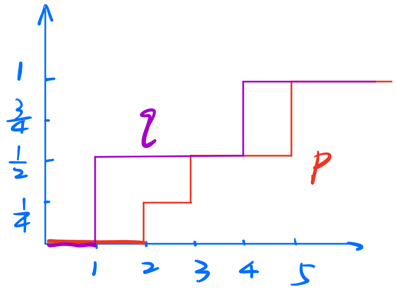
1. Use examples to see FOSD and SOSD.

    (a) p is a lottery of winning 2 and 3 dollars with probability $\frac{1}{4}$, respectively and winning 5 dollars with probability $\frac{1}{2}$; q is an even randomization between 1 and 4 dollars. Show that p FOSD q.

    (b) p is an even randomization between 2 and 3 dollars; q is a lottery of winning 1, 2, 3, and 4 dollars with probability $\frac{1}{4}$, respectively. Show that p SOSD q.

(a)

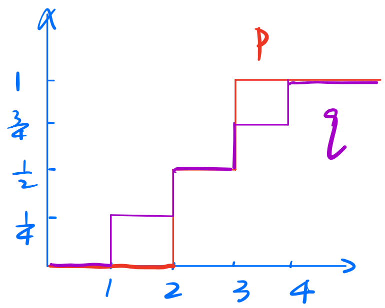
	p	q
1	0	$\frac{1}{2}$
2	$\frac{1}{4}$	0
3	$\frac{1}{4}$	0
4	0	$\frac{1}{2}$
5	$\frac{1}{2}$	0



p FOSD q

(b)

	p	q
1	0	$\frac{1}{4}$
2	$\frac{1}{2}$	$\frac{1}{4}$
3	$\frac{1}{2}$	$\frac{1}{4}$
4	0	$\frac{1}{4}$



p SOSD q

FOSD: p FOSD $q \iff F_p(x) \leq F_q(x) \ \forall x$

SOSD: p SOSD $q \iff \int_{\underline{x}}^{r} F_p(x)\,dx \leq \int_{\underline{x}}^{r} F_q(x)\,dx \ \forall r$

2. The definition of first-order stochastic dominance is that p FOSD q if every expected-utility maximizer with nondecreasing u weakly prefers p to q, i.e., $\forall u$ $s.t.$ $u'(x) \geq 0$, $\int_{\underline{x}}^{\bar{x}} u(x)\mathrm{d}F_p(x) \geq \int_{\underline{x}}^{\bar{x}} u(x)\mathrm{d}F_q(x)$.

   (a) Show that $F_p(x) \leq F_q(x)$ for $\forall x$ is an equivalent definition of FOSD.

   (b) Show that if p FOSD q, then the mean of x under p is larger than or equal to that under q. [Hint: Use the definition of FOSD]

   (c) Show that the converse of (b) is not true, i.e., provide an example where the mean of x under p is larger than or equal to that under q, but p does not FOSD q.

(a) pf: p FOSD $q \iff F_p(x) \leq F_q(x)$, $\forall x$

"$\Leftarrow$": Assume $F_p(x) \leq F_q(x)$ for $\forall x$

$$\int_{\underline{x}}^{\overline{x}} u(x)\, dF_p(x) = \int_{\underline{x}}^{\overline{x}} -u(x)\, d(1-F_p(x))$$

$$= -u(x)(1-F_p(x))\Big|_{\underline{x}}^{\overline{x}} + \int_{\underline{x}}^{\overline{x}} (1-F_p(x))\, du(x)$$

$$= \int_{\underline{x}}^{\overline{x}} (1-F_p(x))\, du(x) \quad \text{by normalizing } u(\underline{x})=0$$

Thus, $\int_{\underline{x}}^{\overline{x}} u(x)\, dF_p(x) - \int_{\underline{x}}^{\overline{x}} u(x)\, dF_q(x)$

$$= \int_{\underline{x}}^{\overline{x}} \big((1-F_p(x)) - (1-F_q(x))\big)\, du(x)$$

$$= \int_{\underline{x}}^{\overline{x}} (F_q(x) - F_p(x))\, u'(x)\, dx \geq 0$$

since by assumption, $F_q(x) \geq F_p(x)$, and $u'(x) \geq 0$, $\forall x$

$$\Rightarrow \int_{\underline{x}}^{\overline{x}} u(x)\, dF_p(x) \geq \int_{\underline{x}}^{\overline{x}} u(x)\, dF_q(x)$$

By definition, p FOSD q.

