Asymptotic Marginal Propensity to Consume*

Qingyin Ma† Alexis Akira Toda‡

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Abstract

We prove that the consumption functions in income fluctuation problems are asymptotically linear if the marginal utility is regularly varying. We also analytically characterize the asymptotic marginal propensities to consume (MPCs) out of wealth and derive necessary and sufficient conditions under which they are 0, 1, or are somewhere in between. When the return process with time-varying volatility is calibrated from data, the asymptotic MPCs can be zero with moderate risk aversion. Our results potentially explain why the saving rates among the rich are positive and increasing in wealth.

Keywords: income fluctuation problem, regular variation, saving rate.

JEL codes: D15, E21.

1 Introduction

How does the consumption of the rich look like? Canonical models in macroeconomics suggest that the marginal propensity to consume out of wealth (MPC) is either independent of wealth (in simple analytically solved models with constant relative risk aversion preferences) or increasing in wealth (in many numerical works). However, recent empirical evidence suggests that the MPC is decreasing in wealth. Understanding the asymptotic behavior of consumption is important because the large observed wealth inequality implies that a significant fraction of aggregate consumption comes from the rich.

This paper theoretically studies the asymptotic properties of the consumption functions in general income fluctuation problems. Under the weak assumption that the marginal utility asymptotically behaves like a power function as consumption increases,[1] we show that the consumption functions are asymptotically linear, or the asymptotic MPCs converge to some constants. Furthermore, we also analytically characterize the asymptotic MPCs and derive necessary and sufficient conditions under which they are equal to 0, 1, or somewhere in

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†International School of Economics and Management, Capital University of Economics and Business. Email: qingyin.ma@cueb.edu.cn.

‡Department of Economics, University of California San Diego. Email: atoda@ucsd.edu.

[1]This specification includes commonly used utility functions such as constant relative risk aversion (CRRA) or hyperbolic absolute risk aversion (HARA).

between. We prove that asymptotic MPCs depend only on risk aversion and the stochastic processes for the discount factor and return on wealth, and are independent of the income process.

Our first result that the MPCs converge as agents become wealthier is not surprising. In simple analytically solvable models that feature homothetic preferences and no income risk as in Samuelson (1969), it is well-known that the MPC is independent of wealth. With HARA preferences and general income shocks, Carroll and Kimball (1996) show that the consumption functions are concave, which implies that the MPCs converge. We show that this asymptotic linearity of consumption functions is a much more general property: instead of special utility functions such as CRRA or HARA, we only require that the marginal utility asymptotically behaves like a power function, which is mathematically defined as *regular variation*. Our results are significantly more general than the existing literature because regular variation is a parametric assumption only at infinity and we do not impose any assumptions on the utility function on compact sets beyond the usual monotonicity and concavity. The asymptotic linearity of consumption functions justifies the textbook affine consumption function as in Keynes (1936) and the linear extrapolation of policy functions when solving models numerically as in Gouin-Bonenfant and Toda (2018). Furthermore, it explains the "approximate aggregation" property of heterogeneous-agent general equilibrium models as in Krusell and Smith (1998).

Our second result that the asymptotic MPCs can be theoretically zero is new and surprising, because it might appear to contradict existing theoretical and numerical works. When the asymptotic MPC is zero, the MPC must be eventually decreasing as a function of wealth. Thus our theoretical results are consistent with the recent empirical evidence that the saving rate is increasing in wealth as documented by Fagereng et al. (2019) using Norwegian administrative data. The possibility of zero asymptotic MPC has an important policy implication. In many countries, consumption tax is a popular tax instrument. However, if the asymptotic MPC is zero, because the rich asymptotically consume nothing, they pay no additional consumption tax. Thus the consumption tax is regressive and may not be desirable from equity perspectives.

The intuition for why the asymptotic MPCs can be zero is as follows. In analytically solvable models with CRRA preferences and no transitory income shocks, one needs a condition of the form $\mathrm{E}\,\beta R^{1-\gamma} < 1$ so that the dynamic programming problem has a solution, where β is the discount factor, γ is the relative risk aversion, and R is the gross return on wealth. Carroll (2009) refers to this condition as the "finite value condition". However, this condition is not necessary for the existence of a solution when agents are subject to transitory income shocks and borrowing constraints as recently shown by Ma et al. (2020). When the classical condition $\mathrm{E}\,\beta R^{1-\gamma} < 1$ is violated, a solution to the income fluctuation problem still exists but the asymptotic MPCs may become zero. In Proposition 3.6 below, we provide conditions under which the asymptotic MPCs are equal to 0, 1, or are in $(0, 1)$.

Note that to violate the finite value condition $\mathrm{E}\,\beta R^{1-\gamma} < 1$ so as to obtain zero asymptotic MPCs, one needs to go beyond canonical models in which discount factor and returns on wealth are constant. This is because with constant discount factor β and constant gross risk-free rate R, equilibrium arguments require that $\beta R < 1$ (see Stachurski and Toda (2019) for a discussion), and with nonnegative interest rate ($R \geq 1$), we obtain $\beta R^{1-\gamma} = (\beta R)R^{-\gamma} < 1$ necessarily. Thus, to obtain zero asymptotic MPC, either random discount factors or capital income risk are necessary. The fact that models with capital income risk are less common may explain why the possibility of zero asymptotic MPC has not been previously recognized.

In Section 4 we calibrate the asset return process with time-varying volatility from data and show that the asymptotic MPCs can be zero with moderate risk aversion. Our results potentially explain why the saving rates among the rich are positive and increasing in wealth.

1.1 Related literature

Our paper is related to the theoretical studies of the income fluctuation problem, which is a key building block of heterogeneous-agent models in modern macroeconomics.[2] Chamberlain and Wilson (2000) study the existence of a solution assuming bounded utility and applying the contraction mapping theorem. Li and Stachurski (2014) relax the boundedness assumption and apply policy function iteration. Ma et al. (2020) allow for stochastic discounting and returns on wealth and discuss the ergodicity, stochastic stability, and tail behavior of wealth. While the main focus of these papers is the existence and uniqueness of a solution, we focus on the asymptotic behavior of consumption. To obtain our results, we draw on the proof technique of Li and Stachurski (2014) and Ma et al. (2020) based on policy function iteration and apply the mathematical theory of regular variation in Bingham et al. (1987).

Carroll and Kimball (1996) show the concavity of consumption functions assuming HARA preferences. Ma et al. (2020) also show the concavity of consumption functions assuming some high-level concavity structure on the utility function, but they explicitly verify this condition only for the CRRA utility. The concavity of consumption functions implies asymptotic linearity, which is related to our results. In contrast, we prove the asymptotic linearity under the much weaker assumption of regularly varying marginal utility. Furthermore, we obtain an exact characterization of asymptotic MPCs, which is absent in the existing literature.

2 Income fluctuation problem

In this section we introduce a general income fluctuation problem following the setting in Ma et al. (2020). Time is discrete and denoted by $t = 0, 1, 2, \dots$. Let a_t be the financial wealth of the agent at the beginning of period t. The agent chooses consumption $c_t \geq 0$ and saves the remaining wealth $a_t - c_t$. The period utility function is u and the discount factor, gross return on wealth, and non-financial income in period t are denoted by β_t, R_t, Y_t, where we normalize

[2] See, for example, Cao (2020) and Açıkgöz (2018) for the existence of equilibrium with and without aggregate shocks, where the theoretical properties of the income fluctuation problem play an important role.

$\beta_0 = 1$. Thus the agent solves

$$\begin{aligned} &\text{maximize} && \mathrm{E}_0 \sum_{t=0}^{\infty} \left(\prod_{i=0}^{t} \beta_i \right) u(c_t) \\ &\text{subject to} && a_{t+1} = R_{t+1}(a_t - c_t) + Y_{t+1}, && \text{(2.1a)} \\ &&& 0 \le c_t \le a_t, && \text{(2.1b)} \end{aligned}$$

where the initial wealth $a_0 = a$ is given, (2.1a) is the budget constraint, and (2.1b) implies that the agent cannot borrow.[3] The stochastic processes $\{\beta_t, R_t, Y_t\}_{t \ge 1}$ obey

$$\beta_t = \beta(Z_t, \varepsilon_t), \quad R_t = R(Z_t, \zeta_t), \quad Y_t = Y(Z_t, \eta_t),$$

where β, R, Y are nonnegative measurable functions, $\{Z_t\}_{t \ge 0}$ is a time-homogeneous finite state Markov chain taking values in $\mathsf{Z} = \{1, \dots, Z\}$ with a transition probability matrix P, and the innovation processes $\{\varepsilon_t\}, \{\zeta_t\}, \{\eta_t\}$ are independent and identically distributed (IID) over time and mutually independent.

Before discussing the properties of the income fluctuation problem (2.1), we note that it is very general despite the fact that it is cast as an infinite-horizon optimization problem in a stationary environment. For example, a finite lifetime is permitted by allowing $\beta(Z_t, \varepsilon_t) = 0$ in some states. Life-cycle features such as age-dependent income and mortality risk in Huggett (1996) are also permitted by supposing that agents have some finite upper bound for age, time is part of the state $z \in \mathsf{Z}$, and that the discount factor $\beta(Z_t, \varepsilon_t)$ includes survival probability.

We introduce the following notations. For a square matrix A, $r(A)$ denotes its spectral radius (largest absolute value of all eigenvalues). The symbols β, R, Y are shorthand of $\beta(Z, \varepsilon)$, $R(Z, \zeta)$, $Y(Z, \eta)$ and $\hat{\beta}, \hat{R}, \hat{Y}$ are shorthand of $\beta(\hat{Z}, \hat{\varepsilon})$, $R(\hat{Z}, \hat{\zeta})$, $Y(\hat{Z}, \hat{\eta})$. Define the diagonal matrix D_β by

$$D_\beta(z, z) = \mathrm{E}_z\, \beta = \mathrm{E}\left[\beta(Z, \varepsilon) \mid Z = z\right] = \mathrm{E}\, \beta(z, \varepsilon).$$

More generally, for any stochastic process $\{X_t\}$ such that the distribution of X_t conditional on all past information and $Z_t = z$ depends only on z, let D_X be the diagonal matrix such that $D_X(z, z) = \mathrm{E}_z\, X = \mathrm{E}\left[X \mid Z = z\right]$. Consider the following assumptions.

Assumption 1. *The utility function* $u : [0, \infty) \to \mathbb{R} \cup \{-\infty\}$ *is twice continuously differentiable on* $(0, \infty)$ *and satisfies* $u' > 0$, $u'' < 0$, $u'(0) = \infty$, *and* $u'(\infty) < 1$.

Assumption 1 is essentially the usual Inada condition together with monotonicity and concavity.

Assumption 2. *The following conditions hold:*

1. $\mathrm{E}_z\, \beta < \infty$ *and* $\mathrm{E}_z\, \beta R < \infty$ *for all* $z \in \mathsf{Z}$,
2. $r(PD_\beta) < 1$ *and* $r(PD_{\beta R}) < 1$,
3. $\mathrm{E}_z\, Y < \infty$ *and* $\mathrm{E}_z\, u'(Y) < \infty$ *for all* $z \in \mathsf{Z}$.

[3] The no-borrowing condition $a_t - c_t \ge 0$ is without loss of generality as discussed in Chamberlain and Wilson (2000) and Li and Stachurski (2014).

The condition $r(PD_\beta) < 1$ generalizes $\beta < 1$ to the case with random discount factors. The condition $r(PD_{\beta R}) < 1$ generalizes the 'impatience' condition[4] $\beta R < 1$ to the stochastic case. Under these two assumptions, the income fluctuation problem (2.1) admits a unique solution.

Theorem 2.1. *Suppose Assumptions 1 and 2 hold. Then the income fluctuation problem* (2.1) *has a unique solution. Furthermore, the consumption function* $c(a, z)$ *can be computed by policy function iteration.*

Proof. See Ma et al. (2020, Theorem 2.2).[5] □

'Policy function iteration' means the following. When the borrowing constraint $c_t \le a_t$ does not bind, the Euler equation implies

$$u'(c_t) = \mathrm{E}_t\, \beta_{t+1} R_{t+1} u'(c_{t+1}).$$

If $c_t = a_t$, then clearly $u'(c_t) = u'(a_t)$. Therefore combining these two cases, we can compactly express the Euler equation as

$$u'(c_t) = \max\left\{\mathrm{E}_t\, \beta_{t+1} R_{t+1} u'(c_{t+1}), u'(a_t)\right\}.$$

Based on this observation, given a candidate consumption function $c(a, z)$, the policy function iteration updates the consumption function by the value $\xi = Tc(a, z)$ that solves

$$u'(\xi) = \max\left\{\mathrm{E}_z\, \hat{\beta}\hat{R}u'(c(\hat{R}(a - \xi) + \hat{Y}, \hat{Z})), u'(a)\right\}. \tag{2.2}$$

Let $\mathcal{C}$ be the space of candidate consumption functions such that $c : (0, \infty) \times \mathsf{Z} \to \mathbb{R}$ is continuous, is increasing in the first element, $0 < c(a, z) \le a$ for all $a > 0$ and $z \in \mathsf{Z}$, and

$$\sup_{(a,z)\in(0,\infty)\times\mathsf{Z}} |u'(c(a, z)) - u'(a)| < \infty.$$

For $c, d \in \mathcal{C}$, define

$$\rho(c, d) = \sup_{(a,z)\in(0,\infty)\times\mathsf{Z}} |u'(c(a, z)) - u'(d(a, z))|. \tag{2.3}$$

When Assumptions 1 and 2 hold, Theorem 2.2 of Ma et al. (2020) shows that $\mathcal{C}$ is a complete metric space with metric ρ and $T : \mathcal{C} \to \mathcal{C}$ defined as $Tc(a, z) = \xi$ that solves (2.2) is a contraction mapping. We call the operator T the *time iteration operator*.[6]

Exploiting policy function iteration, Ma et al. (2020) show several properties such as (i) consumption and savings are increasing in wealth and (ii) consumption is increasing in income. Under further assumptions on the utility function, they also show the concavity and asymptotic linearity of the consumption function. However, these are high-level assumptions that are explicitly verified only for the constant relative risk aversion (CRRA) utility. Below, we significantly weaken this assumption.

