



**UTM**  
UNIVERSITI TEKNOLOGI MALAYSIA

# Chapter 1

Part 4

## Quantifiers & Proof Technique

# QUANTIFIERS

- Most of the statements in mathematics and computer science are not described properly by the propositions.
- Since most of the statements in mathematics and computer science use **variables**, the system of logic must be extended to include **statements with the variables**.

# QUANTIFIERS

- Let  $P(x)$  is a statement with variable  $x$  and  $A$  is a set.
- $P$  a **propositional function** or also known as **predicate** if for each  $x$  in  $A$ ,  $P(x)$  is a proposition.
- Set  $A$  is the **domain of discourse** of  $P$ .
- Domain of discourse  $\rightarrow$  the particular domain of the variable in a propositional function.

# QUANTIFIERS

- A **predicate** is a statement that contains variables.

## Example:

$$P(x) : x > 3$$

$$Q(x, y) : x = y + 3$$

$$R(x, y, z) : x + y = z$$

# Example

$x^2 + 4x$  is an odd integer

(domain of discourse is set of positive numbers).

$$x^2 - x - 6 = 0$$

(domain of discourse is set of real numbers).

x is rated as Research University in Malaysia

(domain of discourse is set of university in Malaysia).

# QUANTIFIERS

A predicate becomes a proposition if the variable(s) contained is(are)

- Assigned specific value(s)
- Quantified

## Example

- $P(x) : x > 3$ .

What are the truth values of  $P(4)$  and  $P(2)$ ?

- $Q(x,y) : x = y + 3$ . (multiple variables)

What are the truth values of  $Q(1,2)$  and  $Q(3,0)$ ?

# Propositional functions

- Let  $P(x)$  : “ $x$  is a multiple of 5”
  - For what values of  $x$  is  $P(x)$  true?
- Let  $P(x)$  :  $x+1 > x$ 
  - For what values of  $x$  is  $P(x)$  true?
- Let  $P(x)$  :  $x + 3 = 0$ 
  - For what values of  $x$  is  $P(x)$  true?

# QUANTIFIERS

- Two types of quantifiers:
  - **Universal**
  - **Existential**



# QUANTIFIERS

- Let  $A$  be a propositional function with domain of discourse  $B$ . The statement

for every  $x$ ,  $A(x)$

is **universally quantified statement**

- Symbol  $\forall$  called a **universal quantifier** is used “**for every**”.
- Can be read as “**for all**”, “**for any**”.

# Universal quantifiers

Represented by an upside-down A:  $\forall$

- It means “for all”

## Example

Let  $P(x) = x+1 > x$

We can state the following:

- $\forall x P(x)$
- English translation: “for all values of  $x$ ,  $P(x)$  is true”
- English translation: “for all values of  $x$ ,  $x+1 > x$  is true”

# QUANTIFIERS

- The statement can be written as

$$\forall x A(x)$$

- Above statement is true if  $A(x)$  is true for every  $x$  in  $B$   
(**false if  $A(x)$  is false for at least one  $x$  in  $B$**  ).
- OR In order to prove that a universal quantification is **true**,  
it must be shown for **ALL cases**  
In order to prove that a universal quantification is **false**, it  
must be shown to be false for only **ONE case**
- A value  $x$  in the domain of discourse that makes the statement  $A(x)$   
*false* is called a **counterexample** to the statement.

# Example

- Let the universally quantified statement is

$$\forall x (x^2 \geq 0)$$

Domain of discourse is the set of real numbers.

- This statement is true** because for every real number  $x$ , it is true that the square of  $x$  is positive or zero.

# Example

- Let the universally quantified statement is

$$\forall x (x^2 \leq 9)$$

Domain of discourse is a set  $B = \{1, 2, 3, 4\}$

- When  $x = 4$ , the statement produce false value.
- Thus, the above statement is false and the **counterexample is 4.**

# QUANTIFIERS

- Easy to prove a universally quantified statement is true or false if the domain of discourse is not too large.
- What happen if the domain of discourse contains a large number of elements?
- For example, a set of integer from 1 to 100, the set of positive integers, the set of real numbers or a set of students in School of Computing. It will be hard to show that every element in the set is *true*.

**Use existential quantifier!!**

# QUANTIFIERS

- Let  $A$  be a propositional function with domain of discourse  $B$ . The statement

There exist  $x, A(x)$

is **existentially quantified statement**

- Symbol  $\exists$  called an **existential quantifier** is used “**there exist**”.
- Can be read as “**for some**”, “**for at least one**”.

# QUANTIFIERS

- The statement can be written as

$$\exists x A(x)$$

- Above statement is true if  $A(x)$  is true for at least one  $x$  in  $B$  (**false if every  $x$  in  $B$  makes the statement  $A(x)$  false**).
- **Just find one  $x$  that makes  $A(x)$  true!**



# Example

- Let the existentially quantified statement is

$$\exists x \left( \frac{x}{x^2 + 1} = \frac{2}{5} \right)$$

Domain of discourse is the set of real numbers.

