# Team Note of Baka Team

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## Contents



# 1 Template

## 1.1 Template

```
#include <bits/stdc++.h>

using namespace std;
using ll = long long;

#define MASK(k) 1LL << (k)
#define BIT(x, k) ((x) >> (k) & 1)
#define all(x) (x).begin(), (x).end()

template<class T> bool minimize(T& a, const T& b) {
    if(a > b) return a = b, true;
    return false;
}

template<class T> bool maximize(T& a, const T& b) {
    if(a < b) return a = b, true;
    return false;
}

ll fdiv(ll a, ll b) {
    assert(b != 0);
    if(b < 0) a *= -1, b *= -1;
    return a >= 0 ? a / b : (a + 1) / b - 1;
}

ll cdiv(ll a, ll b) {
    assert(b != 0);
    if(b < 0) a *= -1, b *= -1;
    return a <= 0 ? a / b : (a - 1) / b + 1;
}
```

## 1.2 Debug

```
void debug_utils() {}

template<class T, class ...U> void debug_utils(T a, U... b) {
    cerr << a;
    if(sizeof...(b)) { cerr << ", "; debug_utils(b...);}
}

#define debug(...) { cerr << #__VA_ARGS__ << " = ";
debug_utils(__VA_ARGS__); cerr << "\n"; }

template<class Tp1, class Tp2>
ostream& operator << (ostream& cout, pair<Tp1, Tp2> val) {
    return cout << val.first << " " << val.second << "\n";
}

template<class Data, class Tp =
decltype(declval<Data>().begin())>
typename enable_if<!is_same<Data, string>::value,
ostream&>::type
operator << (ostream& cout, Data val) {
    cout << "[";
    for(auto i = val.begin(); i != val.end(); ++i)
        cout << (i == val.begin() ? "" : " ") << *i;
    return cout << "]";
}

template<class Data, class = decltype(declval<Data>().top())>
ostream& operator << (ostream& cout, Data val) {
    cout << "[";
    for(; val.size(); val.pop())
        cout << val.top() << (val.size() == 1 ? "" : " ");
    return cout << "]";
}

template<class Tp> ostream& operator << (ostream& cout,
queue<Tp> val) {
    cout << "[";
    for(; val.size(); val.pop())
        cout << val.front() << (val.size() == 1 ? "" : " ");
    return cout << "]";
}
```

## 1.3 Generate

```
mt19937_64
rng(chrono::steady_clock::now().time_since_epoch().count());

ll randint(ll a, ll b) {
    return uniform_int_distribution<ll> (a, b) (rng);
}
```

# 2 Data Structure

## 2.1 Mutidimensional Vector

```
template<class Tp, int D = 1>
struct Tvector : public vector<Tvector<Tp, D - 1>> {
    template <class... Args>
    Tvector(int n = 0, Args... args) : vector<Tvector<Tp, D -
    1>>(n, Tvector<Tp, D - 1>(args...)) {}
};

template <class Tp>
struct Tvector<Tp, 1> : public vector<Tp> {
    Tvector(int n = 0, Tp val = Tp()) : vector<Tp>(n, val) {}
};
```

## 2.2 Rollback

```
vector<pair<int*, int>> event;

void assign(int* u, int v) {
    event.push_back({u, exchange(*u, v)});
}

void rollback(int t) {
    for(; (int) event.size() > t; event.pop_back()) {
        *event.back().first = event.back().second;
    }
}
```

## 2.3 Wavelet Tree

```
struct wavelet {
    wavelet *left, *right;
    vector<ll> pref;
    int wl, wr;

    wavelet() { }
    wavelet(int tl, int tr, int pL, int pR, vector<ll> &v) {
        wl = tl, wr = tr;
        if (wl == wr || pL > pR) return;

        int mid = (wl + wr) >> 1;

        pref.pb(0);
        for (int i = pL; i <= pR; i++)
            pref.pb(pref.back() + (v[i] <= mid));

        ll piv = stable_partition(v.begin() + pL, v.begin() +
        pR + 1
                  , [&](int x){ return x <= mid; }) -
                  v.begin() - 1;

        left = new wavelet(wl, mid, pL, piv, v);
        right = new wavelet(mid + 1, wr, piv + 1, pR, v);
    }

    ll findKth(int k, int l, int r) {
        if (wl == wr) return wl;
        // cout << wl << " " << wr << " " << k << '\n';

        int amt = pref[r] - pref[l - 1];
        int lBound = pref[l - 1];
        int rBound = pref[r];

        if (amt >= k) return left->findKth(k, lBound + 1,
        rBound);

        return right->findKth(k - amt, l - lBound, r -
        rBound);
    }
};
```

## 2.4 Sparse lichao tree

```
struct Line {
  ll m, b;
  Line(ll _m = 0, ll _b = INF * 8) : m(_m), b(_b) {}
};

ll F(Line l, ll x) {
  return l.m * x + l.b;
}
struct lichao_t {
  lichao_t *left = nullptr, *right = nullptr;
  Line mn;
  lichao_t(ll tl = 0, ll tr = 0) {}
  void Update(ll tl, ll tr, Line nLine) {
    ll mid = (tl + tr) >> 1;
    bool pLeft = (F(nLine, tl) < F(mn, tl));
    bool pMid = (F(nLine, mid) < F(mn, mid));
    if (pMid)
      swap(mn, nLine);
    if (tl == tr)
      return;
    if (pLeft != pMid) {
      if (left == nullptr)
        left = new lichao_t();
      left->Update(tl, mid, nLine);
    } else {
      if (right == nullptr)
        right = new lichao_t();
      right->Update(mid + 1, tr, nLine);
    }
  }
  ll Query(ll tl, ll tr, ll x) {
    if (tl == tr)
      return F(mn, x);
    ll mid = (tl + tr) >> 1;
    ll res = F(mn, x);
    if (x <= mid) {
      if (left != nullptr)
        minimize(res, left->Query(tl, mid, x));
    } else {
      if (right != nullptr)
        minimize(res, right->Query(mid + 1, tr, x));
    }
    return res;
  }
};
```

## 2.5 Line Container

```
struct Line {
  mutable ll k, m, p;
  bool operator<(const Line& o) const { return k < o.k; }
  bool operator<(ll x) const { return p < x; }
};

struct LineContainer : multiset<Line, less<>> {
  // (for doubles, use inf = 1/.0, div(a,b) = a/b)
  static const ll inf = LLONG_MAX;
  ll div(ll a, ll b) { // floored division
    return a / b - ((a ^ b) < 0 && a % b); }
  bool isect(iterator x, iterator y) {
    if (y == end()) return x->p = inf, 0;
    if (x->k == y->k) x->p = x->m > y->m ? inf : -inf;
    else x->p = div(y->m - x->m, x->k - y->k);
    return x->p >= y->p;
  }
  void add(ll k, ll m) {
    auto z = insert({k, m, 0}), y = z++, x = y;
    while (isect(y, z)) z = erase(z);
    if (x != begin() && isect(--x, y)) isect(x, y = erase(y));
    while ((y = x) != begin() && (--x)->p >= y->p)
      isect(x, erase(y));
  }
  ll query(ll x) {
    assert(!empty());
    auto l = *lower_bound(x);
    return l.k * x + l.m;
  }
};
```

## 2.6 Ordered Set

```
#include <ext/pb_ds/assoc_container.hpp>
#include <ext/pb_ds/tree_policy.hpp>

using namespace __gnu_pbds;

template<class T>
using OrderedTree = tree<T, null_type, less<T>, rb_tree_tag,
    tree_order_statistics_node_update>;
```

## 2.7 Treap

```
struct Lazy {
    // [...]
};

struct Node {
    // [...]
};

Node operator + (Node u, Node v) {
    // [...]
    return Node{};
}

struct Tnode {
    Tnode *l = NULL, *r = NULL;
    Node node, key;
    Lazy lazy;
    int size = 1, prior = 0;
    Tnode(Node key = Node{}, int prior = randint(-1 << 30, 1
    << 30)): node(key), key(key), prior(prior) {}
};

using Pnode = Tnode*;

Pnode IgnoreNode;

// PreProcess.
void InitIgnoreNode() {
    IgnoreNode = new Tnode{};
    IgnoreNode->size = 0;
}

#define NODE(x) (x ? x : IgnoreNode)

Pnode FIX(Pnode u) {
    if(u) {
        u->size = NODE(u->l)->size + 1 + NODE(u->r)->size;
        u->node = NODE(u->l)->node + NODE(u)->key +
        NODE(u->r)->node;
    }
    return u;
}

void update_node(Pnode u, Lazy val) {
    if(!u) return;
    // [...]
}

void Down(Pnode t) {

}

Pnode merge(Pnode l, Pnode r) {
    if(!l || !r) return (l ? l : r);
    Down(l); Down(r);
    if(l->prior > r->prior) {
        l->r = merge(l->r, r);
        return FIX(l);
    } else {
        r->l = merge(l, r->l);
        return FIX(r);
    }
}

pair<Pnode, Pnode> split(Pnode t, int k) {
    if(!t) return {NULL, NULL};
    else Down(t);
    Pnode l = NULL, r = NULL;
    if(k <= NODE(t->l)->size) tie(l, t->l) = split(t->l, k), r
    = t;
    else                tie(t->r, r) = split(t->r, k - 1 -
    NODE(t->l)->size), l = t;
    FIX(t);
    return {l, r};
}

tuple<Pnode, Pnode, Pnode> split(Pnode t, int u, int v) {
    if(!t) return {NULL, NULL, NULL};
    Pnode l = NULL, m = NULL, r = NULL;
    tie(t, r) = split(t, v + 1);
    tie(l, m) = split(t, u);
    return {l, m, r};
}

void DFS(Pnode t) {
    if(!t) return;
    Down(t);
    DFS(t->l);
    // [...]
    DFS(t->r);
}
```

# 3 Graphs

## 3.1 Max Matching (Hopcroft)

```
/*
hopcroft karp for finding maximum matching on bipartite graphs
time complexity : O(E.sqrt(V))
layL[i] is the bfs layer of the ith vertex of left partition
layR[i] is for the ith vertex of the right partition
mtR[i] is the vertex matched with the ith vertex of right
partition, -1 if unmatched
adj[i] is list of neighbours of ith vertex of left partition
*/

struct hopcroft {
    ll nl, nr;

    // adj list of the left partition
    vector<vector<int>> adj;
    vector<int> layL, layR, mtR, cur, nxt;

    vector<bool> vis[2], mark;

    hopcroft(int n, int m) : nl(n), nr(m) {
        adj.assign(n, {});
    }

    bool dfs(int u, int len) {
        if(layL[u] != len) return 0;

        layL[u] = -1;
        for (int v : adj[u]) {
            if (layR[v] == len + 1) {
                layR[v] = 0;
                if (mtR[v] == -1 || dfs(mtR[v], len + 1))
                    return mtR[v] = u, 1;
            }
        }

        return 0;
    }

    ll max_matching() {
        layL.assign(nl, 0);
        layR.assign(nr, 0);
        mtR.assign(nr, -1);

        ll res = 0;

        while (true) {
            fill(all(layL), 0);
            fill(all(layR), 0);
            cur.clear();
```

```
        for (int u : mtR)
            if(u != -1) layL[u] = -1;

        for (int i = 0; i < sz(adj); i++)
            if (layL[i] != -1)
                cur.pb(i);

        bool isLast = false;
        for (int lay = 1; ; lay++) {
            nxt.clear();

            for (int u : cur) {
                for (int v : adj[u]) {
                    if (mtR[v] == -1) {
                        layR[v] = lay;
                        isLast = true;
                    }
                    else if (mtR[v] != u && !layR[v]) {
                        layR[v] = lay;
                        nxt.pb(mtR[v]);
                    }
                }
            }

            if (isLast) break;

            if (nxt.empty()) return res;

            for (int u : nxt)
                layL[u] = lay;

            swap(cur, nxt);
        }

        for (int i = 0; i < sz(adj); i++)
            res += dfs(i, 0);
    }
}

void dfs2(int u, int l) {
    vis[l][u] = true;

    if (!l) {
        for (int v : adj[u]) {
            if (!vis[1][v])
                dfs2(v, 1);
        }
    }
    else {
        if (mtR[u] != -1 && !vis[0][mtR[u]])
            dfs2(mtR[u], 0);
    }
}

//edges in matching -> right to left, else left to right
//return {left/right, index} of minimum cover
vector<pll> minCover() {
    vis[0].assign(nl, false);
    vis[1].assign(nr, false);
    mark.assign(nl, false);

    vector<pll> res;
    for (int i = 0; i < nr; i++)
        if (mtR[i] != -1)
            mark[mtR[i]] = true;

    for (int i = 0; i < nl; i++)
        if (!mark[i])
            dfs2(i, 0);

    //unvisited of the left and visited of the right is in min cover
    for (int i = 0; i < nl; i++)
        if (!vis[0][i])
            res.pb({0, i});

    for (int i = 0; i < nr; i++)
        if (vis[1][i])
            res.pb({1, i});
            return res;
    }
};
```

### 3.2 Max Flow (Push Relabel)

```
/**
 * Author: Simon Lindholm
 * Date: 2015-02-24
 * License: CC0
 * Source: Wikipedia, tinyKACTL
 * Description: Push-relabel using the highest label selection rule and the gap heuristic. Quite fast in practice.
 *  To obtain the actual flow, look at positive values only.
 * Time: $O(V^2\sqrt E)$
 * Status: Tested on Kattis and SPOJ, and stress-tested
 */
#pragma once


struct PushRelabel {
  struct Edge {
    int dest, back;
    ll f, c;
  };
  vector<vector<Edge>> g;
  vector<ll> ec;
  vector<Edge*> cur;
  vector<vi> hs; vi H;
  PushRelabel(int n) : g(n), ec(n), cur(n), hs(2*n), H(n) {}

  void addEdge(int s, int t, ll cap, ll rcap=0) {
    if (s == t) return;
    g[s].push_back({t, sz(g[t]), 0, cap});
    g[t].push_back({s, sz(g[s])-1, 0, rcap});
  }

  void addFlow(Edge& e, ll f) {
    Edge &back = g[e.dest][e.back];
    if (!ec[e.dest] && f) hs[H[e.dest]].push_back(e.dest);
    e.f += f; e.c -= f; ec[e.dest] += f;
    back.f -= f; back.c += f; ec[back.dest] -= f;
  }
  ll calc(int s, int t) {
    int v = sz(g); H[s] = v; ec[t] = 1;
    vi co(2*v); co[0] = v-1;
    rep(i,0,v) cur[i] = g[i].data();
    for (Edge& e : g[s]) addFlow(e, e.c);

    for (int hi = 0;;) {
      while (hs[hi].empty()) if (!hi--) return -ec[s];
      int u = hs[hi].back(); hs[hi].pop_back();
      while (ec[u] > 0)  // discharge u
        if (cur[u] == g[u].data() + sz(g[u])) {
          H[u] = 1e9;
          for (Edge& e : g[u]) if (e.c && H[u] > H[e.dest]+1)
            H[u] = H[e.dest]+1, cur[u] = &e;
          if (++co[H[u]], !--co[hi] && hi < v)
            rep(i,0,v) if (hi < H[i] && H[i] < v)
              --co[H[i]], H[i] = v + 1;
          hi = H[u];
        } else if (cur[u]->c && H[u] == H[cur[u]->dest]+1)
          addFlow(*cur[u], min(ec[u], cur[u]->c));
        else ++cur[u];
    }
  }
  bool leftOfMinCut(int a) { return H[a] >= sz(g); }
};
```

### 3.3 Min Cost Max Flow

```
int n, m, k, source, sink;
struct FlowEdge {
    int to, rev, id, flow, cap, cost;
};
vector<FlowEdge> adj[MAX_N];
int dist[MAX_N];
```

```
bool inQueue[MAX_N];
pii trc[MAX_N];
queue<int> q;
int ans;

void addEdge(int u, int v, int cost, int cap) {
    int szU = adj[u].size();
    int szV = adj[v].size();
    adj[u].pb({v, szV, szU, 0, cap, cost});
    adj[v].pb({u, szU, szV, 0, cap, cost});
}

bool BellmanFord() {
    for (int i = 1; i <= n; i++) {
        dist[i] = inf;
    }
    dist[source] = 0;
    q.push(source);
    inQueue[source] = true;
    while (!q.empty()) {
        int u = q.front();
        q.pop();
        inQueue[u] = false;
        for (auto e : adj[u]) {
            int v = e.to;
            int c = (e.flow >= 0 ? 1 : -1) * e.cost;
            if (e.flow < e.cap && dist[u] + c < dist[v]) {
                dist[v] = dist[u] + c;
                trc[v] = {u, e.id};
                if (!inQueue[v]) {
                    q.push(v);
                }
            }
        }
    }
    return dist[sink] < inf;
}

void inc() {
    int incFlow = inf;

    for (int i = sink; i != source; i = trc[i].fi) {
        int u = trc[i].fi;
        int id = trc[i].se;
        minimize(incFlow, (adj[u][id].flow >= 0 ?
        adj[u][id].cap - adj[u][id].flow : -adj[u][id].flow));
    }

    minimize(incFlow, k);


    for (int i = sink; i != source; i = trc[i].fi) {
        int u = trc[i].fi;
        int id = trc[i].se;
        adj[u][id].flow += incFlow;
        adj[i][adj[u][id].rev].flow -= incFlow;
    }

    ans += incFlow * dist[sink];
    k -= incFlow;

    if (!k) {
        cout << ans << '\n';
        for (int i = 1; i <= n; i++) {
            for (auto e : adj[i]) {
                if (e.flow > 0) {
                    cout << i << " " << e.to << " " << e.flow
                    << '\n';
                }
            }
        }
        cout << "0 0 0";
        exit(0);
    }
}

signed main() {
    ios_base::sync_with_stdio(false);
    cin.tie(nullptr);
    cin >> n >> m >> k >> source >> sink;
    for (int i = 1; i <= m; i++) {
        int u, v, c, d;
        cin >> u >> v >> c >> d;
        addEdge(u, v, c, d);
    }

    while (BellmanFord()) {
        inc();
    }

    cout << -1;

    return 0;
}
```

