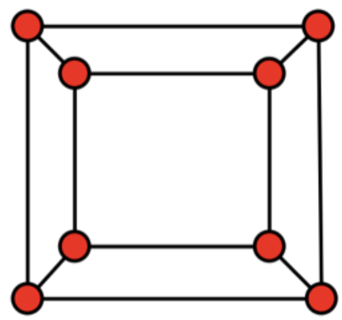
1 On a special intersection graph of class 2

Our interest here is to find a graph of class 2 with $\Delta = 3$ and $\omega = 2$. Moreover, we want to ask ourselves if such a graph can be an intersection graph of squares, and an intersection graph of unit squares.
Let's take a look at the following graph:

Figure 1: Graph G used as a base.

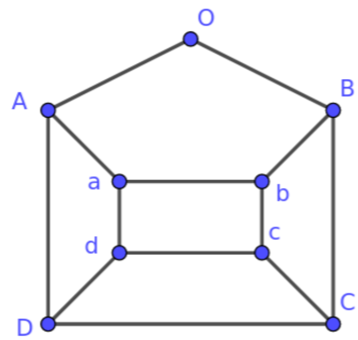


Lemma 1.1. A graph obtained by replacing an edge of G by a vertex of degree 2 is not 3-edge colorable.

Proof. Notice that the previous embedding of G is consisting of two rectangles, one inside the other, with edges linking the cornes. Call a, b, c, d the corners of the inner rectangle, and A, B, C, D the corner of the outer rectangle.
The edges we consider are the obvious ab, bc, cd, da (same for capitals) and the aA (same for other letters.).
Then we obtain an other graph, G' by supressing AB, adding a vertex O, then adding two edges: AO and OB.

We start by coloring the 5-cycle $AOBba$. Without loss of generality, consider that a 3-coloring of it uses two times colors 1 and 2, and one time color 3. Consider the three following cases:

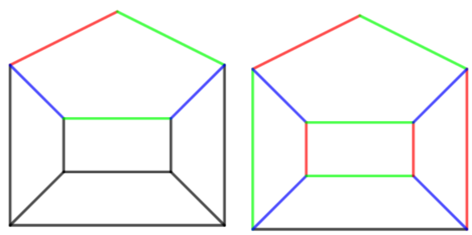
- 1. color 3 is on the AO edge.
- 2. color 3 is on the ab edge.
- 3. color 3 is on the Aa edge.



case 1:

Suppose Aa is in color 1 and AO is in 3. . Then, by trying to use only 3 colors you end up with, in the following order:
ad 3; bc 3; BC 3; AD 2; Dd 1. Then you are forced to color DC in a 4th color.

Figure 2: Case 1: we end up needing a 4th color.



case 2:

Suppose Aa is in color 1. Then you must color in the following order:
ad 2; AD 3; Dd 1; dc 3; bc 1; cC 2. Then you have to use a 4th color for Dd.

case 3:

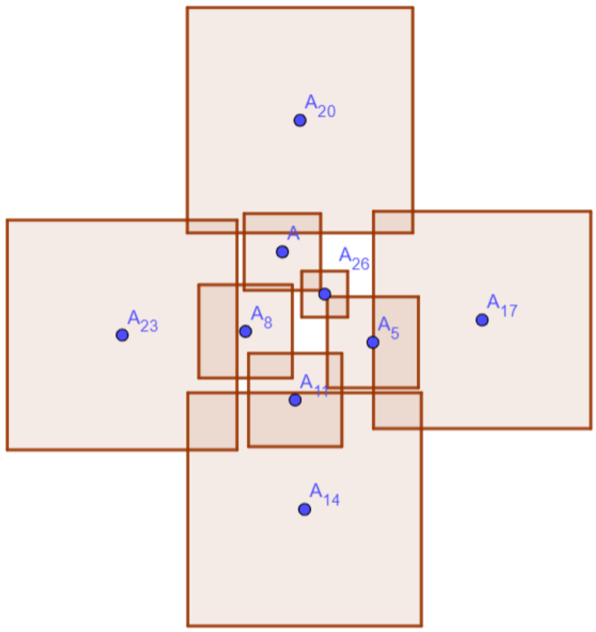
Suppose Aa is in 3 and ab is in 1. The coloring order is:
ad 2; bc 3; dc 1; Dd 3; Cc 2; DC 1. Then a 4th color is necessary for BC and AD.