[4]The impatience condition is essentially necessary in a general equilibrium model as discussed in Stachurski and Toda (2019).

[5]In addition to Assumptions 1 and 2, Ma et al. (2020) assume that the transition probability matrix P is irreducible. However, irreducibility is required only for their ergodicity result, not for existence and uniqueness of a solution.

[6]The time iteration operator was introduced by Coleman (1990).

3 Asymptotic linearity of consumption functions

In this section we show that the consumption functions in the income fluctuation problem (2.1) are asymptotically linear when the marginal utility is regularly varying. To state the main results, we introduce several notions. A positive measurable function $\ell : (0, \infty) \to (0, \infty)$ is *slowly varying* if $\ell(\lambda x)/\ell(x) \to 1$ as $x \to \infty$ for all $\lambda > 0$. A positive measurable function $f : (0, \infty) \to (0, \infty)$ is *regularly varying* with index $\rho \in \mathbb{R}$ if $f(\lambda x)/f(x) \to \lambda^\rho$ as $x \to \infty$ for all $\lambda > 0$. Bingham et al. (1987, Theorem 1.4.1) show that if f is a positive measurable function such that $g(\lambda) = \lim_{x\to\infty} f(\lambda x)/f(x) \in (0, \infty)$ exists for $\lambda > 0$ in a set of positive measure, then the limit function must be of the form $g(\lambda) = \lambda^\rho$ for some $\rho \in \mathbb{R}$. Furthermore, $\lim_{x\to\infty} f(\lambda x)/f(x) = \lambda^\rho$ holds for all $\lambda > 0$ and hence f is regularly varying with index ρ.

Assumption 3. *The marginal utility function is regularly varying with index $-\gamma < 0$: there exists a slowly varying function ℓ such that $u'(c) = c^{-\gamma}\ell(c)$.*

The assumption that marginal utility is regularly varying is related to several assumptions in the literature. Following Brock and Gale (1969) and Schechtman and Escudero (1977), we say that u' has an *asymptotic exponent* $-\gamma$ if $\log u'(c)/\log c \to -\gamma$ as $c \to \infty$. We say that u is *asymptotically CRRA* with coefficient γ if $-cu''(c)/u'(c) \to \gamma$ as $c \to \infty$.

Proposition 3.1. *Let* $\mathrm{aCRRA}(\gamma)$*,* $\mathrm{RV}(-\gamma)$*, and* $\mathrm{aE}(-\gamma)$ *be respectively the class of utility functions u such that u is asymptotically CRRA with coefficient γ, u' is regularly varying with index $-\gamma$, and u' has an asymptotic exponent $-\gamma$. Then*

$$\mathrm{aCRRA}(\gamma) \subsetneq \mathrm{RV}(-\gamma) \subsetneq \mathrm{aE}(-\gamma).$$

It is clear from Proposition 3.1 that Assumption 3 is significantly weaker than CRRA because it imposes a parametric assumption only at infinity ($c \to \infty$). Furthermore, the parameter $\gamma > 0$ can be interpreted as the asymptotic relative risk aversion of the agent.

To prove our main results, we introduce a technical condition that permits us to apply the Dominated Convergence Theorem.

Assumption 4. *There exists $\delta > 0$ such that $R \in \{0\} \cup [\delta, \infty)$ almost surely.*

This assumption holds, for example, if R takes finitely many values, which is almost always the case in applied numerical works. Note that we allow the possibility $R = 0$ with positive probability. Throughout the rest of the paper, we introduce the following conventions to simplify the notations. "1" denotes either the real number 1 or the vector $(1, \dots, 1)' \in \mathbb{R}^Z$ depending on the context. We interpret $0 \cdot \infty = 0$ and $\beta R^{1-\gamma} = (\beta R)R^{-\gamma}$, so $\beta R^{1-\gamma} = 0$ whenever $\beta = 0$ or $R = 0$ regardless of the value of $\gamma > 0$.

Although the following property is an immediate implication of the above assumptions and convention, we state it as a lemma since we frequently refer to it.

Lemma 3.2. *Suppose Assumptions 2(1) and 4 hold. Then $\mathrm{E}_z\, \beta R^{1-\gamma} < \infty$ for all $z \in \mathsf{Z}$.*

Under the maintained assumptions, we can show that the consumption functions are asymptotically linear, which is our main result.

Theorem 3.3 (Asymptotic linearity)**.** *Suppose Assumptions 1–4 hold and let $c(a, z)$ be the consumption function established in Theorem 2.1. Let $D = D_{\beta R^{1-\gamma}}$ be the diagonal matrix whose (z, z)-th element is $\mathrm{E}_z\, \beta R^{1-\gamma} < \infty$ and define $F : \mathbb{R}_+^Z \to \mathbb{R}_+^Z$ by*

$$(Fx)(z) = \left(1 + (PDx)(z)^{1/\gamma}\right)^{\gamma}. \tag{3.1}$$

Then F has a fixed point $x^* \in \mathbb{R}_+^Z$ if and only if $r(PD) < 1$, in which case it is unique. Furthermore, the followings are true.

1. *If $r(PD) < 1$, then*

$$0 \le \liminf_{a \to \infty} \frac{c(a, z)}{a} \le \limsup_{a \to \infty} \frac{c(a, z)}{a} \le x^*(z)^{-1/\gamma}. \tag{3.2}$$

   If $\liminf_{a \to \infty} c(a, z)/a > 0$ for all $z \in \mathsf{Z}$, then

$$\bar{c}(z) := \lim_{a \to \infty} \frac{c(a, z)}{a} = x^*(z)^{-1/\gamma} \in (0, 1]. \tag{3.3}$$

2. *If $r(PD) \ge 1$ and PD is irreducible, then*

$$\bar{c}(z) := \lim_{a \to \infty} \frac{c(a, z)}{a} = 0.$$

An immediate implication of Theorem 3.3 is that, when $r(PD) < 1$, either $\liminf_{a \to \infty} c(a, z)/a = 0$ for some $z \in \mathsf{Z}$, or

$$\lim_{a \to \infty} \frac{c(a, z)}{a} = \bar{c}(z) \tag{3.4}$$

exists for all $z \in \mathsf{Z}$. However, the theorem does not tell which case occurs. The following proposition provides easily testable sufficient conditions for the limit (3.3) to hold.

Proposition 3.4. *Let everything be as in Theorem 3.3. Suppose that either*

1. *$r(PD) < 1$ and u exhibits constant relative risk aversion (CRRA), so $u'(c) = c^{-\gamma}$, or*
2. *u exhibits bounded relative risk aversion (BRRA), so*

$$0 \le \underline{\gamma} \le -\frac{xu''(c)}{u'(c)} \le \bar{\gamma} < \infty \tag{3.5}$$

   for all $c > 0$ and

$$\max_{z \in \mathsf{Z}} \mathrm{E}_z\, \hat{\beta} \hat{R} \max\left\{\hat{R}^{-\underline{\gamma}}, \hat{R}^{-\bar{\gamma}}\right\} < 1. \tag{3.6}$$

Then the limit (3.3) *holds.*

The limit $\bar{c}(z) = \lim_{a \to \infty} c(a, z)/a$ is the asymptotic *average* propensity to consume out of wealth as $a \to \infty$. However, from now on we refer to it as the asymptotic *marginal* propensity to consume (asymptotic MPC) to be consistent with the literature. This language can be justified because

$$\lim_{a\to\infty}\frac{c(a,z)}{a}=\lim_{a\to\infty}\frac{\partial}{\partial a}c(a,z)$$

by l'Hôpital's theorem whenever the limit in the right-hand side exists.

Theorem 3.3 and Proposition 3.4 have several implications. First, they justify the textbook affine consumption function as in Keynes (1936). Second, they justify linearly extrapolating policy functions outside the grid when solving models numerically as in Gouin-Bonenfant and Toda (2018). Finally, they explain the "approximate aggregation" property in heterogeneous-agent general equilibrium models as in Krusell and Smith (1998). Approximate aggregation refers to the observation that, when solving heterogeneous-agent general equilibrium models, keeping track of just the first moment of the wealth distribution is nearly sufficient, despite the fact that the entire wealth distribution is a state variable. Because the market clearing condition involves aggregate savings, aggregation would be possible if saving is linear in wealth. Our results show that consumption (hence saving) is approximately linear in wealth, which explains the approximate aggregation property.

An interesting implication of Proposition 3.4 is that the asymptotic MPCs $\bar{c}(z)$ depend only on the matrix PD, which in turn depends only on the asymptotic relative risk aversion γ as well as "multiplicative shocks" β and R, and not on "additive shocks" Y. The following corollary verifies the intuition in Gouin-Bonenfant and Toda (2018) that only multiplicative shocks matter for characterizing the behavior of wealthy agents.

Corollary 3.5 (Irrelevance of additive shocks)**.** *Let everything be as in Theorem 3.3. The asymptotic MPCs $\bar{c}(z)$ depend only on the asymptotic relative risk aversion γ, transition probability matrix P, the discount factor β, and the return on wealth R, and not on income Y.*

The system of fixed point equations

$$x(z)=(Fx)(z)=\left(1+(PDx)(z)^{1/\gamma}\right)^{\gamma},\quad z=1,\dots,Z \tag{3.7}$$

is in general nonlinear and does not admit a closed-form solution. Below, we discuss several examples with explicit solutions. Throughout, we assume that some sufficient conditions for asymptotic linearity (as in Proposition 3.4) hold.

Example 3.1. If $\gamma=1$, then (3.7) becomes

$$x^*=1+PDx^*\iff x^*=(I-PD)^{-1}1,$$

where $D=D_\beta=\operatorname{diag}(\dots,\mathrm{E}_z\,\beta,\dots)$.[7] This example generalizes the log utility case discussed in the Online Appendix of Toda (2019).[8] A corollary is that with log utility, we always have $\bar{c}(z)>0$.

[7] Note that since $r(PD)=r(PD_\beta)<1$ by Assumption 2, $(I-PD)^{-1}=\sum_{k=0}^{\infty}(PD)^k$ exists and is nonnegative.

[8] In Toda (2019), there is an expression $I-BP$, not $I-PB$, where B plays the role of the diagonal matrix D in Theorem 3.3. This difference comes from the fact that the discount factors are predetermined in Toda (2019), whereas they are (potentially) stochastic in our more general setting.

Example 3.2. If $b = b(z) = \mathrm{E}_z\, \beta R^{1-\gamma}$ does not depend on z, then $D = bI$. If $x = k1$ is a multiple of the vector 1, then $PDx = bPk1 = bk1$ because P is a transition probability matrix. Thus if $b < 1$, (3.7) reduces to

$$x^*(z) = (1 + (bx^*(z))^{1/\gamma})^\gamma \iff x^*(z) = (1 - b^{1/\gamma})^{-\gamma} \iff \bar{c}(z) = 1 - b^{1/\gamma}.$$

This example shows that with constant discounting ($\beta(z, \varepsilon) \equiv \beta$) and risk-free saving ($R(z, \zeta) \equiv R$), the asymptotic MPC is constant regardless of the income shocks:

$$\bar{c}(z) = \begin{cases} 1 - (\beta R^{1-\gamma})^{1/\gamma} & \text{if } \beta R^{1-\gamma} < 1, \\ 0 & \text{otherwise.} \end{cases}$$

(The argument related to $\bar{c}(z) = 0$ is a little rough but can be formally justified using Proposition 3.6(2) below.)

Example 3.3. Suppose the return on wealth $R_t = R(Z_t, \zeta_t)$ does not depend on Z_t, so $R_t = R(\zeta_t)$. Assume further that $\log R_t$ is normally distributed with standard deviation σ and mean $\mu - \sigma^2/2$, so $\mathrm{E}\, R = \mathrm{e}^\mu$.[9] Let the discount factor $\beta = \mathrm{e}^{-\delta}$ be constant, where $\delta > 0$ is the discount rate. Then using the property of the normal distribution, we obtain

$$\begin{aligned} &1 > \mathrm{E}\, \beta R = \mathrm{e}^{-\delta + \mu} \iff \delta > \mu, \\ &1 > \mathrm{E}\, \beta R^{1-\gamma} = \mathrm{e}^{-\delta + (1-\gamma)(\mu - \gamma\sigma^2/2)} \iff \delta > (1 - \gamma) \left(\mu - \frac{1}{2}\gamma\sigma^2 \right). \end{aligned}$$

Therefore assuming $\delta > \mu$ for Assumption 2 to hold, it follows from Example 3.2 that

$$\bar{c}(z) = \begin{cases} 1 - \mathrm{e}^{-\psi\delta - (1-\psi)(\mu - \gamma\sigma^2/2)} > 0 & \text{if } \delta > (1 - \gamma)\left(\mu - \frac{1}{2}\gamma\sigma^2\right), \\ 0 & \text{otherwise,} \end{cases}$$

where $\psi = 1/\gamma$ is the elasticity of intertemporal substitution. If $\gamma > 1$, then $(1 - \gamma)(\mu - \gamma\sigma^2/2) \to \infty$ as $\gamma, \sigma \to \infty$, so the asymptotic MPC is 0 if risk aversion or volatility is sufficiently high.

