisfy our conditions.

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1. Introduction

Reward functions that are unbounded below have long been a stumbling block for recursive solution methods, due to a failure of the standard contraction mapping arguments first developed by Blackwell (1965). At the same time, such specifications are popular in economics and finance, due to their convenience and well-established properties. This issue is more than esoteric, since the Bellman equation for such problems can have multiple solutions that confound the search for optima. Computation of solutions, already challenging when the state space is large, becomes even more so when rewards are unbounded.

Here we propose a new approach to handling problems with values that are unbounded below. Instead of creating a new optimality theory, our approach proceeds by transforming the Bellman equation to convert these unbounded problems into bounded

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ones. The main advantage of this approach is that, when the conditions stated below are satisfied, the transformed problem can be solved using standard methods based around contraction mappings. The technical contribution of our paper lies obtaining suitable conditions and providing a proof that the solution to the transformed problem is equal to the solution to the original one. While the method is not universal, we show by example that many well-known decision problems do satisfy our conditions.

Our work contributes to a substantial existing literature on dynamic choice with unbounded rewards. The best known approach to such problems is the weighted supremum norm method, originally developed by Wessels (1977) and connected to economic modeling by Boyd (1990). This approach has been successful in treating many maximization problems where rewards are unbounded above. Unfortunately, as noted by many authors, this same approach typically fails when rewards are unbounded below.[2]

This failure was a major motivation behind the development of the local contraction approach to dynamic programming, due to Rincón-Zapatero and Rodríguez-Palmero (2003), Martins-da Rocha and Vailakis (2010) and, for the stochastic case Matkowski and Nowak (2011). This local contraction method, which requires contractions on successively larger subsets of the state space, is ingenious and elegant but also relatively technical, which might be the cause of slow uptake on the part of applied economists. A second disadvantage in terms of applications is that the convergence results for value function iteration are not as sharp as with traditional dynamic programming.

Another valuable contribution is Jaśkiewicz and Nowak (2011), which explicitly admits problems with rewards that are unbounded below. In this setting, they show that the value function of a Markov decision process is a solution to the Bellman equation. We strengthen their results by adding a uniqueness result and proving that value function iteration leads to an optimal policy. Both of these results are significant from an applied and computational perspective. Like Jaśkiewicz and Nowak (2011), we combine our methodology with the weighted supremum norm approach, so that we can handle problems that are both unbounded above and unbounded below.

Many other researchers have used transformations of the Bellman equation, including Rust (1987), Jovanovic (1982), Bertsekas (2017), Ma and Stachurski (2018) and

[2]See, for example, the discussions in Le Van and Vailakis (2005) or Jaśkiewicz and Nowak (2011). Alvarez and Stokey (1998) find some success handling certain problems that are unbounded below using weighted supremum norm methods, although they require a form of homogeneity that fails to hold in the applications we consider. Bäuerle and Jaśkiewicz (2018) extend the weighted supremum norm technique to risk sensitive preferences in a setting where utility is bounded below.

Abbring et al. (2018). These transformations are typically aimed at improving economic intuition, estimation properties or computational efficiency. The present paper is, to the best of our knowledge, the first to consider transformations of the Bellman equation designed to solving dynamic programming problems with unbounded rewards.

The rest of our paper is structured as follows. Section 2 starts the exposition with typical examples. Section 3 presents theory and Section 4 gives additional applications. Most proofs are deferred to the appendix.

2. Example Applications

We first illustrate the methodology for converting unbounded problems to bounded ones in some common settings.

2.1. **Application 1: Optimal Savings.** Consider an optimal savings problem where a borrowing constrained agent seeks to solve

$$\sup \, \mathbb{E} \sum_{t=0}^{\infty} \beta^t u(c_t)$$

subject to the constraints

$$0 \leqslant c_t \leqslant w_t, \quad w_{t+1} = R(w_t - c_t) + y_{t+1} \quad \text{and} \quad (w_0, y_0) \text{ given.} \tag{1}$$

Here $\beta \in (0,1)$ is the discount factor, c_t, w_t and y_t are respectively consumption, wealth and non-financial income at time t, R is the rate of return on financial income,[3] and u is the CRRA utility function defined by

$$u(c) = \frac{c^{1-\gamma} - 1}{1 - \gamma} \quad \text{with} \quad \gamma > 1. \tag{2}$$

We are focusing on the case $\gamma > 1$ because it is the most empirically relevant and, at the same time, the most challenging for dynamic programming.

Assume that $\{y_t\}$ is a Markov process with state space $\mathsf{Y} \subset \mathbb{R}_+$ and stochastic kernel P satisfying[4]

$$\bar{u} := \inf_{y \in \mathsf{Y}} \int u(y') P(y, \mathrm{d}y') > -\infty. \tag{3}$$

[3]The timing associated with the wealth constraint in (1) is such that y_{t+1} is excluded from the time t information set, as in, say Benhabib et al. (2015). One can modify the second constraint in (1) to an alternative timing such as $w_{t+1} = R(w_t - c_t + y_t)$ and the arguments below still go through after suitable modifications. An application along these lines is given in Section 2.3.

[4]Here $P(y, \cdot)$ can be interpreted as the transition probability. In particular, $P(y, A)$ represents the probability of transitioning from y to set A in one step. See Section 3.1 for formal definition.

Condition (3) holds if, say,

- $\{y_t\}$ is a finite state Markov chain taking positive values (see, e.g., Açıkgöz (2018) and Cao (2018)), or
- $\{y_t\}$ is IID and $\mathbb{E}\, u(y_t) > -\infty$ (see, e.g., Benhabib et al. (2015)), or
- $\{y_t\}$ is a Markov switching process, say, $y_t = \mu_t + \sigma_t \varepsilon_t$, where $\{\varepsilon_t\} \overset{\text{IID}}{\sim} N(0,1)$, while $\{\mu_t\}$ and $\{\sigma_t\}$ are positive and driven by finite state Markov chains (see, e.g., Heathcote et al. (2010) and Kaplan and Violante (2010)).

