Matching Covered Graphs and Subdivision

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Objective

Theorem 6

Every non-bipartite **matching covered** graphs contains a **nice** subgraph that is an odd **subdivision** of K_4 or $\bar{C}_6$.

Matching Covered Graphs

Theorem 1

A graph G has a perfect matching if and only if $c1(G - X) \leq |X|$, for each set X of vertices.

Matching Covered Graphs

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A graph G has a perfect matching if and only if $c1(G - X) \leq |X|$, for each set X of vertices.

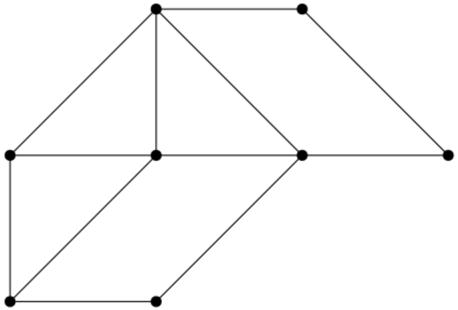
Definition

If the equality holds for some set X of vertices in the previous theorem, then X is a barrier.

Matching Covered Graphs

Definition

A connected graph is matching covered if each of its edges lies in some perfect matching.



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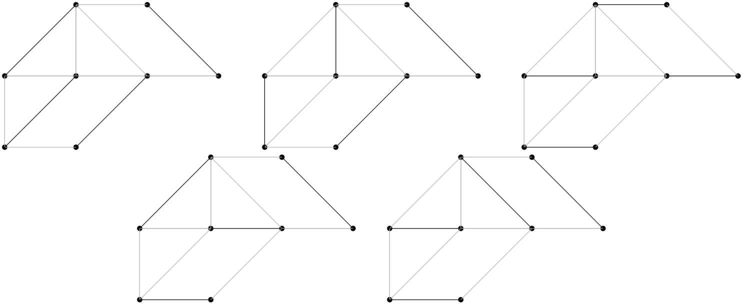
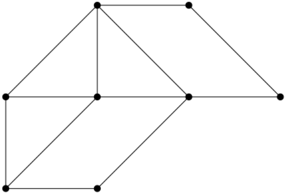
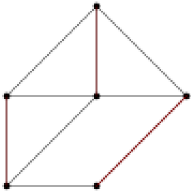


Nice graphs

Definition

A subgraph H of matching covered graph G is nice if G-H has a perfect matching.

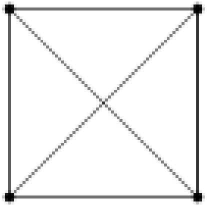




Bicritical Graphs

Definition

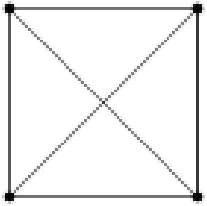
A graph is bicritical if deletion of any two of its vertices yields a graph having a perfect matching.

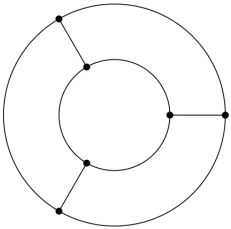


Brick

Definition

A brick is a 3-connected bicritical graph.

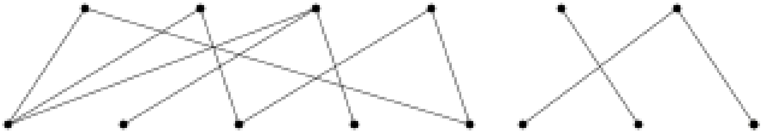




Brace

Definition

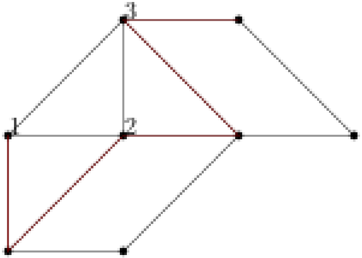
A bipartite graph is a brace if deletion of any four vertices, two from each color class, yields a graph having a perfect matching.



