Order-d ($d \geq 4$) T-SVD Algebraic Framework—Special Case

Remark .1. *When we utilize the FFT as the invertible linear transform, the block circulant matrix of $\mathcal{A}$ and the block diagonal matrix of $\bar{\mathcal{A}}$ has the following relation:*

$$(\tilde{\mathrm{F}} \otimes \mathrm{I}_{n_1}) \cdot \mathrm{bcirc}(\mathcal{A}) \cdot (\tilde{\mathrm{F}}^{-1} \otimes \mathrm{I}_{n_2}) = \mathrm{bdiag}(\bar{\mathcal{A}}), \tag{1}$$

where $\tilde{\mathrm{F}} = \mathrm{F}_{n_d} \otimes \mathrm{F}_{n_{d-1}} \otimes \cdots \otimes \mathrm{F}_{n_3}$, $\tilde{\mathrm{F}}^{-1} = \mathrm{F}_{n_d}^{-1} \otimes \mathrm{F}_{n_{d-1}}^{-1} \otimes \cdots \otimes \mathrm{F}_{n_3}^{-1}$, *symbol* $\otimes$ *denotes the Kronecker product and* $(\mathrm{F}_{n_d} \otimes \mathrm{F}_{n_{d-1}} \otimes \cdots \otimes \mathrm{F}_{n_3})/\sqrt{n_d n_{d-1} \cdots n_3}$ *is orthogonal. By using the property of real symmetric circulant matrix (See the Definition 1 in [1]), we have*

Algorithm 1 FFT based order-d t-product

Input: $\mathcal{A} \in \mathbb{R}^{n_1 \times n_2 \times n_3 \times \cdots \times n_d}$,$\mathcal{B} \in \mathbb{R}^{n_2 \times l \times n_3 \times \cdots \times n_d}$.

Output: $\mathcal{C} = \mathcal{A} * \mathcal{B} \in \mathbb{R}^{n_1 \times l \times n_3 \times \cdots \times n_d}$.

1. Compute the result of FFT on $\mathcal{A}$ and $\mathcal{B}$

for $i = 3, 4, \cdots, d$ **do**

$\bar{\mathcal{A}} \leftarrow \mathrm{fft}(\mathcal{A}, [\,], i)$, $\bar{\mathcal{B}} \leftarrow \mathrm{fft}(\mathcal{B}, [\,], i)$;

end for

2. Apply (2) and (3) to compute each matrix slice of $\bar{\mathcal{C}}$ with $\bar{\mathrm{A}}$ in (3) replaced by $\bar{\mathrm{C}}$

for $i_3 = 1, 2, \cdots, \lceil \frac{n_3+1}{2} \rceil; i_4 = 2, 3, \cdots, \lceil \frac{n_4+1}{2} \rceil; \cdots; i_d = 2, 3, \cdots, \lceil \frac{n_d+1}{2} \rceil$
$i_3' = 1, 2, \cdots, \lceil \frac{n_3+1}{2} \rceil; i_4' = 1, 2, \cdots, \lceil \frac{n_4+1}{2} \rceil; \cdots; i_{d-1}' = 1, 2, \cdots, \lceil \frac{n_{d-1}+1}{2} \rceil$ **do**

$$\bar{\mathrm{C}}^{(i_3, i_4, \cdots, i_{d-1}, i_d)} = \begin{cases} \bar{\mathrm{A}}^{(i_3,1,1,\cdots,1,1)} \cdot \bar{\mathrm{B}}^{(i_3,1,1,\cdots,1,1)}, \\ \bar{\mathrm{A}}^{(i_3',i_4,1,\cdots,1,1)} \cdot \bar{\mathrm{B}}^{(i_3',i_4,1,\cdots,1,1)}, \\ \bar{\mathrm{A}}^{(i_3',i_4',i_5,\cdots,1,1)} \cdot \bar{\mathrm{B}}^{(i_3',i_4',i_5,\cdots,1,1)}, \\ \vdots \\ \bar{\mathrm{A}}^{(i_3',i_4',i_5',\cdots,i_{d-1}',i_d)} \cdot \bar{\mathrm{B}}^{(i_3',i_4',i_5',\cdots,i_{d-1}',i_d)}. \end{cases} \tag{2}$$

end for

3. Compute the result of inverse FFT on $\bar{\mathcal{C}}$

for $i = d, d-1, \cdots, 3$ **do**

$\mathcal{C} \leftarrow \mathrm{ifft}(\bar{\mathcal{C}}, [\,], i)$.