So far, we know from Theorem 3.3 that if $r(PD) < 1$, then the bound (3.2) holds, and if $r(PD) \geq 1$ and PD is irreducible, then $\lim_{a\to\infty} c(a, z)/a = 0$ for all $z \in \mathsf{Z}$. Thus the remaining case is when $r(PD) \geq 1$ and PD is reducible. We can characterize the asymptotic MPC as follows in this case.

First, by the proof of Theorem 3.3 (see (A.18) in Appendix A) we always have

$$\limsup_{a\to\infty} \frac{c(a, z)}{a} \leq x_n(z)^{-1/\gamma}$$

regardless of the reducibility of PD, where $\{x_n\}_{n=1}^\infty \subset \mathbb{R}_+^Z$ is defined by $x_0 = 1$ and iterating $x_n = Fx_{n-1}$. Because the function F in (3.1) is monotonic, the sequence $\{x_n\}_{n=0}^\infty$ is monotonically increasing and so converges to some $x^* \in [1, \infty]^Z$. Therefore the upper bound (3.2) is still valid.

Under what conditions does the agent asymptotically consume nothing ($\bar{c}(z) = 0$), everything ($\bar{c}(z) = 1$), or a nontrivial amount ($\bar{c}(z) \in (0, 1)$)? To answer

[9] Strictly speaking, the lognormal specification for R_t violates Assumption 4. We can instead use a truncated lognormal distribution with an arbitrarily small truncation point to satisfy Assumption 4.

this question, let $K = PD$ be as in Theorem 3.3. By relabeling the states $z = 1, \dots, Z$ if necessary, without loss of generality we may assume that K is block upper triangular,

$$K = \begin{bmatrix} K_1 & \cdots & * \\ \vdots & \ddots & \vdots \\ 0 & \cdots & K_J \end{bmatrix}, \tag{3.8}$$

where each diagonal block K_j is irreducible. Partition Z as $\mathsf{Z} = \mathsf{Z}_1 \cup \cdots \cup \mathsf{Z}_J$ accordingly. Then we have the following complete characterization.

Proposition 3.6. *Let everything be as in Theorem 3.3 and express $K = PD$ as in* (3.8)*. Let $x^* := \lim_{n\to\infty} x_n(z) \in [1,\infty]^Z$ be the limit of $\{x_n\}_{n=0}^{\infty}$ defined by $x_0 = 1$ and iterating $x_n = Fx_{n-1}$, and set $\bar{c}(z) = x^*(z)^{-1/\gamma}$. Then the followings are true.*

1. *$\bar{c}(z) = 1$ if and only if $\hat{\beta}\hat{R} = 0$ almost surely conditional on $Z = z$.*
2. *$\bar{c}(z) = 0$ if and only if there exist j, $\hat{z} \in \mathsf{Z}_j$, and $m \in \mathbb{N}$ such that $K^m(z, \hat{z}) > 0$ and $r(K_j) \geq 1$.*

Theorem 3.3 and Proposition 3.6 are surprising. In typical income fluctuation problems, we require a condition of the form $\mathrm{E}_z \hat{\beta}\hat{R}^{1-\gamma} < 1$.[10] However, Ma et al. (2020) show that for a general income fluctuation problem to have a solution, no conditions on risk aversion are required beyond Assumptions 1 and 2 as we can see from Theorem 2.1. When the marginal utility is regularly varying, Theorem 3.3 implies that the consumption functions are asymptotically linear with no further conditions on γ. Thus the classical "finite value condition" (Carroll, 2009) of the form $\mathrm{E}_z \hat{\beta}\hat{R}^{1-\gamma} < 1$ is not necessary. We can understand this phenomenon through Proposition 3.6. When $\mathrm{E}_z \hat{\beta}\hat{R}^{1-\gamma} \geq 1$ for some z, it may be $r(K_j) \geq 1$, and the asymptotic MPC $\bar{c}(z)$ may become 0. This is not a contradiction: it simply reflects the fact that the existing literature has only studied cases that guarantee $\bar{c}(z) > 0$.

4 Asymptotic MPCs and saving rates

Empirically, it is known that the rich save more. For example, Quadrini (1999) documents that entrepreneurs (who tend to be rich) have high saving rates. Dynan et al. (2004) document that there is a positive association between saving rates and lifetime income. More recently, using Norwegian administrative data, Fagereng et al. (2019) show that among households with positive net worth, saving rates are increasing in wealth. In this section we apply our theory of asymptotic MPCs to shed light on these empirical findings.

Following Fagereng et al. (2019), we define an agent's saving rate by the change in net worth divided by total income. Therefore using the budget constraint (2.1a), the saving rate is

$$s_{t+1} = \frac{a_{t+1} - a_t}{(R_{t+1} - 1)(a_t - c_t) + Y_{t+1}} = 1 - \frac{c_t/a_t}{(R_{t+1} - 1)(1 - c_t/a_t) + Y_{t+1}/a_t}. \tag{4.1}$$

[10] See, for example, the discussion on p. 244 of Samuelson (1969), Equation (3) of Carroll (2009), Equation (18) of Toda (2014), or Equation (3) of Toda (2019).

Letting $a \to \infty$, the saving rate of an infinitely wealthy agent is

$$s = 1 - \frac{\bar{c}}{(\hat{R}-1)(1-\bar{c})}, \tag{4.2}$$

where $\bar{c}$ is the asymptotic MPC. Under what conditions can the saving rate (4.1) be increasing in wealth, and in particular, can the asymptotic saving rate (4.2) become positive?

The following proposition shows that standard models are unable to explain empirical facts.

Proposition 4.1. *Consider a classical Bewley (1977) model in which agents are infinitely-lived, discount factor β and return on wealth $R > 1$ are constant, and marginal utility is regularly varying (e.g., CRRA utility). Then in the stationary equilibrium the asymptotic saving rate* (4.2) *is negative.*

De Nardi and Fella (2017, Figure 1) numerically show that the saving rate in a Bewley model is decreasing in wealth and becomes negative as agents become wealthier. Proposition 4.1 proves that in any such model, the asymptotic saving rate is negative. The following proposition shows that just by allowing β and R to be stochastic need not solve the problem when $\bar{c} > 0$.

Proposition 4.2. *Consider a Bewley (1977) model in which agents are infinitely-lived, $\{\beta_t, R_t\}_{t\ge 1}$ is* IID *with $\mathrm{E}\, R > 1$ and $\mathrm{E}\, \beta R^{1-\gamma} < 1$, and marginal utility is regularly varying (e.g., CRRA utility) with index $-\gamma$. If the stationary equilibrium wealth distribution has an unbounded support, then the asymptotic saving rate* (4.2) *evaluated at $\hat{R} = \mathrm{E}\, R$ is nonpositive.*

One possible explanation for the positive and increasing saving rates is to consider models with discount factor or return heterogeneity. If $r(PD_{\beta R^{1-\gamma}}) \ge 1$, then by Theorem 3.3 we have $\bar{c} = 0$ and hence the asymptotic saving rate becomes $s = 1 > 0$ using (4.2).[11]

To show the possibility of positive and increasing savinge rates, we consider a numerical example. The agent has CRRA utility with relative risk aversion $\gamma > 0$ and constant discount factor $\beta = \mathrm{e}^{-\delta}$ ($\delta > 0$ is the discount rate). We suppose that the return process $\{R_t\}$ exhibits constant expected return $\mathrm{E}\, R_t = \mathrm{e}^{\mu}$ with GARCH$(1,1)$ innovations:

$$\log R_t = \mu - \frac{1}{2}\sigma_t^2 + \epsilon_t, \tag{4.3a}$$

$$\epsilon_t = \sigma_t \zeta_t, \quad \zeta_t \sim \text{IID} N(0,1) \tag{4.3b}$$

$$\sigma_t^2 = \omega + \alpha \epsilon_{t-1}^2 + \rho \sigma_{t-1}^2, \tag{4.3c}$$

where $\sigma_t > 0$ is conditional volatility, ϵ_t is a zero mean innovation, and we assume $\omega, \alpha, \rho > 0$ and $\alpha + \rho < 1$ to ensure stationarity.

To calibrate the parameters, we use the 1947–2018 monthly data for U.S. stock market returns from the updated spreadsheet of Welch and Goyal (2008).[12] We first construct $\widehat{\epsilon}_t$ by demeaning the log excess returns $\log R_t - \log R_{f,t}$ and

[11] Another possibility is to consider overlapping generations models. Stachurski and Toda (2019, Theorem 9) present a model with random birth/death and show that it is possible to have $\beta R > 1$ in equilibrium. In this case, by the proof of Proposition 4.1, we have $s > 0$.

[12] `http://www.hec.unil.ch/agoyal/docs/PredictorData2018.xlsx`.

estimate the GARCH parameters $\omega = 9.1297 \times 10^{-5}$, $\alpha = 0.8354$, and $\rho = 0.1188$. We then estimate μ in (4.3a) as $\mu = \log(\mathrm{E}\, R_t) = 6.8011 \times 10^{-3}$ from the real gross stock market returns. Because our model requires a finite state Markov chain, we discretize the GARCH$(1,1)$ process (4.3) using the Farmer and Toda (2017) method as described in Appendix B with $N_v = 3$ points for the volatility state and $N_\epsilon = 7$ for the return state. Moreover, we set $\beta = 0.9^{1/12} = 0.9913$ so that the annual discounting is around 10%.

To solve the income fluctuation problem (2.1), we need to specify the income process. Because the U.S. economy has been growing, and by Corollary 3.5 the details on the income process is not important for the asmyptotic MPCs, for simplicity we assume that income grows at a constant rate g, so $Y_t = \mathrm{e}^{gt}$. We calibrate the growth rate g from the U.S. real per capita GDP in 1947–2018 and obtain $g = 1.6208 \times 10^{-3}$ at the monthly frequency. Although the theory in Ma et al. (2020) requires a stationary process for income, it is straightforward to allow for constant growth in income by detrending the model when the utility is CRRA. After simple algebra, it suffices to use

$$
\begin{aligned}
\log \tilde{R}_t &= \log R_t - g = \mu - g - \frac{1}{2}\sigma_t^2 + \epsilon_t, \\
\tilde{\beta} &= \beta \mathrm{e}^{(1-\gamma)g} = \mathrm{e}^{-\delta + (1-\gamma)g}, \\
\tilde{Y}_t &= Y_t \mathrm{e}^{-gt} = 1.
\end{aligned}
$$

In the current setting, Assumptions 1, 3, 4, and conditions 1 and 3 of Assumption 2 obviously hold. To apply Theorems 2.1, 3.3, and Proposition 3.4, it remains to determine $r(PD_{\beta R}), r(PD_{\beta R^{1-\gamma}}) \gtrless 1$. Figure 1 shows the determination of the asymptotic MPC $\bar{c}(z)$ when we change the relative risk aversion γ and the discount rate δ. We see that the asymptotic MPCs can be zero if relative risk aversion is moderately high (above 3).

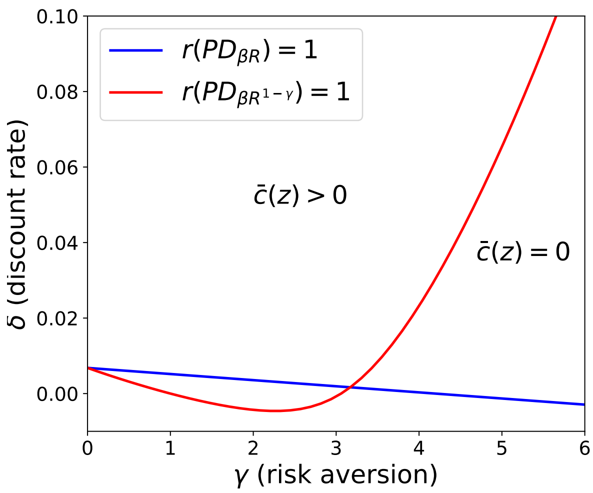

Figure 1: Determination of asymptotic MPCs with GARCH$(1,1)$ returns.

Next, we explore properties of the optimal consumption rule, asymptotic MPC, and saving rate. To that end, we solve the model for $\gamma = 2, 4$. According to Figure 1 and Theorem 2.1, a unique solution exists in each scenario given the calibrated discount factor. Figure 2 shows the optimal consumption rule. Consistent with our theory, for $\gamma = 2$ the consumption functions are approximately linear with a positive slope for high asset level. When $\gamma = 4$, the consumption functions show a more concave pattern.