Tight-cut

Definition

For any subset X of the set of vertices of a graph G, the set of all edges of G with exactly one end in X is denoted by $\partial(X)$, and is referred to as a cut of G. If G is connected and $C := \partial(X)$. A cut $C := \partial(X)$ of G is tight if $|C \cap M| = 1$, for every perfect matching M of G.



Link between Brick and Brace

Definition

A matching covered graph which is free of nontrivial tight cuts is a brace if it is bipartite, and a brick if it is nonbipartite.

Ear

Definition

Let H be a subgraph of G. A path P in G - E(H) is an ear of H if both ends of P lies in H and P is internally disjoint of H.

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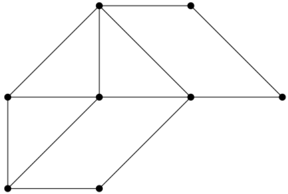
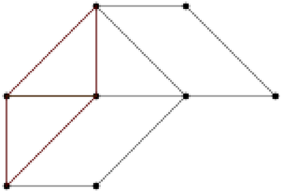
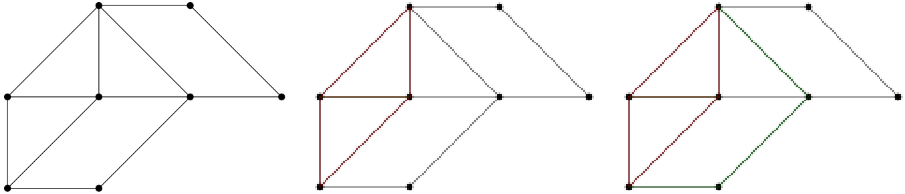




Definition

Let H be a subgraph of G. A path P in G - E(H) is an ear of H if both ends of P lies in H and P is internally disjoint of H.



Ear Decomposition

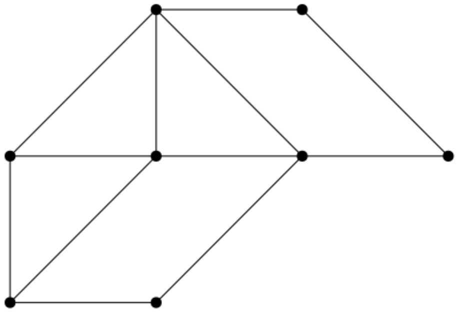
Definition

An ear-decomposition of a matching covered graph G is a sequence $K_2 = G_0 \subset G_1 \subset ... \subset G_k = G$ of matching covered subgraphs of G_{i+1} where for $0 \leq i < k$, G_{i+1} is the union of G_i and one or two vertex-disjoint ears of G_i.

Ear decomposition

Theorem 2

Every matching covered graph has an ear decomposition.



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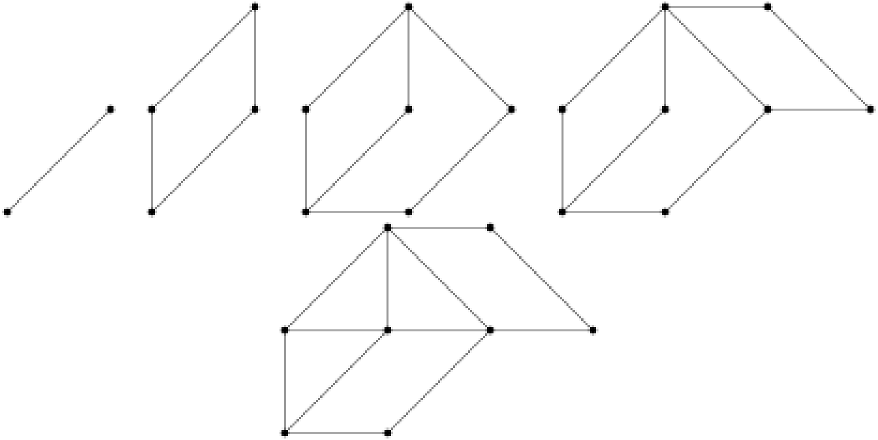
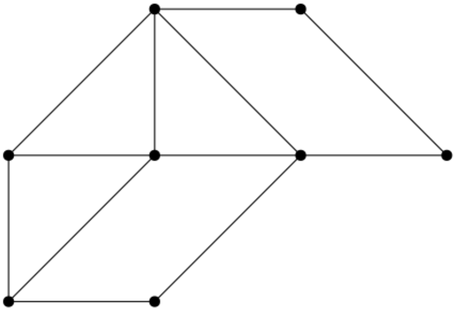


Ear decomposition

Theorem 3

Every matching covered graph has an ear decomposition starting with any nice matching covered subgraph.



