



Chapter 2

(Part 2)

Functions



Functions

- Let X and Y be nonempty sets
- A function f from X to Y is a relation from X to Y having the properties.
 - The domain of f is X
 - If $(x,y), (x,y') \in f$, then $y=y'$

(e.g. $f(1)=b, f(2)=b$ is a function, but $f(1)=a, f(1)=b$ is NOT a function)



Functions

- A function from X to Y is denoted, $f: X \rightarrow Y$
- The domain of f is the set X .
- The set Y is called the codomain or target of f .
- The set $\{ y \mid (x,y) \in f \}$ is called the range.

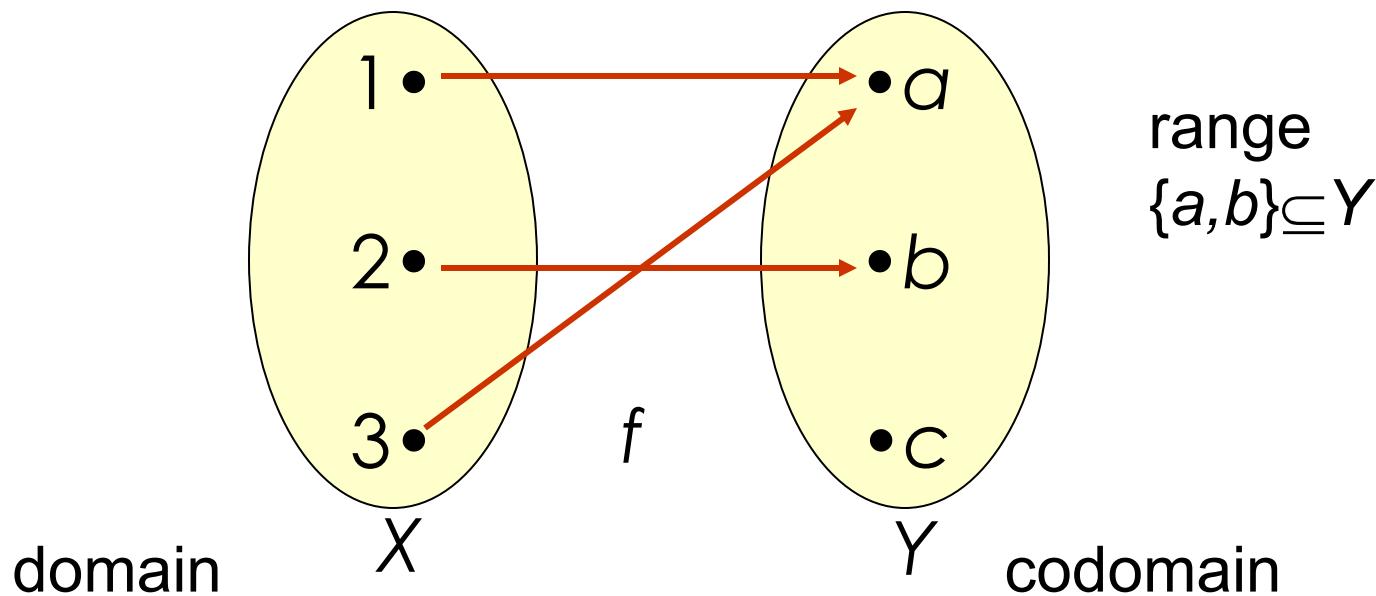


example

- The relation, $f = \{ (1,a), (2,b), (3,a) \}$ from $X = \{ 1, 2, 3 \}$ to $Y= \{ a, b, c \}$ is a function from X to Y.
- The domain of f is X
- The range of f is $\{a, b\}$

