ECO4010 Tutorial 12

1. Consider the following simultaneous game of incomplete information where player i's type is t_i, which follows uniform distribution on $[0, x]$ for $x > 0$.

	Opera	*Fight*
Opera	$2 + t_1, 1$	$0, 0$
Fight	$0, 0$	$1, 2 + t_2$

   Construct a Bayesian Nash equilibrium. Hint: assume threshold values that divide the types into the two actions.

2. Find the symmetric equilibrium in 2-bidder "losers-pay" auction, where the highest bidder wins the object and the loser must pay his bid. The winner pays nothing. Using the general bidding strategies, find the seller's expected revenue in the 2-bidder, $U[0, 1]$ case.

3. (Optional) There are N bidders with their valuation v_i i.i.d. distributed on F($\cdot$), $v_i \in [\underline{v}, \overline{v}]$. Find the symmetric equilibrium in an all-pay auction directly.

1. Consider the following simultaneous game of incomplete information where player i's type is t_i, which follows uniform distribution on $[0, x]$ for $x > 0$.

		2	
		Opera	*Fight*
1	*Opera*	$2 + t_1, 1$	$0, 0$
	Fight	$0, 0$	$1, 2 + t_2$

Construct a Bayesian Nash equilibrium. Hint: assume threshold values that divide the types into the two actions.

Let k_i be the threshold for player i.

Suppose in BNE. 1 chooses O if $t_1 \geq k_1$, F o/w.

2 chooses F if $t_2 \geq k_2$, O o/w.

① Given k_1, 2's BR is F if

$$0 \cdot \left(1 - \frac{k_1}{x}\right) + (2+t_2) \cdot \frac{k_1}{x} \geq 1 \cdot \left(1 - \frac{k_1}{x}\right) + 0 \cdot \frac{k_1}{x}$$

$$t_2 \geq \frac{x - 3k_1}{k_1} \quad \Rightarrow k_2 = \frac{x - 3k_1}{k_1} \quad \text{--(1)}$$

② Given k_2, 1's BR is O if

$$(2+t_1) \frac{k_2}{x} + 0 \cdot \left(1 - \frac{k_2}{x}\right) \geq 0 \cdot \frac{k_2}{x} + 1 \cdot \left(1 - \frac{k_2}{x}\right)$$

$$t_1 \geq \frac{x - 3k_2}{k_2} \quad \Rightarrow k_1 = \frac{x - 3k_2}{k_2} \quad \text{--(2)}$$

By (1) & (2), $k_1 = k_2 = \frac{-3 + \sqrt{9 + 4x}}{2}$

$\Rightarrow$ BNE: 1 chooses O if $t_1 \geq \frac{-3+\sqrt{9+4x}}{2}$, F o/w.

2 chooses F if $t_2 \geq \frac{-3+\sqrt{9+4x}}{2}$, O o/w.

2. Find the symmetric equilibrium in 2-bidder "losers-pay" auction, where the highest bidder wins the object and the loser must pay his bid. The winner pays nothing. Using the general bidding strategies, find the seller's expected revenue in the 2-bidder, $U[0,1]$ case.

① Assume the bidding function $\beta(\cdot)$ is strictly increasing and differentiable, and $v \sim G(\cdot)$.

$\Rightarrow \max \quad P(\text{win}) \cdot v_i + P(\text{lose}) \cdot (-b_i)$

WLOG, consider bidder 1:

$$P(\text{win}) = P(b_1 > b_2) = P(\beta(v_1') > \beta(v_2))$$
$$= P(v_1' > v_2) = G(v_1') = G(\beta^{-1}(b_1))$$

$$\Rightarrow \max_{v_1'} \quad G(v_1') \cdot v_1 + (1 - G(v_1'))(-\beta(v_1'))$$

FOC: $g(v_1')v_1 + g(v_1')\beta(v_1') - (1 - G(v_1'))\beta'(v_1') = 0$

In BNE, $v_1' = v_1$

$$\Rightarrow \quad g(v_1)v_1 + g(v_1)\beta(v_1) - (1 - G(v_1))\beta'(v_1) = 0$$
$$g(v_1)v_1 = -g(v_1)\beta(v_1) + (1 - G(v_1))\beta'(v_1)$$
$$= [(1 - G(v_1))\beta(v_1)]'$$

$$\Rightarrow \int_{\underline{v}}^{v_1} x\, g(x)\,dx = \beta(v_1)(1 - G(v_1)) + C$$

