Monte Carlo Tree Search in *Hex*

Broderick Arneson, Ryan B. Hayward, and Philip Henderson

***Abstract*—*Hex*, the classic board game invented by Piet Hein in 1942 and independently by John Nash in 1948, has been a domain of AI research since Claude Shannon's seminal work in the 1950s. Until the Monte Carlo *Go* revolution a few years ago, the best computer *Hex* players used knowledge-intensive alpha–beta search. Since that time, strong Monte Carlo *Hex* players have appeared that are on par with the best alpha–beta *Hex* players. In this paper, we describe MoHex, the Monte Carlo tree search *Hex* player that won gold at the 2009 Computer Olympiad. Our main contributions to Monte Carlo tree search include using inferior cell analysis and connection strategy computation to prune the search tree. In particular, we run our random game simulations not on the actual game position, but on a reduced equivalent board.**

***Index Terms*— Artificial intelligence, computational intelligence, computational and artificial intelligence, games, *Hex*.**

I. Introduction

MONTE CARLO TREE SEARCH (MCTS) is an exciting new development in the field of automated game players, attaining great success in a variety of games, most notably *Go* [9]–[11], [13], [25].

Hex is the classic board game invented by Piet Hein in 1942 and independently by John Nash in 1948 [19], [31]. *Hex* has been a domain of artificial intelligence research since Claude Shannon's seminal work in the 1950s [34].

Like *Go*, *Hex* has a large branching factor that limits automated players to shallow or selective search. Unlike *Go* programs, *Hex* programs have reasonably strong evaluation functions, and therefore straightforward alpha–beta techniques have been successful [3], [6], [28].

We began experimenting with the possibility of an effective automated *Hex* player based on the MCTS framework in early 2007. The resulting player, named MoHex, won silver and gold, respectively, at the 2008 and 2009 Hex Computer Olympiads [5], [6]. In the latter tournament, MoHex was undefeated.

In this paper, we describe the framework of MoHex, with particular emphasis on its algorithmic contributions which exploit mathematical properties of *Hex*:

- MoHex uses proven knowledge in the MCTS tree to prune inferior children;
- MoHex computes Monte Carlo simulations from a reduced equivalent board, rather than the actual game board.

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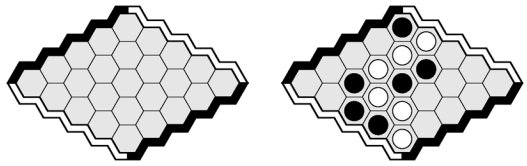


Fig. 1. An empty 5 × 5 *Hex* board and a game won by white.

In Section II, we review the rules of *Hex*, and algorithms for finding inferior cells and connection strategies. In Section III, we discuss previous automated *Hex* players and review the success of alpha–beta-based programs. In Section IV, we describe the basic framework of MoHex. In Section V, we describe our enhancements of MoHex, especially *Hex*-specific techniques for pruning children in the MCTS tree, and for constructing reduced but equivalent boards which improve and accelerate MoHex's game simulations. In Section VI, we analyze experimental data which measure the effectiveness of our techniques and the scaling of MoHex's strength with greater computing power. In Section VII, we analyze MoHex's performance against other automated *Hex* players. In Section VIII, we discuss the potential application of these techniques to related games, such as *Y* and *Havannah*. In Section IX, we review MoHex's current limitations, and avenues for future research.

II. Rules and Algorithms of *Hex*

A. Rules of Hex

Hex has simple rules: black and white alternate turns, and on each turn a player places a single stone of their color on any unoccupied cell. The winner is the player who forms a chain of their stones connecting their two opposing board sides. See Fig. 1.

It is never disadvantageous for players to have an extra stone of their color on the board, and *Hex* can never end in a draw, so by Nash's strategy-stealing argument *Hex* is a first-player win on all $n \times n$ boards [31]. The first-player advantage is considerable and when unrestricted usually leads to an easy win, so *Hex* is often played with the swap rule: the first player makes black's first move, and the second player then chooses whether to play as black or white. White makes the next move, and the players alternate turns thereafter. This variation is a second-player win, but in practice produces closer games than without the swap rule.

Like *Go*, *Hex* can be played on any $n \times n$ board. The Computer Olympiad Hex tournaments use 11×11 boards. Humans typically play on board sizes ranging from 11×11 to 19×19 [24], [27], with beginners often starting out on smaller boards. Automated *Hex* solvers have solved all opening moves on all board sizes up to and including 8×8 [21].