example

- $f = \{ (1,a), (2,b), (3,a) \}$



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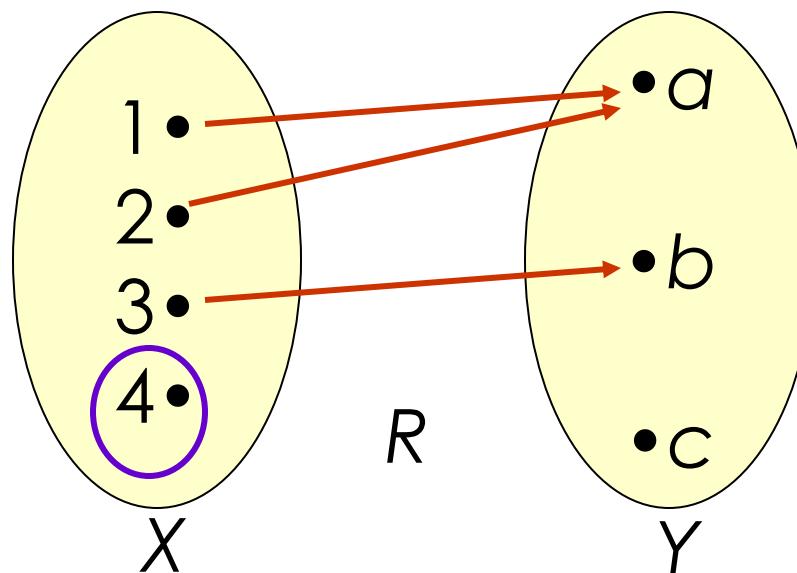


example

- The relation, $R = \{(1,a), (2,a), (3,b)\}$ from $X = \{1, 2, 3, 4\}$ to $Y = \{a, b, c\}$ is NOT a function from X to Y .
- The domain of R , { 1,2,3 } is not equal to X .

example

$$R= \{(1,a), (2,a), (3,b)\}$$



There is no arrow from 4



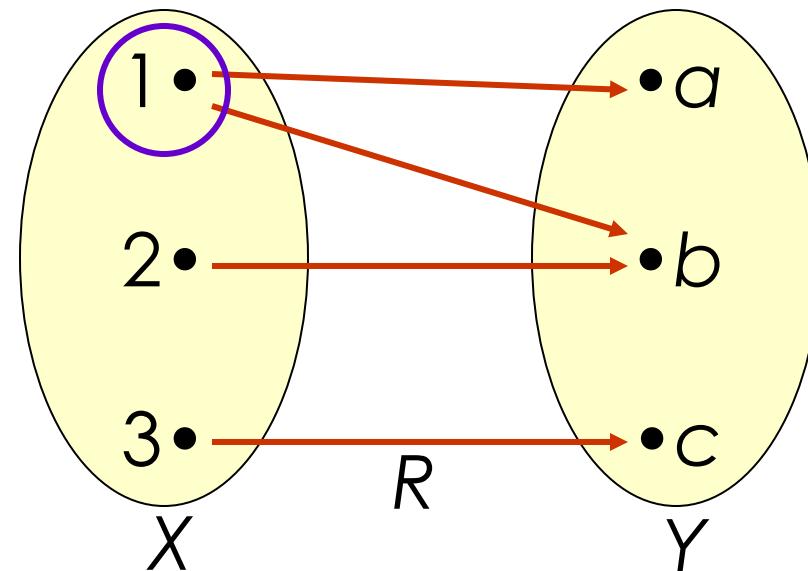
example

- The relation, $R= \{(1,a), (2,b), (3,c), (1,b)\}$ from $X= \{1, 2, 3\}$ to $Y= \{a, b, c\}$ is NOT a function from X to Y
- (1,a) and (1,b) in R but $a \neq b$.

