Review Exercises for Final Exam

The following exercises should help your preparations for the final exam, but do not necessarily guarantee you will pass it. Since answers to most of them can be found in your class notes or the textbook, solutions will not be posted here.

1. All questions in Assignment #3 and #4.

NP-Complete Problems and Polynomial-Time Reductions

2. Consider the polynomial-time reduction f: $\phi \rightarrow (G_\phi, k)$ used to prove 3SAT $\leq_p$ VERTEX-COVER. Prove that ϕ is satisfiable iff G_ϕ has a k-vertex cover.

3. Consider the polynomial-time reduction used to prove 3SAT $\leq_p$ HAMPATH. Let:

$$\phi = (\neg x_1 \vee x_2 \vee \neg x_2) \wedge (x_1 \vee \neg x_2 \vee \neg x_1)$$

Here, $\neg x_i$ is the negation of x_i.

   a. Give (G_ϕ, s, t) constructed from ϕ by the reduction.
   b. Give one satisfying assignment A for ϕ, and a corresponding HAM path from s to t in G_ϕ produced by the proof.
   c. Give one HAM path from s to t in G_ϕ that is distinct from the one you gave in (b), and give a corresponding assignment for ϕ produced by the proof.

4. Consider the polynomial-time reduction from HAMPATH to UHAMPATH.

   a. Give the undirected graph G' constructed from the following directed graph G:
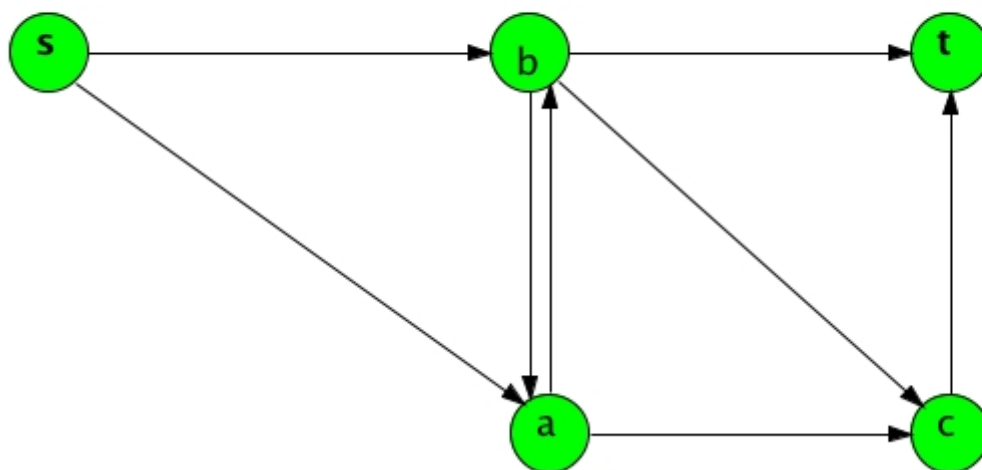
   b. Give one HAM path in G, and the corresponding UHAM path in G'.
   c. Give one UHAM path in G' that is distinct from the one you gave in (b), and give the corresponding HAM path in G.

Space Complexity and Its Relation to Time Complexity

5. Give a precise definition of worst-case space complexity functions of DTMs and NTMs that halt on all input strings.

6. Let M be any DTM or NTM that halts on all input strings. Prove the following:

    a. $W_M^{space}(n) \le W_M^{time}(n) + 1$.

    b. $W_M^{time}(n) \le (|\Gamma|+|Q|)^{W_M^{space}(n)+1}$, where Γ is the tape alphabet of M and Q is the state set of M. Hint: Use the possible configurations of M.