end for

$$\begin{cases} \bar{\mathrm{A}}^{(1,1,1,\cdots,1,1)} \in \mathbb{R}^{n_1 \times n_2}, \\ \mathrm{conj}(\bar{\mathrm{A}}^{(i_3,1,\cdots,1)}) = \bar{\mathrm{A}}^{(n_3-i_3+2,1,\cdots,1)}, \\ \mathrm{conj}(\bar{\mathrm{A}}^{(i_3',i_4,1,\cdots,1)}) = \bar{\mathrm{A}}^{(n_3',n_4-i_4+2,1,\cdots,1)}, \\ \mathrm{conj}(\bar{\mathrm{A}}^{(i_3',i_4',i_5,1,\cdots,1)}) = \bar{\mathrm{A}}^{(n_3',n_4',n_5-i_5+2,1,\cdots,1)}, \\ \cdots \\ \mathrm{conj}(\bar{\mathrm{A}}^{(i_3',i_4',i_5',\cdots,i_{d-1}',i_d)}) = \bar{\mathrm{A}}^{(n_3',n_4',n_5',\cdots,n_{d-1}',n_d-i_d+2)}, \end{cases} \tag{3}$$

for $i_3 = 2, 3, \cdots, \lceil \frac{n_3+1}{2} \rceil; \cdots; i_d = 2, 3, \cdots, \lceil \frac{n_d+1}{2} \rceil$; $i_3' = 1, 2, \cdots, \lceil \frac{n_3+1}{2} \rceil; \cdots; i_{d-1}' = 1, 2, \cdots, \lceil \frac{n_{d-1}+1}{2} \rceil$. *Here, conj(·) is the conjugate operator. The explicit expressions of* $n_j'(j = 3, 4, 5, \cdots, d)$ *can be written as follow:*

$$n_j' = \begin{cases} n_j' - i_j' + 2, & i_j' \neq 1 \\ 1, & i_j' = 1 \end{cases}.$$

On the contrary, for any given $\bar{A} \in \mathbb{C}^{n_1 \times \cdots \times n_d}$ satisfying (3), there exists a real tensor $\mathcal{A} \in \mathbb{R}^{n_1 \times \cdots \times n_d}$ such that (1) holds. Leveraging on the property (3), the computational cost for the order-d WTSN operator can be further reduced.

Remark .2. *For $i_3 = 1, 2, \cdots, \lceil \frac{n_3+1}{2} \rceil; i_4 = 2, 3, \cdots, \lceil \frac{n_4+1}{2} \rceil; \cdots; i_d = 2, 3, \cdots, \lceil \frac{n_d+1}{2} \rceil$; $i_3^{'} = 1, 2, 3, \cdots, \lceil \frac{n_3+1}{2} \rceil; i_4^{'} = 1, 2, 3, \cdots, \lceil \frac{n_4+1}{2} \rceil; \cdots; i_{d-1}^{'} = 1, 2, 3, \cdots, \lceil \frac{n_{d-1}+1}{2} \rceil$; we let*

