Optimal Timing of Decisions: A General Theory Based on Continuation Values[1]

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ABSTRACT. By extending a methodology dating back to Jovanovic (1982), we develop a comprehensive theory of optimal timing of decisions based on continuation value functions and operators that act on them. Rewards can be bounded or unbounded. One advantage of this approach over standard Bellman methods is that continuation value functions are smoother than value functions. Another is that, for a range of problems, the continuation value function exists in a lower dimensional space than the value function. We exploit these advantages to obtain a range of new results on optimality, optimal behavior and efficient computation.

Keywords: Continuation values, dynamic programming, optimal timing

1. INTRODUCTION

A large variety of decision making problems involve choosing when to act in the face of risk and uncertainty. Examples include deciding if or when to accept a job offer, exit or enter a market, default on a loan, bring a new product to market, exploit some new technology or business opportunity, or exercise a real or financial option. See, for example, McCall (1970), Jovanovic (1982), Hopenhayn (1992), Dixit and Pindyck (1994), Ericson and Pakes (1995), Peskir and Shiryaev (2006), Arellano (2008), Perla and Tonetti (2014), and Fajgelbaum et al. (2017).

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The most general and robust techniques for solving these kinds of problems revolve around dynamic programming. The standard machinery centers on the Bellman equation, which identifies current value in terms of a trade off between current rewards and the discounted value of future states. The Bellman equation is traditionally solved by framing the solution as a fixed point of the Bellman operator. Standard references include Bellman (1969) and Stokey et al. (1989). Applications to optimal timing can be found in Dixit and Pindyck (1994), Albuquerque and Hopenhayn (2004), Ljungqvist and Sargent (2012), and many other sources.

At the same time, economists have initiated development of an alternative method, based around operations on continuation value functions, that parallels the traditional method and yet differs in certain asymmetric ways described in detail below. The earliest technically sophisticated example is Jovanovic (1982). In the context of an incumbent firm's exit decision, he studies a contractive operator with a unique fixed point representing the value of staying in the industry for the current period and then behave optimally. Subsequent papers in a similar vein include Burdett and Vishwanath (1988), Gomes et al. (2001), Ljungqvist and Sargent (2008), Lise (2013), Dunne et al. (2013), Moscarini and Postel-Vinay (2013), and Menzio and Trachter (2015).

Most of these papers focus on continuation values as a function of the state rather than traditional value functions because they provide sharper economic intuition in some specific context. For example, in a job search setting, the continuation value—the value of rejecting the current offer—is the value of unemployment, and of direct interest for policy.

There are, however, deeper advantages of this methodology that are not generally recognized or understood. To clarify, recall that, for a given problem, the value function provides the value of optimally choosing to either act today or wait, given the current environment. The continuation value is the value associated with choosing to wait today and then reoptimize next period, again taking into account the current environment. One key asymmetry arising here is that, if one chooses to wait, then certain aspects of the current environment become irrelevant, and hence need not be considered as arguments to the continuation value function.

Consider, for example, a potential entrant to a market who must consider fixed costs of entry, the evolution of prices, productivity dynamics, and so on. Some aspects of the environment will be persistent, while others are transitory (e.g., in Fajgelbaum et al. (2017), prices and beliefs are persistent while fixed costs are transitory). All relevant state components must be included in the value function, whether persistent or transitory, since all affect the choice of whether to enter or wait today. On the other hand, purely transitory components do not affect continuation values, since, in that scenario, the decision to wait has already been made.

Such asymmetries place the continuation value function in a lower dimensional space than the value function whenever they exist. As we show, this greatly simplifies challenging problems associated with unbounded rewards, continuity and differentiability arguments, parametric monotonicity results, and so on.[2] On the computational side, reduction of the state space by one or more dimensions can radically increase computational speed.[3] For example, computation time falls from more than 7 days to less than 3 minutes in a standard job search model considered in section 5.1.

Another asymmetry between value functions and continuation value functions is that the latter are typically smoother. For example, in job search problems, the value function is usually kinked at the reservation wage, while the continuation

[2]One might imagine that this difference in dimensionality between the two approaches could, in some circumstances, be reversed, with the value function existing in a strictly lower dimensional space than the continuation value function. However, for the broad class of decision problems we consider, the dimensionality of the value function is always at least as large. Intuitively, the value function includes information form both the continuation value function and the exit reward function. The latter is a primitive, so we can reduce dimensionality (at least weakly) by not carrying information needed to calculate it across time periods.

[3]An alternative approach to reducing the cost of dimensionality is found in Rust (1997). While that paper concerns numerical methods and is based on value function iteration, the same idea could potentially be applied in continuation value function space.

value function is smooth. In this and other settings, the relative smoothness comes from taking expectations over stochastic transitions, since integration is a smoothing operation. Like lower dimensionality, increased smoothness helps on both the analytical and the computational side. On the computational side, smoother functions are easier to approximate. On the analytical side, greater smoothness lends itself to sharper results based on derivatives, as elaborated on below.

In this paper we undertake the first systematic study of optimal timing of decisions based around continuation value functions and the operators that act on them, heavily expoiting the advantages discussed above. As well as providing a general optimality theory, we obtain (i) conditions under which continuation values are (a) continous, (b) monotone, and (c) differentiable as functions of the economic environment, (ii) conditions under which parametric continuity holds (often required for proofs of existence of recursive equilibria in many-agent environments), and (iii) conditions under which threshold policies are (a) continuous, (b) monotone, and (c) differentiable. In the latter case we derive an expression for the derivative of the threshold relative to other aspects of the economic environment and show how it contributes to economic intuition.

The closest counterparts to these results in the existing literature are those concerning individual applications. For example, Jovanovic (1982) shows that the continuation value function associated with an incumbent firm's exit decision is monotone and continuous (theorem 1). Chatterjee and Rossi-Hansberg (2012) show that the continuation value function (the value of not using a project) is continuous and increasing in the average revenue (lemma 1, section 2.1). In a model of oil drilling investment, Kellogg (2014) provides sufficient conditions for the existence of a threshold policy (a reservation productivity at which the firm is indifferent between drilling a well or not), and conditions under which this policy is decreasing in the average oil price and increasing in the dayrate (conditions (i)–(v), section B). Our theory generalizes and extends these results in a unified framework. Some results, such as differentiability and related properties, are entirely new.

All of our theory is developed in a setting that accommodates both bounded rewards and the kinds of unbounded rewards routinely encountered in optimal timing problems.[4] This is achieved by building on the approach to unbounded rewards based on weighted supremum norms pioneered by Boyd (1990) and used in numerous other studies (see, e.g., Alvarez and Stokey (1998) and Le Van and Vailakis (2005)). The underlying idea is to introduce a weighted norm in a space of candidate functions and then establish the contraction property for the relevant operator under this norm. We apply this idea in continuation value function space.

In many applications of optimal timing, a subset of states are conditionally independent of the future states, so future transitions of the reward functions are defined on a space that is lower dimensional than the state space. To exploit this fact we use n-step Markov transitions to build weight functions. For mean-reverting state processes, this has the additional advantage that initial effects tend to die out over time, making the future transitions relatively flat. As a result, in the context of optimal timing, our assumptions on primitives are weaker than those found in existing work using weighted supremum norms.

An alternative line of research on unbounded dynamic programming uses local contractions on increasing sequences of compact subsets (see, e.g., Rincón-Zapatero and Rodríguez-Palmero (2003) or Martins-da Rocha and Vailakis (2010)). This idea exploits the underlying structure of the technological correspondence related to the state process, which, in optimal growth models, provides natural bounds on the growth rate of the state process and, through these bounds, a suitable sequence of compact subsets to construct local contractions. Unfortunately, such structures

[4] Typical examples include applications with AR(1) state processes, possibly with unit roots, and standard reward functions (CRRA, log, etc.—see, e.g, Low et al. (2010), Bagger et al. (2014), Kellogg (2014)). In models with learning (see, e.g., Burdett and Vishwanath (1988), Mitchell (2000), Crawford and Shum (2005), Nagypál (2007), Timoshenko (2015)), unbounded state spaces and rewards are routinely used to work with tractable prior-posterior structures.

are missing in most sequential decision settings we study, making the local contraction approach inapplicable.[5]

The paper is structured as follows. Section 2 outlines the method and provides general optimality results. Section 3 discusses the properties of the continuation value function, such as monotonicity and differentiability. Section 4 explores the connections between the continuation value and the optimal policy. Section 5 provides a list of economic applications and compares the computational efficiency of the continuation value approach and traditional approach. Section 6 provides extensions and section 7 concludes. Proofs are deferred to the appendix.

2. Optimality Results

This section provides optimality results. Prior to this task, we introduce some mathematical techniques and notations used in this paper.

2.1. **Preliminaries.** Let $\mathbb{N} := \{1, 2, \dots\}$, $\mathbb{N}_0 := \{0\} \cup \mathbb{N}$, $\mathbb{R}_{++} := (0, \infty)$ and $\mathbb{R}_+ := [0, \infty)$. For $a, b \in \mathbb{R}$, let $a \vee b := \max\{a, b\}$. If f and g are functions, then $(f \vee g)(x) := f(x) \vee g(x)$. If $(\mathsf{Z}, \mathscr{Z})$ is a measurable space, then $b\mathsf{Z}$ is the set of $\mathscr{Z}$-measurable bounded functions from Z to $\mathbb{R}$, with norm $\|f\| := \sup_{z \in \mathsf{Z}} |f(z)|$. Given a function $\kappa \colon \mathsf{Z} \to \mathbb{R}_{++}$, the *$\kappa$-weighted supremum norm* of $f \colon \mathsf{Z} \to \mathbb{R}$ is

$$\|f\|_\kappa := \sup_{z \in \mathsf{Z}} \frac{|f(z)|}{\kappa(z)}.$$

If $\|f\|_\kappa < \infty$, we say that f is *κ-bounded*. The symbol $b_\kappa \mathsf{Z}$ will denote the set of all functions from Z to $\mathbb{R}$ that are both $\mathscr{Z}$-measurable and κ-bounded. The pair $(b_\kappa \mathsf{Z}, \| \cdot \|_\kappa)$ forms a Banach space (see, e.g., Boyd (1990), page 331).

A *stochastic kernel* P on $(\mathsf{Z}, \mathscr{Z})$ is a map $P \colon \mathsf{Z} \times \mathscr{Z} \to [0, 1]$ such that $z \mapsto P(z, B)$ is $\mathscr{Z}$-measurable for each $B \in \mathscr{Z}$ and $B \mapsto P(z, B)$ is a probability measure for each $z \in \mathsf{Z}$. We understand $P(z, B)$ as the probability of a state transition from

[5]There are also a number of applied studies that treat unbounded problems, such as Poschke (2010), Chatterjee and Rossi-Hansberg (2012)) and Kellogg (2014). In these problems the techniques are specialized to the application in question and not readily applicable in other settings.

$z \in \mathsf{Z}$ to $B \in \mathscr{Z}$ in one step. For all $n \in \mathbb{N}$, $P^n(z, B) := \int P(z', B) P^{n-1}(z, \mathrm{d}z')$ is the probability of a state transition from z to $B \in \mathscr{Z}$ in n steps. Given a $\mathscr{Z}$-measurable function $h : \mathsf{Z} \to \mathbb{R}$, let

$$(P^n h)(z) =: \mathbb{E}_z h(Z_n) := \int h(z') P^n(z, \mathrm{d}z') \text{ for all } n \in \mathbb{N}_0,$$

with $P^0 h := h$ and $Ph := P^1 h$. We say that P is *stochastically increasing* if Ph is increasing for all increasing function $h \in b\mathsf{Z}$. When Z is a Borel subset of $\mathbb{R}^m$, a *density kernel* on Z is a measurable map $f : \mathsf{Z} \times \mathsf{Z} \to \mathbb{R}_+$ such that $\int_{\mathsf{Z}} f(z'|z) dz' = 1$ for all $z \in \mathsf{Z}$. We say that P *has a density representation* if there exists a density kernel f such that $P(z, B) = \int_B f(z'|z)\, \mathrm{d}z'$ for all $z \in \mathsf{Z}$ and $B \in \mathscr{Z}$.

2.2. **Set Up.** Let $(Z_n)_{n \geq 0}$ be a Markov process on probability space $(\Omega, \mathscr{F}, \mathbb{P})$ taking values in measurable space $(\mathsf{Z}, \mathscr{Z})$. Let P be the corresponding stochastic kernel. Let $\{\mathscr{F}_n\}_{n \geq 0}$ be a filtration in $\mathscr{F}$ to which $(Z_n)_{n \geq 0}$ is adapted. Let $\mathbb{P}_z$ indicate probability conditioned on $Z_0 = z$, while $\mathbb{E}_z$ is expectation conditioned on the same event.[6] A random variable τ taking values in $\mathbb{N}_0$ is called a (finite) *stopping time* with respect to $\{\mathscr{F}_n\}_{n \geq 0}$ if $\mathbb{P}\{\tau < \infty\} = 1$ and $\{\tau \leq n\} \in \mathscr{F}_n$ for all $n \geq 0$. Below, $\tau = n$ has the interpretation of choosing to act at time n. Let $\mathscr{M}$ denote the set of all stopping times with respect to the filtration $\{\mathscr{F}_n\}_{n \geq 0}$.