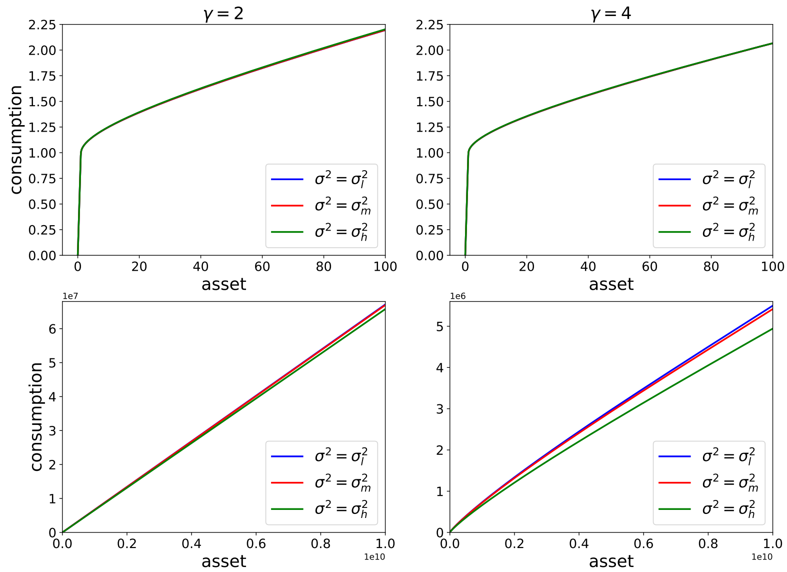

Figure 2: Optimal consumption rule.

Note: The top and bottom panels plot the consumption functions in the range $a \in [0, 100]$ and $a \in [0, 10^{10}]$, respectively. Here and in other figures, the left (right) panels correspond to $\gamma = 2$ ($\gamma = 4$). For visibility, we plot across asset and the three volatility states $\sigma_l^2 < \sigma_m^2 < \sigma_h^2$ holding $\epsilon = 0$ constant.

To see the asymptotic properties further, Figure 3 plots the consumption rate (solid lines) in log-log scale. We see that the consumption rate is decreasing in wealth for each realized volatility. For $\gamma = 2$, as asset level gets large, the asymptotic MPC approaches a positive constant that coincides with the theoretical level asymptotic MPC calculated based on Theorem 3.3 (dotted lines), indicating that the consumption function is asymptotically linear, which is consistent with Proposition 3.4. For $\gamma = 4$, the consumption rate exhibits a clear decreasing trend even when asset is extremely large ($a \approx 10^{10}$), which is consistent with zero asymptotic MPC established in Theorem 3.3.

Finally, Figure 4 shows the saving rate assuming $\sigma_t^2 = \sigma_{t+1}^2 \in \left\{\sigma_l^2, \sigma_m^2, \sigma_h^2\right\}$ and $\epsilon = 0$. When wealth is low, the borrowing constraint binds and labor income is the only source of income and net worth accumulation, i.e., $s_{t+1} = (Y_{t+1} - a)/Y_{t+1} = 1 - \mathrm{e}^{-g}a$, which is decreasing in asset. A moderately greater wealth implies lower saving rates because capital income is used to finance disproportionately large consumption. The saving rate starts to increase when wealth is relatively high (≈ 100). Importantly, when $\gamma = 4$ and $\sigma^2 \in \left\{\sigma_l^2, \sigma_m^2\right\}$,

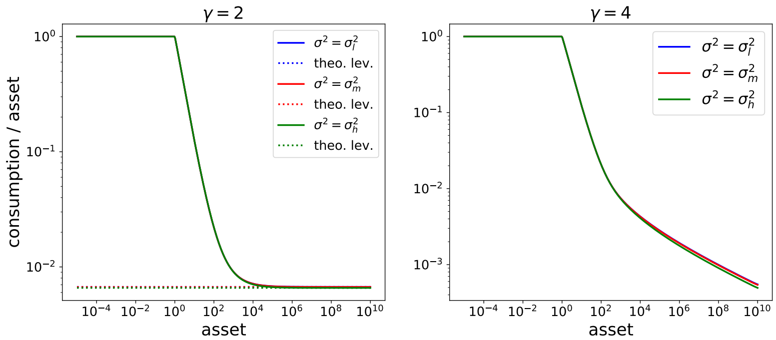

Figure 3: Consumption rate.

the saving rate is increasing in wealth among agents with large asset and the asymptotic saving rate is positive,[13] as opposed to the increasing but negative saving rate when $\gamma = 2$. This example illustrates that the empirically observed positive and increasing saving rate could potentially be explained by models with capital income risk, particularly those with zero asymptotic MPC in certain states.

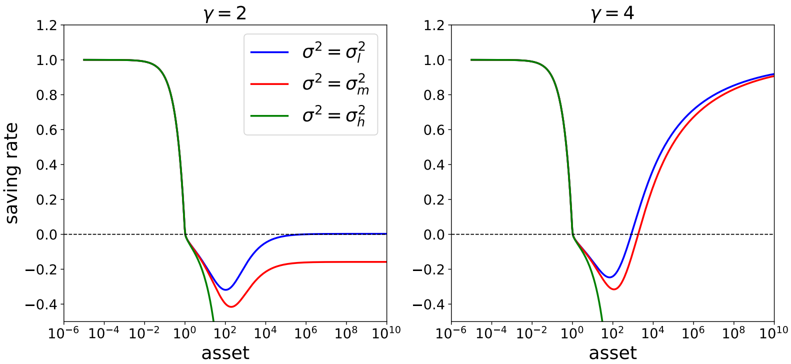

Figure 4: Saving rate.

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[13]When $\sigma^2 = \sigma_h^2$, the saving rate is discontinuous because $\hat{R} < 1$ when $\epsilon = 0$.

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A Proofs

We need the following result to prove Proposition 3.1.

Theorem A.1 (Representation Theorem)**.** *Let $\ell : (0,\infty) \to (0,\infty)$ be measurable. Then ℓ is slowly varying if and only if it may be written in the form*

$$\ell(x) = \exp\left(\eta(x) + \int_a^x \frac{\varepsilon(t)}{t}\,\mathrm{d}t\right) \tag{A.1}$$

for $x \geq a$ for some $a > 0$, where η, ε are measurable and $\eta(x) \to \eta \in \mathbb{R}$, $\varepsilon(x) \to 0$ as $x \to \infty$.

Proof. See Bingham et al. (1987, Theorem 1.3.1). □

Proof of Proposition 3.1.

Step 1. aCRRA(γ) $\subset$ RV($-\gamma$).

If u is asymptotically CRRA with coefficient γ, by definition we can write

$$\frac{cu''(c)}{u'(c)} = -\gamma + \varepsilon(c),$$

where $\varepsilon(c) \to 0$ as $c \to \infty$. Dividing both sides by $c > 1$ and integrating on $[1, c]$, we obtain

$$\log\frac{u'(c)}{u'(1)} = -\gamma \log c + \int_1^c \frac{\varepsilon(t)}{t}\,\mathrm{d}t \iff u'(c) = c^{-\gamma}\exp\left(\log u'(1) + \int_1^c \frac{\varepsilon(t)}{t}\,\mathrm{d}t\right).$$

Therefore by Theorem A.1, u' is regularly varying with index $-\gamma$ by setting $u'(c) = c^{-\gamma}\ell(c)$ with ℓ defined by (A.1) with $\eta(x) \equiv \log u'(1)$ and $a = 1$.

Step 2. RV($-\gamma$) $\not\subset$ aCRRA(γ).

Consider the marginal utility function $u'(c) = c^{-\gamma}\ell(c)$ for ℓ in (A.1), where

$$\eta(x) = \delta \int_0^x \frac{\sin t}{t}\,\mathrm{d}t$$

for some $\delta \in (0, \gamma)$ and $\varepsilon(x) \equiv 0$. Noting that $\int_0^\infty \sin t/t\,\mathrm{d}t = \pi/2$,[14] we have $\eta(x) \to \pi\delta/2$ as $x \to \infty$, so by Theorem A.1, ℓ is slowly varying and $u \in$ RV($-\gamma$). However, log differentiating u', we obtain

$$\frac{u''(c)}{u'(c)} = -\frac{\gamma}{c} + \delta\frac{\sin c}{c} \iff -\frac{cu''(c)}{u'(c)} = \gamma - \delta \sin c > 0,$$

so $u'' < 0$ always but $-cu''(c)/u'(c)$ does not converge as $c \to \infty$. Therefore $u \notin$ aCRRA(γ).

[14]See Hardy (1909) for an interesting discussion of this integral.

Step 3. $\mathrm{RV}(-\gamma) \subset \mathrm{aE}(-\gamma)$.

If u' is regularly varying with index $-\gamma$, by Theorem A.1 we can write $u'(c) = c^{-\gamma}\ell(c)$ with ℓ as in (A.1). Take any $\delta > 0$. Since $\varepsilon(c) \to 0$ as $c \to \infty$, we can take $\bar{c} > \max\{a, 1\}$ such that $|\varepsilon(c)| \le \delta$ for $c \ge \bar{c}$. Then

$$\begin{aligned}
|\log \ell(c)| &= \left|\eta(c) + \int_a^c \frac{\varepsilon(t)}{t} \,\mathrm{d}t\right| \le |\eta(c)| + \int_a^c \frac{|\varepsilon(t)|}{t} \,\mathrm{d}t \\
&\le |\eta(c)| + \int_a^{\bar{c}} \frac{|\varepsilon(t)|}{t} \,\mathrm{d}t + \int_{\bar{c}}^c \frac{\delta}{t} \,\mathrm{d}t \\
&= |\eta(c)| + \int_a^{\bar{c}} \frac{|\varepsilon(t)|}{t} \,\mathrm{d}t + \delta \log \frac{c}{\bar{c}}.
\end{aligned}$$

Dividing both sides by $\log c > \log \bar{c} > 0$ and letting $c \to \infty$, noting that $\eta(c) \to \eta$ as $c \to \infty$, we obtain

$$\limsup_{c\to\infty} \left| \frac{\log \ell(c)}{\log c} \right| \le \delta.$$

Since $\delta > 0$ is arbitrary, letting $\delta \downarrow 0$, we obtain

$$\lim_{c\to\infty} \frac{\log \ell(c)}{\log c} = 0.$$

Therefore $\log u'(c)/\log c \to -\gamma$ because $u'(c) = c^{-\gamma}\ell(c)$, and by definition u' has an asymptotic exponent $-\gamma$.

Step 4. $\mathrm{aE}(-\gamma) \not\subset \mathrm{RV}(-\gamma)$.

Consider the marginal utility function $u'(c) = c^{-\gamma} \exp(\delta \sin \log c)$, where $\delta \in (0, \gamma)$. Then

$$\frac{\log u'(c)}{\log c} = -\gamma + \delta \frac{\sin \log c}{\log c} \to -\gamma$$

as $c \to \infty$, so $u \in \mathrm{aE}(-\gamma)$. Furthermore, by log differentiating u', we obtain

$$\frac{u''(c)}{u'(c)} = -\frac{\gamma}{c} + \delta \frac{\cos \log c}{c} \iff -\frac{cu''(c)}{u'(c)} = \gamma - \delta \cos \log c > 0,$$

so $u'' < 0$ always. To show $u \notin \mathrm{RV}(-\gamma)$, it suffices to show that $\ell(c) := \exp(\delta \sin \log c)$ is not slowly varying. Take $\lambda = \mathrm{e}^{\pi}$ and $c_n = \mathrm{e}^{\pi(n-1/2)}$. Then

$$\begin{aligned}
\log \frac{\ell(\lambda c_n)}{\ell(c_n)} &= \delta(\sin \log(\lambda c_n) - \sin \log c_n) \\
&= \delta(\sin(\pi(n + 1/2)) - \sin(\pi(n - 1/2))) \\
&= 2(-1)^n \delta,
\end{aligned}$$

which does not converge as $n \to \infty$. Therefore ℓ is not slowly varying. □

Proof of Lemma 3.2. If $R = 0$, then $\beta R^{1-\gamma} = (\beta R)R^{-\gamma} = 0$ by convention. If $R > 0$, then $R \ge \delta$ almost surely by Assumption 4. In either case $\beta R^{1-\gamma} \le \beta R \delta^{-\gamma}$, so

$$\mathrm{E}_z \beta R^{1-\gamma} \le \mathrm{E}_z \beta R \delta^{-\gamma} = \delta^{-\gamma} \mathrm{E}_z \beta R < \infty$$

by Assumption 2(1). □

Before proving Theorem 3.3, we first proceed heuristically to motivate what the value of $\bar{c}(z) = \lim_{a\to\infty} c(a,z)/a$ should be if it exists. Assuming that the borrowing constraint does not bind, the Euler equation (2.2) implies

$$u'(\xi) = \mathrm{E}_z\, \hat{\beta}\hat{R}u'(c(\hat{R}(a-\xi)+\hat{Y},\hat{Z})),$$

where $\xi = c(a,z)$. Multiplying both sides by a^γ, setting $c(a,z) = \bar{c}(z)a$ motivated by (3.4), letting $a \to \infty$, using Assumption 3 (regular variation), and interchanging expectations and limits, it must be

$$\bar{c}(z)^{-\gamma} = \mathrm{E}_z\, \hat{\beta}\hat{R}^{1-\gamma}\bar{c}(\hat{Z})^{-\gamma}(1-\bar{c}(z))^{-\gamma}.$$

Dividing both sides by $(1-\bar{c}(z))^{-\gamma}$ and setting $x(z) = \bar{c}(z)^{-\gamma}$, we obtain

$$x(z) = \left(1 + \left(\mathrm{E}_z\, \hat{\beta}\hat{R}^{1-\gamma}x(\hat{Z})\right)^{1/\gamma}\right)^{\gamma}, \quad z = 1,\dots,Z. \tag{A.2}$$

Noting that $\hat{\beta},\hat{R}$ depend only on $\hat{Z}$ and IID innovations, we have

$$\mathrm{E}_z\, \hat{\beta}\hat{R}^{1-\gamma}x(\hat{Z}) = \sum_{\hat{z}=1}^{Z} P(z,\hat{z})\, \mathrm{E}_{\hat{z}}\, \hat{\beta}\hat{R}^{1-\gamma}x(\hat{z}). \tag{A.3}$$

Therefore letting P be the transition probability matrix and $D = D_{\beta R^{1-\gamma}}$ be the diagonal matrix whose (z,z)-th element is $\mathrm{E}_z\, \beta R^{1-\gamma} < \infty$ by Lemma 3.2, we can rewrite (A.2) as (3.7). The following proposition provides a necessary and sufficient condition for the existence of a fixed point of F.

