1 Maximum Matching - greedy algorithm

Theorem 1.1. The maximum matching greedy algorithm for interval graphs is correct.

Proof. Let $I := \{i_1, ...i_n\}$; with the intervals sorted by their right-end point. The proof is done by induction on $|I|$.
For $|I| = 1$, the algorithm gives $M = \emptyset$; which is correct.

Induction step
Suppose the algorithm gives the correct answer for $n-1$ intervals. Let $|I| = n$, and M be the matching computed by the greedy algorithm for $I - i_n$. Let U be the set of unmatched intervals.
If U is empty, when treating I the algorithm will give the same answer: M.
Let U be non empty. Let $i_k \in U$. If $i_k \cap i_n \neq \emptyset$, the algorithm will return $M + (i_k, i_n)$. Indeed, for $i < k$:

- If i_i could be matched with something in $I - i_n$, it would never choose to match it with i_n as it is the worst choice.
- If i_i could not be matched in $I - i_n$, then i_i will be left unmatched; because if it was possible to match it with i_n; then i_i and i_k would have not be unmatched $I - i_n$.

Augmentating path
Suppose now that U is non empty, and that for some $i_k \in U$ we can find a shortest augmentating path $P : (i_k, j_1, j_1', j_2, j_2', ..., j_m', i_n)$, namely:

- $i_k \cap j_1 \neq \emptyset$ and $j_m' \cap i_n \neq \emptyset$.
- $\forall x < m, j_x' \cap j_{x+1} \neq \emptyset$.
- $(j_x,\, j_x') \in M$.

Until the end of the proof, we will use the notation r_k, r_x, r_x' for the right-end points of respectively i_k, j_x, j_x', and l_k, l_x, l_x' for left-end points.

Decreasing property
We shall now prove by a second induction that $\forall x, r_{x+1}' < r_x'$.
First, observe than $r_1 < r_k$ and $r_1' < r_k$ as otherwise, i_k would not be left unmatched in the first place.
As for the couple j_2, j_2', we have either:

- case 1: $r'_1 < l_k$ and $r'_1 < r_2$. As j'_2 and j'_1 are not matched, they must be disconnected, meaning $r'_2 < l'_1 \Rightarrow r'_2 < r'_1$.

- case 2: $r'_1 > l_k$. As P is the shortest, we have $r_2 < l_k$. As j_2 and j'_2 are matched, $r'_2 < min(r_1, r'_1) \Rightarrow r'_2 < r'_1$.

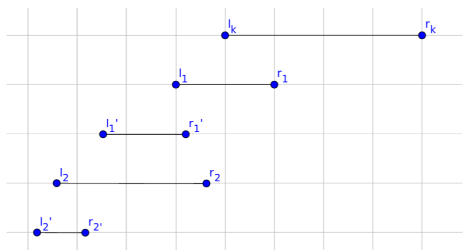

Figure 1: case 1.

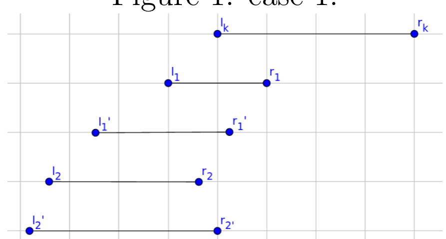

Figure 2: case 2.

Induction step
The induction step is proved in exactly the same way, by replacing i_k by j'_p; j_1, j'_1 by j_{p+1}, j'_{p+1}, etc.

Conclusion
In definitive, we have $r'_m < r'_1 < r_k$. It is thus a contradiction with the fact that $j'_m \cap i_n \neq \emptyset$.

$\square$