Continuation Value Methods for Sequential Decisions: Optimality and Efficiency[1]

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ABSTRACT. This paper provides systematic analysis of a solution method for sequential decision problems originally due to Jovanovic (1982). We show that the operator employed by this method is semiconjugate to the Bellman operator associated with the corresponding dynamic program and has essentially equivalent dynamic properties. At the same time, for a broad class of sequential problems, the effective state space for this operator is strictly smaller. We characterize the difference in terms of model primitives and provide a range of examples.

Keywords: Continuation values, dynamic programming, sequential decisions

1. INTRODUCTION

Many decision making problems involve choosing when to act in the face of risk and uncertainty. Examples include deciding if or when to accept a job offer, exit or enter a market, default on a loan, bring a new product to market, exploit some new technology, or exercise an option (see, e.g., McCall (1970), Jovanovic (1982), Hopenhayn (1992), Dixit and Pindyck (1994), Ericson and Pakes (1995), Arellano (2008), Perla and Tonetti (2014), Fajgelbaum et al. (2017), and Schaal (2017)).

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Sequential decision problems that involve choosing when to act can be solved using standard dynamic programming methods based around the Bellman equation. There is, however, an alternative approach—introduced by Jovanovic (1982) in the context of industry dynamics—that focuses on continuation values. The idea involves calculating the continuation value directly, using an operator referred to below as the Jovanovic operator. This technique is now well-known to economists and routinely employed in a large variety of economic applications (see, e.g., Gomes et al. (2001), Ljungqvist and Sargent (2008), Lise (2013), Moscarini and Postel-Vinay (2013), Fajgelbaum et al. (2017), and Schaal (2017)).

Despite the existence of these two parallel and commonly used methods, their theoretical connections and relative efficiency have hitherto received no general investigation. One cost of this status quo is that studies using continuation value methods have been compelled to provide their own optimality analysis piecemeal in individual applications (see, e.g., Jovanovic (1982), Moscarini and Postel-Vinay (2013), or Fajgelbaum et al. (2017)). A second cost is that the most effective choice of method vis-a-vis a given application is often unknown ex-ante, and revealed only by experimentation in particular settings.

Here we begin a systematic analysis of the relationship between these two methods in a generic optimal stopping setting. As a first step, we show that the Bellman operator and the Jovanovic operator are semiconjugate, implying that any fixed point of one of the operators is a direct translation of a fixed point of the other. Iterative sequences generated by the operators are also simple translations. Second, we add topological structure to the generic setting and show that, the Bellman operator and Jovanovic operator are both contraction mappings under identical assumptions, and that convergence to the respective fixed points occurs at the same rate.

The results stated above all elucidate the natural similarity between Bellman and Jovanovic operators. Despite these similarities, there do however remain important differences in terms of efficiency and analytical convenience. These differences concern the dimensionality of the effective state spaces associated with each operator. In particular, for an important class of problems, referred to below as continuation decomposable problems, the effective state space of the continuation value function is strictly lower than that of the value function. We characterize this class in terms of the structure of reward and state transition functions and provide a range of examples.

Lower dimensionality simplifies both theory and computation. As one illustration we study the time complexity of iteration with the Jovanovic and Bellman operators and quantify the difference analytically. These large efficiency gains—typically measured in orders of magnitude—arise because, in the presence of continuation decomposability, continuation value based methods mitigate the curse of dimensionality, one of the primary stumbling blocks for dynamic programming (Rust (1997)).

To ensure sufficient generality for economic applications, we embed our optimality and symmetry arguments in (a) a space of potentially unbounded functions endowed with the weighted supremum norm distance, and (b) a space of integrable functions with divergence measured by L_p norm. In particular, unbounded rewards are permitted provided that they do not cause continuation values to diverge. In doing so we draw on and extend work on dynamic programming with unbounded rewards found in several economic studies, including Boyd (1990), Rincón-Zapatero and Rodríguez-Palmero (2003), Martins-da Rocha and Vailakis (2010), Kamihigashi (2014) and Bäuerle and Jaśkiewicz (2018).

The paper is structured as follows. Section 2 outlines the problem. Section 3 explores the symmetric theoretical properties of the Bellman and Jovanovic operators in terms of fixed points and convergence. Section 4 discusses the asymmetries of their relative efficiency. Section 5 provides applications. Longer proofs are deferred to the appendix.

2. Set Up

This section presents a generic optimal stopping problem and the key operators and optimality concepts. As a first step, we introduce some mathematical techniques and notation used in this paper.

2.1. **Preliminaries.** Let $\mathbb{N}$ be the set of natural numbers and $\mathbb{N}_0 := \{0\} \cup \mathbb{N}$. For $a, b \in \mathbb{R}$, let $a \vee b := \max\{a, b\}$. If f and g are functions, then $(f \vee g)(x) := f(x) \vee g(x)$. Given a Polish space Z and Borel sets $\mathscr{B}$, let $m\mathscr{B}$ be all $\mathscr{B}$-measurable functions from Z to $\mathbb{R}$. Given $\kappa \colon \mathsf{Z} \to (0, \infty)$, the *$\kappa$-weighted supremum norm* of $f \colon \mathsf{Z} \to \mathbb{R}$ is

$$\|f\|_\kappa := \sup_{z \in \mathsf{Z}} \frac{|f(z)|}{\kappa(z)}.$$

If $\|f\|_\kappa < \infty$, we say that f is *κ-bounded*. The symbol $b_\kappa\mathsf{Z}$ denotes all $\mathscr{B}$-measurable functions from Z to $\mathbb{R}$ that are κ-bounded.

Given a probability measure π on $(\mathsf{Z}, \mathscr{B})$ and a constant $p \geq 1$, let

$$\|f\|_p := \left(\int |f|^p \, \mathrm{d}\pi \right)^{1/p}.$$

Let $L_p(\pi)$ be all (equivalence classes of) functions $f \in m\mathscr{B}$ for which $\|f\|_p < \infty$.

Both $(b_\kappa\mathsf{Z}, \|\cdot\|_\kappa)$ and $(L_p(\pi), \|\cdot\|_p)$ form Banach spaces.

A *stochastic kernel* P on Z is a map $P \colon \mathsf{Z} \times \mathscr{B} \to [0,1]$ such that $z \mapsto P(z, B)$ is $\mathscr{B}$-measurable for each $B \in \mathscr{B}$ and $B \mapsto P(z, B)$ is a probability measure for each $z \in \mathsf{Z}$. For all $t \in \mathbb{N}$, $P^t(z, B) := \int P(z', B) P^{t-1}(z, \mathrm{d}z')$ is the probability of a state transition from z to $B \in \mathscr{B}$ in t steps, where $P^1(z, B) := P(z, B)$. A Z-valued stochastic process $\{Z_t\}$ on some probability space $(\Omega, \mathscr{F}, \mathbb{P})$ is called *P-Markov* if

$$\mathbb{P}\{Z_{t+1} \in B \mid \mathscr{F}_t\} = \mathbb{P}\{Z_{t+1} \in B \mid Z_t\} = P(Z_t, B)$$

$\mathbb{P}$-almost surely for all $t \in \mathbb{N}_0$ and all $B \in \mathscr{B}$. Here $\{\mathscr{F}_t\}$ is the natural filtration induced by $\{Z_t\}$. In what follows, $\mathbb{P}_z$ evaluates probabilities conditional on $Z_0 = z$ and $\mathbb{E}_z$ is the corresponding expectations operator.

2.2. **Optimal Stopping.** Let $(\mathsf{Z}, \mathscr{B})$ be a measurable space. For the purposes of this paper, an optimal stopping problem is a tuple (β, c, P, r) where

- $\beta \in (0, 1)$ is discount factor,
- $c \in m\mathscr{B}$ is a *flow continuation reward* function,
- P is a stochastic kernel on $(\mathsf{Z}, \mathscr{B})$, and
- $r \in m\mathscr{B}$ is a *terminal reward function*.

The interpretation is as follows: At time t an agent observes Z_t, the current realization of a Z-valued P-Markov process $\{Z_t\}_{t\geq 0}$, and chooses between stopping and continuing. Stopping generates terminal reward $r(Z_t)$ while continuing yields flow continuation reward $c(Z_t)$. If the agent continues, the time $t+1$ state Z_{t+1} is observed and the process repeats. Future rewards are discounted at rate β.

An $\mathbb{N}_0$-valued random variable τ is called a (finite) *stopping time* if $\mathbb{P}\{\tau < \infty\} = 1$ and $\{\tau \leq t\} \in \mathscr{F}_t$ for all $t \geq 0$. Let $\mathscr{M}$ denote all such stopping times. The *value function* v^* for (β, c, P, r) is defined at $z \in \mathsf{Z}$ by

$$v^*(z) := \sup_{\tau \in \mathscr{M}} \mathbb{E}_z \left\{ \sum_{t=0}^{\tau-1} \beta^t c(Z_t) + \beta^\tau r(Z_\tau) \right\}. \tag{1}$$

A stopping time $\tau \in \mathscr{M}$ is called *optimal* if it attains the supremum in (1). Assume that the value function solves the Bellman equation[2]

$$v^*(z) = \max \left\{ r(z),\, c(z) + \beta \int v^*(z') P(z, dz') \right\}. \tag{2}$$

The corresponding Bellman operator is

$$Tv(z) = \max \left\{ r(z), c(z) + \beta \int v(z') P(z, \mathrm{d}z') \right\}. \tag{3}$$

The *continuation value function* associated with this problem is defined at $z \in \mathsf{Z}$ by

$$\psi^*(z) := c(z) + \beta \int v^*(z') P(z, dz'). \tag{4}$$

Using (2) and (4), we observe that ψ^* satisfies

$$\psi^*(z) = c(z) + \beta \int \max \left\{ r(z'), \psi^*(z') \right\} P(z, \mathrm{d}z'). \tag{5}$$

Analogous to the Bellman operator, the *continuation value operator* or *Jovanovic operator* Q is constructed such that the continuation value function ψ^* is a fixed point:

$$Q\psi(z) = c(z) + \beta \int \max\{r(z'), \psi(z')\} P(z, \mathrm{d}z'). \tag{6}$$

[2]A sufficient condition is that $\mathbb{E}_z \left(\sup_{k \geq 0} \left| \sum_{t=0}^{k-1} \beta^t c(Z_t) + \beta^k r(Z_k) \right| \right) < \infty$ for all $z \in \mathsf{Z}$, as can be shown by applying theorem 1.11 (claim 1) of Peskir and Shiryaev (2006). Later we add assumptions under which this condition is guaranteed.

3. Symmetries Between the Operators

In this section we show that Bellman and Jovanovic operators are semiconjugate and discuss the implications.[3] The semiconjugate relationship is most easily shown using operator-theoretic notation. To this end, let $Ph(z) := \int h(z')P(z, \mathrm{d}z')$ for all integrable $h \in m\mathscr{B}$ and observe that the Bellman operator T can then be expressed as $T = RL$ where

$$R\psi := r \vee \psi \quad \text{and} \quad Lv := c + \beta P v. \tag{7}$$

(For any two operators we write the composition $A \circ B$ more simply as AB.)

