Please see the back of this handout for set notations and equivalences. A printable document will be provided on Canvas as for our logical equivalences.

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1. Consider $A = \{1, 3, 6, 8, 10\}$ and $B = \{2, 4, 6, 7, 10\}$.
   (a) Determine the following sets: $A \cap B$, $A \cup B$, and $A - B$.
   (b) Compute the cardinalities: $|A|$, $|B|$, $|A \cap B|$, $|A \cup B|$, $|A - B|$.

2. Let $U = \{x \in \mathbb{Z} \mid 1 \leq x \leq 100\}$. Consider $A = \{x \in U \mid x \text{ is divisible by } 3\}$ and $B = \{x \in U \mid x \text{ is divisible by } 5\}$.
   (a) Describe the following sets in set-builder notation: $A \cap B$, $A \cup B$, and $A - B$.
   (b) Compute the cardinalities: $|A|$, $|B|$, $|A \cap B|$, $|A \cup B|$, $|A - B|$.

3. Prove or disprove the following about any sets A, B, and C.
   (a) $A \setminus B \subseteq A$
   (b) $(A \cap C) \setminus (B \cap C) = (A \setminus B) \cap C$
   (c) $(A \cup C) \setminus (B \cup C) = (A \setminus B) \cup C$

4. The ***symmetric difference*** of two sets A and B is denoted by $A \Delta B$ and defined as

$$A \Delta B = (A \setminus B) \cup (B \setminus A).$$

   Prove the following:
   (a) $A \Delta B = (A \cup B) \setminus (A \cap B)$
   (b) $\overline{A \cup B}$ is disjoint from $A \Delta B$
   (c) $A \cap B$ is disjoint from $A \Delta B$

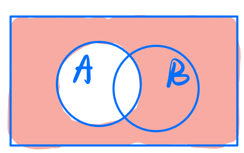
Set Notation

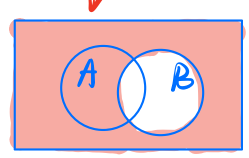
$$
\begin{aligned}
x \in A &\Leftrightarrow x \text{ is an element of } A \\
x \notin A &\Leftrightarrow \neg(x \in A) \\
\emptyset &\Leftrightarrow \{\}, \text{the empty set} \\
A \cup B &= \{x \mid x \in A \cup B \Leftrightarrow x \in A \vee x \in B\} \\
A \cap B &= \{x \mid x \in A \cap B \Leftrightarrow x \in A \wedge x \in B\} \\
\overline{A} &= \{x \mid x \in \overline{A} \Leftrightarrow x \notin A\} \\
A \setminus B &= \{x \mid x \in A \setminus B \Leftrightarrow x \in A \wedge x \notin B\} \\
A \subseteq B &\Leftrightarrow \forall x.\, x \in A \rightarrow x \in B
\end{aligned}
$$

Set Equivalences

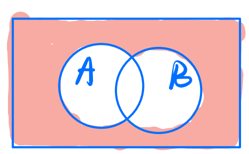
$$
\begin{aligned}
A \cap B &= B \cap A && \text{(commutativity of } \cap) \\
A \cup B &= B \cup A && \text{(commutativity of } \cup) \\
(A \cap B) \cap C &= A \cap (B \cap C) && \text{(associativity of } \cap) \\
(A \cup B) \cup C &= A \cup (B \cup C) && \text{(associativity of } \cup) \\
A \cap A &= A && \text{(idempotence for } \cap) \\
A \cup A &= A && \text{(idempotence for } \cup) \\
A \cap (B \cup C) &= (A \cap B) \cup (A \cap C) && \text{(distributivity of } \cap \text{ over } \cup) \\
A \cup (B \cap C) &= (A \cup B) \cap (A \cup C) && \text{(distributivity of } \cup \text{ over } \cap) \\
\overline{\overline{A}} &= A && \text{(double complement)} \\
A \setminus B &= A \cap \overline{B} && \text{(subtraction in terms of } \cap) \\
\overline{A \cap B} &= \overline{A} \cup \overline{B} && \text{(De Morgan for } \cap) \\
\overline{A \cup B} &= \overline{A} \cap \overline{B} && \text{(De Morgan for } \cup) \\
A \cup \mathcal{U} &= \mathcal{U} \\
A \cap \mathcal{U} &= A \\
A \cup \overline{A} &= \mathcal{U} \\
A \cap \overline{A} &= \emptyset \\
A \cup \emptyset &= A \\
A \cap \emptyset &= \emptyset
\end{aligned}
$$