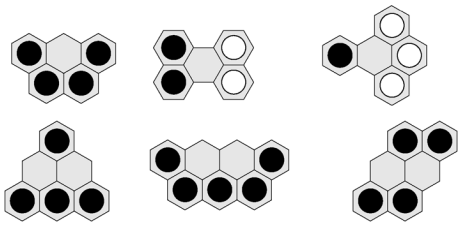


Fig. 2. Local fill-in patterns. Top three patterns identify dead cells, and can be filled in with stones of either player. Bottom three patterns identify black-captured regions, and can be filled in with black stones.

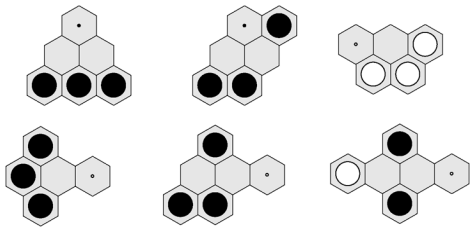


Fig. 3. Local inferior cell patterns. Empty cells can be pruned from consideration by black based on their reversible/dominated relation to the corresponding dotted cells.

B. Inferior Cell Analysis

There are two main techniques in *Hex* inferior cell analysis: fill-in and move pruning.

Fill-in is the process of adding to a *Hex* position a set of stones that is guaranteed not to alter the position's minimax value. There are two main categories of fill-in: dead cells and captured regions. A *dead cell* is a cell that is provably useless for both players. A *captured region* is a set of cells on which one player has a second-player strategy that negates any benefit their opponent might gain from playing in the region. MoHex uses 19 local patterns for identifying dead cells and captured regions; some are shown in Fig. 2. MoHex also uses graph-theoretic algorithms and board decompositions to find larger classes of fill-in configurations [21].

Move pruning is the process of omitting from consideration a legal move in the current game position. Combinatorial game theory allows reversible moves to be bypassed, and dominated moves to be pruned as long as some dominating move is considered [8]. This theory forms the basis for pruning various forms of *Hex* cell, including vulnerable, captured-reversible, fill-in-dominated, and induced-path-dominated cells [15], [16], [18], [22], [23]. As with fill-in, MoHex uses (about 250) local patterns to find such cells. See Fig. 3.

C. Computing Connection Strategies

H-search is an algorithm that finds cell-to-cell connection strategies in a given *Hex* position [2], [4]. Starting with the base case of trivially connected adjacent cell pairs, H-search inductively builds larger connections by combining smaller ones in series and parallel until no further connection strategies are found. H-search is known to be incomplete; in particular, it misses some relatively simple connections with overlapping substrategies that humans can easily detect. Thus, minor H-search extensions have been developed [20], [32].

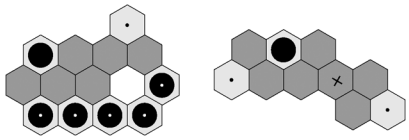


Fig. 4. A black VC and a black SC. Carriers are shaded, endpoints are dotted, and the first move of the SC strategy is $\times$.

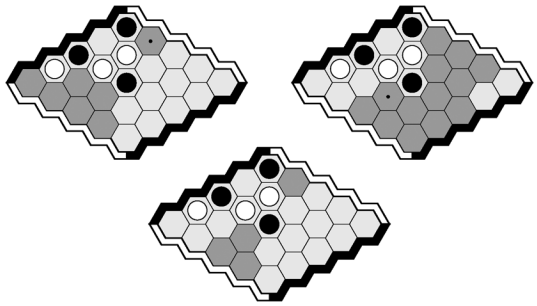


Fig. 5. Two white winning SC carriers and the corresponding mustplay for black.

A virtual connection (VC) is a second-player strategy for connecting two endpoints, and a virtual semiconnection (SC) is a first-player strategy for connecting two endpoints. A *winning connection* is a strategy whose two endpoints are opposing board sides. The *carrier* of a connection strategy is the (minimal) set of empty cells required to carry out this strategy. See Fig. 4.

If H-search finds a winning VC for either player, or a winning SC for the player to move, then this player can win by following the discovered connection strategy.

If H-search finds a winning SC for the player who just moved, then the player to move need only consider moves inside the carrier of this SC, since all other moves are provably losing; that is, all moves outside the winning SC's carrier leave the opponent a winning SC on their next turn.