$$\bar{\mathrm{A}}^{(i_3,i_4,\cdots,i_{d-1},i_d)} = \begin{cases} \bar{\mathrm{U}}^{(i_3,1,1,\cdots,1,1)} \cdot \bar{\mathrm{S}}^{(i_3,1,1,\cdots,1,1)} \cdot (\bar{\mathrm{V}}^{(i_3,1,1,\cdots,1,1)})^*, \\ \bar{\mathrm{U}}^{(i_3^{'},i_4,1,\cdots,1,1)} \cdot \bar{\mathrm{S}}^{(i_3^{'},i_4,1,\cdots,1,1)} \cdot (\bar{\mathrm{V}}^{(i_3^{'},i_4,1,\cdots,1,1)})^*, \\ \bar{\mathrm{U}}^{(i_3^{'},i_4^{'},i_5,\cdots,1,1)} \cdot \bar{\mathrm{S}}^{(i_3^{'},i_4^{'},i_5,\cdots,1,1)} \cdot (\bar{\mathrm{V}}^{(i_3^{'},i_4^{'},i_5,\cdots,1,1)})^*, \\ \cdots \\ \bar{\mathrm{U}}^{(i_3^{'},i_4^{'},i_5^{'},\cdots,i_{d-1}^{'},i_d)} \cdot \bar{\mathrm{S}}^{(i_3^{'},i_4^{'},i_5^{'},\cdots,i_{d-1}^{'},i_d)} \cdot (\bar{\mathrm{V}}^{(i_3^{'},i_4^{'},i_5^{'},\cdots,i_{d-1}^{'},i_d)})^*. \end{cases} \tag{4}$$

be the be the full SVD of $\bar{\mathrm{A}}^{(i_3,i_4,\cdots,i_{d-1},i_d)}$, where the singular values in $\bar{\mathrm{S}}^{(i_3,i_4,\cdots,i_d)}$ are real. Besides, for $i_3 = \lceil \frac{n_3+1}{2} \rceil + 1, \cdots, n_3$; $i_4 = \lceil \frac{n_4+1}{2} \rceil + 1, \cdots, n_4$; $i_5 = \lceil \frac{n_5+1}{2} \rceil + 1, \cdots, n_5$; $i_d = \lceil \frac{n_d+1}{2} \rceil + 1, \cdots, n_d$; we let

$$\begin{cases} \bar{\mathrm{U}}^{(1,1,1,\cdots,1,1)} \in \mathbb{R}^{n_1 \times n_2}, \\ \bar{\mathrm{U}}^{(i_3,1,1,\cdots,1,1)} = \mathrm{conj}(\bar{\mathrm{U}}^{(n_3-i_3+2,1,1,\cdots,1,1)}), \\ \bar{\mathrm{U}}^{(i_3,i_4,1,\cdots,1,1)} = \mathrm{conj}(\bar{\mathrm{U}}^{(n_3^{'},n_4-i_4+2,1,\cdots,1,1)}), \\ \bar{\mathrm{U}}^{(i_3,i_4,i_5,1,\cdots,1)} = \mathrm{conj}(\bar{\mathrm{U}}^{(n_3^{'},n_4^{'},n_5-i_5+2,1,\cdots,1)}), \\ \vdots \\ \bar{\mathrm{U}}^{(i_3,i_4,i_5,\cdots,i_{d-1},i_d)} = \mathrm{conj}(\bar{\mathrm{U}}^{(n_3^{'},n_4^{'},n_5^{'},\cdots,n_{d-1}^{'},n_d-i_d+2)}). \end{cases} \tag{5}$$

$$\begin{cases} \bar{\mathrm{S}}^{(1,1,1,\cdots,1,1,1)} \in \mathbb{R}^{n_1 \times n_2}, \\ \bar{\mathrm{S}}^{(i_3,1,1,\cdots,1,1,1)} = (\bar{\mathrm{S}}^{(n_3-i_3+2,1,1,\cdots,1,1,1)}), \\ \bar{\mathrm{S}}^{(i_3,i_4,1,\cdots,1,1,1)} = (\bar{\mathrm{S}}^{(n_3^{'},n_4-i_4+2,1,\cdots,1,1,1)}), \\ \bar{\mathrm{S}}^{(i_3,i_4,i_5,1,\cdots,1,1)} = (\bar{\mathrm{S}}^{(n_3^{'},n_4^{'},n_5-i_5+2,1,\cdots,1,1)}), \\ \vdots \\ \bar{\mathrm{S}}^{(i_3,i_4,i_5,\cdots,i_{d-2},i_{d-1},i_d)} = (\bar{\mathrm{S}}^{(n_3^{'},n_4^{'},n_5^{'},\cdots,n_{d-2}^{'},n_{d-1}^{'},n_d-i_d+2)}). \end{cases} \tag{6}$$