□

1.1 Case of squares intersection graph

Lemma 1.2. G' can be the intersection graph of axis parralel squares.

Observe an example of such a graph:



1.2 Case of unit squares intersection graph

Definition 1.1 (direction of an intersection for unit squares)**.** We say that a square a intersect b in the direction "UL" if the upper-left corner of a lies in b. Consequently, it would means b intersect a in the "DR" direction. There is 4 different directions: "UL", "UR", "DR", "DL"

Remark. Right now, we will suppose that a square a can't intersect a square b in two adjacent directions (see definition 1.2) at the same time. Later we will prove that in the case of our problem, this is not possible: see lemma 1.5.

Lemma 1.3. If a intersect b in a direction and a intersect c in the same direction, then b and c must intersect.

Remark. Thus, if a intersect b in a direction, and a intersect c; such that b and c do not intersect, then the intersection of a and c must be in a different direction.

Lemma 1.4. Let a and b two unit squares with affixes respectively z_a and z_b and p_x, p_y the projections on axis x, y. Then:

$$a \cap b = \emptyset \Leftrightarrow (|p_x(z_a) - p_x(z_b)| > 1) \text{ or } (|p_y(z_a) - p_y(z_b)| > 1)$$

Proof. Trivial. □

Definition 1.2 (adjacent directions)**.** a intersect b and c in distinct adjacent directions if the corners of a that are inside b and c lies on an edge of a. For instance, UR and DL are not adjacent directions.

Lemma 1.5. If a intersect b in direction UL and c in directions UR and DL at the same time, then:

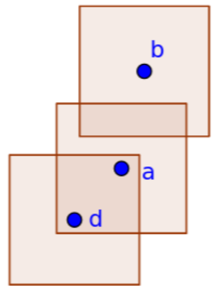
$$\forall d \text{ such that } d \cap b \neq \emptyset \text{ and } d \cap c \neq \emptyset \Rightarrow d \cap a \neq \emptyset$$

Proof. As a and c share a's upper and down right corners, the only possibility for d and c to intersect without d intersection a would be that c interest d in direction UL or DL. Applying lemma 1.4 on d and c on axis x and noticing that $p_x(z_c) < p_x(z_d)$ would give us that $p_x(z_b) < 1 + p_x(z_d)$, ie $d \cap b = \emptyset$. □

Remark. This implies that in an intersection graph of unit squares that is a C_4, we can't find two squares with one having two of it's corners inside the other.

Lemma 1.6. Let G be C_4, an intersection graph of unit squares. Then; $\forall v \in V(G)$, v intersects it's neighbours in adjacent directions.

Proof. Let p_x and p_y the projections on respectively x and y axis.
Let a, b, c and d be unit squares such that their corresponding intersection graph is C_4. Without loss of generality, suppose that a intersect b in direction UR and a intersect d in direction DL.
Call z_a, z_b, z_c and z_d the affixes of the center of the squares.
If c intersect d in direction DL, then a, c and d would share a common point in the upper right corner of d, contradicting the fact we have C_4.
Suppose c intersect d in a direction different than DL such that a and c share no common point; then $c \cap b = \emptyset$, and we shall prove this using projections. As a and c share no commont point, in at least one of the projections, we



must have an empty intersection. Call it p. Thus, we have $p(z_c) < 1 + p(z_a)$. as a intersect b in UR direction, then $p_x(z_a) < p_x(z_b)$ and $p_y(z_a) < p_y(z_b)$. By transitivity; we must have $p(z_c) < 1 + p(z_b)$; or, in other words; $c \cap b = \emptyset$. □

Theorem 1.7. $G^{'}$ can't be a intersection graph of unit squares.

Proof. We will actually prove that a subgraph of $G^{'}$ can't be done with unit squares. Let $G^{''} := G'[U]$ with $U := \{a, b, d, c, C, D, A\}$ The proof is done by trying to construct the squares according to the edges of G''.
First, place d. It belongs to three cycles of size 4 at the same time through it's three neighbours: a, c and D. Suppose, without loss of generality, that d intersect a in direction UL; then d must intersect D in direction either DL or UR in order to complete the cycle $daAD$. For the same reason, d must intersect c in direction either DL or UR to complete $dcCD$. Thus, completing the cycle $abcd$ becomes impossible as d is intersecting a and c in non-adjacent directions. □