The *mustplay* for the player to move is the intersection of their opponent's winning SC carriers. By the same reasoning, all moves outside the mustplay are provably losing. See Fig. 5.

H-search prunes many losing moves and produces perfect endgame play, but is time costly: efficient implementations can compute the connection strategies for about 25 positions per second on tournament-sized 11×11 boards.

III. Alpha–Beta *Hex* Players

Together with E. F. Moore, Claude Shannon developed the first automated *Hex* player in the 1950s, an electronic circuit network which set positive charges for one player's stones, negative charges for the other player's stones, and then played at a certain saddle point. The computer played strong opening moves but sometimes erred in tactical situations [34]. Shannon also developed a computer to play *Bird Cage*, now known as *Bridg-it*, a game similar to *Hex*. This circuit network set the resistance of one player's stones to zero, the resistance of the other player's stones to infinity, and then played at a cell with greatest voltage drop [12].

In 2000, Hexy won the First Computer Olympiad Hex competition [3]. Hexy's evaluation function uses an augmentation

of Shannon's *Bird Cage* circuit in which extra wires are added which correspond to connections found by H-search [2]. Hexy uses this evaluation function in a straightforward alpha–beta search.

In 2003, 2004, and 2006, Six won the next three Computer Olympiad Hex competitions [17], [29], [35]. Six significantly refines the Hexy framework by improving H-search efficiency via heuristic limits, restricting the alpha–beta branching factor, tuning the evaluation function, and using a small amount of inferior cell analysis [28]. Six uses a truncated two-ply alpha–beta search. Six is open source and has been played by many humans. It is generally considered to be a very strong, although not quite expert, player on board sizes up to 11×11, with near-perfect play on board sizes up to 8×8 [27].

In 2008, Wolve won the Fifth Computer Olympiad Hex competition, defeating the previous champion four games to zero [6]. Wolve was developed in conjunction with MoHex and shares the same codebase. Wolve improves on Six's design by modifying the evaluation function, improving H-search efficiency, using much more inferior cell analysis, using an opening book generated by self-play, and using a solver in parallel with the player. Like Six, the 2008 version uses a two-ply search. The 2009 version of Wolve searches to four-ply when time allows.

In summary, computer players using alpha–beta search have proven to be strong competitors against skilled human players, and until 2008 dominated computer *Hex* competitions.

IV. Basic MoHex Framework

A. Monte Carlo Tree Search

MCTS is a best-first search algorithm that is guided by the outcome of random game simulations. The algorithm is composed of three basic phases:

1) *tree traversal* from the root to some leaf node;
2) *random game simulation* from the leaf node's corresponding game position;
3) *tree update* using information from the simulated game.

The basic algorithm is anytime, repeating these steps until no more time remains. After the tree traversal phase, the search tree is expanded by adding the children of the selected leaf node. When MCTS terminates, the child with the largest subtree (i.e., which produced the most simulations) is selected as the best move.

MoHex's MCTS is built on the codebase of Fuego, the *Go* program developed by Enzenberger and Müller at the University of Alberta, Edmonton, AB, Canada [10].

B. Tree Traversal and Update

MoHex uses the upper confidence bounds applied to trees (UCT) framework combined with the all-moves-as-first (AMAF) heuristic to select the best child during tree traversal [14], [25].

The UCT framework tracks a value for each node, based on the sum of its average win/loss performance (based on all random simulated games that started in its subtree) plus an exploration term (that increases the value for less explored nodes).

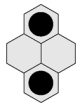


Fig. 6. Bridge rollout pattern: if white plays one empty cell, black plays the other.

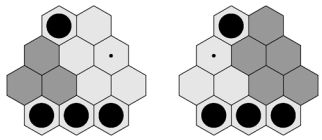


Fig. 7. 4-3-2 VC rollout pattern: if white plays a shaded cell, black maintains the connection by playing the corresponding dotted cell.

The tree traversal starts at the root, recursively proceeding to the child of highest value until it reaches a leaf node. This formula is designed to balance the complementary concerns of exploitation (applying the best performing move) versus exploration (trying moves that have largely been ignored).

