Notes on Lower Bounds on Ground-State Energies of Local Hamiltonians through the Renormalization Group

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Relaxation method:Using locality to obtain lower bound

(1) First relaxation: Truncating the hierarchy

(2) Second relaxation: Applying coarse-graining to the constraints

$$
\begin{aligned}
&\min_{\psi\in\mathcal{H}}\langle\psi|H|\psi\rangle\\
&=\min_{\psi\in\mathcal{H}}\sum_i\langle\psi|h_i|\psi\rangle\\
&=\min_{\{\rho_i\}\leftarrow\psi}\sum_i\operatorname{Tr}(h_i\rho_i)
\end{aligned}
$$

plugging in the locality,the H is the sum of local terms,with identity act on the rest of the system.Then write down it in terms of the reduced density matrix on the corresponding subsets of particles.

Constraint:"$\{\rho_i\}\leftarrow\psi$" $\Leftrightarrow$ there exists a state $\psi\in\mathcal{H}$ such that ρ_i are its reduced states.**(Intractable)**

Specify a weaker constraint,erase some constraints and get **non-physical states**.So the solution is not physical but mathematically a rigorous lower bound

$$
\geq\min_{\{\rho_i\}\in\mathcal{S}_?}\sum_i\operatorname{Tr}(h_i\rho_i)\quad\text{(Relaxation)}
$$

where

$$
H=\sum_i h_i,h_i=h_{a_i}\otimes\mathbb{I}_{a_i^c}
$$

(ρ_i is local reduced density matrix on the patch where h_i acts)

In order to obtain the exact local energy,there is a problem, because all these states on the different patches should be compatible with the global quantum state.

Are they compatible with global quantum state?It is a hard to verify and it is a problem called quantum marginal problem.

Rephrasing the ground-state-energy problem as a hierarchy of semidefinite constraints（We want to find the minimum energy density of a translation-invariant Hamiltonian consisting of an identical nearest-neighbor interaction term h, acting on every pair of consecutive spins on an infinite chain）:

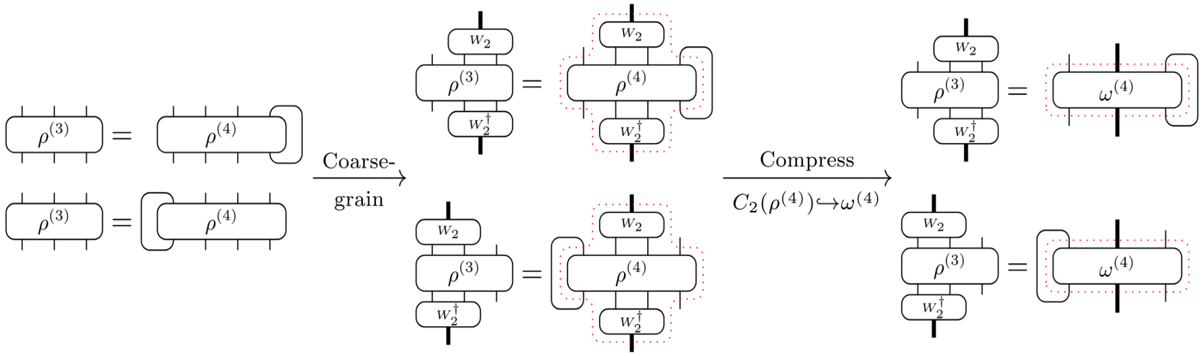
$$
\begin{aligned}
&E_{\mathrm{TI}}:=\min_{\rho^{(2)},\psi_{\mathrm{TI}}}\operatorname{Tr}\left(h\rho^{(2)}\right)\\
&\text{s.t. }\rho^{(2)}\geq 0,\quad\operatorname{Tr}\left(\rho^{(2)}\right)=1,\\
&\rho^{(2)}\leftarrow\psi_{\mathrm{TI}}
\end{aligned}
$$

restrict $\rho^{(2)}$ to be compatible with a global translation-invariant (TI) state

$$
\rho^{(m-1)}\leftarrow\rho^{(m)}\Leftrightarrow\rho^{(m-1)}=\operatorname{Tr}_{\mathrm{L}}\left(\rho^{(m)}\right)=\operatorname{Tr}_{\mathrm{R}}\left(\rho^{(m)}\right)
$$

$$
\operatorname{Tr}_{\mathrm{L}}\left(\rho^{(m)}\right)=\operatorname{Tr}_{\mathrm{R}}\left(\rho^{(m)}\right)\text{(LTI condition)}
$$

Iterative Coarse-Graining scheme



Notice that after the coarse-graining, $\rho^{(3)}$ is related only to the image of $\rho^{(4)}$ under the coarse-graining map $C_2\left(\rho^{(4)}\right) = (\mathbf{I} \otimes W_2 \otimes \mathbf{I})\rho^{(4)}\left(\mathbf{I} \otimes W_2^{\dagger} \otimes \mathbf{I}\right)$. We can therefore encode the coarse-grained constraints using a smaller-dimensional variable $\omega^{(4)}$.

According to the spirit of coarse-graining, a site s' after renormalization should correspond to several sites $\{s\} \equiv \mathcal{B} \subset \mathcal{L}$ (referred to as a block) before renormalization.

