Supplementary Materials for "Hyperspectral Anomaly Detection Fused Unified Nonconvex Tensor Ring Factors Regularization "

Wenjin Qin, Hailin Wang, *Student Member, IEEE,* Hao Shu, Feng Zhang, Jianjun Wang, *Member, IEEE,* Xiangyong Cao, *Member, IEEE,* Xi-Le Zhao, and Gemine Vivone, *Senior Member, IEEE*

In this supplementary material, we provide the theoretical proofs for the convergence of the proposed nonconvex HAD algorithm. The main results are given in Theorem .1 and Theorem .2 below. Before giving the convergence analysis, we first show that the sequences $\left\{(\mathcal{L}^{(n,k)})^{\{\nu\}}\right\}$, $\left\{(\mathcal{S}^{(n,k)})^{\{\nu\}}\right\}$, $\left\{(\mathcal{G}^{(n)})^{\{\nu\}}\right\}$, $\mathcal{E}^{\{\nu\}}$, $\left\{\mathcal{Y}^{\{\nu\}}\right\}$ and $\left\{\mathcal{Q}^{(n,k)^{\{\nu\}}}\right\}$ $(n,k=1,2,3)$ generated by the proposed HAD algorithm are bounded (*please see Lemmas .3-.4 for more details*).

Lemma .1. ***(Bolzano-Weierstrass theorem*** *[1]) Every bounded sequence of real numbers has a convergent subsequence.*

Lemma .2. *[2] Suppose $F:\mathbb{R}^{n_1\times n_2}\rightarrow\mathbb{R}$ is represented as $F(\boldsymbol{X})=f\circ\vec{\sigma}(\boldsymbol{X})$, where $\boldsymbol{X}\in\mathbb{R}^{n_1\times n_2}$ with SVD $\boldsymbol{X}=\boldsymbol{U}\operatorname{diag}(\sigma_1,\cdots,\sigma_n)\boldsymbol{V}^T$, $n=\max(n_1,n_2)$, and f is differentiable. Then, the gradient of $F(\boldsymbol{X})$ at $\boldsymbol{X}$ is*

$$\frac{\partial F(\boldsymbol{X})}{\partial \boldsymbol{X}}=\boldsymbol{U}\operatorname{diag}(\vec{\theta})\boldsymbol{V}^T,$$

where $\vec{\theta}=\left.\frac{\partial f(\vec{y})}{\partial\vec{y}}\right|_{\vec{y}=\vec{\sigma}(\boldsymbol{X})}$.

Lemma .3. *The sequences $\left\{\mathcal{Y}^{\{\nu\}}\right\}$ and $\left\{\mathcal{Q}^{(n,k)^{\{\nu\}}}\right\}$ $(n,k=1,2,3)$ generated by the proposed nonconvex HAD Algorithm are bounded.*

Proof. **1) The proof of boundedness of sequence $\left\{\mathcal{Q}^{(n,k)^{\{\nu\}}}\right\}$ $(n,k=1,2,3)$:** By the first order necessary optimality condition of $\mathcal{L}^{(n,k)}$-subproblem, for each $n,k=1,2,3$ we obtain

$$\begin{aligned}0&\in\frac{1}{\gamma}\frac{\partial}{\partial\mathcal{L}^{(n,k)}}\bigg|_{\mathcal{L}^{(n,k)}=\left\{\mathcal{L}^{(n,k)}\right\}^{\{\nu+1\}}}\|\mathcal{L}^{(n,k)}\|_{\Phi,\mathfrak{L}}-(\mathcal{Q}^{(n,k)})^{\{\nu\}}-\mu^{\{\nu\}}\left(\nabla_k(\mathcal{G}^{(n)})^{\{\nu+1\}}-(\mathcal{L}^{(n,k)})^{\{\nu+1\}}-(\mathcal{S}^{(n,k)})^{\{\nu+1\}}\right)\\&=\frac{1}{\gamma}\frac{\partial}{\partial\mathcal{L}^{(n,k)}}\bigg|_{\mathcal{L}^{(n,k)}=\left\{\mathcal{L}^{(n,k)}\right\}^{\{\nu+1\}}}\|\mathcal{L}^{(n,k)}\|_{\Phi,\mathfrak{L}}-(\mathcal{Q}^{(n,k)})^{\{\nu+1\}}\end{aligned}\tag{1}$$

Now, we let $\mathcal{L}^{(n,k)}=\mathcal{U}^{(n,k)}*_{\mathfrak{L}}\mathcal{K}^{(n,k)}*_{\mathfrak{L}}(\mathcal{V}^{(n,k)})^T$, $\sigma_{ij}^{(n,k)}=\mathfrak{L}(\mathcal{K}^{(n,k)})^{<j>}(i,i)$. According to previous Lemma .2, it then follows that

$$\frac{1}{\gamma}\frac{\partial}{\partial\mathfrak{L}(\mathcal{L}^{(n,k)})^{<j>}}\left\|\mathfrak{L}(\mathcal{L}^{(n,k)})^{<j>}\right\|_{\Phi}=\frac{1}{\gamma}\cdot\mathfrak{L}(\mathcal{U}^{(n,k)})^{<j>}\cdot\operatorname{diag}\left\{\partial\Phi(\sigma_{ij}^{(n,k)})\right\}\cdot\left(\mathfrak{L}(\mathcal{V}^{(n,k)})^{<j>}\right)^T,$$

and then one can obtain

$$\left\|\frac{\partial}{\partial\mathfrak{L}(\mathcal{L}^{(n,k)})^{<j>}}\left\|\mathfrak{L}(\mathcal{L}^{(n,k)})^{<j>}\right\|_{\Phi}\right\|_F=\sqrt{\sum_{i=1}^{r}\left(\partial\Phi(\sigma_{ij}^{(n,k)})\right)^2},$$

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Wenjin Qin, Feng Zhang, and Jianjun Wang are with the School of Mathematics and Statistics, Southwest University, Chongqing 400715, China (e-mail: qinwenjin2021@163.com, zfmath@swu.edu.cn, wjj@swu.edu.cn).

Hailin Wang and Hao Shu are with the School of Mathematics and Statistics, Xi'an Jiaotong University, Xi'an 710049, China (e-mail: wanghailin97@163.com, haoshu812@gmail.com).

Xiangyong Cao is with School of Computer Science and Technology and Ministry of Education Key Lab For Intelligent Networks and Network Security, Xi'an Jiaotong University, Xi'an 710049, China (e-mail: caoxiangyong@mail.xjtu.edu.cn).

Xi-Le Zhao is with the School of Mathematical Sciences/Research Center for Image and Vision Computing, University of Electronic Science and Technology of China, Chengdu 611731, China (e-mail: xlzhao122003@163.com).

Gemine Vivone is with the Institute of Methodologies for Environmental Analysis, CNR-IMAA, 85050 Tito Scalo, Italy, and also with the National Biodiversity Future Center (NBFC), 90133 Palermo, Italy (e-mail: gemine.vivone@imaa.cnr.it).

where r denotes the tubal rank of $\mathcal{L}^{(n,k)}$. Below, we will prove that $\partial\Phi\big(\sigma_{ij}^{(n,k)}\big)$ is bounded. Here, we only discuss the parameters of the nonconvex function $\Phi(\cdot)$ on $[0,\infty)$ since they are symmetrical with respect to y-axis. Specifically,

Case 1: $\Phi(\cdot)$ is set to be Firm penalty function. $\partial\Phi\big(\sigma_{ij}^{(n,k)}\big) \leq 1$ is bounded.

Case 2: $\Phi(\cdot)$ is set to be SCAD penalty function. $\partial\Phi\big(\sigma_{ij}^{(n,k)}\big) \leq 1$ is bounded.

Case 3: $\Phi(\cdot)$ is set to be MCP penalty function. $\partial\Phi\big(\sigma_{ij}^{(n,k)}\big) \leq 1$ is bounded.

Case 4: $\Phi(\cdot)$ is set to be logarithmic function. $\partial\Phi\big(\sigma_{ij}^{(n,k)}\big) \leq \frac{1}{\theta+\sigma_{ij}^{(n,k)}} \leq \frac{1}{\theta}$ is bounded.

Case 5: $\Phi(\cdot)$ is set to be ℓ_q penalty function. In order to overcome the singularity of $(x^q)' = \frac{q}{x^{1-q}}$ near ∞ as x near 0, we consider for $0 < \varrho \ll 1$ the approximation, i.e.,

$$(x^q)' \approx \frac{q}{\max(x^{1-q}, \varrho^{1-q})}.$$

Thus, $\partial\Phi\big(\sigma_{ij}^{(n,k)}\big) = \frac{q}{\max((\sigma_{ij}^{(n,k)})^{1-q}, \varrho^{1-q})} \leq \frac{q}{\varrho^{1-q}}$ is bounded.

Case 6: $\Phi(\cdot)$ is set to be Capped-type penalty function. From the above five cases, we obtain the boundedness of $\partial\Phi\big(\sigma_{ij}^{(n,k)}\big)$.

