Recall the proof structure for weak induction: To prove a infinite sequence of statements $P(m), P(m+1), P(m+2), \ldots$ for some integer m and predicate $P(\cdot)$:

- Prove $P(m)$.
- Prove that $P(n) \Rightarrow P(n+1)$ for all integers $n \geq m$.

Finally, conclude that, by weak induction, $P(m), P(m+1), P(m+2), \ldots$ are all true; that is, for all integers $n \geq m$, $P(n)$ is true.

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1. Prove or disprove: There is a real number x such that $x^3 < x < x^2$.

2. Prove or disprove: There is a real number x such that $x^4 < x < x^2$.

3. Prove by weak induction that, for any positive integer n, $\sum_{i=0}^{n} 2^i = 2^{n+1} - 1$. Note that the notation $\sum_{i=0}^{n} 2^i$ denotes the finite sum $2^0 + 2^1 + \ldots + 2^n$.

4. Prove by weak induction that, for any natural number n, $\sum_{i=1}^{n} i^2 = n(n+1)(2n+1)/6$. Note that the notation $\sum_{i=0}^{n} i^2$ denotes the finite sum $1^2 + 2^2 + 3^2 + \ldots + n^2$.

5. Prove by weak induction that, for any natural number $n \geq 12$, that n can be written as a sum of (non-negative multiples of) 4s and 5s. *[Hint: Involving cases may be helpful.]*

6. **Fun with Geometry**: Consider a collection of n lines in the plane that are in *general position*, meaning that no two lines are parallel and no three lines intersect at the same point. The n lines subdivide the plane into regions.

   - Prove that, for any collection of n lines in general position, the regions induced by the lines can be colored with only two colors so that any two adjacent regions have different colors.
   - **To think about later**: How many regions do the n lines create? Prove your answer using weak induction.

Review:

* Proof $\begin{cases} \forall: & \forall x, p(x) \\ \exists: & \exists x, p(x) \Rightarrow \text{easier, find an example.} \end{cases}$

Disproof $\begin{cases} \forall: & \text{Prove } \neg(\forall x, p(x)) = \exists x, \neg p(x) \Rightarrow \text{easier, find a counter-example} \\ \exists: & \text{Prove } \neg(\exists x, p(x)) = \forall x, \neg p(x) \end{cases}$

* Weak induction for $P(n)$, $n \geq m$

- Base case: Prove $P(m)$
- IH: Assume $P(n)$ holds.
- IS: Prove $P(n) \Rightarrow P(n+1)$ for $\forall n \geq m$
- Conclusion: $P(n)$ holds for $\forall n \geq m$

1. Prove or disprove: There is a real number x such that $x^3 < x < x^2$.

Pf: Find an example: $x = -2$ $(-8 < -2 < 4)$

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2. Prove or disprove: There is a real number x such that $x^4 < x < x^2$.

Pf: Method 1:

negation

WTS: $\forall x,\ x^4 \geq x$ or $x \geq x^2$ --- $(*)$

Direct proof: $x^4 - x \geq 0$ or $x^2 - x \leq 0$

$x(x^3-1) \geq 0$ or $x(x-1) \leq 0$

$x(x-1)(x^2+x+1) \geq 0$ or $x(x-1) \leq 0$

$\left(x+\frac{1}{4}\right)^2 + \frac{3}{4} > 0$

$x(x-1) \geq 0$

must hold

Method 2:

Prove $(*)$ by contradiction.

Suppose not, $\exists x$ s.t. $x^4 < x < x^2$

Since $x^4 < x$, we must have $x > 0$

(o/w $x \leq 0$, $x^4 \geq 0 \geq x$ which is a contradiction)

Thus, $x^3 < 1 < x$, but $x^3 > x$ for $x > 1$ $\therefore\!\!\times$

So $(*)$ holds

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5. Prove by weak induction that, for any natural number $n \geq 12$, that n can be written as a sum of (non-negative multiples of) 4s and 5s. *[Hint: Involving cases may be helpful.]*

Pf: Let $P(n)$ denote

"$\forall n \in \mathbb{N},\ n \geq 12,\ \exists a, b \in \mathbb{Z}^+$ s.t. $n = 4a + 5b$"

① Base case: Prove $P(12)$

Note that $12 = 4 \cdot 3$, $a=3$, $b=0$ so $P(12)$ holds.

② IH: Assume $P(n)$ holds for arbitrary $n \geq 12$

③ IS: By IH, $\exists a, b \in \mathbb{Z}^+$ s.t. $n = 4a + 5b$

We prove $P(n+1)$ by cases.