3.1. General Theory.

Let $\mathcal{V}$ be a subset of $m\mathscr{B}$ such that $v^* \in \mathcal{V}$ and $T\mathcal{V} \subset \mathcal{V}$. The set $\mathcal{V}$ is understood as a set of candidate value functions. (Specific classes of functions are considered in the next section.) Let $\mathcal{C}$ be defined by

$$\mathcal{C} := L\mathcal{V} = \{\psi \in m\mathscr{B} \colon \; \psi = c + \beta P v \; \text{ for some } v \in \mathcal{V}\}. \tag{8}$$

By definition, L is a surjective mapping from $\mathcal{V}$ onto $\mathcal{C}$. It is also true that R maps $\mathcal{C}$ into $\mathcal{V}$. Indeed, if $\psi \in \mathcal{C}$, then there exists a $v \in \mathcal{V}$ such that $\psi = Lv$, and $R\psi = RLv = Tv$, which lies in $\mathcal{V}$ by assumption.

Lemma 3.1. *On $\mathcal{C}$, the operator Q satisfies $Q = LR$, and $Q\mathcal{C} \subset \mathcal{C}$.*

Proof. The first claim is immediate from the definitions. The second follows from the claims just established (i.e., R maps $\mathcal{C}$ to $\mathcal{V}$ and L maps $\mathcal{V}$ to $\mathcal{C}$). $\square$

The preceding discussion implies that Q and T are *semiconjugate*, in the sense that $LT = QL$ on $\mathcal{V}$ and $TR = RQ$ on $\mathcal{C}$. Indeed, since $T = RL$ and $Q = LR$, we have $LT = LRL = QL$ and $TR = RLR = RQ$ as claimed. This leads to the next result:

Proposition 3.1. *The following statements are true:*

1. *If v is a fixed point of T in $\mathcal{V}$, then Lv is a fixed point of Q in $\mathcal{C}$.*
2. *If ψ is a fixed point of Q in $\mathcal{C}$, then $R\psi$ is a fixed point of T in $\mathcal{V}$.*

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[3] Notably, the general theory developed in section 3.1 has no restriction on β values.

Proof. To prove the first claim, fix $v \in \mathcal{V}$. By the definition of $\mathcal{C}$, $Lv \in \mathcal{C}$. Moreover, since $v = Tv$, we have $QLv = LTv = Lv$. Hence, Lv is a fixed point of Q in $\mathcal{C}$. Regarding the second claim, fix $\psi \in \mathcal{C}$. Since R maps $\mathcal{C}$ into $\mathcal{V}$ as shown above, $R\psi \in \mathcal{V}$. Since $\psi = Q\psi$, we have $TR\psi = RQ\psi = R\psi$. Hence, $R\psi$ is a fixed point of T in $\mathcal{V}$. $\square$

The following result says that, at least on a theoretical level, iterating with either T or Q is essentially equivalent.

Proposition 3.2. *$T^{t+1} = RQ^tL$ on $\mathcal{V}$ and $Q^{t+1} = LT^tR$ on $\mathcal{C}$ for all $t \in \mathbb{N}_0$.*

Proof. That the claim holds when $t = 0$ has already been established. Now suppose the claim is true for arbitrary t. By the induction hypothesis we have $T^t = RQ^{t-1}L$ and $Q^t = LT^{t-1}R$. Since Q and T are semiconjugate as shown above, we have $T^{t+1} = TT^t = TRQ^{t-1}L = RQQ^{t-1}L = RQ^tL$ and $Q^{t+1} = QQ^t = QLT^{t-1}R = LTT^{t-1}R = LT^tR$. Hence, the claim holds by induction. $\square$

The theory above is based on the primitive assumption of a candidate value function space $\mathcal{V}$ with properties $v^* \in \mathcal{V}$ and $T\mathcal{V} \subset \mathcal{V}$. Similar results can be established if we start with a generic candidate continuation value function space $\mathscr{C}$ that satisfies $\psi^* \in \mathscr{C}$ and $Q\mathscr{C} \subset \mathscr{C}$. Appendix A gives details.

3.2. **Symmetry under Weighted Supremum Norm.** Next we impose a weighted supremum norm on the domain of T and Q in order to compare contractivity, optimality and related properties. The following assumption generalizes the standard weighted supremum norm assumption of Boyd (1990).

Assumption 3.1. There exist a $\mathscr{B}$-measurable function $g \colon \mathsf{Z} \to \mathbb{R}_+$ and constants $n \in \mathbb{N}_0$ and $a_1, \cdots, a_4, m, d \in \mathbb{R}_+$ such that $\beta m < 1$, and, for all $z \in \mathsf{Z}$,

$$\int |r(z')| P^n(z, \mathrm{d}z') \leq a_1 g(z) + a_2, \tag{9}$$

$$\int |c(z')| P^n(z, \mathrm{d}z') \leq a_3 g(z) + a_4, \tag{10}$$

$$\text{and} \int g(z') P(z, \mathrm{d}z') \leq m g(z) + d. \tag{11}$$

The interpretation is that both $\mathbb{E}_z|r(Z_n)|$ and $\mathbb{E}_z|c(Z_n)|$ are small relative to some function g such that $\mathbb{E}_z g(Z_t)$ does not grow too fast.[4] Slow growth in $\mathbb{E}_z g(Z_t)$ is imposed by (11), which can be understood as a geometric drift condition (see, e.g., Meyn and Tweedie (2009), chapter 15). Note that if both r and c are bounded, then assumption 3.1 holds for $n := 0$, $g := \|r\| \vee \|c\|$, $m := 1$ and $d := 0$.

Assumption 3.1 reduces to that of Boyd (1990) if we set $n = 0$. Here we admit consideration of future transitions to enlarge the set of possible weight functions. The value of this generalization is illustrated in section 5.

Theorem 3.1. *Let assumption 3.1 hold. Then there exist positive constants m' and d' such that for $\ell, \kappa \colon \mathsf{Z} \to \mathbb{R}$ defined by*[5]

$$\ell(z) := m' \left(\sum_{t=1}^{n-1} \mathbb{E}_z |r(Z_t)| + \sum_{t=0}^{n-1} \mathbb{E}_z |c(Z_t)| \right) + g(z) + d' \tag{12}$$

and $\kappa(z) := \ell(z) + m'|r(z)|$, the following statements hold:

(1) Q is a contraction mapping on $(b_\ell \mathsf{Z}, \|\cdot\|_\ell)$, with unique fixed point $\psi^* \in b_\ell \mathsf{Z}$.
(2) T is a contraction mapping on $(b_\kappa \mathsf{Z}, \|\cdot\|_\kappa)$, with unique fixed point $v^* \in b_\kappa \mathsf{Z}$.

The next result shows that the convergence rates of Q and T are the same. In stating it, L and R are as defined in (7), while $\rho \in (0,1)$ is the contraction coefficient of T derived in theorem 3.1 (see (23) in appendix B for details).

Proposition 3.3. *If assumption 3.1 holds, then $R(b_\ell \mathsf{Z}) \subset b_\kappa \mathsf{Z}$, $L(b_\kappa \mathsf{Z}) \subset b_\ell \mathsf{Z}$, and for all $t \in \mathbb{N}_0$, the following statements are true:*

(1) $\left\|Q^{t+1}\psi - \psi^*\right\|_\ell \leq \rho \left\|T^t R\psi - v^*\right\|_\kappa$ for all $\psi \in b_\ell \mathsf{Z}$.
(2) $\left\|T^{t+1}v - v^*\right\|_\kappa \leq \left\|Q^t L v - \psi^*\right\|_\ell$ for all $v \in b_\kappa \mathsf{Z}$.

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[4]One can show that if assumption 3.1 holds for some n, then it must hold for all integer $n' > n$. Hence, to verify assumption 3.1, it suffices to find $n_1 \in \mathbb{N}_0$ for which (9) holds, $n_2 \in \mathbb{N}_0$ for which (10) holds, and that the measurable map g satisfies (11).

[5]If assumption 3.1 holds for $n = 0$, then $\ell(z) = g(z) + d'$ and $\kappa(z) = m'|r(z)| + g(z) + d'$. To guarantee that ℓ and κ are real-valued, here and below, we assume that $\mathbb{E}_z|r(Z_t)|$, $\mathbb{E}_z|c(Z_t)| < \infty$ for $t = 1, \cdots, n-1$, which holds trivially in most applications of interest.

Proposition 3.3 extends proposition 3.2, and their connections can be seen by letting $\mathcal{V} := b_\kappa \mathsf{Z}$. Notably, claim (1) implies that Q converges as fast as T, even when its convergence is weighted by a smaller function (since $\ell \leq \kappa$).

The two operators are also symmetric in terms of continuity of fixed points. The next result illustrates this, when Z is any separable and completely metrizable topological space (e.g., any G_δ subset of $\mathbb{R}^n$) and $\mathscr{B}$ is its Borel sets.

Assumption 3.2. (1) The stochastic kernel P is Feller; that is, $z \mapsto \int h(z')P(z, \mathrm{d}z')$ is continuous and bounded on Z whenever h is. (2) c, r, ℓ, $z \mapsto \int |r(z')|P(z, \mathrm{d}z')$, and $z \mapsto \int \ell(z')P(z, \mathrm{d}z')$ are continuous.[6]

Proposition 3.4. *If assumptions 3.1–3.2 hold, then ψ^* and v^* are continuous.*

3.3. **Symmetry in L_p.** The results of the preceding section for the most part carry over if we switch the underlying space to L_p. This section outlines the main ideas.

Assumption 3.3. The state process $\{Z_t\}$ admits a stationary distribution π and the reward functions r, c are in $L_q(\pi)$ for some $q \geq 1$.

Theorem 3.2. *If assumption 3.3 holds, then for all $1 \leq p \leq q$, we have*[7]

(1) Q is a contraction mapping on $\left(L_p(\pi), \|\cdot\|_p\right)$ of modulus β, and the unique fixed point of Q in $L_p(\pi)$ is ψ^*.
(2) T is a contraction mapping on $\left(L_p(\pi), \|\cdot\|_p\right)$ of modulus β, and the unique fixed point of T in $L_p(\pi)$ is v^*.

The following result implies that Q and T have the same rate of convergence in terms of L_p-norm distance.

Proposition 3.5. *If assumption 3.3 holds, then for all $1 \leq p \leq q$, both R and L map $L_p(\pi)$ into itself. Moreover, for all $1 \leq p \leq q$ and $t \in \mathbb{N}_0$, the following statements hold:*

[6] A sufficient condition for assumption 3.2-(2) is: g and $z \mapsto \mathbb{E}_z g(Z_1)$ are continuous, and $z \mapsto \mathbb{E}_z |r(Z_t)|, \mathbb{E}_z |c(Z_t)|$ are continuous for $t = 0, ..., n$ (with n as defined in assumption 3.1).

[7] We typically omit phrases such as "with probability one" or "almost surely" in what follows. Uniqueness of fixed points is up to a π-null set.

(1) $\left\|Q^{t+1}\psi - \psi^*\right\|_p \leq \beta \left\|T^t R\psi - v^*\right\|_p$ *for all* $\psi \in L_p(\pi)$.
(2) $\left\|T^{t+1}v - v^*\right\|_p \leq \left\|Q^t L v - \psi^*\right\|_p$ *for all* $v \in L_p(\pi)$.