The Bellman equation of this problem is

$$v(w,y) = \sup_{0 \leqslant c \leqslant w} \left\{ u(c) + \beta \int v(R(w-c) + y', y') P(y, \mathrm{d}y') \right\}, \tag{4}$$

where $w \in \mathbb{R}_+$ and $y \in \mathsf{Y}$. Since $c_t \leqslant w_t$, it is clear that the value function is unbounded below. Put differently, if v is a candidate value function, then even if v is bounded, its image

$$Tv(w,y) = \sup_{0 \leqslant c \leqslant w} \left\{ u(c) + \beta \int v(R(w-c) + y', y') P(y, \mathrm{d}y') \right\} \tag{5}$$

under the Bellman operator is dominated by $u(w)$ plus some finite constant, and hence $v(w,y) \to -\infty$ as $w \to 0$ for any $y \in \mathsf{Y}$.

Consider, however, the following transformation. Let $s := w - c$ and

$$g(y,s) := \beta \int v(Rs + y', y') P(y, \mathrm{d}y') \tag{6}$$

so that

$$v(w,y) = \sup_{0 \leqslant s \leqslant w} \left\{ u(w-s) + g(y,s) \right\}. \tag{7}$$

We can eliminate the function v from (7) by using the definition of g. The first step is to evaluate v in (7) at $(Rs + y', y')$, which gives

$$v(Rs + y', y') = \sup_{0 \leqslant s' \leqslant Rs + y'} \left\{ u(Rs + y' - s') + g(y', s') \right\}.$$

Now we take expectations on both sides of the last equality and multiply by β to get

$$g(y,s) = \beta \int \sup_{0 \leqslant s' \leqslant Rs + y'} \left\{ u(Rs + y' - s') + g(y', s') \right\} P(y, \mathrm{d}y'). \tag{8}$$

This is a functional equation in g. We now introduce a modified Bellman operator S such that any solution g of (8) is a fixed point of S:

$$Sg(y,s) = \beta \int \sup_{0 \leqslant s' \leqslant Rs + y'} \left\{ u(Rs + y' - s') + g(y', s') \right\} P(y, \mathrm{d}y'). \tag{9}$$

Let $\mathcal{G}$ be the set of bounded measurable functions on $\mathsf{Y} \times \mathbb{R}_+$. We claim that S maps $\mathcal{G}$ into itself and, moreover, is a contraction of modulus β with respect to the supremum norm.

To see that this is so, pick any $g \in \mathcal{G}$. Then Sg is bounded above, since $\gamma > 1$ implies

$$Sg(y, s) \leqslant \beta(\sup_{c \geqslant 0} u(c) + \|g\|) \leqslant \beta\|g\|,$$

where $\| \cdot \|$ is the supremum norm. More importantly, Sg is bounded below. Indeed,

$$\begin{aligned} Sg(y, s) &\geqslant \beta \int \sup_{0 \leqslant s' \leqslant Rs+y'} \{u(Rs + y' - s') - \|g\|\} \, P(y, \mathrm{d}y') \\ &= \beta \int \{u(Rs + y') - \|g\|\} \, P(y, \mathrm{d}y') \geqslant \beta \int u(y') P(y, \mathrm{d}y') - \beta\|g\| \geqslant \beta \, \bar{u} - \beta\|g\|. \end{aligned}$$

Finally, S is obviously a contraction mapping, since, for any $g, h \in \mathbb{G}$, we have

$$\begin{aligned} &\left| \sup_{s'} \{u(Rs + y' - s') + g(y', s')\} - \sup_{s'} \{u(Rs + y' - s') + h(y', s')\} \right| \\ &\qquad\qquad\qquad\qquad \leqslant \sup_{s'} |g(y', s') - h(y', s')| \end{aligned}$$

and hence

$$|Sg(y, s) - Sh(y, s)| \leqslant \beta \int \sup_{0 \leqslant s' \leqslant Rs+y'} |g(y', s') - h(y', s')| P(y, \mathrm{d}y') \leqslant \beta\|g - h\|.$$

Taking the supremum over all $(y, s) \in \mathsf{Y} \times \mathbb{R}_+$ yields

$$\|Sg - Sh\| \leqslant \beta\|g - h\|.$$

We have now shown that S is a contractive self-map on $\mathcal{G}$. Most significant here is that $\mathcal{G}$ is a space of bounded functions. By Banach's contraction mapping theorem, S has a unique fixed point g^* in $\mathcal{G}$. Presumably, we can insert g^* into the right hand side of the "Bellman equation" (8), compute the maximizer at each state and obtain the optimal savings policy. If a version of Bellman's principle of optimality applies to this modified Bellman equation, we also know that policies obtained in this way exactly coincide with optimal policies, so, if all of these conjectures are correct, we have a complete characterization of optimality.

A significant amount of theory must be put in place to make the proceeding arguments work. In particular, the conjectures discussed immediately above regarding the validity of Bellman's principle of optimality vis-a-vis the modified Bellman equation are nontrivial, since the transformation in (6) that maps v to g is not bijective. As a result, some careful analysis is required before we can make firm conclusions regarding optimality. This is the task of Section 3.

A final comment on this application is that, for this particular problem, we can also use Euler equation methods, which circumvent some of the issues associated with unbounded rewards (see, e.g., Li and Stachurski (2014)). However, these methods are not applicable in many other settings, due to factors such as existence of discrete choices. The next two applications illustrate this point.

