



Counting Methods (Part 2- Permutation & Combination)

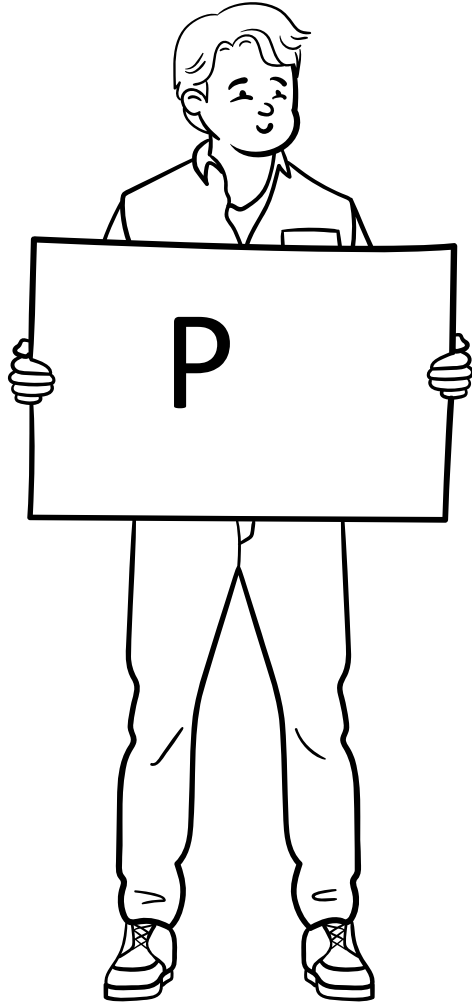
Permutation & Combination

- What's the difference?
- Consider these situations:
 - ***“My fruit salad is a combination of apples, grapes and bananas”***
 - We don't care what order the fruits are in, they could also be "bananas, grapes and apples" or "grapes, apples and bananas", its the same fruit salad.
 - ***“The combination to the safe was 472”***
 - Now we **do** care about the order. "724" would not work, nor would "247". It has to be exactly **4-7-2**.

So, in Mathematics we use more **precise** language:

- If the order **doesn't** matter, it is a **Combination**.
 - Combination means **selection** of things.
 - The word **selection** is used, when the order of things has *no importance*.

- If the order **does** matter, it is a **Permutation**.
 - Permutation means **arrangement** of things.
 - The word **arrangement** is used, if the order of things *is considered*.

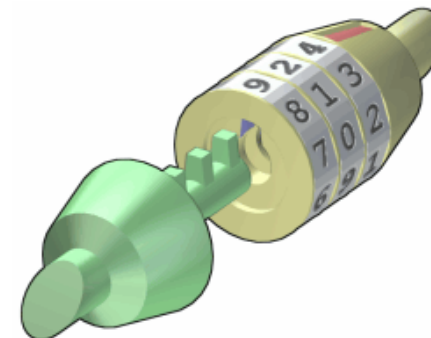


To help you to
remember, think
"**P**ermutation ...
Position"

Permutation

There are basically two types of permutation:

- **Repetition is Allowed:** such as the permutation lock (in picture). It could be "333".
- **No Repetition:** for example, the first three people in a running race. You can't be first and second.



Permutation – No Repetition

- In this case, you have to **reduce** the number of available choices each time.
- For example, what order could 16 pool balls be in?
 - After choosing, say, number "14" you can't choose it again.



- So, your first choice would have 16 possibilities, and your next choice would then have 15 possibilities, then 14, 13, etc.
- And the total permutations would be:

$$16 \times 15 \times 14 \times 13 \times \dots = 20,922,789,888,000$$



- But maybe you don't want to choose them all, just 3 of them, so that would be only:

$$16 \times 15 \times 14 = 3,360$$

- In other words, there are 3,360 different ways that 3 pool balls could be selected out of 16 balls.



