Recall the following properties of relations $R$ from a set $A$ to a set $B$:

- **Function**: For all $a \in A$, $a$ has out-degree $= 1$; i.e.,

$$\forall a \in A.(\exists b \in B.\ aRb) \wedge (\forall b_1, b_2 \in B.\ aRb_1 \wedge aRb_2 \Rightarrow b_1 = b_2).$$

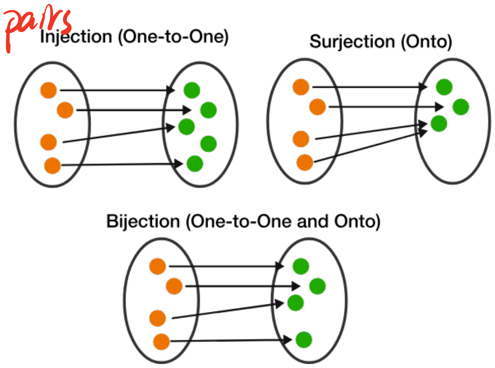
$\forall a \in A$, can find unique $b \in B$ s.t. $aRb$

- **Injective**: For all $b \in B$, $b$ has in-degree $\leq 1$; i.e., $\forall b \in B. \forall a_1, a_2 \in A.\ a_1Rb \wedge a_2Rb \Rightarrow a_1 = a_2$. If $R$ is a function, this is equivalently expressed as $\forall a_1, a_2 \in A.\ R(a_1) = R(a_2) \Rightarrow a_1 = a_2$.
- **Surjective**: For all $b \in B$, $b$ has in-degree $\geq 1$; i.e., $\forall b \in B. \exists a \in A.\ aRb$. If $R$ is a function, this is equivalently expressed as $\forall b \in B. \exists a \in A.\ R(a) = b$.
- **Bijective**: For all $b \in B$, $b$ has in-degree $= 1$; i.e., $R$ is surjective and injective.

Recall the properties of relations $R$ on a set $A$ (i.e., relations from $A$ to $A$):

↳ a set of ordered pairs

- **Reflexive**: $\forall a \in A.\, aRa$. $\Leftrightarrow$ $(a,a) \in R$
- **Irreflexive**: $\forall a \in A.\, \neg(aRa)$.
- **Transitive**: $\forall a, b, c \in A.\, aRb \wedge bRc \Rightarrow aRc$.
- **Symmetric**: $\forall a, b \in A.\, aRb \Rightarrow bRa$.
- **Asymmetric**: $\forall a, b \in A.\, aRb \Rightarrow \neg(bRa)$.
- **Antisymmetric**: $\forall a, b \in A.\, aRb \wedge bRa \Rightarrow a = b$.

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1. For any sets $A$, $B$, $C$, and $D$, what can be said about set $L = (A \cup B) \times (C \cup D)$ and $R = (A \times C) \cup (B \times D)$? Are they equal, is exactly one a subset of the other, etc.?

Sym & Asym : Cannot co-exist (except the trivial $\phi$)

Sym & Antisym: Can co-exist ( $\{(a,a), (b,b), \ldots\}$ )

Asym & Antisym : Asym $\Rightarrow$ Antisym
(stronger) (weaker)

2. Provide functions $f : \mathbb{Z} \to \mathbb{Z}^+$ with the following properties. Prove the correctness of your functions.

   (a) $f$ is **neither** surjective **nor** injective.

   (b) $f$ is surjective and **not** injective.

   (c) $f$ is surjective and injective.

   (d) $f$ is injective and **not** surjective.

3. For each of the following relations, determine whether the relations are 1) reflexive, 2) irreflexive, 3) transitive, 4) symmetric, 5) antisymmetric, and 6) asymmetric. For parts d and e, list the elements of the relation.

   (a) $\varnothing$ on any non-empty set $A$.

   (b) $A \times A$ on any non-empty set $A$.

   (c) $\{(a,a),(a,b),(b,b),(b,c),(c,c)\}$ on set $\{a,b,c,d\}$.

   (d) $<$ on set $\{1,2,3,4\}$.

   (e) $\leq$ on set $\{1,2,3,4\}$.

   (f) $\subseteq$ on set $2^A$ for any non-empty set $A$.