Combining all cases, we can find that

$$\frac{\partial}{\partial\mathfrak{L}(\mathcal{L}^{(n,k)})^{<j>}}\|\mathfrak{L}(\mathcal{L}^{(n,k)})^{<j>}\|_{\Phi}$$

is bounded. Therefore,

$$\frac{\partial}{\partial\mathfrak{L}(\mathcal{L}^{(n,k)})}\|\mathcal{L}^{(n,k)}\|_{\Phi,\mathfrak{L}} = \left[\frac{\partial}{\partial\mathfrak{L}(\mathcal{L}^{(n,k)})^{<1>}}\|\mathfrak{L}(\mathcal{L}^{(n,k)})^{<1>}\|_{\Phi} \;\; |\cdots| \;\; \frac{\partial}{\partial\mathfrak{L}(\mathcal{L}^{(n,k)})^{<J>}}\|\mathfrak{L}(\mathcal{L}^{(n,k)})^{<J>}\|_{\Phi}\right], \tag{2}$$

is bounded, where J denotes the total number of total slices of $\mathfrak{L}(\mathcal{L}^{(n,k)})$. For $\mathfrak{L}(\mathcal{L}^{(n,k)}) = \mathcal{L}^{(n,k)} \times_3 \boldsymbol{U}_{n_3} \times_4 \boldsymbol{U}_{n_4} \cdots \times_d \boldsymbol{U}_{n_d}$ and using the chain rule in matrix calculus, one can obtain that $\frac{\partial}{\partial(\mathcal{L}^{(n,k)})}\|\mathcal{L}^{(n,k)}\|_{\Phi,\mathfrak{L}}$ is bounded. Therefore,

$$(\mathcal{Q}^{(n,k)})^{\{\nu+1\}} \in \frac{1}{\gamma}\frac{\partial}{\partial\mathcal{L}^{(n,k)}}\bigg|_{\mathcal{L}^{(n,k)}=\left\{\mathcal{L}^{(n,k)}\right\}^{\{\nu+1\}}}\|\mathcal{L}^{(n,k)}\|_{\Phi,\mathfrak{L}}$$

is bounded.

2) The proof of boundedness of sequence $\{\mathcal{Y}^{\{\nu\}}\}$**:** According to the equation $\mathcal{Y}^{\{\nu+1\}} = \mathcal{Y}^{\{\nu\}} + \mu^{\{\nu\}}\Big(\mathcal{M} - \Re([\mathcal{G}]) - \mathcal{E}^{\{\nu+1\}}\Big)$, at the $(\nu+1)$-th iteration, we have

$$\begin{aligned}
&\big\|\mathcal{Y}^{\{\nu+1\}}\big\|_F^2 \\
&= \big\|\mathcal{Y}^{\{\nu\}} + \mu^{\{\nu\}}(\mathcal{M} - \Re([\mathcal{G}]) - \mathcal{E}^{\{\nu+1\}})\big\|_F^2 \\
&= (\mu^{\{\nu\}})^2\big\|\frac{1}{\mu^{\{\nu\}}}\mathcal{Y}^{\{\nu\}} + \mathcal{M} - \Re([\mathcal{G}]) - \mathcal{E}^{\{\nu+1\}}\big\|_F^2 \\
&= (\mu^{\{\nu\}})^2\big\|\mathcal{E}^{\{\nu+1\}} - (\mathcal{M} - \Re([\mathcal{G}]) + \mathcal{Y}^{\{\nu\}}\frac{1}{\mu^{\{\nu\}}})\big\|_F^2.
\end{aligned}$$

Let $\mathcal{H}^{\{\nu\}} := \mathcal{M} - \Re([\mathcal{G}]) + \mathcal{Y}^{\{\nu\}}\frac{1}{\mu^{\{\nu\}}}$, we have

$$\big\|\mathcal{Y}^{\{\nu+1\}}\big\|_F^2 = (\mu^{\{\nu\}})^2\big\|\mathcal{E}^{\{\nu+1\}} - \mathcal{H}^{\{\nu\}}\big\|_F^2.$$

When the anomaly tensor $\mathcal{E}$ has structured sparsity on the tubes, i.e., $h(\cdot) = \|\cdot\|_{F,1}$, then we have

$$\big\|\mathcal{Y}^{\{\nu+1\}}\big\|_F^2 = (\mu^{\{\nu\}})^2\sum_{i_1=1}^{I_1}\sum_{i_2=1}^{I_2}\big\|\mathcal{E}^{\{\nu+1\}}{}_{i_1,i_2,:,\cdots,:} - \mathcal{H}^{\{\nu\}}{}_{i_1,i_2,:,\cdots,:}\big\|_F^2, \tag{3}$$

Furthermore, in virtue of the subproblems with respect to $\mathcal{E}^{\{\nu+1\}}$, for $\forall i_1 \in \{1,2,\cdots,n_1\}, i_2 \in \{1,2,\cdots,n_2\}$, we obtain

$$\begin{aligned}
&0 \in \beta\cdot\partial\psi\Big(\big\|(\mathcal{E}^{\{\nu+1\}})_{i_1,i_2,:,\cdots,:}\big\|_F\Big) + \mu^{\{\nu\}}\cdot\Big((\mathcal{E}^{\{\nu+1\}})_{i_1,i_2,:,\cdots,:} - (\mathcal{H}^{\{\nu\}})_{i_1,i_2,:,\cdots,:}\Big) \\
&\Rightarrow \mu^{\{\nu\}}\cdot\Big((\mathcal{E}^{\{\nu+1\}})_{i_1,i_2,:,\cdots,:} - (\mathcal{H}^{\{\nu\}})_{i_1,i_2,:,\cdots,:}\Big) \in -\beta\cdot\partial\psi\Big(\big\|(\mathcal{E}^{\{\nu+1\}})_{i_1,i_2,:,\cdots,:}\big\|_F\Big).
\end{aligned} \tag{4}$$

In the light of (4), $\big(\mathcal{E}^{\{\nu+1\}}{}_{i_1,i_2,:,\cdots,:} - \mathcal{H}^{\{\nu\}}{}_{i_1,i_2,:,\cdots,:}\big)$ can be further expressed as follows:

$$(\mathcal{E}^{\{\nu+1\}}{}_{i_1,i_2,:,\cdots,:} - \mathcal{H}^{\{\nu\}}{}_{i_1,i_2,:,\cdots,:}) = \frac{-\beta\cdot\partial\psi\Big(\big\|(\mathcal{E}^{\{\nu+1\}})_{i_1,i_2,:,\cdots,:}\big\|_F\Big)}{\mu^{\{\nu\}}}. \tag{5}$$

Combining the equation (3) and (5), we obtain

$$\|\mathcal{Y}^{\{\nu+1\}}\|_F^2 = (\mu^{\{\nu\}})^2 \sum_{i_1=1}^{I_1} \sum_{i_2=1}^{I_2} \left(\frac{-\beta \cdot \partial\psi\Big(\big\|(\mathcal{E}^{\{\nu+1\}})_{i_1,i_2,:,\cdots,:}\big\|_F\Big)}{\mu^{\{\nu\}}} \right)^2 \\ \leq I_1 I_2 \cdot \bigg(-\beta \cdot \partial\psi\Big(\big\|(\mathcal{E}^{\{\nu+1\}})_{i_1,i_2,:,\cdots,:}\big\|_F\Big)\bigg)^2.$$

Similar to the proof of the boundedness of $\partial\Phi(\sigma_{ij}^{(n,k)})$, we can deduce that $\partial\psi\Big(\big\|(\mathcal{E}^{\{\nu+1\}})_{i_1,i_2,:,\cdots,:}\big\|_F\Big)$ is bounded. Therefore, the sequence $\mathcal{Y}^{\{\nu+1\}}$ is bounded. Note that similar proofs of the boundedness of the sequence $\mathcal{Y}^{\{\nu+1\}}$ can be analysed for the cases where the tensor $\mathcal{E}^{\{\nu+1\}}$ has structured sparsity on the slices or the tensor $\mathcal{E}^{\{\nu+1\}}$ is an entry-wise anomaly tensor. □

Lemma .4. *Suppose that the sequences $\{\mathcal{Y}^{\{\nu\}}\}$ and $\{\mathcal{Q}^{(n,k)\{\nu\}}\}$ $(n,k=1,2,3)$ generated by the proposed HAD Algorithm are bounded, then the sequences $\{(\mathcal{L}^{(n,k)})^{\{\nu\}}\}$, $\{(\mathcal{S}^{(n,k)})^{\{\nu\}}\}$, $\{(\mathcal{G}^{(n)})^{\{\nu\}}\}$, and $\mathcal{E}^{\{\nu\}}$ are bounded.*

Proof. According to

$$\mathcal{Y}^{\{\nu+1\}} = \mathcal{Y}^{\{\nu\}} + \mu^{\{\nu\}}\Big(\mathcal{M} - \Re([\mathcal{G}]) - \mathcal{E}^{\{\nu+1\}}\Big),$$

$$(\mathcal{Q}^{(n,k)})^{\{\nu+1\}} = (\mathcal{Q}^{(n,k)})^{\{\nu\}} + \mu^{\{\nu\}} \cdot \Big(\nabla_k(\mathcal{G}^{(n)})^{\{\nu+1\}} - (\mathcal{L}^{(n,k)})^{\{\nu+1\}} - (\mathcal{S}^{(n,k)})^{\{\nu+1\}}\Big),$$

and

$$\mathcal{F}(\mathcal{L}^{(n,k)}, \mathcal{S}^{(n,k)}, \mathcal{Q}^{(n,k)}, [\mathcal{G}], \mathcal{E}, \mathcal{Y}) = \sum_{n=1}^{3}\sum_{k=1}^{3}\Big\{\frac{1}{\gamma}\|\mathcal{L}^{(n,k)}\|_{\Phi,\mathfrak{L}} + \alpha \cdot \|\mathcal{S}^{(n,k)}\|_{\ell_1^{\psi}} + \langle \mathcal{Q}^{(n,k)}, \nabla_k(\mathcal{G}^{(n)}) - \mathcal{L}^{(n,k)} - \mathcal{S}^{(n,k)}\rangle \\ + \frac{\mu}{2}\|\nabla_k(\mathcal{G}^{(n)}) - \mathcal{L}^{(n,k)} - \mathcal{S}^{(n,k)}\|_F^2\Big\} + \beta \cdot \|\mathcal{E}\|_{\ell_{F,1}^{\psi}} + \langle \mathcal{Y}, \mathcal{M} - \Re([\mathcal{G}]) - \mathcal{E}\rangle + \frac{\mu}{2}\|\mathcal{M} - \Re([\mathcal{G}]) - \mathcal{E}\|_F^2, \tag{6}$$

