COMPSCI 230 Problem Sets - Proof

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1 Direct Proof

1. Use a direct proof to show that every odd integer is the difference of two squares.

2 Proof by Contrapositive

1. (a) Show that for an integer n, n is even if and only if n^2 is even. (Hint: Using contrapositive for the "if" direction)

   (b) Show that for an integer n, n is odd if and only if n^2 is odd.

2. For integers x and y, use a proof by contrapositive to show that:

   (a) if xy and $x + y$ are both even, then both x and y are even.

   (b) if $x^2 + y^2$ and $3xy$ are both even numbers, then x and y are both even.

3 Proof by Contradiction

1. Prove by contradiction that if r is irrational, then $\sqrt{r}$ is also irrational.

2. Prove by contradiction that among any real numbers $x_1, x_2, \ldots, x_n$, there is at least one number greater or equal to the average $\frac{x_1+x_2+\ldots+x_n}{n}$.

3. Show that if x is irrational and y is any real number then at least one of $x + y$ and $x - y$ must be irrational.

4. Use a proof by contradiction to show that there is no rational number r for which $r^3 + r + 1 = 0$.

5. (a) For integers $p, a > 1$, show that if p divides a, then p does not divide $a + 1$.

   (b) Using the result from part (a), show that there are infinitely many prime numbers. You may use the fact that any integer $n > 1$ is divisible by a prime number.

4 Proof by Cases

1. Show that you can select two out of the three real numbers (which can be any arbitrary real number) such that their product is nonnegative.

2. Prove by cases that any group of 6 people either has 3 people that know each other or 3 people that don't know each other.

3. Use a proof by cases to show that $|xy| = |x||y|$, where x and y are real numbers.

4. Prove that for any odd number n, there exists an integer m such that $n^2 = 8m + 1$.

5. (Hard) Prove that there exists irrational number a and b such that a^b is rational. *Hint: consider* $\sqrt{2}^{\sqrt{2}}$.

5 Disproof

1. Prove or disprove: There exists some integer k such that $4k + 2$ is the difference between two integers, both of which are perfect squares.

2. Prove or disprove: The product of a nonzero rational number and an irrational number is irrational.