Suppose, at each $t \geq 0$, an agent observes Z_t and chooses between stopping (e.g., accepting a job, exercising an option) and continuing. Stopping generates *exit reward* $r(Z_t)$. Continuing yields *flow continuation reward* $c(Z_t)$ and transition to $t+1$, where the agent observes Z_{t+1} and the process repeats. Here $r \colon \mathsf{Z} \to \mathbb{R}$ and $c \colon \mathsf{Z} \to \mathbb{R}$ are measurable functions. Future rewards are discounted at rate $\beta \in (0, 1)$. The value function v^* for this problem is defined at $z \in \mathsf{Z}$ by

$$v^*(z) := \sup_{\tau \in \mathscr{M}} \mathbb{E}_z \left\{ \sum_{t=0}^{\tau-1} \beta^t c(Z_t) + \beta^\tau r(Z_\tau) \right\}. \tag{1}$$

[6]In proofs we take $(\Omega, \mathscr{F})$ to be the canonical sequence space, so that $\Omega = \times_{n=0}^{\infty} \mathsf{Z}$ and $\mathscr{F}$ is the product σ-algebra generated by $\mathscr{Z}$. For background see section 3.4 of Meyn and Tweedie (2012) or section 8.2 of Stokey et al. (1989).

A stopping time $\tau \in \mathscr{M}$ is called an *optimal stopping time* if it attains the supremum in (1). A *policy* is a map σ from Z to $\{0,1\}$, with 0 indicating the decision to continue and 1 indicating the decision to stop. A policy σ is called *optimal* if $\tau^* := \inf\{t \geq 0 \,|\, \sigma(Z_t) = 1\}$ is an optimal stopping time. The *continuation value function* associated with the sequential decision problem (1) is defined at $z \in \mathsf{Z}$ by

$$\psi^*(z) := c(z) + \beta \int v^*(z') P(z, dz'). \tag{2}$$

2.3. **The Continuation Value Operator.** To provide optimality results without insisting that rewards are bounded, we adopt the next assumption:

Assumption 2.1. There exist a $\mathscr{Z}$-measurable function $g \colon \mathsf{Z} \to \mathbb{R}_+$ and constants $n \in \mathbb{N}_0$ and $a_1, \cdots, a_4, m, d \in \mathbb{R}_+$ such that $\beta m < 1$, and, for all $z \in \mathsf{Z}$,

$$\int |r(z')| P^n(z, \mathrm{d}z') \leq a_1 g(z) + a_2, \tag{3}$$

$$\int |c(z')| P^n(z, \mathrm{d}z') \leq a_3 g(z) + a_4, \tag{4}$$

$$\text{and} \int g(z') P(z, \mathrm{d}z') \leq m g(z) + d. \tag{5}$$

The interpretation is that both $\mathbb{E}_z |r(Z_n)|$ and $\mathbb{E}_z |c(Z_n)|$ are small relative to some function g such that $\mathbb{E}_z g(Z_t)$ does not grow too fast.[7] Slow growth in $\mathbb{E}_z g(Z_t)$ is imposed by (5), which can be understood as a geometric drift condition (see, e.g., Meyn and Tweedie (2012), chapter 15). Note that if both r and c are bounded, then assumption 2.1 holds for $n := 0$, $g := \|r\| \vee \|c\|$, $m := 1$ and $d := 0$.

Lemma 2.1. *Under assumption 2.1, the value function solves the Bellman equation*

$$v^*(z) = \max \left\{ r(z),\, c(z) + \beta \int v^*(z') P(z, dz') \right\} = \max \left\{ r(z),\, \psi^*(z) \right\}. \tag{6}$$

[7]One can show that if assumption 2.1 holds for some n, then it must hold for all integer $n' > n$. Hence, to verify assumption 2.1, it suffices to find $n_1 \in \mathbb{N}_0$ for which (3) holds, $n_2 \in \mathbb{N}_0$ for which (4) holds, and that the measurable map g satisfies (5).

Using (2) and (6), we observe that ψ^* satisfies

$$\psi^*(z) = c(z) + \int \max\left\{r(z'), \psi^*(z')\right\} P(z, \mathrm{d}z'). \tag{7}$$

Define Q by

$$Q\psi(z) = c(z) + \beta \int \max\{r(z'), \psi(z')\} P(z, \mathrm{d}z'). \tag{8}$$

We call Q the *continuation value operator* or *Jovanovic operator*. In view of (7), the function ψ^* is a fixed point of Q.

Theorem 2.1. *Let assumption 2.1 hold. Then there exist positive constants m' and d' such that for $\ell\colon \mathsf{Z} \to \mathbb{R}$ defined by*

$$\ell(z) := m' \left(\sum_{t=1}^{n-1} \mathbb{E}_z |r(Z_t)| + \sum_{t=0}^{n-1} \mathbb{E}_z |c(Z_t)| \right) + g(z) + d', \tag{9}$$

the following statements hold:

1. *Q is a contraction mapping on $(b_\ell \mathsf{Z}, \|\cdot\|_\ell)$.*
2. *The unique fixed point of Q in $b_\ell \mathsf{Z}$ is ψ^*.*
3. *The policy defined by $\sigma^*(z) = \mathbb{1}\{r(z) \geq \psi^*(z)\}$ is an optimal policy.*

Note that if assumption 2.1 holds for $n = 0$, then $\ell(z) = g(z) + d'$. If it holds for $n = 1$, then $\ell(z) = m'|c(z)| + g(z) + d'$.

Example 2.1. Consider a job search problem where a worker aims to maximize expected lifetime rewards (see, e.g., Jovanovic (1987), Cooper et al. (2007), Ljungqvist and Sargent (2008), Robin (2011), Moscarini and Postel-Vinay (2013), Bagger et al. (2014)). She can accept current wage offer w_t and work permanently at that wage, or reject the offer, receive unemployment compensation $\tilde{c}_0$ and reconsider next period. Let $w_t = w(Z_t)$ for some idiosyncratic or aggregate state process $(Z_t)_{t\geq 0}$. The exit reward is $r(z) = u(w(z))/(1-\beta)$, the lifetime reward associated with stopping at state z. Here u is a utility function and β is the discount factor. The flow continuation reward is the constant $c_0 := u(\tilde{c}_0)$.

A common specification for the state process $(Z_t)_{t\geq 0} \subset \mathsf{Z} := \mathbb{R}$ is

$$Z_{t+1} = \rho Z_t + b + \varepsilon_{t+1}, \quad (\varepsilon_t)_{t\geq 1} \overset{\text{IID}}{\sim} N(0, \sigma^2), \quad \rho \in [-1, 1]. \tag{10}$$

The Jovanovic operator for this problem is

$$Q\psi(z) = c_0 + \beta \int \max\left\{\frac{u(w(z'))}{1-\beta}, \psi(z')\right\} f(z'|z)\,\mathrm{d}z'. \tag{11}$$

where $f(z'|z) = N(\rho z + b, \sigma^2)$. Let $w(z) := \mathrm{e}^z$ and let utility have the CRRA form

$$u(w) = \frac{w^{1-\delta}}{1-\delta} \quad (\text{if } \delta \geq 0 \text{ and } \delta \neq 1) \quad \text{and} \quad u(w) = \ln w \quad (\text{if } \delta = 1). \tag{12}$$

Since the exit reward is unbounded, traditional solution method based on the Bellman operator and sup norm contractions are not generally valid. Moreover, since $\{\varepsilon_t\}$ has unbounded support, the local contraction method fails.[8] But theorem 2.1 can be applied. Consider, for example, $\delta \geq 0$, $\delta \neq 1$ and $\rho \in [0,1)$. Choose $n \in \mathbb{N}_0$ such that $\beta\,\mathrm{e}^{\rho^n \xi} < 1$, where $\xi := \xi_1 + \xi_2$ with $\xi_1 := (1-\delta)b$ and $\xi_2 := (1-\delta)^2\sigma^2/2$. Observe that[9]

$$\int \mathrm{e}^{(1-\delta)z'} P^n(z, \mathrm{d}z') = b_n\,\mathrm{e}^{\rho^n(1-\delta)z}, \text{ where } b_n := \mathrm{e}^{\xi_1 \sum_{i=0}^{t-1}\rho^i + \xi_2 \sum_{i=0}^{t-1}\rho^{2i}}. \tag{13}$$

It follows that assumption 2.1 holds when $g(z) = \mathrm{e}^{\rho^n(1-\delta)z}$,

$$m = d = \mathrm{e}^{\rho^n \xi}, \quad a_1 = \frac{b_n}{(1-\beta)(1-\delta)}, \quad a_2 = a_3 = 0, \text{ and } a_4 = c_0.$$

Specifically, since $r(z) = \mathrm{e}^{(1-\delta)z}/((1-\beta)(1-\delta))$, an application of (13) gives

$$\int |r(z')| P^n(z, \mathrm{d}z') = b_n\,\mathrm{e}^{\rho^n(1-\delta)z}\frac{1}{(1-\beta)(1-\delta)} = a_1 g(z) + a_2,$$

which is (3). Condition (4) is trivial. Condition (5) holds because

$$\int g(z')P(z,\mathrm{d}z') = \mathrm{e}^{\rho^{n+1}(1-\delta)z}\mathrm{e}^{\rho^n\xi_1 + \rho^{2n}\xi_2} \leq (g(z)+1)\,\mathrm{e}^{\rho^n\xi} = mg(z) + d. \tag{14}$$

Hence, theorem 2.1 applies. The cases $\rho = 1$, $\delta = 1$ and $\rho \in [-1, 0]$ can be treated using similar methods.[10]

[8]The method requires an increasing sequence of compact sets $\{K_j\}$ such that $\mathsf{Z} = \cup_{j=1}^{\infty} K_j$ and $\Gamma(K_j) \subset K_{j+1}$ with probability one, where $\Gamma : \mathsf{Z} \mapsto 2^{\mathsf{Z}}$ is the feasibility correspondence of the state process $\{Z_t\}$ (see, e.g., Rincón-Zapatero and Rodríguez-Palmero (2003), theorems 3–4). This fails in the current case, since Γ corresponds to (10) and shocks have unbounded support.

[9]Recall that for $X \sim N(\mu, \sigma^2)$, we have $\mathbb{E}\,\mathrm{e}^{sX} = \mathrm{e}^{s\mu + s^2\sigma^2/2}$ for any $s \in \mathbb{R}$. Based on (10), the distribution of Z_t given $Z_0 = z$ follows $N\left(b\sum_{i=0}^{t-1}\rho^i, \sigma^2\sum_{i=0}^{t-1}\rho^{2i}\right)$.

[10]See the working paper version (Ma and Stachurski, 2017) for a detailed proof of all cases.

Example 2.2. Consider a job search problem with learning, as in, say, McCall (1970), Pries and Rogerson (2005), Nagypál (2007), or Ljungqvist and Sargent (2012). The set up is as in example 2.1, except that $(w_t)_{t\geq 0}$ follows

$$\ln w_t = \xi + \varepsilon_t, \quad (\varepsilon_t)_{t\geq 0} \overset{\text{IID}}{\sim} N(0, \gamma_\varepsilon).$$

Here ξ is the unobservable mean of the wage process, over which the worker has prior $\xi \sim N(\mu, \gamma)$. The worker's current estimate of the next period wage distribution is $f(w'|\mu,\gamma) = LN(\mu, \gamma + \gamma_\varepsilon)$. If the current offer is turned down, the worker updates his belief after observing w'. By the Bayes' rule, the posterior satisfies $\xi|w' \sim N(\mu', \gamma')$, where

$$\gamma' = 1/\left(1/\gamma + 1/\gamma_\varepsilon\right) \quad \text{and} \quad \mu' = \gamma'\left(\mu/\gamma + \ln w'/\gamma_\varepsilon\right). \tag{15}$$

Let the utility of the worker be defined by (12). As in example 2.1, solution methods based on the contractions with respect to supremum norm or local contractions are not generally valid due to unbounded rewards and shocks. However, theorem 2.1 applies: For any integrable function h, the stochastic kernel satisfies

$$\int h(z')P(z, \mathrm{d}z') = \int h(w', \mu', \gamma') f(w'|\mu, \gamma)\, \mathrm{d}w', \tag{16}$$

where μ' and γ' are defined by (15). Hence, the Jovanovic operator takes form of

$$Q\psi(\mu, \gamma) = c_0 + \beta \int \max\left\{\frac{u(w')}{1-\beta}, \psi(\mu', \gamma')\right\} f(w'|\mu, \gamma)\, \mathrm{d}w'. \tag{17}$$

Consider, for example, $\delta \geq 0$ and $\delta \neq 1$. We claim that assumption 2.1 holds. Let $n := 1$, $g(\mu, \gamma) := \mathrm{e}^{(1-\delta)\mu + (1-\delta)^2\gamma/2}$, $m := 1$ and $d := 0$. By footnote 9,

$$\int w'^{1-\delta} f(w'|\mu, \gamma)\, \mathrm{d}w' = \mathrm{e}^{(1-\delta)^2\gamma_\varepsilon/2} \cdot \mathrm{e}^{(1-\delta)\mu + (1-\delta)^2\gamma/2}. \tag{18}$$

Since the rewards are $r(w) := w^{1-\delta}/((1-\delta)(1-\beta))$ and $c \equiv c_0$, conditions (3)–(4) hold.[11] Moreover, condition (5) holds since by (15) and footnote 9,

$$\mathbb{E}_{\mu,\gamma} g(\mu', \gamma') := \int g(\mu', \gamma') f(w'|\mu, \gamma)\, \mathrm{d}w' = g(\mu, \gamma) = m g(\mu, \gamma) + d. \tag{19}$$

[11] Since $\int r(w') f(w'|\mu, \gamma)\, \mathrm{d}w' = \mathrm{e}^{(1-\delta)^2\gamma_\varepsilon/2} \cdot \mathrm{e}^{(1-\delta)\mu + (1-\delta)^2\gamma/2}/((1-\beta)(1-\delta))$, (3) holds with $a_1 := \mathrm{e}^{(1-\delta)^2\gamma_\varepsilon/2}/((1-\beta)(1-\delta))$ and $a_2 := 0$. Obviously, (4) holds with $a_3 := 0$ and $a_4 := c_0$.