Proposition 3.5 is an extension of proposition 3.2 in an L_p space, and their connections can be seen by letting $\mathcal{V} := L_p(\pi)$.

4. Asymmetries Between the Operators

The preceding results show that T and Q exhibit dynamics that are in many senses symmetric. However, for a large number of economic models, the effective state space for Q is lower dimensional than that of T. This section provides definitions and analysis, with examples deferred to section 5. Throughout, we write

$$\mathsf{Z} = \mathsf{X} \times \mathsf{Y} \quad \text{and} \quad Z_t = (X_t, Y_t)$$

where X is a Borel subset of $\mathbb{R}^\ell$ and Y is a Borel subset of $\mathbb{R}^n$.

4.1. **Continuation Decomposability.**

We call an optimal stopping problem (β, c, P, r) *continuation decomposable* if c and P are such that

(a) (X_{t+1}, Y_{t+1}) and X_t are independent given Y_t and
(b) c is a function of Y_t but not X_t.

Condition (a) implies that $P(z, \mathrm{d}z')$ can be represented by the conditional distribution of (x', y') given y, denoted below by $F_y(x', y')$. On an intuitive level, continuation decomposable problems are those where some state variables matter only for terminal rewards.

The significance of continuation decomposability is that, for such models, the Jovanovic operator can be written as

$$Q\psi(y) = c(y) + \beta \int \max\left\{r(x', y'), \psi(y')\right\} \mathrm{d}F_y(x', y'). \tag{13}$$

Thus, Q acts on functions defined over Y alone. In contrast, assuming that all state variables are non-trivial in the sense that they impact on the value function, T continues to act on functions defined over all of $\mathsf{Z} = \mathsf{X} \times \mathsf{Y}$. The set Y is ℓ dimensions lower than Z.

4.2. **Complexity Analysis.** One way to compare the efficiency of Q and T is to consider the time complexity of continuation value function iteration (CVI) and value function iteration (VFI). Both finite and infinite space approximations are considered.

4.2.1. *Finite Space.* Let $\mathsf{X} = \times_{i=1}^{\ell} \mathsf{X}^i$ and $\mathsf{Y} = \times_{j=1}^{n} \mathsf{Y}^j$, where X^i and Y^j are subsets of $\mathbb{R}$. Each X^i (resp., Y^j) is represented by a grid of K_i (resp., M_j) points. Integration operations in both VFI and CVI are replaced by summations. We use $\hat{P}$ and $\hat{F}$ to denote the transition matrices (i.e., discretized stochastic kernels) for VFI and CVI respectively.[8]

Let $K := \Pi_{i=1}^{\ell} K_i$ and $M := \Pi_{j=1}^{n} M_j$ with $K = 1$ for $\ell = 0$. Let $n > 0$. There are KM grid points on $\mathsf{Z} = \mathsf{X} \times \mathsf{Y}$ and M grid points on Y. The matrix $\hat{P}$ is $(KM) \times (KM)$ and $\hat{F}$ is $M \times (KM)$. VFI and CVI are implemented by the operators $\hat{T}$ and $\hat{Q}$ defined respectively by

$$\hat{T}\vec{v} := \vec{r} \vee (\vec{c} + \beta \hat{P} \vec{v}) \quad \text{and} \quad \hat{Q}\vec{\psi}_y := \vec{c}_y + \beta \hat{F}(\vec{r} \vee \vec{\psi}).$$

Here $\vec{q}$ represents a column vector with i-th element equal to $q(x_i, y_i)$, where (x_i, y_i) is the i-th element of the list of grid points on $\mathsf{X} \times \mathsf{Y}$. Let $\vec{q}_y$ denote the column vector with the j-th element equal to $q(y_j)$, where y_j is the j-th element of the list of grid points on Y. The vectors $\vec{v}$, $\vec{r}$, $\vec{c}$ and $\vec{\psi}$ are $(KM) \times 1$, while $\vec{c}_y$ and $\vec{\psi}_y$ are $M \times 1$.

4.2.2. *Infinite Space.* We use the same number of grid points as before, but now for continuous state function approximation rather than discretization. In particular, we replace the discrete state summation with Monte Carlo integration. Assume that the transition function of the state process follows

$$X_{t+1} = f_1(Y_t, W_{t+1}), \quad Y_{t+1} = f_2(Y_t, W_{t+1}), \quad \{W_t\} \overset{\text{IID}}{\sim} \Phi.$$

After drawing $U_1, \cdots, U_N \overset{\text{IID}}{\sim} \Phi$, with N being the MC sample size, CVI and VFI are implemented by

$$\hat{Q}\psi(y) := c(y) + \beta \frac{1}{N} \sum_{i=1}^{N} \max \left\{ r\left(f_1(y, U_i), f_2(y, U_i)\right), h\langle\psi\rangle\left(f_2(y, U_i)\right) \right\}$$

[8] See Tauchen and Hussey (1991) for a general discussion of discretization methods.

$$\text{and} \quad \hat{T}v(x,y) := \max\left\{ r(x,y),\, c(y) + \beta \frac{1}{N} \sum_{i=1}^{N} g\langle v \rangle \left(f_1(y, U_i), f_2(y, U_i) \right) \right\}.$$

Here $\psi = \{\psi(y)\}$, with y in the set of grid points on Y, and $v = \{v(x,y)\}$, with (x,y) in the set of grid points on $\mathsf{X} \times \mathsf{Y}$. Moreover, $h\langle \cdot \rangle$ and $g\langle \cdot \rangle$ are interpolating functions for CVI and VFI respectively. For example, $h\langle \psi \rangle(z)$ can be understood as interpolating the vector ψ to obtain a function $h\langle \psi \rangle$ and then evaluating at z.

4.2.3. *Time Complexity.* Table 1 provides the time complexity of CVI and VFI, estimated by counting the number of floating point operations. Each such operation is assumed to have complexity $\mathcal{O}(1)$.[9] Function evaluations associated with the model primitives are also assumed to be of order $\mathcal{O}(1)$.

TABLE 1. Time complexity: VFI v.s CVI

Cmplx.	VFI: 1-loop	CVI: 1-loop	VFI: n-loop	CVI: n-loop
FS	$\mathcal{O}(K^2M^2)$	$\mathcal{O}(KM^2)$	$\mathcal{O}(nK^2M^2)$	$\mathcal{O}(nKM^2)$
IS	$\mathcal{O}(NKM\log(KM))$	$\mathcal{O}(NM\log(M))$	$\mathcal{O}(nNKM\log(KM))$	$\mathcal{O}(nNM\log(M))$

Note: For IS approximation, binary search is used when we evaluate the interpolating function at a given point. The results hold for linear, quadratic, cubic, and k-nearest neighbors interpolations.

For both finite space (FS) and infinite space (IS) approximations, CVI provides better performance than VFI. For FS, CVI is more efficient than VFI by order $\mathcal{O}(K)$, while for IS, CVI is more efficient than VFI by order $\mathcal{O}\left(K\log(KM)/\log(M)\right)$. For example, if we have 250 grid points in each dimension, then in the FS case, evaluating a given number of loops will take around 250^{ℓ} times longer via CVI than via VFI, after adjusting for order approximations.

See appendix C for a proof of the results in table 1.

[9]Floating point operations are any elementary actions (e.g., $+$, $\times$, $\vee$, $\wedge$) on or assignments with floating point numbers. If f and g are scalar functions on $\mathbb{R}^n$, we write $f(x) = \mathcal{O}(g(x))$ whenever there exist $C, M > 0$ such that $\|x\| \geq M$ implies $|f(x)| \leq C|g(x)|$, where $\|\cdot\|$ is the sup norm.

5. Applications

We consider six applications. For the first two cases, we discuss both optimality and continuation decomposability. For the remaining cases, we discuss only the latter.

5.1. **Job Search.** Consider a worker who receives current wage offer w_t and chooses to either accept and work permanently at that wage, or reject the offer, receive unemployment compensation c_0 and reconsider next period (see, e.g., McCall (1970) or Pissarides (2000)). The wage process $\{w_t\}_{t \geq 0}$ is assumed to be

$$w_t = \eta_t + \theta_t \xi_t, \quad \text{where} \quad \ln \theta_t = \rho \ln \theta_{t-1} + \ln \varepsilon_t \tag{14}$$

and $\{\xi_t\}$, $\{\varepsilon_t\}$ and $\{\eta_t\}$ are positive IID innovations that are mutually independent. We interpret θ_t as the persistent component of labor income and allow it to be nonstationary. When η_t is constant it can be interpreted as social security.[10] Viewed as an optimal stopping problem,

- the state is $z = (w, \theta)$, with stochastic kernel P defined by (14),
- the terminal reward is $r(w) = u(w)/(1-\beta)$, where u is a utility function,
- and the flow continuation reward c is the constant $u(c_0)$.

The model is continuation decomposable, as can be seen by letting $X_t := w_t$ and $Y_t := \theta_t$. In particular, c does not depend on w_t and (w_{t+1}, θ_{t+1}) is independent of w_t given θ_t. Hence the effective state space for Q is one-dimensional while that of T is two-dimensional. Letting $F_\theta(w', \theta')$ be the distribution of (w_{t+1}, θ_{t+1}) given θ_t, the Bellman operator satisfies

$$Tv(w, \theta) = \max \left\{ \frac{u(w)}{1-\beta}, \; u(c_0) + \beta \int v(w', \theta') \, \mathrm{d}F_\theta(w', \theta') \right\},$$

while the Jovanovic operator is

$$Q\psi(\theta) = u(c_0) + \beta \int \max \left\{ \frac{u(w')}{1-\beta}, \; \psi(\theta') \right\} \mathrm{d}F_\theta(w', \theta').$$

[10] Similar dynamics appear in many labor market, search-theoretic and real options studies (see e.g., Gomes et al. (2001), Low et al. (2010), Chatterjee and Eyigungor (2012), Bagger et al. (2014), Kellogg (2014)).

Whether or not assumptions 3.1–3.3 hold depends on the primitives. Suppose for example that

$$u(w) = \frac{w^{1-\gamma}}{1-\gamma} \quad \text{with} \quad u(w) = \ln w \quad \text{when } \gamma = 1. \tag{15}$$

We focus here on the case $\gamma = 1$ and $0 \leq \rho < 1$, although other cases such as $\gamma > 1$ and $-1 < \rho < 0$ can be treated with similar arguments.[11] We take $\varepsilon_t \sim LN(0, \sigma^2)$. Regarding assumptions 3.1–3.2, we assume that $\{\eta_t\}$, $\{\eta_t^{-1}\}$ and $\{\xi_t\}$ have finite first moments.

The reward function for this dynamic program is unbounded above and below, and the state space is likewise unbounded. Nevertheless, we can establish the key optimality results from section 3 as follows. First, choose $n \in \mathbb{N}_0$ such that $\beta \exp(\rho^{2n}\sigma^2) < 1$, and let

$$g(z) = g(w, \theta) = \theta^{\rho^n}.$$

To verify (9), we make use of the following technical lemma, which is obtained from the law of motion (14), and provides a bound on expected time n wages in terms of initial condition $\theta_0 = \theta$. The proof is in appendix B.