2.2. **Application 2: Job Search.** As in McCall (1970), an unemployed worker can either accept current job offer $w_t = z_t + \xi_t$ and work at that wage forever or choose an outside option (e.g., irregular work in the informal sector) yielding $c_t = z_t + \zeta_t$ and continue to the next period. Here z_t is a persistent component, while ξ_t and ζ_t are transient components. We assume that $\{\xi_t\}$ and $\{\zeta_t\}$ are IID and lognormal, and

$$\ln z_{t+1} = \rho \ln z_t + \sigma \varepsilon_{t+1}, \quad \{\varepsilon_t\} \overset{\text{IID}}{\sim} N(0,1). \tag{10}$$

The worker's value function satisfies the Bellman equation

$$v(w,c,z) = \max\left\{\frac{u(w)}{1-\beta},\ u(c) + \beta \mathbb{E}_z\, v(w',c',z')\right\}. \tag{11}$$

Let u be increasing, continuous, and unbounded below with $u(w) = -\infty$ as $w \to 0$. For now, let u be bounded above. Moreover, we assume that

$$\text{either} \quad \inf_{z>0} \mathbb{E}_z u(w') > -\infty \quad \text{or} \quad \inf_{z>0} \mathbb{E}_z u(c') > -\infty. \tag{12}$$

Condition (12) is satisfied if u is CRRA, say, since then $\mathbb{E}\, u(\xi_t)$ and $\mathbb{E}\, u(\zeta_t)$ are finite. Note that $v(w,c,z)$ is unbounded below since utility can be arbitrarily close to $-\infty$.

To shift to a bounded problem, we can proceed in a similar vein to our manipulation of the Bellman equation in the optimal savings case. First we set

$$g(z) := \beta \mathbb{E}_z\, v(w',c',z'),$$

so that (11) can be written as

$$v(w,c,z) = \max\left\{\frac{u(w)}{1-\beta},\ u(c) + g(z)\right\}.$$

Next we use the definition of g to eliminate v from this last expression, which leads to the functional equation

$$g(z) = \beta \mathbb{E}_z \max\left\{\frac{u(w')}{1-\beta},\ u(c') + g(z')\right\}. \tag{13}$$

The corresponding fixed point operator is

$$Sg(z) = \beta \mathbb{E}_z \max\left\{\frac{u(w')}{1-\beta},\ u(c') + g(z')\right\}. \tag{14}$$

If g is bounded above then clearly so is Sg. Moreover, if g is bounded below by some constant M, then, by Jensen's inequality,

$$Sg(z) \geqslant \beta \max \left\{ \mathbb{E}_z \frac{u(w')}{1-\beta},\ \mathbb{E}_z u(c') + M \right\}.$$

Condition (12) then implies that Sg is also bounded below.

An argument similar to the one adopted above for the optimal savings model proves that S is a contraction mapping with respect to the supremum norm on a space of bounded functions (Section 3 gives details). Thus, we can proceed down essentially the same path we used for the optimal savings problem, with the same caveat that the modified Bellman operator S and the original Bellman operator need to have the same connection to optimality, and all computational issues need to be clarified.

2.3. **Application 3: Optimal Default.** Consider an infinite horizon optimal savings problem with default, in the spirit of Arellano (2008) and a large related literature.[5] A country with current assets w_t chooses between continuing to participate in international financial markets and default. Output

$$y_t = y(z_t, \xi_t)$$

is a function of a persistent component $\{z_t\}$ and an innovation $\{\xi_t\}$. The persistent component is a Markov process such as the one in (10) and the transient component $\{\xi_t\}$ is IID. To simplify the exposition, we assume that default leads to permanent exclusion from financial markets, with lifetime value

$$v^d(y, z) = \mathbb{E} \sum_{t=0}^{\infty} \beta^t u(y_t).$$

Notice that v^d satisfies the functional equation

$$v^d(y, z) = u(y) + \beta \mathbb{E}_z v^d(y', z').$$

The value of continued participation in financial markets is

$$v^c(w, y, z) = \sup_{-b \leqslant w' \leqslant R(w+y)} \left\{ u(w + y - w'/R) + \beta \mathbb{E}_z v(w', y', z') \right\},$$

where $b > 0$ is a constant borrowing constraint and v is the value function satisfying

$$v(w, y, z) = \max \left\{ v^d(y, z),\ v^c(w, y, z) \right\}.$$

The utility function u has the same properties as Section 2.2. It is easy to see that v is unbounded below since u can be arbitrarily close to $-\infty$. However, we can convert this into a bounded problem, as the following analysis shows.

[5]Recent examples include Aguiar and Amador (2019) and Aguiar et al. (2019).

Let i be a discrete choice variable taking values in $\{0,1\}$, with 0 indicating default and 1 indicating continued participation. We define

$$g(z, w', i) := \begin{cases} \beta \mathbb{E}_z v^d(y', z') & \text{if } \ i = 0 \\ \beta \mathbb{E}_z v(w', y', z') & \text{if } \ i = 1 \end{cases}$$

so that for $-b \leqslant w' \leqslant R(w+y)$, we have

$$v(w, y, z) = \max \left\{ u(y) + g(z, w', 0), \sup_{w'} \{u(w + y - w'/R) + g(z, w', 1)\} \right\}.$$

Eliminating the value function v yields

$$g(z, w', 0) = \beta \mathbb{E}_z \{u(y') + g(z', w', 0)\} \quad \text{and}$$

$$g(z, w', 1) = \beta \mathbb{E}_z \max \left\{ u(y') + g(z', w', 0), \sup_{w''} \{u(w' + y' - w''/R) + g(z', w'', 1)\} \right\},$$

where $-b \leqslant w'' \leqslant R(w' + y')$. We can then define the fixed point operator S corresponding to these functional equations.