The AMAF heuristic uses each random simulated game to update win/loss statistics for all moves in the simulated game, rather than for only the first move in the simulated game. Each move played by the winner is assigned a win, and each move played by the loser is assigned a loss. Thus, the AMAF heuristic accelerates the rate at which MCTS accumulates data, although the resulting data may be less accurate. Tree updates occur at each node along the path from the leaf to the root, thereby influencing leaf node selection in future tree traversals.

Like Fuego, MoHex plays strongest when it uses an exploration constant of zero, effectively turning off UCT exploration and relying solely on the AMAF heuristic to find strong candidate moves.

C. Random Game Simulation

Playing *Hex* until the board is completely filled results in the same win/loss outcome as stopping once one player has a winning chain, so the random game simulation need not check for game termination after each move. Hence, *Hex* game simulations can be efficiently implemented: add all empty cells to an array, shuffle them randomly, and play the remaining moves in order. A consequence of this particular implementation is that each legal move's AMAF statistics are updated after each game simulation.

As with *Go*, it is beneficial to apply some knowledge during simulated games [9]. MoHex uses a single pattern during random game simulation: if a player probes any opponent bridge, then the opponent always replies so as to maintain the connection. See Fig. 6. If multiple bridges are probed simultaneously, then one such bridge is randomly selected and then maintained.

Yopt, another MCTS *Hex* program, uses additional patterns based on another commonly occurring VC [33]. See Fig. 7. However, in our tests, MoHex showed no strength gain from such patterns.

V. MoHex Enhancements

A. Tree Knowledge

Like many other MCTS players, MoHex uses knowledge-intensive algorithms in important parts of the tree, as well as flags to indicate solved states [36].

Using a fixed "knowledge threshold," if any node is visited often enough during tree traversal, then both inferior cell analysis and the H-search algorithm are run on that position. There are two possible outcomes: either fill-in or H-search solves the position, or the position value is still unknown.

In the former case, all child subtrees are deleted, and the tree node is marked such that any tree traversal that encounters this node omits the random game simulation, and simply updates its ancestor nodes using the correct outcome.

In the latter case, subtrees corresponding to moves that can be eliminated via inferior cell analysis or mustplay results are pruned from the tree.

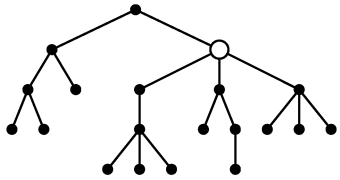
Furthermore, for every tree node that surpasses the knowledge threshold, its fill-in computation is stored permanently and applied to every tree traversal. Since fill-in is computed anew for each tree node, there can be disagreement between the fill-in of a node and its child, so the descendant node's fill-in takes precedence, and any prior fill-in knowledge is ignored.

This fill-in produces two benefits. First, the random game simulations are shorter (since the number of empty cells has decreased), and thus allows more game simulations per second. Second, the accuracy of the game simulations should improve, since any resolved regions of the board are guaranteed to have the correct outcome.

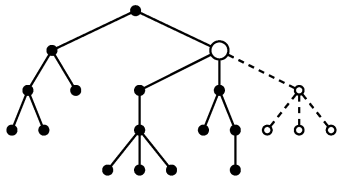
Although each child node corresponding to a fill-in move is deleted, a fill-in move might still be available in some child's subtree, possibly yielding an illegal game sequence in which a fill-in move is played. To avoid this, each unpruned child's subtree is also deleted excepting their roots and all relevant statistics (e.g., UCT and AMAF data). Note that any subsequent tree expansions below the parent node will not conflict with the fill-in. See Fig. 8.

The knowledge threshold is typically small (e.g., the 2009 Olympiad version had threshold 50), so the size of any truncated subtree is small, and the subsequent loss of information is apparently more than compensated by the gain in performance.

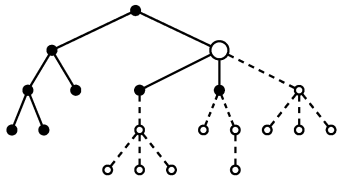
For instance, consider the top *Hex* position in Fig. 9. If MoHex evaluates this position for black without knowledge, it consistently gets a score of 71%–73% for 30-s searches. By applying knowledge in the Monte Carlo tree, fill-in produces the bottom *Hex* position in Fig. 9, and the evaluation scores plummet to 1%–8% for 30-s searches. This behavior also holds for other MCTS *Hex* players: this position was taken from an olympiad game between Yopt and Wolve, where Yopt's evaluation score surpassed 90% in the endgame despite its losing position.