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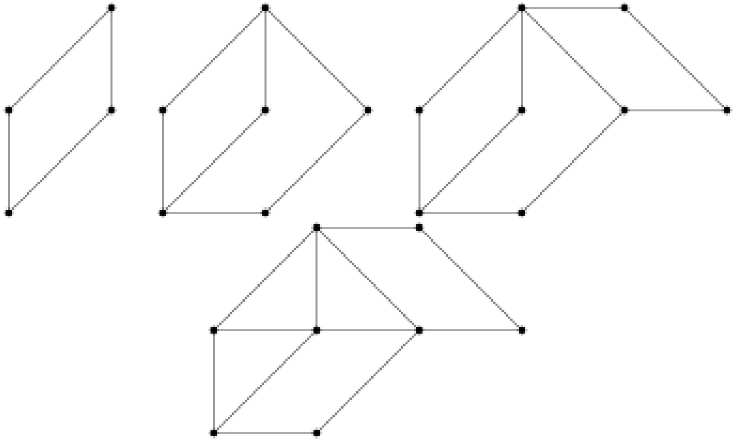
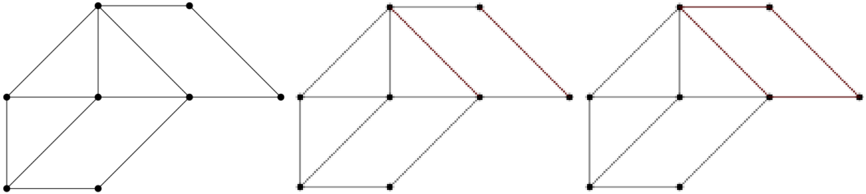


Theorem 4

Any two edges of a matching covered graph lie in a nice circuit.



Ear decomposition

Theorem 4

Any two edges of a matching covered graph lie in a nice circuit C.

Proposition

Let C a nice circuit of G as in theorem 4. We can find a perfect matching M such that C is M-alternating.

Odd subdivision

Definition

An odd subdivision of a graph G is a graph obtained from G by subdividing each edge in an odd number of edges.

Main theorem

Theorem 6

Every non-bipartite matching covered graphs contains a nice subgraph that is an odd subdivision of K_4 *or* $\bar{C}_6$.

Proof

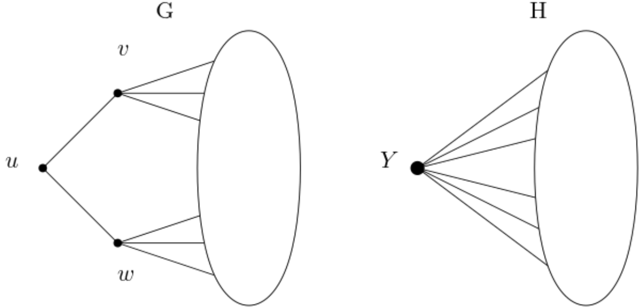
By induction on $|V(G)| + |E(G)|$.

Proof: case 1

- Suppose that G has proper subgraph H non bipartite, matching covered and nice.
- By induction, G verifies Theorem 6.

Proof: case 2

Suppose G has a vertex u of degree 2.



- Contract u and it's neighbours to obtain H.

Proof: case 2

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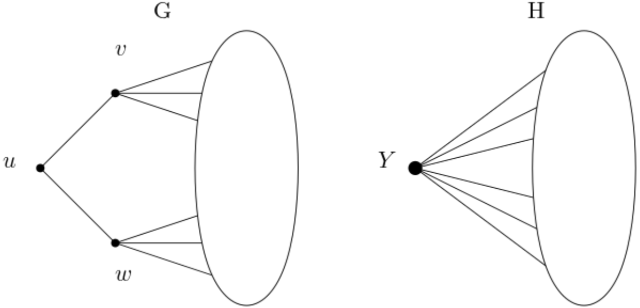

- Contract u and it's neighbours to obtain H.
- G non bipartite and matching covered $\Rightarrow$ H non bipartite and matching covered.

