1. This problem helps you review the basic concepts of utility representation and continuous preferences. Prove or disprove the following:

   (a) If $f : \mathbb{R} \to \mathbb{R}$ is a strictly increasing function and $u : X \to \mathbb{R}$ is a utility function representing the preference relation $\succsim$, then the function $v : X \to \mathbb{R}$ defined by $v(x) = f(u(x))$ is also a utility function representing $\succsim$.

True.

pf: Take $\forall x, y \in X$. Since $u(\cdot)$ represents $\succsim$,

by def, $x \succsim y \iff u(x) \geq u(y)$

$\iff f(u(x)) \geq f(u(y))$ since f is strictly increasing

$\iff v(x) \geq v(y)$

$\Rightarrow v$ represents $\succsim$ by def.

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Remark: A strictly increasing transformation of a utility function is still a utility function $\Rightarrow$ not change order, but only magnitude

   (b) If both u and v represent $\succsim$, then there is a strictly monotonic function $f : \mathbb{R} \to \mathbb{R}$ such that $v(x) = f(u(x))$.

False.

Counter-example: $v(x) = x$, $u(x) = \begin{cases} x, & \text{if } x \leq 0 \\ x+1, & \text{if } x > 0 \end{cases}$

$v(x) = f(u(x)) \Rightarrow f(x) = \begin{cases} x, & \text{if } x \leq 0 \\ x-1, & \text{if } x > 0 \end{cases}$

$\Rightarrow$ not monotone

(c) A continuous preference relation can be represented by a discontinuous utility function. [See HW2 Q4 as the same kind of problems.]

True.

See the example in (b)

By Representation thm, $\succsim$ is conti. since $\exists v(x)$ conti. and $v(x)$ represents $\succsim$.

But $u(x)$ disconti. and $u(x)$ represents $\succsim$.

(d) In the case of $X = \mathbb{R}$, the preference relation that is represented by the discontinuous function $u(x) = \lfloor x \rfloor$ (the largest integer n such that $x \geq n$) is not a continuous relation.

Def: $\succsim$ conti. $\iff$ $\forall x \in X$, both $U(x)$, $L(x)$ are closed

True.

pf: Take $x = 0$, $u(x) = 0$. Consider $L(x)$:

$$L(x) = \{y \in X : x \succsim y\} = \{y \in X : u(x) \geq u(y)\}$$
$$= \{y \in \mathbb{R} : 0 \geq \lfloor y \rfloor\} = (-\infty, 1)$$

which is not closed.

To see this, take $x_n = 1 - \frac{1}{n} \in (-\infty, 1)$. $x_n \to 1 \notin (-\infty, 1)$

$\Rightarrow$ $\succsim$ not conti.

(e) In the case of $X = \mathbb{N}$, any preference relation can be represented by a utility function that returns only integers as values.

False.

Counter-example:

Define the preference relation to be:

$1 \succ 3 \succ 5 \succ \cdots \succ 2 \succ 4 \succ 6 \succ \cdots$

(i.e., the agent strictly prefers odds in ascending order, followed by the evens in ascending order)

Suppose $\exists\, u: X \to \mathbb{Z}$ representing $\succsim$

$\Rightarrow \exists\, N, n \in \mathbb{N}$ s.t. $u(1) = N$, $u(2) = n$

For $\forall$ odd number d, $u(d) \in \{n+1, n+2, \cdots, N\}$

with # of elements $= N - n < \infty$

However, the set of d is countably infinite

$\Rightarrow u(\cdot)$ maps countably infinite numbers to a finite set $\times$

$\Rightarrow \nexists\, u(\cdot)$ for the defined preference relation

2. Consider the following UMP with the utility function in a three-good setting:

$$u(x_1, x_2, x_3) = (x_1 - \beta_1)^{\alpha_1}(x_2 - \beta_2)^{\alpha_2}(x_3 - \beta_3)^{\alpha_3} \quad \text{s.t. } p \cdot x \leq w,$$

where $\beta_i \geq 0$, $\alpha_i > 0$ for all i, $p \gg 0$, and $w > 0$.

(a) Explain why there is no loss of generality to assume that $\alpha_1 + \alpha_2 + \alpha_3 = 1$.