1) Node reaches knowledge threshold; perform inferior cell analysis and H-search computations.



2) Prune children that are inferior or outside of mustplay.



3) Remove subtrees of all remaining children.

Fig. 8. Applying knowledge to the Monte Carlo tree.

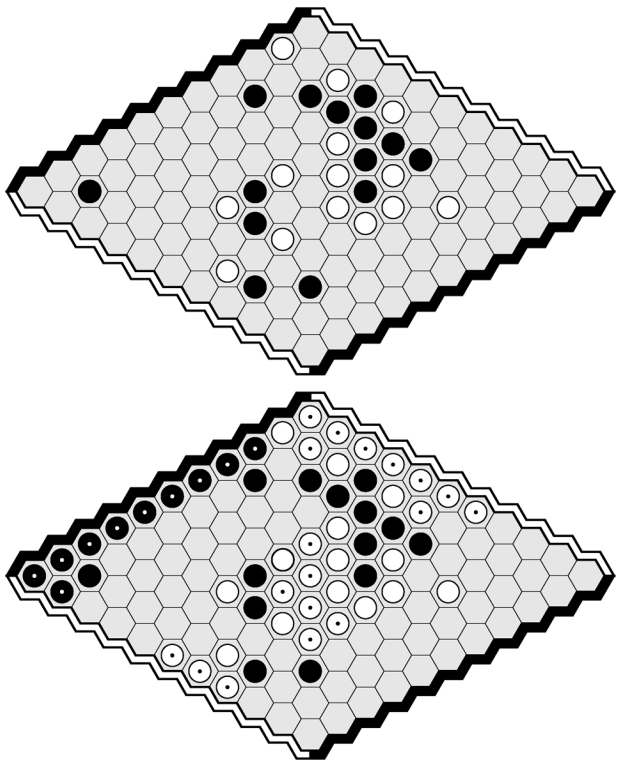


Fig. 9. A *Hex* position and its fill-in reduced position.

B. Lock-Free Parallelization

MoHex uses the Fuego codebase, and so benefits from Fuego's lock-free parallel MCTS [10]. MoHex's knowledge computations are handled within this lock-free framework. Thus, it is possible for different threads to perform duplicate knowledge computations concurrently, but this is extremely rare in practice.

C. Parallel Solver

MoHex runs a *Hex* solver concurrently with the search. This produces perfect play whenever the game position can be solved quickly. This solver uses inferior cell analysis and H-search in the same way as MoHex: to identify state values, and prune inferior moves from consideration.

The solver is based on depth-first proof number (DFPN) search, and so also relies on these knowledge computations to restrict its branching factor and guide it to the strongest moves [1], [30]. It also uses a move ordering heuristic to temporarily prune the weaker moves of each node in the DFPN search tree, gradually revealing them as their sibling moves are proven to be losing. This technique reduces the combinatorial explosion of large branching factors, and also prevents the winning player from spending time exploring (heuristically) weaker moves.

Our solver can solve all 8×8 openings in roughly 30 h. It has also solved over half of the 9×9 openings [7].

D. Time Management

Over thousands of games between strong players (i.e., any of MoHex, Wolve, Six), the average 11×11 game length is about 60 moves. Within a 1-min search, MoHex's solver usually computes the game's value by move 35, and almost always by move 45. This bounds the number of moves MoHex needs to generate in a game. *Hex* tournaments allow 30 min/player, so MoHex can easily allot 1 min/move. In the 2009 Computer Olympiad, MoHex allotted 96 s/move.

VI. Experimental Results

Because of the swap rule, an automated *Hex* player needs to be able to respond competently to every opening move the opponent might select. Thus, our testing iterates over all 11×11 openings with the swap rule off, with each program playing once as black and once as white for every opening. Thus, one round consists of $11 \times 11 \times 2 = 242$ games. In order to reduce the standard error to a small percentage, we typically run several rounds.

While this testing format is helpful in identifying weaknesses in our algorithm's performance (e.g., openings where we perform poorly as both black and white), it significantly dampens any strength gains obtained, as polarized openings (i.e., openings easily won by black or white) are played twice, and essentially guarantee at least some wins for the weaker player. Thus, the Elo gains reported in our experiments underrepresent the expected tournament play Elo improvement.