If g is bounded above by some constant K, then $Sg \leqslant \sup_c u(c) + K$. More importantly, if g is bounded below by some constant M, we obtain

$$Sg(z, w', 0) \geqslant \beta \mathbb{E}_z u(y') + \beta M \quad \text{and}$$

$$\begin{aligned} Sg(z, w', 1) &\geqslant \beta \mathbb{E}_z \max \{u(y') + M, \, u(w' + y' + b/R) + M\} \\ &= \beta \mathbb{E}_z \max \{u(y'), \, u(w' + y' + b/R)\} + \beta M. \end{aligned}$$

Hence, Sg is bounded below by a finite constant if

$$\inf_z \mathbb{E}_z u(y') > -\infty. \tag{15}$$

For example, (15) holds if $y_t = z_t + \xi_t$ where $\{z_t\}$ is positive and $\mathbb{E}\, u(\xi_t) > -\infty$. An argument similar to the one in Section 2.1 now proves that S is a contraction with respect to the supremum norm (Section 3 gives details).

3. General Formulation

The preceding section showed how some unbounded problems can be converted to bounded problems by modifying the Bellman equation. The next step is to confirm the validity of such a modification in terms of the connection between the modified Bellman equation and optimal policies. We do this in a generic dynamic programming setting that contains the applications given above.

3.1. **Theory.** For a given set E, let $\mathscr{B}(E)$ be the Borel subsets of E. For our purpose, a dynamic program consists of

- a nonempty set X called the *state space*,
- a nonempty set A called the *action space*,
- a nonempty correspondence Γ from X to A called the *feasible correspondence*, along with the associated set of *state action pairs*
$$\mathsf{D} := \{(x, a) \in \mathsf{X} \times \mathsf{A} : a \in \Gamma(x)\},$$
- a measurable map $r : \mathsf{D} \to \mathbb{R} \cup \{-\infty\}$ called the *reward function*,
- a constant $\beta \in (0, 1)$ called the *discount factor*, and
- a *stochastic kernel* Q governing the evolution of states.[6]

Each period, an agent observes a state $x_t \in \mathsf{X}$ and responds with an action $a_t \in \Gamma(x_t) \subset \mathsf{A}$. The agent then obtains a reward $r(x_t, a_t)$, moves to the next period with a new state x_{t+1}, and repeats the process by choosing a_{t+1} and so on. The state process updates according to $x_{t+1} \sim Q(x_t, a_t, \cdot\,)$.

Let Σ denote the set of *feasible policies*, which we assume to be nonempty and define as all measurable maps $\sigma : \mathsf{X} \to \mathsf{A}$ satisfying $\sigma(x) \in \Gamma(x)$ for all $x \in \mathsf{X}$. Given any policy $\sigma \in \Sigma$ and initial state $x_0 = x \in \mathsf{X}$, the σ*-value function* v_σ is defined by

$$v_\sigma(x) = \sum_{t=0}^{\infty} \beta^t \mathbb{E}_x r(x_t, \sigma(x_t)).$$

We understand $v_\sigma(x)$ as the lifetime value of following policy σ now and forever, starting from current state x.

The *value function* associated with this dynamic program is defined at each $x \in \mathsf{X}$ by

$$v^*(x) = \sup_{\sigma \in \Sigma} v_\sigma(x). \tag{16}$$

A feasible policy σ^* is called *optimal* if $v_{\sigma^*} = v^*$ on X. The objective of the agent is to find an optimal policy that attains the maximum lifetime value.

To handle rewards that are unbounded above as well as below, we introduce a weighting function κ, which is a measurable function mapping X to $[1, \infty)$. Let $\mathcal{G}$ be the set of measurable functions $g : \mathsf{D} \to \mathbb{R}$ such that g is bounded below and

$$\|g\|_\kappa := \sup_{(x,a) \in \mathsf{F}} \frac{|g(x, a)|}{\kappa(x)} < \infty. \tag{17}$$

[6]Here a *stochastic kernel* corresponding to our controlled Markov process $\{(x_t, a_t)\}$ is a mapping $Q : \mathsf{D} \times \mathscr{B}(\mathsf{X}) \to [0, 1]$ such that (i) for each $(x, a) \in \mathsf{D}$, $A \mapsto Q(x, a, A)$ is a probability measure on $\mathscr{B}(\mathsf{X})$, and (ii) for each $A \in \mathscr{B}(\mathsf{X})$, $(x, a) \mapsto Q(x, a, A)$ is a measurable function.

The pair $(\mathcal{G}, \|\cdot\|_\kappa)$ is a Banach space (see, e.g., Bertsekas (2013)). Moreover, at each $x \in \mathsf{X}$ and $(x, a) \in \mathsf{D}$, we define

$$\bar{r}(x) := \sup_{a \in \Gamma(x)} r(x, a) \quad \text{and} \quad \ell(x, a) := \mathbb{E}_{x,a} \bar{r}(x'). \tag{18}$$

Assumption 3.1. There exist constants $d \in \mathbb{R}_+$ and $\alpha \in (0, 1/\beta)$ such that $\bar{r}(x) \leqslant d\kappa(x)$ and $\mathbb{E}_{x,a}\, \kappa(x') \leqslant \alpha\kappa(x)$ for all $(x, a) \in \mathsf{D}$.

Assumption 3.1 relaxes the standard weighted supremum norm assumptions (see, e.g., Wessels (1977) or Bertsekas (2013)), in the sense that the reward function is allowed to be unbounded from below.

Next, we define S on $\mathcal{G}$ as

$$Sg(x, a) := \beta \mathbb{E}_{x,a} \sup_{a' \in \Gamma(x')} \left\{ r\left(x', a'\right) + g(x', a') \right\}. \tag{19}$$

Given $g \in \mathcal{G}$, a feasible policy σ is called *g-greedy* if

$$r(x, \sigma(x)) + g(x, \sigma(x)) = \sup_{a \in \Gamma(x)} \{r(x, a) + g(x, a)\} \quad \text{for all } x \in \mathsf{X}. \tag{20}$$

Although the reward function is potentially unbounded below, the dynamic program can be solved by the operator S, as the following theorem shows.

