1. Can weakly dominated strategies be played in a trembling hand perfect equilibrium? Why or why not?

Ans: No

pf: Suppose a_i is a weakly dominated strategy by α_i

By def, $\exists a_{-i} \in A_{-i}$, $u_i(\alpha_i, a_{-i}) > u_i(a_i, a_{-i})$

Take any completely mixed strategy profile $\{\alpha^k\}$

$\Rightarrow$ a_{-i} yields positive weight in α_{-i}^k, $\forall k$

Given α_{-i}^k, $u(\alpha_i, \alpha_{-i}^k) > u(a_i, \alpha_{-i}^k)$ $\forall k$

$\Rightarrow$ a_i is not optimal against any mixture

$\Rightarrow$ a_i cannot be played in THPE.

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Def. δ_i is completely/totally mixed strategy if $\delta_i(s_i) > 0$ for $\forall s_i \in S_i$.

Def. a_i is weakly dominated if $\exists \alpha_i$ s.t. $u_i(\alpha_i, a_{-i}) \geq u_i(a_i, a_{-i})$ $\forall a_{-i} \in A_{-i}$

and $u_i(\alpha_i, a_{-i}) > u_i(a_i, a_{-i})$ for some $a_{-i} \in A_{-i}$

2. A strategy profile σ is trembling hand perfect if there exists a sequence of strategy profiles $\sigma^n \to \sigma$ for all i and $s_i \in S_i$ such that $\sigma_i(s_i) > 0$ implies that δ_i $\cancel{s_i}$ is a best response to σ^n_{-i}. Prove that every trembling hand perfect profile is a Nash equilibrium.

pf: Let δ be a THPE.

$\Rightarrow \exists \{\delta^n\}$ s.t. $\delta^n \to \delta$. $\forall i$. $\forall s_i \in S_i$

and $\delta_i \in BR_i(\delta^n_{-i})$. $\forall n$

(WTS: δ_i is the BR to δ_{-i}, $\forall i$)

Take $\forall\ \delta_i' \in \Delta(S_i)$, we have

$$u_i(\delta_i, \delta_{-i}^n) \geq u_i(\delta_i', \delta_{-i}^n) \quad \forall n$$

$$\lim_{n\to\infty} u_i(\delta_i, \delta_{-i}^n) \geq \lim_{n\to\infty} (\delta_i', \delta_{-i}^n)$$

By continuity of $u_i(\cdot)$,

$$u_i(\delta_i, \delta_{-i}) \geq u_i(\delta_i', \delta_{-i}) \text{ for } \forall \delta_i' \in \Delta(S_i)$$

$\Rightarrow$ $\delta_i \in BR_i(\delta_{-i})$ for $\forall i \in I$

By def, δ is also a NE.

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3. For the following game, find the pure strategy NEs. Show whether or not they are tembling-hand perfect.

Player 2

		L_2	R_2
Player 1:	L_1	1, 6	0, 5
	R_1	1, 1	1, 2

pure NE:

(L_1, L_2) (R_1, R_2)

pf: ① For (L_1, L_2), consider any seq. for 2.

$\delta_2^k = \varepsilon_k \cdot L_2 + (1-\varepsilon_k) R_2 \to L_2$ where $\varepsilon_k \to 1$

$\Rightarrow BR_1(\delta_2^k) = R_1 . \ \forall k$

$\Rightarrow L_1 \notin BR_1(\delta_2^k) . \ \forall k$

$\Rightarrow (L_1, L_2)$ is not THPE

② For (R_1, R_2), consider any seq. for 1.