- Statement is true** because it is **possible to find at least one real number  $x$**  to make the proposition true.
- For example, if  $x = 2$ , we obtain the true proposition as below

$$\left( \frac{x}{x^2 + 1} = \frac{2}{5} \right) = \left( \frac{2}{2^2 + 1} = \frac{2}{5} \right)$$

# Negation of Quantifiers

- Distributing a negation operator across a quantifier changes a universal to an existential and vice versa.

$$\neg (\forall x P(x)) ; \exists x \neg P(x)$$

$$\neg (\exists x P(x)) ; \forall x \neg P(x)$$

## Example

- Let  $P(x)$  :  $x$  is taking Discrete Structure course with the domain of discourse is the set of all students.

$$\forall x P(x)$$

All students are taking Discrete Structure course.

$$\exists x P(x)$$

There is some students who are taking Discrete Structure course.

# Example

$$\neg (\exists x P(x)) ; \forall x \neg P(x)$$

$$\neg \exists x P(x)$$

None of the students are taking Discrete Structure course.

$$\forall x \neg P(x)$$

All students are not taking Discrete Structure course.

# Example

$$\neg (\forall x P(x)) ; \exists x \neg P(x)$$

$$\neg \forall x P(x)$$

Not all students are taking Discrete Structure course.

$$\exists x \neg P(x)$$

There is some students who are not taking Discrete Structure course

- Consider “For every student in this class, that student has studied calculus”
- Rephrased: “For every student  $x$  in this class,  $x$  has studied calculus”
  - Let  $C(x)$  be “ $x$  has studied calculus”
  - Let  $S(x)$  be “ $x$  is a student in this class”

$$\forall x C(x)$$

- True if the universe of discourse is **all students in this class**

- What about if the universe of discourse is **all students (or all people?)**
  - $\forall x (S(x) \wedge C(x))$  **X**
    - This is wrong! Why? (because this statement says that all people are students in this class and have studied calculus)
  - $\forall x (S(x) \rightarrow C(x))$  **✓**

$C(x)$  : “x has studied calculus”

$S(x)$  : “x is a student in this class”

- Consider:
  - “Some students have visited Mexico”
  - Rephrasing: “There exists a student who has visited Mexico”
  
- Let:
  - $S(x)$  be “**x is a student**”
  - $M(x)$  be “**x has visited Mexico**”

$\exists x M(x)$

  - True if the universe of discourse is all students



- What about if the universe of discourse is all people?

$$\exists x (S(x) \wedge M(x))$$

$$\exists x (S(x) \rightarrow M(x)) \quad \text{This is wrong! Why?}$$

suppose someone is not student =  $F \rightarrow T$  or  $F \rightarrow F$ , both make the statement true (**refer to truth table  $p \rightarrow q$** )

$S(x)$  : “x is a student”

$M(x)$  : “x has visited Mexico”

# Translating from English

- Consider “Every student in this class has visited Canada or Mexico”
- Let,  $S(x)$  be “**x is a student in this class**”  
 $M(x)$  be “**x has visited Mexico**”  
 $C(x)$  be “**x has visited Canada**”

$$\forall x (M(x) \vee C(x))$$

(**When the universe of discourse is all students in this class**)

$$\forall x (S(x) \rightarrow (M(x) \vee C(x)))$$

(**When the universe of discourse is all people or all students**)

## Mathematical systems consists:

- **Axioms**: assumed to be true.
- **Definitions**: used to create new concepts.
- **Undefined terms**: some terms that are not explicitly defined.
- **Theorem**
  - Statement that can be shown to be true (under certain conditions)
  - Typically stated in one of three ways:
    - As Facts
    - As Implications
    - As Bi-implications

## Direct Proof (Direct Method)

- Proof of those theorems that can be expressed in the form  $\forall x (P(x) \rightarrow Q(x))$ ,  $D$  is the domain of discourse.
- Select a particular, but arbitrarily chosen, member  $a$  of the domain  $D$ .
- Show that the statement  $P(a) \rightarrow Q(a)$  is true. (Assume that  $P(a)$  is true).
- Show that  $Q(a)$  is true.
- By the rule of Universal Generalization (UG),  $\forall x (P(x) \rightarrow Q(x))$  is true.

# Example

“For all integer  $x$ , if  $x$  is odd, then  $x^2$  is odd”

Or  $P(x) : x$  is an odd integer

$Q(x) : x^2$  is an odd integer

$$\forall x(P(x) \rightarrow Q(x))$$

The domain of discourse is set  $Z$  of all integer.

