# The Chain Reaction of Endogenous Population Growth: A Theoretical Deduction

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## 1 Introduction

Malthus (1798) first linked the impact of economic factors to fertility and proposed the classical dynamic population economic growth model. Becker (1973) extended population and economic theory to the analysis of household behavior based on microeconomic perspectives. Their theoretical constructions were developed by later scholars to verify the mutual effects between population growth and other variables in economic growth.

Neoclassical models simply ignore some significant relationships between population and economic growth for assuming population growth is a stable exogenous factor. To fill this blank, this paper attempts to regard population growth rate as an endogenous variable and investigate its chain reaction in a traditional growth model. To be specific, a transmission mechanism through population is explored, where population growth rate depends on consumption per capita and people's decisions about fertility further affect their work effort. Population growth rate functions as an intermediate transmitter to keep the economic outcome in a dynamic balance.

## 2 Theoretical Framework

### 2.1 Introducing the growth model

We adopt the Cobb-Douglas production function. Normally, $L_t$ can denote either aggregate population or the total suppliable working hours due to their proportional relationship. But here with the hypothesis that fertility will affect working hours, we make a clear division:

$$Y_t = K_t^{\alpha}[A_t L_t]^{1-\alpha}$$

$$K_{t+1} = sY_t + (1-\delta)K_t$$

$$A_{t+1} = (1+g)A_t$$

$$N_{t+1} = (1+n)N_t$$

$$y_t = \frac{Y_t}{N_t}, k_t = \frac{K_t}{N_t}, l_t = \frac{L_t}{N_t}$$

where $L_t$ denotes the total effective labor hours and $N_t$ denotes population. Let $Y_t$= output, $K_t$ = capital stock, $A_t$ = technological progress, $s$ = the savings rate, $\delta$ = depreciation rate, $n$ = population growth rate. After the procedure to find the balanced growth path when $g_K = \frac{\Delta K_t}{K_t}$, $g_Y = \alpha g_K + (1-\alpha)(n+g)$, and $g_K = g_Y = n+g$, we obtain:

$$Y_t = (\frac{s}{n+g+\delta})^{\frac{\alpha}{1-\alpha}} A_t L_t$$

Divide both sides by the population $N_t$ and calculate the equilibrium on per capita level:

$$y_t = (\frac{s}{n+g+\delta})^{\frac{\alpha}{1-\alpha}} A_t l_t \tag{1}$$

### 2.2 Population growth rate as an influencing factor

First we consider the impact of fertility choice on labor hours. Following the model set up by Palivos (1995) and Yip and Zhang (1996), an agent faces the time allocation constraint:

$$l + \phi(n) = T$$

where $l$ is individual effective working hours, $\phi(n)$ is the time cost of raising children. This equation implies individual allocates his time between production ($l$) and child-rearing ($\phi(n)$) given the same time endowment (T). In general, $\phi(n) \in C^2$ is a strictly increasing function with $\phi'(n) > 0$, $\phi(0) = 0$, exhibiting a positive marginal time cost of child-rearing. But the sign of its second-order derivative is uncertain, illustrating that the marginal cost can be increasing, constant, or decreasing. Without loss of generality, we assume a linear cost function for child-rearing, i.e., $\phi(n) = \lambda n$, where $\lambda > 0$. Thus,

$$l = T - \lambda n$$

Then we plug this relationship into equation (1):

$$y_t = (\frac{s}{n+g+\delta})^{\frac{\alpha}{1-\alpha}} A_t(T - \lambda n) \tag{2}$$

Equation (2) reveals a negative relationship between $n$ and $y_t$, suggesting the rise of fertility will crowd out effective labor hours to a certain extent. But this response only possesses a temporary in-between effect.