1. For any sets $A$, $B$, $C$, and $D$, what can be said about set $L = (A \cup B) \times (C \cup D)$ and $R = (A \times C) \cup (B \times D)$? Are they equal, is exactly one a subset of the other, etc.?

Guess and verify: $A = \{a\}$, $B = \{b\}$, $C = \{c\}$, $D = \{d\}$

$\Rightarrow L = \{a, b\} \times \{c, d\} = \{(a,c), (a,d), (b,c), (b,d)\}$

$R = \{(a,c)\} \cup \{(b,d)\} = \{(a,c), (b,d)\}$

$\Rightarrow R \subseteq L$ but $L \nsubseteq R$

Pf of $R \subseteq L$:

Consider $\forall (x,y) \in R = (A \times C) \cup (B \times D)$

By def of "$\cup$", $(x,y) \in A \times C$ or $(x,y) \in B \times D$

$\Rightarrow x \in A, y \in C$ or $x \in B, y \in D$

$\Rightarrow x \in A \cup B, \; y \in C \cup D$

$\Rightarrow (x,y) \in (A \cup B) \times (C \cup D) = L$

$\Rightarrow R \subseteq L$

Pf of $L \nsubseteq R$:

Use the counter-example above. $\blacksquare$

3. For each of the following relations, determine whether the relations are 1) reflexive, 2) irreflexive, 3) transitive, 4) symmetric, 5) antisymmetric, and 6) asymmetric. For parts d and e, list the elements of the relation.

   (a) $\varnothing$ on any non-empty set $A$.
   (b) $A \times A$ on any non-empty set $A$.
   (c) $\{(a,a),(a,b),(b,b),(b,c),(c,c)\}$ on set $\{a,b,c,d\}$.
   (d) $<$ on set $\{1,2,3,4\}$.
   (e) $\leq$ on set $\{1,2,3,4\}$.
   (f) $\subseteq$ on set $2^A$ for any non-empty set $A$.

(a) X 1) Reflexive: $\forall a \in A,\ aRa$

Since $R=\phi$. $\forall a \in A$. $(a,a) \notin A$ / $\neg(aRa)$

✓ 2) Irreflexive: $\forall a \in A,\ \neg(aRa)$

By 1).

✓ 3) Transitive: $\forall a,b,c \in A.\ \underbrace{\underbrace{aRb \wedge bRc}_{F} \Rightarrow aRc}_{T}$

$R=\phi$

✓ 4) Symmetric: $\forall a,b \in A,\ \underbrace{\underbrace{aRb}_{F} \Rightarrow bRa}_{T}$

✓ 5) Antisymmetric: $\forall a,b \in A.\ \underbrace{\underbrace{aRb \wedge bRa}_{F} \Rightarrow a=b}_{T}$

✓ 6) Asymmetric: $\forall a,b \in A\ \underbrace{\underbrace{aRb}_{F} \Rightarrow \neg(bRa)}_{T}$

3. For each of the following relations, determine whether the relations are 1) reflexive, 2) irreflexive, 3) transitive, 4) symmetric, 5) antisymmetric, and 6) asymmetric. For parts d and e, list the elements of the relation.

   (a) $\varnothing$ on any non-empty set $A$.
   (b) $A \times A$ on any non-empty set $A$. ⇒ all possible ordered pairs ⇒ universal relation on $A$ ⇒ $\forall a,b \in A,\ aRb$.
   (c) $\{(a,a),(a,b),(b,b),(b,c),(c,c)\}$ on set $\{a,b,c,d\}$.
   (d) $<$ on set $\{1,2,3,4\}$.
   (e) $\leq$ on set $\{1,2,3,4\}$.
   (f) $\subseteq$ on set $2^A$ for any non-empty set $A$.