Proposition A.2. *Let everything be as in Theorem 3.3. Then F in (3.1) has a fixed point $x^* \in \mathbb{R}_+^Z$ if and only if $r(PD) < 1$, in which case the fixed point is unique.*

Take any $x_0 \in \mathbb{R}_+^Z$ and define the sequence $\{x_n\}_{n=1}^\infty \subset \mathbb{R}_+^Z$ by

$$x_n = Fx_{n-1} \tag{A.4}$$

for all $n \in \mathbb{N}$. Then the followings are true.

1. *If $r(PD) < 1$, then $\{x_n\}_{n=1}^\infty$ converges to x^*.*
2. *If $r(PD) \geq 1$ and PD is irreducible, then $x_n(z) \to x^*(z) = \infty$ as $n \to \infty$ for all $z \in \mathsf{Z}$.*

Proof. Immediate from Lemmas A.3 and A.4 below. □

Lemma A.3. *Let $\gamma > 0$ and define $\phi : \mathbb{R}_+ \to \mathbb{R}_+$ by $\phi(t) = (1+t^{1/\gamma})^\gamma$. Then there exist $a \geq 1$ and $b \geq 0$ such that $\phi(t) \leq at + b$. Furthermore, we can take $a \geq 1$ arbitrarily close to 1. (The choice of b may depend on a.)*

Proof. The proof depends on $\gamma \gtrless 1$.

Case 1: $\gamma \leq 1$. Let us show that we can take $a = b = 1$. Let $f(t) = 1+t-\phi(t)$. Then $f(0) = 0$ and

$$f'(t) = 1 - \phi'(t) = 1 - \gamma(1+t^{1/\gamma})^{\gamma-1}\frac{1}{\gamma}t^{1/\gamma-1} = 1 - (t^{-1/\gamma}+1)^{\gamma-1} \geq 0,$$

so $f(t) \geq 0$ for all $t \geq 0$. Therefore $\phi(t) \leq 1+t$.

Case 2: $\gamma > 1$. By simple algebra we obtain

$$\phi''(t) = (\gamma - 1)(t^{-1/\gamma} + 1)^{\gamma-2} \left(-\frac{1}{\gamma} t^{-1/\gamma-1} \right) < 0, \tag{A.5}$$

so ϕ is increasing and concave. Therefore $\phi(t) \le \phi(u) + \phi'(u)(t - u)$ for all t, u. Letting $a = \phi'(u)$ and $b = \max\{0, \phi(u) - \phi'(u)u\}$, we obtain $\phi(t) \le at + b$. Furthermore, since $\phi'(t) = (t^{-1/\gamma} + 1)^{\gamma-1} \to 1$ as $t \to \infty$, we can take $a = \phi'(u)$ arbitrarily close to 1 by taking u large enough. □

The following lemma slightly generalizes a result in Toda (2019).

Lemma A.4. *Let $\gamma > 0$ and K be a $Z \times Z$ nonnegative matrix. Define $F : \mathbb{R}^Z_+ \to \mathbb{R}^Z_+$ by $Fx = \phi(Kx)$, where ϕ is as in Lemma A.3 and is applied element-wise. Then F has a fixed point $x^* \in \mathbb{R}^Z_+$ if and only if $r(K) < 1$, in which case x^* is unique.*

Take any $x_0 \in \mathbb{R}^Z_+$ and define the sequence $\{x_n\}_{n=1}^\infty \subset \mathbb{R}^Z_+$ by $x_n = Fx_{n-1}$ for all $n \in \mathbb{N}$. Then the followings are true.

1. *If $r(K) < 1$, then $\{x_n\}_{n=1}^\infty$ converges to x^*.*
2. *If $r(K) \ge 1$ and K is irreducible, then $x_n(z) \to x^*(z) = \infty$ as $n \to \infty$ for all $z \in \mathsf{Z}$.*

Proof. We divide the proof into several steps.

Step 1. If $r(K) \ge 1$, then F does not have a fixed point. If in addition K is irreducible, then $x_n(z) \to \infty$ for all $z \in \mathbb{Z}$.

We prove the contrapositive. Suppose that F has a fixed point $x^* \in \mathbb{R}^Z_+$. Since $\phi > 0$, we have $x^* \gg 0$. Since clearly $\phi(t) > t$ for all $t \ge 0$, we have $x^* = \phi(Kx^*) \gg Kx^*$. Since K is a nonnegative matrix, by the Perron-Frobenius theorem, we can take a right eigenvector $y > 0$ such that $y'K = r(K)y'$. Since $x^* \gg Kx^*$ and $y > 0$, we obtain

$$0 < y'(x^* - Kx^*) \implies r(K)y'x^* < y'x^*.$$

Dividing both sides by $y'x^* > 0$, we obtain $r(K) < 1$.

Suppose that $r(K) \ge 1$ and K is irreducible. Since K is nonnegative and ϕ is strictly increasing, $F = \phi \circ K$ is a monotone map. Therefore to show $x_n(z) \to \infty$, it suffices to show this when $x_0 = 0$. Since $x_1 = Fx_0 = F0 = 1 \ge 0$, applying F^{n-1} we obtain $x_n \ge x_{n-1}$ for all n. Since $\{x_n\}_{n=0}^\infty$ is an increasing sequence in $\mathbb{R}^Z_+$, if it is bounded, then it converges to some $x^* \in \mathbb{R}^Z_+$. By continuity, x^* is a fixed point of F, which is a contradiction. Therefore $\{x_n\}_{n=0}^\infty$ is unbounded, so $x_n(\hat{z}) \to \infty$ for at least one $\hat{z} \in \mathsf{Z}$. Since by assumption K is irreducible, for each $(z, \hat{z}) \in \mathsf{Z}^2$, there exists $m \in \mathbb{N}$ such that $K^m(z, \hat{z}) > 0$. Therefore

$$x_{m+n}(z) \ge K^m(z, \hat{z})x_n(\hat{z}) \to \infty$$

as $n \to \infty$, so $x_n(z) \to \infty$ for all $z \in \mathsf{Z}$.

Step 2. If $r(K) < 1$, then F has a unique fixed point x^* in $\mathbb{R}^Z_+$. If we take $a \in [1, 1/r(K))$ and $b > 0$ as in Lemma A.3, then

$$1 \le x^* \ll (I - aK)^{-1}b1. \tag{A.6}$$

Take any fixed point $x^* \in \mathbb{R}_+^Z$ of F. Since $\phi(t) \geq 1$ for all $t \geq 0$, clearly $x^* \geq 1$. Since K is nonnegative and $ar(K) < 1$, the inverse $(I - aK)^{-1} = \sum_{k=0}^{\infty}(aK)^k$ exists and is nonnegative. Therefore

$$x^* = Fx^* \ll aKx^* + b1 \implies x^* \ll (I - aK)^{-1}b1,$$

which is (A.6).

The proof of existence and uniqueness uses a similar strategy to Borovička and Stachurski (2020). Clearly F is a monotone map. Using (A.5), it follows that F is convex if $\gamma \leq 1$ and concave if $\gamma \geq 1$. Define $u_0 = 0$ and $v_0 = (I - aK)^{-1}b1 \gg 0$. Then $Fu_0 = 1 \gg 0 = u_0$ and $Fv_0 = \phi(Kv_0) \ll aKv_0 + b1 = v_0$. Hence by Theorem 2.1.2 of Zhang (2013), which is based on Theorem 3.1 of Du (1990), F has a unique fixed point in $[u_0, v_0] = [0, v_0]$. Since by (A.6) any fixed point x^* must be in this interval, it follows that F has a unique fixed point in $\mathbb{R}_+^Z$.

Step 3. If $r(K) < 1$, then $\{x_n\}_{n=1}^{\infty}$ converges to x^*.

Let $a \in [1, 1/r(K))$, $b > 0$, and $v_0 \gg 0$ be as in the previous step. Since $Fx = \phi(Kx)$, we obtain

$$x_n = Fx_{n-1} = \phi(Kx_{n-1}) \ll aKx_{n-1} + b1.$$

Iterating, we obtain

$$\begin{aligned} x_n &\ll (aK)^n x_0 + \sum_{k=0}^{n-1}(aK)^k(b1) \\ &= (aK)^n x_0 + \sum_{k=0}^{\infty}(aK)^k(b1) - \sum_{k=n}^{\infty}(aK)^k(b1) \\ &= (aK)^n(x_0 - v_0) + v_0. \end{aligned}$$

Since $r(aK) = ar(K) < 1$, we have $(aK)^n(x_0 - v_0) \to 0$ as $n \to \infty$. Therefore $0 = u_0 \ll x_n \ll v_0$ for large enough n. Again by Theorem 2.1.2 of Zhang (2013), we have $x_n \to x^*$ as $n \to \infty$. □

We apply policy function iteration to prove Theorem 3.3. Let $\mathcal{C}$ be the space of candidate consumption functions as defined in Section 2. We further restrict the candidate space to satisfy asymptotic linearity:

$$\mathcal{C}_1 = \left\{ c \in \mathcal{C} \,\middle|\, (\forall z \in \mathsf{Z})\exists \bar{c}(z) = \lim_{a \to \infty} \frac{c(a,z)}{a} \in [0,1] \right\}. \tag{A.7}$$

The following proposition shows that the time iteration operator T defined in Section 2 maps the candidate space $\mathcal{C}_1$ into itself and also shows how the asymptotic MPCs of c and Tc are related.

Proposition A.5. *Let everything be as in Proposition A.2. Then $T\mathcal{C}_1 \subset \mathcal{C}_1$. For $c \in \mathcal{C}_1$, let $\bar{c}(z) = \lim_{a \to \infty} c(a,z)/a$ and $x(z) = \bar{c}(z)^{-\gamma} \in [1, \infty]$. Then*

$$\lim_{a \to \infty} \frac{Tc(a,z)}{a} = (Fx)(z)^{-1/\gamma}, \tag{A.8}$$

where F is as in (3.1)*.*

Proof. Immediate from Lemmas A.6, A.7, and A.8 below. ☐

Lemma A.6. *Let $f : (0, \infty) \to (0, \infty)$ be a positive measurable function such that $\lambda = \lim_{c\to\infty} f(c) \in [0, \infty)$ exists. If Assumptions 1 and 3 hold, then*

$$\lim_{c\to\infty} \frac{u'(f(c)c)}{u'(c)} = \lambda^{-\gamma}. \tag{A.9}$$

Proof. Suppose $\lambda > 0$. Take any numbers $\underline{\lambda}, \bar{\lambda}$ such that $0 < \underline{\lambda} < \lambda < \bar{\lambda}$. Since $f(c) \to \lambda$ as $c \to \infty$, there exists $\underline{c} > 0$ such that $f(c) \in [\underline{\lambda}, \bar{\lambda}]$ for $c \geq \underline{c}$. Since u' is strictly decreasing by Assumption 1, it follows that $u'(\underline{\lambda}c) \geq u'(f(c)c) \geq u'(\bar{\lambda}c)$ for $c \geq \underline{c}$. Dividing both sides by $u'(c)$, letting $c \to \infty$, and using Assumption 3, we obtain

$$\underline{\lambda}^{-\gamma} = \lim_{c\to\infty} \frac{u'(\underline{\lambda}c)}{u'(c)} \geq \limsup_{c\to\infty} \frac{u'(f(c)c)}{u'(c)} \geq \liminf_{c\to\infty} \frac{u'(f(c)c)}{u'(c)} \geq \lim_{c\to\infty} \frac{u'(\bar{\lambda}c)}{u'(c)} = \bar{\lambda}^{-\gamma}.$$

Letting $\underline{\lambda}, \bar{\lambda} \to \lambda$, we obtain (A.9).

If $\lambda = 0$, take any $\bar{\lambda} > 0$. By the same argument as above, we obtain

$$\liminf_{c\to\infty} \frac{u'(f(c)c)}{u'(c)} \geq \lim_{c\to\infty} \frac{u'(\bar{\lambda}c)}{u'(c)} = \bar{\lambda}^{-\gamma},$$

so letting $\bar{\lambda} \downarrow 0$, we obtain (A.9) (and both sides are ∞). ☐

Lemma A.7. *Suppose Assumptions 1, 2(1), 3, and 4 hold. Then $T\mathcal{C}_1 \subset \mathcal{C}_1$.*

Proof. Let $c \in \mathcal{C}_1$ and $\bar{c}(z) = \lim_{a\to\infty} c(a, z)/a \in [0, 1]$. Since $0 < Tc(a, z) \leq a$, it suffices to show that $\lim_{a\to\infty} Tc(a, z)/a$ exists for all $z \in \mathsf{Z}$.

