In lecture, we described how to "count" the following sets of objects:

- *Permutations* of n (out of n) items, where $n \geq 0$: $n!$
  n-length sequences of all n items (order matters, no repetition)
- *Permutations* of k out of n items, where $0 \leq k \leq n$: $\frac{n!}{(n-k)!}$
  k-length sequences whose elements are out of the n items (order matters, no repetition)
- *Combinations* of k (out of n) items, where $0 \leq k \leq n$: $\binom{n}{k} = \frac{n!}{(n-k)!k!}$
  Subsets of size k of a set of size n (order does not matter, no repetition)
- *Distributions* of n identical items among k entities, where $n + k \geq 0$: $\binom{n+k-1}{k-1}$
  "Stars and bars": Distinct sequences of n stars ($\star$) and $k - 1$ bars (|) (we only care about the number of items distributed to each entity)

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1. Prove the following equalities by "counting two ways," meaning that for a fixed set of objects of your design (think sequences, combinations, distributions of items above), express its cardinality as both the left- and right-hand sides of the equality. That is, for each of the given equalities of the form $L = R$ below, define a set S such that $|S| = L$ and $|S| = R$, then finally conclude that indeed $L = R$.

   (a) $\binom{n}{k} = \binom{n}{n-k}$ for any integers n, k with $n \geq k \geq 0$.

   (b) $\binom{n}{k} = \binom{n-1}{k-1} + \binom{n-1}{k}$ for any integers n, k with $n > k > 1$.

   (c) $\sum_{k=0}^{n} \binom{n}{k} = 2^n$ any integers n, k with $n \geq k \geq 0$.

2. Charlie owns a pizza parlor that offers 9 distinct toppings.

   (a) How many different pizzas can Charlie make with exactly 3 distinct toppings?

   (b) How many pizzas can be made with any number of distinct toppings?

   (c) What is the total number of pizzas that with any number of distinct toppings between 3 and 6.

3. A (7-digit) phone number is *non-increasing* if the digits are in non-increasing order. For example, 996-5442 is a non-increasing phone number, while 996-5224 is not.

   (a) How many distinct phone numbers are there?

   (b) How many non-increasing phone numbers are there?

   (c) How many non-increasing phone numbers start with an 8?

   (d) What is the probability that a phone number chosen uniformly at random is non-increasing ***and*** ends with a 5?

   (e) What is the probability that a phone number chosen uniformly at random is non-increasing ***or*** ends with a 5 (inclusive)?

4. Kate bought an assortment of 17 chocolate-covered treats — 6 strawberries, 4 pretzels, and 7 cherries — to divide up among the three grad TAs.

   (a) How many ways can Kate divide up the treats?

   (b) To divide them up, Kate rolls a fair "three-sided" die (d3) independently for each treat. (We will discuss independence in detail next lecture, but for now: Assume that any *sequence* of the 17 outcomes are equally likely.) If the outcome of the die roll is 1, 2, or 3, then Alex, Erin, or Kevin receives the treat, respectively. What is the probability that:

      i. Kevin gets no treats?

      ii. Kevin gets exactly 2 treats?

      iii. Erin gets exactly 1 or 3 pretzels?

      iv. Alex receives exactly 2 fruits and a pretzel?

Counting & Prob.

* $P(A \cup B) = P(A) + P(B) - P(A \cap B)$

Independent Events A, B

$$\underline{P(A \cap B) = P(A) \cdot P(B)}$$

Disjoint Events. A, B

$$\underline{P(A \cap B) = 0}$$

$$\Rightarrow P(A \cup B) = P(A) + P(B)$$

* $P(\bar{A}) = 1 - P(A)$

$P(A \cup B) \leq P(A) + P(B)$ ("=" holds when disjoint)

$P(A \setminus B) = P(A) - P(A \cap B)$

$P(A) \leq P(B)$ if $A \subseteq B$

* Useful Identities:

① $\binom{n}{k} = \binom{n}{n-k}$

② $\binom{n}{k} + \binom{n}{k+1} = \binom{n+1}{k+1}$

③ $\sum_{k=0}^{n} \binom{n}{k} = 2^n$

2. Charlie owns a pizza parlor that offers 9 distinct toppings.

(a) How many different pizzas can Charlie make with exactly 3 distinct toppings?