Hence, theorem 2.1 is applicable. Define ℓ by (9). Then theorem 2.1 implies that Q is a contraction mapping on $(b_\ell \mathsf{Y}, \|\cdot\|_\ell)$ with unique fixed point ψ^*, where $\mathsf{Y} := \mathbb{R} \times \mathbb{R}_{++}$. The case $\delta = 1$ can be analyzed similarly.[12]

Remark 2.1. From (17), we see that the continuation value is a function of (μ, γ). On the other hand, current rewards depend on w, so the value function depends on (w, μ, γ). Thus, the former is lower dimensional than the latter.

With unbounded rewards and shocks, solution methods based on the Bellman operator with respect to the supremum norm or local contractions fail in the next few examples. However, theorem 2.1 can be applied.

Example 2.3. Consider an infinite-horizon American call option (see, e.g., Peskir and Shiryaev (2006) or Duffie (2010)) with state process be as in (10) and state space $\mathsf{Z} := \mathbb{R}$. Let $p_t = p(Z_t) = \mathrm{e}^{Z_t}$ be the current price of the underlying asset, and $\gamma > 0$ be the riskless rate of return (i.e., $\beta = \mathrm{e}^{-\gamma}$). With strike price K, the exit reward is $r(z) = (p(z) - K)^+$, the reward of exercising the option, while the flow continuation reward is $c \equiv 0$. The Jovanovic operator for the option satisfies

$$Q\psi(z) = \mathrm{e}^{-\gamma} \int \max\{(p(z') - K)^+, \psi(z')\} f(z'|z) \, \mathrm{d}z'.$$

If $\rho \in (-1, 1)$, we can let $\xi := |b| + \sigma^2/2$ and fix $n \in \mathbb{N}_0$ such that $\mathrm{e}^{-\gamma + |\rho^n|\xi} < 1$, so assumption 2.1 holds with $g(z) := \mathrm{e}^{\rho^n z} + \mathrm{e}^{-\rho^n z}$ and $m := d := \mathrm{e}^{|\rho^n|\xi}$. (If $\mathrm{e}^{-\gamma+\xi} < 1$, then assumption 2.1 holds with $n = 0$ for all $\rho \in [-1, 1]$.)

Example 2.4. Suppose that, in each period, a firm observes an idea with value $Z_t \in \mathsf{Z} := \mathbb{R}_+$ and decides whether to put this idea into productive use or develop it further, by investing in R&D (see, e.g., Jovanovic and Rob (1989), Bental and Peled (1996), Perla and Tonetti (2014)). The first choice gives reward $r(Z_t) = Z_t$. The latter incurs fixed cost $c_0 > 0$. Let the R&D process be governed by the exponential law (with rate parameter $\theta > 0$)

$$F(z'|z) := \mathbb{P}(Z_{t+1} \leq z' | Z_t = z) = 1 - \mathrm{e}^{-\theta(z'-z)} \quad (z' \geq z). \tag{20}$$

[12]See the working paper version (Ma and Stachurski, 2017) for a detailed proof of this case.

The Jovanovic operator satisfies

$$Q\psi(z) = -c_0 + \beta \int \max\{z',\ \psi(z')\} \, \mathrm{d}F(z'|z).$$

In this case, assumption 2.1 is satisfied with $n := 0$, $g(z) := z$, $m := 1$ and $d := 1/\theta$.

Example 2.5. Consider a firm exit problem (see, e.g., Hopenhayn (1992), Ericson and Pakes (1995), Asplund and Nocke (2006), Dinlersoz and Yorukoglu (2012), Coşar et al. (2016)). In each period, an incumbent firm observes a productivity shock a_t, where $a_t = a(Z_t) = \mathrm{e}^{Z_t}$ and $Z_t \in \mathsf{Z} := \mathbb{R}$ obeys (10), and decides whether or not to exit the market next period. A fixed cost $c_f > 0$ is paid each period and the firm's output is $q(a, l) = al^{\alpha}$, where $\alpha \in (0, 1)$ and l is labor demand. Given output and input prices p and w, the reward functions are $r(z) = c(z) = Ga(z)^{\frac{1}{1-\alpha}} - c_f$, where $G = (\alpha p/w)^{\frac{1}{1-\alpha}} (1-\alpha)w/\alpha$. The Jovanovic operator is

$$Q\psi(z) = \left(Ga(z)^{\frac{1}{1-\alpha}} - c_f \right) + \beta \int \max \left\{ Ga(z')^{\frac{1}{1-\alpha}} - c_f, \psi(z') \right\} f(z'|z) \, \mathrm{d}z'.$$

For $\rho \in [0, 1)$, choose $n \in \mathbb{N}_0$ such that $\beta \mathrm{e}^{\rho^n \xi} < 1$, where $\xi := \frac{b}{1-\alpha} + \frac{\sigma^2}{2(1-\alpha)^2}$. Then assumption 2.1 holds with $g(z) := \mathrm{e}^{\rho^n z/(1-\alpha)}$ and $m := d := \mathrm{e}^{\rho^n \xi}$. Other parameterizations (such as the unit root case $\rho = 1$) can also be handled.[13]

3. Properties of Continuation Values

This section studies some further properties of the continuation value function.

3.1. **Continuity.**

We begin by stating conditions under which the continuation value function is continuous. We require the following assumptions.

Assumption 3.1. (1) The stochastic kernel P is Feller; that is, $z \mapsto \int h(z') P(z, \mathrm{d}z')$ is continuous and bounded on Z whenever h is. (2) c, r, ℓ, $z \mapsto \int |r(z')| P(z, \mathrm{d}z')$, and $z \mapsto \int \ell(z') P(z, \mathrm{d}z')$ are continuous.

[13] See the working paper version (Ma and Stachurski, 2017) for details.

Remark 3.1. If assumption 2.1 holds for $n = 0$ and P is Feller, then assumption 3.1-(2) is equivalent to: c, r, g and $z \mapsto \mathbb{E}_z g(Z_1)$ are continuous.[14] Moreover, a general sufficient condition for assumption 3.1-(2) is: g and $z \mapsto \mathbb{E}_z g(Z_1)$ are continuous, and $z \mapsto \mathbb{E}_z |r(Z_t)|, \mathbb{E}_z |c(Z_t)|$ are continuous for $t = 0, ..., n$.

Proposition 3.1. *If assumptions 2.1 and 3.1 hold, then ψ^* is continuous.*

The next result treats the special case when P admits a density representation. Note that continuity of r is not required.

Corollary 3.1. *If assumption 2.1 holds, P admits a density representation $f(z'|z)$ that is continuous in z, and that c, ℓ and $z \mapsto \int |r(z')| f(z'|z) \, \mathrm{d}z'$, $\int \ell(z') f(z'|z) \, \mathrm{d}z'$ are continuous, then ψ^* is continuous.*

Remark 3.2. If r and c are bounded, then assumption 3.1-(1) and the continuity of r and c are sufficient for the continuity of ψ^* (by proposition 3.1). If in addition P has a density representation f, then the continuity of c and $z \mapsto f(z'|z)$ is sufficient for ψ^* to be continuous by corollary 3.1.

Example 3.1. In the job search model of example 2.1, ψ^* is continuous. Assumption 2.1 holds, as shown. P has a density representation $f(z'|z) = N(\rho z + b, \sigma^2)$ that is continuous in z. Moreover, $c \equiv c_0$, g and $z \mapsto \mathbb{E}_z g(Z_1)$ are continuous, and $z \mapsto \mathbb{E}_z |r(Z_t)|$ is continuous for all $t \in \mathbb{N}$ (see (13)–(14)). Hence, ℓ and $z \mapsto \mathbb{E}_z \ell(Z_1)$ are continuous, and the conditions of corollary 3.1 are satisfied.

Example 3.2. In the adaptive search model of example 2.2, assumption 2.1 holds for $n = 1$, as shown. By (16) and lemma 7.2, P is Feller. Moreover, $c \equiv c_0$, and r, g and $(\mu, \gamma) \mapsto \mathbb{E}_{\mu,\gamma} |r(w')|, \mathbb{E}_{\mu,\gamma} g(\mu', \gamma')$ are continuous (see (18)–(19)), where we define $\mathbb{E}_{\mu,\gamma} |r(w')| := \int |r(w')| f(w'|\mu, \gamma) \, \mathrm{d}w'$. Hence, assumption 3.1 holds (recall remark 3.1). By proposition 3.1, ψ^* is continuous.

[14] If $n = 0$ in assumption 2.1, then $|r(z)| \leq a_1 g(z) + a_2$. Since r, g and $z \mapsto \mathbb{E}_z g(Z_1)$ are continuous, theorem 1.1 of Feinberg et al. (2014) implies that $z \mapsto \mathbb{E}_z |r(Z_1)|$ is continuous.

Example 3.3. Recall the option pricing model of example 2.3. By corollary 3.1, we can show that ψ^* is continuous. The proof is similar to example 3.1, except that we use $|r(z)| \leq \mathrm{e}^z + K$, the continuity of $z \mapsto \int (\mathrm{e}^{z'} + K) f(z'|z)\, \mathrm{d}z'$, and lemma 7.2 to show that $z \mapsto \mathbb{E}_z |r(Z_1)|$ is continuous.

Example 3.4. Recall the R&D decision problem of example 2.4. Assumption 2.1 holds for $n = 0$. For all bounded continuous function $h : \mathsf{Z} \to \mathbb{R}$, lemma 7.2 shows that $z \mapsto \int h(z')\, \mathrm{d}F(z'|z)$ is continuous, so P is Feller. Moreover, r, c and g are continuous, and $z \mapsto \mathbb{E}_z g(Z_1)$ is continuous since

$$\int |z'| P(z, \mathrm{d}z') = \int_{[z,\infty)} z' \theta \mathrm{e}^{-\theta(z'-z)}\, \mathrm{d}z' = z + 1/\theta.$$

Hence, assumption 3.1 holds. By proposition 3.1, ψ^* is continuous.

Example 3.5. Recall the firm exit model of example 2.5. Through similar analysis to examples 3.1 and 3.3, we can show that ψ^* is continuous.

3.2. **Monotonicity.** We now study monotonicity under the following assumption.[15] In the assumption, the second statement holds whenever r is increasing and P is stochastically increasing (recall section 2.1).

Assumption 3.2. The function c is increasing, as is $z \mapsto \int \max\{r(z'), \psi(z')\} P(z, \mathrm{d}z')$ for all increasing $\psi \in b_\ell \mathsf{Z}$.

Proposition 3.2. *If assumptions 2.1 and 3.2 hold, then ψ^* is increasing.*

Example 3.6. In example 2.1, assumption 2.1 holds. If $\rho \geq 0$, the density kernel $f(z'|z) = N(\rho z + b, \sigma^2)$ is stochastically increasing in z. Since r and c are increasing, assumption 3.2 holds. By proposition 3.2, ψ^* is increasing.

Similarly, for the option pricing model of example 2.3 and the firm exit model of example 2.5, if $\rho \geq 0$, then ψ^* is increasing. Moreover, ψ^* is increasing in example 2.4. The details are omitted.

[15] We focus on the monotone increasing case. The monotone decreasing case is similar.

Example 3.7. For the adaptive search model of example 2.2, $r(w)$ is increasing, μ' is increasing in μ, and $f(w'|\mu,\gamma) = N(\mu, \gamma + \gamma_\varepsilon)$ is stochastically increasing in μ, so $\mathbb{E}_{\mu,\gamma}(r(w') \vee \psi(\mu', \gamma'))$ is increasing in μ for all candidate ψ that is increasing in μ. Since $c \equiv c_0$, by proposition 3.2, ψ^* is increasing in μ.

3.3. **Differentiability.** Suppose $\mathsf{Z} = \mathsf{Z}^1 \times \cdots \times \mathsf{Z}^m \subset \mathbb{R}^m$, with typical element $z = (z^1, ..., z^m)$. Given $h : \mathsf{Z} \to \mathbb{R}$, let $D_i^j h(z)$ be the j-th partial derivative of h with respect to z^i. For a density kernel f, let $D_i^j f(z'|z) := \partial^j f(z'|z)/\partial(z^i)^j$.

Assumption 3.3. *$D_i c(z)$ exists for all $z \in \text{int}(\mathsf{Z})$ and $i = 1, ..., m$.*

Let $z^{-i} := (z^1, ..., z^{i-1}, z^{i+1}, ..., z^m)$. Given $z_0 \in \mathsf{Z}$ and $\delta > 0$, let $B_\delta(z_0^i) := \{z^i \in \mathsf{Z}^i : |z^i - z_0^i| < \delta\}$ and let $\bar{B}_\delta(z_0^i)$ be its closure.

Assumption 3.4. *P has a density representation f, and, for $i = 1, \ldots, m$,*

1. *$D_i^2 f(z'|z)$ exits for all $(z, z') \in \text{int}(\mathsf{Z}) \times \mathsf{Z}$;*
2. *$(z, z') \mapsto D_i f(z'|z)$ is continuous;*
3. *There are finite solutions of z^i to $D_i^2 f(z'|z) = 0$ (denoted by $z_i^*(z', z^{-i})$), and, for all $z_0 \in \text{int}(\mathsf{Z})$, there exist $\delta > 0$ and a compact subset $A \subset \mathsf{Z}$ such that $z' \notin A$ implies $z_i^*(z', z_0^{-i}) \notin B_\delta(z_0^i)$.*

Remark 3.3. When the state space is unbounded above and below, a sufficient condition for assumption 3.4-(3) is: there are finite solutions of z^i to $D_i^2 f(z'|z) = 0$, and, for all $z_0 \in \text{int}(\mathsf{Z})$, $\|z'\| \to \infty$ implies $|z_i^*(z', z_0^{-i})| \to \infty$.

