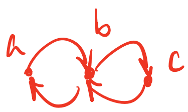
Recall that a graph G is defined as an ordered pair (V, E) where V is a finite set of ***vertices*** and E is a set of ***edges***. If G is ***undirected***, then $E \subseteq \{\{a, b\} \in 2^V \mid a \neq b\}$ and each edge is a two-element subset of V, and if G is ***directed*** then $E \subseteq \{(a, b) \in V \times V \mid a \neq b\}$ and each edge is an ordered pair of distinct vertices. It is convention to refer to $|V|$ as n and $|E|$ as m.

A pair of vertices u, v is *strongly-connected* if there is a path from u to v and a path from v to u. A directed graph $G = (V, E)$ is *strongly-connected* if each pair of vertices is strongly-connected. In lecture, we defined the strongly-connected components (SCCs) of a directed graph by the blocks of the partition induced by the strongly-connected relation.

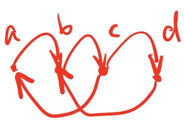
Equivalently, they are maximal strongly-connected subgraphs of G. That is, for any strongly-connected subgraph C of G, if there exists no subgraph C' of G for which C is a subgraph of C' then C is a maximal strongly-connected subgraph of G, and thus C is an SCC of G. (Informally, if there is no "larger" strongly-connected subgraph C' of G that contains C within it, C is an SCC.)



1. The following problems ask you to draw strongly-connected directed graphs.

   a, c ; b

   (a) Draw a strongly-connected directed graph so that there exists two distinct vertices u, v such that any path from u to v and any path from v to u share a vertex (other than u or v).



   a, d ; (b, c)

   (b) Draw a strongly-connected directed graph so that there exists two distinct vertices u, v such that any path from u to v and any path from v to u share **an edge**.

   (c) Draw a strongly-connected directed graph with no cycles.

2. For any directed graph $G = (V, E)$, if there is a path from a vertex u to a distinct vertex v and a path from v to u, then there is a cycle containing u. (Note, this is not an immediate consequence of any results from lecture.)

   [Hint: Consider a proof by induction on the combined lengths of the paths (by number of edges).]

3. A *tournament* graph is a directed graph where exactly one of (u, v) or (v, u) is in E for every pair of distinct vertices $u, v \in V$. A *champion* of the graph is a vertex from which every other vertex is reachable by a path of length at most two from the champion. That is, every other vertex is an out-neighbor of the champion, or it is the out-neighbor of an out-neighbor of the champion.

   (a) How many different tournament graphs are there with n vertices? How does this compare to the number of directed graphs with n vertices? (While we are at it, how many undirected graphs are there with n vertices?)

   (b) Prove that any vertex in G with largest out-degree is a champion.
   [Hint: Consider a proof by contradiction.]

   (c) Prove that any tournament has a Hamiltonian path; that is, a path that visits every vertex exactly once.
   [Hint: Consider a proof by induction on the number of vertices. (The number of edges is determined exactly by the number of vertices, so inducting on edges is similar but arguably more complicated for unclear gains.)]

4. Let $G = (V, E)$ be a directed **acyclic** graph (DAG), and let R be the relation on V where uRv if and only if there is a path in G from u to v, for any vertices $u, v \in R$. That is, uRv if and only if v is *reachable* from u.

   (a) Show that G has $|V|$ strongly-connected components.

   (b) Show that G has a ***source*** vertex, a vertex with in-degree 0. By symmetry, show that G has a ***sink*** vertex, a vertex with out-degree 0.
   [Hint: Consider a maximal path in G; that is, a path P for which no other path P' contains P.]

   (c) One can show that R is a weak partial order, which you may assume for this problem. (Consider verifying this for yourself!)

   Recall that a partial order is *total* if, for any two distinct elements, one element is related to the other (but not vice versa). Prove that R is a *total* order if and only if G contains a Hamiltonian path. That is, prove that for any pair of distinct vertices $u, v \in V$, either uRv or vRu.

   [Hint: Use induction on the number of vertices for the forward direction.]

2. For any directed graph $G = (V, E)$, if there is a path from a vertex u to a distinct vertex v and a path from v to u, then there is a cycle containing u. (Note, this is not an immediate consequence of any results from lecture.)

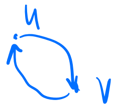
[Hint: Consider a proof by induction on the combined lengths of the paths (by number of edges).]

pf: (Don't fix $u, v \Rightarrow$ arbitrary)

$P(z)$: If $\exists$ a path $u \to v$, and a path $v \to u$ with combined length z, $\exists$ cycle containing u.

Note $z \geq 2$ for distinct vertices u, v.

Base: $z = 2$.



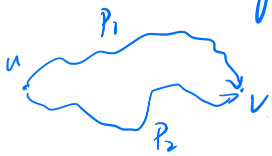
(u,v), (v,u)

$\Rightarrow u, v, u$ is a cycle in G including u.