---

### 3.4 Weighted Matching (Hungarian)

```
/**
 * Author: Benjamin Qi, chilli
 * Date: 2020-04-04
 * License: CC0
 * Description: Given a weighted bipartite graph, matches
every node on
 * the left with a node on the right such that no
 * nodes are in two matchings and the sum of the edge weights
is minimal. Takes
 * cost[N][M], where cost[i][j] = cost for L[i] to be matched
with R[j] and
 * returns (min cost, match), where L[i] is matched with
 * R[match[i]]. Negate costs for max cost. Requires N <= M.
 * Time: O(N^2M)
 * Status: Tested on kattis:cordonbleu, stress-tested
 */
#pragma once

pair<int, vi> hungarian(const vector<vi> &a) {
  if (a.empty()) return {0, {}};
  int n = sz(a) + 1, m = sz(a[0]) + 1;
  vi u(n), v(m), p(m), ans(n - 1);
  rep(i,1,n) {
    p[0] = i;
    int j0 = 0; // add "dummy" worker 0
    vi dist(m, INT_MAX), pre(m, -1);
    vector<bool> done(m + 1);
    do { // dijkstra
      done[j0] = true;
      int i0 = p[j0], j1, delta = INT_MAX;
      rep(j,1,m) if (!done[j]) {
        auto cur = a[i0 - 1][j - 1] - u[i0] - v[j];
        if (cur < dist[j]) dist[j] = cur, pre[j] = j0;
        if (dist[j] < delta) delta = dist[j], j1 = j;
      }
      rep(j,0,m) {
        if (done[j]) u[p[j]] += delta, v[j] -= delta;
        else dist[j] -= delta;
      }
      j0 = j1;
    } while (p[j0]);
    while (j0) { // update alternating path
      int j1 = pre[j0];
      p[j0] = p[j1], j0 = j1;
    }
  }
  rep(j,1,m) if (p[j]) ans[p[j] - 1] = j - 1;
  return {-v[0], ans}; // min cost
}
```

---

### 3.5 2 SAT

```
//source: https://wiki.vnoi.info/vi/algo/graph-theory/2-SAT

#include <bits/stdc++.h>

using namespace std;

const int maxN = 500500;
```

```
int n, m;

// Lu đ th G
vector<int> G[maxN << 1];

// Ly giá tr ph đnh ca x
int NOT(int x) {
    return x + (x <= n ? n : -n); // -x
}

// Thêm điu kin u OR v
void add_clause(int u, int v) {
    G[NOT(u)].push_back(v); // -u -> v
    G[NOT(v)].push_back(u); // -v -> u
}

// Tìm thành phn liên thông mnh
int id[maxN << 1];
int num[maxN << 1], low[maxN << 1];
int timeDFS = 0, scc = 0;
int st[maxN << 1];

void dfs(int u) {
    num[u] = low[u] = ++timeDFS;
    st[++st[0]] = u;
    for(const int& v : G[u]) {
        if(id[v] != 0) continue;
        if(num[v] == 0) {
            dfs(v);
            low[u] = min(low[u], low[v]);
        } else low[u] = min(low[u], num[v]);
    }

    if(num[u] == low[u]) {
        for(++scc; true; ) {
            int v = st[st[0]--];
            id[v] = scc;
            if(v == u) break;
        }
    }
}

int main() {
    cin.tie(0)->sync_with_stdio(0);

    cin >> n >> m;
    for(int i = 1; i <= m; ++i) {
        int u, v; cin >> u >> v;
        add_clause(u, v);
    }

    // Thut toán Tarjan
    for(int i = 1; i <= 2 * n; ++i) {
        if(!id[i]) dfs(i);
    }

    bool answer = 1;
    for(int i = 1; i <= n; ++i) {
        // Kim tra điu kin tn ti phng án
        if(id[i] == id[NOT(i)]) answer = 0;
    }
    if(!answer) {
        cout << "IMPOSSIBLE"; // Thông báo bài toán vô nghim
        return 0;
    }
    // In tp giá tr a1, a2, ..., an
    for(int i = 1; i <= n; ++i) cout << (id[i] < id[NOT(i)])
    << " ";
    return 0;
}
```

# 4 DP

## 4.1 Divide And Conquer Optimization

```
void compute(int l, int r, int optl, int optr) {
    if (l > r)
        return;
    int mid = (l + r) >> 1;
    pair<long long, int> best = {LLONG_MAX, -1};

    for (int k = optl; k <= min(mid, optr); k++) {
        best = min(best, {(k ? dp_before[k - 1] : 0) + C(k,
        mid), k});
    }

    dp_cur[mid] = best.first;
    int opt = best.second;

    compute(l, mid - 1, optl, opt);
    compute(mid + 1, r, opt, optr);
}
```

## 4.2 Matrix Multiplication Optimization

```
namespace Matrix_Exponentiation {

    const int MAX_ROW = x; // Change Max_row here
    const int MAX_COL = x; // Change Max_col here
    int64_t mod = 1e9 + 7; // Change MOD here
    int64_t mxmod = (int64_t)(7e18 / mod) * mod;

    void change_mod(int _mod) {
        mod = _mod;
        mxmod = (int64_t)(7e18 / mod) * mod;
    }

    int64_t multi(int64_t a, int64_t b) {
        int64_t ret = 0;
        for(int i = 0 ; MASK(i) <= b ; i ++, a = (a + a) %
        mod) {
            if(MASK(i) & b) ret = (ret + a) % mod;
        }
        return ret;
    }

    struct Matrix {
        int r,c;
        int64_t a[MAX_ROW][MAX_COL];
        void Resize(int _r,int _c) {
            for (int i = 0; i < r; i ++) {
                for (int j = 0; j < c; j ++) {
                    a[i][j] = 0;
                }
            }
        }

        auto & operator [] (int i) { return a[i]; }

        const auto & operator[] (int i) const { return a[i]; }

        Matrix operator *(const Matrix& other) {
            Matrix product, tmp;
            product.Resize(r, other.c);
            tmp.Resize(r, other.c);
            for (int i = 0; i < product.r; i ++) {
                for (int j = 0; j < c; j ++) {
                    for (int k = 0; k < product.c; k ++) {
//                      product[i][k] += multi(a[i][j] ,
other.a[j][k]);
                        tmp[i][k] += a[i][j] * other[j][k];
                        if(tmp[i][k] >= mxmod)
                            tmp[i][k] -= mxmod;
                    }
                }
            }
            for (int i = 0; i < product.r; i ++) {
                for (int j = 0; j < product.c; j ++) {
                    product[i][j] = tmp[i][j] % mod;
                }
            }
            return product;
        }

        void operator *= (const Matrix& other) {
            *this = *this * other;
        }
```

```cpp
        Matrix operator ^ (const int64_t& b) {
            Matrix ret;
            Matrix m = *this;
            ret.Resize(m.r, m.c);
            for (int i = 0; i < ret.r; i ++) {
                ret[i][i] = 1;
            }
            for(int i = 0 ; MASK(i) <= b ; i ++, m *= m) {
                if (b & MASK(i)) {
                    ret*=m;
                }
            }
            return ret;
        }

        void operator ^= (const int64_t& b) {
            *this = *this ^ b;
        }

        friend ostream& operator << (ostream& os, const
        Matrix& M) {
            for (int i = 0; i < M.r; i ++) {
                for (int j = 0; j < M.c; j ++) {
                    os << M.a[i][j] << " \n"[j == M.c - 1];
                }
            }
            return os;
        }
    };
}

using namespace Matrix_Exponentiation;
```

# 5 Strings

## 5.1 KMP

```cpp
vector<int> prefix_function(string s) {
    int n = (int)s.length();
    vector<int> pi(n);
    for (int i = 1; i < n; i++) {
        int j = pi[i-1];
        while (j > 0 && s[i] != s[j])
            j = pi[j-1];
        if (s[i] == s[j])
            j++;
        pi[i] = j;
    }
    return pi;
}
```

## 5.2 Z Function

```cpp
vector<int> zfunction(const string& s) {
    int n = (int) s.size();
    vector<int> z(n);
    for(int i = 1, l = 0, r = 0; i < n; ++i) {
         if(i <= r) z[i] = min(r - i, z[i - l]);
         while(i + z[i] < n && s[z[i]] == s[i + z[i]]) ++z[i];
         if(i + z[i] - 1 > r) l = i, r = i + z[i] - 1;
    } return z;
}
```

## 5.3 Aho Corasick (static)

```cpp
const int CAP = 1003, ALPHABET = 26;
int cntTrie = 1;
int fail[CAP], to[CAP][ALPHABET];
bool ending[CAP];

void add_string(const string& s) {
    int u = 1;
    for(const char& c : s) {
        int x = c - 'a';
        if(!to[u][x]) {
            to[u][x] = ++cntTrie;
        }
        u = to[u][x];
    }
    ending[u] = true;
}

void aho_corasick() {
    queue<int> q; q.push(1);
    while(q.size()) {
        int u = q.front(); q.pop();
        for(int x = 0; x < ALPHABET; ++x) {
            int& v = to[u][x];
            if(!v) {
                v = u == 1 ? 1 : to[fail[u]][x];
            } else {
                if(!fail[v]) fail[v] = fail[u];
                fail[v] = u == 1 ? 1 : to[fail[v]][x];
                ending[v] |= ending[fail[v]];
                q.push(v);
            }
        }
    }
}
```

## 5.4 Aho Corasick (vector)

```cpp
struct TrieNode {
  int pi = 0;
  int child[26] = {0};
};
vector<TrieNode> trie;
vector<vector<int>> adj;

int TrieInsert(const string& s) {
  int p = 0;
  for (int i = 0; i < s.size(); i++) {
    if (!trie[p].child[s[i] - 'a']) {
      trie[p].child[s[i] - 'a'] = trie.size();
      trie.pb(TrieNode());
    }
    p = trie[p].child[s[i] - 'a'];
  }
  return p;
}

void AhoCorasickBuild() {
  queue<int> q;
  for (int i = 0; i < 26; i++) {
    if (trie[0].child[i]) {
      q.push(trie[0].child[i]);
    }
  }
  while (!q.empty()) {
    int u = q.front();
    q.pop();
    for (int i = 0; i < 26; i++) {
      if (!trie[u].child[i]) continue;
      int j = trie[u].pi;
      while (!trie[j].child[i]) {
        if (!j) break;
        j = trie[j].pi;
      }
      trie[trie[u].child[i]].pi = trie[j].child[i];
      q.push(trie[u].child[i]);
    }
    adj[trie[u].pi].pb(u);
  }
}

signed main() {
  trie.pb(TrieNode());


  adj.resize(trie.size());
  AhoCorasickBuild();
}
```

# 6 Math

## 6.1 Chinese Remainder Theorem

```
struct gcd_t { ll x, y, d; };

gcd_t e_gcd(ll a, ll b) {
    if (b == 0) return {1, 0, a};

    gcd_t res = e_gcd(b, a % b);

    return {res.y, res.x - res.y * (a / b), res.d};
}

pll crt(vector<ll> r, vector<ll> m) {
    //find x such that for (1 <= i <= n): x = r[i] (mod m[i])
    //return {y, z} where x = y (mod z), z = lcm of vector m
    //all solutions are congruent modulo z

    ll y = r[0], z = m[0];
    for (int i = 1; i < sz(r); i++) {
        gcd_t cur = e_gcd(z, m[i]);

        ll x = cur.x, d = cur.d;

        if((r[i] - y) % d != 0) return {-1, -1};

        //ka = kb (mod kc)   =>   a = b (mod c) if (gcd(k, c)
        = 1)
        //add (x * (r[i] - y) / d * z) to result (with moduli
        lcm(z, m[i]))
        ll tmp = (x * (r[i] - y) / d) % (m[i] / d);
        y = y + tmp * z;
        z = z / d * m[i];

        y %= z;
        if (y < 0) y += z;
    }

    return {y, z};
}

ll inverse(ll a, ll m) {
    gcd_t cur = e_gcd(a, m);

    return (cur.x % m + m) % m;
}
```

## 6.2 Miller Rabin

```
// From
https://github.com/SnapDragon64/ContestLibrary/blob/master/math.h
// which also has specialized versions for 32-bit and 42-bit
//
// Tested:
// - https://oj.vnoi.info/problem/icpc22_national_c (fastest
solution)
// - https://www.spoj.com/problems/PON/

// Rabin miller {{{
inline uint64_t mod_mult64(uint64_t a, uint64_t b, uint64_t m)
{
    return __int128_t(a) * b % m;
}
uint64_t mod_pow64(uint64_t a, uint64_t b, uint64_t m) {
    uint64_t ret = (m > 1);
    for (;;) {
        if (b & 1) ret = mod_mult64(ret, a, m);
        if (!(b >>= 1)) return ret;
        a = mod_mult64(a, a, m);
    }
}

// Works for all primes p < 2^64
bool is_prime(uint64_t n) {
    if (n <= 3) return (n >= 2);
    static const uint64_t small[] = {
        2, 3, 5, 7, 11, 13, 17, 19, 23, 29, 31, 37, 41, 43,
        47, 53, 59, 61, 67,
        71, 73, 79, 83, 89, 97, 101, 103, 107, 109, 113, 127,
        131, 137, 139,
        149, 151, 157, 163, 167, 173, 179, 181, 191, 193, 197,
        199,
    };
    for (size_t i = 0; i < sizeof(small) / sizeof(uint64_t);
    ++i) {
        if (n % small[i] == 0) return n == small[i];
    }

    // Makes use of the known bounds for Miller-Rabin
    pseudoprimes.
    static const uint64_t millerrabin[] = {
        2, 3, 5, 7, 11, 13, 17, 19, 23, 29, 31, 37,
    };
    static const uint64_t A014233[] = {  // From OEIS.
        2047LL, 1373653LL, 25326001LL, 3215031751LL,
        2152302898747LL,
        3474749660383LL, 341550071728321LL, 341550071728321LL,
        3825123056546413051LL, 3825123056546413051LL,
        3825123056546413051LL, 0,
    };
    uint64_t s = n-1, r = 0;
    while (s % 2 == 0) {
        s /= 2;
        r++;
    }
    for (size_t i = 0, j; i < sizeof(millerrabin) /
    sizeof(uint64_t); i++) {
        uint64_t md = mod_pow64(millerrabin[i], s, n);
        if (md != 1) {
            for (j = 1; j < r; j++) {
                if (md == n-1) break;
                md = mod_mult64(md, md, n);
            }
            if (md != n-1) return false;
        }
        if (n < A014233[i]) return true;
    }
    return true;
}
// }}}
```

## 6.3 Discrete Logarithm

```
// Computes x which a ^ x = b mod n.

long long d_log(long long a, long long b, long long n) {
    long long m = ceil(sqrt(n));
    long long aj = 1;
    map<long long, long long> M;
    for (int i = 0; i < m; ++i) {
        if (!M.count(aj))
            M[aj] = i;
        aj = (aj * a) % n;
    }

    long long coef = mod_pow(a, n - 2, n);
    coef = mod_pow(coef, m, n);
    // coef =  a ^ (-m)
    long long gamma = b;
    for (int i = 0; i < m; ++i) {
        if (M.count(gamma)) {
            return i * m + M[gamma];
        } else {
            gamma = (gamma * coef) % n;
        }
    }
    return -1;
}
```

## 6.4 Fast Fourier Transform

```
typedef complex<double> cmplx;
typedef vector<complex<double> > VC;
const double PI = acos(-1);
struct FFT {
    static void fft(VC &u, int sign) {
```

```
        int n = u.size();
        double theta = 2. * PI * sign / n;
        for (int m = n; m >= 2; m >>= 1, theta *= 2.) {
            cmplx w(1, 0), wDelta = polar(1., theta);
            for (int i = 0, mh = m >> 1; i < mh; i++) {
                for (int j = i; j < n; j += m) {
                    int k = j + mh;
                    cmplx temp = u[j] - u[k];
                    u[j] += u[k];
                    u[k] = w * temp;
                }
                w *= wDelta;
            }
        }
        for (int i = 1, j = 0; i < n; i++) {
            for (int k = n >> 1; k > (j ^= k); k >>= 1);
            if (j < i) {
                swap(u[i], u[j]);
            }
        }
    }

    static vector<ll> mul(const vector<int> &a, const
    vector<int> &b) {
        int newSz = a.size() + b.size() - 1;
        int fftSz = 1;
        while (fftSz < newSz) fftSz <<= 1;
        VC aa(fftSz, 0.), bb(fftSz, 0.);
        for (int i = 0; i < a.size(); i++) aa[i] = a[i];
        for (int i = 0; i < b.size(); i++) bb[i] = b[i];
        fft(aa, 1), fft(bb, 1);
        for (int i = 0; i < fftSz; i++) aa[i] *= bb[i];
        fft(aa, -1);
        vector<ll> res(newSz);
        for (int i = 0; i < newSz; i++)
            res[i] = (ll)(aa[i].real() / fftSz + 0.5);
        return res;
    }
};
```

