Review ***conditional probability***, where $\Pr[A|B]$ is the probability of event A ***given*** event B:

- $\Pr[A|B] = \Pr[A \cap B] / \Pr[B]$ and $\Pr[A \cap B] = \Pr[A|B] \Pr[B] = \Pr[B|A] \Pr[A]$
- (Bayes' Rule) $\Pr[A|B] = \Pr[B|A] \Pr[A] / \Pr[B]$
- (Law of Total Probability) $\Pr[A] = \Pr[A|B] \Pr[B] + \Pr[A|\overline{B}] \Pr[\overline{B}]$

A, B are ***independent*** if and only if (all three are equivalent):

- $\Pr[A|B] = \Pr[A]$
- $\Pr[B|A] = \Pr[B]$
- $\Pr[A \cap B] = \Pr[A] \cdot \Pr[B]$

The ***expected value*** $\mathrm{E}[X]$ of a ***discrete random variable*** X is defined as $\sum_{x \in X} x \cdot \Pr[X = x]$, where "$x \in X$" is (an abuse of notation) read as "for every x in the ***codomain*** of X." The ***linearity of expectation*** states that for any random variables $X_1, X_2, \ldots, X_n$, we have $\mathrm{E}[\sum_i X_i] = \sum_i \mathrm{E}[X_i]$.

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1. Jeff makes two independent flips of a fair coin.

   (a) The first flip came up heads. What is the probability that the second flip came up heads?

   (b) The second flip came up heads. What is the probability that the first flip came up tails?

   (c) One of the flips came up heads. What is the probability that the other flip came up heads?

2. Suppose we toss two fair dice. Let E_1 denote the event that the sum of the dice is six, E_2 denote the event that the sum of the dice is seven, and F denote the event that the first die equals four.

   (a) Is E_1 independent of F? Prove or disprove.

   (b) Is E_2 independent of F? Prove or disprove.

3. Prove or disprove: If events A, B, and C are such that $\Pr(A)\Pr(B)\Pr(C) = \Pr(A \cap B \cap C)$, then the events are pairwise independent.

4. Suppose there are two bowls of candy. The first bowl has 10 Skittles and 30 M&Ms, and the second bowl has 20 of each. Steve picks a bowl uniformly at random, then takes a random candy from that bowl, which turns out to be an M&M. What is the probability that Steve picked the first bowl?

5. Suppose we have a deck of 10 cards composed of the 2 through 6 in both spades and diamonds which is perfectly shuffled (any permutation is equally probable). Consider drawing a hand of four cards from the shuffled deck. What is the ***expected*** number of (distinct) pairs in the hand? Answer this question multiple ways using different random variables:

   (a) For each $i \in \{2, \ldots, 6\}$, let X_i be the random variable where $X_i = 1$ if the pair of i's is drawn and $X_i = 0$ otherwise. Then let X be the random variable that is the total number of pairs drawn; $X = X_2 + X_3 + X_4 + X_5 + X_6$. What is $\mathrm{E}[X]$? Show your work.

   (b) For any i, j with $1 \leq i < j \leq 4$, let Y_{ij} be the random variable such that $Y_{ij} = 1$ if the i-th card drawn and the j-th card drawn are a pair and $Y_{ij} = 0$ otherwise. Then let Y be the random variable that is the total number of pairs drawn;

   $$Y = \sum_{1 \leq i < j \leq 4} Y_{ij}.$$

   What is $\mathrm{E}[Y]$? Show your work.

