Recall a proof structure for **strong** induction: To prove a sequence of statements $P(m), P(m+1), P(m+2), \ldots$ for some integer m and predicate $P(\cdot)$:

- Prove $P(m)$.
- Prove that $[P(m) \wedge P(m-1) \wedge \ldots \wedge P(n-1)] \Rightarrow P(n)$ for all integers $n > m$. That is, for all integers $n > m$, prove $P(n)$ assuming $P(k)$ for all k with $m \leq k < n$, for all $n > m$.

Finally, conclude that, by weak induction, $P(m), P(m+1), P(m+2), \ldots$ are all true; that is, for all integers $n \geq m$, $P(n)$ is true.

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1. Finish the TODOs in the following proof by strong induction that every positive integer n can be written in binary; that is, n can be written as a sum of **distinct**, non-negative integer powers of two.

   Proof. We prove the claim by strong induction, as follows. Let $P(n)$ be the predicate "n can be written as a sum of **distinct**, non-negative integer powers of two." We wish to show $P(n)$ is true for all integers $n \geq 1$.

   Base case: *TODO:*

   Inductive hypothesis: Let n be an arbitrary integer with $n > 1$. Assume $P(k)$ is true for all k with $1 \leq k < n$.

   Inductive step: We wish to show $P(n)$ holds. We consider two cases for n: either n is even or n is odd.

   If n is even, then $n/2$ is an integer. *TODO: Justify that the induction hypothesis implies something useful about $n/2$, then finish this case.*

   Otherwise, n is odd. *TODO: Justify that the induction hypothesis implies something useful about $n/2$, then finish this case. For sake of time, identify what is different about the argument for this case than for the above.*

   Our case analysis is exhaustive, and in each case, $P(n)$ holds. By strong induction, we conclude that $P(n)$ holds for all $n \geq 1$.

2. We are given a rectangular chocolate bar with a rows and b columns that compose $a \times b$ squares of chocolate. Such a bar can be split into two smaller rectangular bars by either a horizontal or vertical break into a (positive integer) number of rows or columns, respectively. For example, a 2×3 bar can be broken by separating the leftmost column from the others, resulting in two smaller bars: a 2×1 bar and a 2×2 bar.

   Use strong induction to prove the following: *Any* process to break a chocolate bar into $a \times b$ squares uses $ab - 1$ breaks.

   To get started, we must formulate the given claim in the proper form for induction: $P(n)$ for all $n \geq m$ for some integer m and predicate $P(\cdot)$. How should the single predicate variable correspond to the dimensions of a bar? *[Hint: Consider the total number of squares of the bar.]*

3. A *continued fraction* is a number expressed as either an integer n or $n + 1/F$, where F is a continued fraction. (Note the recursive definition!) Prove that any non-negative rational number can be represented as a finite continued fraction. For example,

$$\frac{415}{93} = 4 + \cfrac{1}{2 + \cfrac{1}{6 + \cfrac{1}{7}}}.$$

   You may use the *division theorem*, which states: For any integer a and positive integer b, there exists non-negative integers q, r such that $a = qb + r$ and $r < b$. That is, q is the quotient and r is the remainder of a divided by b.

   [Hint: As in the previous problem, it is not immediately clear how to formulate this as a sequence of statements to be proven by strong induction. Consider the predicate $P(n)$ as "any rational number a/n for non-negative integer a can be written as a continued fraction.", then prove $P(n)$ for all integers $n \geq 1$.]

   To think about later: Suppose one formulates the claim with predicate $P(n)$ as "any rational number n/b for positive integer b can be written as a continued fraction.". Besides the base case changing from $n = 1$ to $n = 0$, where else are there issues in reusing the proof using the hinted predicate above?