IH: $P(k)$ holds for $2 \leq k < z$.

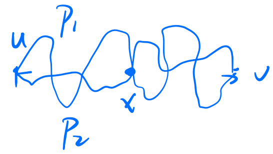
IS: Suppose $\exists P_1 : u \to v$, $P_2 : v \to u$ with combined length z.

① No vertex (except u,v) shared between P_1, P_2.



$\Rightarrow P_1 . P_2$ forms a cycle including u.

② $\exists$ vertex x shared between P_1, P_2



Consider subgraph P_1' of $P_1 : u \to x$, P_2' of $P_2 : x \to u$

$\Rightarrow \exists P_1', P_2'$ with combined length $< z$ (since x is distinct from u,v)

By IH, $\exists$ cycle containing u.

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4. Let $G = (V, E)$ be a directed **acyclic** graph (DAG), and let R be the relation on V where uRv if and only if there is a path in G from u to v, for any vertices $u, v \in R$. That is, uRv if and only if v is *reachable* from u.

(a) Show that G has $|V|$ strongly-connected components.

(b) Show that G has a ***source*** vertex, a vertex with in-degree 0. By symmetry, show that G has a ***sink*** vertex, a vertex with out-degree 0.
[Hint: Consider a maximal path in G; that is, a path P for which no other path P′ contains P.]

(c) One can show that R is a weak partial order, which you may assume for this problem. (Consider verifying this for yourself!)

Recall that a partial order is *total* if, for any two distinct elements, one element is related to the other (but not vice versa). Prove that R is a *total* order if and only if G contains a Hamiltonian path. That is, prove that for any pair of distinct vertices $u, v \in V$, either uRv or vRu.

[Hint: Use induction on the number of vertices for the forward direction.]

(a) pf: Consider SCC: distinct u, v with $u \to v$, $v \to u$.
$\Rightarrow$ two paths form a cycle. contradicting DAG
Thus, no two distinct vertices in SCC
$\Rightarrow$ Only SCC: $\{v\}$. $\forall v \in V$
$\Rightarrow$ $|V|$ SCC
#

(b) pf: Suppose $\nexists$ source in G.
Let $P = v_1, \dots, v_k$ be a maximal path in G.
Since v_1 is not a source, it has in-deg ≥ 1
① $\exists v_i$, $2 \leq i \leq k$ s.t. $v_i \to v_1$ $\Rightarrow$ contradict acyclicity
② $\exists u$ s.t. $u \neq v_i$, $2 \leq i \leq k$ s.t. $u \to v_1$
$\Rightarrow P' = u, v_1, \dots, v_k$ is longer than P

contradicting that P is maximal,

Thus, there must $\exists$ source in G.

For sink, consider $v_k \to v_i$ or $v_k \to u$ by symmetry

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(c) pf: "$\Rightarrow$": Suppose R is total in G

($\exists$ $a \to b$ or $b \to a$ (not both) for $\forall a, b \in V$)

Goal: G with $n = |V|$ contains a Hamiltonian path.

Base: $n = 1$. $V = \{v\}$. trivial

IH: $P(k)$ holds for $1 \leq k < n$.

IS: By (b), $\exists$ source u in G

Let $G' = G \setminus \{u\}$ and R' be the reachability in G'.

$V' = V \setminus \{u\}$. $E' = E \cap (V' \times V')$

$\Rightarrow$ G' is still DAG. $|V'| = n-1$

Consider $\forall$ distinct $x, y \in V'$. any path between x, y in G

cannot pass u (as a source)

$\Rightarrow$ same path persists in G'

Since R is total (xRy or yRx) in G,

R' is total ($xR'y$ or $yR'x$) in G'

By IH, G' has a Hamiltonian path:

$P' = v_1, v_2, \dots, v_{n-1}$

Since R is total, either uRv_i or v_iRu

Since u is a source, $uRv_i \Rightarrow uRv_1$

$\Rightarrow \exists$ edge $u \to v_1$

$\Rightarrow P = u, v_1, v_2, \dots, v_{n-1}$ is a Hamiltonian path in G

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"$\Leftarrow$": Suppose G has a Hamiltonian path:

$P = v_1, \dots, v_n$

WTS: $\forall$ distinct $u, v \in V$, either uRv or vRu

Consider $\forall$ distinct $u, v \in V$

Since P visits every vertex exactly once,

$\exists$ unique i, j s.t. $v_i = u$, $v_j = v$

① $i < j$: $v_i \to v_{i+1} \to \dots \to v_j \Rightarrow uRv$

② $j < i$: $v_j \to v_{j+1} \to \dots \to v_i \Rightarrow vRu$

Thus, R is total.

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