# 7 Geometry

## 7.1 Geomtry Point

```
struct Point{
    typedef ll T;
    T x, y;
    Point(T _x = 0, T _y = 0) : x(_x), y(_y) {}

    bool operator < (Point p) const { return tie(x, y) <
    tie(p.x, p.y); }
    bool operator > (Point p) const { return tie(x, y) >
    tie(p.x, p.y); }
    bool operator == (Point p) const { return tie(x, y) ==
    tie(p.x, p.y); }
    bool operator != (Point p) const { return tie(x, y) !=
    tie(p.x, p.y); }

    Point operator + (Point p) const { return Point(x + p.x, y
    + p.y); }
    Point operator - (Point p) const { return Point(x - p.x, y
    - p.y); }
    T operator * (Point p) const { return x * p.x + y * p.y; }
    T operator ^ (Point p) const { return x * p.y - y * p.x; }

    Point operator * (T d) const { return Point(x * d, y * d);
    }
    Point operator / (T d) const { return Point(x / d, y / d);
    }

    T len2() const { return x * x + y * y; }
    double len() const { return sqrt((double)len2()); }
    Point perp() { return Point(-y, x); }
    friend ostream& operator << (ostream &os, const Point &p)
    {
        return os << "(" << p.x << ", " << p.y << ")"; }
};

ll ccw(const Point &P0, const Point &P1, const Point &P2){
    return (P1 - P0) ^ (P2 - P1);
}

ll sgn(const ll &x) { return (x >= 0 ? (x ? 1 : 0) : -1); }
```

## 7.2 Convex Hull

```
vector<Point> convexHull(vector<Point> dots) {
    sort(dots.begin(), dots.end());
    vector<Point> A(1, dots[0]);
    const int sz = dots.size();
    for(int c = 0; c < 2; reverse(all(dots)), c++)
    for(int i = 1, t = A.size(); i < sz;
    A.emplace_back(dots[i++]))
      while (A.size() > t and ccw(A[A.size()-2], A.back(),
      dots[i]) < 0)
        A.pop_back();
    A.pop_back(); return A;
}
```

## 7.3 Manhattan MST

```
vector<array<ll, 3>> manhattanMST(vector<Point> ps) {
    vector<int> id(sz(ps));
    iota(all(id), 0);
    vector<array<ll, 3>> edges;
    for (int k = 0; k < 4; k++) {
            sort(all(id), [&](int i, int j) { return (ps[i] -
            ps[j]).x < (ps[j] - ps[i]).y; });
            map<int, int> sweep;
            for (int i : id) {
                for (auto it = sweep.lower_bound(-ps[i].y);
                    it != sweep.end(); sweep.erase(it++)) {
                    int j = it->second;
                    Point d = ps[i] - ps[j];
                    if (d.y > d.x) break;
                    edges.push_back({d.y + d.x, i, j});
                }
                sweep[-ps[i].y] = i;
            }
            for (Point &p : ps)
                if (k & 1)
                    p.x = -p.x;
                else
                    swap(p.x, p.y);
    }
    return edges;
}
```

## 7.4 Some Common Geometry Operations

```
ll ccw(const Point &P0, const Point &P1, const Point &P2){
    return (P1 - P0) ^ (P2 - P1);
}

bool on_segment(Point &p, Point &p0, Point &p1){
    if((p1 - p0) * (p - p1) > 0) return false;
    if((p0 - p1) * (p - p0) > 0) return false;
    return (ccw(p, p0, p1) == 0);
}

db dist_segment(Point &p, Point &p0, Point &p1){
    if((p1 - p0) * (p - p1) >= 0) return (p - p1).len();
    if((p0 - p1) * (p - p0) >= 0) return (p - p0).len();
    return abs((db)((p1 - p0) ^ (p - p0)) / (p1 - p0).len());
}

bool insideConvex(Point p, vector<Point> &poly){
    // clock wise
    int n = sz(poly);
    if(ccw(poly[0], poly[1], p) >= 0) return false;
    if(ccw(poly[n - 1], poly[0], p) >= 0) return false;

    ll l = 1, r = n-1;
    while(l < r){
        ll mid = (l+r+1)/2;
        if(ccw(poly[0], p, poly[mid]) >= 0) l = mid;
```

```
        else r = mid - 1;
    }
    r = l + 1;

    return (ccw(poly[l], p, poly[r]) > 0);
}

ll wn_poly(Point p, vector<Point> &poly){
    // 1 if inside 0 if outside, INF if on boundary
    // counter clock wise
    const ll on_boundary = INF;
    ll wn = 0;

    int n = sz(poly);
    for(int i = 0; i < n; i++){
        if(p == poly[i]) return on_boundary;

        int j = (i + 1 != n ? i + 1 : 0);

        if(poly[i].y == p.y && poly[j].y == p.y){
            if(min(poly[i].x, poly[j].x) <= p.x
                 && p.x <= max(poly[i].x, poly[j].x))
                return on_boundary;
        }
        else{
            bool below = (poly[i].y <= p.y);
            //different sides of horizontal ray
            if (below != (poly[j].y <= p.y)){
                ll orientation = ccw(p, poly[i], poly[j]);

                if (orientation == 0) return on_boundary;
                if (below == (orientation > 0)) wn += (below ?
                1 : -1);
            }
        }
    }

    return wn;
}

bool line_intersect(pii a, pii b, pii c, pii d) {
  if (!ccw(c, a, b) || !ccw(d, a, b) || !ccw(a, c, d) ||
  !ccw(b, c, d)) {
    if (!ccw(c, a, b) && dot_product(a, c, b) <= 0) {
      return true;
    }
    if (!ccw(d, a, b) && dot_product(a, d, b) <= 0) {
      return true;
    }
    if (!ccw(a, c, d) && dot_product(c, a, d) <= 0) {
      return true;
    }
    if (!ccw(b, c, d) && dot_product(c, b, d) <= 0) {
      return true;
    }
    return false;
  }
  return (ccw(a, b, c) * ccw(a, b, d) < 0 && ccw(c, d, a) *
  ccw(c, d, b) < 0);
```

# 8 Miscellaneous

## 8.1 Hilber Order for Mo's

```
// From
https://github.com/SnapDragon64/ContestLibrary/blob/master/math.h
// which also has specialized versions for 32-bit and 42-bit
//
// Tested:
// - https://oj.vnoi.info/problem/icpc22_national_c (fastest
solution)
// - https://www.spoj.com/problems/PON/

// Rabin miller {{{
inline uint64_t mod_mult64(uint64_t a, uint64_t b, uint64_t m)
{
    return __int128_t(a) * b % m;
}
uint64_t mod_pow64(uint64_t a, uint64_t b, uint64_t m) {
    uint64_t ret = (m > 1);
    for (;;) {
        if (b & 1) ret = mod_mult64(ret, a, m);
        if (!(b >>= 1)) return ret;
        a = mod_mult64(a, a, m);
    }
}

// Works for all primes p < 2^64
bool is_prime(uint64_t n) {
    if (n <= 3) return (n >= 2);
    static const uint64_t small[] = {
        2, 3, 5, 7, 11, 13, 17, 19, 23, 29, 31, 37, 41, 43,
        47, 53, 59, 61, 67,
        71, 73, 79, 83, 89, 97, 101, 103, 107, 109, 113, 127,
        131, 137, 139,
        149, 151, 157, 163, 167, 173, 179, 181, 191, 193, 197,
        199,
    };
    for (size_t i = 0; i < sizeof(small) / sizeof(uint64_t);
    ++i) {
        if (n % small[i] == 0) return n == small[i];
    }

    // Makes use of the known bounds for Miller-Rabin
    pseudoprimes.
    static const uint64_t millerrabin[] = {
        2, 3, 5, 7, 11, 13, 17, 19, 23, 29, 31, 37,
    };
    static const uint64_t A014233[] = {  // From OEIS.
        2047LL, 1373653LL, 25326001LL, 3215031751LL,
        2152302898747LL,
        3474749660383LL, 341550071728321LL, 341550071728321LL,
        3825123056546413051LL, 3825123056546413051LL,
        3825123056546413051LL, 0,
    };
    uint64_t s = n-1, r = 0;
    while (s % 2 == 0) {
        s /= 2;
        r++;
    }
    for (size_t i = 0, j; i < sizeof(millerrabin) /
    sizeof(uint64_t); i++) {
        uint64_t md = mod_pow64(millerrabin[i], s, n);
        if (md != 1) {
            for (j = 1; j < r; j++) {
                if (md == n-1) break;
                md = mod_mult64(md, md, n);
            }
            if (md != n-1) return false;
        }
        if (n < A014233[i]) return true;
    }
    return true;
}
// }}}
```

| OEIS link | Name | First elements | Short description |
|---|---|---|---|
| A000010 | Euler's totient function φ($n$) | 1, 1, 2, 2, 4, 2, 6, 4, 6, 4 | φ($n$) is the number of the positive integers not greater than n that are prime to n |
| A000027 | Natural number | 1, 2, 3, 4, 5, 6, 7, 8, 9, 10 | The natural numbers |
| A000032 | Lucas number | 2, 1, 3, 4, 7, 11, 18, 29, 47, 76 | $L(n) = L(n-1) + L(n-2)$ |
| A000040 | Prime number | 2, 3, 5, 7, 11, 13, 17, 19, 23, 29 | The prime numbers |
| A000045 | Fibonacci number | 0, 1, 1, 2, 3, 5, 8, 13, 21, 34 | $F(n) = F(n-1) + F(n-2)$ with $F(0) = 0$ and $F(1) = 1$ |
| A000058 | Sylvester's sequence | 2, 3, 7, 43, 1807, 3263443, 10650056950807, 113423713055421844361000443 | $a(n+1) = a(n)^2 - a(n) + 1$, with $a(0) = 2$ |
| A000073 | Tribonacci number | 0, 1, 1, 2, 4, 7, 13, 24, 44, 81 | $T(n) = T(n-1) + T(n-2) + T(n-3)$ with $T(0) = 0$, $T(1) = T(2) = 1$ |
| A000108 | Catalan number | 1, 1, 2, 5, 14, 42, 132, 429, 1430, 4862 | $C_n = \frac{1}{n+1}\binom{2n}{n} = \frac{(2n)!}{(n+1)!\,n!} = \prod_{k=2}^{n} \frac{n+k}{k}$ for $n \geq 0$. |
| A000110 | Bell number | 1, 1, 2, 5, 15, 52, 203, 877, 4140, 21147 | The number of partitions of a set with $n$ elements |
| A000111 | Euler number | 1, 1, 1, 2, 5, 16, 61, 272, 1385, 7936 | The number of linear extensions of the "zig-zag" poset |
| A000124 | Lazy caterer's sequence | 1, 2, 4, 7, 11, 16, 22, 29, 37, 46 | The maximal number of pieces formed when slicing a pancake with $n$ cuts |
| A000129 | Pell number | 0, 1, 2, 5, 12, 29, 70, 169, 408, 985 | $a(0) = 0$, $a(1) = 1$; for $n > 1$, $a(n) = 2a(n-1) + a(n-2)$ |
| A000142 | Factorial | 1, 1, 2, 6, 24, 120, 720, 5040, 40320, 362880 | $n! = 1 \cdot 2 \cdot 3 \cdot 4 \cdot \ldots \cdot n$ |
| A000217 | Triangular number | 0, 1, 3, 6, 10, 15, 21, 28, 36, 45 | $a(n) = C(n+1, 2) = n(n+1)/2 = 0 + 1 + 2 + \ldots + n$ |
| A000292 | Tetrahedral number | 0, 1, 4, 10, 20, 35, 56, 84, 120, 165 | The sum of the first $n$ triangular numbers |
| A000330 | Square pyramidal number | 0, 1, 5, 14, 30, 55, 91, 140, 204, 285 | (n(n+1)(2n+1)) / 6<br><br>The number of stacked spheres in a pyramid with a square base |
| A000396 | Perfect number | 6, 28, 496, 8128, 33550336, 8589869056, 137438691328, 2305843008139952128 | $n$ is equal to the sum of the proper divisors of $n$ |
| A000668 | Mersenne prime | 3, 7, 31, 127, 8191, 131071, 524287, 2147483647, 2305843009213693951, 618970019642690137449562111 | $2^p - 1$ if p is a prime |
| A007588 | Stella octangula number | 0, 1, 14, 51, 124, 245, 426, 679, 1016, 1449, 1990, 2651, 3444, 4381, ... | Stella octangula numbers: $n*(2*n^2 - 1)$. |
| A000793 | Landau's function | 1, 1, 2, 3, 4, 6, 6, 12, 15, 20 | The largest order of permutation of $n$ elements |
| A000796 | Decimal expansion of Pi | 3, 1, 4, 1, 5, 9, 2, 6, 5, 3 | |
| A000931 | Padovan sequence | 1, 1, 1, 2, 2, 3, 4, 5, 7, 9 | $P(0) = P(1) = P(2) = 1$, $P(n) = P(n-2) + P(n-3)$ |
| A000945 | Euclid–Mullin sequence | 2, 3, 7, 43, 13, 53, 5, 6221671, 38709183810571, 139 | $a(1) = 2$, $a(n+1)$ is smallest prime factor of $a(1)a(2)\ldots a(n)+1$. |
| A000959 | Lucky number | 1, 3, 7, 9, 13, 15, 21, 25, 31, 33 | A natural number in a set that is filtered by a sieve |

| OEIS link | Name | First elements | Short description |
|---|---|---|---|
| A001006 | Motzkin number | 1, 1, 2, 4, 9, 21, 51, 127, 323, 835 | The number of ways of drawing any number of nonintersecting chords joining *n* (labeled) points on a circle |
| A001045 | Jacobsthal number | 0, 1, 1, 3, 5, 11, 21, 43, 85, 171, 341 | $a(n) = a(n - 1) + 2a(n - 2)$, with $a(0) = 0$, $a(1) = 1$ |
| A001065 | sequence of Aliquot sums s(n) | 0, 1, 1, 3, 1, 6, 1, 7, 4, 8 | s(n) is the sum of the proper divisors of the integer n |
| A001113 | Decimal expansion of e (mathematical constant) | 2, 7, 1, 8, 2, 8, 1, 8, 2, 8 | |
| A001190 | Wedderburn–Etherington number | 0, 1, 1, 1, 2, 3, 6, 11, 23, 46 | The number of binary rooted trees (every node has out-degree 0 or 2) with n endpoints (and $2n - 1$ nodes in all) |
| A001358 | Semiprime | 4, 6, 9, 10, 14, 15, 21, 22, 25, 26 | Products of two primes |
| A001462 | Golomb sequence | 1, 2, 2, 3, 3, 4, 4, 4, 5, 5 | $a(n)$ is the number of times $n$ occurs, starting with $a(1) = 1$ |
| A001608 | Perrin number | 3, 0, 2, 3, 2, 5, 5, 7, 10, 12 | $P(0) = 3$, $P(1) = 0$, $P(2) = 2$; $P(n) = P(n-2) + P(n-3)$ for $n > 2$ |
| A001620 | Euler–Mascheroni constant | 5, 7, 7, 2, 1, 5, 6, 6, 4, 9 | $\gamma = \lim_{n\to\infty}\left(\sum_{k=1}^{n}\frac{1}{k} - \ln(n)\right) = \lim_{b\to\infty}\int_1^b\left(\frac{1}{\lfloor x\rfloor} - \frac{1}{x}\right)\,dx.$ |
| A001622 | Decimal expansion of the golden ratio | 1, 6, 1, 8, 0, 3, 3, 9, 8, 8 | $\varphi = \frac{1+\sqrt{5}}{2} = 1.6180339887\ldots.$ |
| A002064 | Cullen number | 1, 3, 9, 25, 65, 161, 385, 897, 2049, 4609, 10241, 22529, 49153, 106497 | $n\,2^n + 1$ |
| A002110 | Primorial | 1, 2, 6, 30, 210, 2310, 30030, 510510, 9699690, 223092870 | The product of first *n* primes |
| A002113 | Palindromic number | 0, 1, 2, 3, 4, 5, 6, 7, 8, 9 | A number that remains the same when its digits are reversed |
| A002182 | Highly composite number | 1, 2, 4, 6, 12, 24, 36, 48, 60, 120 | A positive integer with more divisors than any smaller positive integer |
| A002193 | Decimal expansion of square root of 2 | 1, 4, 1, 4, 2, 1, 3, 5, 6, 2 | |
| A002201 | Superior highly composite number | 2, 6, 12, 60, 120, 360, 2520, 5040, 55440, 720720 | A positive integer $n$ for which there is an $e>0$ such that $d(n)/n^e \geq d(k)/k^e$ for all $k>1$ |
| A002378 | Pronic number | 0, 2, 6, 12, 20, 30, 42, 56, 72, 90 | $n(n+1)$ |
| A002808 | Composite number | 4, 6, 8, 9, 10, 12, 14, 15, 16, 18 | The numbers $n$ of the form $xy$ for $x > 1$ and $y > 1$ |
| A002858 | Ulam number | 1, 2, 3, 4, 6, 8, 11, 13, 16, 18 | $a(1) = 1$; $a(2) = 2$; for $n>2$, $a(n)$ = least number > $a(n-1)$ which is a unique sum of two distinct earlier terms; semiperfect |
| A002997 | Carmichael number | 561, 1105, 1729, 2465, 2821, 6601, 8911, 10585, 15841, 29341 | Composite numbers $n$ such that $a^{(n-1)} == 1 \pmod{n}$ if $a$ is prime to $n$ |
| A003261 | Woodall number | 1, 7, 23, 63, 159, 383, 895, 2047, 4607 | $n\,2^n - 1$ |