1) $a \geq 1$.

$$
\begin{aligned}
n+1 &= n + 5 - 4 \\
&= 4a + 5b + 5 - 4 \\
&= 4\underbrace{(a-1)}_{a' \geq 0} + 5\underbrace{(b+1)}_{b' \geq 0}
\end{aligned}
$$

$\Rightarrow P(n+1)$ holds

2) $a = 0$. $n = 5b$ for $b \geq 0$

$n+1 = 5b + 1 = 5b \cancel{+ 5 - 4}$ $+ 16 - 15 = 5(b-3) + 4 \cdot 4$

b' a'

Since $n \geq 12$, we must have $b \geq 3$. $b - 3 \geq 0$

$\Rightarrow P(n+1)$ holds

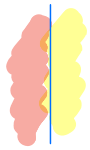
By weak induction, $P(n)$ holds for $\forall n \geq 12$, $n \in \mathbb{N}$ #

6. **Fun with Geometry**: Consider a collection of n lines in the plane that are in *general position*, meaning that no two lines are parallel and no three lines intersect at the same point. The n lines subdivide the plane into regions.

- Prove that, for any collection of n lines in general position, the regions induced by the lines can be colored with only two colors so that any two adjacent regions have different colors.
- **To think about later**: How many regions do the n lines create? Prove your answer using weak induction.

pf: Let $P(n)$ denote " _ _ _ "

① Base case:



$P(1)$ holds

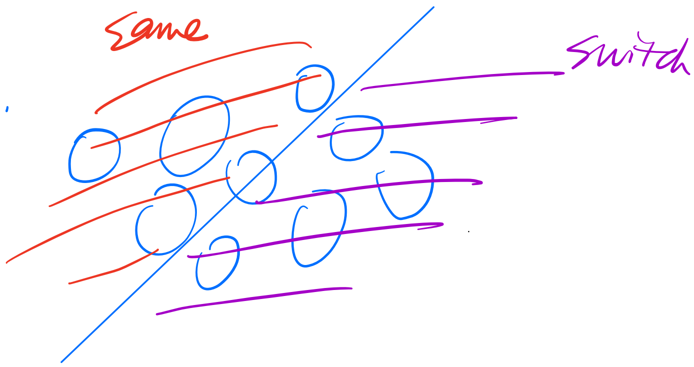
② IH: For arbitrary $n \in \mathbb{Z}^+$, assume $P(n)$ holds.

③ IS: We introduce the $(n+1)$-th line L.

Consider the following coloring method:

$$\begin{cases} \text{all new regions left of } L \Rightarrow \text{color the same before } L \\ \text{all new regions right of } L \Rightarrow \text{color different before } L \end{cases}$$

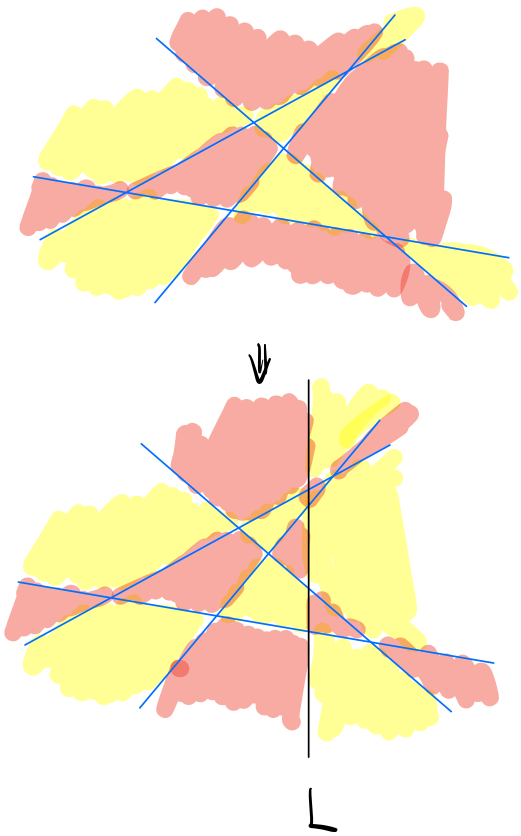
e.g.:



Check: Any two adjacent regions

$\begin{cases} \text{all on the left of } L & \checkmark \\ \text{all on the right of } L & \checkmark \\ \text{one left of } L, \text{ one right of } L & \checkmark \end{cases}$

e.g.,



$\Rightarrow$ $P(n+1)$ holds

By induction, $P(n)$ holds for $\forall n \in \mathbb{Z}^+$.

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(b) Recurrence relation: $R(n+1) = R(n) + (n+1)$

$$R(0) = 1$$
$$R(1) = 2$$
$$R(2) = 4$$
$$R(3) = 7$$
$$R(4) = 11$$
$$\vdots$$
$$R(n) = \frac{n(n+1)}{2} + 1$$

① Guess & verify + Induction

② Solve by recurrence relation