- But how do we write that mathematically?
- Answer: we use the "factorial function"
- The **factorial function** (symbol: !) just means to multiply a series of **descending** natural numbers. Examples:
 - $4! = 4 \times 3 \times 2 \times 1 = 24$
 - $7! = 7 \times 6 \times 5 \times 4 \times 3 \times 2 \times 1 = 5,040$
 - $1! = 1$

- So, if you wanted to select **all** of the billiard balls, the permutations would be:

$$16! = 20,922,789,888,000$$



- A permutation of n distinct elements x_1, \dots, x_n is an ordering of the n elements x_1, \dots, x_n ,

- There are $n!$ permutations of n elements

$$p(n) = n!$$

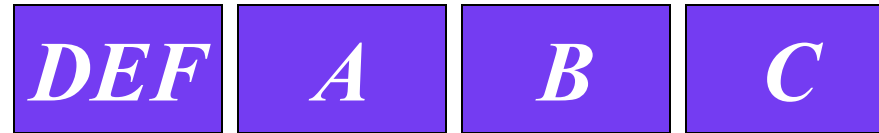
$$= n.(n-1).(n-2)...2.1$$

Example 1

- There are 6 permutations of three elements
- If the elements are denoted A, B, C, the six permutations are
 - ABC, ACB, BAC, BCA, CAB, CBA
 - $3! = 3 \cdot 2 \cdot 1 = 6$

Example 2

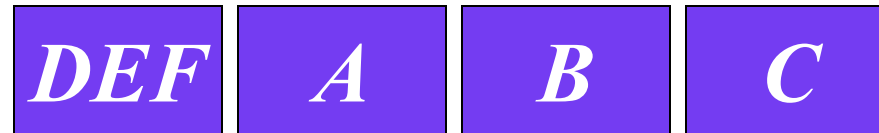
- How many permutations of letters *ABCDEF* contain the substring *DEF*?



- $4! = 24$

Example 3

- How many permutations of letters *ABCDEF* contain the letters *DEF* together in any order?



- DEF* $3! = 6$
- DEF, A, B, C* $4! = 24$ $6.24=144$

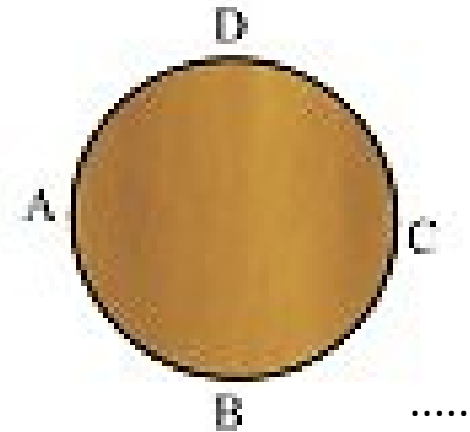
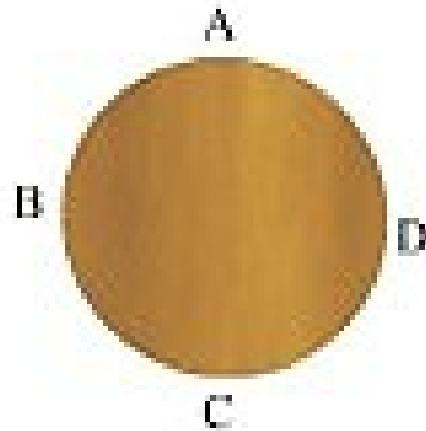
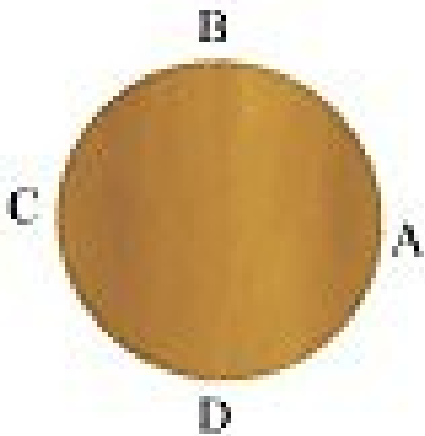
Permutations - Circular Arrangement

- The permutation in a row or along a line has a **beginning** and an **end**.
- But there is nothing like beginning or end or first and last in a **circular permutation**.
- In circular permutations, we consider one of the objects as fixed and the remaining objects are arranged as in linear permutation.