### 2.3 Population growth rate as an affected factor

Next we consider the impact of consumption and output level on population growth. Nerlove and Rault (1997) interpreted the Solow-Swan model by treating $n$ as an endogenous factor, with $n(k) = n[(1-s)f(k)] = n[(1-s)y_t]$. To simplify, we take the following example as illustration:

$$\frac{N_{t+1}}{N_t} = 1 + n(y_t) = h(y_t) = \frac{(1-s)y_t}{c_m} \tag{3}$$

where population growth is equal to the ratio of actual consumption to some minimal, positive level of consumption per capita, $c_m$ (see Nerlove and Rault, 1997, p.1129 for the formula derivation). Combining equation (2) and (3) together, we obtain the result:

$$y_t = A_t[\frac{sc_m}{(1-s)y_t + c_m(g+\delta-1)}]^{\frac{\alpha}{1-\alpha}}(T + \lambda - \frac{\lambda(1-s)y_t}{c_m})$$

Moreover, by taking log we have:

$$\begin{aligned} ln(y_t) + \frac{\alpha}{1-\alpha} ln[(1-s)y_t + c_m(g+\delta-1)] - ln[(T+\lambda) - \frac{\lambda(1-s)y_t}{c_m}] \\ = ln(A_0) + gt + \frac{\alpha}{1-\alpha} ln(s) + \frac{\alpha}{1-\alpha} ln(c_m) \end{aligned}$$

which can be empirically estimated by OLS. Furthermore, even though this complex mathematical equation cannot be directly solved, the monotonicity shown in the graph demonstration (see Figure. I) guarantees there exists a locally stable unique nonzero steady state.

## 3 Conclusion and Limitations

The above framework derives one of the transmission chains through population in economic growth, assuming the effects in both transmitting directions are simultaneous. Compared to Neoclassical models, endogenous population growth rate interacts with other economic factors in a general dynamic equilibrium. As variables reinforced or canceled out by each other, a dynamic balanced growth path is eventually achieved.

For limitations, the functional form of $h(y_t)$ in equation (3) can be expanded, which may entail the possibility of multiple equilibria and instability of some equilibria. Except the time allocation constraint, a standard budget or resource constraint can be added in Section 2.2 to improve the adaptability of this model. Robust evidence of the final formulation needs to be provided by econometric examination. In future studies, more transmission routes are yet to be explored. Razen and Benzion's dynamic population model (1975) with utility function considering parent's progeny Altruism can be combined to optimize the model.

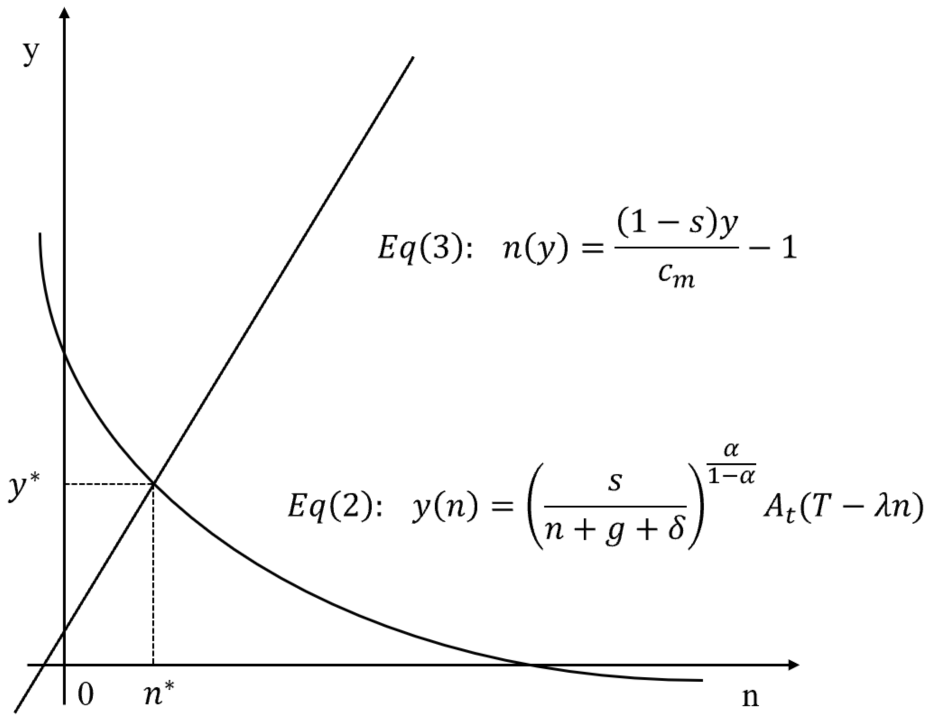


**Figure 1: The General Equilibrium in the Dynamic Population Model**

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