Proof: case 2

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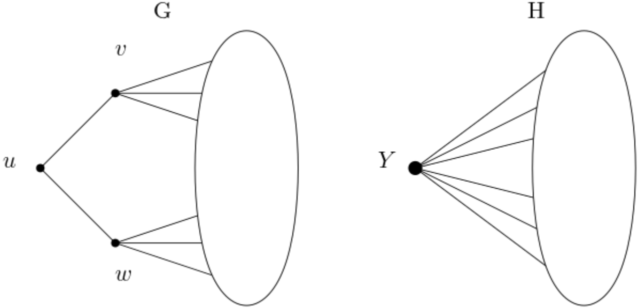
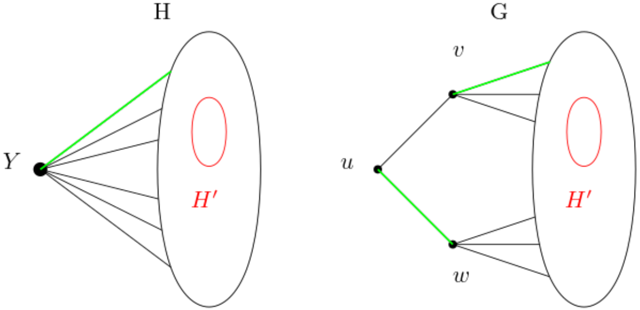


- Contract u and it's neighbours to obtain H.
- G non bipartite and matching covered $\Rightarrow$ H non bipartite and matching covered.
- Find nice subgraph $H' \subset H$, odd subdivision of K_4 / $\bar{C}_6$, and a perfect matching M of $H - H'$.
- Extend the H' or M in G depending on the situation.

Proof: case 2.1

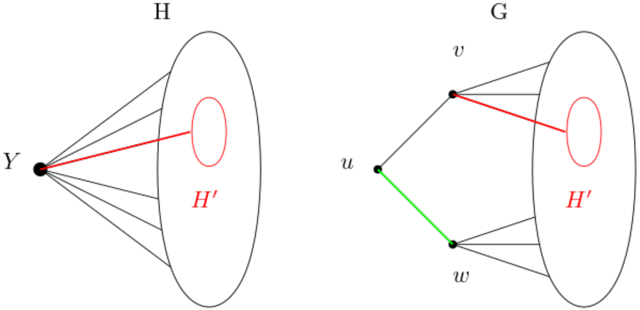
Suppose neither v and w belongs to H'.



- The subgraph H' exists in G.
- Y is matched in $H - H'$; and we can extend M on G by adding either uv or uw.

Proof: case 2.2

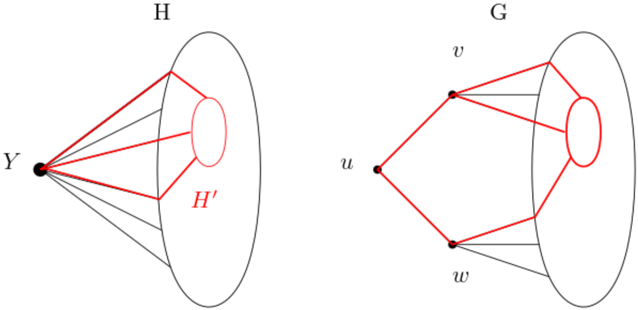
Suppose exactly one of v and w is in H'.



- The subgraph H' exists in G.
- Y is not matched in $H - H'$; and we can extend M on G by adding either uv or uw.

Proof: case 2.3

Suppose both of v and w belongs to H'.