Theorem 3.1. *If Assumption 3.1 holds and ℓ is bounded below, then*

(1) *$S\mathcal{G} \subset \mathcal{G}$ and S is a contraction mapping on $(\mathcal{G}, \|\cdot\|_\kappa)$.*
(2) *S admits a unique fixed point g^* in $\mathcal{G}$.*
(3) *$S^k g$ converges to g^* at rate $O((\alpha\beta)^k)$ under $\|\cdot\|_\kappa$.*
(4) *If there exists a closed subset $\mathbb{G}$ of $\mathcal{G}$ such that $S\mathbb{G} \subset \mathbb{G}$ and a g-greedy policy exists for each $g \in \mathbb{G}$, then, in addition,*
   (a) *g^* is an element of $\mathbb{G}$ and satisfies*

$$g^*(x, a) = \beta \mathbb{E}_{x,a} v^*(x') \quad \text{and} \quad v^*(x) = \max_{a \in \Gamma(x)} \left\{ r(x, a) + g^*(x, a) \right\}.$$

   (b) *At least one optimal policy exists.*
   (c) *A feasible policy is optimal if and only if it is g^*-greedy.*

3.2. **Sufficient Conditions.** Consider a dynamic programming problem

$$\max \mathbb{E} \sum_{t=0}^{\infty} \beta^t r(w_t, s_t) \tag{21}$$

subject to

$$0 \leqslant s_t \leqslant w_t, \quad w_{t+1} = f(s_t, \eta_{t+1}), \quad \eta_t = h(z_t, \varepsilon_t) \quad \text{and} \quad (w_0, z_0) \text{ given.} \tag{22}$$

Here z and ε correspond respectively to a Markov process $\{z_t\}$ on Z and an IID process $\{\varepsilon_t\}$, f and h are nonnegative continuous functions, and f is increasing in s. Furthermore, Z and the range space of $\{\eta_t\}$ are Borel subsets of finite-dimensional Euclidean spaces, and the stochastic kernel P corresponding to $\{z_t\}$ is Feller.[7]

This problem can be placed in our framework by setting

$$x := (w, z), \quad a := s, \quad \mathsf{X} := \mathbb{R}_+ \times \mathsf{Z}, \quad \mathsf{A} := \mathbb{R}_+, \quad \Gamma(x) := [0, w]$$
$$\text{and} \quad \mathsf{D} := \{(w, z, s) \in \mathbb{R}_+ \times \mathsf{Z} \times \mathbb{R}_+ : 0 \leqslant s \leqslant w\}.$$

Suppose that the reward function $r : \mathsf{D} \to \mathbb{R} \cup \{-\infty\}$ is increasing in w and decreasing in s, r is continuous on the interior of D and, if r is bounded below, it is continuous.

Recall κ defined in Assumption 3.1. Let

$$\underline{\ell}(z) := \mathbb{E}_z r(f(0, \eta'), 0) \quad \text{and} \quad \kappa_e(z, s) := \mathbb{E}_{z,s} \kappa(w', z').$$

Let $\mathbb{G}$ be the set of functions g in $\mathcal{G}$ that is increasing in its last argument and continuous. Notice that, in the current setting, S defined on $\mathbb{G}$ is given by

$$Sg(z, s) = \beta \mathbb{E}_{z,s} \max_{s' \in [0, w']} \{r(w', s') + g(z', s')\}.$$

Theorem 3.1 is applicable in the current setting, as the following result illustrates.

Proposition 3.2. *If Assumption 3.1 holds for some continuous functions κ and κ_e, and $\underline{\ell}$ is continuous and bounded below, then S is a contraction mapping on $(\mathbb{G}, \|\cdot\|_\kappa)$ and the conclusions of Theorem 3.1 hold.*

4. Applications

In this section, we complete the discussion of all applications in Section 2. We also extend the optimality results of Benhabib et al. (2015) by adding a persistent component to labor income and returns.

4.1. **Optimal Savings (Continued).**

Recall the optimal savings problem of Section 2.1. This problem can be placed into the framework of Section 3.2 by letting

$$\eta = z := y, \quad r(w, s) := u(w - s), \quad f(s, \eta') := Rs + \eta' \quad \text{and} \quad h(z, \varepsilon) := z.$$

To establish the desired properties, it remains to verify the conditions of Proposition 3.2. Since we have shown that $S\mathcal{G} \subset \mathcal{G}$, where $\mathcal{G}$ is the set of bounded measurable

[7] In other words, $z \mapsto \int h(z') P(z, \mathrm{d}z')$ is bounded and continuous whenever h is.

functions on $\mathsf{Y} \times \mathbb{R}_+$, we can simply set $\kappa \equiv 1$ such that Assumption 3.1 holds. In this case, both κ and κ_e are continuous functions. Moreover, note that

$$\underline{\ell}(y) = \mathbb{E}_y u(y') = \int u(y') P(y, \mathrm{d}y'),$$

which is bounded below by (3). As a result, all the conclusions of Theorem 3.1 hold as long as $y \mapsto \int u(y') P(y, \mathrm{d}y')$ is continuous. In particular, when this further condition holds, S is a contraction mapping on $(\mathbb{G}, \|\cdot\|)$ with unique fixed point g^*, and a feasible policy is optimal if and only if it is g^*-greedy. Here $\mathbb{G}$ is the set of bounded continuous functions on $\mathsf{Y} \times \mathbb{R}_+$ that is increasing in its last argument.