Lemma 5.1. *For all $n \in \mathbb{N}_0$, (a) there exist a pair $A_n, B \in \mathbb{R}$ such that $\mathbb{E}_\theta |\ln w_n| \leq A_n \theta^{\rho^n} + B$, and (b) $\theta \mapsto \mathbb{E}_\theta |\ln w_n|$ is continuous.*

Now (9) can be established, since, conditioning on $\theta_0 = \theta$,

$$\mathbb{E}_\theta |r(w_n)| = \frac{\mathbb{E}_\theta |\ln w_n|}{1-\beta} \leq \frac{A_n}{1-\beta} \theta^{\rho^n} + \frac{B}{1-\beta} = \frac{A_n}{1-\beta} g(w, \theta) + \frac{B}{1-\beta}.$$

Condition (10) is trivial because c is constant. To see that condition (11) holds, note that $\rho \in [0, 1)$, so, conditioning on $\theta_0 = \theta$ once more,

$$\mathbb{E}_\theta \, g(w_1, \theta_1) = \mathbb{E} \, (\theta^\rho \varepsilon_1)^{\rho^n} = \theta^{\rho^{n+1}} \exp(\rho^{2n}\sigma^2/2) \leq (\theta^{\rho^n} + 1) \exp(\rho^{2n}\sigma^2).$$

Hence (11) holds with $m = d = \exp(\rho^{2n}\sigma^2)$. Assumption 3.1 has now been established. By theorem 3.1 and proposition 3.3, Q and T are contraction mappings

[11] One can also treat the nonstationary case $\rho = \pm 1$ under some further parametric assumptions using the weighted supremum norm techniques developed above. Details are available from the authors on request.

with the same rate of convergence. The above analysis also implies that assumption 3.2 holds (see footnote 6), so proposition 3.4 implies that both v^* and ψ^* are continuous.

We can also embed this problem in $L_p(\pi)$. To verify assumption 3.3, we assume that the distributions of $\{\eta_t\}$ and $\{\xi_t\}$ are represented respectively by densities μ and ν, and that $\{\eta_t\}$, $\{\eta_t^{-1}\}$ and $\{\xi_t\}$ have finite q-th moments.

Since $\rho \in [0, 1)$, the state process $\{(w_t, \theta_t)\}$ has stationary density

$$\pi(w, \theta) = f^*(\theta)\, p(w|\theta),$$

where $f^*(\theta) = LN(0, \sigma^2/(1-\rho^2))$ and $\int_A p(w|\theta)\, \mathrm{d}w = \int_{\{\eta+\theta\xi \in A\}} \mu(\eta)\nu(\xi)\, \mathrm{d}(\eta, \xi)$. Then the next lemma (proved in appendix B) implies that assumption 3.3 holds.

Lemma 5.2. *The reward functions r and c are in $L_q(\pi)$.*

By theorem 3.2 and proposition 3.5, Q and T are both contraction mappings with the same rate of convergence (in L_p norm distances, for all $1 \leq p \leq q$).

5.2. **Search with Learning.** Consider a job search problem with learning (see, e.g., McCall (1970), Pries and Rogerson (2005), Nagypál (2007), or Ljungqvist and Sargent (2012)). The setup is as in section 5.1, except that $\{w_t\}_{t \geq 0}$ follows

$$\ln w_t = \xi + \varepsilon_t, \quad \text{where} \quad \{\varepsilon_t\}_{t\geq 0} \overset{\text{IID}}{\sim} N(0, \delta_\varepsilon).$$

Here ξ is an unobservable mean over which the worker has prior $\xi \sim N(\mu, \delta)$. The worker's current estimate of the next period wage distribution is $f(w'|\mu, \delta) = LN(\mu, \delta + \delta_\varepsilon)$. If the current offer is turned down, the worker updates his belief after observing w'. By Bayes' rule, the posterior satisfies $\xi|w' \sim N(\mu', \delta')$, where $\delta' = \nu(\delta) := 1/\left(1/\delta + 1/\delta_\varepsilon\right)$ and $\mu' = \phi(\mu, \delta, w') := \delta'\left(\mu/\delta + \ln w'/\delta_\varepsilon\right)$. Viewed as an optimal stopping problem,

- the state is $z = (w, \mu, \delta)$, and for each map h, the stochastic kernel P satisfies
$$\int h(z')P(z, \mathrm{d}z') = \int h\left(w', \phi(\mu, \delta, w'), \nu(\delta)\right) f(w'|\mu, \delta)\, \mathrm{d}w'$$
- the reward functions are $r(w) = u(w)/(1-\beta)$ and $c \equiv u(c_0)$.

The model is continuation decomposable with $X_t := w_t$ and $Y_t := (\mu_t, \delta_t)$, since r does not depend on (μ_t, δ_t) and the next period state $(w_{t+1}, \mu_{t+1}, \delta_{t+1})$ is independent of w_t once μ_t and δ_t are known. Letting $F_{\mu,\delta}(w', \mu', \delta')$ be the distribution of $(w_{t+1}, \mu_{t+1}, \delta_{t+1})$ given (μ_t, δ_t), the Bellman and Jovanovic operators are, respectively,

$$Tv(w, \mu, \delta) = \max\left\{\frac{u(w)}{1-\beta},\ u(c_0) + \beta \int v(w', \mu', \delta')\, \mathrm{d}F_{\mu,\delta}(w', \mu', \delta')\right\}$$

$$\text{and} \quad Q\psi(\mu, \delta) = u(c_0) + \beta \int \max\left\{\frac{u(w')}{1-\beta},\ \psi(\mu', \delta')\right\} \mathrm{d}F_{\mu,\delta}(w', \mu', \delta').$$

Again, the domain of the candidate function space is one dimension lower for Q than T.

Regarding optimality, suppose, for example, that the CRRA parameter γ is greater than 1. (The case $\gamma = 1$ can be treated along similar lines.) Let $n = 1$ and let

$$g(w, \mu, \delta) = \mathrm{e}^{(1-\gamma)\mu + (1-\gamma)^2 \delta / 2}.$$

Condition (9) holds, since, conditioning on $(\mu_0, \delta_0) = (\mu, \delta)$,

$$\mathbb{E}_{\mu,\delta} |r(w_1)| = \frac{\mathbb{E}_{\mu,\delta}\, w_1^{1-\gamma}}{1-\beta} = \frac{\mathrm{e}^{(1-\gamma)^2 \delta_\varepsilon / 2}}{1-\beta}\, g(w, \mu, \delta).$$

Condition (10) is trivial since c is constant. Condition (11) holds, since, conditioning on $(\mu_0, \delta_0) = (\mu, \delta)$, the expressions of μ' and δ' imply that

$$\mathbb{E}_{\mu,\delta}\, g(w_1, \mu_1, \delta_1) = \mathrm{e}^{(1-\gamma)^2 \delta_1 / 2 + (1-\gamma)\delta_1 \mu / \delta}\, \mathbb{E}_{\mu,\delta}\, w_1^{(1-\gamma)\delta_1/\delta_\varepsilon} = g(w, \mu, \delta).$$

Hence assumption 3.1 holds. Theorem 3.1 and proposition 3.3 imply that Q and T are contraction mappings with the same rate of convergence. The analysis above also implies that assumption 3.2 holds (see footnote 6), so v^* and ψ^* are continuous by proposition 3.4.

5.3. **Firm Entry.** Consider a condensed version of the firm entry problem in Fajgelbaum et al. (2017). At the start of period t, a firm observes a fixed cost f_t and then decides whether to incur this cost and enter a market, earning stochastic payoff π_t, or wait and reconsider next period. The sequence $\{f_t\}$ is IID, while the current payoff π_t is unknown prior to entry. The firm has prior belief $\phi(\pi; \theta_t)$, where ϕ is a distribution over payoffs that is parameterized by a vector θ_t. If the firm does not enter then θ_t is updated via Bayesian learning. In an optimal stopping format,

- the state is $z = (f, \theta)$, with stochastic kernel P defined by the distribution of $\{f_t\}$ and the Bayesian updating mechanism of $\{\theta_t\}$,
- the terminal reward is the entry payoff $r(f, \theta) = \int \pi\phi(\mathrm{d}\pi; \theta) - f$,
- and the flow continuation reward $c \equiv 0$.

This model is continuation decomposable, as can be seen by letting $X_t := f_t$ and $Y_t := \theta_t$. In particular, since $\{f_t\}$ is IID, (f_{t+1}, θ_{t+1}) is independent of f_t given θ_t. Let $F_\theta(f', \theta')$ be the distribution of (f_{t+1}, θ_{t+1}) given θ_t. The Bellman operator is

$$Tv(f, \theta) = \max\left\{\int \pi\phi(\mathrm{d}\pi; \theta) - f,\ \beta \int v(f', \theta')\,\mathrm{d}F_\theta(f', \theta')\right\},$$

while the Jovanovic operator is

$$Q\psi(\theta) = \beta \int \max\left\{\int \pi\phi(\mathrm{d}\pi; \theta') - f',\ \psi(\theta')\right\} \mathrm{d}F_\theta(f', \theta').$$

5.4. **Research and Development.** Firm's R&D decisions are often modeled as a sequential search process for better technologies (see, e.g., Jovanovic and Rob (1989), Bental and Peled (1996), Perla and Tonetti (2014)). Each period, an idea of value s_t is observed, and the firm decides whether to put this idea into productive use, or develop it further by investing in R&D. The former choice yields a payoff $r(s_t, k_t)$, where k_t is the amount of capital input. The latter incurs a fixed cost $c_0 > 0$ (that renders $k_{t+1} = k_t - c_0$) and creates a new technology s_{t+1} next period. Let $\{s_t\} \overset{\text{IID}}{\sim} \mu$. Viewed as an optimal stopping problem,

- the state is $z = (s, k)$, and for given map h, the stochastic kernel P satisfies
$$\int h(z') P(z, \mathrm{d}z') = \int h\left(s', k - c_0\right) \mu(\mathrm{d}s'),$$
- the terminal reward is $r(s, k)$ and the flow continuation reward is $c \equiv -c_0$.

This model is also continuation decomposable, as can be seen by letting $X_t := s_t$ and $Y_t := k_t$. Let $F_k(s', k')$ be the distribution of (s_{t+1}, k_{t+1}) given k_t. The Bellman and Jovanovic operators are respectively

$$Tv(s,k) = \max\left\{ r(s,k), -c_0 + \beta \int v(s',k')\, \mathrm{d}F_k(s',k') \right\}$$

$$\text{and} \quad Q\psi(s) = -c_0 + \beta \int \max\left\{ r(s',k'), \psi(s') \right\} \mathrm{d}F_k(s',k').$$

5.5. **Real Options.** Consider a general financial/real option framework (see, e.g., Dixit and Pindyck (1994), Alvarez and Dixit (2014), and Kellogg (2014)). Let p_t be the current price of a certain financial/real asset and λ_t another state variable. The process $\{\lambda_t\}$ is Φ-Markov and affects $\{p_t\}$ via $p_t = f(\lambda_t, \varepsilon_t)$, where $\{\varepsilon_t\} \overset{\text{IID}}{\sim} \mu$ and is independent of $\{\lambda_t\}$. Let K be the strike price of the asset. Each period, the agent decides whether to exercise the option now (i.e., purchase the asset at price K), or wait and reconsider next period. In an optimal stopping format,

- the state is $z = (p, \lambda)$, and for given map h, the stochastic kernel P satisfies
$$\int h(z') P(z, \mathrm{d}z') = \int h\left(f(\lambda', \varepsilon'),\, \lambda'\right) \mu(\mathrm{d}\varepsilon')\, \Phi(\lambda, \mathrm{d}\lambda'),$$
- the terminal reward (exercise the option now) is $r(p) = (p - K)^+$,
- and the flow continuation reward is $c \equiv 0$.