1. Jeff makes two independent flips of a fair coin.

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(a) A: first flip (H), B: second flip (H)

Since A, B independent, $Pr(B|A) = P(B) = \frac{1}{2}$

(b) B: second flip (H), C: first flip (T)

Since B, C independent, $Pr(C|B) = P(C) = \frac{1}{2}$

(c) D: both flips (H), E: at least one flip (H)

$$\Rightarrow Pr(D|E) = \frac{Pr(D \cap E)}{Pr(E)}$$

HH, TH, HT, TT

$\underbrace{\frac{1}{4} \quad \frac{1}{4} \quad \frac{1}{4}}_{\frac{3}{4}} \quad \frac{1}{4}$

$$= \frac{\frac{1}{4}}{\frac{3}{4}} = \frac{1}{3}$$

3. Prove or disprove: If events A, B, and C are such that $\Pr(A)\Pr(B)\Pr(C) = \Pr(A \cap B \cap C)$, then the events are pairwise independent.

Counter-example:

① A: H, B: T. $C = A \cap B$

$\Pr(A)\Pr(B)\Pr(C) = \frac{1}{2} \cdot \frac{1}{2} \cdot 0 = 0$

$\Pr(A \cap B \cap C) = \Pr(C \cap C) = \Pr(C) = 0$

$\Rightarrow \Pr(A)\Pr(B)\Pr(C) = \Pr(A \cap B \cap C)$

But $\Pr(A) \cdot \Pr(B) = \frac{1}{4} \neq 0 = \Pr(A \cap B)$

② Roll 2 distinct dice

A: first 1 2 3

B: first 3 4 5

C: sum 9 $\left((3,6), (4,5), (5,4), (6,3)\right) \Rightarrow \frac{4}{36} = \frac{1}{9}$

$\Pr(A)\Pr(B)\Pr(C) = \frac{1}{2} \cdot \frac{1}{2} \cdot \frac{1}{9} = \frac{1}{36}$

$\Pr(A \cap B \cap C) = \Pr(3, 6) = \frac{1}{6} \times \frac{1}{6} = \frac{1}{36}$

But $\Pr(A) \cdot \Pr(B) = \frac{1}{2} \cdot \frac{1}{2} = \frac{1}{4} \neq \frac{1}{6} = P(A \cap B)$

5. Suppose we have a deck of 10 cards composed of the 2 through 6 in both spades and diamonds which is perfectly shuffled (any permutation is equally probable). Consider drawing a hand of four cards from the shuffled deck. What is the ***expected*** number of (distinct) pairs in the hand? Answer this question multiple ways using different random variables:

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(b) For any i, j with $1 \le i < j \le 4$, let Y_{ij} be the random variable such that $Y_{ij} = 1$ if the i-th card drawn and the j-th card drawn are a pair and $Y_{ij} = 0$ otherwise. Then let Y be the random variable that is the total number of pairs drawn;

$$Y = \sum_{1 \le i < j \le 4} Y_{ij}.$$

What is $\mathrm{E}[Y]$? Show your work.

(a) $X_i = 1$ if both copies (spade & diamond) of $i \in \{2,3,4,5,6\}$ appear in the 4 cards, $X_i = 0$ o/w.

$$\begin{aligned} E[X] &= E[X_2 + X_3 + \cdots + X_6] \\ &= E[X_2] + \cdots + E[X_6] \\ &= \sum_{i=2}^{6} E[X_i] \\ &= \sum_{i=2}^{6} \Pr(\text{both cards of } i \text{ appear}) \quad X_i = 1 \\ &= 5 \times \frac{\binom{8}{2}}{\binom{10}{4}} = 5 \times \frac{28}{210} = 5 \times \frac{2}{15} = \frac{2}{3} \end{aligned}$$

(b) $Y_{ij} = 1$ if i^{th}, j^{th} cards form a pair,

$Y_{ij} = 0$ o/w

$$E[Y] = E\left[\sum_{1\le i<j\le 4} Y_{ij}\right]$$

$$= \sum_{1\le i<j\le 4} E[Y_{ij}]$$

$$= \sum_{1\le i<j\le 4} \Pr(Y_{ij}=1)$$

$$= \binom{4}{2} \times \frac{1}{9}$$

$$= 6 \times \frac{1}{9}$$

$$= \frac{2}{3}$$

1 2
1 3
1 4
2 3
2 4
3 4