It is easy to check $C = 0$ since $\beta(\underline{v}) = 0$

$$\Rightarrow \quad \beta(v_1) = \frac{\int_{\underline{v}}^{v_1} x\, g(x)\,dx}{1 - G(v_1)}$$

② when $G = U$, $\underline{v} = 0$ $\Rightarrow V_1 \sim U[0,1]$

$\Rightarrow G(v_1) = v_1$, $g(x) = 1$

$$\Rightarrow \beta(v_1) = \frac{\int_0^{v_1} x \cdot 1 \, dx}{1 - v_1} = \frac{v_1^2}{2(1-v_1)}$$

BNE: $\beta(v_i) = \dfrac{v_i^2}{2(1-v_i)}$. $\forall i \in \{1,2\}$

$\Rightarrow$ Seller's expected revenue

$$= E_{v_i}\left[P(v_1 < v_2)\beta(v_1) + P(v_2 < v_1)\beta(v_2) \right]$$

$$= E_{v_i}\left[\sum_{i=1,2} P(v_i < v_{-i}) \cdot \beta(v_i) \right]$$

$$= \int_0^1 \left(\sum_{i=1,2} P(v_i < v_{-i}) \cdot \beta(v_i) \right) g(v_i) \, dv_i$$

$$= \sum_{i=1,2} \int_0^1 P(v_i < v_{-i}) \cdot \beta(v_i) \, g(v_i) \, dv_i$$

$$= \sum_{i=1,2} \int_0^1 \cancel{(1 - \underbrace{G(v_i)}_{= v_i})} \cdot \frac{v_i^2}{\cancel{2(1-v_i)}} \, 1 \, dv_i$$

$$= \sum_{i=1,2} \frac{1}{6} v_i^3 \Big|_0^1$$

$$= \frac{1}{3}$$

3. (Optional) There are N bidders with their valuation v_i i.i.d. distributed on F(·), $v_i \in [\underline{v}, \overline{v}]$. Find the symmetric equilibrium in an all-pay auction directly.

All-pay auction: Everyone pays what he bids regardless of winning or losing.

Assume the bidding function $\beta^{AP}(\cdot)$ is strictly increasing & differentiable.

$\Rightarrow$ $\beta^{AP}(v)$ is the expected payment. $\beta^{AP}(\underline{v}) = 0$

WLOG, consider bidder 1:

$$P(\text{win}) = P(b_1 > \max_{j \neq 1} b_j) = P(\beta(v_1') > \beta(\max_{j \neq 1} v_j))$$

$$= P(v_1' > \max_{j \neq 1} v_j) \overset{i.i.d.}{=} P(v_1' > v_2) \cdots P(v_1' > v_N)$$

$$= F^{N-1}(v_1')$$

Assume $F^{N-1}(\cdot) \triangleq G(\cdot)$, $\max_{j \neq 1} v_j \sim G$

$$\Rightarrow \max_{v_1'} \; P(\text{win}) \cdot v_1 - \beta^{AP}(v_1')$$

$$= G(v_1') \cdot v_1 - \beta^{AP}(v_1')$$

FOC: $g(v_1') v_1 - [\beta^{AP}(v_1')]' = 0$

In BNE, $v_1' = v_1$

$$\Rightarrow g(v_1) \cdot v_1 - [\beta^{AP}(v_1)]' = 0$$

$$\beta^{AP}(v_1) = \int_{\underline{v}}^{v_1} g(y) y \, dy + C$$

Since $\beta^{AP}(\underline{v}) = 0$, $C = 0$

$$\Rightarrow \beta^{AP}(v_1) = \int_{\underline{v}}^{v_1} g(y)\, y\, dy$$

$$\Rightarrow \text{BNE: } \beta^{AP}(v_i) = \int_{\underline{v}}^{v_i} g(y)\, y\, dy \quad \forall i \in \{1, 2, \ldots N\}$$

Ex 1. Suppose there are n bidders with v_i i.i.d distributed on $F(\cdot)$. $v_i \in [\underline{v}, \bar{v}]$. Show that $b_i(v_i) = v_i$ is a BNE in SPA.

pf: Suppose the bidding function $\beta(\cdot)$ strictly $\uparrow$ & differentiable, $F^{N-1} \overset{d}{=} G$.