- In H', every vertex is at most of degree 3

Proof: case 2.3

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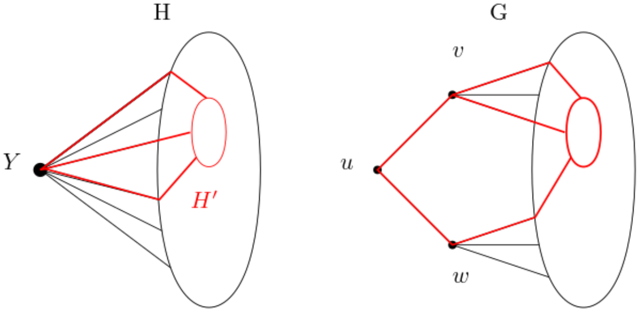

- In H', every vertex is at most of degree 3, hence one of v, w is of degree 1.

Proof: case 2.3

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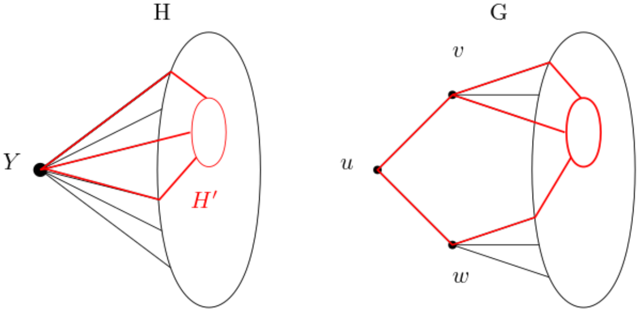

- In H', every vertex is at most of degree 3, hence one of v, w is of degree 1.
- Extend H' on uv and uw. This extension is still an odd subdivision of K_4 or $\bar{C}_6$: we replaced one edge by 3 edges.
- M do not need to be extended.

Proof: case 3

Suppose neither of case 1 and 2 applies. What is the objective?

- Find $H' \subset G$, odd subdivision of K_4 or $\bar{C}_6$.

Proof: case 3

Suppose neither of case 1 and 2 applies. What is the objective?

- Find $H' \subset G$, odd subdivision of K_4 or $\bar{C}_6$.
- All odd subdivisions of K_4 or $\bar{C}_6$ are non bipartite and matching covered

Proof: case 3

Suppose neither of case 1 and 2 applies. What is the objective?

- Find $H' \subset G$, odd subdivision of K_4 or $\bar{C}_6$.
- All odd subdivisions of K_4 or $\bar{C}_6$ are non bipartite and matching covered $\Rightarrow H' = G$.

Proof: case 3

Suppose neither of case 1 and 2 applies. What is the objective?

- Find $H' \subset G$, odd subdivision of K_4 or $\bar{C}_6$.
- All odd subdivisions of K_4 or $\bar{C}_6$ are non bipartite and matching covered $\Rightarrow H' = G$.
- No vertex of degree 2 $\Rightarrow G$ is K_4 or $\bar{C}_6$ itself.

Proof: case 3

Suppose that none of the previous cases applies.

- G is non bipartite and matching covered $\Rightarrow$ we can find an ear decomposition $K_2 := G_0, ... G_n := G$. (Theorem 2)

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Suppose that none of the previous cases applies.

- G is non bipartite and matching covered $\Rightarrow$ we can find an ear decomposition $K_2 := G_0, \dots G_n := G$. (Theorem 2)
- G_{n-1} is a proper subgraph of G; so it must be bipartite, with sets A and B.

Proof: case 3

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- No vertex of degree 2 in G so the ear we add from G_{n-1} to G_n are edges.

Proof: case 3

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- To break bipartite property: add an edge with both ends in the same vertex group.

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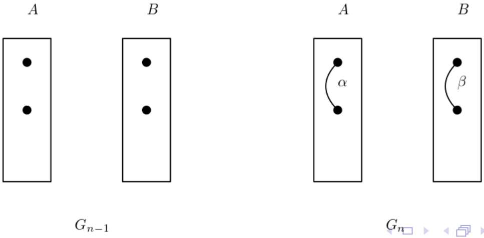
- No vertex of degree 2 in G so the ear we add from G_{n-1} to G_n are edges.
- To break bipartite property: add an edge with both ends in the same vertex group.
- We need to add at least two edges as Gn is matching covered.



