Recall the properties of relations R on a set A (i.e., relations from A to A):

- **Reflexive**: $\forall a \in A.\, aRa$.
- **Irreflexive**: $\forall a \in A.\, \neg(aRa)$.
- **Transitive**: $\forall a, b, c \in A.\, aRb \wedge bRc \Rightarrow aRc$.
- **Symmetric**: $\forall a, b \in A.\, aRb \Rightarrow bRa$.
- **Asymmetric**: $\forall a, b \in A.\, aRb \Rightarrow \neg(bRa)$.
- **Antisymmetric**: $\forall a, b \in A.\, aRb \wedge bRa \Rightarrow a = b$.

A relation R on a set A is:

- a *partial order* if R is reflexive, antisymmetric, and transitive.
- a *total order* if R is a partial order and for all distinct $a, b \in A$, aRb or bRa.
- an *equivalence relation* if R is reflexive, symmetric, and transitive.

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1. Consider the relation R on $\{1, 2, 3, 4, 5\}$ is an equivalence relation:

$$R = \{(1,1), (1,4), (4,1), (4,4), (5,5), (2,2), (2,3), (3,2), (3,3)\}$$

   (a) Verify R is an equivalence relation.
   (b) What is the equivalence class of 3, [3]?
   (c) What is the partition induced by R?

2. Consider the relation R on set $A = \{n \in \mathbb{Z} \mid 1 \leq n \leq 10\}$:

$$R = \{(x, y) \in A \times A \mid x = y \vee (x \text{ is odd} \wedge x < y)\}$$

   (a) Verify that R is a partial order.
   (b) What is the size of the largest chain in R?
   (c) What is the size of the largest antichain in R?
   (d) At least how many chains must any chain decomposition of R have?

3. Let A be a non-empty set and let R be a relation on A.

   (a) Prove or disprove that there exists an equivalence relation S on A such that $R \subseteq S$.
   (b) Prove or disprove that there exists a partial order S on A such that $S \subseteq R$.

4. Let $f : A \to A$ and $g : A \to A$ be functions on a set A. On HW5, you showed that if $g \circ f$ is a bijection and A **is finite**, then g and f are bijections.

   Prove that does not extend to the case when A is **is infinite**. That is, show that there exists an infinite set A and functions f, g on A such that $g \circ f$ is a bijection but at least one of g and f **are not** bijections.

5. Prove that the set of even integers is *countably infinite* by showing a bijection from the set of positive integers. (A surjection suffices to show it is countably infinite, but make it injective as well.)

6. Show that $|\mathbb{R}| = |\mathbb{R} \setminus \{0\}|$ by proving that the following function $f : \mathbb{R} \to \mathbb{R} \setminus \{0\}$ is a bijection. Recall that $\mathbb{Z}_{\geq 0}$ denotes the set of all non-negative integers.

$$f(x) = \begin{cases} x + 1 & \text{if } x \in \mathbb{Z}_{\geq 0} \\ x & \text{otherwise} \end{cases}$$

7. Let D be the set of odd integers. Show that $|\mathbb{R}| = |\mathbb{R} \setminus D|$ by proving there exists a bijection $f : \mathbb{R} \to \mathbb{R} \setminus D$. (Hint: Unlike the previous, we now have a countably infinite number of "holes" to avoid. Use the fact from problem 5 that the evens (and similarly, the odds) are countably infinite to map all integers differently than to themselves.)

8. Let A be a countably infinite set, and let R be any infinite relation on A; that is, $R \subseteq A \times A$. Prove that R is countably infinite by exhibiting a bijection from A to R.

2. Consider the relation R on set $A = \{n \in \mathbb{Z} \mid 1 \le n \le 10\}$:

$$R = \{(x,y) \in A \times A \mid x = y \vee (x \text{ is odd} \wedge x < y)\}$$

(a) Verify that R is a partial order. ref + Antisym + Trans

(b) What is the size of the largest chain in R?

(c) What is the size of the largest antichain in R?

(d) At least how many chains must any chain decomposition of R have?

(a) ① Reflexive: $\forall x \in A$, $x = x \Rightarrow xRx$ ✓

② Antisymmetric: Assume $(x,y) \in R$, $(y,x) \in R$ (WTS: $x=y$)

1) $x=y$. trivially done

2) $x \neq y$ $\quad$ x is odd, $x<y$

$\quad$ y is odd, $y<x$ $\quad$ $\times$

By 1), 2), $x=y$ ✓

③ Transitivity: Assume $(x,y) \in R$, $(y,z) \in R$ (WTS: $(x,z) \in R$)

