# COMPSCI 230 Problem Sets - Logic

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## 1 Propositional Logic

1. Let $A$ and $B$ be propositions. Use truth tables to prove the de Morgan rules:
   (a) $\neg(A \wedge B) \iff (\neg A \vee \neg B)$
   (b) $\neg(A \vee B) \iff (\neg A \wedge \neg B)$
2. Prove that the following are tautologies:
   (a) $((x \Rightarrow \neg y) \Rightarrow z) \iff ((x \wedge y) \vee z)$
   (b) $(x \wedge (x \Rightarrow y)) \Rightarrow y$
3. Show that $\neg(p \vee (\neg p \wedge q))$ and $\neg p \wedge \neg q$ are logically equivalent.
4. Show that $(p \wedge q) \Rightarrow r$ and $(p \Rightarrow r) \wedge (q \Rightarrow r)$ are not logically equivalent.

   *Hint: Find an assignment of truth values that makes one of these propositions true and the other false.*
5. The proposition $p$ NOR $q$ is true when both $p$ and $q$ are false, and it is false otherwise. Let $p$ NOR $q$ be denoted by $p \downarrow q$ (called Peirce arrow after C.S. Peirce). Show that
   (a) use truth table to prove $p \downarrow q$ is logically equivalent to $\neg(p \vee q)$
   (b) $p \downarrow p \iff \neg p$
   (c) $(p \downarrow q) \downarrow (p \downarrow q) \iff (p \vee q)$
6. Let $p, q, r$ be the propositions:
   - $p$: You get an A on the final exam.
   - $q$: You do every exercise in this book.
   - $r$: You get an A in this class.

   Write these propositions using $p, q, r$ and logical connectives (including negations).

   (a) Getting an A on the final and doing every exercise in this book is sufficient for getting an A in this class.
   (b) You will get an A in this class if and only if you finish at least one of them: do every exercise in this book, get an A on the final exam.

7. Given two assumptions:

   - Logic is difficult or not many students like logic.
   - If mathematics is easy, then logic is not difficult.

   Determine whether each of the following are valid conclusions of these assumptions:

   (a) That mathematics is not easy, if many students like logic.

   (b) That not many students like logic, if mathematics is not easy.

   (c) That mathematics is not easy or logic is difficult.

   (d) That logic is not difficult or mathematics is not easy.

   (e) That if not many students like logic, then either mathematics is not easy or logic is not difficult.

   by translating these assumptions into statements involving propositional variables and logical connectives, e.g., $d$ for "logic is difficult"; $s$ for "many students like logic"; and $e$ for "mathematics is easy."

# 2 Predicate Logic

1. Prove that the following are tautologies:

   (a) $(\exists x P(x) \Rightarrow \exists x Q(x)) \Rightarrow \exists x(P(x) \Rightarrow Q(x))$

   (b) $(\exists x(P(x) \Rightarrow Q(x))) \iff (\forall x(P(x) \Rightarrow (\exists x Q(x))))$

2. Let $C(x)$ be the statement $x$ has a cat, let $D(x)$ be the statement $x$ has a dog, and let $F(x)$ be the statement $x$ has a ferret. Express each of these statements in terms of $C(x), D(x), F(x)$, quantifiers, and logical connectives. Let the domain consist of all students in your class.

   (a) A student in your class has a cat, a dog, and a ferret.

   (b) No student in your class has a cat, a dog, and a ferret.

   (c) For each of the three animals, cats, dogs, and ferrets, there is a student in your class who has this animal as a pet.

3. Write the proposition "There is at most one ball in every urn" using logical connectives and quantifiers. Use the symbols $b_1, b_2$ for balls, $u_1, u_2$ for urns and $IN(b, u)$ for "ball $b$ is in urn $u$".

4. Use predicates, quantifiers, logical connectives, and mathematical operators to translate the following statements. For instance, "The sum of two positive integers is always positive" into $\forall x \in \mathbb{Z}, \forall y \in \mathbb{Z}, ((x > 0) \wedge (y > 0) \Rightarrow (x + y > 0))$.

(a) Every real number except zero has a multiplicative inverse. (A multiplicative inverse of a real number $x$ is a real number $y$ such that $xy = 1$.)
(b) Every positive integer is the sum of the squares of four integers.
(c) The sum of the squares of two integers is greater than or equal to the square of their sum.

5. (a) Find a common domain for the variables $x, y$, and $z$ for which the statement $\forall x \forall y((x \neq y) \Rightarrow \forall z((z = x) \vee (z = y)))$ is true and another domain for which it is false.
   (b) Let $Q(x, y)$ be the statement $x + y = 0$. Are the following quantifications $\exists y \forall x Q(x, y)$ and $\forall x \exists y Q(x, y)$ respectively, true or false, where the domain for all variables consists of all real numbers.

6. Express the negations of each of these statements so that all negation symbols immediately precede the predicates.
   (a) $\exists x \exists y (Q(x, y) \iff Q(y, x))$
   (b) $\forall y \exists x \exists z (T(x, y, z) \vee Q(x, y))$

7. Let $P(x, y)$ be a predicate with variables $x, y \in \mathbb{Z}$. Let $A = \forall x \exists y : P(x, y)$ and $B = \exists y \forall x : P(x, y)$.
   (a) Let $P(x, y)$ be $x = y$. Show that $A \Rightarrow B$ is false.
   (b) Show that $B$ implies $A$.

8. Let $n \geq 1$ be an even integer and $x_1, x_2, \ldots, x_n$ be statements. Suppose that $(x_1 \Rightarrow x_2) \wedge (x_2 \Rightarrow x_3) \wedge \cdots \wedge (x_{n-1} \Rightarrow x_n) \wedge (x_n \Rightarrow x_1)$ is true. Show that

$$x_1 \operatorname{xor} x_2 \operatorname{xor} \ldots \operatorname{xor} x_n = \text{false},$$

   where `xor` denotes exclusive or.

9. For $x, y \in \mathbb{Z}$, let predicate $P(x, y) = (|x| < |y|)$ or $(|x| = |y|$ and $x \leq y)$.
   (a) Show that for $x \in \mathbb{Z}$, $P(x, x)$ is true.
   (b) Show that for $x, y \in \mathbb{Z}$, $x = y$ if and only if $P(x, y) \wedge P(y, x)$.
   (c) Show that for $x, y, z \in \mathbb{Z}$, $P(x, y) \wedge P(y, z)$ implies $P(x, z)$.
   (d) (Hard) Show that there exists a predicate $R(x, y)$ for $x, y \in \mathbb{Q}$ such that the following properties hold simultaneously:
      - For $x \in \mathbb{Q}$, $R(x, x)$ is true;
      - For $x, y \in \mathbb{Q}$, $x = y$ if and only if $R(x, y) \wedge R(y, x)$;
      - For $x, y, z \in \mathbb{Q}$, $R(x, y) \wedge R(y, z)$ implies $R(x, z)$;
      - For an arbitrary nonempty subset $B \subseteq \mathbb{Q}$ of rational numbers, there exists a unique element $x^* \in B$ such that for every $y \in B$ the predicate $R(x^*, y)$ is true.