$\overline{A}$





$\overline{B}$



$\overline{A \cup B} = \overline{A} \cap \overline{B}$

Set Theory

e.g., $\{1,2\}=\{2,1\}$

$\{1,2,2\}=\{1,2\}$

$\{a,b,1,2\}$

$\{\{a\},a\}$

* Set: an unordered collection of distinct objects
* Classical sets: $\mathbb{Z}$, $\mathbb{N}$, $\mathbb{R}$, $\mathbb{Q}$, $\mathbb{C}$
* Define sets by properties: $S=\{x\in A \mid P(x)\}$
* Membership:

  - Element & Sets: $x\in A$, $x\notin A \to \neg(x\in A)$
  - Sets & Sets:
    - $A\subseteq B$: $\forall x\in A \Rightarrow x\in B$
    - $A=B$: $A\subseteq B \Leftrightarrow B\subseteq A$
    - $A\subset B$: $A\subseteq B$, $A\neq B$

* Set Operations:

① $A\cap B=\{x\in U \mid x\in A \text{ and } x\in B\}$

② $A\cup B=\{x\in U \mid x\in A \text{ or } x\in B\}$

③ $\overline{A}=A^c=\{x\in U \mid x\notin A\}$

④ $A\setminus B=A-B=\{x\in U \mid x\in A \text{ and } x\notin B\}=A\cap\overline{B}$

Other useful facts:

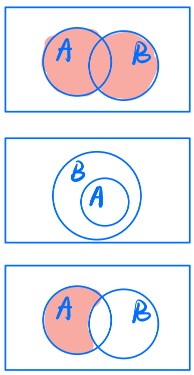
① $|A\cup B|=|A|+|B|-|A\cap B|$

② $A\subseteq B \Rightarrow \overline{B}\subseteq\overline{A}$

③ $|A-B|=|A|-|A\cap B|$

Prove?

Venn diagram:



Also, $A_1 \cup A_2 \cup \cdots \cup A_n \iff \bigcup_{i=1}^{n} A_i$

$A_1 \cap A_2 \cap \cdots \cap A_n \iff \bigcap_{i=1}^{n} A_i$

* $A \cdot B$ disjoint : $\forall x,\ x \notin A$ or $x \notin B$

$\iff \forall x,\ \neg(x \in A) \vee \neg(x \in B)$

$\iff \forall x,\ \neg(x \in A \wedge x \in B)$

$\iff \forall x,\ \neg(x \in A \cap B)$

$\iff \forall x,\ x \notin A \cap B$

$\iff A \cap B = \phi$

Def using set operator : $\boxed{A \cap B = \phi}$

* Set Equivalences ($A \setminus B = A \cap \bar{B}$, dM, distributivity)

* proof :
  - "$\subseteq$" : By definition
  - "$=$" :
    - By definition ($\subseteq$, $\supseteq$)
    - Set equivalences
* Disproof : Counter-example (venn diagram)

3. Prove or disprove the following about any sets A, B, and C.

(a) $A \setminus B \subseteq A$

(b) $(A \cap C) \setminus (B \cap C) = (A \setminus B) \cap C$

(c) $(A \cup C) \setminus (B \cup C) = (A \setminus B) \cup C$

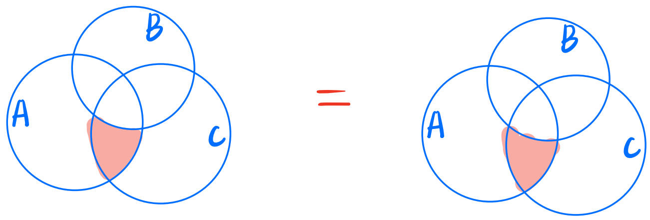
Pf: (a) By def. WTS: $\forall x,\ x \in A \setminus B \Rightarrow x \in A$

Consider any $x \in A \setminus B$.

By def of "$\setminus$". $x \in A$ and $x \notin B \Rightarrow x \in A$

Thus, $A \setminus B \subseteq A$

(b)



$(A \cap C) \setminus (B \cap C)$

$= (A \cap C) \cap \overline{(B \cap C)}$ (subtraction in terms of $\cap$)

$= (A \cap C) \cap (\bar{B} \cup \bar{C})$ (dM)

$= ((A \cap C) \cap \bar{B}) \cup ((A \cap C) \cap \bar{C})$ (distribution of $\cap$ over $\cup$)

$= \cdots \cup (A \cap (C \cap \bar{C}))$ (associativity of $\cap$)

$= \cdots \cup (A \cap \phi)$ ($X \cap \bar{X} = \phi$)