1) $x=y$, $y=z$ $\Rightarrow$ $x=z$ $\Rightarrow$ $(x,z) \in R$

2) $x=y$, $y \neq z$ $\Rightarrow$ $x=y$, y odd, $y<z$

$\Rightarrow$ x odd, $x<z$ $\Rightarrow$ $(x,z) \in R$

3) $x \neq y$, $y=z$ $\Rightarrow$ x odd, $x<y=z$ $\Rightarrow$ $(x,z) \in R$

4) $x \neq y$, $y \neq z$ $\Rightarrow$ x odd $x<y$, y odd, $y<z$

$\Rightarrow$ x odd $x<y<z$ $\Rightarrow$ $(x,z) \in R$

(b) Every pair of distinct elements is comparable under R

$C = \{1,3,5,7,9,10\} \Rightarrow |C| = 6$ $\qquad$ $(o,m) \in R$

Proof of why $\nexists$ longer chain:

A has 5 odd and 5 even numbers

$\Rightarrow$ If $|C| \geq 7$, C has more than 2 even numbers

$\Rightarrow$ not comparable

(c) Any two distinct even numbers are incomparable

$D_1 = \{2, 4, 6, 8, 9\}$ $D_2 = \{2, 4, 6, 8, 10\}$

$\Rightarrow |D_1| = |D_2| = 5$

+ 10 to D_1 (9 R 10) + odd number to D_2 (o R 10)

+ odd number to D_1 (o R 9)

(d) Dilworth's thm: min # of chains in the chain decomposition

= the size of longest antichain = 5

(Each element in D_1 / D_2 forces its own chain)

4. Let $f : A \to A$ and $g : A \to A$ be functions on a set A. On HW5, you showed that if $g \circ f$ is a bijection and A **is finite**, then g and f are bijections.

Prove that does not extend to the case when A is **is infinite**. That is, show that there exists an infinite set A and functions f, g on A such that $g \circ f$ is a bijection but at least one of g and f **are not** bijections.

Classic example: $A = \mathbb{N} = \{0, 1, 2, \ldots\}$.

Define $f(n) = n+1$. $g(n) = \begin{cases} 0, & n = 0 \\ n-1, & n \geq 1 \end{cases}$

$\Rightarrow f$ not surjective. no n s.t. $f(n) = 0$

g not injective. $g(0) = g(1) = 0$

$\Rightarrow f$ & g not bijective,

but $(g \circ f)(n) = g(f(n)) = g(n+1) = (n+1) - 1 = n$ (identity func)

which is clearly bijective.

$\boxed{}$

6. Show that $|\mathbb{R}| = |\mathbb{R} \setminus \{0\}|$ by proving that the following function $f : \mathbb{R} \to \mathbb{R} \setminus \{0\}$ is a bijection. Recall that $\mathbb{Z}_{\geq 0}$ denotes the set of all non-negative integers.

$$f(x) = \begin{cases} x+1 & \text{if } x \in \mathbb{Z}_{\geq 0} \\ x & \text{otherwise} \end{cases}$$

Pf: ① f surjective: $\forall y \in \mathbb{R}\setminus\{0\}$, $\exists x \in \mathbb{R}$ s.t. $f(x) = y$

1) $y \notin \mathbb{Z}^+$, let $x = y \notin \mathbb{Z}^+$ and $x = y \neq 0$, $f(x) = x = y$

2) $y \in \mathbb{Z}^+$, let $x = y - 1 \in \mathbb{Z}_{\geq 0}$, $f(x) = x+1 = y-1+1 = y$

② f injective: Suppose $f(x) = f(y)$ WTS: $x = y$

1) $x, y \in \mathbb{Z}_{\geq 0}$, $x+1 = y+1 \Rightarrow x = y$

2) $x, y \notin \mathbb{Z}_{\geq 0}$, $x = y$

3) WLOG, $x \in \mathbb{Z}_{\geq 0}$, $y \notin \mathbb{Z}_{\geq 0}$, $x+1 = y$.

$\Rightarrow$ $x+1$ is a positive integer, but y is not $\times$!

$\Rightarrow x = y$

By ① & ②, f is bijective.

$\boxtimes$

8. Let A be a countably infinite set, and let R be any infinite relation on A; that is, $R \subseteq A \times A$. Prove that R is countably infinite by exhibiting a bijection from A to R.

Pf: A countably infinite $\Rightarrow$ $\exists$ bijection $b: \mathbb{Z}^+ \to A$

$\Rightarrow$ construct an infinite matrix whose (i,j)-entry is $(b(i), b(j)) \in A \times A$

$$
\begin{array}{c|cccc}
 & b(1) & b(2) & b(3) & \\
\hline
b(1) & 1,1 & 1,2 & 1,3 & \cdots \\
b(2) & 2,1 & 2,2 & 2,3 & \cdots \\
b(3) & 3,1 & 3,2 & 3,3 & \cdots \\
\vdots & \vdots & \vdots & \vdots &
\end{array}
$$

$\Rightarrow$ $A \times A$ countable.

Since $R \subseteq A$, R countable

Also, R infinite $\Rightarrow$ R countably infinite