"$\Rightarrow$": Assume p FOSD q.

Suppose for contradiction. $\exists\, y$ s.t. $F_p(y) > F_q(y)$

Consider the non-decreasing u defined as:

$$u(x) = 1\{x \geq y\} = \begin{cases} 0, & \text{if } x \leq y \\ 1, & \text{if } x > y \end{cases}$$

$$\begin{aligned} E_p[u(x)] &= 0 \times P_r(x \leq y) + 1 \times P_r(x > y) \\ &= P_r(x > y) \\ &= 1 - P_r(x \leq y) \\ &= 1 - F_p(y) \end{aligned}$$

Similarly. $E_q[u(x)] = 1 - F_q(y)$

$\Rightarrow E_p[u(x)] < E_q[u(x)]$

$\Rightarrow \exists\, u(x)$ defined above, s.t.

$$\int_{\underline{x}}^{\overline{x}} u(x)\, dF_p(x) < \int_{\underline{x}}^{\overline{x}} u(x)\, dF_q(x)$$

$\Rightarrow p$ does not FOSD q $\quad$ ✗.

Thus, we must have $F_p(x) \leq F_q(x)$ for $\forall x$

Above all. p FOSD $q \Leftrightarrow F_p(x) \leq F_q(x)$, $\forall x$

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(b) pf: Since p FOSD q, by definition,

$$\int_{\underline{x}}^{\overline{x}} u(x)\, dF_p(x) \geq \int_{\underline{x}}^{\overline{x}} u(x)\, dF_q(x)$$

for $\forall$ non-decreasing $u(\cdot)$

Take $u(x) = x$. $u'(x) = 1 > 0$

$$\Rightarrow \int_{\underline{x}}^{\overline{x}} x\, dF_p(x) \geq \int_{\underline{x}}^{\overline{x}} x\, dF_q(x)$$

$$E_p[x] \geq E_q[x]$$

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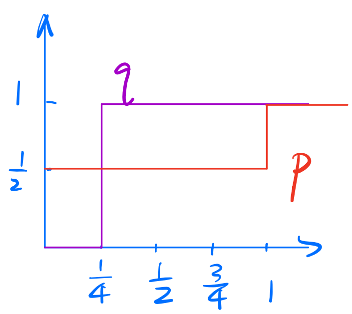
(c) A counter-example:

$$F_p(x) = \begin{cases} 0, & x < 0 \\ \frac{1}{2}, & 0 \leq x < 1 \\ 1, & x \geq 1 \end{cases} \qquad E_p[x] = \frac{0+1}{2} = \frac{1}{2}$$

$$F_q(x) = \begin{cases} 0, & x < \frac{1}{4} \\ 1, & x \geq \frac{1}{4} \end{cases} \qquad E_q[x] = \frac{1}{4}$$

$$\Rightarrow E_p[x] \geq E_q[x]$$

However,



$F_p(x) > F_q(x)$ for $x < \frac{1}{2}$

$\Rightarrow$ p does not FOSD q

3. (Optional) The definition of second-order stochastic dominance is that p SOSD q if every expected-utility maximizer with nondecreasing and concave u weakly prefers p to q, i.e., $\forall u$ s.t. $u'(x) \geq 0$ and $u''(x) \leq 0$, $\int_{\underline{x}}^{\bar{x}} u(x)\mathrm{d}F_p(x) \geq \int_{\underline{x}}^{\bar{x}} u(x)\mathrm{d}F_q(x)$. Rothschild and Stiglitz (1970) showed that there are two alternative definitions of SOSD, as displayed in the lecture note, and proved the equivalence of the three definitions. Assume $E_p = E_q$, i.e., p and q have the same expected value.

(a) Show that $\int_{\underline{x}}^{r} F_p(x)\mathrm{d}x \leq \int_{\underline{x}}^{r} F_q(x)\mathrm{d}x$ for $\forall r \geq 0$ is an equivalent definition of SOSD.