Assumption 3.5. *k is continuous and $\int |k(z') D_i f(z'|z)| \, \mathrm{d}z' < \infty$ for all $z \in \text{int}(\mathsf{Z})$, $k \in \{r, \ell\}$ and $i = 1, ..., m$.*

The following provides a general result for the differentiability of ψ^*.

Proposition 3.3. *If assumptions 2.1 and 3.3–3.5 hold, then ψ^* is differentiable at interior points, and, for all $z \in \text{int}(\mathsf{Z})$ and $i = 1, \ldots, m$,*

$$D_i \psi^*(z) = D_i c(z) + \int \max\{r(z'), \psi^*(z')\} D_i f(z'|z) \, \mathrm{d}z'.$$

To obtain continuous differentiability we add the following:

Assumption 3.6. For $i = 1, \ldots, m$, the following conditions hold:

(1) $z \mapsto D_i c(z)$ is continuous on $\text{int}(\mathsf{Z})$;
(2) k and $z \mapsto \int |k(z') D_i f(z'|z)| \, \mathrm{d}z'$ are continuous on $\text{int}(\mathsf{Z})$ for $k \in \{r, \ell\}$.

Proposition 3.4. *If assumptions 2.1, 3.4 and 3.6 hold, then $z \mapsto D_i \psi^*(z)$ is continuous on* $\text{int}(\mathsf{Z})$ *for $i = 1, \ldots, m$.*

Example 3.8. Recall the job search model of example 2.1. It can be shown that, with $h(z, a) := \mathrm{e}^{a(\rho z + b) + a^2 \sigma^2 / 2} / \sqrt{2\pi\sigma^2}$,

(a) the two solutions of $\frac{\partial^2 f(z'|z)}{\partial z^2} = 0$ are $z^*(z') := \frac{z' - b \pm \sigma}{\rho}$;
(b) $\int \left| \frac{\partial f(z'|z)}{\partial z} \right| \mathrm{d}z' = \frac{|\rho|}{\sigma} \sqrt{\frac{2}{\pi}}$;
(c) $\mathrm{e}^{az'} \left| \frac{\partial f(z'|z)}{\partial z} \right| \leq h(z, a) \exp\left\{ -\frac{[z' - (\rho z + b + a\sigma^2)]^2}{2\sigma^2} \right\} \frac{|\rho z'| + |\rho(\rho z + b)|}{\sigma^2}$;
(d) the two terms on both sides of (c) are continuous in z;
(e) the integration (w.r.t. z') of the right side of (c) is continuous in z.

By remark 3.3 and (a), assumption 3.4-(3) holds. Based on (13), conditions (b)–(e), and lemma 7.2, one can show that assumption 3.6-(2) holds. The other conditions of proposition 3.4 are straightforward. Hence, ψ^* is continuously differentiable.

Example 3.9. Recall the option pricing problem of example 2.3. Through similiar analysis, once can show that ψ^* is continuously differentiable. Figure 1 illustrates. While ψ^* is smooth, v^* is kinked at around $z = 3$ in both cases.[16]

Example 3.10. Recall the firm exit model of example 2.5. Through similar analysis to examples 3.8-3.9, one can show that ψ^* is continuously differentiable.

3.4. **Parametric Continuity.** Consider the parameter space $\Theta \subset \mathbb{R}^k$. Let $P_\theta, r_\theta, c_\theta$, v_θ^* and ψ_θ^* denote the stochastic kernel, exit and flow continuation rewards, value and continuation value functions with respect to the parameter $\theta \in \Theta$, respectively.

[16]We set $\gamma = 0.04$, $K = 20$, $b = -0.2$, $\sigma = 1$, and consider $\rho = \pm 0.65$.

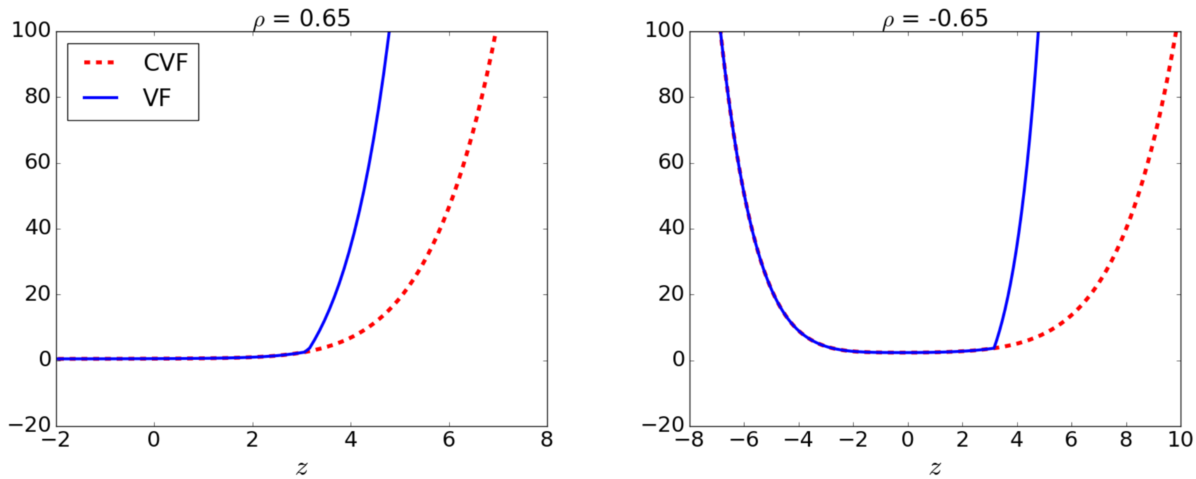

FIGURE 1. Comparison of ψ^* (CVF) and v^* (VF)

Similarly, let n_θ, $a_{i\theta}$, m_θ, d_θ and g_θ denote the key elements of assumption 2.1 with respect to θ. Define $n := \sup_{\theta \in \Theta} n_\theta$, $m := \sup_{\theta \in \Theta} m_\theta$, $d := \sup_{\theta \in \Theta} d_\theta$ and $\bar{a} := \sum_{i=1}^4 \sup_{\theta \in \Theta} a_{i\theta}$.

Assumption 3.7. Assumption 2.1 holds at all $\theta \in \Theta$, with $\beta m < 1$ and $n, d, \bar{a} < \infty$.

Similar to theorem 2.1, one can show that if assumption 3.7 holds, then there exist positive constants m' and d' such that for $\ell : \mathsf{Z} \times \Theta \to \mathbb{R}$ defined by

$$\ell(z, \theta) := m' \left(\sum_{t=1}^{n-1} \mathbb{E}_z^\theta |r_\theta(Z_t)| + \sum_{t=0}^{n-1} \mathbb{E}_z^\theta |c_\theta(Z_t)| \right) + g_\theta(z) + d',$$

Q is a contraction mapping on $b_\ell(\mathsf{Z} \times \Theta)$ with unique fixed point $(z, \theta) \mapsto \psi_\theta^*(z)$, where $\mathbb{E}_z^\theta$ denotes the conditional expectation with respect to $P_\theta(z, \cdot)$.

Assumption 3.8. (1) $P_\theta(z, \cdot)$ is Feller; that is, $(z, \theta) \mapsto \int h(z') P_\theta(z, \mathrm{d}z')$ is continuous for all bounded continuous function $h : \mathsf{Z} \to \mathbb{R}$. (2) $(z, \theta) \mapsto c_\theta(z), r_\theta(z), \ell(z, \theta)$, $\int |r_\theta(z')| P_\theta(z, \mathrm{d}z')$, $\int \ell(z', \theta) P_\theta(z, \mathrm{d}z')$ are continuous.

The following result is an extension of proposition 3.1. The proofs are similar.

Proposition 3.5. *If assumptions 3.7–3.8 hold, then $(z, \theta) \mapsto \psi_\theta^*(z)$ is continuous.*

Example 3.11. Recall the job search problem of example 2.1. Let $\Theta := [-1, 1] \times A \times B \times C$, where A, B are bounded subsets of $\mathbb{R}_{++}, \mathbb{R}$, respectively, and $C \subset \mathbb{R}$. A typical element of Θ is $\theta = (\rho, \sigma, b, c_0)$. Proposition 3.5 implies that $(\theta, z) \mapsto \psi^*_\theta(z)$ is continuous. The proof is similar to example 3.1.

4. Optimal Policies

This section provides a systematic study of optimal timing of decisions when there are threshold states, and explores the key properties of the optimal policies. We begin in the next section by imposing assumptions under which the optimal policy follows a reservation rule.

4.1. **Set Up.** Let Z be a subset of $\mathbb{R}^m$ with $\mathsf{Z} = \mathsf{X} \times \mathsf{Y}$, where X is a convex subset of $\mathbb{R}$, Y is a convex subset of $\mathbb{R}^{m-1}$, and the state process $(Z_t)_{t \geq 0}$ takes the form $\{(X_t, Y_t)\}_{t \geq 0}$. Here $(X_t)_{t \geq 0}$ and $(Y_t)_{t \geq 0}$ are stochastic processes on X and Y respectively. Assume that $(X_t)_{t \geq 0}$ and $(Y_t)_{t \geq 0}$ are conditionally independent: given Y_t, the next period states (X_{t+1}, Y_{t+1}) and X_t are independent. The stochastic kernel $P(z, \mathrm{d}z')$ can then be represented by the conditional distribution of (x', y') on y, denoted as $\mathbb{F}_y(x', y')$, i.e., $P(z, \mathrm{d}z') = P((x, y), \mathrm{d}(x', y')) = \mathrm{d}\mathbb{F}_y(x', y')$. Finally, assume that $c : \mathsf{Y} \to \mathbb{R}$.

Assumption 4.1. *r is strictly monotone on X. Moreover, for all $y \in \mathsf{Y}$, there exists $x \in \mathsf{X}$ such that $r(x, y) = c(y) + \beta \int v^*(x', y') \, \mathrm{d}\mathbb{F}_y(x', y')$.*

With assumption 4.1 in force, we call X_t the *threshold state* and Y_t the *environment*. We call X the *threshold state space* and Y the *environment space*. Under assumption 4.1, the reservation rule property holds: there is a *decision threshold* $\bar{x} : \mathsf{Y} \to \mathsf{X}$ such that when x attains $\bar{x}$, the agent is indifferent between stopping and continuing, i.e., $r(\bar{x}(y), y) = \psi^*(y)$ for all $y \in \mathsf{Y}$. The optimal policy then follows

$$\sigma^*(x, y) = \begin{cases} \mathbb{1}\{x \geq \bar{x}(y)\}, & \text{if } r \text{ is strictly increasing in } x \\ \mathbb{1}\{x \leq \bar{x}(y)\}, & \text{if } r \text{ is strictly decreasing in } x \end{cases} \tag{21}$$

4.2. **Results.** The next few results mainly rely on the implicit function theorem. See the working paper version (Ma and Stachurski, 2017) for proofs.

Proposition 4.1. *Suppose assumption 4.1 holds, and that either the assumptions of proposition 3.1 or of corollary 3.1 (plus the continuity of r) hold. Then $\bar{x}$ is continuous.*

Regarding parametric continuity, let $\bar{x}_\theta$ be the decision threshold w.r.t. $\theta \in \Theta$.

Proposition 4.2. *If assumptions 3.7–3.8 and 4.1 hold, then $(y, \theta) \mapsto \bar{x}_\theta(y)$ is continuous.*

A typical element of Y is $y = \left(y^1, ..., y^{m-1}\right)$. Given $h : \mathsf{Y} \to \mathbb{R}$ and $l : \mathsf{X} \times \mathsf{Y} \to \mathbb{R}$, we define $D_i h(y) := \partial h(y) / \partial y^i$, $D_i l(x, y) := \partial l(x, y) / \partial y^i$ and $D_x l(x, y) := \partial l(x, y) / \partial x$. The next result shows that the decision threshold is continuously differentiable with respect to the environment under certain assumptions, and provides an expression for the derivative.

Proposition 4.3. *Let assumptions 2.1, 3.4, 3.6 and 4.1 hold. If r is continuously differentiable on $\mathrm{int}(\mathsf{Z})$, then $\bar{x}$ is continuously differentiable on $\mathrm{int}(\mathsf{Y})$, with*

$$D_i \bar{x}(y) = -\frac{D_i r(\bar{x}(y), y) - D_i \psi^*(y)}{D_x r(\bar{x}(y), y)} \quad \textit{for all } y \in \mathrm{int}(\mathsf{Y}) \textit{ and } i = 1, ..., m.$$

The intuition behind this expression is as follows: Since $(x, y) \mapsto r(x, y) - \psi^*(y)$ is the premium of terminating the sequential decision process, which is null at the decision threshold, the change in this premium in response to instantaneous changes of x and y cancel out. Hence, the rate of change of $\bar{x}(y)$ with respect to y^i is equivalent to minus the ratio of the marginal premiums of y^i and x. See (24) for an application in the context of job search.

The next result considers monotonicity and applications are given below.