The model is continuation decomposable, with $X_t := p_t$ and $Y_t := \lambda_t$. Let $F_\lambda(p', \lambda')$ be the distribution of (p_{t+1}, λ_{t+1}) conditional on λ_t. The Bellman and Jovanovic operators are, respectively,

$$Tv(p,\lambda) = \max\left\{ (p-K)^+, \beta \int v(p',\lambda')\, \mathrm{d}F_\lambda(p',\lambda') \right\}$$

$$\text{and} \quad Q\psi(\lambda) = \beta \int \max\left\{ (p'-K)^+, \psi(\lambda') \right\} \mathrm{d}F_\lambda(p',\lambda').$$

5.6. **Transplants.** In health economics, a well-known problem concerns the decision of a surgeon to accept/reject a transplantable organ for the patient (see, e.g., Alagoz et al., 2004). The surgeon aims to maximize the reward of the patient. Each period, she receives an organ offer of quality q_t, where $\{q_t\} \overset{\text{IID}}{\sim} G$. The patient's health h_t evolves according to a H-Markov process if the surgeon rejects the organ. If she accepts this organ for transplant, the operation succeeds with probability $p(q_t, h_t)$, and confers benefit $B(h_t)$ to the patient, while a failed operation results in death. The patient's single period utility when alive is $u(h_t)$. Viewed as an optimal stopping problem,

- the state is $z = (q, h)$, and for a given map f, the stochastic kernel P satisfies
$$\int f(z')P(z, \mathrm{d}z') = \int f(q', h')G(\mathrm{d}q')H(h, \mathrm{d}h'),$$
- the terminal reward (accept the offer) is $r(q, h) = u(h) + p(q, h)B(h)$,
- and the flow continuation reward is $c(h) = u(h)$.

This model is continuation decomposable by letting $X_t := q_t$ and $Y_t := h_t$. Let $F_h(q', h')$ be the distribution of (q_{t+1}, h_{t+1}) given h_t. The Bellman and Jovanovic operators are respectively

$$Tv(q, h) = \max \left\{ u(h) + p(q, h)B(h),\, u(h) + \beta \int v(q', h')\, \mathrm{d}F_h(q', h') \right\}$$

$$\text{and} \quad Q\psi(h) = u(h) + \beta \int \max \left\{ u(h') + p(q', h')B(h'),\, \psi(h') \right\} \mathrm{d}F_h(q', h').$$

Appendix A: Some Lemmas

To see the symmetric properties of Q and T from an alternative perspective, we start our analysis with a generic candidate continuation value function space. Let $\mathscr{C}$ be a subset of $m\mathscr{B}$ such that $\psi^* \in \mathscr{C}$ and $Q\mathscr{C} \subset \mathscr{C}$. Let $\mathscr{V}$ be defined by

$$\mathscr{V} := R\mathscr{C} = \{ v \in m\mathscr{B} \colon v = r \vee \psi \ \text{ for some } \psi \in \mathscr{C} \}. \tag{16}$$

Then R is a surjective map from $\mathscr{C}$ onto $\mathscr{V}$, $Q = LR$ on $\mathscr{C}$ and $T = RL$ on $\mathscr{V}$. The following result parallels the theory of section 3.1, and is helpful for deriving important convergence properties once topological structure is added to the generic setting, as to be shown.

Lemma 5.3. *The following statements are true:*

1. $L\mathscr{V} \subset \mathscr{C}$ *and* $T\mathscr{V} \subset \mathscr{V}$.
2. *If* v *is a fixed point of* T *in* $\mathscr{V}$, *then* Lv *is a fixed point of* Q *in* $\mathscr{C}$.
3. *If* ψ *is a fixed point of* Q *in* $\mathscr{C}$, *then* $R\psi$ *is a fixed point of* T *in* $\mathscr{V}$.
4. $T^{t+1} = RQ^tL$ *on* $\mathscr{V}$ *and* $Q^{t+1} = LT^tR$ *on* $\mathscr{C}$ *for all* $t \in \mathbb{N}_0$.

Proof. The proof is similar to that of propositions 3.1–3.2 and thus omitted. □

Lemma 5.4. *Under assumption 3.1, there exist* $b_1, b_2 \in \mathbb{R}_+$ *such that for all* $z \in \mathsf{Z}$, *(1)* $|v^*(z)| \leq \sum_{t=0}^{n-1} \beta^t \mathbb{E}_z[|r(Z_t)| + |c(Z_t)|] + b_1 g(z) + b_2$.

(2) $|\psi^*(z)| \leq \sum_{t=1}^{n-1} \beta^t \mathbb{E}_z|r(Z_t)| + \sum_{t=0}^{n-1} \beta^t \mathbb{E}_z|c(Z_t)| + b_1 g(z) + b_2$.

Proof. Without loss of generality, we assume $m \neq 1$ in assumption 3.1. By that assumption, $\mathbb{E}_z|r(Z_n)| \leq a_1 g(z) + a_2$, $\mathbb{E}_z|c(Z_n)| \leq a_3 g(z) + a_4$ and $\mathbb{E}_z g(Z_1) \leq m g(z) + d$ for all $z \in \mathsf{Z}$. For all $t \geq 1$, by the Markov property (see, e.g., Meyn and Tweedie (2009), section 3.4.3),

$$\mathbb{E}_z g(Z_t) = \mathbb{E}_z \left[\mathbb{E}_z \left(g(Z_t)|\mathscr{F}_{t-1}\right)\right] = \mathbb{E}_z \left(\mathbb{E}_{Z_{t-1}} g(Z_1)\right) \leq m\, \mathbb{E}_z g(Z_{t-1}) + d.$$

Induction shows that for all $t \geq 0$,

$$\mathbb{E}_z g(Z_t) \leq m^t g(z) + \frac{1-m^t}{1-m} d. \tag{17}$$

Moreover, for all $t \geq n$, applying the Markov property again yields

$$\mathbb{E}_z|r(Z_t)| = \mathbb{E}_z \left[\mathbb{E}_z \left(|r(Z_t)||\mathscr{F}_{t-n}\right)\right] = \mathbb{E}_z \left(\mathbb{E}_{Z_{t-n}}|r(Z_n)|\right) \leq a_1\, \mathbb{E}_z g(Z_{t-n}) + a_2.$$

By (17), for all $t \geq n$, we have

$$\mathbb{E}_z|r(Z_t)| \leq a_1 \left(m^{t-n} g(z) + \frac{1-m^{t-n}}{1-m} d\right) + a_2. \tag{18}$$

Similarly, for all $t \geq n$, we have

$$\mathbb{E}_z|c(Z_t)| \leq a_3\, \mathbb{E}_z g(Z_{t-n}) + a_4 \leq a_3 \left(m^{t-n} g(z) + \frac{1-m^{t-n}}{1-m} d\right) + a_4. \tag{19}$$

Let $S(z) := \sum_{t \geq 1} \beta^t \mathbb{E}_z \left[|r(Z_t)| + |c(Z_t)|\right]$. Based on (17)–(19), we can show that

$$S(z) \leq \sum_{t=1}^{n-1} \beta^t \mathbb{E}_z[|r(Z_t)| + |c(Z_t)|] + \frac{a_1+a_3}{1-\beta m} g(z) + \frac{(a_1+a_3)d + a_2 + a_4}{(1-\beta m)(1-\beta)}. \tag{20}$$

Since $|v^*| \leq |r| + |c| + S$ and $|\psi^*| \leq |c| + S$, the two claims hold by letting $b_1 := \frac{a_1+a_3}{1-\beta m}$ and $b_2 := \frac{(a_1+a_3)d+a_2+a_4}{(1-\beta m)(1-\beta)}$. □

Lemma 5.5. *Under assumption 3.1, the value function solves the Bellman equation*

$$v^*(z) = \max\left\{r(z),\, c(z) + \beta \int v^*(z') P(z, dz')\right\} = \max\left\{r(z),\, \psi^*(z)\right\}. \tag{21}$$

Proof of lemma 5.5. By theorem 1.11 of Peskir and Shiryaev (2006), it suffices to show that $\mathbb{E}_z \left(\sup_{k\geq 0} \left|\sum_{t=0}^{k-1} \beta^t c(Z_t) + \beta^k r(Z_k)\right|\right) < \infty$ for all $z \in \mathsf{Z}$. This is true since

$$\sup_{k \geq 0} \left|\sum_{t=0}^{k-1} \beta^t c(Z_t) + \beta^k r(Z_k)\right| \leq \sum_{t \geq 0} \beta^t [|r(Z_t)| + |c(Z_t)|] \tag{22}$$

with probability one, and by the monotone convergence theorem and lemma 5.4 (see (20) in appendix A), the right hand side of (22) is $\mathbb{P}_z$-integrable for all $z \in \mathsf{Z}$. □

Appendix B : Main Proofs

Proof of theorem 3.1. Let $d_1 := a_1 + a_3$ and $d_2 := a_2 + a_4$. Since $\beta m < 1$ by assumption 3.1, we can choose positive constants m' and d' such that

$$m + d_1 m' > 1, \quad \rho := \beta(m + d_1 m') < 1 \quad \text{and} \quad d' \geq (d_2 m' + d)/(m + d_1 m' - 1). \tag{23}$$

Regarding claim (1), we first show that Q is a contraction mapping on $b_\ell \mathsf{Z}$ with modulus ρ. By the weighted contraction mapping theorem (see, e.g., Boyd (1990), section 3), it suffices to verify: (a) Q is monotone, i.e., $Q\psi \leq Q\phi$ if $\psi, \phi \in b_\ell \mathsf{Z}$ and $\psi \leq \phi$; (b) $Q0 \in b_\ell \mathsf{Z}$ and $Q\psi$ is $\mathscr{B}$-measurable for all $\psi \in b_\ell \mathsf{Z}$; and (c) $Q(\psi + a\ell) \leq Q\psi + a\rho\ell$ for all $a \in \mathbb{R}_+$ and $\psi \in b_\ell \mathsf{Z}$. Obviously, condition (a) holds. By (6) and (12), we have

$$\frac{|(Q0)(z)|}{\ell(z)} \leq \frac{|c(z)|}{\ell(z)} + \beta \int \frac{|r(z')|}{\ell(z)} P(z, dz') \leq (1+\beta)/m' < \infty$$

for all $z \in \mathsf{Z}$, so $\|Q0\|_\ell < \infty$. The measurability of $Q\psi$ follows immediately from our primitive assumptions. Hence, condition (b) holds. By the Markov property (see, e.g., Meyn and Tweedie (2009), section 3.4.3), we have

$$\int \mathbb{E}_{z'} |r(Z_t)| P(z, \mathrm{d}z') = \mathbb{E}_z |r(Z_{t+1})| \;\text{ and }\; \int \mathbb{E}_{z'} |c(Z_t)| P(z, \mathrm{d}z') = \mathbb{E}_z |c(Z_{t+1})|.$$