- There are two cases of circular-permutations:
 - If clockwise and anti clockwise orders **are different**, then total number of circular-permutations is given by **$(n-1)!$**
 - If clockwise and anti clockwise orders are taken as **not different**, then total number of circular-permutations is given by

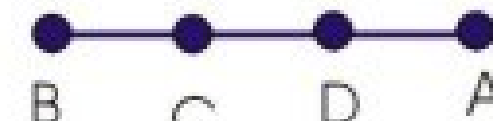
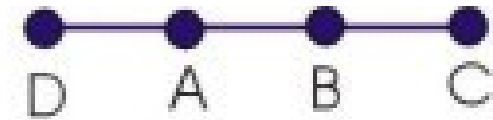
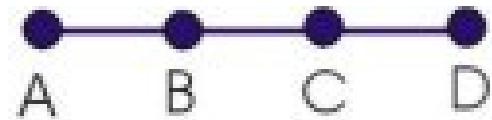
$$\frac{(n-1)!}{2!}$$

- Let's consider that 4 persons A,B,C, and D are sitting around a round table
- Shifting A, B, C, D, one position in anticlock-wise direction, we get the following arrangements:-



- Thus, we use that if 4 persons are sitting at a round table, then they can be shifted four times.
- But these four arrangements will be the same, because the sequence of A, B, C, D, is same.

- But if A, B, C, D, are sitting in a row, and they are shifted, then the four **linear-arrangement** will be different.



- Hence, if we have '4' things, then for each circular-arrangement, number of linear-arrangements is 4.
- Similarly, if we have ' n ' things, then for each circular-arrangement, number of linear-arrangement is n .

- Hence, the number of circular permutations is

$$P_n = (n-1)!$$

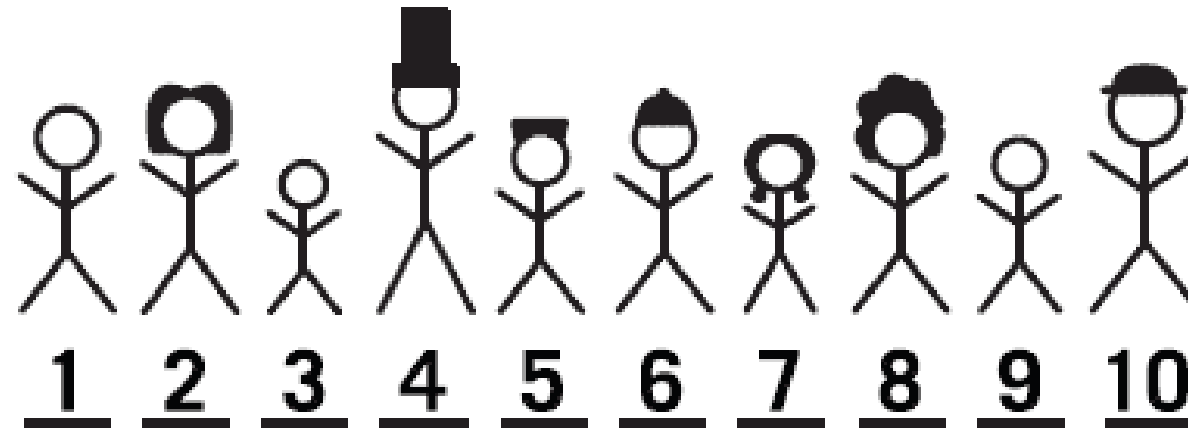
- The number is $(n-1)!$ instead of the usual factorial, $n!$ since all cyclic permutations of objects are equivalent because the circle can be rotated.

- Thus, the number of permutations of 4 objects in a row = $4!$
- Where as the number of circular permutations of 4 objects is $(4-1)! = 3!$

Example 1

- Suppose we are expecting ten people for dinner.
- How many ways can we seat them around a circular table?