Proof: case 3

- G_{n-1} is matching covered so for every two edges e_1, e_2 we can find a nice circuit including them. (theorem 4)

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- Choose e_1 and e_2 both sharing an end with α (or β)
- Build this way the smallest circuit C. α (or β) is now a chord of C.

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- Choose e_1 and e_2 both sharing an end with α (or β)
- Build this way the smallest circuit C. α (or β) is now a chord of C.
- Let M_1 be a perfect matching of G_{n-1} such that C is M_1-alternating.

Proof: case 3

Proposition 7

If γ is a chord of C different from α (or β) then γ crosses α (or β).

Proof: case 3

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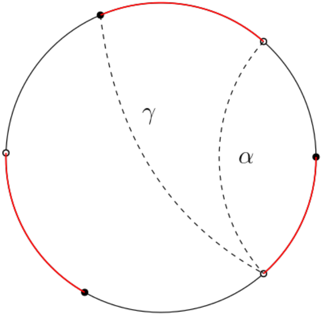
If γ is a chord of C different from α (or β) then γ crosses α (or β).

Proof

Suppose α do not cross γ. Then the circuit $C' \cup \{\gamma\}$ (or $C'' \cup \{\gamma\}$) is nice, and strictly smaller than C.



Proof: case 3

Let M be a perfect matching of G containing α and β. Build D a M_1 alternating circuit of $M \triangle M_1$ containing α.

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- G' is nice:

Proof: case 3

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- G' is nice: C and D are both M_1 alternating and M_1 is a perfect matching of G.
- G' is matching covered

Proof: case 3

Let M be a perfect matching of G containing α and β. Build D a M_1 alternating circuit of $M \triangle M_1$ containing α. Note that D also contains β as $|D|$ is even. Let $G' := G[C \cup D]$

- G' is nice: C and D are both M_1 alternating and M_1 is a perfect matching of G.
- G' is matching covered by $C \backslash M_1; D \backslash M_1$ and M_1.
- G' not bipartite: ($\alpha \in D$ induces an odd cycle)

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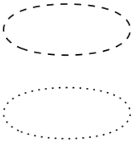
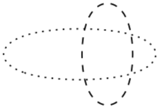
- G' is matching covered by $C \backslash M_1$; $D \backslash M_1$ and M_1.
- G' not bipartite: ($\alpha \in D$ induces an odd cycle)

Conclusion: $G = G[C \cup D]$.

Proof: case 3



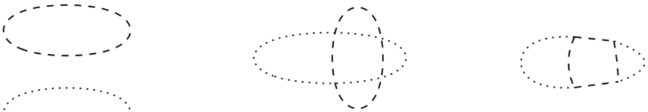
Case 1



Case 2

Case 3

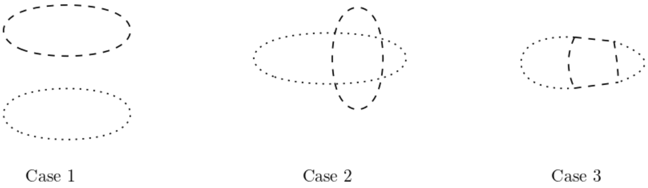
Proof: case 3



Case 1 Case 2 Case 3

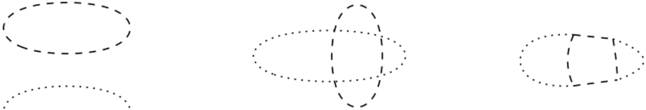
- Case 1: not possible: $\alpha \in D$ is a chord of C.

Proof: case 3



- Case 1: not possible: $\alpha \in D$ is a chord of C.
- Case 2: D which is M_1 and M alternating makes it impossible.

Proof: case 3

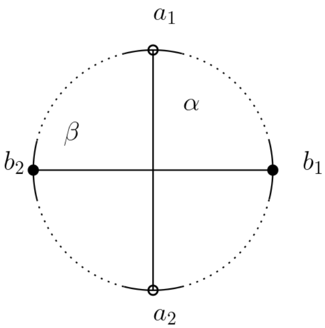


Case 1 Case 2 Case 3

- Case 1: not possible: $\alpha \in D$ is a chord of C.
- Case 2: D which is M_1 and M alternating makes it impossible.
- So G is cubic.