| OEIS link | Name | First elements | Short description |
|---|---|---|---|
| A003459 | Permutable prime | 2, 3, 5, 7, 11, 13, 17, 31, 37, 71 | The numbers for which every permutation of digits is a prime |
| A005044 | Alcuin's sequence | 0, 0, 0, 1, 0, 1, 1, 2, 1, 3, 2, 4, 3, 5, 4, 7, 5, 8, 7, 10, 8, 12, 10, 14 | number of triangles with integer sides and perimeter *n* |
| A005100 | Deficient number | 1, 2, 3, 4, 5, 7, 8, 9, 10, 11 | The numbers *n* such that $\sigma(n) < 2n$ |
| A005101 | Abundant number | 12, 18, 20, 24, 30, 36, 40, 42, 48, 54 | The sum of divisors of *n* exceeds $2n$ |
| A005150 | Look-and-say sequence | 1, 11, 21, 1211, 111221, 312211, 13112221, 1113213211, 31131211131221, 13211311123113112211, | A = 'frequency' followed by 'digit'-indication |
| A005224 | Aronson's sequence | 1, 4, 11, 16, 24, 29, 33, 35, 39, 45 | "t" is the first, fourth, eleventh, ... letter in this sentence, not counting spaces or commas |
| A005235 | Fortunate number | 3, 5, 7, 13, 23, 17, 19, 23, 37, 61 | The smallest integer m > 1 such that pn# + m is a prime number, where the primorial pn# is the product of the first n prime numbers |
| A005349 | Harshad numbers in base 10 | 1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 12 | a Harshad number in base 10 is an integer that is divisible by the sum of its digits (when written in base 10) |
| A005384 | Sophie Germain prime | 2, 3, 5, 11, 23, 29, 41, 53, 83, 89 | A prime number *p* such that $2p+1$ is also prime |
| A005835 | Semiperfect number | 6, 12, 18, 20, 24, 28, 30, 36, 40, 42 | A natural number *n* that is equal to the sum of all or some of its proper divisors |
| A006037 | Weird number | 70, 836, 4030, 5830, 7192, 7912, 9272, 10430, 10570, 10792 | A natural number that is abundant but not semiperfect |
| A006842 | Farey sequence numerators | 0, 1, 0, 1, 1, 0, 1, 1, 2, 1 | |
| A006843 | Farey sequence denominators | 1, 1, 1, 2, 1, 1, 3, 2, 3, 1 | |
| A006862 | Euclid number | 2, 3, 7, 31, 211, 2311, 30031, 510511, 9699691, 223092871 | 1 + product of first *n* consecutive primes |
| A006886 | Kaprekar number | 1, 9, 45, 55, 99, 297, 703, 999, 2223, 2728 | $X^2 = Ab^n + B$, where $0 < B < b^n$ $X = A + B$ |
| A007304 | Sphenic number | 30, 42, 66, 70, 78, 102, 105, 110, 114, 130 | Products of 3 distinct primes |
| A007318 | Pascal's triangle | 1, 1, 1, 1, 2, 1, 1, 3, 3, 1 | Pascal's triangle read by rows |
| A007770 | Happy number | 1, 7, 10, 13, 19, 23, 28, 31, 32, 44 | The numbers whose trajectory under iteration of sum of squares of digits map includes 1 |
| A010060 | Prouhet–Thue–Morse constant | 0, 1, 1, 0, 1, 0, 0, 1, 1, 0 | $\tau = \sum_{i=0}^{\infty} \frac{t_i}{2^{i+1}}$ |
| A014080 | Factorion | 1, 2, 145, 40585 | A natural number that equals the sum of the factorials of its decimal digits |
| A014577 | Regular paperfolding sequence | 1, 1, 0, 1, 1, 0, 0, 1, 1, 1 | At each stage an alternating sequence of 1s and 0s is inserted between the terms of the previous sequence |
| A016114 | Circular prime | 2, 3, 5, 7, 11, 13, 17, 37, 79, 113 | The numbers which remain prime under cyclic shifts of digits |
| A018226 | Magic number (physics) | 2, 8, 20, 28, 50, 82, 126 | A number of nucleons (either protons or neutrons) such that they are arranged into complete shells within the atomic nucleus. |

| OEIS link | Name | First elements | Short description |
|---|---|---|---|
| A019279 | Superperfect number | 2, 4, 16, 64, 4096, 65536, 262144, 1073741824, 1152921504606846976, 309485009821345068724781056 | $\sigma^2(n) = \sigma(\sigma(n)) = 2n\,,$ |
| A027641 | Bernoulli number | 1, -1, 1, 0, -1, 0, 1, 0, -1, 0, 5, 0, -691, 0, 7, 0, -3617, 0, 43867, 0 | |
| A031214 | First elements in all OEIS sequences | 1, 1, 1, 0, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, | One of sequences referring to the OEIS itself |
| A033307 | Decimal expansion of Champernowne constant | 1, 2, 3, 4, 5, 6, 7, 8, 9, 1 | formed by concatenating the positive integers |
| A035513 | Wythoff array | 1, 2, 4, 3, 7, 6, 5, 11, 10, 9 | A matrix of integers derived from the Fibonacci sequence |
| A036262 | Gilbreath's conjecture | 2, 1, 3, 1, 2, 5, 1, 0, 2, 7 | Triangle of numbers arising from Gilbreath's conjecture |
| A037274 | Home prime | 1, 2, 3, 211, 5, 23, 7, 3331113965338635107, 311, 773 | For $n \geq 2$, $a(n)$ = the prime that is finally reached when you start with $n$, concatenate its prime factors (A037276) and repeat until a prime is reached; $a(n) = -1$ if no prime is ever reached |
| A046075 | Undulating number | 101, 121, 131, 141, 151, 161, 171, 181, 191, 202 | A number that has the digit form *ababab* |
| A050278 | Pandigital number | 1023456789, 1023456798, 1023456879, 1023456897, 1023456978, 1023456987, 1023457689, 1023457698, 1023457869, 1023457896 | Numbers containing the digits 0-9 such that each digit appears exactly once |
| A052486 | Achilles number | 72, 108, 200, 288, 392, 432, 500, 648, 675, 800 | Powerful but imperfect |
| A060006 | Decimal expansion of Pisot–Vijayaraghavan number | 1, 3, 2, 4, 7, 1, 7, 9, 5, 7 | real root of $x^3-x-1$ |
| A076336 | Sierpinski number | 78557, 271129, 271577, 322523, 327739, 482719, 575041, 603713, 903983, 934909 | Odd $k$ for which $\left\{\, k2^n+1 : n \in \mathbb{N} \,\right\}$ consists only of composite numbers |
| A076337 | Riesel number | 509203, 762701, 777149, 790841, 992077 | Odd $k$ for which $\left\{\, k2^n-1 : n \in \mathbb{N} \,\right\}$ consists only of composite numbers |
| A086747 | Baum–Sweet sequence | 1, 1, 0, 1, 1, 0, 0, 1, 0, 1 | $a(n) = 1$ if binary representation of $n$ contains no block of consecutive zeros of odd length; otherwise $a(n) = 0$ |
| A094683 | Juggler sequence | 0, 1, 1, 5, 2, 11, 2, 18, 2, 27 | If $n$ mod 2 = 0 then floor($\sqrt{n}$) else floor($n^{3/2}$) |
| A097942 | Highly totient number | 1, 2, 4, 8, 12, 24, 48, 72, 144, 240 | Each number $k$ on this list has more solutions to the equation $\varphi(x) = k$ than any preceding $k$ |
| A100264 | Decimal expansion of Chaitin's constant | 0, 0, 7, 8, 7, 4, 9, 9, 6, 9 | |
| A104272 | Ramanujan prime | 2, 11, 17, 29, 41, 47, 59, 67 | The $n$th Ramanujan prime is the least integer $R_n$ for which $\pi(x) - \pi(x/2) \geq n$, for all $x \geq R_n$. |
| A122045 | Euler number | 1, 0, −1, 0, 5, 0, −61, 0, 1385, 0 | $\frac{1}{\cosh t} = \frac{2}{e^t + e^{-t}} = \sum_{n=0}^{\infty} \frac{E_n}{n!} \cdot t^n$ |

# Theoretical Computer Science Cheat Sheet

## Definitions

| | |
|---|---|
| $f(n)=O(g(n))$ | iff $\exists$ positive $c, n_0$ such that $0 \le f(n) \le cg(n)\ \forall n \ge n_0$. |
| $f(n)=\Omega(g(n))$ | iff $\exists$ positive $c, n_0$ such that $f(n) \ge cg(n) \ge 0\ \forall n \ge n_0$. |
| $f(n)=\Theta(g(n))$ | iff $f(n)=O(g(n))$ and $f(n)=\Omega(g(n))$. |
| $f(n)=o(g(n))$ | iff $\lim_{n\to\infty} f(n)/g(n)=0$. |
| $\lim\limits_{n\to\infty} a_n = a$ | iff $\forall \epsilon > 0$, $\exists n_0$ such that $\lvert a_n - a\rvert < \epsilon$, $\forall n \ge n_0$. |
| $\sup S$ | least $b \in \mathbb{R}$ such that $b \ge s$, $\forall s \in S$. |
| $\inf S$ | greatest $b \in \mathbb{R}$ such that $b \le s$, $\forall s \in S$. |
| $\liminf\limits_{n\to\infty} a_n$ | $\lim\limits_{n\to\infty} \inf\{a_i \mid i \ge n, i \in \mathbb{N}\}$. |
| $\limsup\limits_{n\to\infty} a_n$ | $\lim\limits_{n\to\infty} \sup\{a_i \mid i \ge n, i \in \mathbb{N}\}$. |
| $\binom{n}{k}$ | Combinations: Size $k$ subsets of a size $n$ set. |
| $\left[{n \atop k}\right]$ | Stirling numbers (1st kind): Arrangements of an $n$ element set into $k$ cycles. |
| $\left\{{n \atop k}\right\}$ | Stirling numbers (2nd kind): Partitions of an $n$ element set into $k$ non-empty sets. |
| $\left\langle{n \atop k}\right\rangle$ | 1st order Eulerian numbers: Permutations $\pi_1\pi_2\ldots\pi_n$ on $\{1,2,\ldots,n\}$ with $k$ ascents. |
| $\left\langle\!\!\left\langle{n \atop k}\right\rangle\!\!\right\rangle$ | 2nd order Eulerian numbers. |
| $C_n$ | Catalan Numbers: Binary trees with $n+1$ vertices. |

## Series

$$\sum_{i=1}^{n} i = \frac{n(n+1)}{2}, \quad \sum_{i=1}^{n} i^2 = \frac{n(n+1)(2n+1)}{6}, \quad \sum_{i=1}^{n} i^3 = \frac{n^2(n+1)^2}{4}.$$

In general:

$$\sum_{i=1}^{n} i^m = \frac{1}{m+1}\left[(n+1)^{m+1} - 1 - \sum_{i=1}^{n}\left((i+1)^{m+1} - i^{m+1} - (m+1)i^m\right)\right]$$

$$\sum_{i=1}^{n-1} i^m = \frac{1}{m+1}\sum_{k=0}^{m}\binom{m+1}{k} B_k n^{m+1-k}.$$

Geometric series:

$$\sum_{i=0}^{n} c^i = \frac{c^{n+1}-1}{c-1}, \quad c \ne 1, \quad \sum_{i=0}^{\infty} c^i = \frac{1}{1-c}, \quad \sum_{i=1}^{\infty} c^i = \frac{c}{1-c}, \quad |c| < 1,$$

$$\sum_{i=0}^{n} ic^i = \frac{nc^{n+2} - (n+1)c^{n+1} + c}{(c-1)^2}, \quad c \ne 1, \quad \sum_{i=0}^{\infty} ic^i = \frac{c}{(1-c)^2}, \quad |c| < 1.$$

Harmonic series:

$$H_n = \sum_{i=1}^{n} \frac{1}{i}, \quad \sum_{i=1}^{n} iH_i = \frac{n(n+1)}{2}H_n - \frac{n(n-1)}{4}.$$

$$\sum_{i=1}^{n} H_i = (n+1)H_n - n, \quad \sum_{i=1}^{n}\binom{i}{m}H_i = \binom{n+1}{m+1}\left(H_{n+1} - \frac{1}{m+1}\right).$$

**1.** $\displaystyle \binom{n}{k} = \frac{n!}{(n-k)!k!}$, **2.** $\displaystyle \sum_{k=0}^{n}\binom{n}{k} = 2^n$, **3.** $\displaystyle \binom{n}{k} = \binom{n}{n-k}$,

**4.** $\displaystyle \binom{n}{k} = \frac{n}{k}\binom{n-1}{k-1}$, **5.** $\displaystyle \binom{n}{k} = \binom{n-1}{k} + \binom{n-1}{k-1}$,

**6.** $\displaystyle \binom{n}{m}\binom{m}{k} = \binom{n}{k}\binom{n-k}{m-k}$, **7.** $\displaystyle \sum_{k=0}^{n}\binom{r+k}{k} = \binom{r+n+1}{n}$,

**8.** $\displaystyle \sum_{k=0}^{n}\binom{k}{m} = \binom{n+1}{m+1}$, **9.** $\displaystyle \sum_{k=0}^{n}\binom{r}{k}\binom{s}{n-k} = \binom{r+s}{n}$,

**10.** $\displaystyle \binom{n}{k} = (-1)^k\binom{k-n-1}{k}$, **11.** $\displaystyle \left\{{n \atop 1}\right\} = \left\{{n \atop n}\right\} = 1$,

**12.** $\displaystyle \left\{{n \atop 2}\right\} = 2^{n-1} - 1$, **13.** $\displaystyle \left\{{n \atop k}\right\} = k\left\{{n-1 \atop k}\right\} + \left\{{n-1 \atop k-1}\right\}$,

**14.** $\displaystyle \left[{n \atop 1}\right] = (n-1)!$, **15.** $\displaystyle \left[{n \atop 2}\right] = (n-1)!H_{n-1}$, **16.** $\displaystyle \left[{n \atop n}\right] = 1$, **17.** $\displaystyle \left[{n \atop k}\right] \ge \left\{{n \atop k}\right\}$,

**18.** $\displaystyle \left[{n \atop k}\right] = (n-1)\left[{n-1 \atop k}\right] + \left[{n-1 \atop k-1}\right]$, **19.** $\displaystyle \left\{{n \atop n-1}\right\} = \left[{n \atop n-1}\right] = \binom{n}{2}$, **20.** $\displaystyle \sum_{k=0}^{n}\left[{n \atop k}\right] = n!$, **21.** $\displaystyle C_n = \frac{1}{n+1}\binom{2n}{n}$,

**22.** $\displaystyle \left\langle{n \atop 0}\right\rangle = \left\langle{n \atop n-1}\right\rangle = 1$, **23.** $\displaystyle \left\langle{n \atop k}\right\rangle = \left\langle{n \atop n-1-k}\right\rangle$, **24.** $\displaystyle \left\langle{n \atop k}\right\rangle = (k+1)\left\langle{n-1 \atop k}\right\rangle + (n-k)\left\langle{n-1 \atop k-1}\right\rangle$,

**25.** $\displaystyle \left\langle{0 \atop k}\right\rangle = \begin{cases} 1 & \text{if } k = 0, \\ 0 & \text{otherwise} \end{cases}$ **26.** $\displaystyle \left\langle{n \atop 1}\right\rangle = 2^n - n - 1$, **27.** $\displaystyle \left\langle{n \atop 2}\right\rangle = 3^n - (n+1)2^n + \binom{n+1}{2}$,

**28.** $\displaystyle x^n = \sum_{k=0}^{n}\left\langle{n \atop k}\right\rangle\binom{x+k}{n}$, **29.** $\displaystyle \left\langle{n \atop m}\right\rangle = \sum_{k=0}^{m}\binom{n+1}{k}(m+1-k)^n(-1)^k$, **30.** $\displaystyle m!\left\{{n \atop m}\right\} = \sum_{k=0}^{n}\left\langle{n \atop k}\right\rangle\binom{k}{n-m}$,

**31.** $\displaystyle \left\langle{n \atop m}\right\rangle = \sum_{k=0}^{n}\left\{{n \atop k}\right\}\binom{n-k}{m}(-1)^{n-k-m}k!$, **32.** $\displaystyle \left\langle\!\!\left\langle{n \atop 0}\right\rangle\!\!\right\rangle = 1$, **33.** $\displaystyle \left\langle\!\!\left\langle{n \atop n}\right\rangle\!\!\right\rangle = 0 \quad \text{for } n \ne 0$,

**34.** $\displaystyle \left\langle\!\!\left\langle{n \atop k}\right\rangle\!\!\right\rangle = (k+1)\left\langle\!\!\left\langle{n-1 \atop k}\right\rangle\!\!\right\rangle + (2n-1-k)\left\langle\!\!\left\langle{n-1 \atop k-1}\right\rangle\!\!\right\rangle$, **35.** $\displaystyle \sum_{k=0}^{n}\left\langle\!\!\left\langle{n \atop k}\right\rangle\!\!\right\rangle = \frac{(2n)^{\underline{n}}}{2^n}$,

**36.** $\displaystyle \left\{{x \atop x-n}\right\} = \sum_{k=0}^{n}\left\langle\!\!\left\langle{n \atop k}\right\rangle\!\!\right\rangle\binom{x+n-1-k}{2n}$, **37.** $\displaystyle \left\{{n+1 \atop m+1}\right\} = \sum_{k}\binom{n}{k}\left\{{k \atop m}\right\} = \sum_{k=0}^{n}\left\{{k \atop m}\right\}(m+1)^{n-k}$,

## Theoretical Computer Science Cheat Sheet

### Identities Cont.

**38.** $\genfrac{[}{]}{0pt}{}{n+1}{m+1} = \sum_k \genfrac{[}{]}{0pt}{}{n}{k}\binom{k}{m} = \sum_{k=0}^{n} \genfrac{[}{]}{0pt}{}{k}{m} n^{\underline{n-k}} = n! \sum_{k=0}^{n} \frac{1}{k!}\genfrac{[}{]}{0pt}{}{k}{m},$

**39.** $\genfrac{[}{]}{0pt}{}{x}{x-n} = \sum_{k=0}^{n} \left\langle\!\!\left\langle {n \atop k} \right\rangle\!\!\right\rangle \binom{x+k}{2n},$

**40.** $\genfrac{\{}{\}}{0pt}{}{n}{m} = \sum_k \binom{n}{k}\genfrac{\{}{\}}{0pt}{}{k+1}{m+1}(-1)^{n-k},$

**41.** $\genfrac{[}{]}{0pt}{}{n}{m} = \sum_k \genfrac{[}{]}{0pt}{}{n+1}{k+1}\binom{k}{m}(-1)^{m-k},$

**42.** $\genfrac{\{}{\}}{0pt}{}{m+n+1}{m} = \sum_{k=0}^{m} k\genfrac{\{}{\}}{0pt}{}{n+k}{k},$

**43.** $\genfrac{[}{]}{0pt}{}{m+n+1}{m} = \sum_{k=0}^{m} k(n+k)\genfrac{[}{]}{0pt}{}{n+k}{k},$

**44.** $\binom{n}{m} = \sum_k \genfrac{\{}{\}}{0pt}{}{n+1}{k+1}\genfrac{[}{]}{0pt}{}{k}{m}(-1)^{m-k},$

**45.** $(n-m)!\binom{n}{m} = \sum_k \genfrac{[}{]}{0pt}{}{n+1}{k+1}\genfrac{\{}{\}}{0pt}{}{k}{m}(-1)^{m-k}, \quad \text{for } n \ge m,$

**46.** $\genfrac{\{}{\}}{0pt}{}{n}{n-m} = \sum_k \binom{m-n}{m+k}\binom{m+n}{n+k}\genfrac{[}{]}{0pt}{}{m+k}{k},$

**47.** $\genfrac{[}{]}{0pt}{}{n}{n-m} = \sum_k \binom{m-n}{m+k}\binom{m+n}{n+k}\genfrac{\{}{\}}{0pt}{}{m+k}{k},$

**48.** $\genfrac{\{}{\}}{0pt}{}{n}{\ell+m}\binom{\ell+m}{\ell} = \sum_k \genfrac{\{}{\}}{0pt}{}{k}{\ell}\genfrac{\{}{\}}{0pt}{}{n-k}{m}\binom{n}{k},$

**49.** $\genfrac{[}{]}{0pt}{}{n}{\ell+m}\binom{\ell+m}{\ell} = \sum_k \genfrac{[}{]}{0pt}{}{k}{\ell}\genfrac{[}{]}{0pt}{}{n-k}{m}\binom{n}{k}.$

### Trees

Every tree with $n$ vertices has $n-1$ edges.