Proposition 4.4. *Let assumptions 2.1, 3.2 and 4.1 hold. If r is defined on X and is strictly increasing, then $\bar{x}$ is increasing.*

5. Applications

Let us now look at applications in some more detail, including how the preceding results can be applied and what their implications are.[17]

5.1. **Search with Learning.** Recall the adaptive search model of example 2.2 (see also examples 3.2 and 3.7). By lemma 2.1, the value function satisfies

$$v^*(w, \mu, \gamma) = \max \left\{ \frac{u(w)}{1-\beta}, c_0 + \beta \int v^*(w', \mu', \gamma') f(w'|\mu, \gamma) \, \mathrm{d}w' \right\},$$

while the Jovanovic operator is given by (17). This is a threshold state sequential decision problem, with threshold state $x := w \in \mathbb{R}_{++} =: \mathsf{X}$ and environment $y := (\mu, \gamma) \in \mathbb{R} \times \mathbb{R}_{++} =: \mathsf{Y}$. By the intermediate value theorem, assumption 4.1 holds. Hence, the optimal policy is represented by a reservation wage $\bar{w} : \mathsf{Y} \to \mathbb{R}$ at which the worker is indifferent between accepting and rejecting the offer. By examples 3.2 and 3.7 and propositions 4.1 and 4.4, $\bar{w}$ is increasing in μ and continuous. The intuition behind this monotonicity is that more optimistic agent (higher μ) expects higher offers to be realized.

By theorem 2.1, we can compute the reservation wage by iterating with Q. In doing so we set $\beta = 0.95$, $\gamma_\varepsilon = 1.0$, $\tilde{c}_0 = 0.6$, and consider different levels of risk aversion: $\delta = 3, 4, 5, 6$. The grid points of (μ, γ) lie in $[-10, 10] \times [10^{-4}, 10]$ with 200 points for the μ grid and 100 points for the γ grid. Here and below, integration is computed via Monte Carlo with 1000 draws, and function approximation is via linear interpolation.[18]

In Figure 2, the reservation wage is increasing in μ, which agrees with our analysis (see example 3.7). The reservation wage is increasing in γ when μ is small and decreasing in γ when μ is large. Intuitively, although a pessimistic worker (low μ) expects to obtain low offers on average, the downside risks are mitigated because

[17] The code needed to replicate all of the applications discussed in this section can be found at `https://github.com/jstac/continuation_values_public`.

[18] Changing the number of Monte Carlo samples, the grid range and grid density produces similar results in this and later examples.

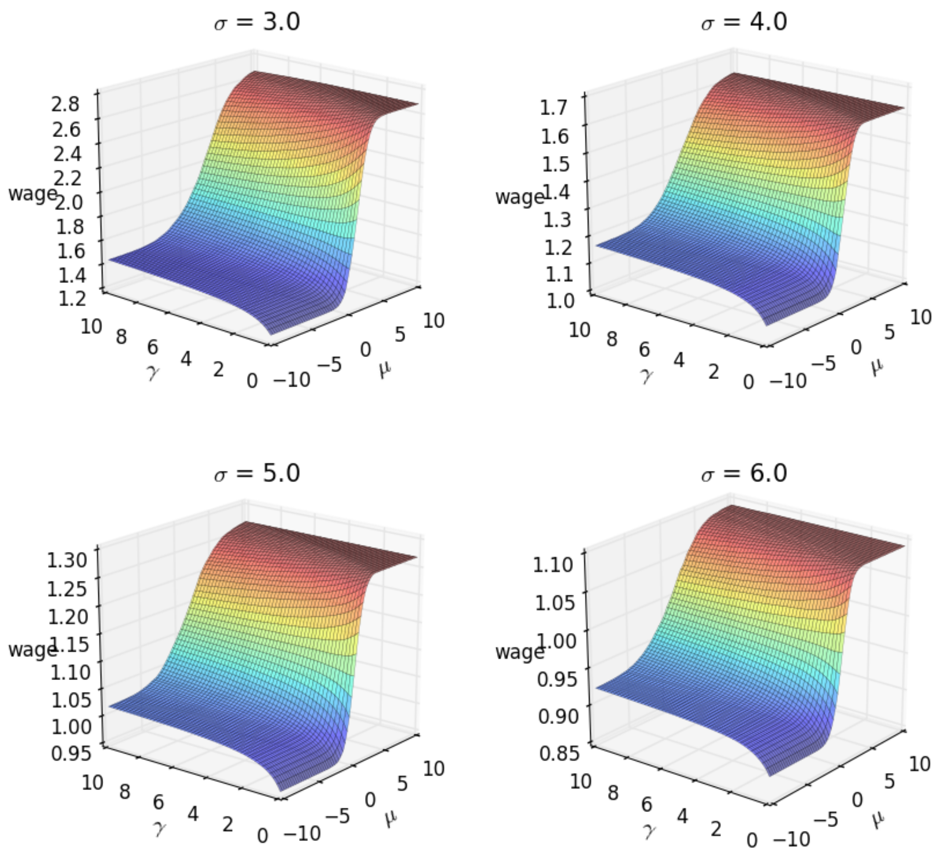

FIGURE 2. The reservation wage

compensation is obtained when the offer is turned down. A higher level of uncertainty (higher γ) provides a better chance of high offers. For an optimistic (high μ) worker, however, the insurance of compensation has less impact. With higher level uncertainty, the risk-averse worker has incentive to enter the labor market at an earlier stage so as to avoid downside risks.

In computation, value function iteration (VFI) takes more than one week, while continuation value iteration (CVI), being only 2-dimensional, takes 178 seconds.[19]

5.2. **Firm Entry.** Consider a firm entry problem in the style of Fajgelbaum et al. (2017). Each period, an investment cost $f_t \in \mathbb{R}$ is observed, where $\{f_t\} \overset{\text{IID}}{\sim} h$ with $\int |f| h(f) \, \mathrm{d}f < \infty$.[20] The firm then decides whether to incur this cost and enter the market to earn a stochastic dividend x_t or wait and reconsider. Let $x_t = \xi_t + \varepsilon_t^x$, $\{\varepsilon_t^x\} \overset{\text{IID}}{\sim} N(0, \gamma_x)$, where ξ_t and ε_t^x are respectively a persistent and a transient component, with

$$\xi_t = \rho \xi_{t-1} + \varepsilon_t^{\xi}, \quad \{\varepsilon_t^{\xi}\} \overset{\text{IID}}{\sim} N(0, \gamma_{\xi}).$$

A public signal y_{t+1} is released at the end of each period t, where $y_t = \xi_t + \varepsilon_t^y$, $\{\varepsilon_t^y\} \overset{\text{IID}}{\sim} N(0, \gamma_y)$. The firm has prior $\xi \sim N(\mu, \gamma)$ that is updated after observing y' if the firm chooses to wait. The posterior satisfies $\xi | y' \sim N(\mu', \gamma')$, with

$$\gamma' = 1 / \left(1/\gamma + \rho^2 / (\gamma_{\xi} + \gamma_y) \right) \quad \text{and} \quad \mu' = \gamma' \left(\mu/\gamma + \rho y' / (\gamma_{\xi} + \gamma_y) \right). \tag{22}$$

The firm has utility $u(x) = \left(1 - \mathrm{e}^{-ax}\right)/a$, where $a > 0$ is the absolute risk aversion coefficient. The rewards are $r(f, \mu, \gamma) := \mathbb{E}_{\mu,\gamma}[u(x)] - f = \left(1 - \mathrm{e}^{-a\mu + a^2(\gamma + \gamma_x)/2}\right)/a - f$ and $c \equiv 0$.[21] This is a threshold state problem, with threshold state $x := f \in \mathbb{R} =: \mathsf{X}$ and environment $y := (\mu, \gamma) \in \mathbb{R} \times \mathbb{R}_{++} =: \mathsf{Y}$. The Jovanovic operator is

$$Q\psi(\mu, \gamma) = \beta \int \max \left\{ \mathbb{E}_{\mu', \gamma'}[u(x')] - f', \psi(\mu', \gamma') \right\} p(f', y' | \mu, \gamma) \, \mathrm{d}(f', y'),$$

[19]We terminate the iteration at precision 10^{-4}. The time of CVI is calculated as the average of the four cases ($\sigma = 3, 4, 5, 6$). Moreover, to implement VFI, we set the grid points of w in $[10^{-4}, 10]$ with 50 points, and combine them with the grid points for μ and γ to run the simulation. Indeed, with this additional state, VFI spends more than one week in each of our four cases. All simulations are processed in a standard Python environment on a laptop with a 2.9 GHz Intel Core i7 and 32GB RAM.

[20]Generally, we allow for $f_t < 0$, which can be interpreted as investment compensation.

[21]Since in general $\{f_t\}$ can be supported on $\mathbb{R}$ and the exit reward is unbounded, solution methods based on the Bellman operator with respect to the supremum norm or local contractions fail on the theoretical side. However, the theory we develop can be applied.

where $p(f', y'|\mu, \gamma) = h(f')l(y'|\mu, \gamma)$ with $l(y'|\mu, \gamma) = N(\rho\mu, \rho^2\gamma + \gamma_\xi + \gamma_y)$. Let $n := 1$, $g(\mu, \gamma) := \mathrm{e}^{-\mu + a^2\gamma/2}$, $m := 1$ and $d := 0$. Define ℓ according to (9), and let $\bar{f} : \mathsf{Y} \to \mathbb{R}$ be the reservation cost.

Proposition 5.1. *The following statements are true:*

1. *Q is a contraction mapping on $(b_\ell\mathsf{Y}, \|\cdot\|_\ell)$ with unique fixed point ψ^*.*
2. *The reservation cost is $\bar{f}(\mu, \gamma) = \mathbb{E}_{\mu,\gamma}[u(x)] - \psi^*(\mu, \gamma)$.*
3. *ψ^* and $\bar{f}$ are continuous functions.*
4. *If $\rho \geq 0$, then ψ^* is increasing in μ.*

Remark 5.1. Note that the first three claims of proposition 5.1 place no restriction on the range of ρ values, the correlation coefficient of $\{\xi_t\}$.

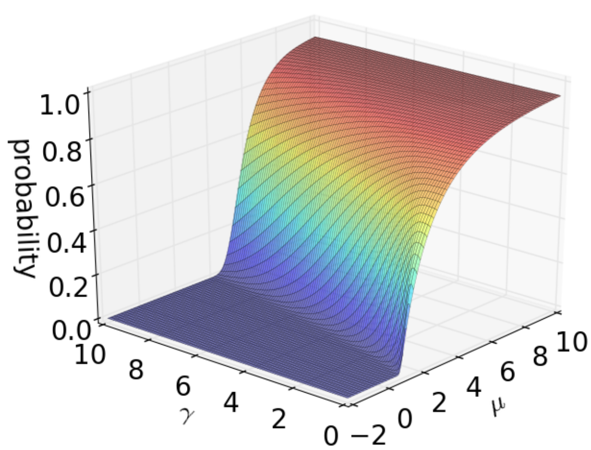

FIGURE 3. The perceived probability of investment

In simulation, we set $\beta = 0.95$, $a = 0.2$, $\gamma_x = 0.1$, $\gamma_y = 0.05$, and $h = LN(0, 0.01)$. Let $\rho = 1$, $\gamma_\xi = 0$, and let the grid points of (μ, γ) lie in $[-2, 10] \times [10^{-4}, 10]$ with 100 points for each variable.

Figure 3 plots the perceived probability of investment $\mathbb{P}\left\{f \leq \bar{f}(\mu,\gamma)\right\}$. As expected, this probability is increasing in μ and decreasing in γ, since a more optimistic firm (higher μ) is more likely to invest, and with higher level uncertainty (higher γ) the risk averse firm prefers to delay investment so as to avoid downside risks.[22] In terms of computation time, VFI takes more than one week, while CVI takes 921 seconds.[23]

5.3. **Search with Permanent and Transitory Components.** Consider a job search problem as in example 2.1, but with state process

$$w_t = \eta_t + \theta_t \xi_t, \quad \ln \theta_t = \rho \ln \theta_{t-1} + \ln u_t, \quad \rho \in [-1, 1]. \tag{23}$$

Here $\{\xi_t\} \overset{\text{IID}}{\sim} h$ and $\{\eta_t\} \overset{\text{IID}}{\sim} v$ are positive processes with finite first moments, $\int \eta^{-1} v(\eta)\, \mathrm{d}\eta < \infty$, $\{u_t\} \overset{\text{IID}}{\sim} LN(0, \gamma_u)$, and $\{\xi_t\}$, $\{\eta_t\}$ and $\{\theta_t\}$ are independent. Such settings appear in many search-theoretic and real options studies (see e.g., Gomes et al. (2001), Low et al. (2010), Chatterjee and Eyigungor (2012), Bagger et al. (2014), Kellogg (2014)). We interpret θ_t and ξ_t respectively as the persistent and transitory components of income, and η_t as social security.

The Jovanovic operator is

$$Q\psi(\theta) = c_0 + \beta \int \max\left\{\frac{u(w')}{1-\beta}, \psi(\theta')\right\} f(\theta'|\theta) h(\xi') v(\eta')\, \mathrm{d}(\theta', \xi', \eta'),$$

where $w' = \eta' + \theta'\xi'$ and $f(\theta'|\theta) = LN(\rho \ln \theta, \gamma_u)$. This is a threshold state problem, with threshold state $w \in \mathbb{R}_{++} =: \mathsf{X}$ and environment $\theta \in \mathbb{R}_{++} =: \mathsf{Y}$. Let $\bar{w}$ be the reservation wage. Recall the risk aversion coefficient δ in (12). Consider, for example, $\delta = 1$ and $\rho \in (-1, 1)$. Fix $n \in \mathbb{N}_0$ such that $\beta \mathrm{e}^{\rho^{2n}\gamma_u} < 1$. Let $g(\theta) := \theta^{\rho^n} + \theta^{-\rho^n}$ and $m := d := \mathrm{e}^{\rho^{2n}\gamma_u}$.

