Review:

* Negating predicates (dm for quantifiers) + dm for $\wedge$, $\vee$

$$\neg(\forall x\ P(x)) \Leftrightarrow \exists x.\ \neg P(x)$$

$$\neg(\exists x\ P(x)) \Leftrightarrow \forall x\ \neg P(x)$$

$$\Rightarrow \neg(\forall x\ P(x) \vee Q(x)) \Leftrightarrow \exists x.\ \neg(P(x) \vee Q(x))$$

$$\Leftrightarrow \exists x\ \neg P(x) \wedge \neg Q(x) \quad \text{(dm for } \vee\text{)}$$

* Prove $A \Rightarrow B$

① Direct proof:

Assume $A \Rightarrow \sim \Rightarrow \cdots \Rightarrow \cdots \Rightarrow$ Therefore, B

② Contradiction

Assume $A \wedge \neg B \Rightarrow \cdots \Rightarrow (C \wedge \neg C) \Rightarrow F$

therefore, $A \Rightarrow B$

$$\underset{T}{A \Rightarrow B} \Leftrightarrow \underset{T}{\neg A \vee B} \Leftrightarrow \neg \underbrace{(A \wedge \neg B)}_{F}$$

③ Contrapositive

1) want to show (WTS): $A \Rightarrow B$

2) WTS: $\neg B \Rightarrow \neg A$

3) Prove $\neg B \Rightarrow \neg A$ by direct proof

Assume $\neg B \Rightarrow \cdots \Rightarrow \cdots \Rightarrow \cdots \Rightarrow$ therefore $\neg A$.

④ Cases

⑤ Induction

* Prove $A \Leftrightarrow B$

1) $A \Rightarrow B$

2) $B \Rightarrow A$

1. Recall that a *rational number* is any real number of the form a/b for integers a, b, such as $1/3$ and $5/7$. Consider the following formula, where $\mathbb{R}$ is the set of reals, and $\mathbb{Q}$ is the set of rational numbers.

$$\forall a, b \in \mathbb{R}. \exists c \in \mathbb{Q}. \underbrace{(a = b)}_{F1} \vee \underbrace{((\min(a, b) < c) \wedge (c < \max(a, b)))}_{F2}.$$

   (a) What is the formula saying in plain English?

   (b) Write its negation as a predicate formula using De Morgan's laws (for quantifiers and for $\wedge$ and $\vee$).

(a) For any real numbers a, b, there exists a rational number c such that either $a = b$ or $\min(a,b) < c < \max(a,b)$.

$\Rightarrow$ If a and b are different then c lies between a and b.

(b)

$$\neg \left(\forall a, b \in \mathbb{R}, \exists c \in \mathbb{Q}. (a = b) \vee \left((\min(a,b) < c) \wedge (c < \max(a,b)) \right) \right)$$

$$\Leftrightarrow \exists a, b \in \mathbb{R}, \neg \left(\exists c \in \mathbb{Q}. (a = b) \vee \left((\min(a,b) < c) \wedge (c < \max(a,b)) \right) \right)$$

dm for $\forall$

$$\Leftrightarrow \exists a, b \in \mathbb{R}, \forall c \in \mathbb{Q}, \neg \left((a = b) \vee \left((\min(a,b) < c) \wedge (c < \max(a,b)) \right) \right)$$

dm for $\exists$

$$\Leftrightarrow \exists a, b \in \mathbb{R}. \forall c \in \mathbb{Q}, \neg (a = b) \wedge \neg \left((\min(a,b) < c) \wedge (c < \max(a,b)) \right)$$

dm for $\vee$

$$\Leftrightarrow \exists a, b \in \mathbb{R}. \forall c \in \mathbb{Q}, \neg (a = b) \wedge \left(\neg (\min(a,b) < c) \vee \neg (c < \max(a,b)) \right)$$

dm for $\wedge$

$$\Leftrightarrow \exists a, b \in \mathbb{R}. \forall c \in \mathbb{Q}. (a \neq b) \wedge \left((\min(a,b) \geq c) \vee (c \geq \max(a,b)) \right)$$

neg. of $=, <, >$

2. Prove the following claim by direct proof, contradiction, and contrapositive:

$$\text{If } \underbrace{x \text{ is an even and prime integer}}_{A}, \text{ then } \underbrace{x = 2}_{B}.$$

WTS: $A \Rightarrow B$

Def 1: A non-zero integer n is divisible by non-zero integer k iff $n = k \cdot q$ for some integer q. (k is a divisor of n)

Ex: 27 is divisible by 3 ($27 = 3 \times 9$), 3 is a divisor of 27.

Def 2: A prime number is a natural number greater than 1 that is only divisible by 1 and itself. (it has no positive divisors other than 1 and itself.

Ex: 27 (1, 3, 9, 27) not prime, 7 (1, 7), prime

Pf: ① Direct:

Assume x is even and prime

Since x is even, it is divisible by 2.

Since x is prime, it is divisible only by itself and 1

Then, x is divisible by only 2 and 1.

Therefore, $x = 2 \cdot 1 = 2$

② Contradiction:

Assume x is even and prime and $x \neq 2$.

Since x is even, x is divisible by 2.

Since $x \neq 2$, x has a divisor other than 1 and x,

so x is not prime.

This contradicts the assumption x is prime.

Thus, the claim holds.

② Contrapositive.

WTS: If a positive integer $x \neq 2$, then x is not even or not prime. $\neg B \Rightarrow \neg A$.

Direct proof of the contrapositive:

Assume the positive integer $x \neq 2$

1) x is not even.

Trivially, the conclusion of the contrapositive holds.

2) x is even. (hope to show x is not prime)

By definition of an even number, $x = 2y$ for some $y \in \mathbb{Z}^+$.

Since $x \neq 2$, x has a divisor other than 1 and x.

So x is not prime.

Thus the conclusion of the contrapositive holds.

Combining both 1) & 2), the contrapositive holds.

$\Rightarrow$ the original statement holds.

$\blacksquare$

Remark: $A \Rightarrow B$ or C

Pf: Assume A

1) B	1) C
2) $\neg B \wedge C$	2) $\neg C \wedge B$

$\Rightarrow$ prove $\neg B \wedge C$ or $\neg C \wedge B$