we have

$$\mathcal{F}\Big((\mathcal{L}^{(n,k)})^{\{\nu\}}, (\mathcal{S}^{(n,k)})^{\{\nu\}}, [\mathcal{G}], \mathcal{E}^{\{\nu\}}, (\mathcal{Q}^{(n,k)})^{\{\nu\}}, \mathcal{Y}^{\{\nu\}}\Big) \\ = \mathcal{F}\Big((\mathcal{L}^{(n,k)})^{\{\nu\}}, (\mathcal{S}^{(n,k)})^{\{\nu\}}, [\mathcal{G}], \mathcal{E}^{\{\nu\}}, (\mathcal{Q}^{(n,k)})^{\{\nu-1\}}, \mathcal{Y}^{\{\nu-1\}}\Big) + \frac{\mu^{\{\nu\}} - \mu^{\{\nu-1\}}}{2}\Big\{\|\Re([\mathcal{G}]) + \mathcal{E}^{\{\nu\}} - \mathcal{M}\|_F^2 \\ + \sum_{n,k=1}^{3}\Big\|\nabla_k\big((\mathcal{G}^{(n)})^{\{\nu\}}\big) - (\mathcal{L}^{(n,k)})^{\{\nu\}} - (\mathcal{S}^{(n,k)})^{\{\nu\}}\Big\|_F^2\Big\} + \sum_{k,n=1}^{3}\langle(\mathcal{Q}^{(n,k)})^{\{\nu\}} - (\mathcal{Q}^{(n,k)})^{\{\nu-1\}}, \\ \nabla_k\big((\mathcal{G}^{(n)})^{\{\nu\}}\big) - (\mathcal{L}^{(n,k)})^{\{\nu\}} - (\mathcal{S}^{(n,k)})^{\{\nu\}}\rangle + \langle \mathcal{Y}^{\{\nu\}} - \mathcal{Y}^{\{\nu-1\}}, \mathcal{M} - \Re([\mathcal{G}]) - \mathcal{E}^{\{\nu\}}\rangle \\ = \mathcal{F}\Big((\mathcal{L}^{(n,k)})^{\{\nu\}}, (\mathcal{S}^{(n,k)})^{\{\nu\}}, [\mathcal{G}], \mathcal{E}^{\{\nu\}}, (\mathcal{Q}^{(n,k)})^{\{\nu-1\}}, \mathcal{Y}^{\{\nu-1\}}\Big) + \frac{\mu^{\{\nu\}} - \mu^{\{\nu-1\}}}{2(\mu^{\{\nu-1\}})^2}\Big\{\|\mathcal{Y}^{\{\nu\}} - \mathcal{Y}^{\{\nu-1\}}\|_F^2 \\ + \sum_{n,k=1}^{3}\|(\mathcal{Q}^{(n,k)})^{\{\nu\}} - (\mathcal{Q}^{(n,k)})^{\{\nu-1\}}\|_F^2\Big\} + \frac{1}{\mu^{\{\nu-1\}}}\Big\{\sum_{n,k=1}^{3}\|(\mathcal{Q}^{(n,k)})^{\{\nu\}} - (\mathcal{Q}^{(n,k)})^{\{\nu-1\}}\|_F^2 + \|\mathcal{Y}^{\{\nu\}} - \mathcal{Y}^{\{\nu-1\}}\|_F^2\Big\} \\ = \mathcal{F}\Big((\mathcal{L}^{(n,k)})^{\{\nu\}}, (\mathcal{S}^{(n,k)})^{\{\nu\}}, [\mathcal{G}], \mathcal{E}^{\{\nu\}}, (\mathcal{Q}^{(n,k)})^{\{\nu-1\}}, \mathcal{Y}^{\{\nu-1\}}\Big) + \frac{\mu^{\{\nu\}} + \mu^{\{\nu-1\}}}{2(\mu^{\{\nu-1\}})^2}\Big\{\|\mathcal{Y}^{\{\nu\}} - \mathcal{Y}^{\{\nu-1\}}\|_F^2 \\ + \sum_{k,n=1}^{3}\|(\mathcal{Q}^{(n,k)})^{\{\nu\}} - (\mathcal{Q}^{(n,k)})^{\{\nu-1\}}\|_F^2\Big\}. \tag{7}$$

Combined the equation (7) and

$$
\begin{aligned}
(\mathcal{G}^{(n)})^{\{\nu+1\}} &= \arg\min_{\mathcal{G}^{(n)}} \Big\{\mathcal{F}\Big((\mathcal{L}^{(n,k)})^{\{\nu\}}, (\mathcal{S}^{(n,k)})^{\{\nu\}}, [\mathcal{G}], \mathcal{E}^{\{\nu\}}, (\mathcal{Q}^{(n,k)})^{\{\nu\}}, \mathcal{Y}^{\{\nu\}}\Big)\Big\},\\
(\mathcal{S}^{(n,k)})^{\{\nu+1\}} &= \arg\min_{(\mathcal{S}^{(n,k)})} \Big\{\mathcal{F}\Big((\mathcal{L}^{(n,k)})^{\{\nu\}}, (\mathcal{S}^{(n,k)}), [\mathcal{G}], \mathcal{E}^{\{\nu\}}, (\mathcal{Q}^{(n,k)})^{\{\nu\}}, \mathcal{Y}^{\{\nu\}}\Big)\Big\},\\
(\mathcal{L}^{(n,k)})^{\{\nu+1\}} &= \arg\min_{(\mathcal{L}^{(n,k)})} \Big\{\mathcal{F}\Big((\mathcal{L}^{(n,k)}), (\mathcal{S}^{(n,k)})^{\{\nu+1\}}, [\mathcal{G}], \mathcal{E}^{\{\nu\}}, (\mathcal{Q}^{(n,k)})^{\{\nu\}}, \mathcal{Y}^{\{\nu\}}\Big)\Big\},\\
\mathcal{E}^{\{\nu+1\}} &= \arg\min_{\mathcal{E}} \Big\{\mathcal{F}\Big((\mathcal{L}^{(n,k)})^{\{\nu+1\}}, (\mathcal{S}^{(n,k)})^{\{\nu+1\}}, [\mathcal{G}], \mathcal{E}^{\{\nu\}}, (\mathcal{Q}^{(n,k)})^{\{\nu\}}, \mathcal{Y}^{\{\nu\}}\Big)\Big\},
\end{aligned}
$$

we have

$$
\begin{aligned}
&\mathcal{F}\Big((\mathcal{L}^{(n,k)})^{\{\nu+1\}}, (\mathcal{S}^{(n,k)})^{\{\nu+1\}}, [\mathcal{G}], \mathcal{E}^{\{\nu+1\}}, (\mathcal{Q}^{(n,k)})^{\{\nu\}}, \mathcal{Y}^{\{\nu\}}\Big)\\
&\leq \mathcal{F}\Big((\mathcal{L}^{(n,k)})^{\{\nu\}}, (\mathcal{S}^{(n,k)})^{\{\nu\}}, [\mathcal{G}], \mathcal{E}^{\{\nu\}}, (\mathcal{Q}^{(n,k)})^{\{\nu\}}, \mathcal{Y}^{\{\nu\}}\Big)\\
&= \mathcal{F}\Big((\mathcal{L}^{(n,k)})^{\{\nu\}}, (\mathcal{S}^{(n,k)})^{\{\nu\}}, [\mathcal{G}], \mathcal{E}^{\{\nu\}}, (\mathcal{Q}^{(n,k)})^{\{\nu-1\}}, \mathcal{Y}^{\{\nu-1\}}\Big) + \frac{\mu^{\{\nu\}}+\mu^{\{\nu-1\}}}{2(\mu^{\{\nu-1\}})^2}\Big\{\|\mathcal{Y}^{\{\nu\}}-\mathcal{Y}^{\{\nu-1\}}\|_F^2+\\
&\sum_{k,n=1}^{3}\|(\mathcal{Q}^{(n,k)})^{\{\nu\}}-(\mathcal{Q}^{(n,k)})^{\{\nu-1\}}\|_F^2\Big\}.
\end{aligned} \tag{8}
$$