Proof: case 3; α, β cross

Suppose that α and β crosses. Then a_1, b_1, a_2, b_2 appears in this order in C; hence $G = K_4$.



Proof: case 3; α, β cross

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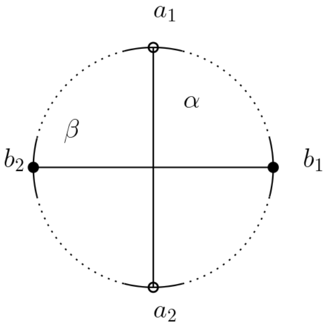
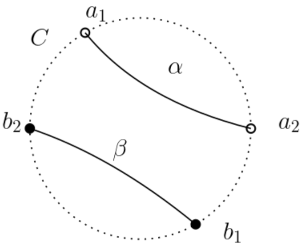


Note that we can't have other vertices (between a_1, b_1 for instance) without forcing G to have bipartite proper subgraph with an odd subdivision ...

Proof: case 3, α, *beta* do not cross

Suppose α and β do not cross.



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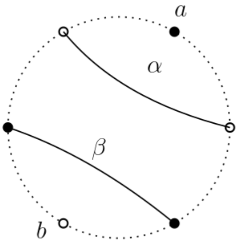
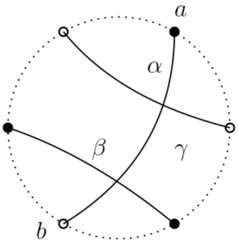


- a_1, a_2, b_1, b_2 appears in this order in C.
- Path from a_1 to a_2 that doesn't contains b_1 must have a vertex $b \in B$.

Proof: case 3, α, *beta* do not cross

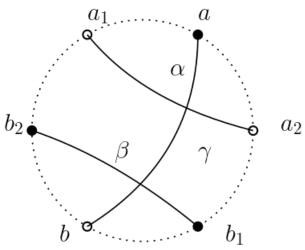
Suppose α and β do not cross.

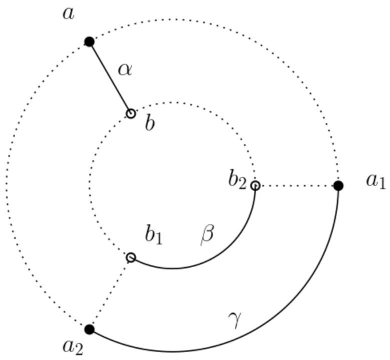


- a_1, a_2, b_1, b_2 appears in this order in C.
- Path from a_1 to a_2 that doesn't contains b_1 must have a vertex $b \in B$.
- Proposition 7: a chord γ starting in b must cross α.

Proof: case 3, α, *beta* do not cross

Suppose α and β do not cross.





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- Path from a_1 to a_2 that doesn't contains b_1 must have a vertex $b \in B$.
- Proposition 7: a chord γ starting in b must cross α.
- Obtained graph is $\bar{C}_6$

Deduced theorem

Lemma 8

Let G be a brick, $e, f \in E(G)$ such that any perfect matching that contains one of these edges also contains the other. Then $G - e - f$ is bipartite.

Deduced theorem

Theorem 9

Let G be a brick different from K_4 and $\bar{C}_6$. Then G has an edge e such that G - e is matching covered.

Deduced theorem

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Let G be a brick different from K_4 and $\bar{C}_6$. Then G has an edge e such that G - e is matching covered.

Proof

1. Theorem 6 and 3 : G has an ear-decomposition in which the first non-bipartite graph is an odd subdivision of K_4 and $\bar{C}_6$
2. Lemma 8 : G_{k-1} is either bipartite or $G_k = G$ arises from G_{k-1} by adding a single edge.
3. G different from K_4 and $\bar{C}_6$ then G_{k-1} can not be bipartite. G_k arises from G_{k-1} by the adjunction of a single ear e.
3. $G_{k-1} = G - e$ is matching covered