Weak Induction

- $P(m)$
- $\forall n \geq m,\ P(n) \Rightarrow P(n+1)$

Strong Induction

- $P(m)$
- $\forall n \geq m,$

$$P(m) \wedge P(m+1) \wedge \cdots \wedge P(n) \Rightarrow P(n+1)$$

$$\Downarrow$$

$$\forall k,\ m \leq k \leq n : P(k)$$

Weak Induction

- $P(m)$
- $\forall n > m,\ P(n-1) \Rightarrow P(n)$

Strong Induction

- $P(m)$
- $\forall n > m$

$$P(m) \wedge P(m+1) \wedge \cdots \wedge P(n-1) \Rightarrow P(n)$$

$$\Downarrow$$

$$\forall k,\ m \leq k < n : P(k)$$

Remark: Strong induction does not need to really use all $P(m) \cdots P(n-1)$ to imply $P(n)$

3. A *continued fraction* is a number expressed as either an integer n or $n + 1/F$, where F is a continued fraction. (Note the recursive definition!) Prove that any non-negative rational number can be represented as a finite continued fraction. For example,

CF $\begin{cases} n \\ n + \frac{1}{F} \end{cases}$

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[Hint: As in the previous problem, it is not immediately clear how to formulate this as a sequence of statements to be proven by strong induction. Consider the predicate $P(n)$ as "any rational number a/n for non-negative integer a can be written as a continued fraction.", then prove $P(n)$ for all integers $n \geq 1$.]

Pf: Two way to construct $P(n)$ $\begin{cases} \frac{a}{n} \text{ for any non-negative } a. ✓ \\ \frac{n}{b} \text{ ?} \end{cases}$

Base case: $n=1$. By def, $\frac{a}{1} = a$ is an integer, thus a CF.

IH: Let n be an arbitrary integer with $n > 1$.
Assume $P(k)$ holds for $\forall k$ with $1 \leq k < n$.

$\frac{a}{1}, \frac{a}{2}, \dots, \frac{a}{n-1}$ are CF $\Rightarrow$ Goal: $P(n) \Rightarrow \frac{a}{n}$ is CF

IS: Consider $\frac{a}{n}$.

Case I: $\frac{a}{n}$ is an integer. By def, $\frac{a}{n}$ is a CF.

Case II: $\frac{a}{n}$ is not an integer.

By division thm, $\exists$ non-negative integer q, r s.t.
$a = qn + r$ with $r < n$

$$\Rightarrow \frac{a}{n} = \frac{qn+r}{n}$$

$$= q + \frac{r}{n}$$

$$= q + \frac{1}{\frac{n}{r}} \Rightarrow \text{CF}$$

$\Rightarrow$ By IH and def, $\frac{a}{n}$ is a CF

By strong induction, $P(n)$ holds for $\forall n \geq 1$.

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To think about later: Suppose one formulates the claim with predicate $P(n)$ as "any rational number n/b for positive integer b can be written as a continued fraction.". Besides the base case changing from $n = 1$ to $n = 0$, where else are there issues in reusing the proof using the hinted predicate above?

BC: $n = 0$. $\frac{n}{b} = \frac{0}{b} = 0 \in \mathbb{Z} \Rightarrow$ CF

IH: $P(k)$ holds for $0 \leq k < n$ (WTS: $P(n)$)

$\frac{0}{b}, \frac{1}{b}, \frac{2}{b}, \ldots, \frac{n-1}{b}$ are CF $\Rightarrow \frac{n}{b}$ is CF

IS: Consider $\frac{n}{b}$.

By division thm, $\exists q, r \in \mathbb{N}$ s.t. $b = qn + r$, $0 < r < n$

$$\Rightarrow \frac{n}{b} = \frac{n}{qn + r} \quad \text{(hard to proceed)}$$

$\Rightarrow$ What if $\exists q, r \in \mathbb{N}$ s.t. $n = qb + r$, $0 < r < b$

$$\Rightarrow \frac{n}{b} = \frac{qb + r}{b}$$

$$= q + \frac{r}{b}$$

$\Rightarrow$ Cannot use induction on n