example

$$R = \{(1,a), (2,b), (3,c), (1,b)\}$$

There are 2 arrows
from 1



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example

- For the function, $f = \{(1,a), (2,b), (3,a)\}$

- We may write

$$f(1)=a, \quad f(2)=b, \quad f(3)=a$$

- Notation $f(x)$ is used to define a function



example

■ $f(x) = x^2$

$$f(2) = 4, \quad f(-3.5) = 12.25, \quad f(0) = 0$$

$$f = \{(x, x^2) \mid x \text{ is a real number}\}$$

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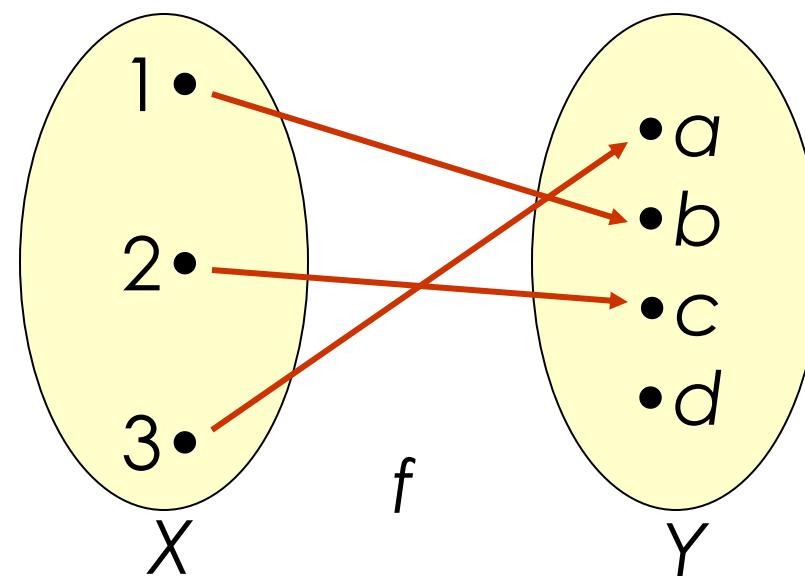
One-to-one

- A function f from X to Y , is said one-to-one (or injective) if for each $y \in Y$, there is at most one $x \in X$, with $f(x)=y$.
- For all x_1, x_2 , if $f(x_1) = f(x_2)$, then $x_1=x_2$.
$$\forall x_1 \forall x_2 ((f(x_1) = f(x_2)) \rightarrow (x_1=x_2))$$

example

- The function, $f = \{ (1,b), (3,a), (2,c) \}$ from $X = \{ 1, 2, 3 \}$ to $Y = \{ a, b, c, d \}$ is one-to-one.

Each element in Y
has at most one
arrow pointing to it



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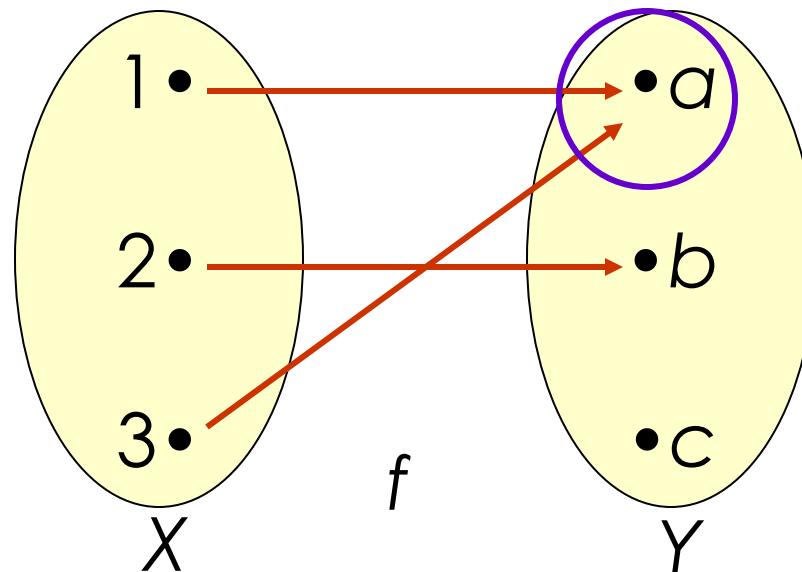


example

- The function, $f = \{ (1,a), (2,b), (3,a) \}$ from $X = \{ 1, 2, 3 \}$ to $Y = \{ a, b, c \}$ is NOT one-to-one.
- $f(1) = a = f(3)$

example

$$f = \{ (1,a), (2,b), (3,a) \}$$



a has 2 arrows
pointing to it



example

Show that the function,

$$f(n) = 2n+1$$

on the set of positive integers is one-to-one.

