ORCO — Optimization Under Uncertainty

Homework on Robust Optimization

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In this homework we investigate Γ-uncertainty in the context of linear programs (LPs). In particular we consider LPs of the following form:

$$\begin{aligned} \max \quad & c^T x \\ \text{s.t.} \quad & Ax \leq b \\ & \ell \leq x \leq u \end{aligned} \tag{LP}$$

where $A \in \mathbb{R}^{m \times n}$, $b \in \mathbb{R}^m$, and $x, \ell, u \in \mathbb{R}^n$. We assume without loss of generality that the objective function is certain. On the other hand, for each row A_i of A we have a set $J_i \subseteq \{1, 2, \ldots, n\}$ of indices of the elements of A_i that are uncertain. The elements $\{1, 2, \ldots, n\} \setminus J_i$ of A_i are certain. Furthermore, for $1 \leq i \leq m$, we have a value $\Gamma_i \in [0, |J_i|]$ that controls how much the uncertainty may affect the elements of A_i indexed by J_i. Suppose that the element a_{ij} of A is uncertain (i.e., $j \in J_i$). Then the value of a_{ij} will be in the interval $[a_{ij} - \xi_{ij}, a_{ij} + \xi_{ij}]$ for some given $\xi_{ij} \geq 0$. Hence the uncertainty is coordinate-wise, that is, $\mathcal{U} = \mathcal{U}_1 \times \ldots \times \mathcal{U}_m$. For a given solution $x \in \mathbb{R}^n$, we quantify the amount of uncertainty that can hit constraint i by

$$\beta_i(x, \Gamma_i) := \max_{\{S_i \cup \{t_i\} | S_i \subseteq J_j,\, |S_i| = \lfloor \Gamma_i \rfloor,\, t_i \in J_i \setminus S_i\}} \left\{ \sum_{j \in S_i} \xi_{ij} |x_j| + (\Gamma_i - \lfloor \Gamma_i \rfloor) \xi_{it_i} |x_{t_i}| \right\}$$

We obtain the following non-linear robust counterpart of (LP):

$$\begin{aligned} \max \quad & c^T x \\ \text{s.t.} \quad & A_i x + \beta_i(x, \Gamma_i) \leq b \qquad (1 \leq i \leq m) \\ & \ell \leq x \leq u \end{aligned} \tag{RLP}$$

Exercise 1. Our first goal is to show that the non-linear optimization problem (RLP) can be written as a linear program.

(*a*) Show that for a feasible solution x of (RLP) and $1 \le i \le m$, the function $\beta_i(x, \Gamma_i)$ is equal to the optimal value of

$$\begin{aligned} \max \quad & \sum_{j \in J_i} \xi_{ij} |x_j| z_{ij} \\ \text{s.t.} \quad & \sum_{j \in J_i} z_{ij} \le \Gamma_i \\ & 0 \le z_{ij} \le 1 \qquad (j \in J_i) \end{aligned} \tag{P_i}$$

(*b*) By using the LP dual of (P_i) show that (RLP) can be written as a linear program of the following form

$$\begin{aligned} \max \quad & c^T x \\ \text{s.t.} \quad & A_i x + z_i \Gamma_i + \sum_{j \in J_i} p_{ij} \le b_i & (1 \le i \le m) \\ & z_i + p_{ij} \ge \xi_{ij} y_j & (1 \le i \le m,\ j \in J_i) \\ & p_{ij} \ge 0 & (1 \le i \le m,\ j \in J_i) \\ & -y \le x \le y \\ & \ell \le x \le u \\ & y \ge 0,\ z, p \ge 0 \end{aligned} \tag{RLP_{lin}}$$

where $x, y, u, \ell \in \mathbb{R}^n$ and $z \in \mathbb{R}^m$.

Exercise 2. The parameters Γ_i control the amount of uncertainty we allow per constraint. We would like to investigate the impact of a very small change $\Delta\Gamma_i$ of Γ_i on the objective function. For this purpose we assume that (RLP_{lin}) has a unique optimal primal solution z^* and a unique optimal dual solution q^*. Furthermore, let B be the unique optimal basis corresponding to z^*. Our goal is to show that the derivative of the objective function with respect to Γ_i is equal to $z_i^* q_i^*$.

(*a*) We consider (RLP_{lin}) written in standard form and depending on the parameter Γ_i:

$$G(\Gamma_i) := \begin{cases} \max & c^T x \\ \text{ s.t.} & Ax + \Gamma_i z_i e_i = b \qquad (1 \le i \le m) \end{cases}$$

where e_i is a unit vector whose ith element is 1. Show that if z_i is a non-basic variable with respect to B then the derivative of the objective function given above is correct.

(*b*) To conclude assume that z_i is a basic variable with respect to B. First, write the basis associated to $G(\Gamma_i + \Delta\Gamma_i)$ as the sum of two matrices and apply the Sherman-Morrison identity (that can be found on wikipedia) to obtain its inverse. Use the result to show that

$$\lim_{\Delta\Gamma_i \to 0} \frac{G(\Gamma_i + \Delta\Gamma_i) - G(\Gamma_i)}{\Delta\Gamma_i} = z_i^* q_i^* \ .$$