# COMPSCI 230 Problem Sets - Induction

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## 1 Weak Induction

1. Prove by induction that
$$1 + 3 + 5 + \cdots + (2n - 1) = n^2 \quad \text{for any } n \geq 1.$$

2. (a) Prove by induction that for any $n \in \mathbb{N}$
$$2^0 + 2^1 + 2^2 + \cdots + 2^n = 2^{n+1} - 1.$$

   (b) Prove by induction that the generalized geometric series summation
$$1 + a + a^2 + \ldots + a^n = \frac{a^{n+1} - 1}{a - 1}, \text{ where } a \neq 1.$$

3. Prove by induction that
$$1 \cdot 2 + 2 \cdot 3 + \cdots + (n-1) \cdot n = \frac{(n-1)n(n+1)}{3}.$$

4. Use induction to find all $n \in \mathbb{N}, n > 0$, satisfying $2^n > 2n + 7$.

5. Using induction, prove the inequality:
$$\forall n \geq 2, \quad \sum_{k=1}^{n} \frac{1}{\sqrt{k}} > \sqrt{n}.$$

6. Let $n \geq 2$ be an integer. Show that
$$\sum_{i=1}^{n-1} \frac{1}{\sqrt{i} + \sqrt{i+1}} + 1 = \sqrt{n}.$$

7. (a) Let $f_n$ be the $n$-th Fibonacci number, i.e. $f_1 = f_2 = 1$, and $f_{n+2} = f_{n+1} + f_n$. Prove that
$$f_1 + f_2 + \cdots + f_n = f_{n+2} - 1.$$

   (b) Let $g_n$ satisfy the same recurrence as the Fibonacci sequence but have different initial values: $g_1 = a$, $g_2 = b$, and $g_{n+2} = g_{n+1} + g_n$. Prove that
$$g_1 + g_2 + \cdots + g_n = g_{n+2} - b.$$

8. (Hard) There are $N$ people standing in a circle and numbered 1 to $N$ clockwise. The game goes as follows.

   - Player 1 starts and eliminates player directly to the left of him.
   - We search clockwise (to the left) for the first non-eliminated player and make him eliminate player directly to the left of him.
   - These moves are repeated until one player is left. He is declared the winner of the game.

   For example, let's say there are 4 players. Initially: $4, 3, 2, 1$. After the first step by 1, we get the position $4, 3, 1$. Next move is done by player 3 (he is first to the left of 1). He eliminates player 4 and the remaining position is $3, 1$. Next move is done by player 1 (he is first to the left of 3). He eliminates player 3 and 1 is the only player left. Thus, for $N = 4$, the winner is 1.

   (a) Let $N = 2^m$ where $m \in \mathbb{N}$. Using induction, show that the winner is 1.

   (b) Let $N = 2^m + r$ where $m \in \mathbb{N}$ and $r < 2^m$. Using part (a), show that the winner is $2r + 1$.

9. (A Fun Paradox) After learning about proof by induction in Discrete Math, Jasper comes up with a self-defined theorem: "All horses in the world have the same color." His friend, Zoe, finds this theorem absurd, but Jasper presents his "rigorous" proof to her:

   *Proof.* Let $P(n)$ denote the statement "any set of $n$ horses have the same color."

   1. Base Case ($n = 1$): If there is exactly one horse in the world, the statement is trivially true, so $P(1)$ holds.
   2. Inductive Hypothesis: Assume $P(n)$ holds for some arbitrary $n \in \mathbb{N}^+$.
   3. Inductive Steps:
      - Take a group of $n + 1$ horses and label them as $\{H_1, H_2, ..., H_{n+1}\}$.
      - Consider the first $n$ horses: $\{H_1, H_2, ..., H_n\}$. By the inductive hypothesis, any $n$ horses have the same color, so the first $n$ horses have the same color.
      - Consider the last $n$ horses: $\{H_2, H_3, ..., H_{n+1}\}$. Then again, by the inductive hypothesis, they have the same color.
      - Since these two groups overlap (i.e., $H_2, H_3, ..., H_n$ are in both groups), the first and last horses must also have the same color.
      - Thus, all $n + 1$ horses must have the same color, so $P(n + 1)$ holds.

   Therefore, by weak induction, $P(n)$ holds for all $n \in \mathbb{N}^+$, proving that all horses in the world must have the same color. $\square$

   Zoe finds this result bizarre, but she cannot spot any flaw in the proof. Can you help her find the mistake?

## 2 Strong Induction

1. Using induction, prove the Cauchy-Schwarz inequality: For any $a_1, \dots, a_n$, and any $b_1, \dots, b_n$,

$$a_1 b_1 + a_2 b_2 + \cdots + a_n b_n \leq \sqrt{a_1^2 + a_2^2 + \cdots + a_n^2} \cdot \sqrt{b_1^2 + b_2^2 + \cdots + b_n^2}.$$

   *Hint: consider $P(2)$.*

2. Show that any integer at least 8 can be formed by combination of 3 and 5.

3. A confectionery company is designing an assorted pack of confectionery consisting of chocolate (15g/bag), marshmallow (6g/bag) and toffee (10g/bag). Show that for any pack with an integer weight at least 61g (i.e., 61g, 62g, 63g, etc.), there is always a way to mix these three kinds of confectionery so that the pack contains some ($\geq 1$ bag) of each confectionery.

4. Prove by induction on $n$ that a square cake can be cut into $n > 5$ square pieces (not necessarily of the same size).

5. Label the first prime number 2 as $P_1$. Label the second prime number 3 as $P_2$. Similarly, label the $n$-th prime number as $P_n$. Use strong induction to prove that $P_n < 2^{2^n}$ for an arbitrary $n \in \mathbb{N}^+$.

   *Hint: consider $P_1 P_2 \cdots P_{n-1} + 1$.*

6. Nim is a famous game in which two players take turns removing items from a pile of $n$ items. For every turn, the player can remove one, two, or three items at a time. The player removing the last match loses. Use strong induction to show that, if each player plays the best strategy possible, the first player wins if $n = 4j, 4j+2$, or $4j+3$ for some non-negative integer $j$ and the second player wins in the remaining case when $n = 4j + 1$ for some nonnegative integer $j$.

7. Let the sequence $a_n$ be defined as $a_1 = a_2 = a_3 = 1$ and $a_n = a_{n-1} + a_{n-2} + a_{n-3}$ for all $n \geq 4$. Prove that $a_n < 2^n$ holds for all $n \in \mathbb{Z}_+$.

8. We define the sequence of numbers

$$a_n = \begin{cases} 1 & \text{if } 0 \leq n \leq 3, \\ a_{n-1} + a_{n-2} + a_{n-3} + a_{n-4} & \text{if } n \geq 4. \end{cases}$$

   Prove that $a_n \equiv 1 \pmod 3$ for all $n \geq 0$.