WLOG, for bidder 1:

$$P(\text{win}) = P(b_1 > \max_{j \neq 1} b_j) = P(\beta(v_1') > \beta(\max_{j \neq 1} v_j))$$

$$= P(v_1' > \max_{j \neq 1} v_j) = F^{N-1}(v_1') = G(v_1')$$

$$\Rightarrow \max_{j \neq 1} v_j \sim G$$

$$\max_{v_1'} P(\text{win})\left(v_1 - E\left[\beta(\max_{j \neq 1} v_j) \,\middle|\, \text{win}\right]\right)$$

$$= G(v_1')\left(v_1 - E\left[\beta(\max_{j \neq 1} v_j) \,\middle|\, v_1' > \max_{j \neq 1} v_j\right]\right)$$

$$= G(v_1')\left(v_1 - \frac{\int_{\underline{v}}^{v_1'} \beta(y) g(y)\, dy}{G(v_1')}\right) \text{ by } \max_{j \neq 1} v_j \sim G$$

$$= G(v_1') \cdot v_1 - \int_{\underline{v}}^{v_1'} \beta(y) g(y)\, dy$$

FOC: $g(v_1') v_1 - \beta(v_1') g(v_1') = 0$

In BNE, $v_1' = v_1$

$$\Rightarrow \quad g(v_1)\, v_1 - \beta(v_1)\, g(v_1) = 0$$

$$\beta(v_1) = v_1$$

$\Rightarrow$ BNE: $\beta(v_i) = v_i, \quad \forall i \in \{1, 2, \cdots n\}$.

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Ex 2 (War of Attrition, PS5 Q2)

Suppose 2 players simultaneously choose $s_i \in \mathbb{R}_+$.
The player with larger s_i wins and gets
$\theta_i \in \mathbb{R}_+$. θ_i i.i.d. $\sim F[0, \bar{\theta}]$. Both pay
smaller s_j. Solve for the BNE.

Suppose $s(\cdot) \uparrow$ & diff.

WLOG, consider player 1:

$$p(\text{win}) = P(s_1 > s_2) = P(s(\theta_1') > s(\theta_2))$$
$$= P(\theta_1' > \theta_2) = F(\theta_1')$$

$$\Rightarrow \max_{\theta_1'} \quad P(\text{lose})(-s(\theta_1')) + P(\text{win})\, E[\theta_1 - s(\theta_2) \mid \underbrace{\text{win}}_{\theta_1' > \theta_2}]$$

$$= -s(\theta_1') \cdot (1 - F(\theta_1')) + \cancel{F(\theta_1')} \frac{\int_0^{\theta_1'} (\theta_1 - s(y)) f(y)\, dy}{\cancel{F(\theta_1')}}$$

$$= -s(\theta_1')(1 - F(\theta_1')) + \int_0^{\theta_1'} (\theta_1 - s(y)) f(y)\, dy$$

FOC: $-s'(\theta_1')(1 - F(\theta_1')) + s(\theta_1') f(\theta_1')$

$$+ (\theta_1 - s(\theta_1')) f(\theta_1') = 0$$

$$-s'(\theta_1')(1-F(\theta_1')) + \cancel{s(\theta_1')f(\theta_1')}$$

$$+\theta_1 f(\theta_1') - \cancel{s(\theta_1')f(\theta_1')} = 0$$

In BNE: $\theta_1' = \theta_1$

$$\Rightarrow -s'(\theta_1)(1-F(\theta_1)) + \theta_1 f(\theta_1) = 0$$

$$s'(\theta_1) = \frac{\theta_1 f(\theta_1)}{1-F(\theta_1)}$$

$$s(\theta_1) = \int_0^{\theta_1} \frac{x f(x)}{1-F(x)}\, dx$$

$$\Rightarrow \text{BNE}: s_i = s(\theta_i) = \int_0^{\theta_i} \frac{x f(x)}{1-F(x)}\, dx$$