Kraft inequality: If the depths of the leaves of a binary tree are $d_1, \ldots, d_n$:

$$\sum_{i=1}^{n} 2^{-d_i} \le 1,$$

and equality holds only if every internal node has 2 sons.

### Recurrences

Master method:

$$T(n) = aT(n/b) + f(n), \quad a \ge 1, b > 1$$

If $\exists \epsilon > 0$ such that $f(n) = O(n^{\log_b a - \epsilon})$ then

$$T(n) = \Theta(n^{\log_b a}).$$

If $f(n) = \Theta(n^{\log_b a})$ then

$$T(n) = \Theta(n^{\log_b a} \log_2 n).$$

If $\exists \epsilon > 0$ such that $f(n) = \Omega(n^{\log_b a + \epsilon})$, and $\exists c < 1$ such that $af(n/b) \le cf(n)$ for large $n$, then

$$T(n) = \Theta(f(n)).$$

Substitution (example): Consider the following recurrence

$$T_{i+1} = 2^{2^i} \cdot T_i^2, \quad T_1 = 2.$$

Note that $T_i$ is always a power of two. Let $t_i = \log_2 T_i$. Then we have

$$t_{i+1} = 2^i + 2t_i, \quad t_1 = 1.$$

Let $u_i = t_i/2^i$. Dividing both sides of the previous equation by $2^{i+1}$ we get

$$\frac{t_{i+1}}{2^{i+1}} = \frac{2^i}{2^{i+1}} + \frac{t_i}{2^i}.$$

Substituting we find

$$u_{i+1} = \tfrac{1}{2} + u_i, \qquad u_1 = \tfrac{1}{2},$$

which is simply $u_i = i/2$. So we find that $T_i$ has the closed form $T_i = 2^{i2^{i-1}}$. Summing factors (example): Consider the following recurrence

$$T(n) = 3T(n/2) + n, \quad T(1) = 1.$$

Rewrite so that all terms involving $T$ are on the left side

$$T(n) - 3T(n/2) = n.$$

Now expand the recurrence, and choose a factor which makes the left side "telescope"

$$\begin{aligned} 1\big(T(n) - 3T(n/2) &= n\big) \\ 3\big(T(n/2) - 3T(n/4) &= n/2\big) \\ \vdots \quad \vdots \quad &\vdots \\ 3^{\log_2 n - 1}\big(T(2) - 3T(1) &= 2\big) \end{aligned}$$

Let $m = \log_2 n$. Summing the left side we get $T(n) - 3^m T(1) = T(n) - 3^m = T(n) - n^k$ where $k = \log_2 3 \approx 1.58496$. Summing the right side we get

$$\sum_{i=0}^{m-1} \frac{n}{2^i} 3^i = n \sum_{i=0}^{m-1} \left(\tfrac{3}{2}\right)^i.$$

Let $c = \frac{3}{2}$. Then we have

$$\begin{aligned} n \sum_{i=0}^{m-1} c^i &= n\left(\frac{c^m - 1}{c - 1}\right) \\ &= 2n(c^{\log_2 n} - 1) \\ &= 2n(c^{(k-1)\log_c n} - 1) \\ &= 2n^k - 2n, \end{aligned}$$

and so $T(n) = 3n^k - 2n$. Full history recurrences can often be changed to limited history ones (example): Consider

$$T_i = 1 + \sum_{j=0}^{i-1} T_j, \quad T_0 = 1.$$

Note that

$$T_{i+1} = 1 + \sum_{j=0}^{i} T_j.$$

Subtracting we find

$$\begin{aligned} T_{i+1} - T_i &= 1 + \sum_{j=0}^{i} T_j - 1 - \sum_{j=0}^{i-1} T_j \\ &= T_i. \end{aligned}$$

And so $T_{i+1} = 2T_i = 2^{i+1}$.

Generating functions:

1. Multiply both sides of the equation by $x^i$.
2. Sum both sides over all $i$ for which the equation is valid.
3. Choose a generating function $G(x)$. Usually $G(x) = \sum_{i=0}^{\infty} x^i g_i$.
3. Rewrite the equation in terms of the generating function $G(x)$.
4. Solve for $G(x)$.
5. The coefficient of $x^i$ in $G(x)$ is $g_i$.

Example:

$$g_{i+1} = 2g_i + 1, \quad g_0 = 0.$$

Multiply and sum:

$$\sum_{i \ge 0} g_{i+1}x^i = \sum_{i \ge 0} 2g_i x^i + \sum_{i \ge 0} x^i.$$

We choose $G(x) = \sum_{i \ge 0} x^i g_i$. Rewrite in terms of $G(x)$:

$$\frac{G(x) - g_0}{x} = 2G(x) + \sum_{i \ge 0} x^i.$$

Simplify:

$$\frac{G(x)}{x} = 2G(x) + \frac{1}{1-x}.$$

Solve for $G(x)$:

$$G(x) = \frac{x}{(1-x)(1-2x)}.$$

Expand this using partial fractions:

$$\begin{aligned} G(x) &= x\left(\frac{2}{1-2x} - \frac{1}{1-x}\right) \\ &= x\left(2\sum_{i \ge 0} 2^i x^i - \sum_{i \ge 0} x^i\right) \\ &= \sum_{i \ge 0} (2^{i+1} - 1)x^{i+1}. \end{aligned}$$

So $g_i = 2^i - 1$.

**Theoretical Computer Science Cheat Sheet**

$\pi \approx 3.14159$, $e \approx 2.71828$, $\gamma \approx 0.57721$, $\phi = \frac{1+\sqrt{5}}{2} \approx 1.61803$, $\hat{\phi} = \frac{1-\sqrt{5}}{2} \approx -.61803$

| $i$ | $2^i$ | $p_i$ |
|---|---|---|
| 1 | 2 | 2 |
| 2 | 4 | 3 |
| 3 | 8 | 5 |
| 4 | 16 | 7 |
| 5 | 32 | 11 |
| 6 | 64 | 13 |
| 7 | 128 | 17 |
| 8 | 256 | 19 |
| 9 | 512 | 23 |
| 10 | 1,024 | 29 |
| 11 | 2,048 | 31 |
| 12 | 4,096 | 37 |
| 13 | 8,192 | 41 |
| 14 | 16,384 | 43 |
| 15 | 32,768 | 47 |
| 16 | 65,536 | 53 |
| 17 | 131,072 | 59 |
| 18 | 262,144 | 61 |
| 19 | 524,288 | 67 |
| 20 | 1,048,576 | 71 |
| 21 | 2,097,152 | 73 |
| 22 | 4,194,304 | 79 |
| 23 | 8,388,608 | 83 |
| 24 | 16,777,216 | 89 |
| 25 | 33,554,432 | 97 |
| 26 | 67,108,864 | 101 |
| 27 | 134,217,728 | 103 |
| 28 | 268,435,456 | 107 |
| 29 | 536,870,912 | 109 |
| 30 | 1,073,741,824 | 113 |
| 31 | 2,147,483,648 | 127 |
| 32 | 4,294,967,296 | 131 |

Pascal's Triangle

1
1 1
1 2 1
1 3 3 1
1 4 6 4 1
1 5 10 10 5 1
1 6 15 20 15 6 1
1 7 21 35 35 21 7 1
1 8 28 56 70 56 28 8 1
1 9 36 84 126 126 84 36 9 1
1 10 45 120 210 252 210 120 45 10 1

## General

Bernoulli Numbers ($B_i = 0$, odd $i \neq 1$):

$$B_0 = 1,\ B_1 = -\tfrac{1}{2},\ B_2 = \tfrac{1}{6},\ B_4 = -\tfrac{1}{30},$$
$$B_6 = \tfrac{1}{42},\ B_8 = -\tfrac{1}{30},\ B_{10} = \tfrac{5}{66}.$$

Change of base, quadratic formula:

$$\log_b x = \frac{\log_a x}{\log_a b}, \qquad \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}.$$

Euler's number $e$:

$$e = 1 + \tfrac{1}{2} + \tfrac{1}{6} + \tfrac{1}{24} + \tfrac{1}{120} + \cdots$$
$$\lim_{n\to\infty}\left(1 + \frac{x}{n}\right)^n = e^x.$$
$$\left(1 + \tfrac{1}{n}\right)^n < e < \left(1 + \tfrac{1}{n}\right)^{n+1}.$$
$$\left(1 + \tfrac{1}{n}\right)^n = e - \frac{e}{2n} + \frac{11e}{24n^2} - O\left(\frac{1}{n^3}\right).$$

Harmonic numbers:

$$1,\ \tfrac{3}{2},\ \tfrac{11}{6},\ \tfrac{25}{12},\ \tfrac{137}{60},\ \tfrac{49}{20},\ \tfrac{363}{140},\ \tfrac{761}{280},\ \tfrac{7129}{2520}, \ldots$$

$$\ln n < H_n < \ln n + 1,$$
$$H_n = \ln n + \gamma + O\left(\frac{1}{n}\right).$$

Factorial, Stirling's approximation:

1, 2, 6, 24, 120, 720, 5040, 40320, 362880, ...

$$n! = \sqrt{2\pi n}\left(\frac{n}{e}\right)^n\left(1 + \Theta\left(\frac{1}{n}\right)\right).$$

Ackermann's function and inverse:

$$a(i,j) = \begin{cases} 2^j & i = 1 \\ a(i-1, 2) & j = 1 \\ a(i-1, a(i, j-1)) & i, j \geq 2 \end{cases}$$
$$\alpha(i) = \min\{j \mid a(j,j) \geq i\}.$$

Binomial distribution:

$$\Pr[X = k] = \binom{n}{k} p^k q^{n-k}, \qquad q = 1 - p,$$
$$\mathrm{E}[X] = \sum_{k=1}^{n} k\binom{n}{k} p^k q^{n-k} = np.$$

Poisson distribution:

$$\Pr[X = k] = \frac{e^{-\lambda}\lambda^k}{k!}, \quad \mathrm{E}[X] = \lambda.$$

Normal (Gaussian) distribution:

$$p(x) = \frac{1}{\sqrt{2\pi}\sigma} e^{-(x-\mu)^2/2\sigma^2}, \quad \mathrm{E}[X] = \mu.$$

The "coupon collector": We are given a random coupon each day, and there are $n$ different types of coupons. The distribution of coupons is uniform. The expected number of days to pass before we to collect all $n$ types is

$$nH_n.$$

## Probability

Continuous distributions: If

$$\Pr[a < X < b] = \int_a^b p(x)\,dx,$$

then $p$ is the probability density function of $X$. If

$$\Pr[X < a] = P(a),$$

then $P$ is the distribution function of $X$. If $P$ and $p$ both exist then

$$P(a) = \int_{-\infty}^{a} p(x)\,dx.$$

Expectation: If $X$ is discrete

$$\mathrm{E}[g(X)] = \sum_x g(x)\Pr[X = x].$$

If $X$ continuous then

$$\mathrm{E}[g(X)] = \int_{-\infty}^{\infty} g(x)p(x)\,dx = \int_{-\infty}^{\infty} g(x)\,dP(x).$$

Variance, standard deviation:

$$\mathrm{VAR}[X] = \mathrm{E}[X^2] - \mathrm{E}[X]^2,$$
$$\sigma = \sqrt{\mathrm{VAR}[X]}.$$

For events $A$ and $B$:

$$\Pr[A \vee B] = \Pr[A] + \Pr[B] - \Pr[A \wedge B]$$
$$\Pr[A \wedge B] = \Pr[A] \cdot \Pr[B],$$

iff $A$ and $B$ are independent.

$$\Pr[A|B] = \frac{\Pr[A \wedge B]}{\Pr[B]}$$

For random variables $X$ and $Y$:

$$\mathrm{E}[X \cdot Y] = \mathrm{E}[X] \cdot \mathrm{E}[Y],$$

if $X$ and $Y$ are independent.

$$\mathrm{E}[X + Y] = \mathrm{E}[X] + \mathrm{E}[Y],$$
$$\mathrm{E}[cX] = c\,\mathrm{E}[X].$$

Bayes' theorem:

$$\Pr[A_i|B] = \frac{\Pr[B|A_i]\Pr[A_i]}{\sum_{j=1}^{n}\Pr[A_j]\Pr[B|A_j]}.$$

Inclusion-exclusion:

$$\Pr\left[\bigvee_{i=1}^{n} X_i\right] = \sum_{i=1}^{n}\Pr[X_i] + \sum_{k=2}^{n}(-1)^{k+1}\sum_{i_i < \cdots < i_k}\Pr\left[\bigwedge_{j=1}^{k} X_{i_j}\right].$$

Moment inequalities:

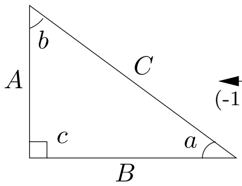
$$\Pr\left[|X| \geq \lambda\,\mathrm{E}[X]\right] \leq \frac{1}{\lambda},$$
$$\Pr\left[\left|X - \mathrm{E}[X]\right| \geq \lambda \cdot \sigma\right] \leq \frac{1}{\lambda^2}.$$

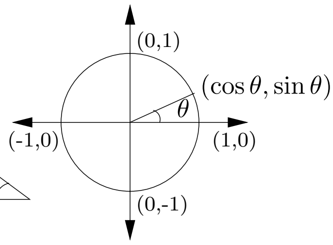
Geometric distribution:

$$\Pr[X = k] = pq^{k-1}, \qquad q = 1 - p,$$
$$\mathrm{E}[X] = \sum_{k=1}^{\infty} kpq^{k-1} = \frac{1}{p}.$$

# Theoretical Computer Science Cheat Sheet

## Trigonometry




Pythagorean theorem:

$$C^2 = A^2 + B^2.$$

Definitions:

$$\sin a = A/C, \quad \cos a = B/C,$$

$$\csc a = C/A, \quad \sec a = C/B,$$

$$\tan a = \frac{\sin a}{\cos a} = \frac{A}{B}, \quad \cot a = \frac{\cos a}{\sin a} = \frac{B}{A}.$$

Area, radius of inscribed circle:

$$\tfrac{1}{2}AB, \quad \frac{AB}{A+B+C}.$$

Identities:

$$\sin x = \frac{1}{\csc x}, \qquad \cos x = \frac{1}{\sec x},$$

$$\tan x = \frac{1}{\cot x}, \qquad \sin^2 x + \cos^2 x = 1,$$

$$1 + \tan^2 x = \sec^2 x, \qquad 1 + \cot^2 x = \csc^2 x,$$

$$\sin x = \cos\left(\tfrac{\pi}{2} - x\right), \qquad \sin x = \sin(\pi - x),$$

$$\cos x = -\cos(\pi - x), \qquad \tan x = \cot\left(\tfrac{\pi}{2} - x\right),$$

$$\cot x = -\cot(\pi - x), \qquad \csc x = \cot\tfrac{x}{2} - \cot x,$$

$$\sin(x \pm y) = \sin x \cos y \pm \cos x \sin y,$$

$$\cos(x \pm y) = \cos x \cos y \mp \sin x \sin y,$$

$$\tan(x \pm y) = \frac{\tan x \pm \tan y}{1 \mp \tan x \tan y},$$

$$\cot(x \pm y) = \frac{\cot x \cot y \mp 1}{\cot x \pm \cot y},$$

$$\sin 2x = 2 \sin x \cos x, \qquad \sin 2x = \frac{2 \tan x}{1 + \tan^2 x},$$

$$\cos 2x = \cos^2 x - \sin^2 x, \qquad \cos 2x = 2\cos^2 x - 1,$$

$$\cos 2x = 1 - 2\sin^2 x, \qquad \cos 2x = \frac{1 - \tan^2 x}{1 + \tan^2 x},$$

$$\tan 2x = \frac{2 \tan x}{1 - \tan^2 x}, \qquad \cot 2x = \frac{\cot^2 x - 1}{2 \cot x},$$

$$\sin(x+y)\sin(x-y) = \sin^2 x - \sin^2 y,$$

$$\cos(x+y)\cos(x-y) = \cos^2 x - \sin^2 y.$$

Euler's equation:

$$e^{ix} = \cos x + i \sin x, \qquad e^{i\pi} = -1.$$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## Matrices