$$\begin{cases} \bar{\mathrm{V}}^{(1,1,1,\cdots,1,1)} \in \mathbb{R}^{n_1 \times n_2}, \\ \bar{\mathrm{V}}^{(i_3,1,1,\cdots,1,1)} = \mathrm{conj}(\bar{\mathrm{V}}^{(n_3-i_3+2,1,1,\cdots,1,1)}), \\ \bar{\mathrm{V}}^{(i_3,i_4,1,\cdots,1,1)} = \mathrm{conj}(\bar{\mathrm{V}}^{(n_3^{'},n_4-i_4+2,1,\cdots,1,1)}), \\ \bar{\mathrm{V}}^{(i_3,i_4,i_5,1,\cdots,1)} = \mathrm{conj}(\bar{\mathrm{V}}^{(n_3^{'},n_4^{'},n_5-i_5+2,1,\cdots,1)}), \\ \vdots \\ \bar{\mathrm{V}}^{(i_3,i_4,i_5,\cdots,i_{d-1},i_d)} = \mathrm{conj}(\bar{\mathrm{V}}^{(n_3^{'},n_4^{'},n_5^{'},\cdots,n_{d-1}^{'},n_d-i_d+2)}). \end{cases} \tag{7}$$

Based on (4)-(7), we present FFT based order-d t-product , order-d t-SVD and order-d WTSN proximal operator in Algorithm 1-3, respectively.

Algorithm 2 FFT based order-d t-SVD

Input: $\mathcal{A} \in \mathbb{R}^{n_1 \times n_2 \times \cdots \times n_{d-1} \times n_d}$.

Output: t-SVD components $\mathcal{U}$,$\mathcal{S}$ and $\mathcal{V}$ of $\mathcal{A}$.

1. Compute the result of FFT on $\mathcal{A}$

for $i = 3, 4, \cdots, d$ **do**

$\bar{\mathcal{A}} \leftarrow \mathrm{fft}(\mathcal{A}, [\,], i)$;

end for

2. Compute each matrix slice of $\bar{\mathcal{U}}$,$\bar{\mathcal{S}}$ and $\bar{\mathcal{V}}$ from $\bar{\mathcal{A}}$ by (4)-(7).

3. Compute the result of inverse FFT on $\bar{\mathcal{U}}$, $\bar{\mathcal{S}}$ and $\bar{\mathcal{V}}$.

for $i = d, d-1, \cdots, 3$ **do**

$\mathcal{U} \leftarrow \mathrm{ifft}(\bar{\mathcal{U}}, [\,], i), \mathcal{S} \leftarrow \mathrm{ifft}(\bar{\mathcal{S}}, [\,], i), \mathcal{V} \leftarrow \mathrm{ifft}(\bar{\mathcal{V}}, [\,], i)$.

end for

Algorithm 3 FFT based order-d WTSN proximal operator

Input: $\mathcal{A} \in \mathbb{R}^{n_1 \times \cdots \times n_d}$, $\tau > 0$, $0 < p < 1$, weight parameter: $\mathcal{W} \in \mathbb{R}^{n_1 \times \cdots \times n_d}$, the number of GST iterations: J and invertible linear transform L.

Output: $\mathcal{D}_{\mathcal{W},p,\tau}(\mathcal{A}) = \mathcal{U} *_L \mathcal{S}_{\mathcal{W},p,\tau} *_L \mathcal{V}^*$.

1. Compute the result of FFT on $\mathcal{A}$

for $i = 3, 4, \cdots, d$ **do**

$\bar{\mathcal{A}} \leftarrow \text{fft}(\mathcal{A}, [\,], i)$;

end for

2. Compute each matrix slice of $\bar{\mathcal{U}}$,$\bar{\mathcal{S}}$ and $\bar{\mathcal{V}}$ from $\bar{\mathcal{A}}$ by (4)-(7).