Iterating (8) ν times, we can obtain

$$
\begin{aligned}
&\mathcal{F}\Big((\mathcal{L}^{(n,k)})^{\{\nu+1\}}, (\mathcal{S}^{(n,k)})^{\{\nu+1\}}, [\mathcal{G}], \mathcal{E}^{\{\nu+1\}}, (\mathcal{Q}^{(n,k)})^{\{\nu\}}, \mathcal{Y}^{\{\nu\}}\Big)\\
&\leq \mathcal{F}\Big((\mathcal{L}^{(n,k)})^{\{1\}}, (\mathcal{S}^{(n,k)})^{\{1\}}, [\mathcal{G}], \mathcal{E}^{\{1\}}, (\mathcal{Q}^{(n,k)})^{\{0\}}, \mathcal{Y}^{\{0\}}\Big) + \sum_{i=1}^{\nu}\frac{\mu^{\{i\}}+\mu^{\{i-1\}}}{2(\mu^{\{i-1\}})^2}\Big\{\|\mathcal{Y}^{\{i\}}-\mathcal{Y}^{\{i-1\}}\|_F^2+\\
&\sum_{k,n=1}^{3}\|(\mathcal{Q}^{(n,k)})^{\{i\}}-(\mathcal{Q}^{(n,k)})^{\{i-1\}}\|_F^2\Big\}\\
&\leq \mathcal{F}\Big((\mathcal{L}^{(n,k)})^{\{0\}}, (\mathcal{S}^{(n,k)})^{\{0\}}, [\mathcal{G}], \mathcal{E}^{\{0\}}, (\mathcal{Q}^{(n,k)})^{\{0\}}, \mathcal{Y}^{\{0\}}\Big) + \sum_{i=1}^{\nu}\frac{\mu^{\{i\}}+\mu^{\{i-1\}}}{2(\mu^{\{i-1\}})^2}\Big\{\|\mathcal{Y}^{\{i\}}-\mathcal{Y}^{\{i-1\}}\|_F^2+\\
&\sum_{k,n=1}^{3}\|(\mathcal{Q}^{(n,k)})^{\{i\}}-(\mathcal{Q}^{(n,k)})^{\{i-1\}}\|_F^2\Big\}\\
&= \frac{\mu^{\{0\}}}{2}\|\mathcal{M}\|_F^2 + \sum_{i=1}^{\nu}\frac{\mu^{\{i\}}+\mu^{\{i-1\}}}{2(\mu^{\{i-1\}})^2}\Big\{\|\mathcal{Y}^{\{i\}}-\mathcal{Y}^{\{i-1\}}\|_F^2+\sum_{k,n=1}^{3}\|(\mathcal{Q}^{(n,k)})^{\{i\}}-(\mathcal{Q}^{(n,k)})^{\{i-1\}}\|_F^2\Big\}\\
&\leq \frac{\mu^{\{0\}}}{2}\|\mathcal{M}\|_F^2 + \sum_{i=1}^{\nu}\frac{\mu^{\{i\}}+\mu^{\{i-1\}}}{2(\mu^{\{i-1\}})^2}\Big\{\max_i\|\mathcal{Y}^{\{i\}}-\mathcal{Y}^{\{i-1\}}\|_F^2+\max_i\sum_{k,n=1}^{3}\|(\mathcal{Q}^{(n,k)})^{\{i\}}-(\mathcal{Q}^{(n,k)})^{\{i-1\}}\|_F^2\Big\}
\end{aligned} \tag{9}
$$

Since $\{\mathcal{Y}^{\{\nu\}}\}$, $\{(\mathcal{Q}^{(n,k)})^{\{\nu\}}, n,k\in\{1,2,3\}$ are bounded, it follows that $\max_i\|\mathcal{Y}^{\{i\}}-\mathcal{Y}^{\{i-1\}}\|_F^2$, $\max_i\sum_{n,k=1}^{3}\|(\mathcal{Q}^{(n,k)})^{\{i\}}-(\mathcal{Q}^{(n,k)})^{\{i-1\}}\|_F^2$ are also bounded. Observed that $\mu^{\{i\}}=\vartheta\mu^{\{i-1\}}=\vartheta^i\mu^{\{0\}}, \mu^{\{0\}}=10^{-3}$, we have

$$
\sum_{i=1}^{\infty}\frac{\mu^{\{i\}}+\mu^{\{i-1\}}}{2(\mu^{\{i-1\}})^2}=\frac{\vartheta+1}{2\mu^{\{0\}}}\sum_{i=1}^{\infty}\frac{1}{\vartheta^{i-1}}=\frac{\vartheta(\vartheta+1)}{2\mu^{\{0\}}(\vartheta-1)}
$$

is bounded, and thus $\mathcal{F}\Big((\mathcal{L}^{(n,k)})^{\{\nu+1\}}, (\mathcal{S}^{(n,k)})^{\{\nu+1\}}, [\mathcal{G}], \mathcal{E}^{\{\nu+1\}}, (\mathcal{Q}^{(n,k)})^{\{\nu\}}, \mathcal{Y}^{\{\nu\}}\Big)$ has upper bound.

On the other hand, we have

$$\begin{aligned}
&\mathcal{F}\Big((\mathcal{L}^{(n,k)})^{\{\nu+1\}},(\mathcal{S}^{(n,k)})^{\{\nu+1\}},[\mathcal{G}],\mathcal{E}^{\{\nu+1\}},(\mathcal{Q}^{(n,k)})^{\{\nu\}},\mathcal{Y}^{\{\nu\}}\Big)+\frac{1}{2\mu^{\{\nu\}}}\|\mathcal{Y}^{\{\nu\}}\|_F^2+\sum_{k,n=1}^{3}\frac{1}{2\mu^{\{\nu\}}}\|(\mathcal{Q}^{(n,k)})^{\{\nu\}}\|_F^2\\
&=\sum_{n=1}^{3}\sum_{k=1}^{3}\Big\{\frac{1}{\gamma}\Big\|(\mathcal{L}^{(n,k)})^{\{\nu+1\}}\Big\|_{\Phi,\mathcal{L}}+\alpha\cdot\Big\|(\mathcal{S}^{(n,k)})^{\{\nu+1\}}\Big\|_{\ell_1^{\psi}}+\\
&\frac{\mu^{\{\nu\}}}{2}\Big\|\nabla_k\big((\mathcal{G}^{(n)})^{\{\nu+1\}}\big)-(\mathcal{L}^{(n,k)})^{\{\nu+1\}}-(\mathcal{S}^{(n,k)})^{\{\nu+1\}}+\frac{(\mathcal{Q}^{(n,k)})^{\{\nu\}}}{\mu^{\{\nu\}}}\Big\|_F^2\Big\}\\
&+\beta\cdot\|\mathcal{E}^{\{\nu+1\}}\|_{\ell_{F,1}^{\psi}}+\frac{\mu^{\{\nu\}}}{2}\|\mathcal{M}-\Re([\mathcal{G}])-\mathcal{E}^{\{\nu+1\}}+\mathcal{Y}^{\{\nu\}}/\mu^{\{\nu\}}\|_F^2.
\end{aligned}\tag{10}$$

Note that the nonconvex functions $\Phi(\cdot)$ and $\psi(\cdot)$ are monotonically increasing on $[0,\infty)$ with $\Phi(0)=\psi(0)=0$. Thus, each term on the right side of the above equation (10) is nonnegative. Since $\mathcal{F}\Big((\mathcal{L}^{(n,k)})^{\{\nu+1\}},(\mathcal{S}^{(n,k)})^{\{\nu+1\}},[\mathcal{G}],\mathcal{E}^{\{\nu+1\}},(\mathcal{Q}^{(n,k)})^{\{\nu\}},\mathcal{Y}^{\{\nu\}}\Big)$, $\{\mathcal{Y}^{\{\nu\}}\}$ and $(\mathcal{Q}^{(n,k)})^{\{\nu\}}$ are bounded, thus we conclude that the sequences $\{(\mathcal{L}^{(n,k)})^{\{\nu\}}\}$, $\{(\mathcal{S}^{(n,k)})^{\{\nu\}}\}$, $\{(\mathcal{G}^{(n)})^{\{\nu\}}\}$, and $\mathcal{E}^{\{\nu\}}$ are bounded. □

Theorem .1. *Suppose that the sequences $\{\mathcal{Y}^{\{\nu\}}\}$ and $\{\mathcal{Q}^{(n,k)\{\nu\}}\}$ $(n,k=1,2,3)$ generated by the proposed HAD Algorithm are bounded. Then, the sequences $\{(\mathcal{L}^{(n,k)})^{\{\nu\}}\}$, $\{(\mathcal{S}^{(n,k)})^{\{\nu\}}\}$, $\{(\mathcal{G}^{(n)})^{\{\nu\}}\}$, and $\mathcal{E}^{\{\nu\}}$ satisfy:*

1) $\lim_{\nu\to\infty}\|\mathcal{M}-\Re([\mathcal{G}])-\mathcal{E}^{\{\nu+1\}}\|_F=0$;
2) $\lim_{\nu\to\infty}\|\nabla_k(\mathcal{G}^{(n)})^{\{\nu+1\}}-(\mathcal{L}^{(n,k)})^{\{\nu+1\}}-(\mathcal{S}^{(n,k)})^{\{\nu+1\}}\|_F=0,\quad n,k=1,2,3$;
3) $\lim_{\nu\to\infty}\|(\mathcal{L}^{(n,k)})^{\{\nu+1\}}-(\mathcal{L}^{(n,k)})^{\{\nu\}}\|_F=\lim_{\nu\to\infty}\|\mathcal{E}^{\{\nu+1\}}-\mathcal{E}^{\{\nu\}}\|_F=0$;
4) $\lim_{\nu\to\infty}\|(\mathcal{S}^{(n,k)})^{\{\nu+1\}}-(\mathcal{S}^{(n,k)})^{\{\nu\}}\|_F=\lim_{\nu\to\infty}\|(\mathcal{G}^{(n)})^{\{\nu+1\}}-(\mathcal{G}^{(n)})^{\{\nu\}}\|_F=0;\quad n,k=1,2,3$.