Suppose $\sum_{i=1}^{3} \alpha_i = k$. consider $\hat{u}(x) = u(x)^{\frac{1}{k}}$

$\Rightarrow \hat{u}(x) \propto u(x)$ since it is a monotone transformation

$\tilde{u}(x) = (x_1 - \beta_1)^{\hat{\alpha}_1}(x_2 - \beta_2)^{\hat{\alpha}_2}(x_3 - \beta_3)^{\hat{\alpha}_3}$

where $\hat{\alpha}_i = \frac{\alpha_i}{k}$

$\hat{\alpha}_1 + \hat{\alpha}_2 + \hat{\alpha}_3 = \sum_{i=1}^{3} \frac{\alpha_i}{k} = \frac{k}{k} = 1$

$\Rightarrow$ WLOG we may assume $\alpha_1 + \alpha_2 + \alpha_3 = 1$

(b) Write down the FOC for the UMP and derive the consumer's Walrasian demand and the indirect utility function.

$$\ln u(x) = \alpha_1 \ln(x_1 - \beta_1) + \alpha_2 \ln(x_2 - \beta_2) + \alpha_3 \ln(x_3 - \beta_3)$$

$$= \sum_{i=1}^{3} \alpha_i \ln(x_i - \beta_i) \quad \text{s.t. } \sum_{i=1}^{3} x_i p_i = w$$

$$\mathcal{L}(x) = \ln u(x) + \lambda \left(w - \sum_{i=1}^{3} x_i p_i\right)$$

FOC:

$$\begin{cases} \frac{\alpha_1}{x_1 - \beta_1} = \lambda p_1 \\ \frac{\alpha_2}{x_2 - \beta_2} = \lambda p_2 \\ \frac{\alpha_3}{x_3 - \beta_3} = \lambda p_3 \\ p_1 x_1 + p_2 x_2 + p_3 x_3 = w \end{cases}$$

$\Rightarrow$

$$\alpha_i = \lambda p_i (x_i - \beta_i)$$

$$\sum_{i=1}^{3} \alpha_i = \sum_{i=1}^{3} \lambda p_i (x_i - \beta_i)$$

$$1 = \lambda \left(w - \sum_{i=1}^{3} p_i \beta_i\right)$$

$$\lambda = \frac{1}{w - \sum_{i=1}^{3} p_i \beta_i}$$

$$\Rightarrow x_i - \beta_i = \frac{\alpha_i}{\lambda p_i}$$

$$x_i = \beta_i + \frac{\alpha_i (w - \sum_{i=1}^{3} p_i \beta_i)}{p_i}$$

$$\Rightarrow x(p, w) = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} \beta_1 + \frac{\alpha_1 (w - p \cdot \beta)}{p_1} \\ \beta_2 + \frac{\alpha_2 (w - p \cdot \beta)}{p_2} \\ \beta_3 + \frac{\alpha_3 (w - p \cdot \beta)}{p_3} \end{pmatrix}$$

where $p \cdot \beta = \sum_{i=1}^{3} p_i \beta_i$

$$\Rightarrow v(p, w) = u(x(p, w))$$
$$= \left(\frac{\alpha_1 (w - p \cdot \beta)}{p_1}\right)^{\alpha_1} \left(\frac{\alpha_2 (w - p \cdot \beta)}{p_2}\right)^{\alpha_2} \left(\frac{\alpha_3 (w - p \cdot \beta)}{p_3}\right)^{\alpha_3}$$

(c) Verify that the derived functions satisfy the following properties:

(1) Walrasian demand $x(p, w)$ is homogeneous of degree zero and satisfies Walras' Law;

Homogeneity of degree 0:

$$x(\lambda p, \lambda w) = \begin{pmatrix} \beta_1 + \frac{\alpha_1 \lambda (w - p \cdot \beta)}{\lambda p_1} \\ \beta_2 + \frac{\alpha_2 \lambda (w - p \cdot \beta)}{p_2} \\ \beta_3 + \frac{\alpha_3 \lambda (w - p \cdot \beta)}{\lambda p_3} \end{pmatrix} = x(p, w)$$

Walras' Law:

$$p \cdot x(p, w) = \sum_{i=1}^{3} \left(p_i \beta_i + p_i \frac{\alpha_i (w - p \beta)}{p_i} \right)$$

$$= p\cdot\beta + (w - p\cdot\beta)\cdot \underbrace{\sum_{i=1}^{3}\alpha_i}_{=1}$$

$$= p\cdot\beta + w - p\cdot\beta$$

$$= w$$

(2) Indirect utility $v(p,w)$ is homogeneous of degree zero;

$$v(\lambda p, \lambda w) = \left(\frac{\alpha_1\lambda(w-p\cdot\beta)}{\lambda p_1}\right)^{\alpha_1}\left(\frac{\alpha_2\lambda(w-p\cdot\beta)}{\lambda p_2}\right)^{\alpha_2}\left(\frac{\alpha_3\lambda(w-p\cdot\beta)}{\lambda p_3}\right)^{\alpha_3}$$

$$= v(p,w)$$

(3) $v(p,w)$ is strictly increasing in w and non-increasing in p_l for all l;