Multiplication:

$$C = A \cdot B, \quad c_{i,j} = \sum_{k=1}^{n} a_{i,k} b_{k,j}.$$

Determinants: $\det A \neq 0$ iff $A$ is non-singular.

$$\det A \cdot B = \det A \cdot \det B,$$

$$\det A = \sum_{\pi} \prod_{i=1}^{n} \operatorname{sign}(\pi) a_{i,\pi(i)}.$$

$2 \times 2$ and $3 \times 3$ determinant:

$$\begin{vmatrix} a & b \\ c & d \end{vmatrix} = ad - bc,$$

$$\begin{vmatrix} a & b & c \\ d & e & f \\ g & h & i \end{vmatrix} = g \begin{vmatrix} b & c \\ e & f \end{vmatrix} - h \begin{vmatrix} a & c \\ d & f \end{vmatrix} + i \begin{vmatrix} a & b \\ d & e \end{vmatrix}$$

$$= \begin{array}{c} aei + bfg + cdh \\ - ceg - fha - ibd. \end{array}$$

Permanents:

$$\operatorname{perm} A = \sum_{\pi} \prod_{i=1}^{n} a_{i,\pi(i)}.$$

## Hyperbolic Functions

Definitions:

$$\sinh x = \frac{e^x - e^{-x}}{2}, \qquad \cosh x = \frac{e^x + e^{-x}}{2},$$

$$\tanh x = \frac{e^x - e^{-x}}{e^x + e^{-x}}, \qquad \operatorname{csch} x = \frac{1}{\sinh x},$$

$$\operatorname{sech} x = \frac{1}{\cosh x}, \qquad \coth x = \frac{1}{\tanh x}.$$

Identities:

$$\cosh^2 x - \sinh^2 x = 1, \qquad \tanh^2 x + \operatorname{sech}^2 x = 1,$$

$$\coth^2 x - \operatorname{csch}^2 x = 1, \qquad \sinh(-x) = -\sinh x,$$

$$\cosh(-x) = \cosh x, \qquad \tanh(-x) = -\tanh x,$$

$$\sinh(x+y) = \sinh x \cosh y + \cosh x \sinh y,$$

$$\cosh(x+y) = \cosh x \cosh y + \sinh x \sinh y,$$

$$\sinh 2x = 2 \sinh x \cosh x,$$

$$\cosh 2x = \cosh^2 x + \sinh^2 x,$$

$$\cosh x + \sinh x = e^x, \qquad \cosh x - \sinh x = e^{-x},$$

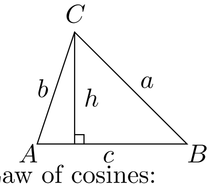
$$(\cosh x + \sinh x)^n = \cosh nx + \sinh nx, \quad n \in \mathbb{Z},$$

$$2\sinh^2 \tfrac{x}{2} = \cosh x - 1, \qquad 2\cosh^2 \tfrac{x}{2} = \cosh x + 1.$$

| $\theta$ | $\sin\theta$ | $\cos\theta$ | $\tan\theta$ |
|---|---|---|---|
| $0$ | $0$ | $1$ | $0$ |
| $\frac{\pi}{6}$ | $\frac{1}{2}$ | $\frac{\sqrt{3}}{2}$ | $\frac{\sqrt{3}}{3}$ |
| $\frac{\pi}{4}$ | $\frac{\sqrt{2}}{2}$ | $\frac{\sqrt{2}}{2}$ | $1$ |
| $\frac{\pi}{3}$ | $\frac{\sqrt{3}}{2}$ | $\frac{1}{2}$ | $\sqrt{3}$ |
| $\frac{\pi}{2}$ | $1$ | $0$ | $\infty$ |

... in mathematics you don't understand things, you just get used to them.
– J. von Neumann

## More Trig.




Law of cosines:

$$c^2 = a^2 + b^2 - 2ab\cos C.$$

Area:

$$A = \tfrac{1}{2}hc,$$

$$= \tfrac{1}{2}ab\sin C,$$

$$= \frac{c^2 \sin A \sin B}{2 \sin C}.$$

Heron's formula:

$$A = \sqrt{s \cdot s_a \cdot s_b \cdot s_c},$$

$$s = \tfrac{1}{2}(a + b + c),$$

$$s_a = s - a,$$

$$s_b = s - b,$$

$$s_c = s - c.$$

More identities:

$$\sin \tfrac{x}{2} = \sqrt{\frac{1 - \cos x}{2}},$$

$$\cos \tfrac{x}{2} = \sqrt{\frac{1 + \cos x}{2}},$$

$$\tan \tfrac{x}{2} = \sqrt{\frac{1 - \cos x}{1 + \cos x}},$$

$$= \frac{1 - \cos x}{\sin x},$$

$$= \frac{\sin x}{1 + \cos x},$$

$$\cot \tfrac{x}{2} = \sqrt{\frac{1 + \cos x}{1 - \cos x}},$$

$$= \frac{1 + \cos x}{\sin x},$$

$$= \frac{\sin x}{1 - \cos x},$$

$$\sin x = \frac{e^{ix} - e^{-ix}}{2i},$$

$$\cos x = \frac{e^{ix} + e^{-ix}}{2},$$

$$\tan x = -i\frac{e^{ix} - e^{-ix}}{e^{ix} + e^{-ix}},$$

$$= -i\frac{e^{2ix} - 1}{e^{2ix} + 1},$$

$$\sin x = \frac{\sinh ix}{i},$$

$$\cos x = \cosh ix,$$

$$\tan x = \frac{\tanh ix}{i}.$$

# Theoretical Computer Science Cheat Sheet

## Number Theory

The Chinese remainder theorem: There exists a number $C$ such that:

$$\begin{aligned} C &\equiv r_1 \bmod m_1 \\ \vdots \quad & \vdots \quad \vdots \\ C &\equiv r_n \bmod m_n \end{aligned}$$

if $m_i$ and $m_j$ are relatively prime for $i \neq j$.

Euler's function: $\phi(x)$ is the number of positive integers less than $x$ relatively prime to $x$. If $\prod_{i=1}^{n} p_i^{e_i}$ is the prime factorization of $x$ then

$$\phi(x) = \prod_{i=1}^{n} p_i^{e_i - 1}(p_i - 1).$$

Euler's theorem: If $a$ and $b$ are relatively prime then

$$1 \equiv a^{\phi(b)} \bmod b.$$

Fermat's theorem:

$$1 \equiv a^{p-1} \bmod p.$$

The Euclidean algorithm: if $a > b$ are integers then

$$\gcd(a, b) = \gcd(a \bmod b, b).$$

If $\prod_{i=1}^{n} p_i^{e_i}$ is the prime factorization of $x$ then

$$S(x) = \sum_{d \mid x} d = \prod_{i=1}^{n} \frac{p_i^{e_i+1} - 1}{p_i - 1}.$$

Perfect Numbers: $x$ is an even perfect number iff $x = 2^{n-1}(2^n - 1)$ and $2^n - 1$ is prime.

Wilson's theorem: $n$ is a prime iff

$$(n-1)! \equiv -1 \bmod n.$$

Möbius inversion:

$$\mu(i) = \begin{cases} 1 & \text{if } i = 1. \\ 0 & \text{if } i \text{ is not square-free.} \\ (-1)^r & \text{if } i \text{ is the product of} \\ & r \text{ distinct primes.} \end{cases}$$

If

$$G(a) = \sum_{d \mid a} F(d),$$

then

$$F(a) = \sum_{d \mid a} \mu(d) G\left(\frac{a}{d}\right).$$

Prime numbers:

$$p_n = n \ln n + n \ln \ln n - n + n \frac{\ln \ln n}{\ln n} + O\left(\frac{n}{\ln n}\right),$$

$$\pi(n) = \frac{n}{\ln n} + \frac{n}{(\ln n)^2} + \frac{2!n}{(\ln n)^3} + O\left(\frac{n}{(\ln n)^4}\right).$$

## Graph Theory

| Definitions: | |
|---|---|
| *Loop* | An edge connecting a vertex to itself. |
| *Directed* | Each edge has a direction. |
| *Simple* | Graph with no loops or multi-edges. |
| *Walk* | A sequence $v_0 e_1 v_1 \ldots e_\ell v_\ell$. |
| *Trail* | A walk with distinct edges. |
| *Path* | A trail with distinct vertices. |
| *Connected* | A graph where there exists a path between any two vertices. |
| *Component* | A maximal connected subgraph. |
| *Tree* | A connected acyclic graph. |
| *Free tree* | A tree with no root. |
| *DAG* | Directed acyclic graph. |
| *Eulerian* | Graph with a trail visiting each edge exactly once. |
| *Hamiltonian* | Graph with a cycle visiting each vertex exactly once. |
| *Cut* | A set of edges whose removal increases the number of components. |
| *Cut-set* | A minimal cut. |
| *Cut edge* | A size 1 cut. |
| *k-Connected* | A graph connected with the removal of any $k-1$ vertices. |
| *k-Tough* | $\forall S \subseteq V, S \neq \emptyset$ we have $k \cdot c(G - S) \leq \lvert S \rvert$. |
| *k-Regular* | A graph where all vertices have degree $k$. |
| *k-Factor* | A $k$-regular spanning subgraph. |
| *Matching* | A set of edges, no two of which are adjacent. |
| *Clique* | A set of vertices, all of which are adjacent. |
| *Ind. set* | A set of vertices, none of which are adjacent. |
| *Vertex cover* | A set of vertices which cover all edges. |
| *Planar graph* | A graph which can be embeded in the plane. |
| *Plane graph* | An embedding of a planar graph. |

$$\sum_{v \in V} \deg(v) = 2m.$$

If $G$ is planar then $n - m + f = 2$, so

$$f \leq 2n - 4, \quad m \leq 3n - 6.$$

Any planar graph has a vertex with degree $\leq 5$.

| Notation: | |
|---|---|
| $E(G)$ | Edge set |
| $V(G)$ | Vertex set |
| $c(G)$ | Number of components |
| $G[S]$ | Induced subgraph |
| $\deg(v)$ | Degree of $v$ |
| $\Delta(G)$ | Maximum degree |
| $\delta(G)$ | Minimum degree |
| $\chi(G)$ | Chromatic number |
| $\chi_E(G)$ | Edge chromatic number |
| $G^c$ | Complement graph |
| $K_n$ | Complete graph |
| $K_{n_1,n_2}$ | Complete bipartite graph |
| $\mathrm{r}(k, \ell)$ | Ramsey number |

## Geometry

Projective coordinates: triples $(x, y, z)$, not all $x$, $y$ and $z$ zero.

$$(x, y, z) = (cx, cy, cz) \quad \forall c \neq 0.$$

| Cartesian | Projective |
|---|---|
| $(x, y)$ | $(x, y, 1)$ |
| $y = mx + b$ | $(m, -1, b)$ |
| $x = c$ | $(1, 0, -c)$ |

Distance formula, $L_p$ and $L_\infty$ metric:

$$\sqrt{(x_1 - x_0)^2 + (y_1 - y_0)^2},$$

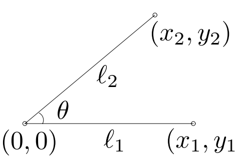
$$\left[\lvert x_1 - x_0 \rvert^p + \lvert y_1 - y_0 \rvert^p\right]^{1/p},$$

$$\lim_{p \to \infty} \left[\lvert x_1 - x_0 \rvert^p + \lvert y_1 - y_0 \rvert^p\right]^{1/p}.$$

Area of triangle $(x_0, y_0)$, $(x_1, y_1)$ and $(x_2, y_2)$:

$$\tfrac{1}{2} \operatorname{abs} \begin{vmatrix} x_1 - x_0 & y_1 - y_0 \\ x_2 - x_0 & y_2 - y_0 \end{vmatrix}.$$

Angle formed by three points:




$$\cos\theta = \frac{(x_1, y_1) \cdot (x_2, y_2)}{\ell_1 \ell_2}.$$

Line through two points $(x_0, y_0)$ and $(x_1, y_1)$:

$$\begin{vmatrix} x & y & 1 \\ x_0 & y_0 & 1 \\ x_1 & y_1 & 1 \end{vmatrix} = 0.$$

Area of circle, volume of sphere:

$$A = \pi r^2, \qquad V = \tfrac{4}{3}\pi r^3.$$

If I have seen farther than others, it is because I have stood on the shoulders of giants.
– Issac Newton

**Theoretical Computer Science Cheat Sheet**

## $\pi$

Wallis' identity:
$$\pi = 2 \cdot \frac{2 \cdot 2 \cdot 4 \cdot 4 \cdot 6 \cdot 6 \cdots}{1 \cdot 3 \cdot 3 \cdot 5 \cdot 5 \cdot 7 \cdots}$$

Brouncker's continued fraction expansion:
$$\tfrac{\pi}{4} = 1 + \cfrac{1^2}{2 + \cfrac{3^2}{2 + \cfrac{5^2}{2 + \cfrac{7^2}{2 + \cdots}}}}$$

Gregrory's series:
$$\tfrac{\pi}{4} = 1 - \tfrac{1}{3} + \tfrac{1}{5} - \tfrac{1}{7} + \tfrac{1}{9} - \cdots$$

Newton's series:
$$\tfrac{\pi}{6} = \frac{1}{2} + \frac{1}{2 \cdot 3 \cdot 2^3} + \frac{1 \cdot 3}{2 \cdot 4 \cdot 5 \cdot 2^5} + \cdots$$

Sharp's series:
$$\tfrac{\pi}{6} = \frac{1}{\sqrt{3}}\left(1 - \frac{1}{3^1 \cdot 3} + \frac{1}{3^2 \cdot 5} - \frac{1}{3^3 \cdot 7} + \cdots\right)$$

Euler's series:
$$\tfrac{\pi^2}{6} = \tfrac{1}{1^2} + \tfrac{1}{2^2} + \tfrac{1}{3^2} + \tfrac{1}{4^2} + \tfrac{1}{5^2} + \cdots$$
$$\tfrac{\pi^2}{8} = \tfrac{1}{1^2} + \tfrac{1}{3^2} + \tfrac{1}{5^2} + \tfrac{1}{7^2} + \tfrac{1}{9^2} + \cdots$$
$$\tfrac{\pi^2}{12} = \tfrac{1}{1^2} - \tfrac{1}{2^2} + \tfrac{1}{3^2} - \tfrac{1}{4^2} + \tfrac{1}{5^2} - \cdots$$

## Partial Fractions

Let $N(x)$ and $D(x)$ be polynomial functions of $x$. We can break down $N(x)/D(x)$ using partial fraction expansion. First, if the degree of $N$ is greater than or equal to the degree of $D$, divide $N$ by $D$, obtaining
$$\frac{N(x)}{D(x)} = Q(x) + \frac{N'(x)}{D(x)},$$
where the degree of $N'$ is less than that of $D$. Second, factor $D(x)$. Use the following rules: For a non-repeated factor:
$$\frac{N(x)}{(x-a)D(x)} = \frac{A}{x-a} + \frac{N'(x)}{D(x)},$$
where
$$A = \left[\frac{N(x)}{D(x)}\right]_{x=a}.$$
For a repeated factor:
$$\frac{N(x)}{(x-a)^m D(x)} = \sum_{k=0}^{m-1} \frac{A_k}{(x-a)^{m-k}} + \frac{N'(x)}{D(x)},$$
where
$$A_k = \frac{1}{k!}\left[\frac{d^k}{dx^k}\left(\frac{N(x)}{D(x)}\right)\right]_{x=a}.$$

The reasonable man adapts himself to the world; the unreasonable persists in trying to adapt the world to himself. Therefore all progress depends on the unreasonable.
– George Bernard Shaw

## Calculus

Derivatives:

**1.** $\dfrac{d(cu)}{dx} = c\dfrac{du}{dx}$, **2.** $\dfrac{d(u+v)}{dx} = \dfrac{du}{dx} + \dfrac{dv}{dx}$, **3.** $\dfrac{d(uv)}{dx} = u\dfrac{dv}{dx} + v\dfrac{du}{dx}$,

**4.** $\dfrac{d(u^n)}{dx} = nu^{n-1}\dfrac{du}{dx}$, **5.** $\dfrac{d(u/v)}{dx} = \dfrac{v\left(\frac{du}{dx}\right) - u\left(\frac{dv}{dx}\right)}{v^2}$, **6.** $\dfrac{d(e^{cu})}{dx} = ce^{cu}\dfrac{du}{dx}$,

**7.** $\dfrac{d(c^u)}{dx} = (\ln c)c^u\dfrac{du}{dx}$, **8.** $\dfrac{d(\ln u)}{dx} = \dfrac{1}{u}\dfrac{du}{dx}$,

**9.** $\dfrac{d(\sin u)}{dx} = \cos u\dfrac{du}{dx}$, **10.** $\dfrac{d(\cos u)}{dx} = -\sin u\dfrac{du}{dx}$,

**11.** $\dfrac{d(\tan u)}{dx} = \sec^2 u\dfrac{du}{dx}$, **12.** $\dfrac{d(\cot u)}{dx} = \csc^2 u\dfrac{du}{dx}$,

**13.** $\dfrac{d(\sec u)}{dx} = \tan u \, \sec u\dfrac{du}{dx}$, **14.** $\dfrac{d(\csc u)}{dx} = -\cot u \, \csc u\dfrac{du}{dx}$,

**15.** $\dfrac{d(\arcsin u)}{dx} = \dfrac{1}{\sqrt{1-u^2}}\dfrac{du}{dx}$, **16.** $\dfrac{d(\arccos u)}{dx} = \dfrac{-1}{\sqrt{1-u^2}}\dfrac{du}{dx}$,

**17.** $\dfrac{d(\arctan u)}{dx} = \dfrac{1}{1+u^2}\dfrac{du}{dx}$, **18.** $\dfrac{d(\operatorname{arccot} u)}{dx} = \dfrac{-1}{1+u^2}\dfrac{du}{dx}$,

**19.** $\dfrac{d(\operatorname{arcsec} u)}{dx} = \dfrac{1}{u\sqrt{1-u^2}}\dfrac{du}{dx}$, **20.** $\dfrac{d(\operatorname{arccsc} u)}{dx} = \dfrac{-1}{u\sqrt{1-u^2}}\dfrac{du}{dx}$,

**21.** $\dfrac{d(\sinh u)}{dx} = \cosh u\dfrac{du}{dx}$, **22.** $\dfrac{d(\cosh u)}{dx} = \sinh u\dfrac{du}{dx}$,

**23.** $\dfrac{d(\tanh u)}{dx} = \operatorname{sech}^2 u\dfrac{du}{dx}$, **24.** $\dfrac{d(\coth u)}{dx} = -\operatorname{csch}^2 u\dfrac{du}{dx}$,

**25.** $\dfrac{d(\operatorname{sech} u)}{dx} = -\operatorname{sech} u \, \tanh u\dfrac{du}{dx}$, **26.** $\dfrac{d(\operatorname{csch} u)}{dx} = -\operatorname{csch} u \, \coth u\dfrac{du}{dx}$,

**27.** $\dfrac{d(\operatorname{arcsinh} u)}{dx} = \dfrac{1}{\sqrt{1+u^2}}\dfrac{du}{dx}$, **28.** $\dfrac{d(\operatorname{arccosh} u)}{dx} = \dfrac{1}{\sqrt{u^2-1}}\dfrac{du}{dx}$,

**29.** $\dfrac{d(\operatorname{arctanh} u)}{dx} = \dfrac{1}{1-u^2}\dfrac{du}{dx}$, **30.** $\dfrac{d(\operatorname{arccoth} u)}{dx} = \dfrac{1}{u^2-1}\dfrac{du}{dx}$,

**31.** $\dfrac{d(\operatorname{arcsech} u)}{dx} = \dfrac{-1}{u\sqrt{1-u^2}}\dfrac{du}{dx}$, **32.** $\dfrac{d(\operatorname{arccsch} u)}{dx} = \dfrac{-1}{|u|\sqrt{1+u^2}}\dfrac{du}{dx}$.