7. Prove $SAT \in SPACE(n)$.

8. Prove { A | A is an NFA that rejects at least one string } $\in NSPACE(n)$.

9. An important procedure in the proof of Savitch's Theorem is $Path_N(C_1, C_2, t, n)$ that decides existence of a path from C_1 to C_2 in the configuration graph of an NTM, N, with space complexity $W_N(n) \le d\cdot f(n)$.

    a. Give precise specifications of the input parameters C_1, C_2, t, n, and the output from the procedure.
    b. Give pseudocode for the body of the procedure.
    c. Show that the space requirement for running $Path_N(C_1, C_2, t, n)$ is $O(f(n)) \times \log_2 t$.

    d. Show that the length of any computation branch of $N \le \alpha^{d\cdot f(n)+1}$, where $\alpha = |\Gamma|+|Q|$.
    e. Using the $Path_N$ procedure, give a DTM equivalent to N.

    f. Show that the space complexity function of the above DTM is $O(f^2(n))$.
    g. Show how to overcome the above DTM's need to compute the value of f(n).

10. Prove $NP \subseteq PSPACE = NPSPACE$.

11.

    a. Define PSPACE-complete problems.
    b. Explain why the definition of PSPACE-complete problems uses polynomial-time reducibility rather than polynomial-space reducibility.

12. Describe the TQBF problem and the Formula Game problem and show that they are effectively the same problem.

13. Describe DTMs and NTMs with a read-only input tape and how their space complexity functions are defined.

14. Define the classes of languages L and NL.

15. Show that $\{ 0^k 1^k \mid k \ge 0 \} \in L$.

16. Show that $PATH \in NL$.

17. Describe log-space transducers.

18. Prove: Let M be any DTMR, NTMR, or LST using at most f(n) work-tape cells. Then $W_M^{time}(n) \le n\ \alpha^{f(n)+1}$ $= n\ 2^{O(f(n))}$, where $\alpha = |\Gamma|+|Q|$.

19.

    a. Define log-space reducibility.

b. Define NL-complete problems.

20. Consider the proof of this theorem: If $A \leq_L B$ and $B \in L$, then $A \in L$. Let M be a log-space reduction transducer from A to B and M_B a log-space DTMR deciding B.

    a. Explain why a sequential simulation of M and M_B does not yield a log-space DTMR deciding A.

    b. Describe how to construct a log-space DTMR deciding A from modified M and M_B.

21. Prove the reflexivity and transitivity of $\leq_L$.

22. Prove:

    a. If $A \in$ NL-Complete and $A \leq_L B$ for $B \in$ NL, then $B \in$ NL-Complete.

    b. If $\exists B \in L \cap$ NL-Complete, then L = NL.

23. Prove PATH $\in$ NL-Complete.

24. Prove NL $\subseteq$ P.

Circuit Complexity and P-Completeness

25. Define each of the following:

    a. a circuit family

    b. what it means for a circuit family to decide a language over $\{0, 1\}$.

    c. equivalence of two circuits

    d. the size of a circuit

    e. size minimality of a circuit and a circuit family

    f. the size complexity of a circuit family

    g. the depth of a circuit

    h. depth minimality of a circuit and a circuit family

    i. the depth complexity of a circuit family

    j. the size and depth complexities of a language over $\{0, 1\}$

26.

    a. Give a proof idea for Theorem: If $A \subseteq \{0, 1\}^*$ is decided by a DTM with worst-case time complexity $O(t(n))$, then A has the circuit size complexity $O(t^2(n))$.

    b. Prove that if $A \in$ P, then A has a polynomial circuit size complexity.

    c. Prove that if A does not have a polynomial circuit size complexity, then $A \notin$ P.

27.

    a. Define P-complete problems.

    b. Give a proof idea for CIRCUIT-VALUE $\in$ P-Complete.

    c. Define the processor and parallel time complexities of a circuit.

    d. Define the class of languages NC.

    e. Describe the evidence suggesting that P-complete problems are inherently sequential.

Approximation Algorithms

28.

    a. Show that the approximation vertex-cover algorithm we studied in class has the ratio bound of 2.

    b. Show that the approximation maximum-cut algorithm we studied in class has the ratio bound of 2.