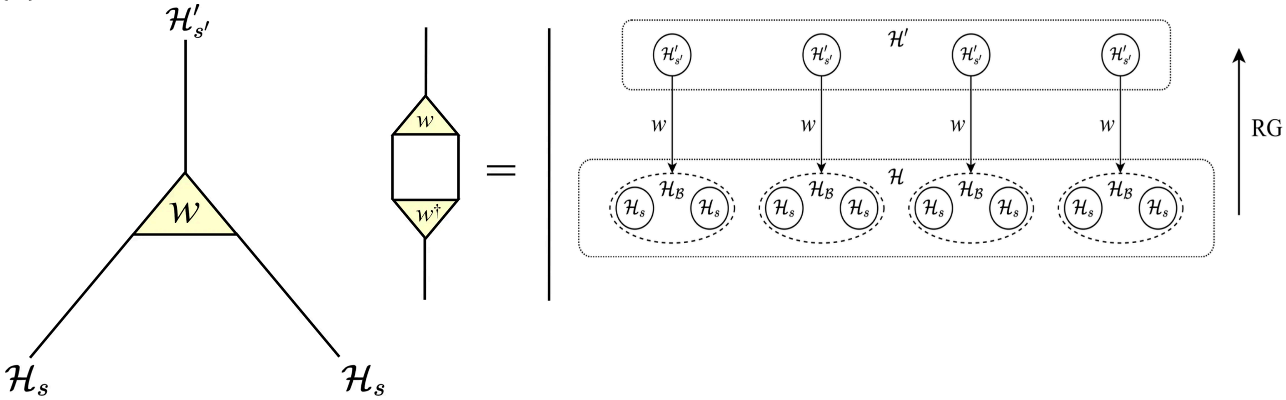


Figure 1

The specific approach is to connect their Hilbert space through a linear mapping.

$$w : \mathcal{H}'_{s'} \to \mathcal{H}_{\mathcal{B}} = \bigotimes_{s \in \mathcal{B}} \mathcal{H}_s$$

It is called an "isometry", such that $w^{\dagger}w = 1$,**Figure 1** is its tensor network graph.

The original lattice is divided into different blocks, where each block $\mathcal{B}$ corresponds to a $w_{\mathcal{B}}$. By mapping each block, the original quantum state can be "coarse-grained" into a new state in a subspace.

$$W = \bigotimes_{\mathcal{B}} w_{\mathcal{B}}$$
$$|\Psi\rangle \to \left|\Psi'\right\rangle = W|\Psi\rangle$$

Therefore, w is to select a subspace reasonably $\mathcal{H}'_{s'}$, such that the projected $\left|\Psi'\right\rangle$ contains all the relevant properties of $|\Psi\rangle$.

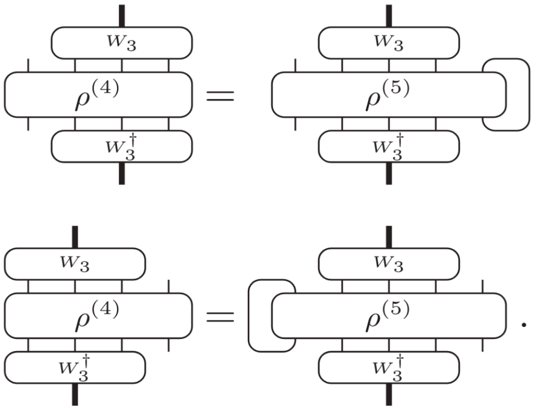


Figure 2

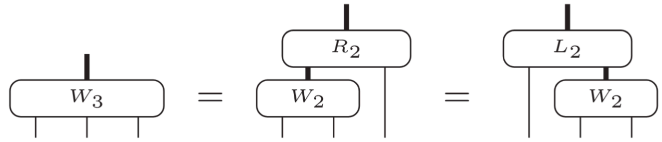
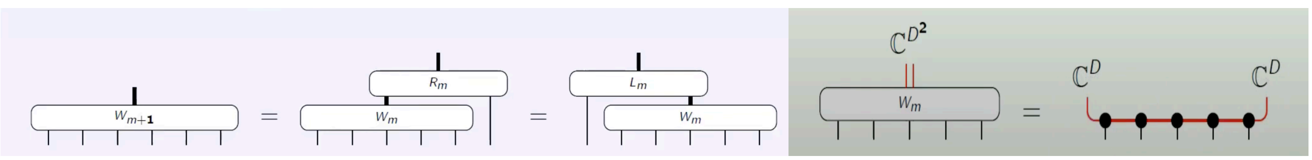


Figure 3

The coarse-graining maps W3 and W2 are related as **Figure 2**,if we require the existence of two additional coarse-graining maps, L_2 and R_2, that act on the output of W_2 together with an additional spin on the left or right side, respectively. Given maps that satisfy **Figure 3.**

An ansatz for the coarse-graining maps: matrix product states



We can use different coarse-graining scheme to construct block spin, here is adding one spin at one time,using MPS to contract m copies of the same rank-3 tensor.

W_{m+1} is equal to add one tensor on the left or right.