Recall that an undirected graph G is defined as an ordered pair (V, E) where V is a finite set and $E \subseteq \{\{a, b\} \in 2^V | a \neq b\}$ is a set of two-element subsets of V. The elements of V are the ***vertices*** of G, and the elements of E are the ***edges*** of G. It is convention to refer to $|V|$ as n and $|E|$ as m (but not always, e.g., if they are defined otherwise or necessarily not when there are multiple graphs of interest with different numbers of vertices or edges.)

An undirected *bipartite* graph G is defined as an ordered pair (V, E) where V can be partitioned into two sets A and B such that $E \subseteq \{\{u, v\} \mid u \in A, v \in B\}$.

Recall that on Tuesday we proved that, for any undirected graph $G = (V, E)$, the sum of vertex degrees is twice the number of edges, i.e.,

$$\sum_{v \in V} d(v) = 2|E|.$$

---

1. A graph has 24 vertices and 30 edges. We are given the degrees of 19 of the 24 vertices: five vertices of degree 4, seven vertices of degree 1, and seven vertices of degree 2. All of the remaining five vertices have degree 3 or 4. How many vertices of degree 4 are there out of those five unknown vertices? Explain your answer.

2. Suppose there are n people at a party, and some pairs are friends with each other. Assume that friendship is symmetric in the sense that if a is friends with b, then b is friends with a, for any distinct people a, b.
   (a) For what values of n is it possible that each person is friends with ***exactly one*** other person at the party?
   (b) For what values of n is it possible that each person is friends with ***exactly three*** other people at the party?
   (c) For what values of n is it possible that each person is friends with ***exactly two*** other people at the party?

3. Prove by induction that, for any undirected bipartite graph $G = (V, E)$ with bipartition A and B, $\sum_{v \in A} d(v) = \sum_{v \in B} d(v) = m$. *[Hint: Induct on the number of edges.]*

4. A rooted binary tree is *full* if every node has either zero or two children.
   (a) Prove that any non-empty rooted full binary tree with i internal nodes (those with at least one child; non-leaf nodes) has $2i + 1$ total nodes. *[Hint: Induct on the number of internal nodes.]*
   (b) How many leaves does any full binary tree have?

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   (c) For what values of n is it possible that each person is friends with ***exactly two*** other people at the party?

(a) Each node : 1 deg.

① n odd : not possible $\Rightarrow$ # deg $= n$ odd ✗.

② n even : possible.

1 2   3 4 — — i $i+1$ — — $n-1$ n

for $\forall$ odd i

$\Rightarrow$ Connected ? not unless $n=2$

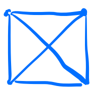
(b) Each node : 3 deg.

① $n \geq 4$: trivial

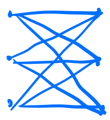
② n odd : not possible $\Rightarrow$ # deg $=3n$ odd ✗.

③ n even : ✓

$n=4$ $n=6$ $n=8$ $n=10$ $n=12$ - -



K_4



$K_{3,3}$

$2K_4$ $K_4 + K_{3,3}$ $2K_6$ —

$3K_4$

$\Rightarrow$ Key : Rigorously prove $\forall n$, $\exists$ combination of $K_4 + K_{3,3}$

pf: Two cases for n:

1) $n = 4k$ for some $k \in \mathbb{Z}$

$\Rightarrow k = \frac{n}{4}$ copies of K_4

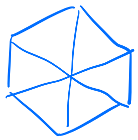
2) $n = 4k+2$ for some $k \in \mathbb{Z}$

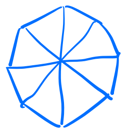
$\Rightarrow n-6 = 4k-4 = 4(k-1)$ is divisible by 4

$\Rightarrow k-1 = \frac{n-6}{4}$ copies of K_4 and a $K_{3,3}$

\#

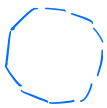
Alternative:





(c) ① $n \geq 3$

② A cycle of n vertices for $\forall n$



4. A rooted binary tree is *full* if every node has either zero or two children.

(a) Prove that any non-empty rooted full binary tree with i internal nodes (those with at least one child; non-leaf nodes) has $2i+1$ total nodes. *[Hint: Induct on the number of internal nodes.]*

(b) How many leaves does any full binary tree have?

(a) pf: $P(i)$: $\forall$ non-empty rooted full binary tree T with i internal nodes has $2i+1$ total nodes.

Base: $P(0)$ $\Rightarrow$ no internal nodes

$\Rightarrow$ a tree without any child

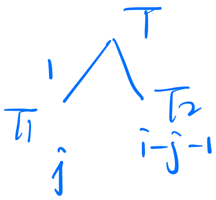
$\Rightarrow$ a single root node

$2 \cdot 0 = 1$ $\Rightarrow$ $P(0)$ holds

IH: $P(k)$ holds for $0, 1, \ldots, i-1$

$\Rightarrow$ To prove $P(i)$

IS: Consider T with $i > 0$



$\Rightarrow$ root node of T has two children (full

$\Rightarrow$ two subtree rooted at the two children: T_1, T_2

Suppose T_1 has j internal nodes.

$\Rightarrow$ T_2 has $i-j-1$ internal nodes

By IH, $0 \le j < i$, $0 \le i-j-1 < i$.

T_1 has $2j+1$ total nodes,

T_2 has $2(i-j-1)+1 = 2i-2j-1$ total nodes

$\Rightarrow T$ has $(2j+1)+(2i-2j-1)+1$

$= 2i+1$ total nodes

$\Rightarrow P(i)$ holds

By strong induction, the claim holds #

(b) By (a), $2i+1$ total nodes

i internal / non-leaf nodes

$\Rightarrow i+1$ leaves.