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example

- For all positive integer, n_1 and n_2 if $f(n_1) = f(n_2)$, then $n_1 = n_2$.

- Let, $f(n_1) = f(n_2)$, $f(n) = 2n+1$
then $2n_1 + 1 = 2n_2 + 1$ (-1)
 $2n_1 = 2n_2$ ($\div 2$)
 $n_1 = n_2$

- This shows that f is one-to-one.



example

- Show that the function,

$$f(n) = 2^n - n^2$$

from the set of positive integers to the set of integers is
NOT one-to-one.



example

- Need to find 2 positive integers, n_1 and n_2
 $n_1 \neq n_2$ with $f(n_1) = f(n_2)$.

- Trial and error,

$$f(2) = f(4)$$

- f is not one-to-one.



Onto

- If f is a function from X to Y and the range of f is Y , f is said to be onto Y
(or an onto function or a surjective function)
- For every $y \in Y$, there exists at least one $x \in X$ such that $f(x)=y$

$$\forall y \in Y \exists x \in X (f(x)=y)$$



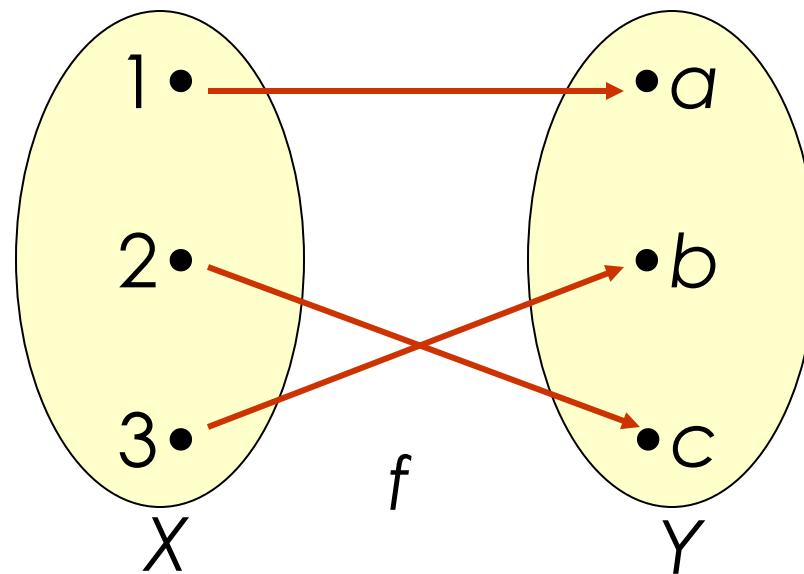
example

- The function, $f = \{ (1,a), (2,c), (3,b) \}$
from $X = \{ 1, 2, 3 \}$ to $Y = \{ a, b, c \}$
is one-to-one and onto Y .