Can verify the theorem for certain value of  $x$ .

$$x=3, x^2=9 ; \text{ odd}$$

Or show that the square of an odd number is an odd number

Rephrased: “if  $n$  is odd, then  $n^2$  is odd”

# Example

- $a$  is an odd integer

$$\Rightarrow a = 2n + 1 \rightarrow \text{for some integer } n$$

$$\Rightarrow a^2 = (2n + 1)^2$$

$$\Rightarrow a^2 = 4n^2 + 4n + 1$$

$$\Rightarrow a^2 = 2(2n^2 + 2n) + 1$$

$$\Rightarrow a^2 = 2m + 1 \rightarrow \text{where } m = 2n^2 + 2n \text{ is an integer}$$

$$\Rightarrow a^2 \rightarrow \text{is an odd integer}$$

## Indirect Proof

- The implication  $p \rightarrow q$  is **equivalent** to the implication  $(\neg q \rightarrow \neg p)$  (**contrapositive**)
- Therefore, in order to show that  $p \rightarrow q$  is true, one can also **show that the implication  $(\neg q \rightarrow \neg p)$  is true.**
- To show that  $(\neg q \rightarrow \neg p)$  is true, assume that the negation of  $q$  is true and prove that the negation of  $p$  is true.

# Example

$P(n) : n^2+3$  is an odd number

$Q(n) : n$  is even number

$$\forall n(P(n) \rightarrow Q(n))$$

$$P(n) \rightarrow Q(n) \equiv \neg Q(n) \rightarrow \neg P(n)$$

$\neg Q(n)$  is true ,  $n$  is not even ( $n$  is odd), so  $n=2k+1$

$$\begin{aligned} n^2 + 3 &= (2k + 1)^2 + 3 \\ &= 4k^2 + 4k + 1 + 3 \\ &= 4k^2 + 4k + 4 \\ &= 2(2k^2 + 2k + 2) \end{aligned}$$



# Example

$$\begin{aligned}n^2 + 3 &= (2k + 1)^2 + 3 \\&= 4k^2 + 4k + 1 + 3 \\&= 4k^2 + 4k + 4 \\&= 2(2k^2 + 2k + 2)\end{aligned}$$

$$t = 2k^2 + 2k + 2$$



*t is integer*

$$n^2 + 3 = 2t$$

**$n^2+3$  is an even integer**, thus  $\neg P(n)$  is true

# Which to use

- When do you use a direct proof versus an indirect proof?
- If it's not clear from the problem, try direct first, then indirect second
  - If indirect fails, try the other proofs

# Example

Prove that “if  $n$  is an integer and  $n^3+5$  is odd, then  $n$  is even”

## Via direct proof

- $n^3+5 = 2k+1$  for some integer  $k$  (definition of odd numbers)
- $n^3 = 2k - 4$
- $n = \sqrt[3]{2k - 4}$
- Umm...

So direct proof didn't work out.

Next up: indirect proof

# Example

Prove that “if  $n$  is an integer and  $n^3+5$  is odd, then  $n$  is even”

## Via indirect proof

- Contrapositive: If  $n$  is odd, then  $n^3+5$  is even
- Assume  $n$  is odd, and show that  $n^3+5$  is even
- $n=2k+1$  for some integer  $k$  (definition of odd numbers)
- $n^3+5 = (2k+1)^3+5 = 8k^3+12k^2+6k+6 = 2(4k^3+6k^2+3k+3)$
- As  $2(4k^3+6k^2+3k+3)$  is 2 times an integer, it is even

# Proof Techniques

## Proof by Contradiction

Assume that the hypothesis is true and that the conclusion is false and then, arrive at a contradiction.

Proposition “**if  $P$  then  $Q$** ”

Proof. **Suppose  $P$  and  $\sim Q$**

Since we have a contradiction, it must be that  $Q$  is true

# Example

Prove that “there are infinitely many prime numbers”.

## Proof:

- Assume there are **not infinitely** many prime numbers, therefore they can be listed, i.e.  $p_1, p_2, \dots, p_n$
- Consider the number  $q = p_1 \times p_2 \times \dots \times p_n + 1$ .
- $q$  is either prime or not divisible, but not listed above.  
Therefore,  $q$  is a prime. However, it was not listed.
- **Contradiction!** Therefore, there are infinitely many primes numbers.

# Example

- For all real numbers  $x$  and  $y$ , if  $x+y \geq 2$ , then either  $x \geq 1$  or  $y \geq 1$ .

## Proof

- Suppose that the conclusion is false. Then

$$x < 1 \text{ and } y < 1$$

Add these inequalities,  $x+y < 1+1 = 2$  ( $x+y < 2$ )

- **Contradiction**
- Thus we conclude that the statement is true.

# Example

Suppose  $a \in \mathbb{Z}$ . If  $a^2$  is even, then  $a$  is even

## Proof

- Contradiction: Suppose  $a^2$  is even and  $a$  is not even.
- Then  $a^2$  is even, and  $a$  is odd
- Let,  $a = 2c + 1$  (odd)

$$a^2 = (2c + 1)^2 = 4c^2 + 4c + 1 = 2(2c^2 + 2c) + 1 \quad (\text{odd})$$

- **Contradiction**
- Thus we conclude that the statement is true.



# Thank You