(b) 1) ✓ Reflexive: $\forall a \in A,\ aRa$

By universal relation

2) ✗ Irreflexive: $\forall a \in A,\ \neg(aRa)$

By 1)

3) ✓ Transitive: $\forall a,b,c \in A.\ aRb \wedge bRc \Rightarrow \underbrace{aRc}_{T}$

$\forall a,c \in A.\ aRc$ T

4) ✓ Symmetric: $\forall a,b \in A,\ \underbrace{\underset{T}{aRb} \Rightarrow \underset{T}{bRa}}_{T}$

5) ✗ Antisymmetric: $\forall a,b \in A.\ aRb \wedge bRa \Rightarrow a=b$

Counter-example: $A = \{1,2\}$

$A \times A = \{(1,1), \exists\ \underline{(1,2),(2,1)}, (2,2)\}$ $1 \neq 2$

6) ✓ Asymmetric: $\forall a,b \in A.\ aRb \Rightarrow \neg(bRa)$

By 5). Asym ⇒ Antisym

Contrapositive: not Antisym ⇒ not Asym

3. For each of the following relations, determine whether the relations are 1) reflexive, 2) irreflexive, 3) transitive, 4) symmetric, 5) antisymmetric, and 6) asymmetric. For parts d and e, list the elements of the relation.

   (a) $\varnothing$ on any non-empty set $A$.

   (b) $A \times A$ on any non-empty set $A$.

   (c) $\{(a,a),(a,b),(b,b),(b,c),(c,c)\}$ on set $\{a,b,c,d\}$.

   (d) $<$ on set $\{1,2,3,4\}$.

   (e) $\leq$ on set $\{1,2,3,4\}$.

   (f) $\subseteq$ on set $2^A$ for any non-empty set $A$.

(c) × 1) Reflexive: $\forall a \in A,\ aRa$

$\exists d \in A,\ (d,d) \notin R \ /\ \neg(dRd)$

× 2) Irreflexive: $\forall a \in A,\ \neg(aRa)$

$\exists a,b,c \in A,\ (a,a),(b,b),(c,c) \in R \ /\ aRa,\ bRb,\ cRc$

× 3) Transitive: $\forall a,b,c \in A,\ \underbrace{aRb \wedge bRc}_{T} \Rightarrow \underbrace{aRc}_{F}$ } F

$\exists (a,b),(b,c) \in R$ but $(a,c) \notin R$

× 4) Symmetric: $\forall a,b \in A,\ \underbrace{aRb}_{T} \Rightarrow \underbrace{bRa}_{F}$ } F

$\exists (a,b) \in R$ but $(b,a) \notin R$

✓ 5) Antisymmetric: $\forall a,b \in A,\ aRb \wedge bRa \Rightarrow a=b$

only $(a,a)$ $(b,b)$ $(c,c)$

× 6) Asymmetric: $\forall a,b \in A,\ \underbrace{aRb}_{T} \Rightarrow \underbrace{\neg(bRa)}_{F}$ } F

$\exists (a,a) \in R \Rightarrow aRa \Rightarrow aRa$

3. For each of the following relations, determine whether the relations are 1) reflexive, 2) irreflexive, 3) transitive, 4) symmetric, 5) antisymmetric, and 6) asymmetric. For parts d and e, list the elements of the relation.

(a) $\varnothing$ on any non-empty set $A$.

(b) $A \times A$ on any non-empty set $A$.

(c) $\{(a,a),(a,b),(b,b),(b,c),(c,c)\}$ on set $\{a,b,c,d\}$.

(d) $<$ on set $\{1,2,3,4\}$. $\{(1,2),(1,3),(1,4),(2,3),(2,4),(3,4)\}$

(e) $\leq$ on set $\{1,2,3,4\}$.

(f) $\subseteq$ on set $2^A$ for any non-empty set $A$.

(d) ✗ 1) Reflexive: $\forall a \in A,\ aRa$

we cannot have $a<a$.

✓ 2) Irreflexive: $\forall a \in A,\ \neg(aRa)$

By 1).

✓ 3) Transitive: $\forall a,b,c \in A.\ aRb \wedge bRc \Rightarrow aRc$

$a<b,\ b<c \Rightarrow a<c$

✗ 4) Symmetric: $\forall a,b \in A,\ a\underset{T}{R b} \Rightarrow bRa_{F}$ } F

$(a<b \not\Rightarrow b<a)$ $\exists\ a=1, b=2,\ 1<2$ $2 \nless 1$

✓ 5) Antisymmetric: $\forall a,b \in A.\ \underbrace{aRb \wedge bRa}_{a<b \text{ and } b<a,\ F} \Rightarrow a=b$ } T

✓ 6) Asymmetric: $\forall a,b \in A.\ aRb \Rightarrow \neg(bRa)$

$a<b \Rightarrow \neg(b<a)$

$\neg(b<a) \Leftrightarrow a \leq b$

3. For each of the following relations, determine whether the relations are 1) reflexive, 2) irreflexive, 3) transitive, 4) symmetric, 5) antisymmetric, and 6) asymmetric. For parts d and e, list the elements of the relation.