A. Scaling

MCTS is a parallelizable anytime algorithm, so the scaling of its performance with respect to time and number of threads is important.

As with many other MCTS programs, MoHex's strength scales logarithmically versus time, with each doubling of the game simulations producing roughly an additional 36 Elo of strength: 8-s/move MoHex defeats 1-s/move MoHex 65.1% of the time. See Fig. 10.

As with Fuego, the lock-free version of multithreaded MoHex scales far better than the locked alternative. Indeed, the change here is even more dramatic than that in *Go*—scaling of the locked version is worse with two threads, and performance actually degrades with only four threads—presumably because the game simulations in *Hex* are so much faster than in *Go*, and the threads spend most of their time in the tree. See Fig. 10.

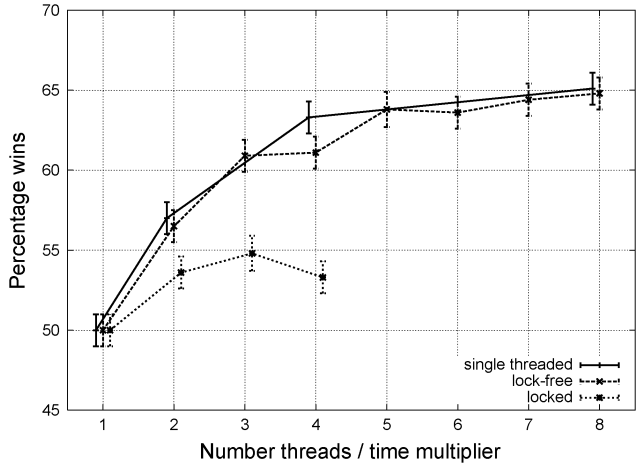


Fig. 10. Performance of locked, lock-free, and time-scaled single threaded MoHex against single threaded 1-s/move MoHex.

Incrementally Added Feature	Win %	Elo gain
Bridge pattern	64.7% ± 1.4%	105
AMAF heuristic	73.9% ± 1.3%	181

Fig. 11. The bridge pattern and AMAF heuristic improve playing strength by 286 Elo.

B. Heuristic Techniques

Both the bridge pattern and AMAF heuristic give major strength gains for MoHex. The bridge pattern produces a 105 Elo strength gain against a naive UCT implementation (with an optimized exploration constant of 0.7), and this improved version is surpassed by another 181 Elo by adding the AMAF heuristic (and turning off UCT exploration). Based on the scaling information above, this total strength gain is roughly equivalent to a 250-fold increase in computing time. See Fig. 11.

We tested many inferior cell analysis patterns as game simulation patterns. Unfortunately, in all cases, these patterns gave MoHex no strength gain. This provides evidence that provably correct information in game simulations can weaken MCTS players.

C. Tree Knowledge

Adding tree knowledge to MoHex is roughly equivalent to doubling the number of game simulations. The optimal knowledge threshold seems to be 400 for the single threaded version, and to decrease proportionally with the number of threads. A very low knowledge threshold can worsen performance, due to the corresponding decrease in the number of rollouts. See Fig. 12.

D. Opening Book

MoHex's opening play can be inconsistent, perhaps because there is so little existing structure to guide the random game simulations. We are investigating the construction of an

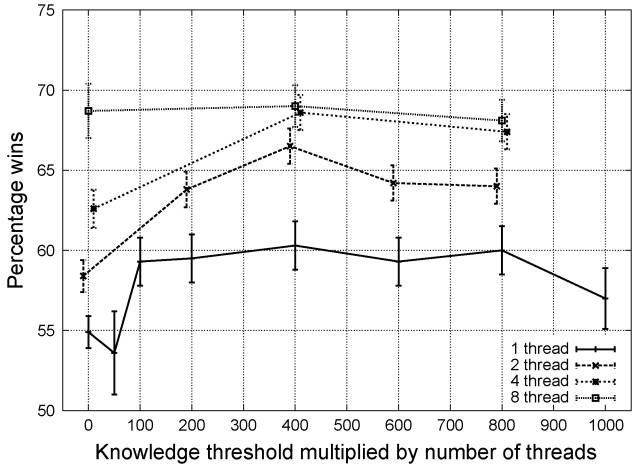


Fig. 12. Threaded 8-s/move MoHex with knowledge against two-ply Wolve. A knowledge threshold of zero means that no knowledge is computed.