example

- $f = \{ (1,a), (2,c), (3,b) \}$

One-to-one
Each element in Y has at most one arrow



Onto
Each element in Y has at least one arrow pointing to it



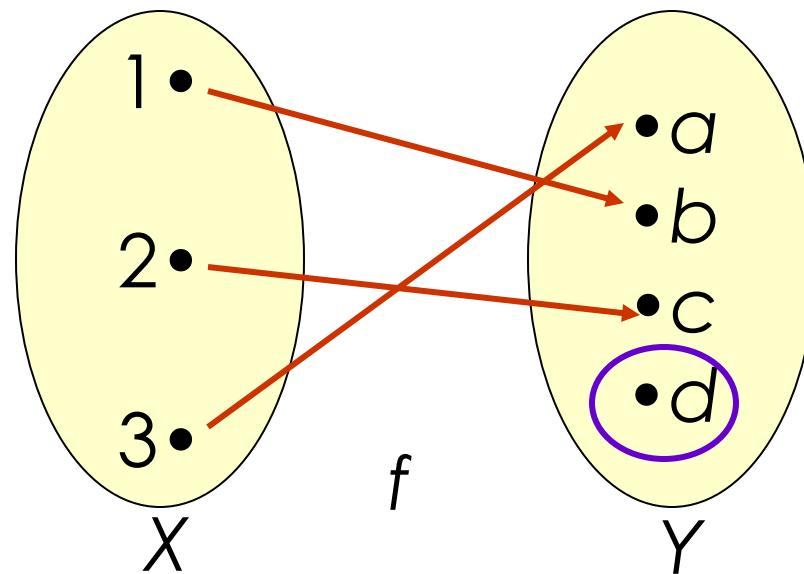
example

- The function, $f = \{ (1,b), (3,a), (2,c) \}$
is not onto $Y = \{a, b, c, d\}$

- It is onto $\{a, b, c\}$

example

$$f = \{ (1,b), (3,a), (2,c) \}$$



not onto
no arrow
pointing to d

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Bijection

- f is called one-to-one correspondence (or bijective or bijection) if f is both one-to-one and onto.



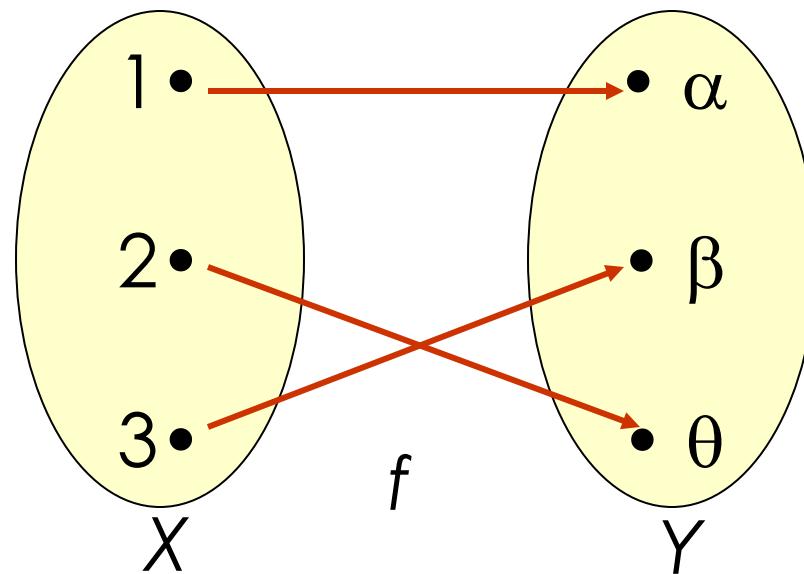
example

- The function, $f = \{ (1,\alpha), (2,\theta), (3,\beta) \}$
from $X = \{ 1, 2, 3 \}$ to $Y = \{\alpha, \beta, \theta \}$
is one-to-one and onto Y .

- The function f is a bijection

example

$$f = \{ (1, \alpha), (2, \theta), (3, \beta) \}$$



One-to-one
and onto Y
-bijection

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exercise

Determine which of the relations f are functions from the set X to the set Y .

a) $X = \{ -2, -1, 0, 1, 2 \}$, $Y = \{ -3, 4, 5 \}$ and

$$f = \{ (-2, -3), (-1, -3), (0, 4), (1, 5), (2, -3) \}$$

b) $X = \{ -2, -1, 0, 1, 2 \}$, $Y = \{ -3, 4, 5 \}$ and

$$f = \{ (-2, -3), (1, 4), (2, 5) \}$$

c) $X = Y = \{ -3, -1, 0, 2 \}$ and

$$f = \{ (-3, -1), (-3, 0), (-1, 2), (0, 2), (2, -1) \}$$

In case any of these relations are functions, determine if they are one-to-one, onto Y , and/or bijection.



