Remark. This statement is true, I guess.

Definition 0.1 (Fibration)**.** A fibration is a mapping between two topological spaces that has the homotopy lifting property for every space X.

Lemma 0.1. Given two line segments whose lengths are a and b respectively there is a real number r such that $b = ra$.

Proof. To prove it by contradiction try and assume that the statement is false, proceed from there and at some point you will arrive to a contradiction. □

1 Super vertex coloring

Definition 1.1 (Super vertex coloring)**.** Let G be an undirected graph. $c : V(G) \to \{1, 2, ..., k\}$ is a super vertex coloring iff:

$$\forall v_1, v_2 \in V(G), c(v_1) = c(v_2) \Rightarrow v_1 \notin N(v_2), \forall u \in N(v_1), v_2 \notin N(u)$$

We say that G is n-super colorable if we can find a super vertex coloring using n colors.

Let K_n be a complete graph. We can clearly find a super vertex coloring by assigning a different color to each vertex. Also by [1] (lemma 2), We can find an edge coloring using n colors (respectively $n-1$) if n is odd (respectively even).

Theorem 1.1. If G admitt a super vertex coloring using n colors, then we can find and n or $n-1$ coloring of it's edges.

Proof. The idea is to use the K_n graph as a tool to decide which color to assign to each edge.
Let G be a n-super colorable graph. Call c_g such a coloration. Let K_n and denote by (k_i) it's vertices. As stated earlier, we can find a n (or $n-1$) edge coloration of it. We shall call this edge coloration e_k.
Then, assign to the k_i a different number in $\{1, ..., n\}$. Call c_k such a coloration.
Let v_iv_j be an edge of G. Find k_1, k_2 the vertices of K_n such that $c_g(v_i) = c_k(k_1)$ and $c_g(v_j) = c_k(k_2)$ assign to v_iv_j the color $e_k(k_1k_2)$.
Now let's prove by contradiction that such a coloring of the edges is proper. Assume the existence of v_1, v_2, v_3 such that $v_1v_2, v_2v_3 \in E(G)$ are assigned to the same color. Let k_1, k_2, k_3 such that $c_g(v_i) = c_k(k_i)$. As e_k is a proper edge coloring, if $e_k(k_1k_2) = e_k(k_2k_3)$, then $k_1 = k_3$. Thus, it would mean that $c_g(v_1) = c_g(v_3)$, ie c_g is not a super vertex coloring of G. □

2 Special graphs

2.1 Trees

Lemma 2.1. Let G be a tree with maximum degree Δ. Then G is $\Delta + 1$ super colorable.

Proof. The proof is done by induction on $|E(G)|$.
When $|E(G)| = 0$, $\Delta = 0$ and a super coloration is simply assigning the same color to each vertex.
Let G be a graph with Δ maximum degree. Find v such that $d(v) = 1$, call v' it's only neighbour. As per the induction property $G - v$ admitts a $\Delta + 1$ super vertex coloring. As the degree of v' in $G - v$ is at most $\Delta - 1$, it means it lacks at least a color among the color of v' and it's neighbours. Choose it for v. □

2.2 Interval graphs

Lemma 2.2. Let G be an interval intersection graph with maximum degree Δ . Then G is $\Delta + 1$ super colorable.

Proof. The proof is done by induction on the number of intervals.
The result is trivial with one interval.

Let G an interval graph. Call I_i it's intervals and v_i it's associated vertices. We can find an ordering of it's intervals. $I_1, I_2, ...$ such that $\forall i, r_i < r_{i+1}$. Consider $G - v_1$. By the induction property, it admitts a $\Delta(G - I_1) + 1$ super vertex coloration.
If $\Delta(G) > \Delta(G - v_1)$, then just assign to v_1 the newly available color.

Suppose $d_G(v_1) = k - 1 < \Delta(G)$. The neighbours of v_1 are $v_2, ..., v_k$. Observe that each two of the neighbours of v_1 are neighbours in G: Indeed, they share at least r_1 as a common point.
Moreover, we can state that $\forall i, j \in \{2, ...k\}, i < j \Rightarrow N(i) \subset N(j)$:
Let i, j such that v_i, v_j and v_1 are neighbours. Let v^* an neighbour of v_i, such as $v^* \notin N(v_1)$

- As v_i and v^* are neighbours, $l^* < r_i$; so $l^* < r_j$ by construction.
- As v^* and v_1 are not neighbours, $r_1 < l^*$
- As v_j and v_1 are neighbours, $l_j < r_1$

Thus, $l_j < l^* < r_j$; so I^* and I_j share a common point in l^*.
So each neighbour of I_1 is also a neighbour of I_k. As $d_{G-v_1}(v_k)$ is at most $\Delta(G) - 1$; we deduce that a least one color has to be missing among the neighbours of v_k (including itself); choose this color for v_1. □

3 References

[1] V.A. Bojarshinov, Edge and total coloring of interval graphs, Disctrete Applied Mathematics 114 (2001) 23-28