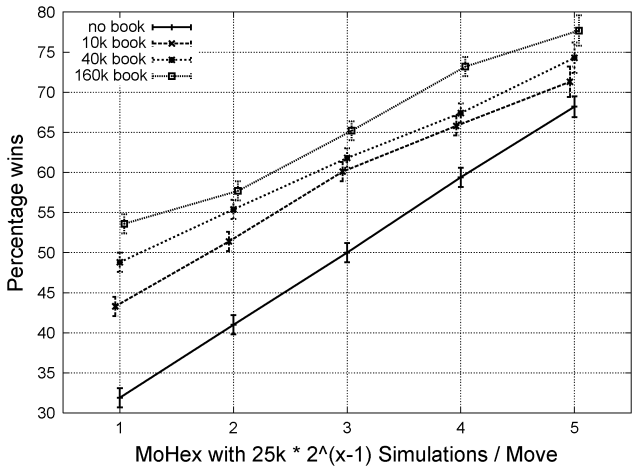


Fig. 13. MoHex with books of increasing size against 100-k/move MoHex with no book.

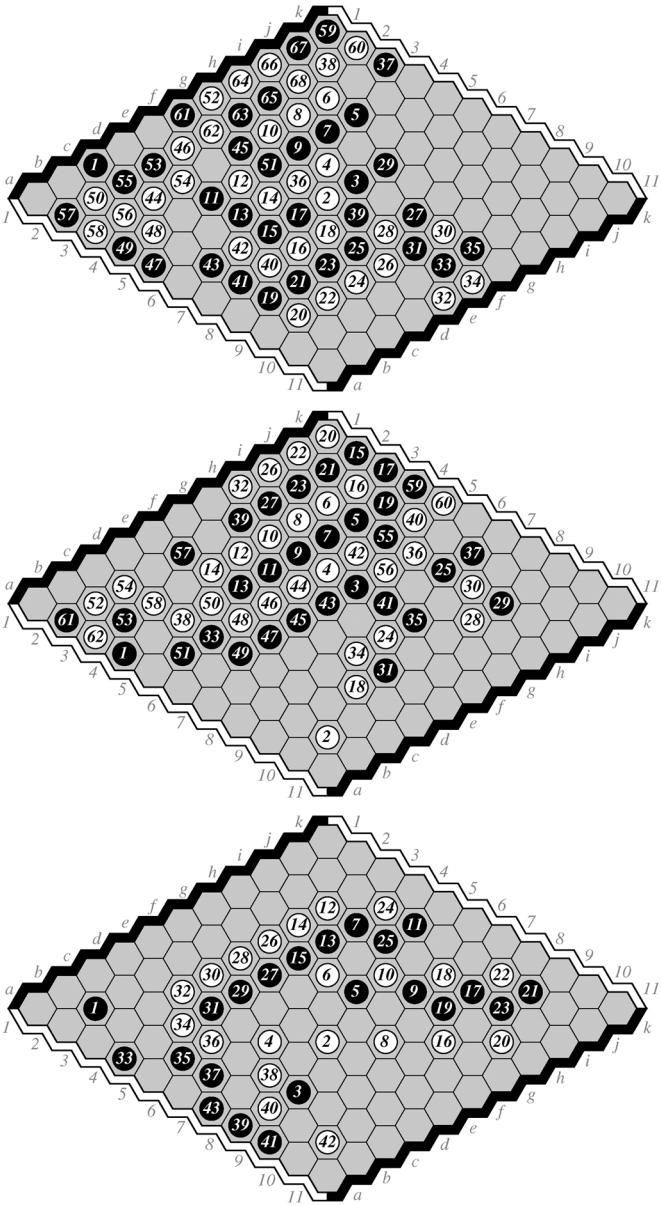


Fig. 14. MoHex 2009 Computer Olympiad games. From top to bottom, opponents are Wolve, Six, and Yopt. MoHex is white in all games.

opening book using Lincke's method [26]. Our initial results are promising, with an opening book for 9×9 *Hex* that was constructed in a day producing gains of 85 Elo, which is worth more than a doubling of simulations. As the book size increases, the strength gains grow logarithmically. See Fig. 13.

VII. Tournament Performance

As mentioned earlier, MoHex won gold at the 2009 Computer Olympiad. Its opponents were Yopt, another MCTS-based player, and the previous alpha–beta-based champions Wolve and Six. See Section III. Although MoHex was undefeated, it nevertheless had a few close games, three of which we briefly examine here. See Fig. 14.