Strictly increasing in w:

$$v(p,w) = (w-p\cdot\beta)\left(\frac{\alpha_1}{p_1}\right)^{\alpha_1}\left(\frac{\alpha_2}{p_2}\right)^{\alpha_2}\left(\frac{\alpha_3}{p_3}\right)^{\alpha_3}$$

$$\frac{\partial v(p,w)}{\partial w} = \left(\frac{\alpha_1}{p_1}\right)^{\alpha_1}\left(\frac{\alpha_2}{p_2}\right)^{\alpha_2}\left(\frac{\alpha_3}{p_3}\right)^{\alpha_3} > 0$$

Non-increasing in p_l for $\forall l$:

$$\frac{\partial v(p,w)}{\partial p_1} = \left(\frac{\alpha_2}{p_2}\right)^{\alpha_2}\left(\frac{\alpha_3}{p_3}\right)^{\alpha_3}\left(-\beta_1\left(\frac{\alpha_1}{p_1}\right)^{\alpha_1} + (w-p\cdot\beta)\alpha_1\left(\frac{\alpha_1}{p_1}\right)^{\alpha_1-1}\left(-\frac{\alpha_1}{p_1^2}\right)\right)$$

$$= \left(\frac{\alpha_1}{p_1}\right)^{\alpha_1}\left(\frac{\alpha_2}{p_2}\right)^{\alpha_2}\left(\frac{\alpha_3}{p_3}\right)^{\alpha_3}\Big(\underbrace{-\beta_1 + (w-p\cdot\beta)\left(-\frac{\alpha_1}{p_1}\right)}_{=-x_1(p,w)}\Big) \leq 0$$

Similar for $\frac{\partial v(p,w)}{\partial p_2}$ and $\frac{\partial v(p,w)}{\partial p_3}$

(4) $v(p, w)$ is continuous in p and w.

The continuity follows directly from the functional form.

(d) Using different prices and incomes (i.e., p and p', w and w') and Walrasian demand to derive a condition (with some inequalities) that violates WARP.

$$\begin{cases} p \cdot x(p', w') \leq w \\ p' \cdot x(p, w) \leq w' \end{cases}$$

3. Assume that $x(p,w)$ is differentiable in $(p,w) \gg 0$ and that $x(p,w) \gg 0$.

(a) Show that whenever a consumer is maximizing a monotone utility function then, at any $(p,w) \gg 0$, there must exist some good k (which may depend on (p,w)) for which

$$\frac{\partial x_k(p,w)}{\partial w} > 0.$$

In other words, there must be some good i whose demand increases with income. [Hint: Walras' Law.]

pf: Suppose not, i.e., $\frac{\partial x_i(p,w)}{\partial w} \le 0$ for $\forall i$

Consider $w' = w + \varepsilon$ where $\varepsilon > 0$

Take $\forall x' \in X(p, w')$, $\forall x \in X(p, w)$

By $\frac{\partial x_i(p,w)}{\partial w} \le 0$ for $\forall i$, $x' \le x$

But $px' \le px = w < w+\varepsilon$, violating Walras' Law $\times$

$\Rightarrow \exists k$ s.t. $\frac{\partial x_k(p,w)}{\partial w} > 0$ #

Or:

Walras' Law gives us $\sum_{i=1}^{n} p_i x_i(p,w) = w$

$\Rightarrow \sum_{i=1}^{n} p_i \frac{\partial x_i(p,w)}{\partial w} = 1$

Since $(p,w) \gg 0$, it is impossible to have

$$\frac{\partial x_i(p,w)}{\partial w} \le 0 \text{ for } \forall i$$

$\Rightarrow \exists k$ s.t. $\frac{\partial x_k(p,w)}{\partial w} > 0$ #

Now further assume a consumer has the additive utility function

$$U(x_1, \ldots x_l) = \sum_{i=1}^{l} u_i(x_i),$$

$u(\cdot)$ strictly concave

where u_i is C^2, with $u_i'(x_i) > 0$ and $u_i''(x_i) < 0$ for all $x_i > 0$ and all i. $\Rightarrow \succsim$ strictly convex

$\Rightarrow x(p,w)$ unique

(b) Show that for the additive utility function, the marginal utility of income diminishes with income, i.e.,

$$\frac{\partial^2 v}{\partial w^2}(p, w) < 0 \text{ for all } (p, w) \gg 0.$$

[Hint: Recall that the marginal utility of income equals the Lagrange multiplier.]

pf: $L = U(x) + \lambda(w - \sum_{i=1}^{l} x_i p_i)$

FOC: $\frac{\partial L}{\partial x_i} = u_i'(x_i) - \lambda p_i = 0$

$u_i'(x_i) = \lambda p_i$ for $\forall i$

Since $\lambda = \frac{\partial v(p,w)}{\partial w}$

$\Rightarrow u_i'(x_i) = \frac{\partial v(p,w)}{\partial w} p_i$ for $\forall i$

Take derivative w.r.t. w on both sides.