$\delta_1^k = \varepsilon_k L_1 + (1-\varepsilon_k) R_1 \to R_1$ where $\varepsilon_k \to 0$

If $BR_2(\delta_1^k) = R_2$, we must have

$$6\varepsilon_k + 1 \cdot (1-\varepsilon_k) \leq 5\varepsilon_k + 2(1-\varepsilon_k)$$

$$\varepsilon_k \leq \frac{1}{2}$$

Take $\varepsilon_k = \frac{1}{2^k}$, $k \in \mathbb{N}$. $\varepsilon_k \to 0$. $\varepsilon_k > 0$

$\Rightarrow (R_1, R_2)$ is THPE

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4. Find NE and ESS in the following game.

	a	b	c
a	2, 2	0, 0	0, 0
b	0, 0	0, 0	1, 1
c	0, 0	1, 1	0, 0

① NE:

pure NE: (a,a) (b,c) (c,b)

MSNE:

1) mix a, b, c: (symmetric)

$$2p = 1-p-q = 1\cdot q \Rightarrow p=\frac{1}{5},\ q=\frac{2}{5}$$

$$\Rightarrow \left(\frac{1}{5}a+\frac{2}{5}b+\frac{2}{5}c,\ \frac{1}{5}a+\frac{2}{5}b+\frac{2}{5}c\right)$$

2) mix b & c: (symmetric)

$$p = 1-p \Rightarrow p=\frac{1}{2}$$

$$\Rightarrow \left(\frac{1}{2}b+\frac{1}{2}c,\ \frac{1}{2}b+\frac{1}{2}c\right)$$

3) mix a & c / a & b: (not symmetric)

$$2p = 1-p \Rightarrow p=\frac{1}{3}$$

$$\Rightarrow \left(\frac{1}{3}a+\frac{2}{3}c,\ \frac{1}{3}a+\frac{2}{3}b\right)$$

$$\left(\frac{1}{3}a+\frac{2}{3}b,\ \frac{1}{3}a+\frac{2}{3}c\right)$$

② ESS:

Since ESS is a symmetric NE.

$\Rightarrow$ only 3 candidates

1) (a,a)

$u(a,a) = 2 > 0 = u(b,a)$

& $u(a,a) = 2 > 0 = u(c,a)$

$\Rightarrow u(a,a) > u(\tau, a) \quad \forall \tau \in \Delta(A)$

By C1, (a,a) is ESS

2) $(\frac{1}{5}a+\frac{2}{5}b+\frac{2}{5}c, \frac{1}{5}a+\frac{2}{5}b+\frac{2}{5}c)$

Let $\alpha^* = (\frac{1}{5}, \frac{2}{5}, \frac{2}{5})$ and $\tau = (1,0,0)$

$\Rightarrow u(\alpha^*, \alpha^*) = \frac{2}{5} = u(\tau, \alpha^*)$

but $u(\tau,\tau) = 2 > \frac{2}{5} = u(\alpha^*, \tau)$

By C2, it is not ESS

3) $(\frac{1}{2}b+\frac{1}{2}c, \frac{1}{2}b+\frac{1}{2}c)$

Let $\alpha^* = (0, \frac{1}{2}, \frac{1}{2})$

$\Rightarrow u(\alpha^*, \alpha^*) = \frac{1}{2}. \quad u(a, \alpha^*) = 0$

(strictly better than any τ including a)

Consider $\tau = (0, p, 1-p) \quad (p \neq \frac{1}{2})$

$u(\tau, \alpha^*) = \frac{1}{2}p + \frac{1}{2}(1-p) = \frac{1}{2} = u(\alpha^*, \alpha^*)$

and $u(\tau,\tau) = p(1-p)\cdot 1 + (1-p)\cdot p\cdot 1$

$$= 2p(1-p)$$

$$= -2(p-\tfrac{1}{2})^2 + \tfrac{1}{2}$$

$$< \tfrac{1}{2} = u(\alpha^*,\tau)$$

By C2. $(\frac{1}{2}b+\frac{1}{2}c, \frac{1}{2}b+\frac{1}{2}c)$ is ESS.

Therefore, among all 7 NEs, only

(a,a) and $(\frac{1}{2}b+\frac{1}{2}c, \frac{1}{2}b+\frac{1}{2}c)$ are ESS.

Def (ESS) $\forall \tau \in \Delta(A)$

C1: $u(\alpha^*,\alpha^*) > u(\tau,\alpha^*)$

C2: $u(\alpha^*,\alpha^*) = u(\tau,\alpha^*)$ and $u(\alpha^*,\tau) > u(\tau,\tau)$