Against Wolve, MoHex's evaluation was close to 50% throughout the game, until its parallel solver indicated that the game was won. Wolve's move 37 looks weak—k2 seems preferable to k3—but Wolve had already lost the game at that point.

Against Six, the game's progression is surprising, since MoHex's opening play is weak and Six develops a strong wall of influence within 19 moves. Nevertheless, MoHex manages to play effectively, with an especially brilliant move 36. MoHex's parallel solver generates moves 40 and beyond. Postgame analysis revealed that the game was close: MoHex blundered on move 38 and Six blundered on move 39.

Against Yopt, the game is straightforward, with parallel bridge structures and many probes of such connections. Postgame analysis indicates that Yopt played winning moves 27 and 29, but that move 31 is a blunder: d4 loses while c4 wins. MoHex maintains its winning position from this point on.

See the Olympiad report for further commentary [5].

A. Experimental Tournaments

Because Olympiad results are based on a very small sample size, we also ran our own tournaments between MoHex and the two previous champions, Six and Wolve. Each tournament game enforced Olympiad time constraints—30 min/player—but

Opponent	MoHex Win %
Six	76.6 ± 3.6
Wolve	49.2 ± 3.2

Fig. 15. MoHex: performance against Six and Wolve.

allowed neither opening book nor parallel solver, so that the results measure only the relative strength of each player's search engine. We did not run tests against Yopt since this program is not publicly available. The results are summarized in Fig. 15. Basically, MoHex dominates Six and is evenly matched with Wolve.

VIII. Application to Other Games

The application of tree knowledge to MCTS, in particular the use of fill-in, can be generalized to other classical board games. Most of *Hex*'s inferior cell analysis applies to *Y* (with only minor modifications relating to the fact that board sides are not owned by players in *Y*), and so a MCTS *Y* player could likewise use fill-in and inferior cell knowledge in its Monte Carlo tree.

Fill-in and inferior cell pruning can also be computed for *Havannah*, but the existence of rings as a winning condition (instead of just connecting paths, such as bridges and forks) makes this process far less straightforward. Our group has performed some initial research on this topic, but no implementation of these ideas has yet been performed. Due to the relative rarity of such patterns in *Havannah*, we expect this knowledge to yield only a minor gain, although it may be more useful in terms of improving a player's parallel solver.

For games like *Go* where such theoretical pruning is extremely difficult and/or rare, perhaps heuristic fill-in—such as preventing simulated games from playing in the territory of live groups and seki regions—could prove useful in improving accuracy and speeding up simulated games.

IX. Limitations and Future Work

Despite MoHex's recent success, there are still many aspects that can be improved.

MoHex does not fully exploit the connection strategies it computes. Although H-search finds connection strategies, MoHex uses this information only for pruning and endgame play. In particular, simulated games do not use known connection properties. As expected, the near-random games usually do not maintain important connections, especially when the carrier is large.

We have tested many algorithms that incorporate connection strategies into the simulations, including the following two: heuristically select connections to maintain during the simulation; at the root node, allow the choice of maintaining some subset of a given list of connection strategies.

Unfortunately, all of these techniques greatly worsened performance. The best variation to date involves using common responses in the Monte Carlo tree to guide the responses in the simulated game; investigations are still ongoing.

Given the strength of our parallel solver, stronger integration of the solver and MCTS could produce even greater benefits. Currently, the solver indicates only whether the root position has been solved; the solver could instead inform MoHex of any tree nodes that it has solved. Furthermore, it might be desirable for the search tree to indicate which moves it currently prefers, thereby encouraging the solver to explore the most pertinent lines.

Other possibilities include using the AMAF heuristic to influence the random game simulations, ensuring that moves perceived to be strong are played early in the simulation. Also, heuristic initialization of moves in the Monte Carlo tree may ensure that strong moves are always explored or speed up the UCT convergence, although such methods have not yet been successful in Monte Carlo *Hex*.

X. Conclusion

We have applied MCTS techniques to produce an automated *Hex* player that is on par with the best players produced via the classical alpha–beta approach. The potential for further improvement remains, especially in terms of better incorporating connection strategies in the tree and game simulations, improved knowledge computation in the tree (i.e., parallel solver updates), development of an opening book, and improving simulation quality.

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