$u_i''(x_i) \cdot \frac{\partial x_i(p,w)}{\partial w} = \frac{\partial^2 v(p,w)}{\partial w^2} \cdot p_i$ for $\forall i$ -- $(*)$

From (a), $\exists k$ s.t. $\frac{\partial x_k(p,w)}{\partial w} > 0$

Consider $(*)$ for x_k.

$$\underbrace{u_k''(x_k)}_{<0} \cdot \underbrace{\frac{\partial x_k(p,w)}{\partial w}}_{>0} = \frac{\partial^2 v(p,w)}{\partial w^2} \underbrace{p_k}_{>0}$$

$$\Rightarrow \quad \frac{\partial^2 v(p,w)}{\partial w^2} < 0 \quad \text{for all } (p,w) \gg 0$$

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(c) Show that for the additive utility function, we may strengthen the conclusion in (a) to the following:

$$\frac{\partial x_i(p,w)}{\partial w} > 0 \text{ for all } (p,w) \gg 0 \text{ and for every good } i.$$

From (b), we have (*) that holds for $\forall i$

$$\underset{<0}{u_i''(x_i)} \cdot \frac{\partial x_i(p,w)}{\partial w} = \underset{<0}{\frac{\partial^2 v(p,w)}{\partial w^2}} \cdot \underset{>0}{p_i} \quad \text{for } \forall i$$

$$\Rightarrow \quad \frac{\partial x_i(p,w)}{\partial w} > 0 \quad \text{for all } (p,w) \gg 0$$

and for $\forall i$

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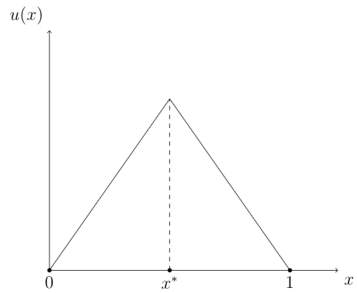
4. (Optional) Show that a preference relation $\succsim$ on $[0,1]$ is continuous and strictly convex iff there exists a continuous utility function u representing $\succsim$ and a point $x^* \in [0,1]$ such that u is strictly increasing on $[0,x^*]$ and strictly decreasing on $[x^*,1]$. Graphically, one example is



pf: "$\Rightarrow$": Assume $\succsim$ conti. strictly convex on $[0,1]$

By Representation Thm, $\exists$ conti. $u(\cdot)$ representing $\succsim$

Since $u(\cdot)$ is conti. on $[0,1]$ which is compact.

By Extreme Value Thm (EVT),

$$\exists\, x^* \in [0,1] \text{ s.t. } u(x^*) = \max_{x\in[0,1]} u(x).$$

Firstly, suppose x^* is not unique, i.e., $\exists\, y^* \in [0,1]$ s.t. $y^* \neq x^*$ and $u(y^*) = u(x^*)$.

By strict convexity of $\succsim$,

$$x^* \prec \frac{x^*+y^*}{2} \Rightarrow u(x^*) < u\left(\frac{x^*+y^*}{2}\right)$$

(take $\alpha = \frac{1}{2}$)

but x^* maximizes $u(\cdot)$ on $[0,1]$ $\times$.

$\Rightarrow$ the maximum value x^* is unique.

Next, assume $0 < x^* < 1$

Let $0 \leq a < b \leq x^*$.

1) If $b = x^*$, $u(b) = u(x^*) = \max_{x \in [0,1]} u(x) \geq u(a)$

2) If $b < x^*$, $\exists \alpha \in (0,1)$ s.t. $b = \alpha a + (1-\alpha) x^*$

By strict convexity, $a \prec \alpha a + (1-\alpha) x^* = b$

$\Rightarrow u(a) < u(b)$

By 1) & 2), u is strictly increasing on $[0, x^*]$

Proving that u is strictly decreasing on $[x^*, 1]$ is analogous

For the case where $x^* = 0$ or $x^* = 1$,

one side is trivial and the other is the same.

"$\Leftarrow$": Assume the conti. u represents $\succsim$ and strictly increasing on $[0, x^*]$ and strictly decreasing on $[x^*, 1]$.

By Representation Thm, $\succsim$ is conti.

Next, let $a, b \succsim c$ with $a \neq b$ and $\alpha \in (0,1)$

$u(c) \leq \min\{u(a), u(b)\} < u(\alpha a + (1-\alpha) b)$

by the strict monotonicity of u on $[0, x^*]$ and $[x^*, 1]$.

$\Rightarrow c \prec \alpha a + (1-\alpha) b$

$\Rightarrow \succsim$ is strictly convex #