4.2. **Job Search (Continued).** Recall the job search problem of Section 2.2. This problem fits into the framework of Section 3.1 if we let the a be a discrete choice variable taking values in $\{0, 1\}$, where 0 denotes the decision to stop and 1 represents the decision to continue,

$$x := (w, z, c), \quad \mathsf{X} := (0, \infty)^3, \quad \mathsf{A} := \{0, 1\}, \quad \Gamma(x) := \{0, 1\}, \quad \mathsf{D} := (0, \infty)^3 \times \{0, 1\}$$

and the reward function $r(x, a)$ be

$$r(w, c, a) := \frac{u(w)}{1 - \beta} \;\text{ if }\; a = 0 \quad \text{and} \quad r(w, c, a) := u(c) \;\text{ if }\; a = 1.$$

We have shown that $S\mathcal{G} \subset \mathcal{G}$, where $\mathcal{G}$ is the set of bounded measurable functions on $(0, \infty)$. Hence, Assumption 3.1 holds with $\kappa \equiv 1$. Note that in this case, the function $\ell(x, a)$ reduces to

$$\ell(z) = \mathbb{E}_z \max \left\{ u(w')/(1 - \beta), u(c') \right\}.$$

Then ℓ is bounded below by Jensen's inequality and (12). Since in addition the action set is finite, a g-greedy policy always exists for all $g \in \mathcal{G}$. Let $\mathbb{G} := \mathcal{G}$. The analysis above implies that all the conclusions of Theorem 3.1 hold.

4.3. **Optimal Default (Continued).** Recall the optimal default problem studied in Section 2.3. This setting is a special case of our framework. In particular,

$$x := (w, y, z), \quad a := (w', i), \quad \mathsf{X} := [-b, \infty) \times \mathsf{Y} \times \mathsf{Z} \quad \text{and} \quad \mathsf{A} := [-b, \infty) \times \{0, 1\},$$

where i is a discrete choice variable taking values in $\{0, 1\}$, and Y and Z are respectively the range spaces of $\{y_t\}$ and $\{z_t\}$. The reward function r reduces to

$$r(w, y, w', i) := \begin{cases} u(y) & \text{if } i = 0, \\ u(w + y - w'/R) & \text{if } i = 1. \end{cases}$$

Since $S\mathcal{G} \subset \mathcal{G}$, where $\mathcal{G}$ is the set of bounded measurable functions on $\mathsf{Z} \times [-b, \infty) \times \{0, 1\}$, Assumption 3.1 holds for $\kappa \equiv 1$. Moreover, ℓ satisfies

$$\ell(z, w') = \mathbb{E}_z \max \{u(y'), u(w' + y' + b/R)\} \geqslant \mathbb{E}_z u(y'),$$

which is bounded below by (15). Let $\mathbb{G}$ be the set of functions in $\mathcal{G}$ that is increasing in its second-to-last argument and continuous. Through similar steps to the proof of Proposition 3.2, one can show that $S\mathbb{G} \subset \mathbb{G}$ and a g-greedy policy exists for all $g \in \mathbb{G}$. As a result, all the conclusions of Theorem 3.1 are true.

4.4. **Optimal Savings with Capital Income Risk.** Consider an optimal savings problem with capital income risk (see, e.g., Benhabib et al. (2015)). The setting is similar to that of Section 2.1, except that the rate of return to wealth is stochastic. In particular, the constraint (1) now becomes

$$0 \leqslant s_t \leqslant w_t, \quad w_{t+1} = R_{t+1} s_t + y_{t+1} \quad \text{and} \quad (w_0, z_0) \text{ given.}$$

where w_t is wealth, s_t is the amount of saving, while R_t and $\{y_t\}$ are respectively the rate of return to wealth and the non-financial income that satisfy

$$R_t = h_R(z_t, \xi_t) \quad \text{and} \quad y_t = h_y(z_t, \zeta_t).$$

Here $\{z_t\}$ is a finite state Markov chain, and $\{\xi_t\}$ and $\{\eta_t\}$ are IID innovation processes. The importance of these features for wealth dynamics is highlighted in Fagereng et al. (2016) and Hubmer et al. (2018), among others.

This problem fits into the framework of Section 3.2 by setting

$$\eta := (R, y), \quad \varepsilon_t := (\xi_t, \zeta_t), \quad r(w, a) := u(w - s) \quad \text{and} \quad f(s, \eta') := Rs + y'.$$

In this case, $\underline{\ell}(z) = \mathbb{E}_z u(y')$ and

$$Sg(z, s) = \beta \mathbb{E}_{z,s} \max_{s' \in [0, w']} \{u(w' - s') + g(z', s')\}.$$

Consider, for example, the CRRA utility in (2). In this case, Assumption 3.1 holds with $\kappa \equiv 1$, and $\mathbb{G}$ reduces to the set of bounded continuous functions on $\mathsf{Z} \times \mathbb{R}_+$ that is increasing in its last argument. The conclusions of Theorem 3.1 hold if $z \mapsto \mathbb{E}_z u(y')$ is continuous and bounded below.

5. Appendix

Let $\mathcal{V}$ (resp., $\mathbb{V}$) be the set of measurable functions $v : \mathsf{X} \to \mathbb{R} \cup \{-\infty\}$ such that $(x, a) \mapsto \beta \mathbb{E}_{x,a} v(x')$ is in $\mathcal{G}$ (resp., $\mathbb{G}$), and let $\mathcal{H}$ (resp., $\mathbb{H}$) be the set of measurable functions $h : \mathsf{D} \to \mathbb{R} \cup \{-\infty\}$ such that $h = r + g$ for some g in $\mathcal{G}$ (resp., $\mathbb{G}$). Next, we define the operators W_0, W_1 and M respectively on $\mathcal{V}$, $\mathcal{G}$ and $\mathcal{H}$ as

$$W_0 v(x,a) := \beta \mathbb{E}_{x,a} v(x'), \quad W_1 g(x,a) := r(x,a) + g(x,a),$$
$$\text{and} \quad Mh(x) := \sup_{a \in \Gamma(x)} h(x,a).$$

Then S in (19) satisfies $S = W_0 M W_1$ on $\mathcal{G}$.