(b) Show that mean-preserving spread implies SOSD. We say F_q is a mean-preserving spread of F_p if $x_q \overset{d}{=} x_p + \epsilon$ for some $x_p \sim F_p$, $x_q \sim F_q$ and ϵ such that $E[\epsilon|x_p] = 0$ for every x_p. (The other direction is omitted and you may refer to the paper *Increasing risk: I. A definition* for details)

(a) pf: p SOSD $q \iff \int_{\underline{x}}^{r} F_p(x)dx \le \int_{\underline{x}}^{r} F_q(x)\,dx,\ \forall r$

In 2(a), we already showed that

$$\int_{\underline{x}}^{\bar{x}} u(x)\,dF_p(x) - \int_{\underline{x}}^{\bar{x}} u(x)\,dF_q(x) = \int_{\underline{x}}^{\bar{x}} (F_q(x) - F_p(x))\,u'(x)\,dx$$

Since u is twice differentiable, we have

$$u'(x) = u'(\bar{x}) - \int_{x}^{\bar{x}} u''(s)\,ds$$

$$\Rightarrow \int_{\underline{x}}^{\bar{x}} u(x)\,dF_p(x) - \int_{\underline{x}}^{\bar{x}} u(x)\,dF_q(x)$$

$$= \int_{\underline{x}}^{\bar{x}} (F_q(x) - F_p(x))\left(u'(\bar{x}) - \int_{x}^{\bar{x}} u''(s)\,ds\right)dx$$

$$= u'(\bar{x}) \int_{\underline{x}}^{\bar{x}} (F_q(x) - F_p(x))\,dx$$

$$+ \int_{\underline{x}}^{\bar{x}} (F_q(x) - F_p(x))\left(\int_{x}^{\bar{x}} (-u''(s))\,ds\right)dx$$

Since we assume $E_p = E_q$, $\int_{\underline{x}}^{\bar{x}} x\,dF_p(x) = \int_{\underline{x}}^{\bar{x}} x\,dF_q(x)$.

from integration by parts we have:

$$x F_p(x)\Big|_{\underline{x}}^{\overline{x}} - \int_{\underline{x}}^{\overline{x}} F_p(x)\,dx = x F_q(x)\Big|_{\underline{x}}^{\overline{x}} - \int_{\underline{x}}^{\overline{x}} F_q(x)\,dx$$

$$(\overline{x}\cdot 1 - \underline{x}\cdot 0) - \int_{\underline{x}}^{\overline{x}} F_p(x)\,dx = (\overline{x}\cdot 1 - \underline{x}\cdot 0) - \int_{\underline{x}}^{\overline{x}} F_q(x)\,dx$$

$$\int_{\underline{x}}^{\overline{x}} F_p(x)\,dx = \int_{\underline{x}}^{\overline{x}} F_q(x)\,dx$$

$$\int_{\underline{x}}^{\overline{x}} \big(F_q(x) - F_p(x)\big)\,dx = 0$$

$$\Rightarrow \int_{\underline{x}}^{\overline{x}} u(x)\,dF_p(x) - \int_{\underline{x}}^{\overline{x}} u(x)\,dF_q(x)$$

$$= \int_{\underline{x}}^{\overline{x}} \big(F_q(x) - F_p(x)\big)\left(\int_{x}^{\overline{x}} (-u''(s))\,ds\right) dx$$

$$= \int_{\underline{x}}^{\overline{x}} \left(\int_{\underline{x}}^{s} \big(F_q(x) - F_p(x)\big)\,dx\right)(-u''(s))\,ds$$

By assumption, $u''(x) \leq 0$ for $\forall x$

$$\Rightarrow \int_{\underline{x}}^{\overline{x}} u(x)\,dF_p(x) - \int_{\underline{x}}^{\overline{x}} u(x)\,dF_q(x) \geq 0$$

$$\text{iff } \int_{\underline{x}}^{s} \big(F_q(x) - F_p(x)\big)\,dx \geq 0$$

By def. p SOSD q $\Leftrightarrow \int_{\underline{x}}^{r} F_p(x)\,dx \leq \int_{\underline{x}}^{r} F_q(x)\,dx$

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(b) pf: Assume F_q is a MPS of F_p

$$\Rightarrow \int_{\underline{x}}^{\overline{x}} u(x)\, dF_q(x) = E[u(x_q)]$$

$$= E[E[u(x_q) \mid x_p]]$$

$$= E[E[u(x_p+\varepsilon) \mid x_p]]$$

$$= \int_{\underline{x}}^{\overline{x}} E[u(x+\varepsilon) \mid x]\, dF_p(x)$$

$$\leq \int_{\underline{x}}^{\overline{x}} u(E[x+\varepsilon \mid x])\, dF_p(x)$$

by Jensen's inequality since u is concave

$$= \int_{\underline{x}}^{\overline{x}} u(x)\, dF_p(x)$$

since $E[\varepsilon \mid x_p] = 0$

$\Rightarrow$ By definition, p SOSD q.

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