(b) How many pizzas can be made with any number of distinct toppings?

(c) What is the total number of pizzas that with any number of distinct toppings between 3 and 6.

(a) $\binom{9}{3} = \frac{9!}{3!(9-3)!} = \frac{9!}{3!6!} = \frac{9\times 8\times 7}{3\times 2\times 1} = 84$

(b) Method 1:

$|T| = 9 \Rightarrow |2^T| = 2^9 = 512$

Method 2:

$\binom{9}{0} + \binom{9}{1} + \binom{9}{2} + \cdots + \binom{9}{9}$

$= \sum_{k=0}^{n} \binom{9}{k} = 2^9$

(c) $\binom{9}{3} + \binom{9}{4} + \binom{9}{5} + \binom{9}{6}$

$= \binom{10}{4} + \binom{10}{6} = \binom{10}{4} + \binom{10}{4}$

$= 2 \times \frac{10!}{4!\times 6!} = 2\times \frac{10\times 9\times 8\times 7}{4\times 3\times 2\times 1} = 420$

4. Kate bought an assortment of 17 chocolate-covered treats — 6 strawberries, 4 pretzels, and 7 cherries — to divide up among the three grad TAs.

   (a) How many ways can Kate divide up the treats?

   (b) To divide them up, Kate rolls a fair "three-sided" die (d3) independently for each treat. (We will discuss independence in detail next lecture, but for now: Assume that any *sequence* of the 17 outcomes are equally likely.) If the outcome of the die roll is 1, 2, or 3, then Alex, Erin, or Kevin receives the treat, respectively. What is the probability that:

      i. Kevin gets no treats?
      ii. Kevin gets exactly 2 treats?
      iii. Erin gets exactly 1 or 3 pretzels?
      iv. Alex receives exactly 2 fruits and a pretzel?

(a) $\binom{n+k-1}{k-1}$ distribution ways for each type.

$$\Rightarrow \binom{6+3-1}{3-1}\cdot\binom{4+3-1}{3-1}\cdot\binom{7+3-1}{3-1}$$

$$= \binom{8}{2}\cdot\binom{6}{2}\binom{9}{2} = 28\times 15\times 36 = 15120$$

(b) i. $P(K \text{ gets none}) = \left(\frac{2}{3}\right)^{17}$

ii. $P(K \text{ gets } 2) = \binom{17}{2}\left(\frac{1}{3}\right)^2\left(\frac{2}{3}\right)^{15}$

binomial distribution $(n=17,\ p=\frac{1}{3})$

iii. $P(E \text{ gets } 1 \text{ or } 3 \text{ pretzels})$

$= P(E \text{ gets } 1\ p) + P(E \text{ gets } 3\ p)$

$= \binom{4}{1}\left(\frac{1}{3}\right)\left(\frac{2}{3}\right)^3 + \binom{4}{3}\left(\frac{1}{3}\right)^3\left(\frac{2}{3}\right)$

$$= \frac{4\times 8}{3^4} + \frac{4\times 2}{3^4} = \frac{40}{3^4} = \frac{40}{81}$$

iv. fruits : strawberries + cherries $(6+7=13)$

pretzel : 4

$P(A \text{ gets } 2f \text{ \& } 1p)$.

$= P(A \text{ gets } 2f) \times P(A \text{ gets } 1p)$

$= \binom{13}{2}\left(\frac{1}{3}\right)^2\left(\frac{2}{3}\right)^{11} \times \binom{4}{1}\left(\frac{1}{3}\right)\left(\frac{2}{3}\right)^3$.