Proof of Theorem 3.1. To see claim (1) holds, we first show that $S\mathcal{G} \subset \mathcal{G}$. Fix $g \in \mathcal{G}$. By the definition of $\mathcal{G}$, there is a lower bound $\underline{g} \in \mathbb{R}$ such that $g \geqslant \underline{g}$. Then

$$\begin{aligned} Sg(x,a) &\geqslant \beta \mathbb{E}_{x,a} \sup_{a' \in \Gamma(x')} \left\{ r(x',a') + \underline{g} \right\} \\ &= \beta \left[\mathbb{E}_{x,a} \sup_{a' \in \Gamma(x')} r(x',a') + \underline{g} \right] = \beta \left[\mathbb{E}_{x,a} \bar{r}(x') + \underline{g} \right] = \beta \left[\ell(x,a) + \underline{g} \right]. \end{aligned}$$

Since by assumption ℓ is bounded below, so is Sg. Moreover, by Assumption 3.1,

$$\begin{aligned} Sg(x,a) &\leqslant \beta \mathbb{E}_{x,a} \left\{ \bar{r}(x') + \sup_{a' \in \Gamma(x')} g(x',a') \right\} \\ &\leqslant \beta \mathbb{E}_{x,a} \left\{ (d + \|g\|_\kappa) \kappa(x') \right\} \leqslant \alpha \beta (d + \|g\|_\kappa) \kappa(x) \end{aligned}$$

for all $(x,a) \in \mathsf{D}$. Hence, Sg/κ is bounded above. Since in addition Sg is bounded below and $\kappa \geqslant 1$, we have $\|Sg\|_\kappa < \infty$. We have now shown that $Sg \in \mathcal{G}$.

Next, we show that S is a contraction mapping on $(\mathcal{G}, \|\cdot\|_\kappa)$. Fix $g_1, g_2 \in \mathcal{G}$. Note that for all $(x,a) \in \mathsf{D}$, we have

$$\begin{aligned} &|Sg_1(x,a) - Sg_2(x,a)| \\ &= \left| \beta \mathbb{E}_{x,a} \sup_{a' \in \Gamma(x')} \{ r(x',a') + g_1(x',a') \} - \beta \mathbb{E}_{x,a} \sup_{a' \in \Gamma(x')} \{ r(x',a') + g_2(x',a') \} \right| \\ &\leqslant \beta \mathbb{E}_{x,a} \left| \sup_{a' \in \Gamma(x')} \{ r(x',a') + g_1(x',a') \} - \sup_{a' \in \Gamma(x')} \{ r(x',a') + g_2(x',a') \} \right| \\ &\leqslant \beta \mathbb{E}_{x,a} \sup_{a' \in \Gamma(x')} |g_1(x',a') - g_2(x',a')| \leqslant \beta \|g_1 - g_2\|_\kappa \mathbb{E}_{x,a} \kappa(x') \leqslant \alpha\beta \|g_1 - g_2\|_\kappa \kappa(x), \end{aligned}$$

where the last inequality follows from Assumption 3.1. Then we have $\|Sg_1 - Sg_2\|_\kappa \leqslant \alpha\beta \|g_1 - g_2\|_\kappa$. Since $\alpha\beta < 1$, S is a contraction mapping on $(\mathcal{G}, \|\cdot\|_\kappa)$ and claim (1) is verified.

Claims (2)–(3) follow immediately from claim (1) and the Banach contraction mapping theorem. Regarding claim (4), since $\mathbb{G}$ is a closed subset of $\mathcal{G}$ and $S\mathbb{G} \subset \mathbb{G}$, S is also a contraction mapping on $(\mathbb{G}, \|\cdot\|_\kappa)$ and the unique fixed point g^* of S is indeed in $\mathbb{G}$. Based on Proposition 2 of Ma and Stachurski (2018), the Bellman operator $T := MW_1W_0$ maps elements of $\mathbb{V}$ into itself and has a unique fixed point $\bar{v}$ in $\mathbb{V}$ that satisfies $\bar{v} = MW_1 g^*$ and $g^* = W_0\bar{v}$.

To verify part (a) of claim (4), it remains to show that $\bar{v} = v^*$. For all $x_0 \in \mathsf{X}$ and $\sigma \in \Sigma$, we have

$$
\begin{aligned}
\bar{v}(x_0) &\geqslant r(x_0, \sigma(x_0)) + \beta \mathbb{E}_{x_0,\sigma(x_0)} \bar{v}(x_1) \\
&\geqslant r(x_0, \sigma(x_0)) + \beta \mathbb{E}_{x_0,\sigma(x_0)} \left\{ r(x_1, \sigma(x_1)) + \beta \mathbb{E}_{x_1,\sigma(x_1)} \bar{v}(x_2) \right\} \\
&= r(x_0, \sigma(x_0)) + \beta \mathbb{E}_{x_0,\sigma(x_0)} r(x_1, \sigma(x_1)) + \beta^2 \mathbb{E}_{x_0,\sigma(x_0)} \mathbb{E}_{x_1,\sigma(x_1)} \bar{v}(x_2) \\
&\geqslant \sum_{t=0}^{T} \beta^t \mathbb{E}_{x_0,\sigma(x_0)} \cdots \mathbb{E}_{x_{t-1},\sigma(x_{t-1})} r(x_t, \sigma(x_t)) + \beta^{T+1} \mathbb{E}_{x_0,\sigma(x_0)} \cdots \mathbb{E}_{x_T,\sigma(x_T)} \bar{v}(x_{T+1}) \\
&= \sum_{t=0}^{T} \beta^t \mathbb{E}_{x_0} r(x_t, \sigma(x_t)) + \beta^T \mathbb{E}_{x_0,\sigma(x_0)} \cdots \mathbb{E}_{x_{T-1},\sigma(x_{T-1})} g^*(x_T, \sigma(x_T)). \qquad (23)
\end{aligned}
$$