Proposition 5.2. *If $\rho \in (-1, 1)$, then the following statements hold:*

1. *Q is a contraction mapping on $(b_\ell \mathsf{Y}, \|\cdot\|_\ell)$ with unique fixed point ψ^*.*

[22] This result parallels propositions 1–2 of Fajgelbaum et al. (2017).

[23] We terminate the iteration at precision 10^{-4}. To implement VFI, we set the grid points of f in $[10^{-4}, 10]$ with 50 points, and combine them with the grid points for μ and γ to run the simulation.

2. *The reservation wage is* $\bar{w}(\theta) = e^{(1-\beta)\psi^*(\theta)}$.
3. ψ^* *and* $\bar{w}$ *are continuously differentiable, and*

$$\bar{w}'(\theta) = (1-\beta)\bar{w}(\theta)\psi^{*\prime}(\theta). \tag{24}$$

4. *If* $\rho \geq 0$, *then* ψ^* *and* $\bar{w}$ *are increasing in* θ.

The intuition behind the expression (24) for the derivative of $\bar{w}$ is as follows: Since the terminating premium is zero at the reservation wage, the overall effect of changes in w and θ cancel out. Hence, the rate of change of $\bar{w}$ with respect to θ equals the minus ratio of the marginal premiums of θ and w at the decision threshold, denoted respectively by $-\psi^{*\prime}(\theta)$ and $[(1-\beta)\bar{w}(\theta)]^{-1}$.

Remark 5.2. If $\beta e^{\gamma_u/2} < 1$, claims 1–3 of proposition 5.2 remain true for $|\rho| = 1$, and claim 4 is true for $\rho = 1$. The case $\delta \geq 0$ and $\delta \neq 1$ can be treated similarly.[24]

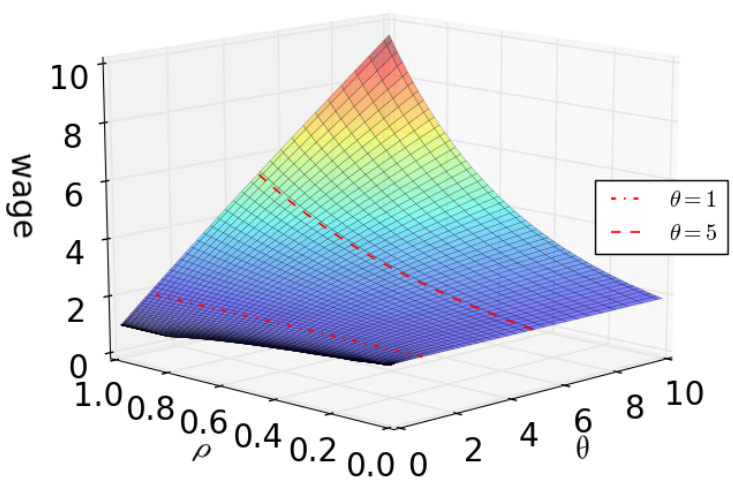

FIGURE 4. The reservation wage

[24]See the working paper version (Ma and Stachurski, 2017) for a detailed proof of these claims.

In simulation, we set $\beta = 0.95$, $\tilde{c}_0 = 0.6$, $\gamma_u = 10^{-4}$, $v = LN(0, 10^{-6})$, $h = LN(0, 5 \times 10^{-4})$, and consider the parametric class problem of $\rho \in [0, 1]$, with 100 grid points. Grid points of θ lie in $[10^{-4}, 10]$ with 200 points, and are scaled to be more dense when θ is smaller.

When $\rho = 0$, $\{\theta_t\} \overset{\text{IID}}{\sim} LN(0, \gamma_u)$, in which case each realized θ will be forgotten at future stages. As a result, the continuation value is independent of θ, yielding a reservation wage parallel to the θ-axis, as shown in figure 4. When $\rho > 0$, the reservation wage is increasing in θ, which is intuitive because higher θ implies a better current situation. Since a higher degree of income persistence (higher ρ) prolongs the mean-reverting process, the reservation wage tends to decrease in ρ in bad states ($\theta < 1$) and increase in ρ in good states ($\theta > 1$).

TABLE 1. Time in seconds under different grid sizes

	Test 1	Test 2	Test 3	Test 4	Test 5	Test 6	Test 7	Test 8	Test 9
CVI	0.141	0.125	0.125	0.813	0.812	0.812	1.062	1.062	1.063
VFI	171.63	284.01	440.75	277.19	1078.27	1488.78	1075.22	1622.09	2696.09

We set $\rho = 0.75$, $\beta = 0.95$, $\tilde{c}_0 = 0.6$, $\gamma_u = 10^{-4}$, $v = LN(0, 10^{-6})$ and $h = LN(0, 5 \times 10^{-4})$. Grid points for (θ, w) lie in $[10^{-4}, 10]^2$, and the grid sizes for (θ, w) in each test are: Test 1: $(200, 200)$, Test 2: $(200, 300)$, Test 3: $(200, 400)$, Test 4: $(300, 200)$, Test 5: $(300, 300)$, Test 6: $(300, 400)$, Test 7: $(400, 200)$, Test 8: $(400, 300)$, and Test 9: $(400, 400)$. For both CVI and VFI, we terminate the iteration at a precision level 10^{-4}.

Table 1 provides a numerical comparison between CVI and VFI under different grid sizes. In tests 1–9, CVI is 1733 times faster than VFI on average, and outperforms VFI more strongly as we increase the grid size. For example, as we increase the grid size of w and π, there is a slight decrease in the speed of CVI, while the speed of VFI drops exponentially (see, e.g., tests 1, 5 and 9).

6. EXTENSIONS: REPEATED SEQUENTIAL DECISIONS

The theory developed above can be extended to accommodate a variety of additional problems, including optimal timing problems with more than two choices,[25]

[25] See the working paper version (Ma and Stachurski, 2017) for a detailed treatment.

or situations where the decision to stop is not permanent. The latter case arises in a number of settings. For example, when a worker accepts a job offer, the resulting job might only be temporary (see, e.g., Ljungqvist and Sargent (2008), Chatterjee and Rossi-Hansberg (2012), Lise (2013), Moscarini and Postel-Vinay (2013), Bagger et al. (2014)). In sovereign default models, default typically leads to a period of exclusion from international financial markets. With positive probability, the country exits autarky and begins borrowing again (see, e.g., Arellano (2008), Chatterjee and Eyigungor (2012), Mendoza and Yue (2012), Hatchondo et al. (2016)).

To put this type of problem in a general setting, suppose that, at date t, an agent is either *active* or *passive*. When active, the agent observes Z_t and chooses whether to continue or to exit. Continuation results in a current reward $c(Z_t)$ and the agent remains active at $t+1$. Exit results in a current reward $s(Z_t)$ and transition to the passive state. From there the agent has no action available, but will return to the active state at $t+1$ and all subsequent period with probability α.

Assumption 6.1. There exist a $\mathscr{Z}$-measurable function $g : \mathsf{Z} \to \mathbb{R}_+$ and constants $n \in \mathbb{N}_0$ and $a_1, \cdots, a_4, m, d \in \mathbb{R}_+$ such that $\beta m < 1$, and, for all $z \in \mathsf{Z}$,

(1) $\int |s(z')| P^n(z, \mathrm{d}z') \leq a_1 g(z) + a_2$;
(2) $\int |c(z')| P^n(z, \mathrm{d}z') \leq a_3 g(z) + a_4$;
(3) $\int g(z') P(z, \mathrm{d}z') \leq m g(z) + d$.

Let $v^*(z)$ and $r^*(z)$ be the maximal discounted value starting at $z \in \mathsf{Z}$ in the active and passive state respectively. The following principle of optimality holds.[26]

Lemma 6.1. *Under assumption 6.1, v^* and r^* satisfy*

$$v^*(z) = \max \left\{ r^*(z), c(z) + \beta \int v^*(z') P(z, \mathrm{d}z') \right\}, \textit{ and}$$

$$r^*(z) = s(z) + \alpha\beta \int v^*(z') P(z, \mathrm{d}z') + (1-\alpha)\beta \int r^*(z') P(z, \mathrm{d}z').$$

[26]See the working paper version (Ma and Stachurski, 2017) for the proof of lemma 6.1.

Let $\psi^* := c + \beta P v^*$. By lemma 6.1, we can write $v^* = r^* \vee \psi^*$, and view ψ^* and r^* as solutions to the functional equations

$$\psi = c + \beta P(r \vee \psi) \quad \text{and} \quad r = s + \alpha\beta P(r \vee \psi) + (1-\alpha)\beta P r. \tag{25}$$

Let m' and d' be positive constants. We consider $\kappa : \mathsf{Z} \to \mathbb{R}_+$ defined by

$$\kappa(z) := m' \sum_{t=0}^{n-1} \mathbb{E}_z \left[|s(Z_t)| + |c(Z_t)| \right] + g(z) + d' \tag{26}$$

and the product space $(b_\kappa \mathsf{Z} \times b_\kappa \mathsf{Z}, \rho_\kappa)$, where ρ_κ is a metric on $b_\kappa \mathsf{Z} \times b_\kappa \mathsf{Z}$ defined by

$$\rho_\kappa((\psi, r), (\psi', r')) = \|\psi - \psi'\|_\kappa \vee \|r - r'\|_\kappa.$$

With this metric, $(b_\kappa \mathsf{Z} \times b_\kappa \mathsf{Z}, \rho_\kappa)$ inherits the completeness of $(b_\kappa \mathsf{Z}, \|\cdot\|_\kappa)$. Now define the Jovanovic operator L on $b_\kappa \mathsf{Z} \times b_\kappa \mathsf{Z}$ by

$$L \begin{pmatrix} \psi \\ r \end{pmatrix} = \begin{pmatrix} c + \beta P(r \vee \psi) \\ s + \alpha\beta P(r \vee \psi) + (1-\alpha)\beta P r \end{pmatrix}.$$

In this setting the following optimality result holds.

Theorem 6.1. *Let assumption 6.1 hold. Then there exist positive constants m' and d' such that for κ defined by (26), the following statements hold:*

1. *L is a contraction mapping on $(b_\kappa \mathsf{Z} \times b_\kappa \mathsf{Z}, \rho_\kappa)$.*
2. *The unique fixed point of L in $b_\kappa \mathsf{Z} \times b_\kappa \mathsf{Z}$ is $h^* := (\psi^*, r^*)$.*

This result closely parallels theorem 2.1 from the permanent exit setting.

7. Conclusion

In this paper we developed a general theory of optimal timing of decisions based around continuation values and operators that act on them. Optimality results were provided under general settings, with bounded or unbounded rewards, as were results on continuity, monotonicity and differentiability of the continuation value function and, when it exists, the threshold optimal policy. In providing these results we exploited the fact that the continuation value function is often smoother and than the value function, and exists in a lower dimensional space. The benefits of these properties were also highly visible in our simulations.

APPENDIX A: SOME LEMMAS

Lemma 7.1. *Under assumption 2.1, there exist $b_1, b_2 \in \mathbb{R}_+$ such that for all $z \in \mathsf{Z}$,*

(1) $|v^*(z)| \leq \sum_{t=0}^{n-1} \beta^t \mathbb{E}_z[|r(Z_t)| + |c(Z_t)|] + b_1 g(z) + b_2$.
(2) $|\psi^*(z)| \leq \sum_{t=1}^{n-1} \beta^t \mathbb{E}_z|r(Z_t)| + \sum_{t=0}^{n-1} \beta^t \mathbb{E}_z|c(Z_t)| + b_1 g(z) + b_2$.

Proof. Without loss of generality, we assume $m \neq 1$ in assumption 2.1. By that assumption, $\mathbb{E}_z|r(Z_n)| \leq a_1 g(z) + a_2$, $\mathbb{E}_z|c(Z_n)| \leq a_3 g(z) + a_4$ and $\mathbb{E}_z g(Z_1) \leq m g(z) + d$ for all $z \in \mathsf{Z}$. For all $t \geq 1$, by the Markov property (see, e.g., Meyn and Tweedie (2012), section 3.4.3),

$$\mathbb{E}_z g(Z_t) = \mathbb{E}_z \left[\mathbb{E}_z \left(g(Z_t)|\mathscr{F}_{t-1}\right)\right] = \mathbb{E}_z \left(\mathbb{E}_{Z_{t-1}} g(Z_1)\right) \leq m \, \mathbb{E}_z g(Z_{t-1}) + d.$$

Induction shows that for all $t \geq 0$,

$$\mathbb{E}_z g(Z_t) \leq m^t g(z) + \frac{1 - m^t}{1 - m} d. \tag{27}$$

Moreover, for all $t \geq n$, applying the Markov property again yields

$$\mathbb{E}_z|r(Z_t)| = \mathbb{E}_z \left[\mathbb{E}_z \left(|r(Z_t)||\mathscr{F}_{t-n}\right)\right] = \mathbb{E}_z \left(\mathbb{E}_{Z_{t-n}}|r(Z_n)|\right) \leq a_1 \, \mathbb{E}_z g(Z_{t-n}) + a_2.$$

By (27), for all $t \geq n$, we have

$$\mathbb{E}_z|r(Z_t)| \leq a_1 \left(m^{t-n} g(z) + \frac{1 - m^{t-n}}{1 - m} d\right) + a_2. \tag{28}$$

Similarly, for all $t \geq n$, we have

$$\mathbb{E}_z|c(Z_t)| \leq a_3 \, \mathbb{E}_z g(Z_{t-n}) + a_4 \leq a_3 \left(m^{t-n} g(z) + \frac{1 - m^{t-n}}{1 - m} d\right) + a_4. \tag{29}$$

Let $S(z) := \sum_{t \geq 1} \beta^t \mathbb{E}_z \left[|r(Z_t)| + |c(Z_t)|\right]$. Based on (27)–(29), we can show that

$$S(z) \leq \sum_{t=1}^{n-1} \beta^t \mathbb{E}_z[|r(Z_t)| + |c(Z_t)|] + \frac{a_1 + a_3}{1 - \beta m} g(z) + \frac{(a_1 + a_3)d + a_2 + a_4}{(1 - \beta m)(1 - \beta)}. \tag{30}$$

Since $|v^*| \leq |r| + |c| + S$ and $|\psi^*| \leq |c| + S$, the two claims hold by letting $b_1 := \frac{a_1 + a_3}{1 - \beta m}$ and $b_2 := \frac{(a_1+a_3)d + a_2 + a_4}{(1 - \beta m)(1 - \beta)}$. □

Let $(X, \mathcal{X}, \nu)$ and $(Y, \mathcal{Y}, u)$ two measure spaces. Lemma 7.2 below can be shown by the Fatou's lemma. The idea of proof is similar to proposition 3.1 below.

