1. Two high-tech firms (1 and 2) are considering a joint venture. Each firm i can invest in a novel technology and can choose a level of investment $x_i \in [0,5]$at a cost of $c_i(x_i) = \frac{x_i^2}{4}$ (think of x_i as how many hours to train employees or how much capital to spend for R & D labs). The revenue of each firm depends on both its investment and the other firm's investment. In particular if firms i and j choose x_i and x_j , respectively, then the gross revenue to firm i is

$$R(x_i, x_j) = \begin{cases} 0, & x_i < 1 \\ 2, & x_i \geq 1, x_j < 2 \\ x_i x_j, & x_i \geq 1, x_j \geq 2 \end{cases} \tag{1}$$

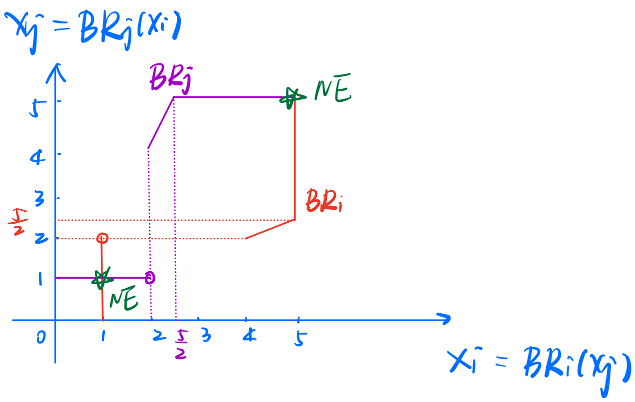
What is the BR for firm i? What are the NE?

$$\pi_i(x_i) = R(x_i, x_j) - C_i(x_i)$$

$$= \begin{cases} -\frac{x_i^2}{4}, & x_i < 1 \\ 2 - \frac{x_i^2}{4}, & x_i \geq 1,\ x_j < 2 \\ x_i x_j - \frac{x_i^2}{4}, & x_i \geq 1,\ x_j \geq 2 \end{cases}$$

$$\Rightarrow BR_i(x_j) = \underset{x_i}{\arg\max}\ \pi_i = \begin{cases} 1, & 0 \leq x_j < 2 \\ \min\{2x_j, 5\}, & 2 \leq x_j \leq 5 \end{cases}$$

$\Rightarrow$

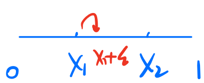


NE: (1,1), (5,5)

2. Consumers are located uniformly along a linear city of length 1. Each consumer wants to buy one unit of good from one existing firm. The transportation cost for the consumer is proportional to the distance to the firm from which he buys. The law prohibits any form of competition through price or service (other than location), so consumers go to the nearest firm. A firm's utility is equal to the number of its customers. Firm located at the same location get the same number of customers.

- (a) There are two firms, and they choose their locations simultaneously. Show that there exists a unique pure strategy Nash equilibrium.
- (b) Show that with three firms there exists no pure-strategy equilibrium.

(a) 1°

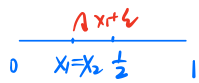


For firm 1, not deviate $\Rightarrow u_1 = x_1 + \frac{x_2 - x_1}{2}$

deviate to $x_1+\varepsilon \Rightarrow u_1' = x_1+\varepsilon + \frac{x_2-x_1-\varepsilon}{2}$

$u_1' > u_1 \Rightarrow$ firm 1 has incentives to deviate $\Rightarrow$ not NE

2° ①

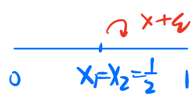


For firm 1, not deviate $\Rightarrow u_1 = \frac{1}{2}$

deviate to $x_1+\varepsilon \Rightarrow u_1' > \frac{1}{2} = u_1$

$\Rightarrow$ firm 1 has incentives to deviate $\Rightarrow$ not NE

②



For both firm 1 & 2. not deviate $\Rightarrow u = \frac{1}{2}$

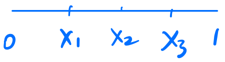
deviate to $x+\varepsilon \Rightarrow u' < \frac{1}{2} = u$

$\Rightarrow$ Both firms have no incentives to deviate

$\Rightarrow$ NE

Therefore, $\exists !$ pure NE : $(\frac{1}{2}, \frac{1}{2})$.

(b) 1°

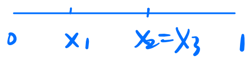


1: $x_1 \to x_1 + \varepsilon$

3: $x_3 \to x_3 - \varepsilon$ $\Rightarrow$ not NE

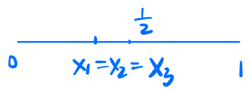
similarly as (a)

2°



1: $x_1 \to x_1 + \varepsilon$ $\Rightarrow$ not NE

3°



1: $x_1 \to x_1 + \varepsilon$

not deviate $\Rightarrow u_1 = \frac{1}{3}$

deviate to $x_1 + \varepsilon$ $\Rightarrow u_1' > \frac{1}{2} > u_1$

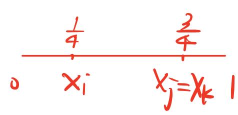
$\Rightarrow$ firm 1 has incentives to deviate

$\Rightarrow$ not NE

Therefore, $\nexists$ pure NE

(Remark: compare the result with the MVT problem with 3 candidates)

	HB	MVT (1 elected)	MVT (2 elected)
2 players	$(\frac{1}{2}, \frac{1}{2})$	$(\frac{1}{2}, \frac{1}{2})$	—
3 players	$\nexists$ NE	$\exists$ NEs	$\exists!$ NE



3. Consider first-price auction with 2 players and $v_2 < v_1$. Find a MSNE.

In MSNE, we must have $b_2 \le V_2$. (o/w $u_2 < 0$)

$\Rightarrow$ Assume 2 has a distributional strategy on $[0, V_2]$

i.e., $F_2(\cdot)$ is the CDF of b_2.

(2 is willing to mix all bids over $[0, V_2]$)

Then consider $b_1 = V_2$.

If $b_1 > V_2$, 1 can lower its bid until $b_1 = V_2$ to gain more.

$\Rightarrow$ Only left to check that 1 has no incentives

to deviate to $b_1 < V_2$ given a proper $F_2(\cdot)$.

$$\Rightarrow \quad E[u_1 \mid b_1 < V_2] \le E[u_1 \mid b_1 = V_2]$$

$$P(b_2 \le x)(V_1 - x) + P(b_2 > x) \cdot 0 \le V_1 - V_2$$

$$F_2(x)(V_1 - x) \le V_1 - V_2$$

$$F_2(x) \le \frac{V_1 - V_2}{V_1 - x}$$

Also, $F_2(0) = 0$, $F_2(V_2) = 1$

$$\Rightarrow \text{Set } F_2(x) = \frac{x}{V_2} \cdot \frac{V_1 - V_2}{V_1 - x}, \quad x \in [0, V_2]$$

Therefore, $b_1 = V_2$, $b_2 \sim F_2$ where $F_2(x) = \frac{x}{V_2} \cdot \frac{V_1 - V_2}{V_1 - x}$, $x \in [0, V_2]$

is a MSNE.