For $\alpha \in [0, 1]$, define

$$g_c(\alpha, a, z) = \frac{u'(\alpha a)}{u'(a)} - \max\left\{ \mathrm{E}_z\, \hat{\beta}\hat{R} \frac{u'(c(\hat{R}(1-\alpha)a + \hat{Y}, \hat{Z}))}{u'(a)}, 1 \right\}. \tag{A.10}$$

By Assumption 1, g_c is continuous and strictly decreasing in $\alpha \in (0, 1]$ with $g_c(0, a, z) = \infty$ and $g_c(1, a, z) \leq 0$. By the intermediate value theorem, for each (a, z), there exists a unique $\alpha \in (0, 1]$ such that $g_c(\alpha, a, z) = 0$. By the definition of the time iteration operator T, we have $g_c(\xi/a, a, z) = 0$, where $\xi = Tc(a, z)$. Therefore $\alpha = Tc(a, z)/a$.

If $\hat{\beta}\hat{R} = 0$ almost surely conditional on $Z = z$, then (A.10) becomes

$$g_c(\alpha, a, z) = \frac{u'(\alpha a)}{u'(a)} - 1.$$

Since $\alpha = Tc(a, z)/a$ solves $g_c(\alpha, a, z) = 0$, it must be $Tc(a, z)/a = \alpha = 1$. Therefore in particular $\lim_{a\to\infty} Tc(a, z)/a = 1$ exists. Below, assume $\hat{\beta}\hat{R} > 0$ with positive probability conditional on $Z = z$.

Take any accumulation point α of $Tc(a, z)/a \in [0, 1]$ as $a \to \infty$, which always exists because $0 < Tc(a, z)/a \leq 1$. Then we can take an increasing sequence $\{a_n\}$ such that $\alpha = \lim_{n\to\infty} Tc(a_n, z)/a_n$. Define $\alpha_n = Tc(a_n, z)/a_n \in (0, 1]$ and

$$\lambda_n = \frac{c(\hat{R}(1-\alpha_n)a_n + \hat{Y}, \hat{Z})}{a_n} > 0. \tag{A.11}$$

By the definitions of α_n and λ_n, we have

$$\begin{aligned} 0 = g_c(\alpha_n, a_n, z) &= \frac{u'(\alpha_n a_n)}{u'(a_n)} - \max\left\{\mathrm{E}_z\, \hat{\beta}\hat{R}\frac{u'(\lambda_n a_n)}{u'(a_n)}, 1\right\} \\ \implies \frac{u'(\alpha_n a_n)}{u'(a_n)} &= \max\left\{\mathrm{E}_z\, \hat{\beta}\hat{R}\frac{u'(\lambda_n a_n)}{u'(a_n)}, 1\right\}. \end{aligned} \tag{A.12}$$

Let us show that

$$\lim_{n\to\infty} \lambda_n = \bar{c}(\hat{Z})\hat{R}(1-\alpha). \tag{A.13}$$

To see this, if $\alpha < 1$ and $\hat{R} > 0$, then since $\hat{R}(1-\alpha_n)a_n \to \hat{R}(1-\alpha)\cdot\infty = \infty$, by the definition of $\bar{c}$ we have

$$\lambda_n = \frac{c(\hat{R}(1-\alpha_n)a_n + \hat{Y}, \hat{Z})}{\hat{R}(1-\alpha_n)a_n + \hat{Y}}\left(\hat{R}(1-\alpha_n) + \frac{\hat{Y}}{a_n}\right) \to \bar{c}(\hat{Z})\hat{R}(1-\alpha),$$

which is (A.13). If $\alpha = 1$ or $\hat{R} = 0$, then since $c(a,z) \le a$, we have

$$\begin{aligned} \lambda_n &= \frac{c(\hat{R}(1-\alpha_n)a_n + \hat{Y}, \hat{Z})}{\hat{R}(1-\alpha_n)a_n + \hat{Y}}\left(\hat{R}(1-\alpha_n) + \frac{\hat{Y}}{a_n}\right) \\ &\le \hat{R}(1-\alpha_n) + \frac{\hat{Y}}{a_n} \to \hat{R}(1-\alpha) = 0, \end{aligned}$$

so again (A.13) holds.

There are two cases to consider. Recall that we interpret $0\cdot\infty = 0$ and $\beta R^{1-\gamma} = (\beta R)R^{-\gamma}$, so $\beta R^{1-\gamma} = 0$ whenever $\beta = 0$ or $R = 0$.

Case 1: $\mathrm{E}_z\, \hat{\beta}\hat{R}^{1-\gamma}\bar{c}(\hat{Z})^{-\gamma} = \infty$. In this case letting $n \to \infty$ in (A.12) and applying Lemma A.6, Fatou's Lemma, and (A.13), we obtain

$$\begin{aligned} \alpha^{-\gamma} = \lim_{n\to\infty} \frac{u'(\alpha_n a_n)}{u'(a_n)} &= \lim_{n\to\infty} \max\left\{\mathrm{E}_z\, \hat{\beta}\hat{R}\frac{u'(\lambda_n a_n)}{u'(a_n)}, 1\right\} \\ &\ge \lim_{n\to\infty} \mathrm{E}_z\, \hat{\beta}\hat{R}\frac{u'(\lambda_n a_n)}{u'(a_n)} \\ &\ge \mathrm{E}_z \lim_{n\to\infty} \hat{\beta}\hat{R}\frac{u'(\lambda_n a_n)}{u'(a_n)} \\ &= \mathrm{E}_z\, \hat{\beta}\hat{R}[\bar{c}(\hat{Z})\hat{R}(1-\alpha)]^{-\gamma} \\ &= \mathrm{E}_z\, \hat{\beta}\hat{R}^{1-\gamma}[\bar{c}(\hat{Z})(1-\alpha)]^{-\gamma} \\ &= \infty\cdot(1-\alpha)^{-\gamma} = \infty. \end{aligned} \tag{A.14}$$

Therefore it must be $\alpha = 0$. Since α is any accumulation point of $Tc(a,z)/a$ as $a \to \infty$, it follows that $\lim_{a\to\infty} Tc(a,z)/a = 0$ exists.

Case 2: $\mathrm{E}_z\, \hat{\beta}\hat{R}^{1-\gamma}\bar{c}(\hat{Z})^{-\gamma} < \infty$. Because $\hat{\beta}\hat{R} > 0$ with positive probability, we have $\mathrm{E}_z\, \hat{\beta}\hat{R}^{1-\gamma}\bar{c}(\hat{Z})^{-\gamma} \in (0,\infty)$. By the same argument as above, we obtain

$$\begin{aligned} &\alpha^{-\gamma} \ge \mathrm{E}_z\, \hat{\beta}\hat{R}^{1-\gamma}[\bar{c}(\hat{Z})(1-\alpha)]^{-\gamma} \\ \implies &\alpha \le \frac{1}{1 + \left(\mathrm{E}_z\, \hat{\beta}\hat{R}^{1-\gamma}\bar{c}(\hat{Z})^{-\gamma}\right)^{1/\gamma}} < 1. \end{aligned} \tag{A.15}$$

Let us show that in (A.14), the limit $n \to \infty$ and the expectation E_z can be interchanged. Noting that $\hat{\beta}, \hat{R}$ depend only on $\hat{Z}$ and IID innovations, we have

$$\mathrm{E}_z \, \hat{\beta}\hat{R}\frac{u'(\lambda_n a_n)}{u'(a_n)} = \sum_{\hat{z}=1}^{Z} P(z,\hat{z}) \, \mathrm{E}_{\hat{z}} \, \hat{\beta}\hat{R}\frac{u'(\lambda_n a_n)}{u'(a_n)}. \tag{A.16}$$

If $\bar{c}(z) = 0$ for some z, then for $\mathrm{E}_z \, \hat{\beta}\hat{R}^{1-\gamma}\bar{c}(\hat{Z})^{-\gamma} < \infty$ to be true, it is necessary that $P(z,\hat{z})\hat{\beta}\hat{R} = 0$ whenever $\bar{c}(\hat{z}) = 0$. In this case the terms in (A.16) corresponding to those $\hat{z}$ are zero and can be ignored. Similarly, if $\hat{R} = R(\hat{z},\hat{\zeta}) = 0$ for some $\hat{z}$ and $\hat{\zeta}$, then $\hat{\beta}\hat{R}u'(\lambda_n a_n)/u'(a_n) = 0$ and we can ignore those terms in (A.16).

Therefore without loss of generality assume $\bar{c}(z) > 0$ for all z and $\hat{R} > 0$. Then by Assumption 4, we have $\hat{R} \geq \delta > 0$. By the definition of λ_n and the monotonicity of consumption functions established in Ma et al. (2020), it follows from the definition of λ_n in (A.11) that

$$\lambda_n \geq \frac{c(\delta(1-\alpha_n)a_n, \hat{Z})}{a_n} \to \bar{c}(\hat{Z})\delta(1-\alpha).$$

Since $\bar{c}(z) > 0$ for all z, $\alpha < 1$ by (A.15), and Z is a finite set, for any

$$\underline{\lambda} \in \left(0, \min_{z \in \mathsf{Z}} \bar{c}(z)\delta(1-\alpha)\right),$$

there exists N such that $\lambda_n \geq \underline{\lambda}$ for all $n \geq N$ and $\hat{Z} \in \mathsf{Z}$. Then by Assumptions 1 and 3, for $n \geq N$ we have

$$\frac{u'(\lambda_n a_n)}{u'(a_n)} \leq \frac{u'(\underline{\lambda} a_n)}{u'(a_n)} \to \underline{\lambda}^{-\gamma}$$

as $n \to \infty$. Therefore for any $M \in (\underline{\lambda}^{-\gamma}, \infty)$, we have

$$\frac{u'(\lambda_n a_n)}{u'(a_n)} \leq M < \infty$$

for large enough n. Since by Assumption 2 we have $\mathrm{E}_z \, \beta R < \infty$ and hence $\mathrm{E}_z \, \hat{\beta}\hat{R} < \infty$, it follows from the Dominated Convergence Theorem and Lemma A.6 that

$$\begin{aligned}
\alpha^{-\gamma} &= \max\left\{\lim_{n\to\infty} \mathrm{E}_z \, \hat{\beta}\hat{R}\frac{u'(\lambda_n a_n)}{u'(a_n)}, 1\right\} \\
&= \max\left\{\mathrm{E}_z \, \hat{\beta}\hat{R} \lim_{n\to\infty} \frac{u'(\lambda_n a_n)}{u'(a_n)}, 1\right\} \\
&= \max\left\{\mathrm{E}_z \, \hat{\beta}\hat{R}^{1-\gamma}[\bar{c}(\hat{Z})(1-\alpha)]^{-\gamma}, 1\right\}. \qquad (A.17)
\end{aligned}$$

Because the left-hand side is strictly decreasing in α and the right-hand side is weakly increasing in α, the number α that solves this equation is unique.

Since α is any accumulation point of $Tc(a,z)/a \in [0,1]$ as $a \to \infty$, it follows that $\lim_{a\to\infty} Tc(a,z)/a$ exists. $\square$

Lemma A.8. *Suppose Assumptions 1–4 hold. For $c \in \mathcal{C}_1$, let $\bar{c}(z) = \lim_{a\to\infty} c(a,z)/a$ and $x(z) = \bar{c}(z)^{-\gamma} \in [1,\infty]$. Then* (A.8) *holds.*

Proof. If $\hat{\beta}\hat{R} = 0$ almost surely conditional on $Z = z$, then by the proof of Lemma A.7, we have $\lim_{a\to\infty} Tc(a,z)/a = 1$. In this case by convention the z-th row of the matrix PD is zero, so

$$(Fx)(z) = \left(1 + (PDx)(z)^{1/\gamma}\right)^{\gamma} = 1$$

by (3.1). Then $\lim_{a\to\infty} Tc(a,z)/a = 1 = (Fx)(z)^{-1/\gamma}$ as desired. Below, assume $\hat{\beta}\hat{R} > 0$ with positive probability conditional on $Z = z$. There are two cases to consider.

Case 1: $\mathbf{E}_z\, \hat{\boldsymbol{\beta}}\hat{\boldsymbol{R}}^{1-\gamma}\bar{\boldsymbol{c}}(\hat{\boldsymbol{Z}})^{-\gamma} = \infty$. By the proof of Lemma A.7, we have $\lim_{a\to\infty} Tc(a,z)/a = 0$. Since

$$(Fx)(z) = \left(1 + (PDx)(z)^{1/\gamma}\right)^{\gamma} = \left(1 + \left(\mathrm{E}_z\, \hat{\beta}\hat{R}^{1-\gamma}\bar{c}(\hat{Z})^{-\gamma}\right)^{1/\gamma}\right)^{\gamma} = \infty$$

by the definition of x and F in (3.1), we obtain

$$\lim_{a\to\infty} Tc(a,z)/a = 0 = (Fx)(z)^{-1/\gamma}$$

as desired.