10 PEOPLE LINED UP



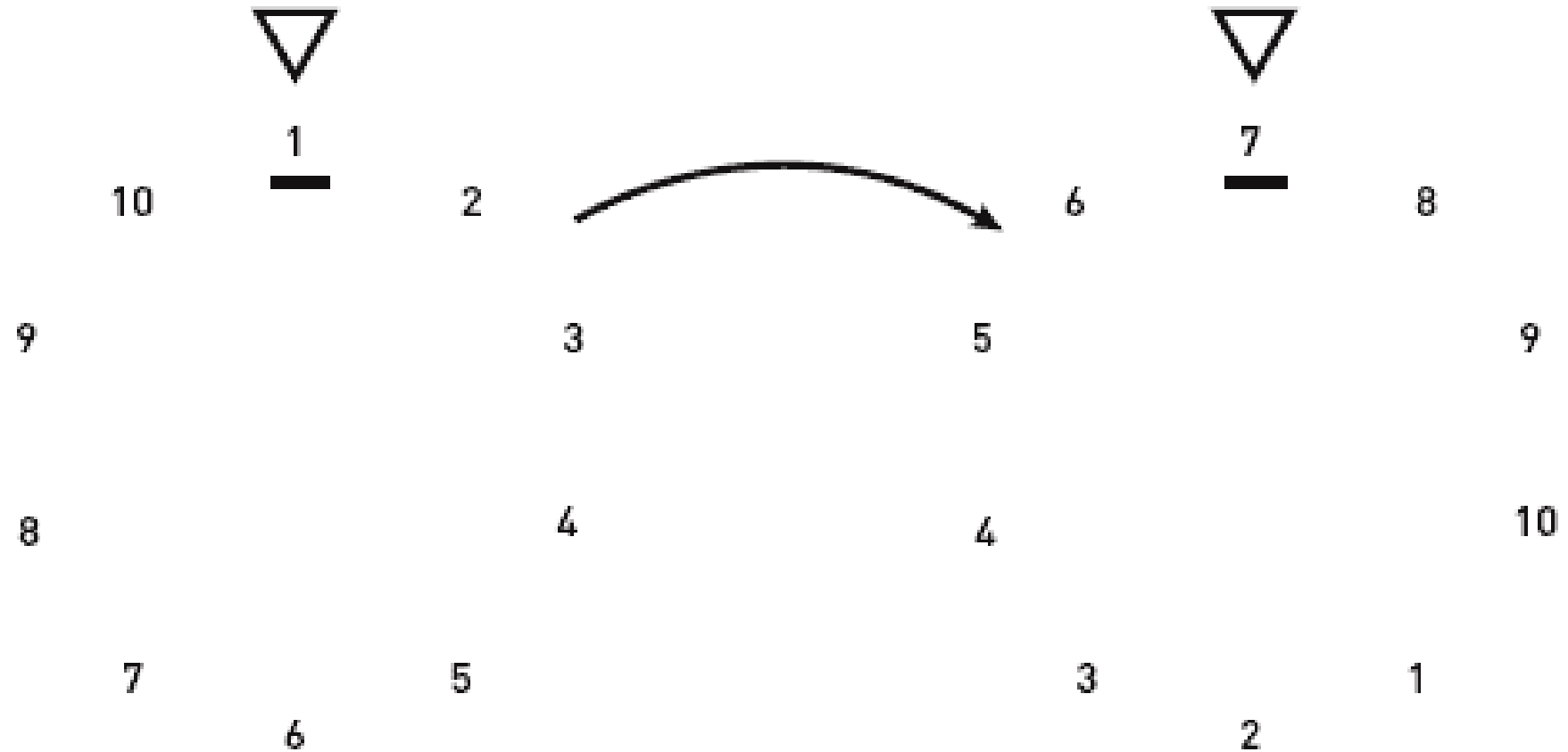
Example 1 - Solution

- First, let's think about how many ways we can line them up.
- As we indicated above, there will be $10!$ ways to line up ten guests
- 10 for the first position, 9 for the second, 8 for the third, and so on.

$$10! = 10 \times 9 \times 8 \times 7 \times 6 \times 5 \times 4 \times 3 \times 2 \times 1$$