Let $h(z) := \sum_{t=1}^{n-1} \mathbb{E}_z |r(Z_t)| + \sum_{t=0}^{n-1} \mathbb{E}_z |c(Z_t)|$, then we have

$$\int h(z') P(z, \mathrm{d}z') = \sum_{t=2}^{n} \mathbb{E}_z |r(Z_t)| + \sum_{t=1}^{n} \mathbb{E}_z |c(Z_t)|. \tag{24}$$

By the construction of m' and d', we have $m + d_1 m' > 1$ and $(d_2 m' + d + d')/(m + d_1 m') \leq d'$. Assumption 3.1 and (24) then imply that

$$\begin{aligned}
\int \kappa(z') P(z, dz') &= m' \sum_{t=1}^{n} \mathbb{E}_z [|r(Z_t)| + |c(Z_t)|] + \int g(z') P(z, dz') + d' \\
&\leq m' \sum_{t=1}^{n-1} \mathbb{E}_z [|r(Z_t)| + |c(Z_t)|] + (m + d_1 m') g(z) + d_2 m' + d + d' \\
&\leq (m + d_1 m') \left(\frac{m'}{m + d_1 m'} h(z) + g(z) + d' \right) \leq (m + d_1 m') \ell(z).
\end{aligned} \tag{25}$$

Since $\ell \leq \kappa$, this implies that for all $z \in \mathsf{Z}$, we have

$$\int \kappa(z') P(z, \mathrm{d}z') \leq (m + d_1 m') \kappa(z) \quad \text{and} \quad \int \ell(z') P(z, \mathrm{d}z') \leq (m + d_1 m') \ell(z). \tag{26}$$

Hence, for all $\psi \in b_\ell \mathsf{Z}$, $a \in \mathbb{R}_+$ and $z \in \mathsf{Z}$, we have

$$\begin{aligned} Q(\psi + a\ell)(z) &= c(z) + \beta \int \max\left\{r(z'), \psi(z') + a\ell(z')\right\} P(z, dz') \\ &\leq c(z) + \beta \int \max\left\{r(z'), \psi(z')\right\} P(z, dz') + a\beta \int \ell(z') P(z, dz') \\ &\leq Q\psi(z) + a\beta(m + d_1 m')\ell(z) = Q\psi(z) + a\rho\ell(z). \end{aligned}$$

So condition (c) holds, and $Q : b_\ell \mathsf{Z} \to b_\ell \mathsf{Z}$ is a contraction mapping of modulus ρ.

Moreover, lemma 5.5 and the analysis related to (5) imply that ψ^* is indeed a fixed point of Q under assumption 3.1. Lemma 5.4 implies that $\psi^* \in b_\ell \mathsf{Z}$. Hence, ψ^* must coincide with the unique fixed point of Q under $b_\ell \mathsf{Z}$, and claim (1) holds.

The proof of claim (2) is similar. In particular, using (26) one can show that $T : b_\kappa \mathsf{Z} \to b_\kappa \mathsf{Z}$ is a contraction mapping with the same modulus. We omit the details. □

Proof of proposition 3.4. Let $b_\ell c\mathsf{Z}$ be the set of continuous functions in $b_\ell \mathsf{Z}$. Since ℓ is continuous by assumption 3.2, $b_\ell c\mathsf{Z}$ is a closed subset of $b_\ell \mathsf{Z}$ (see e.g., Boyd (1990), section 3). To show the continuity of ψ^*, it suffices to verify that $Q(b_\ell c\mathsf{Z}) \subset b_\ell c\mathsf{Z}$ (see, e.g., Stokey et al. (1989), corollary 1 of theorem 3.2). For fixed $\psi \in b_\ell c\mathsf{Z}$, let $h(z) := \max\{r(z), \psi(z)\}$, then there exists $G \in \mathbb{R}_+$ such that $|h(z)| \leq |r(z)| + G\ell(z) =: \tilde{h}(z)$. By assumption 3.2, $z \mapsto \tilde{h}(z) \pm h(z)$ are nonnegative and continuous. For all $z \in \mathsf{Z}$ and $\{z_m\} \subset \mathsf{Z}$ with $z_m \to z$, the generalized Fatou's lemma of Feinberg et al. (2014) (theorem 1.1) implies that

$$\int \left(\tilde{h}(z') \pm h(z')\right) P(z, \mathrm{d}z') \leq \liminf_{m \to \infty} \int \left(\tilde{h}(z') \pm h(z')\right) P(z_m, \mathrm{d}z').$$

Since $\lim_{m \to \infty} \int \tilde{h}(z') P(z_m, \mathrm{d}z') = \int \tilde{h}(z') P(z, \mathrm{d}z')$ by assumption 3.2, we have

$$\pm \int h(z') P(z, \mathrm{d}z') \leq \liminf_{m \to \infty} \left(\pm \int h(z') P(z_m, \mathrm{d}z') \right),$$

where we have used the fact that for all sequences $\{a_m\}$ and $\{b_m\}$ in $\mathbb{R}$ with $\lim\limits_{m \to \infty} a_m$ exists, we have: $\liminf\limits_{m \to \infty}(a_m + b_m) = \lim\limits_{m \to \infty} a_m + \liminf\limits_{m \to \infty} b_m$. Hence,

$$\limsup_{m \to \infty} \int h(z') P(z_m, \mathrm{d}z') \leq \int h(z') P(z, \mathrm{d}z') \leq \liminf_{m \to \infty} \int h(z') P(z_m, \mathrm{d}z'), \tag{27}$$

i.e., $z \mapsto \int h(z') P(z, \mathrm{d}z')$ is continuous. Since c is continuous by assumption, $Q\psi \in b_\ell c\mathsf{Z}$. Hence, $Q(b_\ell c\mathsf{Z}) \subset b_\ell c\mathsf{Z}$ and ψ^* is continuous, as was to be shown. The continuity of v^* follows from the continuity of ψ^* and r and the fact that $v^* = r \vee \psi^*$. □

Proof of proposition 3.3. Let $\mathscr{C} := b_\ell \mathsf{Z}$ and $\mathcal{V} := b_\kappa \mathsf{Z}$. Let $\mathscr{V} := R\mathscr{C}$ and $\mathcal{C} := L\mathcal{V}$ as defined respectively in (16) and (8). Our first goal is to show that $\mathscr{V} \subset b_\kappa \mathsf{Z}$ and $\mathcal{C} \subset b_\ell \mathsf{Z}$.

For all $v \in \mathscr{V}$, by definition, there exists a $\psi \in \mathscr{C}$ such that $v = R\psi = r \vee \psi$. Since $\mathscr{C} = b_\ell \mathsf{Z}$, we have $|\psi| \leq M\ell$ for some constant $M < \infty$. Without loss of generality, we can let $M > 1/m'$, where m' is defined as in theorem 3.1 (see (23) in the proof of theorem 3.1). Hence, $|v| \leq |r| + |\psi| \leq M(m'|r| + \ell) = M\kappa$, i.e., $\|v\|_\kappa < \infty$. Moreover, v is measurable since both r and ψ are. Hence, $v \in b_\kappa \mathsf{Z}$. Since v is arbitrary, we have $\mathscr{V} \subset b_\kappa \mathsf{Z}$.

For all $\psi \in \mathcal{C}$, by definition, there exists $v \in \mathcal{V}$ such that $\psi = Lv = c + \beta P v$. Since $\mathcal{V} = b_\kappa \mathsf{Z}$, we have $|v| \leq M\kappa$ for some constant $M < \infty$. By (25) in the proof of theorem 3.1, $|\psi| \leq |c| + \|v\|_\kappa \ell \leq (1/m' + \|v\|_\kappa)\ell$, i.e., $\|\psi\|_\ell < \infty$. Moreover, ψ is measurable by our primitive assumptions. Hence, $\psi \in b_\ell \mathsf{Z}$. Since ψ is arbitrary, we have $\mathcal{C} \subset b_\ell \mathsf{Z}$.

Regarding claim (1), for all $\psi \in \mathscr{C}$, based on lemma 5.3 and theorem 3.1, we have

$$\left|Q^{t+1}\psi(z) - \psi^*(z)\right| = \left|LT^t R\psi(z) - Lv^*(z)\right| = \beta \left|P(T^t R\psi)(z) - Pv^*(z)\right|.$$

Since we have shown in the proof of theorem 3.1 that $\int \kappa(z')P(z, \mathrm{d}z') \leq (m + m'd_1)\ell(z)$ for all $z \in \mathsf{Z}$ (see equation (25)), by the definition of operator P, for all $z \in \mathsf{Z}$, we have

$$\begin{aligned}\left|P(T^t R\psi)(z) - Pv^*(z)\right| &\leq \int \left|(T^t R\psi)(z') - v^*(z')\right| P(z, \mathrm{d}z') \\ &\leq \|T^t R\psi - v^*\|_\kappa \int \kappa(z')P(z, \mathrm{d}z') \leq (m + m'd_1)\|T^t R\psi - v^*\|_\kappa \, \ell(z).\end{aligned}$$

Recall $\rho := \beta(m + m'd_1) < 1$ defined in (23). The above results imply that

$$\|Q^{t+1}\psi - \psi^*\|_\ell \leq \beta(m + m'd_1) \left\|T^t R\psi - v^*\right\|_\kappa = \rho \left\|T^t R\psi - v^*\right\|_\kappa$$

for all $\psi \in \mathscr{C}$. Hence, claim (1) is verified.

Regarding claim (2), for all $v \in \mathcal{V}$, propositions 3.1–3.2 and theorem 3.1 imply that

$$\left|T^{t+1}v(z) - v^*(z)\right| = \left|(RQ^t L)v(z) - R\psi^*(z)\right| \leq \left|Q^t Lv(z) - \psi^*(z)\right| \leq \|Q^t Lv - \psi^*\|_\ell \, \ell(z)$$

for all $z \in \mathsf{Z}$, where the first inequality is due to the elementary fact that $|a \vee b - c \vee d| \leq |a - c| \vee |b - d|$ for all $a, b, c, d \in \mathbb{R}$. Since $\ell \leq \kappa$ by construction, we have

$$\frac{\left|T^{t+1}v(z) - v^*(z)\right|}{\kappa(z)} \leq \frac{\left|T^{t+1}v(z) - v^*(z)\right|}{\ell(z)} \leq \left\|Q^t Lv - \psi^*\right\|_\ell$$

for all $z \in \mathsf{Z}$. Hence, $\left\|T^{t+1}v - v^*\right\|_\kappa \leq \left\|Q^t Lv - \psi^*\right\|_\ell$ and claim (2) holds. $\square$

Proof of theorem 3.2. Since $r, c \in L_q(\pi)$, by the monotonicity of the L_p-norm, we have $r, c \in L_p(\pi)$ for all $1 \leq p \leq q$. Our first goal is to prove claim (1).

