ECO4010 Tutorial 2

1. This problem helps you review the basic concepts of utility representation and continuous preferences. Prove or disprove the following:

   (a) If $f : \mathbb{R} \to \mathbb{R}$ is a strictly increasing function and $u : X \to \mathbb{R}$ is a utility function representing the preference relation $\succsim$, then the function $v : X \to \mathbb{R}$ defined by $v(x) = f(u(x))$ is also a utility function representing $\succsim$.

   (b) If both u and v represent $\succsim$, then there is a strictly monotonic function $f : \mathbb{R} \to \mathbb{R}$ such that $v(x) = f(u(x))$.

   (c) A continuous preference relation can be represented by a discontinuous utility function. [See HW2 Q4 as the same kind of problems.]

   (d) In the case of $X = \mathbb{R}$, the preference relation that is represented by the discontinuous function $u(x) = \lfloor x \rfloor$ (the largest integer n such that $x \geq n$) is not a continuous relation.

   (e) In the case of $X = \mathbb{N}$, any preference relation can be represented by a utility function that returns only integers as values.

2. Consider the following UMP with the utility function in a three-good setting:

$$u(x_1, x_2, x_3) = (x_1 - \beta_1)^{\alpha_1}(x_2 - \beta_2)^{\alpha_2}(x_3 - \beta_3)^{\alpha_3} \quad \text{s.t. } p \cdot x \leq w,$$

   where $\beta_i \geq 0$, $\alpha_i > 0$ for all i, $p \gg 0$, and $w > 0$.

   (a) Explain why there is no loss of generality to assume that $\alpha_1 + \alpha_2 + \alpha_3 = 1$.

   (b) Write down the FOC for the UMP and derive the consumer's Walrasian demand and the indirect utility function.

   (c) Verify that the derived functions satisfy the following properties:

      (1) Walrasian demand $x(p, w)$ is homogeneous of degree zero and satisfies Walras' Law;

      (2) Indirect utility $v(p, w)$ is homogeneous of degree zero;

      (3) $v(p, w)$ is strictly increasing in w and non-increasing in p_l for all l;

      (4) $v(p, w)$ is continuous in p and w.

   (d) Using different prices and incomes (i.e., p and p', w and w') and Walrasian demand to derive a condition (with some inequalities) that violates WARP.

3. Assume that $x(p,w)$ is differentiable in $(p,w) \gg 0$ and that $x(p,w) \gg 0$.

   (a) Show that whenever a consumer is maximizing a monotone utility function then, at any $(p,w) \gg 0$, there must exist some good k (which may depend on (p,w)) for which

   $$\frac{\partial x_k(p,w)}{\partial w} > 0.$$

   In other words, there must be some good k whose demand increases with income. [Hint: Walras' Law.]

   Now further assume a consumer has the additive utility function

   $$U(x_1, ... x_l) = \sum_{i=1}^{l} u_i(x_i),$$

   where u_i is C^2, with $u_i^{'}(x_i) > 0$ and $u_i^{''}(x_i) < 0$ for all $x_i > 0$ and all i.

   (b) Show that for the additive utility function, the marginal utility of income diminishes with income, i.e.,

   $$\frac{\partial^2 v}{\partial w^2}(p,w) < 0 \text{ for all } (p,w) \gg 0.$$

   [Hint: Recall that the marginal utility of income equals the Lagrange multiplier.]

   (c) Show that for the additive utility function, we may strengthen the conclusion in (a) to the following:

   $$\frac{\partial x_i(p,w)}{\partial w} > 0 \text{ for all } (p,w) \gg 0 \text{ and for every good } i.$$

4. (Optional) Show that a preference relation $\succsim$ on $[0,1]$ is continuous and strictly convex iff there exists a continuous utility function u representing $\succsim$ and a point $x^* \in [0,1]$ such that u is strictly increasing on $[0, x^*]$ and strictly decreasing on $[x^*, 1]$. Graphically, one example is