Integrals:

**1.** $\displaystyle\int cu\,dx = c\int u\,dx$, **2.** $\displaystyle\int (u+v)\,dx = \int u\,dx + \int v\,dx$,

**3.** $\displaystyle\int x^n\,dx = \frac{1}{n+1}x^{n+1}, \quad n \neq -1$, **4.** $\displaystyle\int \frac{1}{x}dx = \ln x$, **5.** $\displaystyle\int e^x\,dx = e^x$,

**6.** $\displaystyle\int \frac{dx}{1+x^2} = \arctan x$, **7.** $\displaystyle\int u\frac{dv}{dx}dx = uv - \int v\frac{du}{dx}dx$,

**8.** $\displaystyle\int \sin x\,dx = -\cos x$, **9.** $\displaystyle\int \cos x\,dx = \sin x$,

**10.** $\displaystyle\int \tan x\,dx = -\ln|\cos x|$, **11.** $\displaystyle\int \cot x\,dx = \ln|\cos x|$,

**12.** $\displaystyle\int \sec x\,dx = \ln|\sec x + \tan x|$, **13.** $\displaystyle\int \csc x\,dx = \ln|\csc x + \cot x|$,

**14.** $\displaystyle\int \arcsin \tfrac{x}{a}dx = \arcsin \tfrac{x}{a} + \sqrt{a^2 - x^2}, \quad a > 0$, Calculus Cont.

**15.** $\int \arccos \frac{x}{a} dx = \arccos \frac{x}{a} - \sqrt{a^2 - x^2}, \quad a > 0,$

**16.** $\int \arctan \frac{x}{a} dx = x \arctan \frac{x}{a} - \frac{a}{2} \ln(a^2 + x^2), \quad a > 0,$

**17.** $\int \sin^2(ax) dx = \frac{1}{2a}\big(ax - \sin(ax)\cos(ax)\big),$

**18.** $\int \cos^2(ax) dx = \frac{1}{2a}\big(ax + \sin(ax)\cos(ax)\big),$

**19.** $\int \sec^2 x \, dx = \tan x,$

**20.** $\int \csc^2 x \, dx = -\cot x,$

**21.** $\int \sin^n x \, dx = -\frac{\sin^{n-1} x \cos x}{n} + \frac{n-1}{n} \int \sin^{n-2} x \, dx,$

**22.** $\int \cos^n x \, dx = \frac{\cos^{n-1} x \sin x}{n} + \frac{n-1}{n} \int \cos^{n-2} x \, dx,$

**23.** $\int \tan^n x \, dx = \frac{\tan^{n-1} x}{n-1} - \int \tan^{n-2} x \, dx, \quad n \neq 1,$

**24.** $\int \cot^n x \, dx = -\frac{\cot^{n-1} x}{n-1} - \int \cot^{n-2} x \, dx, \quad n \neq 1,$

**25.** $\int \sec^n x \, dx = \frac{\tan x \sec^{n-1} x}{n-1} + \frac{n-2}{n-1} \int \sec^{n-2} x \, dx, \quad n \neq 1,$

**26.** $\int \csc^n x \, dx = -\frac{\cot x \csc^{n-1} x}{n-1} + \frac{n-2}{n-1} \int \csc^{n-2} x \, dx, \quad n \neq 1,$

**27.** $\int \sinh x \, dx = \cosh x,$

**28.** $\int \cosh x \, dx = \sinh x,$

**29.** $\int \tanh x \, dx = \ln |\cosh x|,$

**30.** $\int \coth x \, dx = \ln |\sinh x|,$

**31.** $\int \operatorname{sech} x \, dx = \arctan \sinh x,$

**32.** $\int \operatorname{csch} x \, dx = \ln \left|\tanh \frac{x}{2}\right|,$

**33.** $\int \sinh^2 x \, dx = \frac{1}{4}\sinh(2x) - \frac{1}{2}x,$

**34.** $\int \cosh^2 x \, dx = \frac{1}{4}\sinh(2x) + \frac{1}{2}x,$

**35.** $\int \operatorname{sech}^2 x \, dx = \tanh x,$

**36.** $\int \operatorname{arcsinh} \frac{x}{a} dx = x \operatorname{arcsinh} \frac{x}{a} - \sqrt{x^2 + a^2}, \quad a > 0,$

**37.** $\int \operatorname{arctanh} \frac{x}{a} dx = x \operatorname{arctanh} \frac{x}{a} + \frac{a}{2} \ln |a^2 - x^2|,$

**38.** $$\int \operatorname{arccosh} \frac{x}{a} dx = \begin{cases} x \operatorname{arccosh} \dfrac{x}{a} - \sqrt{x^2 + a^2}, & \text{if } \operatorname{arccosh} \frac{x}{a} > 0 \text{ and } a > 0, \\ x \operatorname{arccosh} \dfrac{x}{a} + \sqrt{x^2 + a^2}, & \text{if } \operatorname{arccosh} \frac{x}{a} < 0 \text{ and } a > 0, \end{cases}$$

**39.** $\int \frac{dx}{\sqrt{a^2 + x^2}} = \ln\left(x + \sqrt{a^2 + x^2}\right), \quad a > 0,$

**40.** $\int \frac{dx}{a^2 + x^2} = \frac{1}{a} \arctan \frac{x}{a}, \quad a > 0,$

**41.** $\int \sqrt{a^2 - x^2} \, dx = \frac{x}{2}\sqrt{a^2 - x^2} + \frac{a^2}{2} \arcsin \frac{x}{a}, \quad a > 0,$

**42.** $\int (a^2 - x^2)^{3/2} dx = \frac{x}{8}(5a^2 - 2x^2)\sqrt{a^2 - x^2} + \frac{3a^4}{8} \arcsin \frac{x}{a}, \quad a > 0,$

**43.** $\int \frac{dx}{\sqrt{a^2 - x^2}} = \arcsin \frac{x}{a}, \quad a > 0,$

**44.** $\int \frac{dx}{a^2 - x^2} = \frac{1}{2a} \ln \left|\frac{a + x}{a - x}\right|,$

**45.** $\int \frac{dx}{(a^2 - x^2)^{3/2}} = \frac{x}{a^2\sqrt{a^2 - x^2}},$

**46.** $\int \sqrt{a^2 \pm x^2} \, dx = \frac{x}{2}\sqrt{a^2 \pm x^2} \pm \frac{a^2}{2} \ln \left|x + \sqrt{a^2 \pm x^2}\right|,$

**47.** $\int \frac{dx}{\sqrt{x^2 - a^2}} = \ln \left|x + \sqrt{x^2 - a^2}\right|, \quad a > 0,$

**48.** $\int \frac{dx}{ax^2 + bx} = \frac{1}{a} \ln \left|\frac{x}{a + bx}\right|,$

**49.** $\int x\sqrt{a + bx} \, dx = \frac{2(3bx - 2a)(a + bx)^{3/2}}{15b^2},$

**50.** $\int \frac{\sqrt{a + bx}}{x} dx = 2\sqrt{a + bx} + a \int \frac{1}{x\sqrt{a + bx}} dx,$

**51.** $\int \frac{x}{\sqrt{a + bx}} dx = \frac{1}{\sqrt{2}} \ln \left|\frac{\sqrt{a + bx} - \sqrt{a}}{\sqrt{a + bx} + \sqrt{a}}\right|, \quad a > 0,$

**52.** $\int \frac{\sqrt{a^2 - x^2}}{x} dx = \sqrt{a^2 - x^2} - a \ln \left|\frac{a + \sqrt{a^2 - x^2}}{x}\right|,$

**53.** $\int x\sqrt{a^2 - x^2} \, dx = -\frac{1}{3}(a^2 - x^2)^{3/2},$

**54.** $\int x^2\sqrt{a^2 - x^2} \, dx = \frac{x}{8}(2x^2 - a^2)\sqrt{a^2 - x^2} + \frac{a^4}{8} \arcsin \frac{x}{a}, \quad a > 0,$

**55.** $\int \frac{dx}{\sqrt{a^2 - x^2}} = -\frac{1}{a} \ln \left|\frac{a + \sqrt{a^2 - x^2}}{x}\right|,$

**56.** $\int \frac{x \, dx}{\sqrt{a^2 - x^2}} = -\sqrt{a^2 - x^2},$

**57.** $\int \frac{x^2 \, dx}{\sqrt{a^2 - x^2}} = -\frac{x}{2}\sqrt{a^2 - x^2} + \frac{a^2}{2} \arcsin \frac{x}{a}, \quad a > 0,$

**58.** $\int \frac{\sqrt{a^2 + x^2}}{x} dx = \sqrt{a^2 + x^2} - a \ln \left|\frac{a + \sqrt{a^2 + x^2}}{x}\right|,$

**59.** $\int \frac{\sqrt{x^2 - a^2}}{x} dx = \sqrt{x^2 - a^2} - a \arccos \frac{a}{|x|}, \quad a > 0,$

**60.** $\int x\sqrt{x^2 \pm a^2} \, dx = \frac{1}{3}(x^2 \pm a^2)^{3/2},$

**61.** $\int \frac{dx}{x\sqrt{x^2 + a^2}} = \frac{1}{a} \ln \left|\frac{x}{a + \sqrt{a^2 + x^2}}\right|,$

# Theoretical Computer Science Cheat Sheet

## Calculus Cont.

**62.** $\displaystyle\int \frac{dx}{x\sqrt{x^2-a^2}} = \frac{1}{a}\arccos\frac{a}{|x|}, \quad a>0,$

**63.** $\displaystyle\int \frac{dx}{x^2\sqrt{x^2\pm a^2}} = \mp\frac{\sqrt{x^2\pm a^2}}{a^2x},$

**64.** $\displaystyle\int \frac{x\,dx}{\sqrt{x^2\pm a^2}} = \sqrt{x^2\pm a^2},$

**65.** $\displaystyle\int \frac{\sqrt{x^2\pm a^2}}{x^4}\,dx = \mp\frac{(x^2+a^2)^{3/2}}{3a^2x^3},$

**66.** $$\int \frac{dx}{ax^2+bx+c} = \begin{cases} \dfrac{1}{\sqrt{b^2-4ac}}\ln\left|\dfrac{2ax+b-\sqrt{b^2-4ac}}{2ax+b+\sqrt{b^2-4ac}}\right|, & \text{if } b^2>4ac, \\[2ex] \dfrac{2}{\sqrt{4ac-b^2}}\arctan\dfrac{2ax+b}{\sqrt{4ac-b^2}}, & \text{if } b^2<4ac, \end{cases}$$

**67.** $$\int \frac{dx}{\sqrt{ax^2+bx+c}} = \begin{cases} \dfrac{1}{\sqrt{a}}\ln\left|2ax+b+2\sqrt{a}\sqrt{ax^2+bx+c}\right|, & \text{if } a>0, \\[2ex] \dfrac{1}{\sqrt{-a}}\arcsin\dfrac{-2ax-b}{\sqrt{b^2-4ac}}, & \text{if } a<0, \end{cases}$$

**68.** $$\int \sqrt{ax^2+bx+c}\,dx = \frac{2ax+b}{4a}\sqrt{ax^2+bx+c} + \frac{4ax-b^2}{8a}\int\frac{dx}{\sqrt{ax^2+bx+c}},$$

**69.** $$\int \frac{x\,dx}{\sqrt{ax^2+bx+c}} = \frac{\sqrt{ax^2+bx+c}}{a} - \frac{b}{2a}\int\frac{dx}{\sqrt{ax^2+bx+c}},$$

**70.** $$\int \frac{dx}{x\sqrt{ax^2+bx+c}} = \begin{cases} \dfrac{-1}{\sqrt{c}}\ln\left|\dfrac{2\sqrt{c}\sqrt{ax^2+bx+c}+bx+2c}{x}\right|, & \text{if } c>0, \\[2ex] \dfrac{1}{\sqrt{-c}}\arcsin\dfrac{bx+2c}{|x|\sqrt{b^2-4ac}}, & \text{if } c<0, \end{cases}$$

**71.** $\displaystyle\int x^3\sqrt{x^2+a^2}\,dx = (\tfrac{1}{3}x^2-\tfrac{2}{15}a^2)(x^2+a^2)^{3/2},$

**72.** $\displaystyle\int x^n\sin(ax)\,dx = -\tfrac{1}{a}x^n\cos(ax) + \tfrac{n}{a}\int x^{n-1}\cos(ax)\,dx,$

**73.** $\displaystyle\int x^n\cos(ax)\,dx = \tfrac{1}{a}x^n\sin(ax) - \tfrac{n}{a}\int x^{n-1}\sin(ax)\,dx,$

**74.** $\displaystyle\int x^ne^{ax}\,dx = \frac{x^ne^{ax}}{a} - \tfrac{n}{a}\int x^{n-1}e^{ax}\,dx,$

**75.** $\displaystyle\int x^n\ln(ax)\,dx = x^{n+1}\left(\frac{\ln(ax)}{n+1} - \frac{1}{(n+1)^2}\right),$

**76.** $\displaystyle\int x^n(\ln ax)^m\,dx = \frac{x^{n+1}}{n+1}(\ln ax)^m - \frac{m}{n+1}\int x^n(\ln ax)^{m-1}\,dx.$

$$\begin{array}{lcccc}
x^1 = & x^{\underline{1}} & = & x^{\overline{1}} \\
x^2 = & x^{\underline{2}}+x^{\underline{1}} & = & x^{\overline{2}}-x^{\overline{1}} \\
x^3 = & x^{\underline{3}}+3x^{\underline{2}}+x^{\underline{1}} & = & x^{\overline{3}}-3x^{\overline{2}}+x^{\overline{1}} \\
x^4 = & x^{\underline{4}}+6x^{\underline{3}}+7x^{\underline{2}}+x^{\underline{1}} & = & x^{\overline{4}}-6x^{\overline{3}}+7x^{\overline{2}}-x^{\overline{1}} \\
x^5 = & x^{\underline{5}}+15x^{\underline{4}}+25x^{\underline{3}}+10x^{\underline{2}}+x^{\underline{1}} & = & x^{\underline{5}}-15x^{\overline{4}}+25x^{\overline{3}}-10x^{\overline{2}}+x^{\overline{1}}
\end{array}$$

$$\begin{array}{lclc}
x^{\overline{1}} = & x^1 & x^{\underline{1}} = & x^1 \\
x^{\overline{2}} = & x^2+x^1 & x^{\underline{2}} = & x^2-x^1 \\
x^{\overline{3}} = & x^3+3x^2+2x^1 & x^{\underline{3}} = & x^3-3x^2+2x^1 \\
x^{\overline{4}} = & x^4+6x^3+11x^2+6x^1 & x^{\underline{4}} = & x^4-6x^3+11x^2-6x^1 \\
x^{\overline{5}} = & x^5+10x^4+35x^3+50x^2+24x^1 & x^{\underline{5}} = & x^5-10x^4+35x^3-50x^2+24x^1
\end{array}$$

## Finite Calculus

Difference, shift operators:

$$\Delta f(x) = f(x+1)-f(x),$$
$$\mathrm{E}\,f(x) = f(x+1).$$

Fundamental Theorem:

$$f(x) = \Delta F(x) \Leftrightarrow \sum f(x)\delta x = F(x)+C.$$
$$\sum_a^b f(x)\delta x = \sum_{i=a}^{b-1} f(i).$$