Step 1. We show that $Q\psi \in L_p(\pi)$ for all $\psi \in L_p(\pi)$. Notice that for all $z \in \mathsf{Z}$,

$$\begin{aligned}|Q\psi(z)|^p &\leq 2^p |c(z)|^p + (2\beta)^p \left[\int |r(z')| \vee |\psi(z')| \, P(z, \mathrm{d}z')\right]^p \\ &\leq 2^p |c(z)|^p + (2\beta)^p \int \left[|r(z')| \vee |\psi(z')|\right]^p P(z, \mathrm{d}z') \\ &\leq 2^p |c(z)|^p + (2\beta)^p \left(\int |r(z')|^p \, P(z, \mathrm{d}z') + \int |\psi(z')|^p \, P(z, \mathrm{d}z')\right),\end{aligned}$$

where for the first and the third inequality we have used the elementary fact that $(a+b)^p \leq 2^p (a \vee b)^p \leq 2^p (a^p + b^p)$ for all $a, b, p \geq 0$, and the second inequality is based on Jensen's inequality. Then we have $\|Q\psi\|_p < \infty$, since the above result implies that

$$\begin{aligned}\int |Q\psi(z)|^p \, \pi(\mathrm{d}z) &\leq 2^p \int |c(z)|^p \pi(\mathrm{d}z) + (2\beta)^p \int \int |r(z')|^p \, P(z, \mathrm{d}z') \pi(\mathrm{d}z) \\ &\quad + (2\beta)^p \int \int |\psi(z')|^p \, P(z, \mathrm{d}z') \pi(\mathrm{d}z) \\ &= 2^p \int |c(z)|^p \pi(\mathrm{d}z) + (2\beta)^p \int |r(z')|^p \, \pi(\mathrm{d}z') + (2\beta)^p \int |\psi(z')|^p \, \pi(\mathrm{d}z') \\ &= 2^p \|c\|_p^p + (2\beta)^p \|r\|_p^p + (2\beta)^p \|\psi\|_p^p < \infty,\end{aligned}$$

where the first equality follows from the Fubini theorem and the fact that π is a stationary distribution. We have thus verified that $Q\psi \in L_p(\pi)$.

Step 2. We show that Q is a contraction mapping on $\left(L_p(\pi), \|\cdot\|_p\right)$ of modulus β. For all $\psi, \phi \in L_p(\pi)$, we have

$$\begin{aligned}|Q\psi(z) - Q\phi(z)|^p &= \beta^p \left|\int \left[r(z') \vee \psi(z') - r(z') \vee \phi(z')\right] P(z, \mathrm{d}z')\right|^p \\ &\leq \beta^p \int \left|r(z') \vee \psi(z') - r(z') \vee \phi(z')\right|^p P(z, \mathrm{d}z') \\ &\leq \beta^p \int \left|\psi(z') - \phi(z')\right|^p P(z, \mathrm{d}z'),\end{aligned}$$

where the first inequality holds by Jensen's inequality, and the second follows from the elementary fact that $|a \vee b - c \vee d| \leq |a - c| \vee |b - d|$ for all $a, b, c, d \in \mathbb{R}$. Hence,

$$\begin{aligned}\int |Q\psi(z) - Q\phi(z)|^p \, \pi(\mathrm{d}z) &\leq \beta^p \int \int \left|\psi(z') - \phi(z')\right|^p P(z, \mathrm{d}z') \pi(\mathrm{d}z) \\ &= \beta^p \int \left|\psi(z') - \phi(z')\right|^p \pi(\mathrm{d}z'),\end{aligned}$$

and we have $\|Q\psi - Q\phi\|_p \leq \beta \|\psi - \phi\|_p$. Thus, Q is a contraction on $L_p(\pi)$ of modulus β.

Since $(L_p(\pi), \|\cdot\|_p)$ is a Banach space, based on the contraction mapping theorem, Q admits a unique fixed point in $L_p(\pi)$. In order to prove claim (1), it remains to show that $\psi^* \in L_p(\pi)$ and that ψ^* is a fixed point of Q.

Step 3. We show that $\psi^*, v^* \in L_p(\pi)$. Since $|\psi^*(z)| \vee |v^*(z)| \leq \sum_{t=0}^{\infty} \beta^t \mathbb{E}_z[|r(Z_t)| \vee |c(Z_t)|]$, we have

$$\left[\int |\psi^*(z)|^p \pi(\mathrm{d}z)\right] \vee \left[\int |v^*(z)|^p \pi(\mathrm{d}z)\right] \leq \int \left(\sum_{t=0}^{\infty} \beta^t \mathbb{E}_z[|r(Z_t)| \vee |c(Z_t)|]\right)^p \pi(\mathrm{d}z). \quad (28)$$

Since π is stationary, the Fubini theorem implies that

$$\int \mathbb{E}_z |r(Z_t)|^p \pi(\mathrm{d}z) = \int \int |r(z')|^p P^t(z, \mathrm{d}z') \pi(\mathrm{d}z) = \int |r(z')|^p \pi(\mathrm{d}z') = \|r\|_p^p.$$

Similarly, we have $\int \mathbb{E}_z |c(Z_t)|^p \pi(\mathrm{d}z) = \|c\|_p^p$. The Minkowski and Jensen inequalities then imply that for all $n \in \mathbb{N}$,

$$\begin{aligned}
\left\| \sum_{t=0}^{n} \beta^t \mathbb{E}_z[|r(Z_t)| \vee |c(Z_t)|] \right\|_p &\leq \sum_{t=0}^{n} \beta^t \left\| \mathbb{E}_z[|r(Z_t)| \vee |c(Z_t)|] \right\|_p \\
&\leq \sum_{t=0}^{n} \beta^t \left[\int \mathbb{E}_z[|r(Z_t)| \vee |c(Z_t)|]^p \pi(\mathrm{d}z)\right]^{1/p} \\
&\leq \sum_{t=0}^{n} \beta^t \left[\int \left(\mathbb{E}_z|r(Z_t)|^p + \mathbb{E}_z|c(Z_t)|^p\right) \pi(\mathrm{d}z)\right]^{1/p} \\
&= \sum_{t=0}^{n} \beta^t \left(\|r\|_p^p + \|c\|_p^p\right)^{1/p} \leq \frac{\left(\|r\|_p^p + \|c\|_p^p\right)^{1/p}}{1-\beta} < \infty, \quad (29)
\end{aligned}$$

Moreover, by the monotone convergence theorem, we have

$$\left\| \sum_{t=0}^{n} \beta^t \mathbb{E}_z[|r(Z_t)| \vee |c(Z_t)|] \right\|_p \to \left\| \sum_{t=0}^{\infty} \beta^t \mathbb{E}_z[|r(Z_t)| \vee |c(Z_t)|] \right\|_p. \quad (30)$$

Together, (29)–(30) imply that $\left\|\sum_{t=0}^{\infty} \beta^t \mathbb{E}_z[|r(Z_t)| \vee |c(Z_t)|]\right\|_p < \infty$. By (28), we have $\|\psi^*\|_p \vee \|v^*\|_p < \infty$ and thus $\psi^*, v^* \in L_p(\pi)$.

Step 4. We show that v^* is a fixed point of T and ψ^* is a fixed point of Q, i.e., $\|Tv^* - v^*\|_p = 0$ and $\|Q\psi^* - \psi^*\|_p = 0$. It suffices to show that $Tv^* = v^*$ and $Q\psi^* = \psi^*$ with probability one (measured by π, same below). By theorem 1.11 of Peskir and Shiryaev (2006) and the analysis related to (5), it suffices to show that $\mathbb{E}_z \left(\sup_{k \geq 0} \left|\sum_{t=0}^{k-1} \beta^t c(Z_t) + \beta^k r(Z_k)\right|\right) < \infty$ with probability one. This obviously holds since the left-hand-side term is dominated by $\sum_{t=0}^{\infty} \beta^t \mathbb{E}_z[|r(Z_t)| \vee |c(Z_t)|]$, which is finite with probability one since in step 3 we have shown that it is an object of $L_p(\pi)$.

Steps 1–4 imply that claim (1) holds. The proof of claim (2) is similar and thus omitted. $\square$

Proof of proposition 3.5. Let $\mathscr{C} = \mathcal{V} := L_p(\pi)$, and let $\mathscr{V} := R\mathscr{C}$ and $\mathcal{C} := L\mathcal{V}$ as defined respectively in (16) and (8). Our first goal is to show that $\mathscr{V}, \mathcal{C} \subset L_p(\pi)$.

For all $v \in \mathscr{V}$, there exists a $\psi \in \mathscr{C}$ such that $v = R\psi = r \vee \psi$. Since $\mathscr{C} = L_p(\pi)$ and $r \in L_p(\pi)$ by assumption 3.3, we have $v \in L_p(\pi)$. Hence, $\mathscr{V} \subset L_p(\pi)$. For all $\psi \in \mathcal{C}$, there exists $v \in \mathcal{V}$ such that $\psi = Lv = c + \beta P v$. Since $\mathcal{V} = L_p(\pi)$ and π is stationary, Jensen's inequality implies that $Pv \in L_p(\pi)$. Since $c \in L_p(\pi)$, Minkowski's inequality then implies that $\psi \in L_p(\pi)$. Hence, $\mathcal{C} \subset L_p(\pi)$, as claimed.

Regarding claim (1), for all $\psi \in \mathscr{C}$, based on lemma 5.3, theorem 3.2, Jensen's inequality and Fubini's theorem, we have

$$
\begin{aligned}
\left\| Q^{t+1}\psi - \psi^* \right\|_p &= \left[\int \left| Q^{t+1}\psi(z) - \psi^*(z) \right|^p \pi(\mathrm{d}z) \right]^{1/p} \\
&= \left[\int \left| LT^t R\psi(z) - Lv^*(z) \right|^p \pi(\mathrm{d}z) \right]^{1/p} \\
&= \beta \left[\int \left| PT^t R\psi(z) - Pv^*(z) \right|^p \pi(\mathrm{d}z) \right]^{1/p} \\
&\leq \beta \left[\int \int \left| T^t R\psi(z') - v^*(z') \right|^p P(z, \mathrm{d}z')\pi(\mathrm{d}z) \right]^{1/p} \\
&= \beta \left[\int \left| T^t R\psi(z') - v^*(z') \right|^p \pi(\mathrm{d}z') \right]^{1/p} = \beta \left\| T^t R\psi - v^* \right\|_p .
\end{aligned}
$$

Regarding claim (2), for all $v \in \mathcal{V}$, based on propositions 3.1–3.2 and theorem 3.2, we have

$$
\begin{aligned}
\left\| T^{t+1}v - v^* \right\|_p &= \left[\int \left| T^{t+1}v(z) - v^*(z) \right|^p \pi(\mathrm{d}z) \right]^{1/p} \\
&= \left[\int \left| RQ^t Lv(z) - R\psi^*(z) \right|^p \pi(\mathrm{d}z) \right]^{1/p} \\
&\leq \left[\int \left| Q^t Lv(z) - \psi^*(z) \right|^p \pi(\mathrm{d}z) \right]^{1/p} = \left\| Q^t Lv - \psi^* \right\|_p .
\end{aligned}
$$

Hence, the second claim holds. This concludes the proof. $\square$

Proof of lemma 5.1. Recall that if $X \sim LN(\mu, \sigma^2)$, then $\mathbb{E}\, X^s = \mathrm{e}^{s\mu + s^2\sigma^2/2}$ for all $s \in \mathbb{R}$. By (14), the distribution of θ_n given $\theta_0 = \theta$ follows $LN\left(\rho^n \ln\theta,\, \sigma^2 \sum_{i=0}^{n-1} \rho^{2i}\right)$. Hence, $\mathbb{E}_{\,\theta}\, \theta_n = \theta^{\rho^n} \exp\left[\frac{\sigma^2(1-\rho^{2n})}{2(1-\rho^2)}\right]$.