Case 2: $\mathbf{E}_z\, \hat{\boldsymbol{\beta}}\hat{\boldsymbol{R}}^{1-\gamma}\bar{\boldsymbol{c}}(\hat{\boldsymbol{Z}})^{-\gamma} < \infty$. By the proof of Lemma A.7, we have $\lim_{a\to\infty} Tc(a,z)/a = \alpha$, where α solves (A.17). If $\alpha = 0$, then

$$\infty = \max\left\{\mathrm{E}_z\, \hat{\beta}\hat{R}^{1-\gamma}\bar{c}(\hat{Z})^{-\gamma}, 1\right\} < \infty,$$

which is a contradiction. If $\alpha = 1$, then

$$1 = \max\left\{\mathrm{E}_z\, \hat{\beta}\hat{R}^{1-\gamma}[\bar{c}(\hat{Z})0]^{-\gamma}, 1\right\} = \infty,$$

which is a contradiction. Therefore $\alpha \in (0,1)$, so (A.17) implies

$$\begin{aligned} \alpha^{-\gamma} &= \mathrm{E}_z\, \hat{\beta}\hat{R}^{1-\gamma}[\bar{c}(\hat{Z})(1-\alpha)]^{-\gamma} \\ \iff \alpha &= \frac{1}{1 + \left(\mathrm{E}_z\, \hat{\beta}\hat{R}^{1-\gamma}\bar{c}(\hat{Z})^{-\gamma}\right)^{1/\gamma}} = (Fx)(z)^{-1/\gamma}. \end{aligned}$$

□

With all the above preparations, we can prove Theorem 3.3.

Proof of Theoreom 3.3. Lemma B.4 of Ma et al. (2020) shows that $T : \mathcal{C} \to \mathcal{C}$ is order preserving, that is, $c_1 \leq c_2$ implies $Tc_1 \leq Tc_2$. Define the sequence $\{c_n\} \subset \mathcal{C}$ by $c_0(a,z) = a$ and $c_n = Tc_{n-1}$ for all $n \geq 1$. Since $c_0(a,z)/a = 1$, in particular we have $c_0 \in \mathcal{C}_1$, where $\mathcal{C}_1$ is as in (A.7). Therefore by Proposition A.5, we have $c_n \in \mathcal{C}_1$ for all n, so $\bar{c}_n(z) = \lim_{a\to\infty} c_n(a,z)/a \in [0,1]$ exists. Since $Tc(a,z) \leq a$ for any $c \in \mathcal{C}$, in particular $c_1(a,z) = Tc_0(a,z) \leq a = c_0(a,z)$, so by induction $c_{n+1} \leq c_n$ for all n. Define $c(a,z) = \lim_{n\to\infty} c_n(a,z)$, which exists because $\{c_n\}$ is monotonically decreasing and $c_n \geq 0$. Then by Theorem 2.2 of Ma et al. (2020), this c is the unique fixed point of T and also the unique solution to the income fluctuation problem (2.1). Since $0 \leq c \leq c_n$ point-wise, by Proposition A.5 we have

$$0 \leq \limsup_{a\to\infty} \frac{c(a,z)}{a} \leq \limsup_{a\to\infty} \frac{c_n(a,z)}{a} = x_n(z)^{-1/\gamma}, \tag{A.18}$$

where $\{x_n\}_{n=1}^{\infty} \subset \mathbb{R}_+^Z$ is defined by (A.4) with $x_0 = 1$.

Case 1: $r(PD) \geq 1$ and PD is irreducible. By Proposition A.2 we have $x_n(z) \to \infty$ for all $z \in \mathsf{Z}$. Letting $n \to \infty$ in (A.18), we obtain

$$\lim_{a \to \infty} \frac{c(a,z)}{a} = 0.$$

Case 2: $r(PD) < 1$. By Proposition A.2 we have $x_n(z) \to x^*(z)$, where x^* is the unique fixed point of F in (3.1). Letting $n \to \infty$ in (A.18), we obtain

$$0 \leq \liminf_{a \to \infty} \frac{c(a,z)}{a} \leq \limsup_{a \to \infty} \frac{c(a,z)}{a} \leq x^*(z)^{-1/\gamma},$$

which is (3.2). If $\liminf_{a \to \infty} c(a,z)/a > 0$ for all z, since Z is a finite set we can take $\epsilon > 0$ such that $\liminf_{a \to \infty} c(a,z)/a > \epsilon > 0$. Define $c_0(a,z) = \min\{c(a,z), \epsilon a\}$. Then $c \geq c_0$ and $c_0 \in \mathcal{C}_1$, so by iteration $c \geq c_n := T^n c_0$ for all n. By Proposition A.5, we have

$$\liminf_{a \to \infty} \frac{c(a,z)}{a} \geq \lim_{a \to \infty} \frac{c_n(a,z)}{a} = x_n(z)^{-1/\gamma},$$

where $\{x_n\} \subset \mathbb{R}_+^{\mathsf{Z}}$ is defined by $x_0(z) = \epsilon^{-\gamma} < \infty$ and iterating (A.4). By Proposition A.2, we have $x_n \to x^*$ as $n \to \infty$, so

$$x^*(z)^{-1/\gamma} \geq \limsup_{a \to \infty} \frac{c(a,z)}{a} \geq \liminf_{a \to \infty} \frac{c(a,z)}{a} \geq x^*(z)^{-1/\gamma}.$$

Therefore

$$\lim_{a \to \infty} \frac{c(a,z)}{a} = x^*(z)^{-1/\gamma}. \quad \square$$

We need the following Lemma to prove Proposition 3.4.

Lemma A.9. *Let everything be as in Proposition A.2 and $r(PD) < 1$. If there exist numbers $\{\epsilon(z)\}_{z \in \mathsf{Z}} \subset (0,1]$ such that*

$$u'(\epsilon(z)a) \geq \mathrm{E}_z\, \hat{\beta}\hat{R}u'(\hat{R}\epsilon(\hat{Z})(1-\epsilon(z))a) \tag{A.19}$$

for all $a > 0$ and $z \in \mathsf{Z}$, then the limit (3.3) *holds.*

Proof. Define the candidate consumption function by $c_0(a,z) = \epsilon(z)a$ and let $c_1 = Tc_0$. If we can show $c_1 \geq c_0$ point-wise, then $c_n := T^n c_0 \uparrow c$, and we obtain (3.3) by the same argument as in the proof of Theorem 3.3. To prove $c_1 \geq c_0$ point-wise, suppose on the contrary that $\xi := Tc_0(a,z) < c_0(a,z)$ for some (a,z). Since u' is strictly decreasing and $c_0(a,z) \leq a$, it follows from (2.2) that

$$\begin{aligned} u'(a) \leq u'(\epsilon(z)a) &= u'(c_0(a,z)) < u'(\xi) \\ &= \max\left\{\mathrm{E}_z\, \hat{\beta}\hat{R}u'(c_0(\hat{R}(a-\xi)+\hat{Y},\hat{Z})), u'(a)\right\}. \end{aligned}$$

Therefore it must be $u'(a) < \mathrm{E}_z\, \hat{\beta}\hat{R}u'(c_0(\hat{R}(a-\xi)+\hat{Y},\hat{Z}))$, and using the fact that u' is strictly decreasing and $\hat{Y} \geq 0$ and $\xi < c_0(a,z) = \epsilon(z)a$, we obtain

$$\begin{aligned} u'(\epsilon(z)a) < u'(\xi) &= \mathrm{E}_z\, \hat{\beta}\hat{R}u'(c_0(\hat{R}(a-\xi)+\hat{Y},\hat{Z})) \\ &= \mathrm{E}_z\, \hat{\beta}\hat{R}u'(\epsilon(\hat{Z})(\hat{R}(a-\xi)+\hat{Y})) \\ &\leq \mathrm{E}_z\, \hat{\beta}\hat{R}u'(\hat{R}\epsilon(\hat{Z})(1-\epsilon(z))a), \end{aligned}$$

which contradicts (A.19). $\square$

Proof of Proposition 3.4. We apply Lemma A.9.

Case 1: u is CRRA. Let $x^* \in \mathbb{R}^Z_{++}$ be the unique fixed point of F in (3.1), which necessarily satisfies $x^*(z) \geq 1$ for all z. Then

$$\begin{aligned} x^*(z) &= \left(1 + (PDx^*)(z)^{1/\gamma}\right)^\gamma \\ \iff x^*(z)^{1/\gamma} &= 1 + \left(\mathrm{E}_z\, \hat{\beta}\hat{R}^{1-\gamma} x^*(\hat{Z})\right)^{1/\gamma} \\ \iff x^*(z) &= \mathrm{E}_z\, \hat{\beta}\hat{R} x^*(\hat{Z}) \left(\hat{R}(1 - x^*(z)^{-1/\gamma})\right)^{-\gamma}. \end{aligned}$$

Therefore setting $\epsilon(z) = x^*(z)^{-1/\gamma} \in (0, 1]$, we obtain

$$\epsilon(z)^{-\gamma} = \mathrm{E}_z\, \hat{\beta}\hat{R} \left(\hat{R}\epsilon(\hat{Z})(1 - \epsilon(z))\right)^{-\gamma},$$

which implies (A.19) (with equality) if $u'(c) = c^{-\gamma}$.

Case 2: u is BRRA. We aim to show (A.19) for $\epsilon(z) \equiv \epsilon$ (constant) with sufficiently small $\epsilon > 0$. Fix $a, \epsilon > 0$ and let $x = \epsilon a$ and $y = \hat{R}\epsilon(1 - \epsilon)a$. Let $\gamma(c) = -cu''(c)/u'(c)$ be the local relative risk aversion coefficient of u. If $y \leq x$, then

$$\begin{aligned} \log \frac{u'(y)}{u'(x)} &= \int_x^y (\log u'(c))' \,\mathrm{d}c = \int_x^y \frac{u''(c)}{u'(c)} \,\mathrm{d}c \\ &= \int_y^x \frac{\gamma(c)}{c} \,\mathrm{d}c \leq \int_y^x \frac{\bar{\gamma}}{c} \,\mathrm{d}c = -\bar{\gamma} \log \frac{y}{x} \\ \implies u'(y)/u'(x) &\leq (y/x)^{-\bar{\gamma}}. \end{aligned}$$

Similarly, if $y \geq x$, then $u'(y)/u'(x) \leq (y/x)^{-\underline{\gamma}}$. Since $x = \epsilon a$ and $y = \hat{R}\epsilon(1-\epsilon)a$, we obtain

$$\begin{aligned} \frac{\mathrm{E}_z\, \hat{\beta}\hat{R}u'(\hat{R}\epsilon(1-\epsilon)a)}{u'(\epsilon a)} &\leq \mathrm{E}_z\, \hat{\beta}\hat{R} \max\left\{[\hat{R}(1-\epsilon)]^{-\underline{\gamma}}, [\hat{R}(1-\epsilon)]^{-\bar{\gamma}}\right\} \\ &\to \mathrm{E}_z\, \hat{\beta}\hat{R} \max\left\{\hat{R}^{-\underline{\gamma}}, \hat{R}^{-\bar{\gamma}}\right\} < 1 \end{aligned}$$

as $\epsilon \downarrow 0$ by (3.6). Therefore (A.19) holds for $\epsilon(z) \equiv \epsilon$ (constant) with sufficiently small $\epsilon > 0$. Furthermore, because $\underline{\gamma} \leq \gamma \leq \bar{\gamma}$ and the function $\gamma \mapsto R^{-\gamma}$ is convex, we have

$$\mathrm{E}_z\, \hat{\beta}\hat{R}^{1-\gamma} \leq \mathrm{E}_z\, \hat{\beta}\hat{R} \max\left\{\hat{R}^{-\underline{\gamma}}, \hat{R}^{-\bar{\gamma}}\right\} < 1$$

for all z, so the row sums of PD are less than 1. Hence $r(PD) < 1$. □

Proof of Proposition 3.6. Since $x_n(z) \geq 1$ monotonically converges to $x^*(z)$, we have $\bar{c}(z) = x^*(z)^{-1/\gamma} = 1$ if and only if $x_n(z) = 1$ for all n. If $\hat{\beta}\hat{R} = 0$ almost surely conditional on $Z = z$, then by convention $\hat{\beta}\hat{R}^{1-\gamma} = (\hat{\beta}\hat{R})\hat{R}^{-\gamma} = 0$ almost surely, so $\mathrm{E}_z\, \hat{\beta}\hat{R}^{1-\gamma}x_n(\hat{Z}) = 0$. Therefore $x_n(z) = 1$ for all n by (A.3) and (A.4). Conversely, if $x_n(z) = 1$ for all n, then by (A.3) and (A.4) we have $\mathrm{E}_z\, \hat{\beta}\hat{R}^{1-\gamma}x_n(\hat{Z}) = 0$. Since $x_n(\hat{Z}) \geq 1$ for all $\hat{Z}$, it must be $\hat{\beta}\hat{R}^{1-\gamma} = 0$ and hence $\hat{\beta}\hat{R} = 0$ almost surely conditional on $Z = z$.

Define the block diagonal matrix $\tilde{K} = \operatorname{diag}(K_1, \dots, K_J)$ and the sequence $\{\tilde{x}_n\}_{n=0}^{\infty} \subset [0,\infty]^Z$ by $\tilde{x}_0 = 1$ and (A.4), where K is replaced by $\tilde{K}$ in the definition of F in (3.1). Since $K \geq \tilde{K} \geq 0$, clearly $x_n \geq \tilde{x}_n \geq 1$ for all n. Since by definition $\tilde{K}$ is block diagonal with each diagonal block irreducible, by Lemma A.4 we have $\tilde{x}_n(z) \to \infty$ as $n \to \infty$ if and only if there exists j such that $z \in \mathsf{Z}_j$ and $r(K_j) \geq 1$. Replacing the vector 1 in (3.1) by 0 and iterating, we obtain

$$x_{m+n} \geq K^m x_n \geq K^m \tilde{x}_n.$$

Therefore if there exist j, $\hat{z} \in \mathsf{Z}_j$ and $m \in \mathbb{N}$ such that $K^m(z,\hat{z}) > 0$ and $r(K_j) \geq 1$, then

$$x_{m+n}(z) \geq K^m(z,\hat{z})\tilde{x}_n(\hat{z}) \to \infty$$

as $n \to \infty$, so $x^*(z) = \infty$.