Proof. 1) From the updating formula of $\{(\mathcal{Q}^{(n,k)})^{\{\nu+1\}}\}$ $(n,k=1,2,3)$, and $\mathcal{Y}^{\{\nu+1\}}$, we have

$$\begin{aligned}
&\lim_{\nu\to\infty}\|\mathcal{M}-\Re([\mathcal{G}])-\mathcal{E}^{\{\nu+1\}}\|_F=\lim_{\nu\to\infty}\frac{1}{\mu^{\{\nu\}}}\|\mathcal{Y}^{\{\nu+1\}}-\mathcal{Y}^{\{\nu\}}\|_F=0,\\
&\lim_{\nu\to\infty}\Big\|\nabla_k(\mathcal{G}^{(n)})^{\{\nu+1\}}-(\mathcal{L}^{(n,k)})^{\{\nu+1\}}-(\mathcal{S}^{(n,k)})^{\{\nu+1\}}\Big\|_F=\lim_{\nu\to\infty}\frac{1}{\mu^{\{\nu\}}}\Big\|\{(\mathcal{Q}^{(n,k)})^{\{\nu+1\}}\}-\{(\mathcal{Q}^{(n,k)})^{\{\nu\}}\}\Big\|_F=0.
\end{aligned}$$

2) For the sequence $\{(\mathcal{G}^{(n)})^{\{\nu+1\}}\}$, according to the update rule of $\{\mathcal{Y}^{\{\nu+1\}}\}$ and $\{\mathcal{Q}^{(n,k)\{\nu+1\}}\}$, we have

$$\Re([\mathcal{G}])=\mathcal{M}-\mathcal{E}^{\{\nu+1\}}+\frac{\mathcal{Y}^{\{\nu\}}-\mathcal{Y}^{\{\nu+1\}}}{\mu^{\{\nu\}}},\tag{11}$$

$$\nabla_k(\mathcal{G}^{(n)})^{\{\nu+1\}}=(\mathcal{L}^{(n,k)})^{\{\nu+1\}}+(\mathcal{S}^{(n,k)})^{\{\nu+1\}}+\frac{\{(\mathcal{Q}^{(n,k)})^{\{\nu+1\}}\}-\{(\mathcal{Q}^{(n,k)})^{\{\nu\}}\}}{\mu^{\{\nu\}}}.\tag{12}$$

Through the matrix representation of TR decomposition, Formulas (11) and (12) are equivalent to

$$(\boldsymbol{G}_{(2)}^{(n)})^{\{\nu+1\}}\boldsymbol{R}_1=\boldsymbol{M}_{<n>}-\boldsymbol{E}_{<n>}^{\{\nu+1\}}+\frac{\boldsymbol{Y}_{<n>}^{\{\nu\}}-\boldsymbol{Y}_{<n>}^{\{\nu+1\}}}{\mu^{\{\nu\}}},\tag{13}$$

$$\boldsymbol{D}_n(\boldsymbol{G}_{(2)}^{(n)})^{\{\nu+1\}}=(\boldsymbol{L}_{(2)}^{(n,2)})^{\{\nu+1\}}+(\boldsymbol{S}_{(2)}^{(n,2)})^{\{\nu+1\}}+\frac{(\boldsymbol{Q}_{(2)}^{(n,2)})^{\{\nu+1\}}-(\boldsymbol{Q}_{(2)}^{(n,2)})^{\{\nu\}}}{\mu^{\{\nu\}}},\tag{14}$$

where $\boldsymbol{R}_1=(\boldsymbol{G}_{<2>}^{(\neq n)})^T$.

For the $\{(\mathcal{G}^{(n)})^{\{\nu+1\}}\}$-subproblem, by the first-order optimal condition, we get

$$\begin{aligned}
&\boldsymbol{D}_n^T\boldsymbol{D}_n(\boldsymbol{G}_{(2)}^{(n)})^{\{\nu+1\}}+(\boldsymbol{G}_{(2)}^{(n)})^{\{\nu+1\}}\boldsymbol{R}_1{}^{\{\nu\}}(\boldsymbol{R}_1{}^{\{\nu\}})^T=\Big(\boldsymbol{M}_{<n>}-\boldsymbol{E}_{<n>}^{\{\nu\}}+\frac{\boldsymbol{Y}_{<n>}^{\{\nu\}}}{\mu^{\{\nu\}}}\Big)(\boldsymbol{R}_1{}^{\{\nu\}})^T+\\
&\boldsymbol{D}_n^T\Big((\boldsymbol{L}_{(2)}^{(n,2)})^{\{\nu\}}+(\boldsymbol{S}_{(2)}^{(n,2)})^{\{\nu\}}-(\boldsymbol{Q}_{(2)}^{(n,2)})^{\{\nu\}}/\mu^{\{\nu\}}\Big).
\end{aligned}\tag{15}$$

Furthermore, by the equation (15), we obtain

$$\begin{aligned}&\boldsymbol{D}_n^T\Big[\boldsymbol{D}_n(\boldsymbol{G}_{(2)}^{(n)})^{\{\nu+1\}}-(\boldsymbol{L}_{(2)}^{(n,2)})^{\{\nu\}}-(\boldsymbol{S}_{(2)}^{(n,2)})^{\{\nu\}}+(\boldsymbol{Q}_{(2)}^{(n,2)})^{\{\nu\}}/\mu^{\{\nu\}}\Big]\\&-\Big(\boldsymbol{M}_{<n>}-\boldsymbol{E}_{<n>}^{\{\nu\}}+\frac{\boldsymbol{Y}_{<n>}^{\{\nu\}}}{\mu^{\{\nu\}}}\Big)(\boldsymbol{R}_1)^T+(\boldsymbol{G}_{(2)}^{(n)})^{\{\nu+1\}}\boldsymbol{R}_1(\boldsymbol{R}_1)^T=0\end{aligned}\tag{16}$$

Therefore, combining the equations (13), (14) and (16), we obtain

$$\begin{aligned}&\Big\|\boldsymbol{D}_n^T\boldsymbol{D}_n\Big((\boldsymbol{G}_{(2)}^{(n)})^{\{\nu+1\}}-(\boldsymbol{G}_{(2)}^{(n)})^{\{\nu\}}\Big)+\Big((\boldsymbol{G}_{(2)}^{(n)})^{\{\nu+1\}}-(\boldsymbol{G}_{(2)}^{(n)})^{\{\nu\}}\Big)\boldsymbol{R}_1\boldsymbol{R}_1{}^T\Big\|_F\\
&=\Big\|\boldsymbol{D}_n^T\Big[\boldsymbol{D}_n(\boldsymbol{G}_{(2)}^{(n)})^{\{\nu+1\}}-\boldsymbol{D}_n(\boldsymbol{G}_{(2)}^{(n)})^{\{\nu\}}\Big]+(\boldsymbol{G}_{(2)}^{(n)})^{\{\nu+1\}}\boldsymbol{R}_1\boldsymbol{R}_1{}^T-(\boldsymbol{G}_{(2)}^{(n)})^{\{\nu\}}\boldsymbol{R}_1\boldsymbol{R}_1{}^T\Big\|_F\\
&=\Big\|\boldsymbol{D}_n^T\Big[\boldsymbol{D}_n(\boldsymbol{G}_{(2)}^{(n)})^{\{\nu+1\}}-(\boldsymbol{L}_{(2)}^{(n,2)})^{\{\nu\}}-(\boldsymbol{S}_{(2)}^{(n,2)})^{\{\nu\}}+(\boldsymbol{Q}_{(2)}^{(n,2)})^{\{\nu\}}/\mu^{\{\nu\}}-(\boldsymbol{Q}_{(2)}^{(n,2)})^{\{\nu\}}/\mu^{\{\nu\}}-\frac{(\boldsymbol{Q}_{(2)}^{(n,2)})^{\{\nu\}}-(\boldsymbol{Q}_{(2)}^{(n,2)})^{\{\nu-1\}}}{\mu^{\{\nu-1\}}}\Big]\\
&-\Big(\boldsymbol{M}_{<n>}-\boldsymbol{E}_{<n>}^{\{\nu\}}+\frac{\boldsymbol{Y}_{<n>}^{\{\nu\}}}{\mu^{\{\nu\}}}-(\boldsymbol{G}_{(2)}^{(n)})^{\{\nu+1\}}\boldsymbol{R}_1\Big)\cdot(\boldsymbol{R}_1)^T-\frac{\boldsymbol{Y}_{<n>}^{\{\nu\}}}{\mu^{\{\nu\}}}-\frac{\boldsymbol{Y}_{<n>}^{\{\nu-1\}}+\boldsymbol{Y}_{<n>}^{\{\nu\}}}{\mu^{\{\nu-1\}}}\Big\|_F\\
&\le\Big\|\boldsymbol{D}_n^T(\boldsymbol{Q}_{(2)}^{(n,2)})^{\{\nu\}}/\mu^{\{\nu\}}\Big\|_F+\Big\|\boldsymbol{D}_n^T\frac{(\boldsymbol{Q}_{(2)}^{(n,2)})^{\{\nu\}}-(\boldsymbol{Q}_{(2)}^{(n,2)})^{\{\nu-1\}}}{\mu^{\{\nu-1\}}}\Big\|_F+\Big\|\frac{\boldsymbol{Y}_{<n>}^{\{\nu\}}}{\mu^{\{\nu\}}}\Big\|_F+\Big\|\frac{\boldsymbol{Y}_{<n>}^{\{\nu-1\}}-\boldsymbol{Y}_{<n>}^{\{\nu\}}}{\mu^{\{\nu-1\}}}\Big\|_F.\end{aligned}\tag{17}$$

Note that the sequences $\{(\mathcal{Q}^{(n,k)})^{\{\nu+1\}}\}$ $(n,k=1,2,3)$, and $\mathcal{Y}^{\{\nu+1\}}$ are bounded. Thus, we have

$$\lim_{\nu\to\infty}\|(\mathcal{G}^{(n)})^{\{\nu+1\}}-(\mathcal{G}^{(n)})^{\{\nu\}}\|_F=\lim_{\nu\to\infty}\|(\boldsymbol{G}_{(2)}^{(n)})^{\{\nu+1\}}-(\boldsymbol{G}_{(2)}^{(n)})^{\{\nu\}}\|_F=0.$$

