Assignment 1

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1. a. $f(x) = x^2$ domain: $(-\infty, +\infty)$

$f'(x) = 2x$

$f''(x) = 2 > 0$

$\therefore$ this function is convex in its domain

b. $f(x) = \ln(x)$ domain: $(0, +\infty)$

$f'(x) = \frac{1}{x}$

$f''(x) = -\frac{1}{x^2} < 0$

$\therefore$ this function is concave in its domain

c. $f(x) = \frac{1}{1+e^{-x}}$ domain: $(-\infty, +\infty)$

$f'(x) = -\frac{1}{(1+e^{-x})^2} \cdot (-e^{-x})$

$= \frac{e^{-x}}{(1+e^{-x})^2}$

$f''(x) = \left(\frac{1}{e^x+e^{-x}+2}\right)'$

$= -\frac{1}{(e^x+e^{-x}+2)^2} \cdot (e^x - e^{-x})$

$= \frac{e^{-x} - e^x}{(e^x+e^{-x}+2)^2}$

$\therefore$ if $x \le 0$, then $f''(x) \geq 0$

if $x > 0$, then $f''(x) < 0$

$\therefore$ this function is convex when $x \in (-\infty, 0]$

else, this function is concave when $x \in (0, +\infty)$

2.

a. $x \sim U(0, \theta)$

$f(x) = \frac{1}{\theta - 0}$

$E(x) = \int_{-\infty}^{\infty} x f(x) dx$

$= \int_{0}^{\theta} \frac{x}{\theta - 0} dx$

$= \int_{0}^{\theta} \frac{x}{\theta} dx$

$= \frac{x^2}{2\theta} \Big|_{0}^{\theta}$

$= \frac{\theta}{2}$

b. $Var(x) = E([x - E(x)]^2)$

$= E(x^2 - 2x E(x) + E(x)^2)$

$= E(x^2) + E(-2x E(x)) + E(E(x)^2)$

$= E(x^2) - 2E(x)^2 + E(x)^2$

$= E(x^2) - E(x)^2$

$= \int_{-\infty}^{\infty} x^2 f(x) dx - E(x)^2$

$$= \frac{1}{\theta-0}\int_0^\theta x^2\,dx - \frac{\theta^2}{4}$$

$$= \frac{x^3}{3\theta}\Big|_0^\theta - \frac{\theta^2}{4}$$

$$= \frac{\theta^2}{3} - \frac{\theta^2}{4}$$

$$= \frac{\theta^2}{12}$$

C. $H(x) = -\int f(x)\cdot \ln f(x)\,dx \qquad f(x)=\frac{1}{\theta}$

$$= -\int_0^\theta \frac{1}{\theta}\cdot \ln\left(\frac{1}{\theta}\right)dx$$

$$= -\frac{1}{\theta}\cdot \ln\left(\frac{1}{\theta}\right)\int_0^\theta 1\cdot dx$$

$$= -\frac{1}{\theta}\cdot \ln\left(\frac{1}{\theta}\right)\cdot\theta$$

$$= -\ln\left(\frac{1}{\theta}\right)$$

$$= \ln\theta$$

3. $L(x_1, x_2, \cdots, x_n) = \prod_{i=1}^{n} p(x_i) = \begin{cases} \frac{1}{\theta^n}, & x_1, x_2 \cdots, x_n \in (0,\theta) \\ 0, & \text{others} \end{cases}$

when $\theta$ is smaller, $L$ is larger

$\therefore \theta \geqslant \max(x_1, x_2, \cdots, x_n)$

$\therefore \hat{\theta} = \max(x_1, x_2, \cdots, x_n)$

4.

Let $M$ be the event: the \$1M is behind the door I choose initially

Let $W$ be the event: I win the \$1M by switching the door

$$P(W|M) = 0$$
$$P(W|M^c) = 1$$

$$\therefore P(W) = P(W|M) \cdot P(M) + P(W|M^c) \cdot P(M^c)$$
$$= 0 \times \frac{1}{3} + 1 \times \frac{2}{3}$$
$$= \frac{2}{3}$$

$\therefore$ the probability of wining by switching the door is $\frac{2}{3}$,

So I choose to switch the door.

5. If $\Sigma$ is PSD,

then $\forall\ y = [y_1, y_2, \cdots y_n]^T$, $y^T \Sigma y \geq 0$

$\Big[ E[(X_1-\mu_1)(X_1-\mu_1)] \quad \cdots \quad E[(X_1-\mu_1)(X_n-\mu_n)] \Big]$

$$y^T\Sigma = [y_1, y_2, \cdots y_n]\begin{bmatrix} \cdots & & \cdots \\ E[(X_2-\mu_2)(X_1-\mu_1)] & \cdots & E[(X_2-\mu_2)(X_n-\mu_n)] \\ \vdots & & \vdots \\ E[(X_n-\mu_n)(X_1-\mu_1)] & \cdots & E[(X_n-\mu_n)(X_n-\mu_n)] \end{bmatrix}$$

$$\therefore y^T\Sigma = \Big[E\Big[\Big(\sum_{i=1}^{n} y_i(X_i-\mu_i)\Big)(X_1-\mu_1)\Big].$$
$$E\Big[\Big(\sum_{i=1}^{n} y_i(X_i-\mu_i)\Big)(X_2-\mu_2)\Big] \quad \cdots,$$
$$E\Big[\Big(\sum_{i=1}^{n} y_i(X_i-\mu_i)\Big)(X_n-\mu_n)\Big]\Big]$$

$$\therefore y^T\Sigma y = \sum_{k=1}^{n} E\Big[\Big(\sum_{i=1}^{n} y_i(X_i-\mu_i)\Big)(X_k-\mu_k)\Big]y_k$$
$$= E\Big[\sum_{k=1}^{n}\Big(\sum_{i=1}^{n} y_i(X_i-\mu_i)\Big)(X_k-\mu_k)y_k\Big]$$
$$= E\Big[\Big(\sum_{i=1}^{n} y_i(X_i-\mu_i)\Big)\Big(\sum_{k=1}^{n}(X_k-\mu_k)y_k\Big)\Big]$$

Random Variable

$$Z = \sum_{i=1}^{n} y_i(X_i-\mu_i) = \sum_{k=1}^{n}(X_k-\mu_k)y_k$$

$$\therefore y^T\Sigma y = E(Z^2) \geq 0$$

$\therefore \Sigma$ is PSD.