Notice that, by Assumption 3.1, we have

$$
\begin{aligned}
&\left| \beta^T \mathbb{E}_{x_0,\sigma(x_0)} \cdots \mathbb{E}_{x_{T-1},\sigma(x_{T-1})} g^*(x_T, \sigma(x_T)) \right| \\
&\leqslant \beta^T \mathbb{E}_{x_0,\sigma(x_0)} \cdots \mathbb{E}_{x_{T-1},\sigma(x_{T-1})} \left| g^*(x_T, \sigma(x_T)) \right| \\
&\leqslant \beta^T \mathbb{E}_{x_0,\sigma(x_0)} \cdots \mathbb{E}_{x_{T-1},\sigma(x_{T-1})} \|g^*\|_\kappa \kappa(x_T) \\
&\leqslant \beta^T \alpha^T \|g^*\|_\kappa \kappa(x_0) = (\alpha\beta)^T \|g^*\|_\kappa \kappa(x_0) \to 0 \quad \text{as } \ T \to \infty.
\end{aligned}
$$

Letting $T \to \infty$, (23) then implies that $\bar{v}(x_0) \geqslant v_\sigma(x_0)$. Since $x_0 \in \mathsf{X}$ and $\sigma \in \Sigma$ are arbitrary, we have $\bar{v} \geqslant v^*$. Moreover, since $g^* = W_0\bar{v}$ and there exists a g^*-greedy policy σ^* by assumption, all the inequalities in (23) holds with equality once we let $\sigma = \sigma^*$. In other words, we have $\bar{v} = v_{\sigma^*} \leqslant v^*$. In summary, we have shown that $\bar{v} = v^*$. Hence, $g^* = W_0 v^*$ and $v^* = MW_1 g^*$, and part (a) of claim (4) holds.

Since we have shown that v^* is the unique fixed point of T in $\mathbb{V}$, by Theorem 1 of Ma and Stachurski (2018), the set of optimal policies is nonempty, and a feasible policy is optimal if and only if it is v^*-greedy. Since in addition $g^* = W_0 v^*$, parts (b) and (c) of claim (4) hold. $\square$

Next, we aim to prove Proposition 3.2. For all $g \in \mathbb{G}$ and $(w, z) \in \mathsf{X}$, we define

$$
h_g(w, z) := \max_{0 \leqslant s \leqslant w} \left\{ r(w, s) + g(z, s) \right\} \quad \text{and}
$$

$$
M_g(w, z) := \left\{ s \in [0, w] : h_g(w, z) = r(w, s) + g(z, s) \right\}.
$$

The following result is helpful in applications for verifying $S\mathbb{G} \subset \mathbb{G}$.

Lemma 5.1. *For all $g \in \mathbb{G}$, h_g and M_g satisfy the following properties:*

(1) *h_g is well defined and increasing in w,*
(2) *h_g is continuous on $(0, \infty) \times \mathsf{Z}$,*
(3) *h_g is continuous on X if r is bounded below, and*
(4) *M_g is nonempty, compact-valued, and upper hemicontinuous.*

Proof. Fix $g \in \mathbb{G}$. Since g is bounded below, $h_g(0, z) = r(0, 0) + g(0, z) \in \mathbb{R} \cup \{-\infty\}$ and h_g is well defined at $w = 0$. Now consider $w > 0$. Let D_0 be the interior of D. By assumption, either

(i) r is continuous on D_0 and $\lim_{s \to w} r(w, s) = -\infty$ for some $w \in \mathbb{R}_+$, or
(ii) r is continuous and bounded below.

Each scenario, since g is continuous, the maximum in the definition of h_g can be attained at some $s \in [0, w]$. Hence, h_g is well defined for all $w > 0$. Regarding monotonicity, let $w_1, w_2 \in \mathbb{R}_+$ with $w_1 < w_2$. By the monotonicity of r, we have

$$h_g(w_1, z) \leqslant \max_{s \in [0, w_1]} \{r(w_2, s) + g(s, z)\} \leqslant \max_{s \in [0, w_2]} \{r(w_2, s) + g(s, z)\} = h_g(w_2, z).$$

Hence, claim (a) holds. Claims (b)–(d) follow from Berge's theorem of maximum (adjusted to accommodate possibly negative infinity valued objective functions). □

Proof of Proposition 3.2. ℓ is bounded below since, by the monotonicity of f and r,

$$\ell(x, a) = \mathbb{E}_{z,s} r(w', 0) \geqslant \mathbb{E}_{z} r[f(0, \eta'), 0] = \underline{\ell}(z),$$

which is bounded below by assumption. Moreover, it is obvious that $\mathbb{G}$ is a closed subset of $\mathcal{G}$. Existence of g-greedy policies for g in $\mathbb{G}$ has been verified by Lemma 5.1. It remains to show that $S\mathbb{G} \subset \mathbb{G}$. For fixed $g \in \mathbb{G}$, Theorem 3.1 implies that $Sg \in \mathcal{G}$. To see that Sg is increasing in its last argument and continuous, note that by Lemma 5.1, h_g is continuous on D_0 and increasing in w'. For all $s_1, s_2 \in \mathsf{A}$ with $s_1 \leqslant s_2$, the monotonicity of f implies that

$$\begin{aligned} Sg(z, s_1) &= \beta \mathbb{E}_{z,s_1} h_g(w', z') = \beta \mathbb{E}_{z} h_g(f(s_1, \eta'), z') \\ &\leqslant \beta \mathbb{E}_{z} h_g(f(s_2, \eta'), z') = \beta \mathbb{E}_{z,s_2} h_g(w', z') = Sg(z, s_2). \end{aligned}$$

Hence, Sg is increasing in its last argument. In addition, the definition of $\mathbb{G}$ and the monotonicity of r and f implies that

$$r(f(0, \eta'), 0) - \alpha_1 \leqslant h_g(w', z') \leqslant \alpha_2 \, \kappa(w', z') \quad \text{for some } \alpha_1, \alpha_2 \in \mathbb{R}_+.$$

Since κ_e and $\underline{\ell}$ are continuous and the stochastic kernel P is Feller, Fatou's lemma implies that $Sg(z, s) = \beta \mathbb{E}_{z,s} h_g(w', z')$ is continuous. □

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