Since $w = \eta + \theta\xi$ and $|\ln w| \leq 1/w + w$, we have $|\ln w_n| \leq \eta_n^{-1} + \eta_n + \theta_n \xi_n$. Hence,

$$\mathbb{E}_{\theta} |\ln w_n| \leq \mathbb{E}_{\theta} \left(\eta_n^{-1} + \eta_n + \theta_n \xi_n \right) = \mu_1 + \mu_2 + \mu_3 \, \mathbb{E}_{\theta} \theta_n = A_n \theta^{\rho^n} + B, \tag{31}$$

where μ_1, μ_2 and μ_3 are respectively the mean of η_n^{-1}, η_n and ξ_n, $B := \mu_1 + \mu_2$ and $A_n := \mu_3 \exp\left[\frac{\sigma^2(1-\rho^{2n})}{2(1-\rho^2)}\right]$. Claim (a) is verified.

To verify claim (b), consider a sequence $\theta^{(m)} \to \theta$. By the Fatou's lemma,

$$\mathbb{E}_{\theta} \left(\eta_n^{-1} + \eta_n + \theta_n \xi_n \pm |\ln w_n| \right) \leq \liminf_m \mathbb{E}_{\theta^{(m)}} \left(\eta_n^{-1} + \eta_n + \theta_n \xi_n \pm |\ln w_n| \right).$$

Since $\theta \mapsto \mathbb{E}_{\theta} \left(\eta_n^{-1} + \eta_n + \theta_n \xi_n \right)$ is continuous by (31), the above inequality yields

$$\pm \mathbb{E}_{\theta} |\ln w_n| \leq \liminf_m \left(\pm \mathbb{E}_{\theta^{(m)}} |\ln w_n| \right),$$

i.e., $\lim_m \mathbb{E}_{\theta^{(m)}} |\ln w_n| = \mathbb{E}_{\theta} |\ln w_n|$. Hence, $\theta \mapsto \mathbb{E}_{\theta} |\ln w_n|$ is continuous, as claimed. □

Proof of lemma 5.2. Since $w = \eta + \theta\xi$ and $|\ln w| \leq w + 1/w$, we have $|\ln w|^q \leq 3^q(\eta^{-q} + \eta^q + \theta^q \xi^q)$. By the assumption on $\{\eta_t\}$ and $\{\xi_t\}$, taking expectation (w.r.t π) yields

$$\mathbb{E} |\ln w|^q \leq 3^q (\mathbb{E}\, \eta^{-q} + \mathbb{E}\, \eta^q + \mathbb{E}\, \xi^q \, \mathbb{E}\, \theta^q) < \infty.$$

Since $r(w) = \ln w / (1 - \beta)$, this inequality implies $r \in L_q(\pi)$. Moreover, $c \in L_q(\pi)$ is trivial since c is constant. □

Appendix C: Proof of Table 1 Results

To prove the results of table 1, we introduce some elementary facts on time complexity:

(a) The multiplication of an $m \times n$ matrix and an $n \times p$ matrix has complexity $\mathcal{O}(mnp)$. See, for example, section 2.5.4 of Skiena (2008).

(b) The binary search algorithm finds the index of an element in a given sorted array of length n in $\mathcal{O}(\log(n))$ time. See, for example, section 4.9 of Skiena (2008).

For finite space (FS) approximation, time complexity of VFI (1-loop) reduces to the complexity of matrix multiplication $\hat{P}\vec{v}$, which is of order $\mathcal{O}(K^2 M^2)$ based on the shape of $\hat{P}$ and $\vec{v}$ and fact (a) above. Similarly, time complexity of CVI (1-loop) is determined by $\hat{F}(\vec{r} \vee \vec{\psi})$, which has complexity $\mathcal{O}(KM^2)$. The n-loop complexity is scaled up by $\mathcal{O}(n)$.

For the infinite space (IS) case, let $\mathcal{O}(g)$ and $\mathcal{O}(h)$ denote respectively the complexity (of single point evaluation) of the interpolating functions g and h. Counting the floating point operations associated with all grid points inside the inner loops shows that the one step complexities of VFI and CVI are $\mathcal{O}(NKM)\mathcal{O}(g)$ and $\mathcal{O}(NM)\mathcal{O}(h)$, respectively. Since binary search function evaluation is adopted for the indicated function interpolation mechanisms (see table 1 note), and in particular, since evaluating g at a given point uses binary search $\ell + n$ times, based on fact (b) above, we have

$$\mathcal{O}(g) = \mathcal{O}\left(\sum_{i=1}^{\ell} \log(K_i) + \sum_{j=1}^{n} \log(M_j)\right) = \mathcal{O}(\log(KM)).$$

Similarly, we can show that $\mathcal{O}(h) = \mathcal{O}(\log(M))$. Combining these results, we see that the claims of the IS case hold, concluding our proof of table 1 results.

References

ALAGOZ, O., L. M. MAILLART, A. J. SCHAEFER, AND M. S. ROBERTS (2004): "The optimal timing of living-donor liver transplantation," *Management Science*, 50, 1420–1430.

ALVAREZ, F. AND A. DIXIT (2014): "A real options perspective on the future of the Euro," *Journal of Monetary Economics*, 61, 78–109.

ARELLANO, C. (2008): "Default risk and income fluctuations in emerging economies," *The American Economic Review*, 98, 690–712.

BAGGER, J., F. FONTAINE, F. POSTEL-VINAY, AND J.-M. ROBIN (2014): "Tenure, experience, human capital, and wages: A tractable equilibrium search model of wage dynamics," *The American Economic Review*, 104, 1551–1596.

BÄUERLE, N. AND A. JAŚKIEWICZ (2018): "Stochastic optimal growth with risk sensitive preferences," *Journal of Economic Theory*, 173, 181–200.

BENTAL, B. AND D. PELED (1996): "The accumulation of wealth and the cyclical generation of new technologies: A search theoretic approach," *International Economic Review*, 37, 687–718.

BOYD, J. H. (1990): "Recursive utility and the Ramsey problem," *Journal of Economic Theory*, 50, 326–345.

CHATTERJEE, S. AND B. EYIGUNGOR (2012): "Maturity, indebtedness, and default risk," *The American Economic Review*, 102, 2674–2699.

DIXIT, A. K. AND R. S. PINDYCK (1994): *Investment Under Uncertainty*, Princeton University Press.

ERICSON, R. AND A. PAKES (1995): "Markov-perfect industry dynamics: A framework for empirical work," *The Review of Economic Studies*, 62, 53–82.

FAJGELBAUM, P., E. SCHAAL, AND M. TASCHEREAU-DUMOUCHEL (2017): "Uncertainty traps," *The Quarterly Journal of Economics*, 132, 1641–1692.

FEINBERG, E. A., P. O. KASYANOV, AND N. V. ZADOIANCHUK (2014): "Fatou's lemma for weakly converging probabilities," *Theory of Probability & Its Applications*, 58, 683–689.

GOMES, J., J. GREENWOOD, AND S. REBELO (2001): "Equilibrium unemployment," *Journal of Monetary Economics*, 48, 109–152.

HOPENHAYN, H. A. (1992): "Entry, exit, and firm dynamics in long run equilibrium," *Econometrica*, 1127–1150.

JOVANOVIC, B. (1982): "Selection and the evolution of industry," *Econometrica*, 649–670.

JOVANOVIC, B. AND R. ROB (1989): "The growth and diffusion of knowledge," *The Review of Economic Studies*, 56, 569–582.

KAMIHIGASHI, T. (2014): "Elementary results on solutions to the Bellman equation of dynamic programming: existence, uniqueness, and convergence," *Economic Theory*, 56, 251–273.

KELLOGG, R. (2014): "The effect of uncertainty on investment: evidence from Texas oil drilling," *The American Economic Review*, 104, 1698–1734.

LISE, J. (2013): "On-the-job search and precautionary savings," *The Review of Economic Studies*, 80, 1086–1113.

LJUNGQVIST, L. AND T. J. SARGENT (2008): "Two questions about European unemployment," *Econometrica*, 76, 1–29.

——— (2012): *Recursive Macroeconomic Theory*, MIT Press.

LOW, H., C. MEGHIR, AND L. PISTAFERRI (2010): "Wage risk and employment risk over the life cycle," *The American Economic Review*, 100, 1432–1467.

MARTINS-DA ROCHA, V. F. AND Y. VAILAKIS (2010): "Existence and uniqueness of a fixed point for local contractions," *Econometrica*, 78, 1127–1141.

MCCALL, J. J. (1970): "Economics of information and job search," *The Quarterly Journal of Economics*, 84, 113–126.

MEYN, S. P. AND R. L. TWEEDIE (2009): *Markov Chains and Stochastic Stability*, Springer Science & Business Media.

MOSCARINI, G. AND F. POSTEL-VINAY (2013): "Stochastic search equilibrium," *The Review of Economic Studies*, 80, 1545–1581.

NAGYPÁL, É. (2007): "Learning by doing vs. learning about match quality: Can we tell them apart?" *The Review of Economic Studies*, 74, 537–566.

PERLA, J. AND C. TONETTI (2014): "Equilibrium imitation and growth," *Journal of Political Economy*, 122, 52–76.

PESKIR, G. AND A. SHIRYAEV (2006): *Optimal Stopping and Free-boundary Problems*, Springer.

PISSARIDES, C. A. (2000): *Equilibrium Unemployment Theory*, MIT press.

PRIES, M. AND R. ROGERSON (2005): "Hiring policies, labor market institutions, and labor market flows," *Journal of Political Economy*, 113, 811–839.

RINCÓN-ZAPATERO, J. P. AND C. RODRÍGUEZ-PALMERO (2003): "Existence and uniqueness of solutions to the Bellman equation in the unbounded case," *Econometrica*, 71, 1519–1555.

RUST, J. (1997): "Using randomization to break the curse of dimensionality," *Econometrica*, 487–516.

SCHAAL, E. (2017): "Uncertainty and unemployment," *Econometrica*, 85, 1675–1721.

SKIENA, S. S. (2008): *The Algorithm Design Manual*, Springer, London.

STOKEY, N., R. LUCAS, AND E. PRESCOTT (1989): *Recursive Methods in Economic Dynamics*, Harvard University Press.

TAUCHEN, G. AND R. HUSSEY (1991): "Quadrature-based methods for obtaining approximate solutions to nonlinear asset pricing models," *Econometrica*, 371–396.