3. Apply (3) and (8) to compute each matrix slice of $\bar{\mathcal{Z}}$ with $\bar{\text{A}}$ in (3) replaced by $\bar{\text{Z}}$

for $i_3 = 1, 2, \cdots, \lceil \frac{n_3+1}{2} \rceil; i_4 = 2, 3, \cdots, \lceil \frac{n_4+1}{2} \rceil; \cdots; i_d = 2, 3, \cdots, \lceil \frac{n_d+1}{2} \rceil$
$i_3^{'} = 1, 2, \cdots, \lceil \frac{n_3+1}{2} \rceil; i_4^{'} = 1, 2, \cdots, \lceil \frac{n_4+1}{2} \rceil; \cdots; i_{d-1}^{'} = 1, 2, \cdots, \lceil \frac{n_{d-1}+1}{2} \rceil$ **do**

$$\bar{\text{Z}}^{(i_3,i_4,\cdots,i_{d-1},i_d)} = \begin{cases} \bar{\text{U}}^{(i_3,1,1,\cdots,1,1)} \cdot \text{diag}\left\{ \text{GST}\left(\text{diag}\,(\bar{\text{S}}^{(i_3,1,1,\cdots,1,1)}), \tau\, \text{diag}\,(\text{W}^{(i_3,1,1,\cdots,1,1)}), p, J \right) \right\} \cdot \\ \bar{\text{V}}^{(i_3,1,1,\cdots,1,1)}; \\ \bar{\text{U}}^{(i_3^{'},i_4,1,\cdots,1,1)} \cdot \text{diag}\left\{ \text{GST}\left(\text{diag}\,(\bar{\text{S}}^{(i_3^{'},i_4,1,\cdots,1,1)}), \tau\, \text{diag}\,(\text{W}^{(i_3^{'},i_4,1,\cdots,1,1)}), p, J \right) \right\} \cdot \\ \bar{\text{V}}^{(i_3^{'},i_4,1,\cdots,1,1)}; \\ \bar{\text{U}}^{(i_3^{'},i_4^{'},i_5,\cdots,1,1)} \cdot \text{diag}\left\{ \text{GST}\left(\text{diag}\,(\bar{\text{S}}^{(i_3^{'},i_4^{'},i_5,\cdots,1,1)}), \tau\, \text{diag}\,(\text{W}^{(i_3^{'},i_4^{'},i_5,\cdots,1,1)}), p, J \right) \right\} \cdot \\ \bar{\text{V}}^{(i_3^{'},i_4^{'},i_5,\cdots,1,1)}; \\ \vdots \\ \bar{\text{U}}^{(i_3^{'},i_4^{'},i_5^{'},\cdots,i_{d-1}^{'},i_d)} \cdot \text{diag}\left\{ \text{GST}\left(\text{diag}\,(\bar{\text{S}}^{(i_3^{'},i_4^{'},i_5^{'},\cdots,i_{d-1}^{'},i_d)}), \tau\, \text{diag}\,(\text{W}^{(i_3^{'},i_4^{'},i_5^{'},\cdots,i_{d-1}^{'},i_d)}), p, J \right) \right\} \cdot \\ \bar{\text{V}}^{(i_3^{'},i_4^{'},i_5^{'},\cdots,i_{d-1}^{'},i_d)}. \end{cases} \tag{8}$$

end for

3. Compute the result of inverse FFT on $\bar{\mathcal{Z}}$.

for $i = d, d-1, \cdots, 3$ **do**

$\mathcal{D}_{\mathcal{W},p,\tau}(\mathcal{A}) \leftarrow \text{ifft}(\bar{\mathcal{Z}}, [\,], i)$.

end for

References

[1] O. Rojo and H. Rojo, "Some results on symmetric circulant matrices and on symmetric centrosymmetric matrices," *Linear algebra and its applications*, vol. 392, pp. 211–233, 2004.