3) For the sequence $\{\mathcal{E}^{\{\nu+1\}}\}$, let $\mathcal{H}^{\{\nu\}}:=\mathcal{M}-\Re\Big(\big[(\mathcal{G}^{(n)})^{\{\nu+1\}}\big]_{n=1,2,3}\Big)+\mathcal{Y}^{\{\nu\}}\frac{1}{\mu^{\{\nu\}}}$, we have

$$\begin{aligned}\|\mathcal{E}^{\{\nu+1\}}-\mathcal{E}^{\{\nu\}}\|_F&\le\|\mathcal{E}^{\{\nu+1\}}-\mathcal{H}^{\{\nu\}}\|_F+\|\mathcal{H}^{\{\nu\}}-\mathcal{E}^{\{\nu\}}\|_F\\
&\le\|\mathcal{E}^{\{\nu+1\}}-\mathcal{H}^{\{\nu\}}\|_F+\Big\|\Re\Big(\big[(\mathcal{G}^{(n)})^{\{\nu+1\}}\big]_{n=1,2,3}\Big)-\Re\Big(\big[(\mathcal{G}^{(n)})^{\{\nu\}}\big]_{n=1,2,3}\Big)\Big\|_F\\
&+\Big\|\frac{\mathcal{Y}^{\{\nu\}}}{\mu^{\{\nu\}}}+\frac{\mathcal{Y}^{\{\nu\}}-\mathcal{Y}^{\{\nu-1\}}}{\mu^{\{\nu-1\}}}\Big\|_F.\end{aligned}$$

On the one hand, based on the previous proofs, we can proved that

$$\lim_{\nu\to\infty}\|\mathcal{E}^{\{\nu+1\}}-\mathcal{H}^{\{\nu\}}\|_F=0,\quad\lim_{\nu\to\infty}\|(\mathcal{G}^{(n)})^{\{\nu+1\}}-(\mathcal{G}^{(n)})^{\{\nu\}}\|_F=0.$$

On the other hand, since the boundedness of the sequence $\{\mathcal{Y}^{\{\nu\}}\}$, we have

$$\lim_{\nu\to\infty}\Big\|\frac{\mathcal{Y}^{\{\nu\}}}{\mu^{\{\nu\}}}+\frac{\mathcal{Y}^{\{\nu\}}-\mathcal{Y}^{\{\nu-1\}}}{\mu^{\{\nu-1\}}}\Big\|_F=0.$$

Thus, we obtain $\lim_{\nu\to\infty}\|\mathcal{E}^{\{\nu+1\}}-\mathcal{E}^{\{\nu\}}\|_F=0$.

4) For the sequence $(\mathcal{S}^{(n,k)})^{\{\nu+1\}}$, let $(\mathcal{T}^{(n,k)})^{\{\nu\}}=\nabla_k(\mathcal{G}^{(n)})^{\{\nu+1\}}-(\mathcal{L}^{(n,k)})^{\{\nu\}}+\frac{(\mathcal{Q}^{(n,k)})^{\{\nu\}}}{\mu^{\{\nu\}}}$, we have

$$\begin{aligned}\|(\mathcal{S}^{(n,k)})^{\{\nu+1\}}-(\mathcal{S}^{(n,k)})^{\{\nu\}}\|_F&\le\|(\mathcal{S}^{(n,k)})^{\{\nu+1\}}-(\mathcal{T}^{(n,k)})^{\{\nu\}}\|_F+\|(\mathcal{T}^{(n,k)})^{\{\nu\}}-(\mathcal{S}^{(n,k)})^{\{\nu\}}\|_F\\
&\le\|(\mathcal{S}^{(n,k)})^{\{\nu+1\}}-(\mathcal{T}^{(n,k)})^{\{\nu\}}\|_F+\Big\|\nabla_k(\mathcal{G}^{(n)})^{\{\nu+1\}}-\nabla_k(\mathcal{G}^{(n)})^{\{\nu\}}\Big\|_F+\\
&+\Big\|\frac{(\mathcal{Q}^{(n,k)})^{\{\nu\}}}{\mu^{\{\nu\}}}+\frac{(\mathcal{Q}^{(n,k)})^{\{\nu\}}-(\mathcal{Q}^{(n,k)})^{\{\nu-1\}}}{\mu^{\{\nu-1\}}}\Big\|_F.\end{aligned}$$

Since the boundedness of the sequence $\{(\mathcal{Q}^{(n,k)})^{\{\nu\}}\}$, we have

$$\lim_{\nu\to\infty}\Big\|\frac{(\mathcal{Q}^{(n,k)})^{\{\nu\}}}{\mu^{\{\nu\}}}+\frac{(\mathcal{Q}^{(n,k)})^{\{\nu\}}-(\mathcal{Q}^{(n,k)})^{\{\nu-1\}}}{\mu^{\{\nu-1\}}}\Big\|_F=0.$$

Now, we only need to prove that

$$\lim_{\nu\to\infty}\|(\mathcal{S}^{(n,k)})^{\{\nu+1\}}-(\mathcal{T}^{(n,k)})^{\{\nu\}}\|_F=0.\tag{18}$$

In virtue of the subproblems with respect to $(\mathcal{S}^{(n,k)})^{\{\nu+1\}}$, for $\forall i_1 \in \{1,2,\cdots,n_1\},\cdots,i_d \in \{1,2,\cdots,n_d\}$, we obtain

$$\begin{aligned} &0 \in \alpha\cdot\partial\psi\Big(\Big|\Big((\mathcal{S}^{(n,k)})^{\{\nu+1\}}\Big)_{i_1,i_2,\cdots,i_d}\Big|\Big) + \mu^{\{\nu\}}\cdot\Big\{\Big((\mathcal{S}^{(n,k)})^{\{\nu+1\}}\Big)_{i_1,i_2,\cdots,i_d} - \Big((\mathcal{T}^{(n,k)})^{\{\nu+1\}}\Big)_{i_1,i_2,\cdots,i_d}\Big\} \\ &\Rightarrow \mu^{\{\nu\}}\cdot\Big\{\Big((\mathcal{S}^{(n,k)})^{\{\nu+1\}}\Big)_{i_1,i_2,\cdots,i_d} - \Big((\mathcal{T}^{(n,k)})^{\{\nu+1\}}\Big)_{i_1,i_2,\cdots,i_d}\Big\} \in -\alpha\cdot\partial\psi\Big(\Big|\Big((\mathcal{S}^{(n,k)})^{\{\nu+1\}}\Big)_{i_1,i_2,\cdots,i_d}\Big|\Big). \end{aligned} \tag{19}$$

Furthermore, since $\partial\psi\Big(\Big|\Big((\mathcal{S}^{(n,k)})^{\{\nu+1\}}\Big)_{i_1,i_2,\cdots,i_d}\Big|\Big)$ is bounded, we have

$$\begin{aligned} \lim_{\nu\to\infty}\big\|(\mathcal{S}^{(n,k)})^{\{\nu+1\}} - (\mathcal{T}^{(n,k)})^{\{\nu\}}\big\|_F &= \lim_{\nu\to\infty}\sqrt{\sum_{i_1=1}^{n_1}\cdots\sum_{i_d=1}^{n_d}\Big\{\Big((\mathcal{S}^{(n,k)})^{\{\nu+1\}}\Big)_{i_1,i_2,\cdots,i_d} - \Big((\mathcal{T}^{(n,k)})^{\{\nu+1\}}\Big)_{i_1,i_2,\cdots,i_d}\Big\}^2} \\ &\le \lim_{\nu\to\infty}\sqrt{\Big(-\frac{\alpha}{\mu^{\{\nu\}}}\cdot(n_1\cdots n_d)\cdot\partial\psi\Big(\Big|\Big((\mathcal{S}^{(n,k)})^{\{\nu+1\}}\Big)_{i_1,i_2,\cdots,i_d}\Big|\Big)\Big)^2} = 0. \end{aligned}$$

Therefore, we have

$$\lim_{\nu\to\infty}\|(\mathcal{S}^{(n,k)})^{\{\nu+1\}} - (\mathcal{S}^{(n,k)})^{\{\nu\}}\|_F = 0.$$

5) For the sequence $(\mathcal{L}^{(n,k)})^{\{\nu+1\}}$, let $(\mathcal{P}^{(n,k)})^{\{\nu\}} = \nabla_k(\mathcal{G}^{(n)})^{\{\nu+1\}} - (\mathcal{S}^{(n,k)})^{\{\nu+1\}} + \frac{(\mathcal{Q}^{(n,k)})^{\{\nu\}}}{\mu^{\{\nu\}}}$, we have

$$\begin{aligned} \big\|(\mathcal{L}^{(n,k)})^{\{\nu+1\}} - (\mathcal{L}^{(n,k)})^{\{\nu\}}\big\|_F &\le \big\|(\mathcal{L}^{(n,k)})^{\{\nu+1\}} - (\mathcal{P}^{(n,k)})^{\{\nu\}}\big\|_F + \big\|(\mathcal{P}^{(n,k)})^{\{\nu\}} - (\mathcal{L}^{(n,k)})^{\{\nu\}}\big\|_F \\ &\le \big\|(\mathcal{L}^{(n,k)})^{\{\nu+1\}} - (\mathcal{P}^{(n,k)})^{\{\nu\}}\big\|_F + \Big\|\nabla_k(\mathcal{G}^{(n)})^{\{\nu+1\}} - \nabla_k(\mathcal{G}^{(n)})^{\{\nu\}}\Big\|_F + \\ &\big\|(\mathcal{S}^{(n,k)})^{\{\nu+1\}} - (\mathcal{S}^{(n,k)})^{\{\nu\}}\big\|_F + \Big\|\frac{(\mathcal{Q}^{(n,k)})^{\{\nu\}}}{\mu^{\{\nu\}}} + \frac{(\mathcal{Q}^{(n,k)})^{\{\nu\}} - (\mathcal{Q}^{(n,k)})^{\{\nu-1\}}}{\mu^{\{\nu-1\}}}\Big\|_F. \end{aligned}$$

Since the boundedness of the sequence $\{(\mathcal{Q}^{(n,k)})^{\{\nu\}}\}$, we have

$$\lim_{\nu\to\infty}\Big\|\frac{(\mathcal{Q}^{(n,k)})^{\{\nu\}}}{\mu^{\{\nu\}}} + \frac{(\mathcal{Q}^{(n,k)})^{\{\nu\}} - (\mathcal{Q}^{(n,k)})^{\{\nu-1\}}}{\mu^{\{\nu-1\}}}\Big\|_F = 0.$$

Now, we only need to prove that $\lim_{\nu\to\infty}\big\|(\mathcal{L}^{(n,k)})^{\{\nu+1\}} - (\mathcal{P}^{(n,k)})^{\{\nu\}}\big\|_F = 0$.