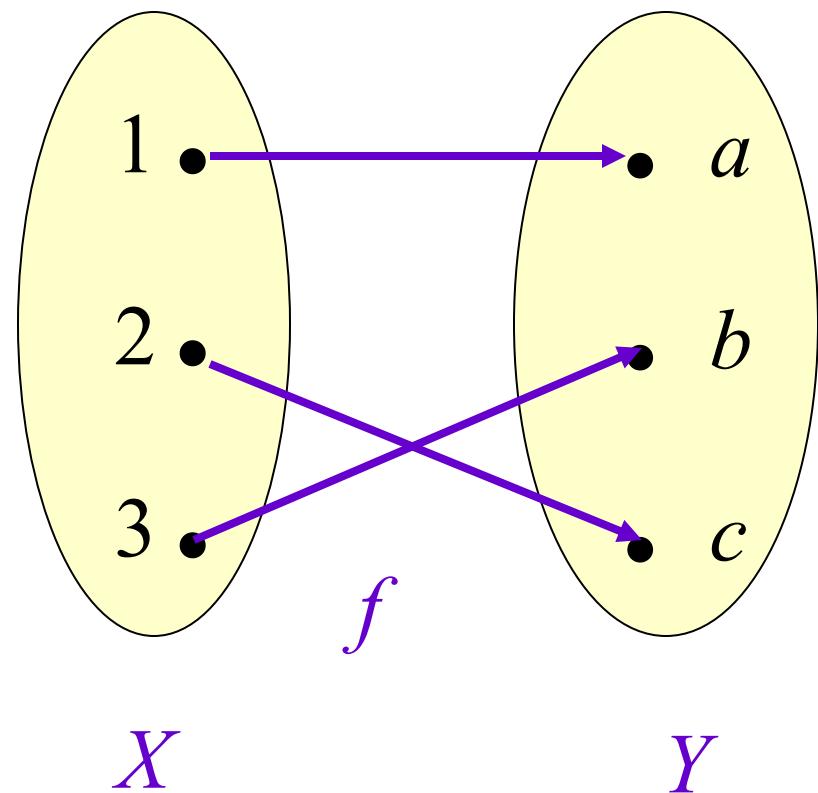
Inverse function

- Let $f: X \rightarrow Y$ be a function.

- The inverse relation $f^{-1} \subseteq Y \times X$ is a function from Y to X , if and only if f is both one-to-one and onto Y .