$= ((A \cap C) \cap \bar{B}) \cup \phi$ ($X \cap \phi = \phi$)

$= (A \cap C) \cap \bar{B}$ ($X \cup \phi = X$)

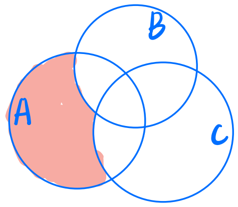
$= A \cap (C \cap \bar{B})$ (associativity of $\cap$)

$= A \cap (\bar{B} \cap C)$ (commutativity of $\cap$)

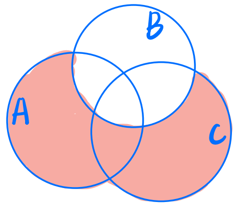
$= (A \cap \bar{B}) \cap C$ (associativity of $\cap$)

$= (A \setminus B) \cap C$ (subtraction in terms of $\cap$)

(e)



$\neq$



Counter-example:

Suppose $A = \{a\}$, $B = \{b\}$, $C = \{c\}$

$(A \setminus B) \cup C = \{a\} \cup \{c\} = \{a, c\}$

$(A \cup C) \setminus (B \cup C) = \{a, c\} \cup \{b, c\} = \{a\}$

Thus, $(A \setminus B) \cup C \neq (A \cup C) \setminus (B \cup C)$

4. The ***symmetric difference*** of two sets A and B is denoted by $A\Delta B$ and defined as

$$A\Delta B = (A \setminus B) \cup (B \setminus A).$$

Prove the following:

(a) $A\Delta B = (A \cup B) \setminus (A \cap B)$

(b) $\overline{A \cup B}$ is disjoint from $A\Delta B$

(c) $A \cap B$ is disjoint from $A\Delta B$

} ⇒ can also use pf by def.

Pf: (a) $A\Delta B = (A\setminus B)\cup(B\setminus A)$ (def of Δ)

$= (A\cap\bar{B})\cup(B\cap\bar{A})$ (subtraction in terms of $\cap$)

$= ((A\cap\bar{B})\cup B)\cap((A\cap\bar{B})\cup\bar{A})$ (distribution of $\cup$ over $\cap$)

$= ((A\cup B)\cap(\bar{B}\cup B))\cap((A\cup\bar{A})\cap(\bar{B}\cup\bar{A}))$ (distribution of $\cup$ over $\cap$)

$= ((A\cup B)\cap U)\cap(U\cap(\bar{B}\cup\bar{A}))$ (law of complement)

$= (A\cup B)\cap(\bar{B}\cup\bar{A})$ (identity of $\cap$)

$= (A\cup B)\cap(\overline{B\cap A})$ (dm)

$= (A\cup B)\setminus(B\cap A)$ (def of set difference)

$= (A\cup B)\setminus(A\cap B)$ (Commutativity of $\cap$)

(b) WTS: $(A\Delta B)\cap(\overline{A\cup B}) = \phi$

From (a), $A\Delta B = (A\cup B)\cap(\overline{B\cap A})$

$= (A\cup B)\cap(\overline{A\cap B})$ (Commutativity of $\cap$)

$\Rightarrow (A\Delta B)\cap(\overline{A\cup B})$

$= \left((A \cup B) \cap (\overline{A \cap B})\right) \cap (\overline{A \cup B})$ (def of Δ)

$= \left((\overline{A \cap B}) \cap (A \cup B)\right) \cap (\overline{A \cup B})$ (commutativity of $\cap$)

$= (\overline{A \cap B}) \cap \left((A \cup B) \cap (\overline{A \cup B})\right)$ (assoc. of $\cap$)

$= (\overline{A \cap B}) \cap \phi$ ($X \cap \overline{X} = \phi$)

$= \phi$

By def. $A \Delta B$ and $\overline{A \cup B}$ are disjoint.

(c) WTS: $(A \Delta B) \cap (A \cap B) = \phi$

From (a), $A \Delta B = (A \cup B) \cap (\overline{B \cap A})$

$= (A \cup B) \cap (\overline{A \cap B})$ (commutativity of $\cap$)

$\Rightarrow (A \Delta B) \cap (A \cap B)$

$= \left((A \cup B) \cap (\overline{A \cap B})\right) \cap (A \cap B)$ (def of Δ)

$= (A \cup B) \cap \left((\overline{A \cap B}) \cap (A \cap B)\right)$ (assoc. of $\cap$)

$= (A \cup B) \cap \phi$ ($X \cap \overline{X} = \phi$)

$= \phi$

By def. $A \Delta B$ and $A \cap B$ are disjoint. $\blacksquare$