Let $(\mathcal{P}^{(n,k)})^{\{\nu\}} = (\mathcal{U}^{(n,k)})^{\{\nu\}} *_{\mathfrak{L}} (\mathcal{S}^{(n,k)})^{\{\nu\}} *_{\mathfrak{L}} ((\mathcal{V}^{(n,k)})^{\{\nu\}})^T$ be the T-SVD components of $(\mathcal{P}^{(n,k)})^{\{\nu\}}$, then we have

$$(\mathcal{L}^{(n,k)})^{\{\nu+1\}} = (\mathcal{U}^{(n,k)})^{\{\nu\}} *_{\mathfrak{L}} (\hat{\mathcal{S}}^{(n,k)})^{\{\nu\}} *_{\mathfrak{L}} ((\mathcal{V}^{(n,k)})^{\{\nu\}})^T,$$

where

$$\Big((\hat{\mathcal{S}}^{(n,k)})^{\{\nu\}}\Big)^{<j>} = \mathrm{diag}(\tilde{\sigma}_{1j}^{(n,k,\nu)}, \tilde{\sigma}_{2j}^{(n,k,\nu)}, \cdots, \tilde{\sigma}_{mj}^{(n,k,\nu)}),$$

in which $\tilde{\sigma}_{ij}^{(n,k,\nu)} = \mathit{Prox}_{\Phi,\frac{1}{\mu^{\{\nu\}}}\cdot\frac{1}{\gamma}}\Big(\sigma_{ij}^{(n,k,\nu)}\Big)$, $\sigma_{ij}^{(n,k,\nu)} = \mathfrak{L}\Big((\mathcal{S}^{(n,k)})^{\{\nu\}}\Big)^{<j>}\Big(i,i\Big)$. Therefore, we have

$$\begin{aligned} &\|(\mathcal{L}^{(n,k)})^{\{\nu+1\}} - (\mathcal{P}^{(n,k)})^{\{\nu\}}\|_F \\ &= \Big\|(\mathcal{U}^{(n,k)})^{\{\nu\}} *_{\mathfrak{L}}\Big\{(\mathcal{S}^{(n,k)})^{\{\nu\}} - (\hat{\mathcal{S}}^{(n,k)})^{\{\nu\}}\Big\} *_{\mathfrak{L}}\big((\mathcal{V}^{(n,k)})^{\{\nu\}}\big)^T\Big\|_F \\ &= \frac{1}{\sqrt{\rho}}\sqrt{\sum_{(i,j)}\Big(\tilde{\sigma}_{ij}^{(n,k,\nu)} - \sigma_{ij}^{(n,k,\nu)}\Big)^2}, \end{aligned} \tag{20}$$

where $\rho > 0$ is a constant determined by transform $\mathfrak{L}$. Besides, $\tilde{\sigma}_{ij}^{(n,k,\nu)}$ and $\sigma_{ij}^{(n,k,\nu)}$ have the following relationship:

$$\tilde{\sigma}_{ij}^{(n,k,\nu)} = \arg\min_{\sigma>0}\frac{1}{\gamma}\Phi(\sigma) + \frac{\mu^{\{\nu\}}}{2}\big(\sigma - \sigma_{ij}^{(n,k,\nu)}\big)^2.$$

By the first order necessary optimality condition, we have

$$\frac{1}{\gamma}\partial\Phi(\tilde{\sigma}_{ij}^{(n,k,\nu)}) + \mu^{\{\nu\}}\big(\tilde{\sigma}_{ij}^{(n,k,\nu)} - \sigma_{ij}^{(n,k,\nu)}\big) = 0. \tag{21}$$

By substituting (21) into (20), we have

$$\begin{aligned}&\left\|(\mathcal{L}^{(n,k)})^{\{\nu+1\}}-(\mathcal{P}^{(n,k)})^{\{\nu\}}\right\|_F^2\\&\leq\frac{1}{\rho}\cdot\frac{1}{(\mu^{\{\nu\}})^2\cdot\gamma^2}\cdot\sharp\{(i,j)\}\cdot\max_{(i,j)}\Big(\partial\Phi\big(\tilde{\sigma}_{ij}^{(n,k,\nu)}\big)\Big)^2.\end{aligned}\tag{22}$$

Since the boundedness of $\partial\Phi\big(\tilde{\sigma}_{ij}^{(n,k,\nu)}\big)$, we can deduce that $\lim_{\nu\to\infty}\big\|(\mathcal{L}^{(n,k)})^{\{\nu+1\}}-(\mathcal{P}^{(n,k)})^{\{\nu\}}\big\|_F=0$. □

Theorem .2. *Let* $\{\mathcal{Y}^{\{\nu\}}\}$, $\{\mathcal{Q}^{(n,k)^{\{\nu\}}}\}$ *(*$n,k=1,2,3$*),* $\{(\mathcal{L}^{(n,k)})^{\{\nu\}}\}$, $\{(\mathcal{S}^{(n,k)})^{\{\nu\}}\}$, $\{(\mathcal{G}^{(n)})^{\{\nu\}}\}$, *and* $\mathcal{E}^{\{\nu\}}$ *be the sequences generated by the proposed nonconvex HAD Algorithm. Suppose that the sequences* $\{\mathcal{Y}^{\{\nu\}}\}$ *and* $\{\mathcal{Q}^{(n,k)^{\{\nu\}}}\}$ *(*$n,k=1,2,3$*) are bounded. Then, any accumulation point of the sequence* $\Big\{\{\mathcal{Y}^{\{\nu\}}\}$, $\{\mathcal{Q}^{(n,k)^{\{\nu\}}}\}$ *(*$n,k=1,2,3$*),* $\{(\mathcal{L}^{(n,k)})^{\{\nu\}}\}$, $\{(\mathcal{S}^{(n,k)})^{\{\nu\}}\}$, $\{(\mathcal{G}^{(n)})^{\{\nu\}}\}$, $\mathcal{E}^{\{\nu\}}\Big\}$ *is a Karush-Kuhn-Tucker (KKT) point of the original optimization problem.*

Proof. If the sequences $\{\mathcal{Y}^{\{\nu\}}\}_{\nu=1}^{\infty}$ and $\{\mathcal{Q}^{(n,k)^{\{\nu\}}}\}_{\nu=1}^{\infty}$ are bounded, we then conclude that the sequence $\Big\{(\mathcal{L}^{(n,k)})^{\{\nu\}}$, $(\mathcal{S}^{(n,k)})^{\{\nu\}}$, $(\mathcal{G}^{(n)})^{\{\nu\}}$, $\mathcal{E}^{\{\nu\}}$, $\mathcal{Y}^{\{\nu\}}$, $(\mathcal{Q}^{(n,k)})^{\{\nu\}}\Big\}_{\nu=1}^{\infty}$ is bound in virtue of Lemma .2-.4. According to the Bolzano-Weierstrass theorem (i.e., Lemma .1), the proposed nonconvex HAD algorithm has at least one accumulation point defined as $\Big\{(\mathcal{L}^{(n,k)})^{\star}$, $(\mathcal{S}^{(n,k)})^{\star}$, $(\mathcal{G}^{(n)})^{\star}$, $\mathcal{E}^{\star}$, $\mathcal{Y}^{\star}$, $(\mathcal{Q}^{(n,k)})^{\star}\Big\}$, and there exists one sequence that converges to this accumulation point. Without loss of generality, we assume that $\Big\{(\mathcal{L}^{(n,k)})^{\{\nu\}}$, $(\mathcal{S}^{(n,k)})^{\{\nu\}}$, $(\mathcal{G}^{(n)})^{\{\nu\}}$, $\mathcal{E}^{\{\nu\}}$, $\mathcal{Y}^{\{\nu\}}$, $(\mathcal{Q}^{(n,k)})^{\{\nu\}}\Big\}_{\nu=1}^{\infty}$ converges to $\Big\{(\mathcal{L}^{(n,k)})^{\star}$, $(\mathcal{S}^{(n,k)})^{\star}$, $(\mathcal{G}^{(n)})^{\star}$, $\mathcal{E}^{\star}$, $\mathcal{Y}^{\star}$, $(\mathcal{Q}^{(n,k)})^{\star}\Big\}$.

