Assignment 3

Daria Davydova

1. **Efficient portfolios (40 points).**

   Consider an investor who wants to invest in N risky assets with return $R_i \;\; \forall i = 1, \ldots, N$ with expected return $E[R_i] = \mu_i$ and variance $V[R_i] = \sigma_i^2$, and in a risk-free asset with return R_f. The investor seeks a N-risky asset portfolio weight vector w (and a weight $1 - w^\top \mathbf{1}$ in the risk-free asset), such that her portfolio return $R_p = R_f + w^\top (R - R_f \mathbf{1})$ maximizes her mean-variance objective function $U(w) = E[R_p] - \frac{\gamma}{2} V[R_p]$.

   (a) Show that an optimal portfolio weight vector w is such that for the corresponding mean-variance efficient portfolio return R_p we have

   $$\mu_i - R_f = \gamma \, cov[R_i, R_p] \;\; \forall i = 1, \ldots N$$

   (b) Show that for any such mean-variance efficient portfolio, we have

   $$\mu_i - R_f = \beta_{i,P}(\mu_P - R_f)$$

   where $\beta_{i,P} = \frac{cov(R_i, R_P)}{\sigma_P^2}$ is the linear regression coefficient of return R_i on the mean-variance efficient portfolio return R_P.

   (c) In turn, show that this implies that, if R_p is the return to a mean-variance efficient portfolio, then for any return i we have

   $$R_i = R_f + \beta_i (R_P - R_f) + \epsilon_i$$

   where $cov(R_P, \epsilon_i) = 0$.

   Hint: use the definition of a linear regression

A linear OLS regression of R_i on R_P decomposes the return of asset i as follows:

$$R_i = \beta_0 + \beta_1(R_P - R_f) + \epsilon_i \tag{1}$$

where $E(\epsilon_i) = 0$ and $Cov(R_P - R_f, \epsilon_i) = Cov(R_P, \epsilon_i) = 0$. The regression coefficient β_1 is given by

$$\beta_1 = \frac{Cov(R_i, R_P - R_f)}{Var(R_P - R_f)}$$

(d) Show that all mean-variance efficient portfolios have the same Sharpe ratio where we define its Sharpe ratio as $SR_p = \frac{\mu_p - R_f}{\sigma_p}$.

2. **Alternative portfolio strategies (60 points).** The optimal mean-variance portfolio is a complex function of estimated means, volatilities, and correlations of asset returns. There are many parameters to estimate. Optimized mean-variance portfolios can be quite unstable due to estimation errors in the inputs.[1]

   In this exercise you will run a horse-race between four different portfolio strategies that invest in several risky assets and a risk-free asset. Diversification is common to all the strategies, but they build a diversified portfolio in different ways. This leads to differences in performance.

   The four portfolios strategies are:

   (a) The tangency portfolio (TAN).

   (b) The global minimum-variance portfolio (GMV). This is a special case of full mean-variance analysis that does not estimate means; it implicitly assumes that all assets have the same mean.

   (c) The risk parity portfolio with weights equal to the inverse of standard deviations of returns (RP). This is also a special case of mean-variance analysis that does not estimate means or correlations; it implicitly assumes that all assets have the same Sharpe ratio and are uncorrelated.

   (d) The equally weighted portfolio (EW). Give conditions on the means and covariance matrix of returns for the EW portfolio to be mean-variance efficient? This

[1] In the situation considered here with 30 assets, there are 495 parameters to estimate. With 100 assets, there are 5,150 parameters to estimate. With 5,000 stocks (approximately the number listed in U.S. markets) the number of parameters to estimate is over 12,000. The potential for errors is enormous.

has nothing to estimate. It is also a special case of mean-variance analysis; it implicitly assumes that all assets have identical mean, variances, and pair-wise correlation.

Download stock prices from January 1978 to December 2021 from the Yahoo finance monthly stock file for the with the following companies: 'AMERICAN EXPRESS CO', 'BOEING CO', 'CATERPILLAR TRACTOR INC', 'CHEVRON CORP NEW', 'DISNEY WALT PRODUCTIONS', 'INTERNATIONAL BUSINESS MACHS CO', 'INTEL CORP', 'JOHNSON & JOHNSON', 'JPMORGAN CHASE & CO', 'COCA COLA CO', 'MCDONALDS CORP', '3M CO', 'MERCK & CO INC', 'PFIZER INC', 'PROCTER & GAMBLE CO', 'TRAVELERS COMPANIES INC', 'UNITED TECHNOLOGIES CORP', 'WALGREENS BOOTS ALLIANCE INC', 'WALMART INC' and 'EXXON MOBIL CORP'.(These companies are in today's Dow 30 index which were already been traded in 1978) For the four different portfolio strategies, track the performance from January 1988 to the end of the sample period.

As the relevant risk-free rate, use the monthly US Treasury Bill rate that can also be downloaded from the Yahoo finance (as US treasury bonds for 5 years). Also, you can use simply 1.5% to simplify the task.

Implement the strategies at time t using data for a ten-year window. Therefore, the first portfolios are formed in the beginning of January 1988 using returns from January 1978 to December 1987. The portfolios are held for one month (i.e, until end of January 1988). The next portfolios are formed at the end of January 1988 using returns from February 1978 to January 1988. The portfolios are held for one month (i.e, until end of February 1988), and so on.

(e) Compute mean and standard deviation of portfolio returns as well as the Sharpe ratio (annualize by multiplying by $\sqrt{12}$)

(f) Interpret the differences in performance

(g) Plot the minimum-variance frontier for the risky assets only in (σ, μ)-space where you estimate the means and covariances using data from January 1988 to December 2018. Plot the location of the four portfolios using their ex-post performance. Comment on the results.

(h) Plot the cumulative performance of one dollar 1 dollar you start investing in January 1988 in each of the strategies (as well as in T-bills)