Differences:

$\Delta(cu) = c\Delta u, \qquad \Delta(u+v) = \Delta u+\Delta v,$

$\Delta(uv) = u\Delta v + \mathrm{E}\,v\Delta u,$

$\Delta(x^{\underline{n}}) = nx^{\underline{n-1}},$

$\Delta(H_x) = x^{\underline{-1}}, \qquad \Delta(2^x) = 2^x,$

$\Delta(c^x) = (c-1)c^x, \qquad \Delta\binom{x}{m} = \binom{x}{m-1}.$

Sums:

$\sum cu\,\delta x = c\sum u\,\delta x,$

$\sum(u+v)\,\delta x = \sum u\,\delta x + \sum v\,\delta x,$

$\sum u\Delta v\,\delta x = uv - \sum \mathrm{E}\,v\Delta u\,\delta x,$

$\sum x^{\underline{n}}\,\delta x = \frac{x^{\underline{n+1}}}{m+1}, \qquad \sum x^{\underline{-1}}\,\delta x = H_x,$

$\sum c^x\,\delta x = \frac{c^x}{c-1}, \qquad \sum\binom{x}{m}\,\delta x = \binom{x}{m+1}.$

Falling Factorial Powers:

$$x^{\underline{n}} = x(x-1)\cdots(x-n+1), \quad n>0,$$
$$x^{\underline{0}} = 1,$$
$$x^{\underline{n}} = \frac{1}{(x+1)\cdots(x+|n|)}, \quad n<0,$$
$$x^{\underline{n+m}} = x^{\underline{m}}(x-m)^{\underline{n}}.$$

Rising Factorial Powers:

$$x^{\overline{n}} = x(x+1)\cdots(x+n-1), \quad n>0,$$
$$x^{\overline{0}} = 1,$$
$$x^{\overline{n}} = \frac{1}{(x-1)\cdots(x-|n|)}, \quad n<0,$$
$$x^{\overline{n+m}} = x^{\overline{m}}(x+m)^{\overline{n}}.$$

Conversion:

$$x^{\underline{n}} = (-1)^n(-x)^{\overline{n}} = (x-n+1)^{\overline{n}}$$
$$= 1/(x+1)^{\overline{-n}},$$
$$x^{\overline{n}} = (-1)^n(-x)^{\underline{n}} = (x+n-1)^{\underline{n}}$$
$$= 1/(x-1)^{\underline{-n}},$$
$$x^n = \sum_{k=1}^n \left\{ n \atop k \right\} x^{\underline{k}} = \sum_{k=1}^n \left\{ n \atop k \right\} (-1)^{n-k}x^{\overline{k}},$$
$$x^{\underline{n}} = \sum_{k=1}^n \left[ n \atop k \right] (-1)^{n-k}x^k,$$
$$x^{\overline{n}} = \sum_{k=1}^n \left[ n \atop k \right] x^k.$$

## Theoretical Computer Science Cheat Sheet

### Series

Taylor's series:

$$f(x) = f(a) + (x-a)f'(a) + \frac{(x-a)^2}{2}f''(a) + \cdots = \sum_{i=0}^{\infty} \frac{(x-a)^i}{i!} f^{(i)}(a).$$

Expansions:

$$\frac{1}{1-x} = 1 + x + x^2 + x^3 + x^4 + \cdots = \sum_{i=0}^{\infty} x^i,$$

$$\frac{1}{1-cx} = 1 + cx + c^2x^2 + c^3x^3 + \cdots = \sum_{i=0}^{\infty} c^i x^i,$$

$$\frac{1}{1-x^n} = 1 + x^n + x^{2n} + x^{3n} + \cdots = \sum_{i=0}^{\infty} x^{ni},$$

$$\frac{x}{(1-x)^2} = x + 2x^2 + 3x^3 + 4x^4 + \cdots = \sum_{i=0}^{\infty} i x^i,$$

$$x^k \frac{d^n}{dx^n}\left(\frac{1}{1-x}\right) = x + 2^n x^2 + 3^n x^3 + 4^n x^4 + \cdots = \sum_{i=0}^{\infty} i^n x^i,$$

$$e^x = 1 + x + \tfrac{1}{2}x^2 + \tfrac{1}{6}x^3 + \cdots = \sum_{i=0}^{\infty} \frac{x^i}{i!},$$

$$\ln(1+x) = x - \tfrac{1}{2}x^2 + \tfrac{1}{3}x^3 - \tfrac{1}{4}x^4 - \cdots = \sum_{i=1}^{\infty} (-1)^{i+1} \frac{x^i}{i},$$

$$\ln \frac{1}{1-x} = x + \tfrac{1}{2}x^2 + \tfrac{1}{3}x^3 + \tfrac{1}{4}x^4 + \cdots = \sum_{i=1}^{\infty} \frac{x^i}{i},$$

$$\sin x = x - \tfrac{1}{3!}x^3 + \tfrac{1}{5!}x^5 - \tfrac{1}{7!}x^7 + \cdots = \sum_{i=0}^{\infty} (-1)^i \frac{x^{2i+1}}{(2i+1)!},$$

$$\cos x = 1 - \tfrac{1}{2!}x^2 + \tfrac{1}{4!}x^4 - \tfrac{1}{6!}x^6 + \cdots = \sum_{i=0}^{\infty} (-1)^i \frac{x^{2i}}{(2i)!},$$

$$\tan^{-1} x = x - \tfrac{1}{3}x^3 + \tfrac{1}{5}x^5 - \tfrac{1}{7}x^7 + \cdots = \sum_{i=0}^{\infty} (-1)^i \frac{x^{2i+1}}{(2i+1)},$$

$$(1+x)^n = 1 + nx + \tfrac{n(n-1)}{2}x^2 + \cdots = \sum_{i=0}^{\infty} \binom{n}{i} x^i,$$

$$\frac{1}{(1-x)^{n+1}} = 1 + (n+1)x + \tbinom{n+2}{2}x^2 + \cdots = \sum_{i=0}^{\infty} \binom{i+n}{i} x^i,$$

$$\frac{x}{e^x - 1} = 1 - \tfrac{1}{2}x + \tfrac{1}{12}x^2 - \tfrac{1}{720}x^4 + \cdots = \sum_{i=0}^{\infty} \frac{B_i x^i}{i!},$$

$$\frac{1}{2x}(1 - \sqrt{1-4x}) = 1 + x + 2x^2 + 5x^3 + \cdots = \sum_{i=0}^{\infty} \frac{1}{i+1}\binom{2i}{i} x^i,$$

$$\frac{1}{\sqrt{1-4x}} = 1 + x + 2x^2 + 6x^3 + \cdots = \sum_{i=0}^{\infty} \binom{2i}{i} x^i,$$

$$\frac{1}{\sqrt{1-4x}}\left(\frac{1-\sqrt{1-4x}}{2x}\right)^n = 1 + (2+n)x + \tbinom{4+n}{2}x^2 + \cdots = \sum_{i=0}^{\infty} \binom{2i+n}{i} x^i,$$

$$\frac{1}{1-x}\ln\frac{1}{1-x} = x + \tfrac{3}{2}x^2 + \tfrac{11}{6}x^3 + \tfrac{25}{12}x^4 + \cdots = \sum_{i=1}^{\infty} H_i x^i,$$

$$\frac{1}{2}\left(\ln\frac{1}{1-x}\right)^2 = \tfrac{1}{2}x^2 + \tfrac{3}{4}x^3 + \tfrac{11}{24}x^4 + \cdots = \sum_{i=2}^{\infty} \frac{H_{i-1} x^i}{i},$$

$$\frac{x}{1-x-x^2} = x + x^2 + 2x^3 + 3x^4 + \cdots = \sum_{i=0}^{\infty} F_i x^i,$$

$$\frac{F_n x}{1 - (F_{n-1} + F_{n+1})x - (-1)^n x^2} = F_n x + F_{2n}x^2 + F_{3n}x^3 + \cdots = \sum_{i=0}^{\infty} F_{ni} x^i.$$

Ordinary power series:

$$A(x) = \sum_{i=0}^{\infty} a_i x^i.$$

Exponential power series:

$$A(x) = \sum_{i=0}^{\infty} a_i \frac{x^i}{i!}.$$

Dirichlet power series:

$$A(x) = \sum_{i=1}^{\infty} \frac{a_i}{i^x}.$$

Binomial theorem:

$$(x+y)^n = \sum_{k=0}^{n} \binom{n}{k} x^{n-k} y^k.$$

Difference of like powers:

$$x^n - y^n = (x-y) \sum_{k=0}^{n-1} x^{n-1-k} y^k.$$

For ordinary power series:

$$\alpha A(x) + \beta B(x) = \sum_{i=0}^{\infty} (\alpha a_i + \beta b_i) x^i,$$

$$x^k A(x) = \sum_{i=k}^{\infty} a_{i-k} x^i,$$

$$\frac{A(x) - \sum_{i=0}^{k-1} a_i x^i}{x^k} = \sum_{i=0}^{\infty} a_{i+k} x^i,$$

$$A(cx) = \sum_{i=0}^{\infty} c^i a_i x^i,$$

$$A'(x) = \sum_{i=0}^{\infty} (i+1) a_{i+1} x^i,$$

$$xA'(x) = \sum_{i=1}^{\infty} i a_i x^i,$$

$$\int A(x)\,dx = \sum_{i=1}^{\infty} \frac{a_{i-1}}{i} x^i,$$

$$\frac{A(x) + A(-x)}{2} = \sum_{i=0}^{\infty} a_{2i} x^{2i},$$

$$\frac{A(x) - A(-x)}{2} = \sum_{i=0}^{\infty} a_{2i+1} x^{2i+1}.$$

Summation: If $b_i = \sum_{j=0}^{i} a_i$ then

$$B(x) = \frac{1}{1-x} A(x).$$

Convolution:

$$A(x)B(x) = \sum_{i=0}^{\infty} \left( \sum_{j=0}^{i} a_j b_{i-j} \right) x^i.$$

God made the natural numbers;
all the rest is the work of man.
– Leopold Kronecker

# Theoretical Computer Science Cheat Sheet

## Series

Expansions:

$$\frac{1}{(1-x)^{n+1}}\ln\frac{1}{1-x} = \sum_{i=0}^{\infty}(H_{n+i}-H_n)\binom{n+i}{i}x^i,$$

$$\left(\frac{1}{x}\right)^{\overline{-n}} = \sum_{i=0}^{\infty}\left\{ i \atop n \right\}x^i,$$

$$x^{\overline{n}} = \sum_{i=0}^{\infty}\left[ n \atop i \right]x^i,$$

$$(e^x-1)^n = \sum_{i=0}^{\infty}\left\{ i \atop n \right\}\frac{n!x^i}{i!},$$

$$\left(\ln\frac{1}{1-x}\right)^n = \sum_{i=0}^{\infty}\left[ i \atop n \right]\frac{n!x^i}{i!},$$

$$x\cot x = \sum_{i=0}^{\infty}\frac{(-4)^iB_{2i}x^{2i}}{(2i)!},$$

$$\tan x = \sum_{i=1}^{\infty}(-1)^{i-1}\frac{2^{2i}(2^{2i}-1)B_{2i}x^{2i-1}}{(2i)!},$$

$$\zeta(x) = \sum_{i=1}^{\infty}\frac{1}{i^x},$$

$$\frac{1}{\zeta(x)} = \sum_{i=1}^{\infty}\frac{\mu(i)}{i^x},$$

$$\frac{\zeta(x-1)}{\zeta(x)} = \sum_{i=1}^{\infty}\frac{\phi(i)}{i^x},$$

$$\zeta(x) = \prod_p\frac{1}{1-p^{-x}},$$

$$\zeta^2(x) = \sum_{i=1}^{\infty}\frac{d(i)}{x^i} \quad \text{where } d(n)=\textstyle\sum_{d|n}1,$$

$$\zeta(x)\zeta(x-1) = \sum_{i=1}^{\infty}\frac{S(i)}{x^i} \quad \text{where } S(n)=\textstyle\sum_{d|n}d,$$

$$\zeta(2n) = \frac{2^{2n-1}|B_{2n}|}{(2n)!}\pi^{2n}, \quad n\in\mathbb{N},$$

$$\frac{x}{\sin x} = \sum_{i=0}^{\infty}(-1)^{i-1}\frac{(4^i-2)B_{2i}x^{2i}}{(2i)!},$$

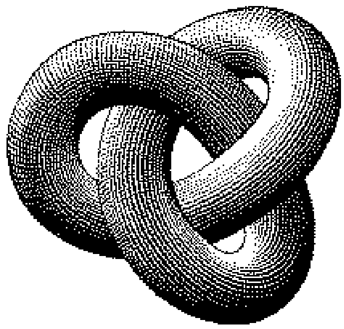
$$\left(\frac{1-\sqrt{1-4x}}{2x}\right)^n = \sum_{i=0}^{\infty}\frac{n(2i+n-1)!}{i!(n+i)!}x^i,$$

$$e^x\sin x = \sum_{i=1}^{\infty}\frac{2^{i/2}\sin\frac{i\pi}{4}}{i!}x^i,$$

$$\sqrt{\frac{1-\sqrt{1-x}}{x}} = \sum_{i=0}^{\infty}\frac{(4i)!}{16^i\sqrt{2}(2i)!(2i+1)!}x^i,$$

$$\left(\frac{\arcsin x}{x}\right)^2 = \sum_{i=0}^{\infty}\frac{4^ii!^2}{(i+1)(2i+1)!}x^{2i}.$$

## Escher's Knot



## Stieltjes Integration

If $G$ is continuous in the interval $[a,b]$ and $F$ is nondecreasing then

$$\int_a^b G(x)\,dF(x)$$

exists. If $a\le b\le c$ then

$$\int_a^c G(x)\,dF(x) = \int_a^b G(x)\,dF(x) + \int_b^c G(x)\,dF(x).$$

If the integrals involved exist

$$\int_a^b \big(G(x)+H(x)\big)\,dF(x) = \int_a^b G(x)\,dF(x) + \int_a^b H(x)\,dF(x),$$

$$\int_a^b G(x)\,d\big(F(x)+H(x)\big) = \int_a^b G(x)\,dF(x) + \int_a^b G(x)\,dH(x),$$

$$\int_a^b c\cdot G(x)\,dF(x) = \int_a^b G(x)\,d\big(c\cdot F(x)\big) = c\int_a^b G(x)\,dF(x),$$

$$\int_a^b G(x)\,dF(x) = G(b)F(b) - G(a)F(a) - \int_a^b F(x)\,dG(x).$$

If the integrals involved exist, and $F$ possesses a derivative $F'$ at every point in $[a,b]$ then

$$\int_a^b G(x)\,dF(x) = \int_a^b G(x)F'(x)\,dx.$$

## Cramer's Rule

If we have equations:

$$\begin{aligned}
a_{1,1}x_1 + a_{1,2}x_2 + \cdots + a_{1,n}x_n &= b_1\\
a_{2,1}x_1 + a_{2,2}x_2 + \cdots + a_{2,n}x_n &= b_2\\
\vdots \qquad\quad \vdots \qquad\qquad\quad \vdots &\\
a_{n,1}x_1 + a_{n,2}x_2 + \cdots + a_{n,n}x_n &= b_n
\end{aligned}$$

Let $A=(a_{i,j})$ and $B$ be the column matrix $(b_i)$. Then there is a unique solution iff $\det A\ne 0$. Let $A_i$ be $A$ with column $i$ replaced by $B$. Then

$$x_i = \frac{\det A_i}{\det A}.$$

| | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|
| 00 | 47 | 18 | 76 | 29 | 93 | 85 | 34 | 61 | 52 |
| 86 | 11 | 57 | 28 | 70 | 39 | 94 | 45 | 02 | 63 |
| 95 | 80 | 22 | 67 | 38 | 71 | 49 | 56 | 13 | 04 |
| 59 | 96 | 81 | 33 | 07 | 48 | 72 | 60 | 24 | 15 |
| 73 | 69 | 90 | 82 | 44 | 17 | 58 | 01 | 35 | 26 |
| 68 | 74 | 09 | 91 | 83 | 55 | 27 | 12 | 46 | 30 |
| 37 | 08 | 75 | 19 | 92 | 84 | 66 | 23 | 50 | 41 |
| 14 | 25 | 36 | 40 | 51 | 62 | 03 | 77 | 88 | 99 |
| 21 | 32 | 43 | 54 | 65 | 06 | 10 | 89 | 97 | 78 |
| 42 | 53 | 64 | 05 | 16 | 20 | 31 | 98 | 79 | 87 |

## Fibonacci Numbers

$1,1,2,3,5,8,13,21,34,55,89,\ldots$

Definitions:

$$F_i = F_{i-1}+F_{i-2}, \quad F_0=F_1=1,$$

$$F_{-i} = (-1)^{i-1}F_i,$$

$$F_i = \frac{1}{\sqrt{5}}\left(\phi^i - \hat{\phi}^i\right),$$

Cassini's identity: for $i>0$:

$$F_{i+1}F_{i-1} - F_i^2 = (-1)^i.$$

Additive rule:

$$F_{n+k} = F_kF_{n+1} + F_{k-1}F_n,$$

$$F_{2n} = F_nF_{n+1} + F_{n-1}F_n.$$

Calculation by matrices:

$$\begin{pmatrix} F_{n-2} & F_{n-1} \\ F_{n-1} & F_n \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ 1 & 1 \end{pmatrix}^n.$$

The Fibonacci number system: Every integer $n$ has a unique representation

$$n = F_{k_1} + F_{k_2} + \cdots + F_{k_m},$$

where $k_i \ge k_{i+1}+2$ for all $i$, $1\le i<m$ and $k_m\ge 2$.

Improvement makes strait roads, but the crooked roads without Improvement, are roads of Genius.
– William Blake (The Marriage of Heaven and Hell)