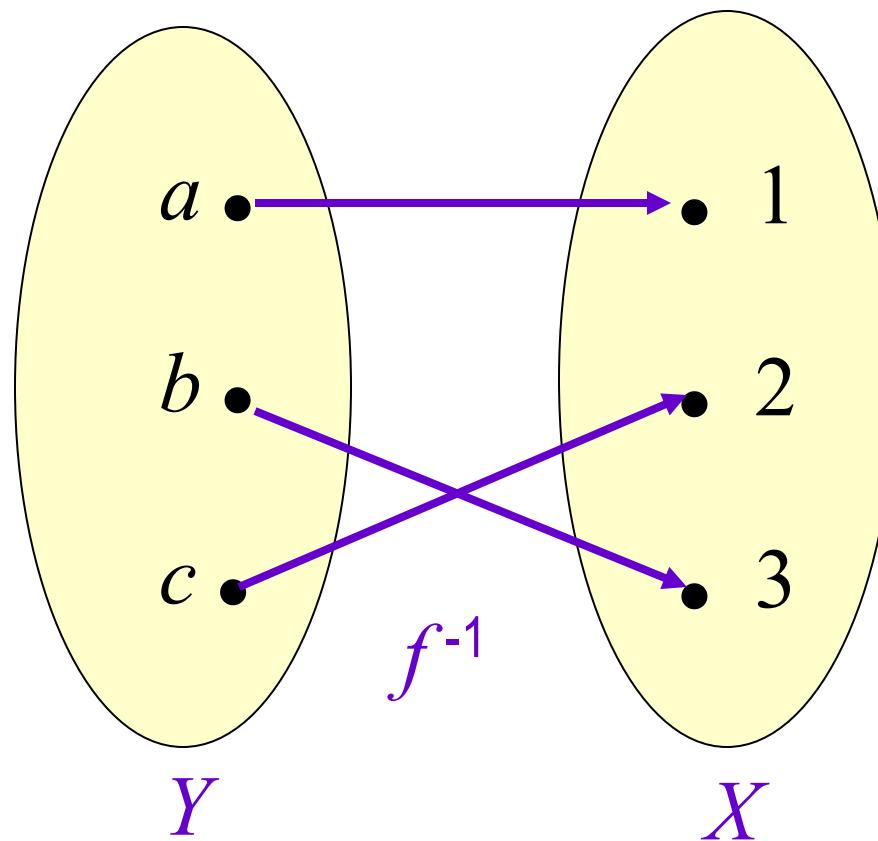
example

- $f = \{(1,a), (2,c), (3,b)\}$
- $f^{-1} = \{(a,1), (c,2), (b,3)\}$



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example



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example

- The function, $f(x) = 9x + 5$ for all $x \in R$
(R is the set of real numbers).
- This function is both one-to-one and onto.
- Hence, f^{-1} exists.
- Let $(y, x) \in f^{-1}$, $f^{-1}(y) = x$
$$(x,y) \in f, \quad y = 9x + 5$$
$$x = (y-5)/9$$
$$f^{-1}(y)=(y-5)/9$$



exercise

- Find each inverse function.

a) $f(x) = 4x + 2, \quad x \in R$

b) $f(x) = 3 + (1/x), \quad x \in R$



Composition

- Suppose that g is a function from X to Y and f is a function from Y to Z .
- The composition of f with g ,

