COMPSCI 230 Midterm Review - Logic

Qinyang Yu

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1. Prove that $(\neg(P \Rightarrow Q)) \Leftrightarrow (P \wedge \neg Q)$ is a tautology.
2. Prove that $((P \Rightarrow Q) \wedge (\neg Q \wedge P))$ is a contradiction.
3. Show that $(p \wedge q) \Rightarrow r$ and $(p \Rightarrow r) \wedge (q \Rightarrow r)$ are not logically equivalent.

   Hint: Find an assignment of truth values that makes one of these propositions true and the other false.
4. The proposition p NOR q is true when both p and q are false, and it is false otherwise. Let p NOR q be denoted by $p \downarrow q$ (called Peirce arrow after C.S. Peirce). Show that
   (a) use truth table to prove $p \downarrow q$ is logically equivalent to $\neg(p \vee q)$
   (b) $p \downarrow p \iff \neg p$
   (c) $(p \downarrow q) \downarrow (p \downarrow q) \iff (p \vee q)$
5. Translate (a) from an English sentence into formulas of predicate logic. Translate (b) from the predicate into an English expression.
   (a) $\forall x \in \mathbb{Z}, \forall y \in \mathbb{Z}, ((x > 0) \wedge (y > 0) \Rightarrow (x + y > 0))$.
   (b) Every real number except zero has a multiplicative inverse. (A multiplicative inverse of a real number x is a real number y such that $xy = 1$.)
6. (a) Find a common domain for the variables x, y, and z for which the statement $\forall x \forall y((x \neq y) \Rightarrow \forall z((z = x) \vee (z = y)))$ is true and another domain for which it is false.
   (b) Let $Q(x, y)$ be the statement $x + y = 0$. Are the following quantifications $\exists y \forall x Q(x, y)$ and $\forall x \exists y Q(x, y)$ respectively, true or false, where the domain for all variables consists of all real numbers.
7. Prove that $(\exists x(P(x) \Rightarrow Q(x))) \iff (\forall x(P(x) \Rightarrow (\exists x Q(x))))$ is a tautology.
8. Let $P(x, y)$ be a predicate with variables $x, y \in \mathbb{Z}$. Let $A = \forall x \exists y : P(x, y)$ and $B = \exists y \forall x : P(x, y)$.
   (a) Let $P(x, y)$ be $x = y$. Show that $A \Rightarrow B$ is false.
   (b) Show that B implies A.

Summary:

* Proposition P vs Predicate $P(x)$

* Logical operators: $\neg$, $\vee$, $\wedge$

  "$\Rightarrow$". "$\Leftrightarrow$" $A \Rightarrow B \Leftrightarrow \neg A \vee B$

* Axiomatic Rules (Distributivity, dM)

* Logic Proofs
  - ① Truth tables
  - ② chain of equivalence.

* Translation & Judgment of predicates

1. Prove that $(\neg(P \Rightarrow Q)) \Leftrightarrow (P \wedge \neg Q)$ is a tautology.

Method 1:

P	Q	$P \Rightarrow Q$	$\neg(P \Rightarrow Q)$	$\neg Q$	$P \wedge \neg Q$	$(\neg(P \Rightarrow Q)) \Leftrightarrow (P \wedge \neg Q)$
T	T	T	F	F	F	T
T	F	F	T	T	T	T
F	T	T	F	F	F	T
F	F	T	F	T	F	T

Method 2:

$$\neg(P \Rightarrow Q)$$

$$\Leftrightarrow \neg(\neg P \vee Q) \qquad \text{(implication)}$$

$$\Leftrightarrow P \wedge \neg Q \qquad \text{(dM)}$$

2. Prove that $((P \Rightarrow Q) \wedge (\neg Q \wedge P))$ is a contradiction.

Method 1:

P	Q	$P \Rightarrow Q$	$\neg Q$	$\neg Q \wedge P$	$(P \Rightarrow Q) \wedge (\neg Q \wedge P)$
T	T	T	F	F	F
T	F	F	T	T	F
F	T	T	F	F	F
F	F	T	T	F	F

Method 2:

$$(P \Rightarrow Q) \wedge (\neg Q \wedge P)$$

$$\Leftrightarrow (\neg P \vee Q) \wedge (\neg Q \wedge P) \quad \text{(implication)}$$

$$\Leftrightarrow \big(\neg P \wedge (\neg Q \wedge P)\big) \vee \big(Q \wedge (\neg Q \wedge P)\big) \quad \text{(distributivity)}$$

$$\Leftrightarrow \big((\neg P \wedge P) \wedge \neg Q\big) \vee \big((Q \wedge \neg Q) \wedge P\big) \quad \text{(associativity)}$$

$$\Leftrightarrow (F \wedge \neg Q) \vee (F \wedge P) \quad \text{(contradiction)}$$

$$\Leftrightarrow F \vee F$$

$$\Leftrightarrow F$$

3. Show that $(p \wedge q) \Rightarrow r$ and $(p \Rightarrow r) \wedge (q \Rightarrow r)$ are not logically equivalent.

Hint: Find an assignment of truth values that makes one of these propositions true and the other false.