   (a) $\varnothing$ on any non-empty set $A$.

   (b) $A \times A$ on any non-empty set $A$.

   (c) $\{(a,a),(a,b),(b,b),(b,c),(c,c)\}$ on set $\{a,b,c,d\}$.

   (d) $<$ on set $\{1,2,3,4\}$.

   (e) $\leq$ on set $\{1,2,3,4\}$. same but add self-loops as (d)

   (f) $\subseteq$ on set $2^A$ for any non-empty set $A$.

(e) 1) Reflexive: $\forall a \in A,\ aRa$ ✓

$a \leq a$

2) Irreflexive: $\forall a \in A,\ \neg(aRa)$ ✗

By 1)

3) Transitive: $\forall a,b,c \in A.\ aRb \wedge bRc \Rightarrow aRc$ ✓

$a \leq b,\ b \leq c \Rightarrow a \leq c$

4) Symmetric: $\forall a,b \in A,\ aRb \Rightarrow bRa$ ✗

$\left.\begin{array}{l} \underset{T}{a \leq b} \Rightarrow b \leq a \quad F \\ \exists a=1, b=3 \quad 1 \leq 3 \quad 3 \nleq 1 \end{array}\right\} F$

5) Antisymmetric: $\forall a,b \in A.\ aRb \wedge bRa \Rightarrow a=b$ ✓

$a \leq b.\ b \leq a \Rightarrow a=b$

6) Asymmetric: $\forall a,b \in A\ aRb \Rightarrow \neg(bRa)$ ✗ $\}F$

$\underset{T}{a \leq b} \Rightarrow \neg(b \leq a)$

$\Updownarrow$

$a < b$ F

$\exists a=b=1,\quad 1 \leq 1.\quad 1 \nleq 1$

3. For each of the following relations, determine whether the relations are 1) reflexive, 2) irreflexive, 3) transitive, 4) symmetric, 5) antisymmetric, and 6) asymmetric. For parts d and e, list the elements of the relation.

   (a) $\varnothing$ on any non-empty set $A$.
   (b) $A \times A$ on any non-empty set $A$.
   (c) $\{(a,a),(a,b),(b,b),(b,c),(c,c)\}$ on set $\{a,b,c,d\}$.
   (d) $<$ on set $\{1,2,3,4\}$.
   (e) $\leq$ on set $\{1,2,3,4\}$.
   (f) $\subseteq$ on set $2^A$ for any non-empty set $A$. power set

(f) 1) Reflexive: $\forall a \in A,\ aRa$ ✓

$\forall$ set $S$: $S \subseteq S$

2) Irreflexive: $\forall a \in A,\ \neg(aRa)$ ✗

By 1)

3) Transitive: $\forall a,b,c \in A.\ aRb \wedge bRc \Rightarrow aRc$ ✓

$\forall S_1, S_2, S_3 \in 2^A,\ S_1 \subseteq S_2,\ S_2 \subseteq S_3 \Rightarrow S_1 \subseteq S_3$

4) Symmetric: $\forall a,b \in A,\ aRb \Rightarrow bRa$ ✗

$\forall S_1, S_2 \in 2^A,\ \underset{T}{S_1 \subseteq S_2} \Rightarrow \underset{F}{S_2 \subseteq S_1}$ $\Big\}$ F

$\exists S_1 \subset S_2\ (S_1 \neq S_2)$

5) Antisymmetric: $\forall a,b \in A.\ aRb \wedge bRa \Rightarrow a = b$ ✓

$\forall S_1, S_2 \in 2^A\ \ S_1 \subseteq S_2,\ S_2 \subseteq S_1 \Rightarrow S_1 = S_2$

6) Asymmetric: $\forall a,b \in A\ \ aRb \Rightarrow \neg(bRa)$ ✗

$\forall S_1, S_2 \in 2^A\ \ \underset{T}{S_1 \subseteq S_2} \Rightarrow \neg(S_2 \subseteq S_1)$ $\Big\}$ F

$\Updownarrow$

$S_1 \subset S_2$

F

Consider $S_1 = S_2 \in 2^A$