1.0 Analytical formula

The price of a put and a call can be calculated using an analytical formula with the Black-Scholes model. The value of S_0 is the initial value of the asset price. The value of K is the strike price. The value of r is the risk-free interest rate and the value σ is the volatility of the stock price. The maturity of the option in years is T. The formula for the put and call respectively are given by :

$$\text{Put} = e^{-rT} E\left[\max\left(K - S_T, 0\right)\right] = Ke^{-rT}\Phi\left(-d_2\right) - S_0\Phi\left(-d_1\right)$$

and

$$\text{Call} = e^{-rT} E\left[\max\left(S_T - K, 0\right)\right] = S_0\Phi\left(d_1\right) - Ke^{-rT}\Phi\left(d_2\right)$$

where

$$d_1 = \frac{\ln\left(\frac{S_0}{K}\right) + \left(r + 0.5\sigma^2\right)T}{\sigma\sqrt{T}},$$

$$d_2 = d_1 - \sigma\sqrt{T},$$

$$\Phi\left(x\right) = \int_{-\infty}^{x} \frac{1}{\sqrt{2\pi}} \exp\left(-0.5y^2\right) dy$$

and

$$E\left[X\right]$$

is the expected value of the random variable X.

2.0 Stochastic simulations

The price of a put and a call can be calculated using an analytical formula with the Black-Scholes model. The value of S_0 is the initial value of the asset price. The value of K is the strike price. The value of r is the risk-free interest rate and the value σ is the volatility of the stock price. The maturity of the option in years is T. The price of a put and a call option can be approximated using stochastic simulations using the following formula

$$\text{Put} = e^{-rT} E\left[\max\left(K - S_T, 0\right)\right] \approx e^{-rT} \frac{1}{M} \sum_{i=1}^{M} \max\left(K - S_T^{(i)}, 0\right)$$

and

$$\text{Call} = e^{-rT} E\left[\max\left(S_T - K, 0\right)\right] \approx e^{-rT} \frac{1}{M} \sum_{i=1}^{M} \max\left(S_T^{(i)} - K, 0\right)$$

where $S_T^{(i)}$ is the simulated price of the simulation i and M is the number of simulations. We can simulate the value of $S_T^{(i)}$ using the following algorithm :

1. **Step 1 :** generate a random number from a standard normal distribution $W^{(i)}$;
2. **Step 2 :** calculate $S_T^{(i)} = \exp\left(\left(r - 0.5\sigma^2\right)T + \sqrt{T}\sigma W^{(i)}\right)$;
3. **Step 3 :** repeat the first and second steps M times;
4. **Step 4 :** approximate the value of the put price using
$$e^{-rT} \frac{1}{M} \sum_{i=1}^{M} \max\left(K - S_T^{(i)}, 0\right)$$
or the value of the call price using
$$e^{-rT} \frac{1}{M} \sum_{i=1}^{M} \max\left(S_T^{(i)} - K, 0\right).$$

The values of $\sqrt{T}W^{(i)}$ can be approximated using a random walk. Also, a high value of M will generate better approximations of the put and call values.

3.0 Binomial tree

The value of S_0 is the initial value of the asset price. The value of K is the strike price. The value of r is the risk-free interest rate and the value σ is the volatility of the stock price. The parameter N is the number of time steps of the binomial tree. The maturity of the option in years is T and the timesteps in the binomial tree is given by

$$\Delta t = \frac{t}{N}.$$

We can approximate the price of a put or call price with the Black-Scholes model using a binomial tree. The approximated prices of the put and call are given, respectively, by

$$\text{Put} = e^{-rT} E\left[\max\left(K - S_T, 0\right)\right] \approx \sum_{i=0}^{N} \binom{N}{i} p^i \left(1-p\right)^{N-i} e^{-N\times\Delta t\times r} \max\left(K - S_0 \times u^i \times d^{N-i}, 0\right)$$

and

$$\text{Call} = e^{-rT} E\left[\max\left(S_T - K, 0\right)\right] \approx \sum_{i=0}^{N} \binom{N}{i} p^i \left(1-p\right)^{N-i} e^{-N\times\Delta t\times r} \max\left(S_0 \times u^i \times d^{N-i} - K, 0\right)$$

where

$$p = \frac{e^{\Delta t\times r} - d}{u-d},$$

$$u = e^{\sqrt{\Delta t}\times\sigma}$$

and

$$d = e^{-\sqrt{\Delta t}\times\sigma}.$$

4.0 Reference

Hull, J. C. (2012). Options, futures, and other derivatives. Pearson. 8th edition.