$$f \circ g$$

is a function

$$(f \circ g)(x) = f(g(x))$$

from X to Z



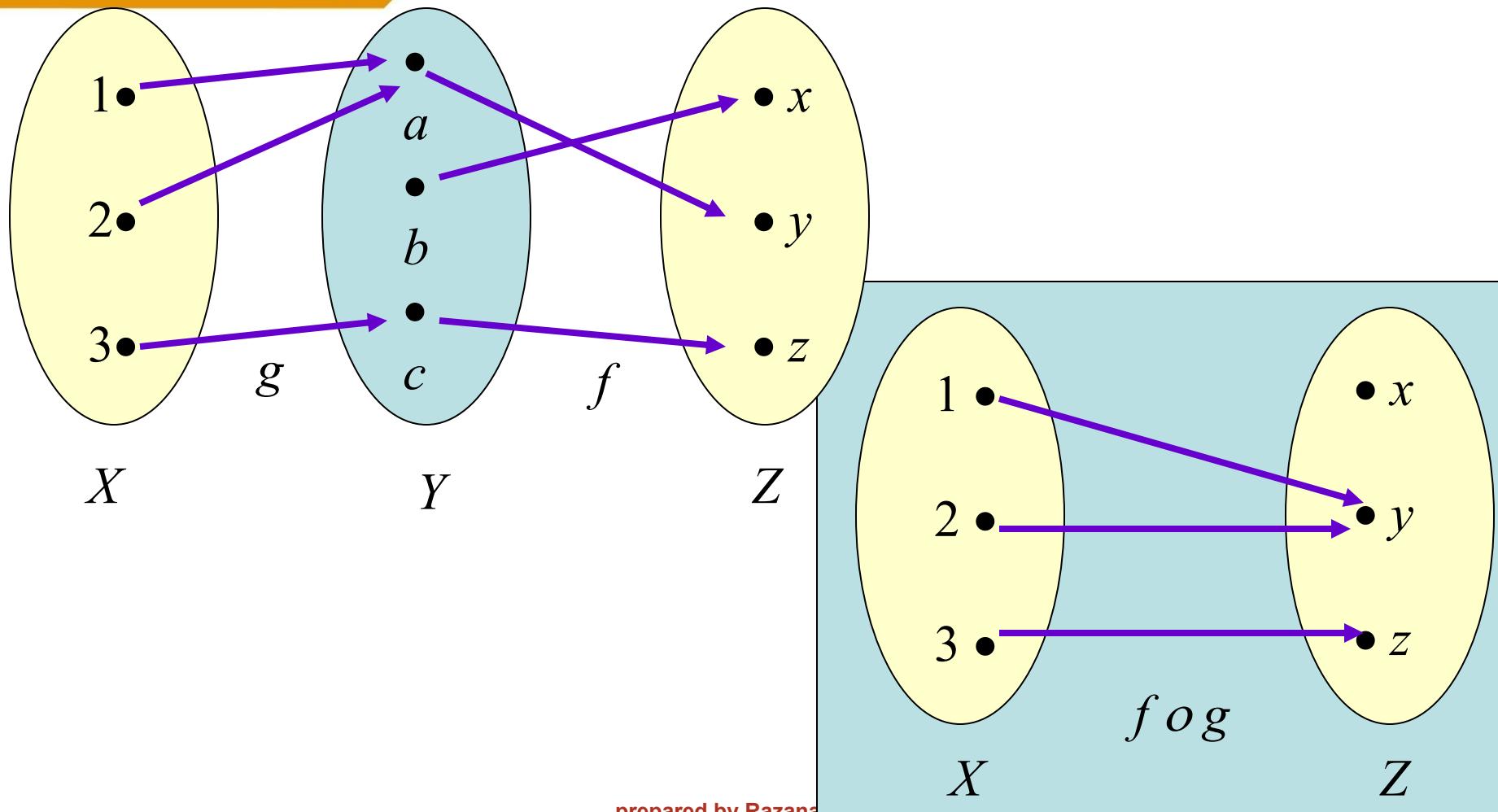
example

- Given, $g = \{ (1,a), (2,a), (3,c) \}$
a function from $X = \{1, 2, 3\}$ to $Y = \{a, b, c\}$ and
 $f = \{ (a,y), (b,x), (c,z) \}$
a function from Y to $Z = \{ x, y, z \}$

- The composition function from X to Z is the function
 $f \circ g = \{ (1,y), (2,y), (3,z) \}$

example

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example

- $f(x) = \log_3 x$ and $g(x) = x^4$

$$f(g(x)) = \log_3(x^4)$$

$$g(f(x)) = (\log_3 x)^4$$

Note: $f \circ g \neq g \circ f$



example

$$f(x) = \frac{1}{5}x$$

$$g(x) = x^2 + 1$$

$$(gof)(x) = g(f(x)) = g\left(\frac{x}{5}\right)$$

$$= \left(\frac{x}{5}\right)^2 + 1 = \frac{x^2}{25} + 1$$

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example

- Composition sometimes allows us to decompose complicated functions into simpler functions.
- example

$$f(x) = \sqrt{\sin 2x}$$

$$g(x) = \sqrt{x} \quad h(x) = \sin x \quad w(x) = 2x$$

$$f(x) = g(h(w(x)))$$

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exercise

- Let f dan g be functions from the positive integers to the positive integers defined by the equations,

$$f(n) = n^2, \quad g(n) = 2^n$$

- Find the compositions

- $f \circ f$
- $g \circ g$
- $f \circ g$
- $g \circ f$