Method 1: ~~Truth tables~~

Method 2:

Consider $p = T$, $q = F$, $r = F$

$\overset{F}{(p \wedge q)} \Rightarrow \overset{F}{r}$ $\qquad T$

$\underset{F}{(p \Rightarrow r)} \wedge \underset{T}{(q \Rightarrow r)}$ $\qquad F$

4. The proposition p NOR q is true when both p and q are false, and it is false otherwise. Let p NOR q be denoted by $p \downarrow q$ (called Peirce arrow after C.S. Peirce). Show that

   (a) use truth table to prove $p \downarrow q$ is logically equivalent to $\neg(p \vee q)$

   (b) $p \downarrow p \iff \neg p$

   (c) $(p \downarrow q) \downarrow (p \downarrow q) \iff (p \vee q)$

(a)

p	q	$p \downarrow q$	$p \vee q$	$\neg(p \vee q)$
T	T	F	T	F
T	F	F	T	F
F	T	F	T	F
F	F	T	F	T

(b)

$$p \downarrow p \iff \neg(p \vee p)$$

$$\iff \neg p$$

(c)

$$(p \downarrow q) \downarrow (p \downarrow q) \iff (\neg(p \vee q)) \downarrow (\neg(p \vee q))$$

$$\iff \neg((\neg(p \vee q)) \vee (\neg(p \vee q)))$$

$$\iff (p \vee q) \wedge (p \vee q)$$

$$\iff p \vee q$$

5. Translate (a) from the predicate into an English sentence. Translate (b) from an English sentence into formulas of predicate logic.

   (a) $\forall x \in \mathbb{Z}, \forall y \in \mathbb{Z}, ((x > 0) \wedge (y > 0) \Rightarrow (x + y > 0))$.

   (b) Every real number except zero has a multiplicative inverse. (A multiplicative inverse of a real number x is a real number y such that $xy = 1$.)

(a) The sum of two positive integers is always positive

(b) $\forall x \in \mathbb{R}, \exists y \in \mathbb{R}, (x=0) \vee (xy=1)$

6. (a) Find a common domain for the variables x, y, and z for which the statement $\forall x \forall y((x \neq y) \Rightarrow \forall z((z = x) \vee (z = y)))$ is true and another domain for which it is false.

(b) Let $Q(x, y)$ be the statement $x + y = 0$. Are the following quantifications $\exists y \forall x Q(x, y)$ and $\forall x \exists y Q(x, y)$ respectively, true or false, where the domain for all variables consists of all real numbers.

(a) ① $x, y, z \in \{a, b\}$. $a \neq b$

② $x, y, z \in \{a, b, c\}$ with a, b, c distinct

(b) ① $\exists y, \forall x, x + y = 0$

False.

② $\forall x, \exists y, x + y = 0$.

True.

7. Prove that $(\exists x(P(x) \Rightarrow Q(x))) \iff (\forall x(P(x) \Rightarrow (\exists x Q(x))))$ is a tautology.

$$\exists x \left(P(x) \Rightarrow Q(x) \right)$$

$$\iff \exists x \left(\neg P(x) \vee Q(x) \right)$$

$$\iff \left(\exists x, \neg P(x) \right) \vee \left(\exists x, Q(x) \right)$$

$$\iff \neg\left(\forall x, P(x) \right) \vee \left(\exists x, Q(x) \right)$$

$$\iff \forall x, P(x) \Rightarrow \exists x, Q(x)$$

8. Let $P(x, y)$ be a predicate with variables $x, y \in \mathbb{Z}$. Let $A = \forall x \exists y : P(x, y)$ and $B = \exists y \forall x : P(x, y)$.

(a) Let $P(x, y)$ be $x = y$. Show that $A \Rightarrow B$ is false.

(b) Show that B implies A. T F

(a) WTS: A is true, B is false.

$A: \forall x, \exists y, x = y \Rightarrow T$

$B: \exists y, \forall x, x = y \Rightarrow F$

(b) Recall HW2 Q1

$$\underbrace{\exists y \forall x : P(x,y)}_{B} \Rightarrow \exists y, \exists x : P(x,y)$$

or

$$\Rightarrow \forall x, \exists y : P(x,y) \quad A$$

Pf: There exists y_0 s.t. $P(x, y_0)$ holds for all x.

$\Rightarrow$ For all x, we can specify $y = y_0$ s.t.

$P(x,y) = P(x, y_0)$ is true.

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