Lemma 7.2. *Let $p : Y \times X \to \mathbb{R}$ be a measurable map that is continuous in x. If there exists a measurable map $q : Y \times X \to \mathbb{R}$ that is continuous in x with $q \geq |p|$ on $Y \times X$, and that $x \mapsto \int q(y, x) u(\mathrm{d}y)$ is continuous, then $x \mapsto \int p(y, x) u(\mathrm{d}y)$ is continuous.*

APPENDIX B : MAIN PROOFS

7.1. Proof of Section 2 Results.

Proof of lemma 2.1. By theorem 1.11 of Peskir and Shiryaev (2006), it suffices to show that $\mathbb{E}_z \left(\sup_{k \geq 0} \left| \sum_{t=0}^{k-1} \beta^t c(Z_t) + \beta^k r(Z_k) \right| \right) < \infty$ for all $z \in \mathsf{Z}$. This is true since

$$\sup_{k \geq 0} \left| \sum_{t=0}^{k-1} \beta^t c(Z_t) + \beta^k r(Z_k) \right| \leq \sum_{t \geq 0} \beta^t [|r(Z_t)| + |c(Z_t)|] \tag{31}$$

with probability one, and by the monotone convergence theorem and lemma 7.1 (see (30) in appendix A), the right hand side of (31) is $\mathbb{P}_z$-integrable for all $z \in \mathsf{Z}$. $\square$

Proof of theorem 2.1. Let $d_1 := a_1 + a_3$ and $d_2 := a_2 + a_4$. Since $\beta m < 1$ by assumption 2.1, we can choose positive constants m' and d' such that $m + d_1 m' > 1$, $\beta(m + d_1 m') < 1$ and $d' \geq (d_2 m' + d)/(m + d_1 m' - 1)$.

To prove claim 1, by the weighted contraction mapping theorem (see, e.g., Boyd (1990), section 3), it suffices to verify: (a) Q is monotone, i.e., $Q\psi \leq Q\phi$ if $\psi, \phi \in b_\ell \mathsf{Z}$ and $\psi \leq \phi$; (b) $Q0 \in b_\ell \mathsf{Z}$ and $Q\psi$ is $\mathscr{Z}$-measurable for all $\psi \in b_\ell \mathsf{Z}$; and (c) $Q(\psi + a\ell) \leq Q\psi + a\beta(m + d_1 m')\ell$ for all $a \in \mathbb{R}_+$ and $\psi \in b_\ell \mathsf{Z}$. Obviously, condition (a) holds. By (8)–(9), we have

$$\frac{|(Q0)(z)|}{\ell(z)} \leq \frac{|c(z)|}{\ell(z)} + \beta \int \frac{|r(z')|}{\ell(z)} P(z, dz') \leq (1 + \beta)/m' < \infty$$

for all $z \in \mathsf{Z}$, so $\|Q0\|_\ell < \infty$. The measurability of $Q\psi$ follows immediately from our primitive assumptions. Hence, condition (b) holds. By the Markov property (see, e.g., Meyn and Tweedie (2012), section 3.4.3), we have

$$\int \mathbb{E}_{z'} |r(Z_t)| P(z, \mathrm{d}z') = \mathbb{E}_z |r(Z_{t+1})| \;\text{ and }\; \int \mathbb{E}_{z'} |c(Z_t)| P(z, \mathrm{d}z') = \mathbb{E}_z |c(Z_{t+1})|.$$

Let $h(z) := \sum_{t=1}^{n-1} \mathbb{E}_z|r(Z_t)| + \sum_{t=0}^{n-1} \mathbb{E}_z|c(Z_t)|$, then we have

$$\int h(z')P(z, \mathrm{d}z') = \sum_{t=2}^{n} \mathbb{E}_z|r(Z_t)| + \sum_{t=1}^{n} \mathbb{E}_z|c(Z_t)|. \tag{32}$$

By the construction of m' and d', we have $m + d_1 m' > 1$ and $(d_2 m' + d + d')/(m + d_1 m') \leq d'$. Assumption 2.1 and (32) then imply that

$$\begin{aligned}
\int \ell(z')P(z, dz') &= m' \left(\sum_{t=2}^{n} \mathbb{E}_z|r(Z_t)| + \sum_{t=1}^{n} \mathbb{E}_z|c(Z_t)| \right) + \int g(z')P(z, dz') + d' \\
&\leq m' \left(\sum_{t=2}^{n-1} \mathbb{E}_z|r(Z_t)| + \sum_{t=1}^{n-1} \mathbb{E}_z|c(Z_t)| \right) + (m + d_1 m')g(z) + d_2 m' + d + d' \\
&\leq (m + d_1 m') \left(\frac{m'}{m + d_1 m'} h(z) + g(z) + \frac{d_2 m' + d + d'}{m + d_1 m'} \right) \leq (m + d_1 m')\ell(z).
\end{aligned}$$

Hence, for all $\psi \in b_\ell \mathsf{Z}$, $a \in \mathbb{R}_+$ and $z \in \mathsf{Z}$, we have

$$\begin{aligned}
Q(\psi + a\ell)(z) &= c(z) + \beta \int \max\left\{r(z'), \psi(z') + a\ell(z')\right\} P(z, dz') \\
&\leq c(z) + \beta \int \max\left\{r(z'), \psi(z')\right\} P(z, dz') + a\beta \int \ell(z')P(z, dz') \\
&\leq Q\psi(z) + a\beta(m + d_1 m')\ell(z).
\end{aligned}$$

So condition (c) holds. Claim 1 is verified.

Regarding claim 2, we have shown that ψ^* is a fixed point of Q under assumption 2.1 (see lemma 2.1 and (7)). Moreover, from lemma 7.1 we know that $\psi^* \in b_\ell \mathsf{Z}$. Hence, ψ^* must coincide with the unique fixed point of Q under $b_\ell \mathsf{Z}$.

Finally, by theorem 1.11 of Peskir and Shiryaev (2006), we can show that $\tilde{\tau} := \inf\{t \geq 0 : v^*(Z_t) = r(Z_t)\}$ is an optimal stopping time. Claim 3 then follows from the definition of the optimal policy and lemma 2.1. □

7.2. **Proof of Section 3 Results.**

Proof of proposition 3.1. Let $b_\ell c\mathsf{Z}$ be the set of continuous functions in $b_\ell \mathsf{Z}$. Since ℓ is continuous by assumption 3.1, $b_\ell c\mathsf{Z}$ is a closed subset of $b_\ell \mathsf{Z}$ (see e.g., Boyd (1990), section 3). To show the continuity of ψ^*, it suffices to verify that $Q(b_\ell c\mathsf{Z}) \subset b_\ell c\mathsf{Z}$ (see, e.g., Stokey et al. (1989), corollary 1 of theorem 3.2). For fixed $\psi \in b_\ell c\mathsf{Z}$, let $h(z) := \max\{r(z), \psi(z)\}$, then there exists $G \in \mathbb{R}_+$ such that $|h(z)| \leq |r(z)| + G\ell(z) =: \tilde{h}(z)$. By assumption 3.1,

$z \mapsto \tilde{h}(z) \pm h(z)$ are nonnegative and continuous. Based on the generalized Fatou's lemma of Feinberg et al. (2014) (theorem 1.1), for all sequence $(z_m)_{m\geq 0}$ of Z with $z_m \to z \in \mathsf{Z}$,

$$\int \left(\tilde{h}(z') \pm h(z')\right) P(z, \mathrm{d}z') \leq \liminf_{m\to\infty} \int \left(\tilde{h}(z') \pm h(z')\right) P(z_m, \mathrm{d}z').$$

Since $\lim_{m\to\infty} \int \tilde{h}(z')P(z_m, \mathrm{d}z') = \int \tilde{h}(z')P(z, \mathrm{d}z')$ by assumption 3.1, we have

$$\pm \int h(z')P(z, \mathrm{d}z') \leq \liminf_{m\to\infty} \left(\pm \int h(z')P(z_m, \mathrm{d}z') \right),$$

where we have used the fact that for given sequences $(a_m)_{m\geq 0}$ and $(b_m)_{m\geq 0}$ of $\mathbb{R}$ with $\lim_{m\to\infty} a_m$ exists, we have: $\liminf_{m\to\infty}(a_m + b_m) = \lim_{m\to\infty} a_m + \liminf_{m\to\infty} b_m$. Hence,

$$\limsup_{m\to\infty} \int h(z')P(z_m, \mathrm{d}z') \leq \int h(z')P(z, \mathrm{d}z') \leq \liminf_{m\to\infty} \int h(z')P(z_m, \mathrm{d}z'), \tag{33}$$

i.e., $z \mapsto \int h(z')P(z, \mathrm{d}z')$ is continuous. Since c is continuous by assumption, $Q\psi \in b_\ell c\mathsf{Z}$. Hence, $Q(b_\ell c\mathsf{Z}) \subset b_\ell c\mathsf{Z}$, and ψ^* is continuous, as was to be shown. □

Proof of proposition 3.2. Standard argument shows that $b_\ell i\mathsf{Z}$, the set of increasing functions in $b_\ell\mathsf{Z}$, is a closed subset. To show that ψ^* is increasing, it suffices to verify that $Q(b_\ell i\mathsf{Z}) \subset b_\ell i\mathsf{Z}$ (see, e.g., Stokey et al. (1989), corollary 1 of theorem 3.2). The assumptions of the proposition guarantee that this is the case. □

Let $\mu(z) := \int \max\{r(z'), \psi^*(z')\} f(z'|z)\, \mathrm{d}z'$ and $\mu_i(z) := \int \max\{r(z'), \psi^*(z')\} D_i f(z'|z)\, \mathrm{d}z'$.

Lemma 7.3. *Suppose assumption 2.1 holds, and, for $i = 1, ..., m$*

1. *P has a density representation f such that $D_i f(z'|z)$ exists, $\forall (z, z') \in \mathrm{int}(\mathsf{Z}) \times \mathsf{Z}$.*
2. *For all $z_0 \in \mathrm{int}(\mathsf{Z})$, there exists $\delta > 0$, such that for $k \in \{r, \ell\}$,*

$$\int |k(z')| \sup_{z^i \in \bar{B}_\delta(z_0^i)} \left|D_i f(z'|z)\right| \mathrm{d}z' < \infty \qquad (z^{-i} = z_0^{-i}).$$

Then: $D_i\mu(z) = \mu_i(z)$ for all $z \in \mathrm{int}(\mathsf{Z})$ and $i = 1, ..., m$.

Proof of lemma 7.3. For all $z_0 \in \mathrm{int}(\mathsf{Z})$, let $\{z_n\}$ be an arbitrary sequence of $\mathrm{int}(\mathsf{Z})$ such that $z_n^i \to z_0^i$, $z_n^i \neq z_0^i$ and $z_n^{-i} = z_0^{-i}$ for all $n \in \mathbb{N}$. For the $\delta > 0$ given by (2), there exists $N \in \mathbb{N}$ such that $z_n^i \in \bar{B}_\delta(z_0^i)$ for all $n \geq N$. Holding $z^{-i} = z_0^{-i}$, by the mean value theorem, there exists $\xi^i(z', z_n, z_0) \in \bar{B}_\delta(z_0^i)$ such that

$$|\triangle^i(z', z_n, z_0)| := \left| \frac{f(z'|z_n) - f(z'|z_0)}{z_n^i - z_0^i} \right| = \left| D_i f(z'|z)_{z^i = \xi^i(z', z_n, z_0)} \right| \leq \sup_{z^i \in \bar{B}_\delta(z_0^i)} \left| D_i f(z'|z) \right|.$$

Since in addition $|\psi^*| \leq G\ell$ for some $G \in \mathbb{R}_+$, we have: for all $n \geq N$,

(a) $\left|\max\{r(z'), \psi^*(z')\}\triangle^i(z', z_n, z_0)\right| \leq (|r(z')| + G\ell(z')) \sup_{z^i \in \bar{B}_\delta(z_0^i)} |D_i f(z'|z)|$,

(b) $\int (|r(z')| + G\ell(z')) \sup_{z^i \in \bar{B}_\delta(z_0^i)} |D_i f(z'|z)| \, dz' < \infty$, and

(c) $\max\{r(z'), \psi^*(z')\}\triangle^i(z', z_n, z_0) \to \max\{r(z'), \psi^*(z')\}D_i f(z'|z_0)$ as $n \to \infty$,

where (b) follows from condition (2). By the dominated convergence theorem,

$$\begin{aligned}\frac{\mu(z_n) - \mu(z_0)}{z_n^i - z_0^i} &= \int \max\{r(z'), \psi^*(z')\}\triangle^i(z', z_n, z_0)\, \mathrm{d}z' \\ &\to \int \max\{r(z'), \psi^*(z')\}D_i f(z'|z_0)\, \mathrm{d}z' = \mu_i(z_0).\end{aligned}$$