1) From the updating formulas of $\{\mathcal{Y}^{\{\nu\}}\}$ and $\{\mathcal{Q}^{(n,k)^{\{\nu\}}}\}$ ($n,k=1,2,3$), we have

$$\begin{aligned}&\lim_{\nu\to\infty}\|\mathcal{M}-\Re([\mathcal{G}])-\mathcal{E}^{\{\nu+1\}}\|_F=\lim_{\nu\to\infty}\frac{1}{\mu^{\{\nu\}}}\|\mathcal{Y}^{\{\nu+1\}}-\mathcal{Y}^{\{\nu\}}\|_F=0,\\&\lim_{\nu\to\infty}\|\nabla_k(\mathcal{G}^{(n)})^{\{\nu+1\}}-(\mathcal{L}^{(n,k)})^{\{\nu+1\}}-(\mathcal{S}^{(n,k)})^{\{\nu+1\}}\|_F=\lim_{\nu\to\infty}\frac{1}{\mu^{\{\nu\}}}\|(\mathcal{Q}^{(n,k)})^{\{\nu+1\}}-(\mathcal{Q}^{(n,k)})^{\{\nu\}}\|_F=0.\end{aligned}$$

Thus, $\lim_{\nu\to\infty}\Big(\nabla_k(\mathcal{G}^{(n)})^{\{\nu\}}-(\mathcal{L}^{(n,k)})^{\{\nu\}}-(\mathcal{S}^{(n,k)})^{\{\nu\}}\Big)=\lim_{\nu\to\infty}(\mathcal{M}-\Re([\mathcal{G}])-\mathcal{E}^{\{\nu\}})=0$. Therefore, we have

$$\mathcal{M}=\Re\Big(\{(\mathcal{G}^{(n)})^{\star}\}\Big)+\mathcal{E}^{\star},\quad\nabla_k(\mathcal{G}^{(n)})^{\star}=(\mathcal{L}^{(n,k)})^{\star}+(\mathcal{S}^{(n,k)})^{\star},\quad n,k=1,2,3.\tag{23}$$

2) For the $(\mathcal{G}^{(n)})^{\{\nu+1\}}$-subproblem, by the first-order optimal condition, we get

$$\begin{aligned}&\boldsymbol{D}_n^T\boldsymbol{D}_n(\boldsymbol{G}_{(2)}^{(n)})^{\{\nu+1\}}+(\boldsymbol{G}_{(2)}^{(n)})^{\{\nu+1\}}\boldsymbol{R}_1(\boldsymbol{R}_1)^T=\Big(\boldsymbol{M}_{<n>}-\boldsymbol{E}_{<n>}^{\{\nu\}}+\frac{\boldsymbol{Y}_{<n>}^{\{\nu\}}}{\mu^{\{\nu\}}}\Big)(\boldsymbol{R}_1)^T+\\&\boldsymbol{D}_n^T\Big((\boldsymbol{L}_{(2)}^{(n,2)})^{\{\nu\}}+(\boldsymbol{S}_{(2)}^{(n,2)})^{\{\nu\}}-(\boldsymbol{Q}_{(2)}^{(n,2)})^{\{\nu\}}/\mu^{\{\nu\}}\Big).\end{aligned}$$

According to the Theorem .1, we have

$$\lim_{\nu\to\infty}\mu^{\{\nu\}}\Big((\mathcal{L}^{(n,k)})^{\{\nu+1\}}-(\mathcal{L}^{(n,k)})^{\{\nu\}}\Big)=\lim_{\nu\to\infty}\mu^{\{\nu\}}\Big((\mathcal{S}^{(n,k)})^{\{\nu+1\}}-(\mathcal{S}^{(n,k)})^{\{\nu\}}\Big)=\lim_{\nu\to\infty}\mu^{\{\nu\}}(\mathcal{E}^{\{\nu+1\}}-\mathcal{E}^{\{\nu\}})=0.$$

Letting $\nu\to\infty$, we further obtain

$$\begin{aligned}&\lim_{\nu\to\infty}\mu^{\{\nu\}}\boldsymbol{D}_n^T\Big\{\boldsymbol{D}_n(\boldsymbol{G}_{(2)}^{(n)})^{\{\nu+1\}}-(\boldsymbol{L}_{(2)}^{(n,2)})^{\{\nu\}}-(\boldsymbol{S}_{(2)}^{(n,2)})^{\{\nu\}}\Big\}\\&=\lim_{\nu\to\infty}\mu^{\{\nu\}}\boldsymbol{D}_n^T\Big\{\boldsymbol{D}_n(\boldsymbol{G}_{(2)}^{(n)})^{\{\nu+1\}}-(\boldsymbol{L}_{(2)}^{(n,2)})^{\{\nu+1\}}-(\boldsymbol{S}_{(2)}^{(n,2)})^{\{\nu+1\}}+\Big((\boldsymbol{L}_{(2)}^{(n,2)})^{\{\nu+1\}}-(\boldsymbol{L}_{(2)}^{(n,2)})^{\{\nu\}}\Big)+\\&\Big((\boldsymbol{S}_{(2)}^{(n,2)})^{\{\nu+\}}-(\boldsymbol{S}_{(2)}^{(n,2)})^{\{\nu\}}\Big)\Big\}=0,\end{aligned}$$

and

$$\lim_{\nu\to\infty}\mu^{\{\nu\}}\left((\boldsymbol{G}_{(2)}^{(n)})^{\{\nu+1\}}\boldsymbol{R}_1+\boldsymbol{E}_{<n>}^{\{\nu\}}-\boldsymbol{M}_{<n>}\right)=\lim_{\nu\to\infty}\mu^{\{\nu\}}\left[\left((\boldsymbol{G}_{(2)}^{(n)})^{\{\nu+1\}}\boldsymbol{R}_1+\boldsymbol{E}_{<n>}^{\{\nu+1\}}-\boldsymbol{M}_{<n>}\right)+(\boldsymbol{E}_{<n>}^{\{\nu\}}-\boldsymbol{E}_{<n>}^{\{\nu+1\}})\right]=0.$$

Therefore, we have

$$\boldsymbol{D}_n^T\cdot(\boldsymbol{Q}_{(2)}^{(n,2)})^{\star}=\boldsymbol{Y}_{<n>}^{\star}\cdot(\boldsymbol{R}_1)^T,\ \ n=1,2,3. \tag{24}$$

3) For the $(\mathcal{L}^{(n,k)})^{\{\nu+1\}}$-subproblem, by the first-order optimal condition, we get

$$\begin{aligned}0\in\ &\frac{1}{\gamma}\frac{\partial}{\partial\mathcal{L}^{(n,k)}}\bigg|_{\mathcal{L}^{(n,k)}=\left\{\mathcal{L}^{(n,k)}\right\}^{\{\nu+1\}}}\|\mathcal{L}^{(n,k)}\|_{\Phi,\mathfrak{L}}-(\mathcal{Q}^{(n,k)})^{\{\nu\}}-\mu^{\{\nu\}}\left(\nabla_k(\mathcal{G}^{(n)})^{\{\nu+1\}}-(\mathcal{L}^{(n,k)})^{\{\nu+1\}}-(\mathcal{S}^{(n,k)})^{\{\nu+1\}}\right)\\ =\ &\frac{1}{\gamma}\frac{\partial}{\partial\mathcal{L}^{(n,k)}}\bigg|_{\mathcal{L}^{(n,k)}=\left\{\mathcal{L}^{(n,k)}\right\}^{\{\nu+1\}}}\|\mathcal{L}^{(n,k)}\|_{\Phi,\mathfrak{L}}-(\mathcal{Q}^{(n,k)})^{\{\nu+1\}}.\end{aligned}$$

Letting $\nu\to\infty$, we have

$$0\in\frac{1}{\gamma}\frac{\partial}{\partial\mathcal{L}^{(n,k)}}\bigg|_{\mathcal{L}^{(n,k)}=\left(\mathcal{L}^{(n,k)}\right)^{\star}}\|\mathcal{L}^{(n,k)}\|_{\Phi,\mathfrak{L}}-(\mathcal{Q}^{(n,k)})^{\star},\ \ n,k=1,2,3. \tag{25}$$

4) For the $(\mathcal{S}^{(n,k)})^{\{\nu+1\}}$-subproblem, by the first-order optimal condition, we get

$$0\in\alpha\frac{\partial}{\partial\mathcal{S}^{(n,k)}}\bigg|_{\mathcal{S}^{(n,k)}=\left\{\mathcal{S}^{(n,k)}\right\}^{\{\nu+1\}}}\|\mathcal{S}^{(n,k)}\|_{\ell_1^{\psi}}-(\mathcal{Q}^{(n,k)})^{\{\nu\}}-\mu^{\{\nu\}}\left(\nabla_k(\mathcal{G}^{(n)})^{\{\nu+1\}}-(\mathcal{L}^{(n,k)})^{\{\nu\}}-(\mathcal{S}^{(n,k)})^{\{\nu+1\}}\right).$$

Letting $\nu\to\infty$, we have

$$0\in\alpha\frac{\partial}{\partial\mathcal{S}^{(n,k)}}\bigg|_{\mathcal{S}^{(n,k)}=\left\{\mathcal{S}^{(n,k)}\right\}^{\star}}\|\mathcal{S}^{(n,k)}\|_{\ell_1^{\psi}}-(\mathcal{Q}^{(n,k)})^{\star},\ \ n,k=1,2,3. \tag{26}$$

5) For the $\mathcal{E}^{\{\nu+1\}}$-subproblem, by the first-order optimal condition, we get

$$\begin{aligned}0\in\ &\beta\frac{\partial}{\partial\mathcal{E}}\bigg|_{\mathcal{E}=\mathcal{E}^{\{\nu+1\}}}\|\mathcal{E}\|_{\ell_{F,1}^{\psi}}-\mathcal{Y}^{\{\nu\}}+\mu^{\{\nu\}}\left(\Re([\mathcal{G}])+\mathcal{E}^{\{\nu+1\}}-\mathcal{M}\right),\\ =\ &\beta\frac{\partial}{\partial\mathcal{E}}\bigg|_{\mathcal{E}=\mathcal{E}^{\{\nu+1\}}}\|\mathcal{E}\|_{\ell_{F,1}^{\psi}}-\mathcal{Y}^{\{\nu+1\}}.\end{aligned}$$

Letting $\nu\to\infty$, we have

$$0\in\beta\frac{\partial}{\partial\mathcal{E}}\bigg|_{\mathcal{E}=\mathcal{E}^{\star}}\|\mathcal{E}\|_{\ell_{F,1}^{\psi}}-\mathcal{Y}^{\star}. \tag{27}$$

By Formulas (23), (24), (25), (26), and (27), we have that $\left\{(\mathcal{L}^{(n,k)})^{\star},\ (\mathcal{S}^{(n,k)})^{\star},\ (\mathcal{G}^{(n)})^{\star},\ \mathcal{E}^{\star},\ \mathcal{Y}^{\star},\ (\mathcal{Q}^{(n,k)})^{\star}\right\}$ is a KKT point of original optimization problem. □

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