Finally, assume that for all j, we have either $r(K_j) < 1$ or $K^m(z,\hat{z}) = 0$ for all $\hat{z} \in \mathsf{Z}_j$ and $m \in \mathbb{N}$. For any $\hat{z}$ such that $K^m(z,\hat{z}) = 0$ for all m, by (3.1) the value of $x_n(z)$ is unaffected by all previous $x_k(\hat{z})$ for $k < n$. Therefore for the purpose of computing $x_n(z)$, we may drop all rows and columns of K corresponding to such $\hat{z}$. The resulting matrix has block diagonal elements K_j with $r(K_j) < 1$ only, so this matrix has spectral radius less than 1. Therefore by Lemma A.4, we have $x_n(z) \to x^*(z) < \infty$ as $n \to \infty$. □

Proof of Proposition 4.1. Stachurski and Toda (2019) show that it must be $\beta R < 1$ in the stationary equilibrium. Since $R > 1$ by assumption, we obtain $\beta R^{1-\gamma} = (\beta R)R^{-\gamma} < 1$. By Example 3.2, the asymptotic MPC is $\bar{c} = 1 - (\beta R^{1-\gamma})^{1/\gamma} \in (0,1)$. Therefore using (4.2), we obtain

$$\begin{aligned} s = 1 - \frac{\bar{c}}{(R-1)(1-\bar{c})} < 0 &\iff (R-1)(1-\bar{c}) < \bar{c} \\ &\iff (R-1)(\beta R^{1-\gamma})^{1/\gamma} < 1 - (\beta R^{1-\gamma})^{1/\gamma} \\ &\iff (\beta R)^{1/\gamma} < 1, \end{aligned}$$

which holds because $\beta R < 1$. □

Proof of Proposition 4.2. Since by assumption $\beta R^{1-\gamma} < 1$, by Example 3.2 the asymptotic MPC is $\bar{c} = 1 - (\mathrm{E}\,\beta R^{1-\gamma})^{1/\gamma} \in (0,1)$. Therefore using (4.2), the asymptotic saving rate evaluated at $\mathrm{E}\,R > 1$ is

$$\begin{aligned} s = 1 - \frac{\bar{c}}{(\mathrm{E}\,R-1)(1-\bar{c})} \leq 0 &\iff (\mathrm{E}\,R-1)(1-\bar{c}) \leq \bar{c} \\ &\iff \mathrm{E}\,R(1-\bar{c}) \leq 1. \end{aligned}$$

Since $\mathrm{E}\,R(1-\bar{c})$ is the expected growth rate of wealth for infinitely wealthy agents, if the wealth distribution is unbounded and $\mathrm{E}\,R(1-\bar{c}) > 1$, then wealth will grow at the top, which violates stationarity. Therefore in a stationary equilibrium, it must be $s \leq 0$. □

B Discretizing the GARCH$(1,1)$ process

In this appendix we explain how to discretize the GARCH$(1,1)$ process (4.3).

B.1 Constructing the grid

Let $v_t = \sigma_t^2$. Using the properties of the GARCH process, it is known that the expected conditional variance is

$$\mathrm{E}[v_t] = \frac{\omega}{1-\alpha-\rho}.$$

Therefore it is natural to take an evenly-spaced grid $\{\bar{\epsilon}_n\}_{n=1}^{N_\epsilon}$, where N_ϵ is an odd number and the largest grid point $\bar{\epsilon} := \bar{\epsilon}_{N_\epsilon}$ is some multiple of $\sqrt{\frac{\omega}{1-\alpha-\rho}}$. Because the conditional variance of the GARCH process can be quite large, it is also natural to choose an exponential grid (as discussed in Appendix B.3) $\{\bar{v}_n\}_{n=1}^{N_v}$ such that the median point of the grid is $\frac{\omega}{1-\alpha-\rho}$.

To determine the end points, let $\underline{v} = v_1$ and $\bar{v} = v_{N_v}$. In principle $v_t = \sigma_t^2$ can be arbitrarily close to ω, so we set $\underline{v} = \omega$. For $v_t = \sigma_t^2$ to remain in the grid when ϵ_{t-1} and σ_{t-1}^2 are at their maximum value, we need

$$\bar{v} \geq \omega + \alpha\bar{\epsilon}^2 + \rho\bar{v} \iff \bar{v} \geq \frac{\omega + \alpha\bar{\epsilon}^2}{1-\rho}.$$

Setting $\bar{\epsilon} = k\sqrt{\frac{\omega}{1-\alpha-\rho}}$ for some $k > 0$, we obtain

$$\bar{v} \geq \frac{1-\rho+(k^2-1)\alpha}{1-\rho}\frac{\omega}{1-\alpha-\rho}. \tag{B.1}$$

In order to be able to match up to the second moments of ϵ_t when $v_t = \bar{v}$, it is necessary and sufficient that

$$\bar{\epsilon} \geq \sqrt{\bar{v}} \iff \bar{v} \leq \bar{\epsilon}^2 = \frac{k^2\omega}{1-\alpha-\rho}. \tag{B.2}$$

We have the following result.

Proposition B.1. *Consider the GARCH$(1,1)$ process (4.3) with $\alpha+\rho<1$ and set $v_t = \sigma_t^2$. Let $N_\epsilon \geq 3$ be an odd number and $N_v \geq 2$. Then there exists a discretization such that*

1. *$\{\bar{\epsilon}_n\}_{n=1}^{N_\epsilon}$ is evenly spaced and centered around 0,*
2. *$\{\bar{v}_n\}_{n=1}^{N_v}$ is exponentially spaced with minimum point ω and median point $\frac{\omega}{1-\alpha-\rho}$, and*
3. *the conditional mean of v_t and the conditional mean and variance of ϵ_t are exact.*

Proof. Set $\bar{\epsilon} = \bar{\epsilon}_{N_\epsilon} = k\sqrt{\frac{\omega}{1-\alpha-\rho}}$ for some $k > 0$. Combining (B.1) and (B.2), we obtain

$$\begin{aligned}
&\frac{1-\rho+(k^2-1)\alpha}{1-\rho}\frac{\omega}{1-\alpha-\rho} \leq \frac{k^2\omega}{1-\alpha-\rho} \\
\iff &1-\rho+(k^2-1)\alpha \leq (1-\rho)k^2 \\
\iff &(k^2-1)(\alpha+\rho-1) \leq 0,
\end{aligned}$$

which always holds if $k \geq 1$ because $\alpha + \rho < 1$. Set

$$(\underline{v}, \bar{v}) = \left(\omega, \frac{1 - \rho + (k^2 - 1)\alpha}{1 - \rho} \frac{\omega}{1 - \alpha - \rho}\right)$$

for some $k \geq 1$ and construct an exponential grid $\{v_n\}_{n=1}^{N_v}$ on $[\underline{v}, \bar{v}]$ with median point $\frac{\omega}{1-\alpha-\rho}$ as explained in Appendix B.3. For the exponential grid to be well-defined, we need

$$\begin{aligned}
&\frac{\omega}{1-\alpha-\rho} < \frac{\underline{v} + \bar{v}}{2} = \frac{1}{2}\left(\omega + \frac{1 - \rho + (k^2 - 1)\alpha}{1 - \rho} \frac{\omega}{1 - \alpha - \rho}\right) \\
\iff &2 < 1 - \alpha - \rho + \frac{1 - \rho + (k^2 - 1)\alpha}{1 - \rho} \\
\iff &(1 - \rho)(1 + \alpha + \rho) < 1 - \rho + (k^2 - 1)\alpha \\
\iff &(1 - \rho)(\alpha + \rho) < (k^2 - 1)\alpha, \qquad \text{(B.3)}
\end{aligned}$$

which holds for large enough $k \geq 1$ because $\alpha > 0$. To make (B.3) true, for example we can set

$$k^2 - 1 = N_\epsilon(1 - \rho)(1 + \rho/\alpha) \iff k = \sqrt{1 + N_\epsilon(1 - \rho)(1 + \rho/\alpha)},$$

which satisfies $k \geq 1$.[15] In this case we have

$$\bar{v} = (1 + N_\epsilon(\alpha + \rho))\frac{\omega}{1 - \alpha - \rho}. \quad \square$$

Applying Proposition B.1 and its proof, we can construct the grid $\{\bar{v}_n\}_{n=1}^{N_v}$ and $\{\bar{\epsilon}_n\}_{n=1}^{N_\epsilon}$ as follows.

Constructing grid for GARCH.

1. Select the number of grid points $N_\epsilon \geq 3$ and $N_v \geq 2$ to discretize the return and variance states.
2. Set $\bar{\epsilon} = \sqrt{(1 + N_\epsilon(1 - \rho)(1 + \rho/\alpha))\frac{\omega}{1-\alpha-\rho}}$ and construct the evenly-spaced grid $\{\bar{\epsilon}_n\}_{n=1}^{N_\epsilon}$ on $[-\bar{\epsilon}, \bar{\epsilon}]$.
3. Set
$$(a, b, c) = \left(\omega, (1 + N_\epsilon(\alpha + \rho))\frac{\omega}{1 - \alpha - \rho}, \frac{\omega}{1 - \alpha - \rho}\right)$$
and construct the exponentially-spaced grid $\{\bar{v}_n\}_{n=1}^{N_v}$ on $[a, b]$ with median point c as in Appendix B.3.

B.2 Constructing transition probabilities

Having constructed the grid, it remains to construct transition probabilities. Let $\mathsf{Z} = \{1, \ldots, N_v\} \times \{1, \ldots, N_\epsilon\}$ be the state space. If $z = (m, n) \in \mathsf{Z}$, then the

[15] Setting $k \sim \sqrt{N_\epsilon}$ is advocated in Farmer and Toda (2017) based on the trapezoidal rule for quadrature.

current conditional variance and return are $(v, \epsilon) = (\bar{v}_m, \bar{\epsilon}_n)$. The next period's conditional variance is then

$$\hat{v} = \omega + \alpha \bar{\epsilon}_n^2 + \rho \bar{v}_m.$$

This $\hat{v}$ will in general not be a grid point. However, we can approximate the transition to $\hat{v}$ by assigning probabilities $1-\theta, \theta$ to the two points $\bar{v}_{m'}, \bar{v}_{m'+1}$ such that

$$\hat{v} = (1-\theta)\bar{v}_{m'} + \theta \bar{v}_{m'+1},$$

where m' is uniquely determined such that $\bar{v}_{m'} < \hat{v} \leq \hat{v}_{m'+1}$.

Because the distribution of $\hat{\epsilon}$ is $N(0, \hat{v})$, we can assign probabilities on the grid points $\{\bar{\epsilon}_n\}_{n=1}^{N_\epsilon}$ such that the mean and variance are exact. For this purpose, we can use the maximum entropy method of Tanaka and Toda (2013, 2015) and Farmer and Toda (2017). If $N_\epsilon = 3$, we can avoid optimizing because there is a closed-form solution as follows. Assign probabilities $p, 1-2p, p$ to points $-\bar{\epsilon}, 0, \bar{\epsilon}$ so that $\mathrm{E}[\epsilon] = 0$ and $\mathrm{Var}[\epsilon] = \hat{v}$. For this purpose, we can set

$$\hat{v} = 2p\bar{\epsilon}^2 \iff p = \frac{\hat{v}}{2\bar{\epsilon}^2},$$

which is always in $(0, 1/2)$ because $\hat{v} \leq \bar{v} < \bar{\epsilon}^2$.

B.3 Exponential grid

In many models, the state variable may become negative (e.g., asset holdings), which causes a problem for constructing an exponentially-spaced grid because we cannot take the logarithm of a negative number. Suppose we would like to construct an N-point exponential grid on a given interval (a, b). A natural idea to deal with such a case is as follows.

Constructing exponential grid.

1. Choose a shift parameter $s > -a$.
2. Construct an N-point evenly-spaced grid on $(\log(a+s), \log(b+s))$.
3. Take the exponential.
4. Subtract s.

The remaining question is how to choose the shift parameter s. Suppose we would like to specify the median grid point as $c \in (a, b)$. Since the median of the evenly-spaced grid on $(\log(a+s), \log(b+s))$ is $\frac{1}{2}(\log(a+s) + \log(b+s))$, we need to take $s > -a$ such that

$$\begin{aligned} c = \exp\left(\frac{1}{2}(\log(a+s) + \log(b+s))\right) - s \\ \iff c + s = \sqrt{(a+s)(b+s)} \\ \iff (c+s)^2 = (a+s)(b+s) \\ \iff c^2 + 2cs + s^2 = ab + (a+b)s + s^2 \\ \iff s = \frac{c^2 - ab}{a+b-2c}. \end{aligned}$$

Note that in this case

$$s + a = \frac{c^2 - ab}{a+b-2c} + a = \frac{(c-a)^2}{a+b-2c},$$

so $s + a$ is positive if and only if $c < \frac{a+b}{2}$. Therefore, for any $c \in \left(a, \frac{a+b}{2}\right)$, it is possible to construct an exponentially-spaced grid with end points (a, b) and median point c.