Hence, $D_i\mu(z_0) = \mu_i(z_0)$, and the claim of the lemma is verified. □

Proof of proposition 3.3. Fix $z_0 \in \text{int}(\mathsf{Z})$. By assumption 3.4 (2)–(3), there exist $\delta > 0$ and a compact subset $A \subset \mathsf{Z}$ such that $z' \notin A$ implies $z_i^*(z', z_0^{-i}) \notin B_\delta(z_0^i)$, hence, for $z^{-i} = z_0^{-i}$,

$$\sup_{z^i \in \bar{B}_\delta(z_0^i)} |D_i f(z'|z)| = |D_i f(z'|z)|_{z^i = z_0^i + \delta} \vee |D_i f(z'|z)|_{z^i = z_0^i - \delta} =: h^\delta(z', z_0).$$

By assumption 3.4-(2), there exists $G \in \mathbb{R}_+$, such that for $z^{-i} = z_0^{-i}$,

$$\begin{aligned}\sup_{z^i \in \bar{B}_\delta(z_0^i)} |D_i f(z'|z)| &\leq \sup_{z' \in A, z^i \in \bar{B}_\delta(z_0^i)} |D_i f(z'|z)| \cdot \mathbb{1}(z' \in A) + h^\delta(z', z_0) \cdot \mathbb{1}(z' \in A^c) \\ &\leq G \cdot \mathbb{1}(z' \in A) + \left(|D_i f(z'|z)|_{z^i = z_0^i + \delta} + |D_i f(z'|z)|_{z^i = z_0^i - \delta}\right) \cdot \mathbb{1}(z' \in A^c).\end{aligned}$$

Assumption 3.5 then shows that condition (2) of lemma 7.3 holds. By assumption 3.3 and lemma 7.3, $D_i\psi^*(z) = D_i c(z) + \mu_i(z)$ for all $z \in \text{int}(\mathsf{Z})$, as was to be shown. □

Proof of proposition 3.4. Since assumption 3.6 implies assumptions 3.3 and 3.5, by proposition 3.3, $D_i\psi^*(z) = D_i c(z) + \mu_i(z)$ on $\text{int}(\mathsf{Z})$. Since $D_i c(z)$ is continuous by assumption 3.6-(1), to show that ψ^* is continuously differentiable, it remains to verify: $z \mapsto \mu_i(z)$ is continuous on $\text{int}(\mathsf{Z})$. Since $|\psi^*| \leq G\ell$ for some $G \in \mathbb{R}_+$,

$$\left|\max\{r(z'), \psi^*(z')\}D_i f(z'|z)\right| \leq (|r(z')| + G\ell(z'))|D_i f(z'|z)|, \ \forall z', z \in \mathsf{Z}. \tag{34}$$

By assumptions 3.4-(2) and 3.6-(2), both sides of (34) are continuous in z, and $z \mapsto \int [|r(z')| + G\ell(z')]|D_i f(z'|z)|\, \mathrm{d}z'$ is continuous. Then $z \mapsto \mu_i(z)$ is continuous by lemma 7.2. □

7.3. **Proof of Section 5 Results.**

Proof of proposition 5.1. Regarding claims 1–2, the exit reward satisfies

$$\left|r(f',\mu',\gamma')\right| \leq 1/a + \left(\mathrm{e}^{a^2\gamma_x/2}/a\right)\cdot \mathrm{e}^{-a\mu'+a^2\gamma'/2} + |f'|. \tag{35}$$

Using (22), we can show that (recall the first claim of footnote 9)

$$\int \mathrm{e}^{-a\mu'+a^2\gamma'/2}P(z,\mathrm{d}z') = \int \mathrm{e}^{-a\mu'+a^2\gamma'/2}l(y'|\mu,\gamma)\,\mathrm{d}y' = \mathrm{e}^{-a\mu+a^2\gamma/2}. \tag{36}$$

Let $\mu_f := \int |f|h(f)\,\mathrm{d}f$. Combining (35)–(36) yields

$$\int \left|r(f',\mu',\gamma')\right| P(z,\mathrm{d}z') \leq (1/a+\mu_f) + \left(\mathrm{e}^{a^2\gamma_x/2}/a\right)\cdot \mathrm{e}^{-a\mu+a^2\gamma/2}. \tag{37}$$

By (36)–(37), assumption 2.1 holds for $n := 1$, $g(\mu,\gamma) := \mathrm{e}^{-a\mu+a^2\gamma/2}$, $m := 1$ and $d := 0$. Moreover, the intermediate value theorem shows that assumption 4.1 holds. By theorem 2.1 and (21) (section 4.1), claims 1–2 hold.

Regarding claim 3, P is Feller by (22) and lemma 7.2. By (22), both sides of (35) are continuous in (μ,γ). By (36), the conditional expectation of the right side of (35) is continuous in (μ,γ). Lemma 7.2 implies that $(\mu,\gamma)\mapsto \mathbb{E}_{\mu,\gamma}|r(Z_1)|$ is continuous. Since in addition g is continuous and $g(\mu,\gamma) = \mathbb{E}_{\mu,\gamma}g(\mu',\gamma')$ by (36), assumption 3.1 holds. Claim 3 then follows from propositions 3.1 and 4.1.

Since l is stochastically increasing in μ for $\rho \geq 0$, claim 4 holds by proposition 3.2. $\square$

Proof of proposition 5.2. For claims 1–2, since $w = \eta + \theta\xi$ and $|\ln w| \leq 1/w + w$, we have

$$\begin{aligned}\int |\ln w'|P(z,\mathrm{d}z') &\leq \int (1/\eta'+\eta')v(\eta')\,\mathrm{d}\eta' + \int \xi' h(\xi')\,\mathrm{d}\xi' \cdot \int \theta' f(\theta'|\theta)\,\mathrm{d}\theta' \\ &= \mu_\eta^- + \mu_\eta^+ + \mu_\xi \cdot e^{\gamma_u/2}\theta^\rho,\end{aligned} \tag{38}$$

where $\mu_\eta^+ := \int \eta v(\eta)\,\mathrm{d}\eta$, $\mu_\eta^- := \int \eta^{-1}v(\eta)\,\mathrm{d}\eta$ and $\mu_\xi := \int \xi h(\xi)\,\mathrm{d}\xi$. Hence,[27]

$$\int |\ln w'|P^t(z,\mathrm{d}z') \leq a_1^{(t)}\,\theta^{\rho^t} + a_2^{(t)} \leq a_1^{(t)}\left(\theta^{\rho^t}+\theta^{-\rho^t}\right) + a_2^{(t)} \tag{39}$$

for some $a_1^{(t)}, a_2^{(t)} > 0$ (do not depend on θ) and all $t \in \mathbb{N}$. Let n, g, m and d be defined as in section 5.3. Then since $\theta^{\rho^{n+1}} + \theta^{-\rho^{n+1}} \leq \theta^{\rho^n} + \theta^{-\rho^n} + 1$ for $\theta > 0$ and $\rho \in [-1,1]$, we have

$$\int g(\theta')f(\theta'|\theta)\,\mathrm{d}\theta' = \left(\theta^{\rho^{n+1}} + \theta^{-\rho^{n+1}}\right)e^{\rho^{2n}\gamma_u/2} \leq mg(\theta) + d. \tag{40}$$

[27]Recall that if $X \sim LN(\mu,\sigma^2)$, then $\mathbb{E}\,X^s = \mathrm{e}^{s\mu+s^2\sigma^2/2}$ for all $s \in \mathbb{R}$.

Hence, assumption 2.1 holds. Assumption 4.1 holds by the intermediate value theorem. Claims 1–2 then follow from theorem 2.1 and (21) (section 4.1).

Regarding claim 3, it is straightforward to show that $\theta \mapsto f(\theta'|\theta)$ is twice differentiable for all θ', that $(\theta, \theta') \mapsto \partial f(\theta'|\theta)/\partial\theta$ is continuous, and that

$$\partial^2 f(\theta'|\theta)/\partial\theta^2 = 0 \text{ has two solutions: } \theta = \theta^*(\theta') = \tilde{a}_i \, \mathrm{e}^{\ln\theta'/\rho}, \; i = 1, 2$$

where $\tilde{a}_1, \tilde{a}_2 > 0$ are constants. If $\rho > 0\,(< 0)$, then $\theta^*(\theta') \to \infty\,(0)$ as $\theta' \to \infty$, and $\theta^*(\theta') \to 0\,(\infty)$ as $\theta' \to 0$. Hence, assumption 3.4 holds. Based on (38)–(40) and lemma 7.2, assumption 3.6 holds. Claim 3 then holds by propositions 3.4 and 4.3.

As $f(\theta'|\theta)$ is stochastically increasing ($\rho > 0$), claim 4 holds by propositions 3.2 and 4.4. $\square$

7.4. **Proof of Section 6 Results.**

Proof of theorem 6.1. Let $d_1 := a_1 + a_3$ and $d_2 := a_2 + a_4$. Since $\beta m < 1$ by assumption 6.1, we can choose $m', d' > 0$ such that $m + d_1 m' > 1$, $\beta(m + d_1 m') < 1$ and $d' \geq \frac{d_2 m' + d}{m + d_1 m' - 1}$.

Regarding claim 1, similar to the proof of theorem 2.1, we have

$$\int \kappa(z') P(z, \mathrm{d}z') \leq (m + d_1 m') \kappa(z). \tag{41}$$

for all $z \in \mathsf{Z}$. We first show that $L\colon (b_\kappa \mathsf{Z} \times b_\kappa \mathsf{Z}, \rho_\kappa) \to (b_\kappa \mathsf{Z} \times b_\kappa \mathsf{Z}, \rho_\kappa)$. For all $h := (\psi, r) \in b_\kappa \mathsf{Z} \times b_\kappa \mathsf{Z}$, define the functions $p(z) := c(z) + \beta \int \max\{r(z'), \psi(z')\} P(z, \mathrm{d}z')$ and $q(z) := s(z) + \alpha\beta \int \max\{r(z'), \psi(z')\} P(z, \mathrm{d}z') + (1 - \alpha)\beta \int r(z') P(z, \mathrm{d}z')$. Obviously, p and q are $\mathscr{Z}$-measurable, and there exists $G \in \mathbb{R}_+$ such that for all $z \in \mathsf{Z}$,

$$\frac{|p(z)| \vee |q(z)|}{\kappa(z)} \leq \frac{|c(z)| \vee |s(z)|}{\kappa(z)} + \frac{\beta G \int \kappa(z') P(z, \mathrm{d}z')}{\kappa(z)} \leq \frac{1}{m'} + \beta(m + d_1 m') G < \infty.$$

These imply that $p \in b_\kappa \mathsf{Z}$ and $q \in b_\kappa \mathsf{Z}$, and hence $Lh \in b_\kappa \mathsf{Z} \times b_\kappa \mathsf{Z}$. Next, we show that L is indeed a contraction mapping on $(b_\kappa \mathsf{Z} \times b_\kappa \mathsf{Z}, \rho_\kappa)$. For all $h_1 := (\psi_1, r_1)$ and $h_2 := (\psi_2, r_2)$ in $b_\kappa \mathsf{Z} \times b_\kappa \mathsf{Z}$, we have $\rho_\kappa(Lh_1, Lh_2) = I \vee J$, with $I := \|\beta P(r_1 \vee \psi_1) - \beta P(r_2 \vee \psi_2)\|_\kappa$ and $J := \|\alpha\beta[P(r_1 \vee \psi_1) - P(r_2 \vee \psi_2)] + (1 - \alpha)\beta(P r_1 - P r_2)\|_\kappa$. For all $z \in \mathsf{Z}$,

$$\begin{aligned} |P(r_1 \vee \psi_1)(z) - P(r_2 \vee \psi_2)(z)| &\leq \int |r_1 \vee \psi_1 - r_2 \vee \psi_2| \, (z') P(z, \mathrm{d}z') \\ &\leq \int (|\psi_1 - \psi_2| \vee |r_1 - r_2|)(z') P(z, \mathrm{d}z') \leq \rho_\kappa(h_1, h_2) \int \kappa(z') P(z, \mathrm{d}z'), \end{aligned} \tag{42}$$

where the second inequality is due to the fact that $|a \vee b - a' \vee b'| \leq |a - a'| \vee |b - b'|$. Combining (41)–(42) yields $I \leq \beta(m + d_1 m')\rho_\kappa(h_1, h_2)$. Similarly, $J \leq \beta(m + d_1 m')\rho_\kappa(h_1, h_2)$ holds. L is then a contraction mapping on $(b_\kappa \mathsf{Z} \times b_\kappa \mathsf{Z}, \rho_\kappa)$ of modulus $\beta(m + d_1 m')$, since

$$\rho_\kappa(Lh_1, Lh_2) = I \vee J \leq \beta(m + d_1 m')\rho_\kappa(h_1, h_2).$$

Regarding claim 2, since ψ^* and r^* solves (25) by lemma 6.1, $h^* := (\psi^*, r^*)$ is indeed a fixed point of L. To verify the claim, it remains to show that $h^* \in b_\kappa \mathsf{Z} \times b_\kappa \mathsf{Z}$. Since

$$\max\{|r^*(z)|, |\psi^*(z)|\} \leq \sum_{t=0}^{\infty} \beta^t \mathbb{E}_z[|s(Z_t)| + |c(Z_t)|],$$

this can be done in a similar way as lemma 7.1. Hence, claim 2 holds. □

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