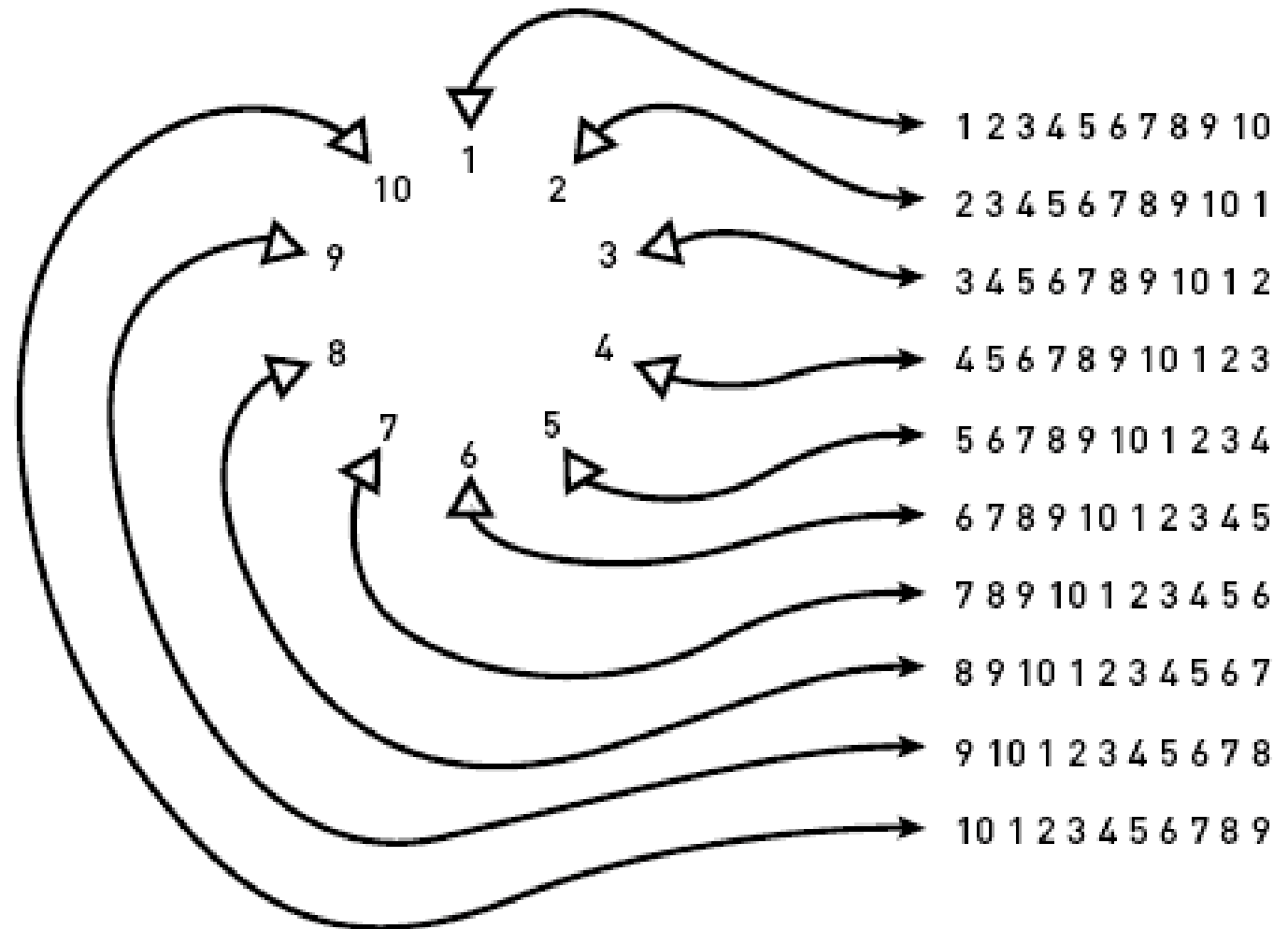
How does this change if they are seated around a circular table?

THESE CIRCULAR PERMUTATIONS ARE EQUIVALENT



ONE CIRCULAR PERMUTATION EQUIVALENT TO TEN LINEAR ONES

- Notice that every **circular arrangement** corresponds to ten different linear arrangements.

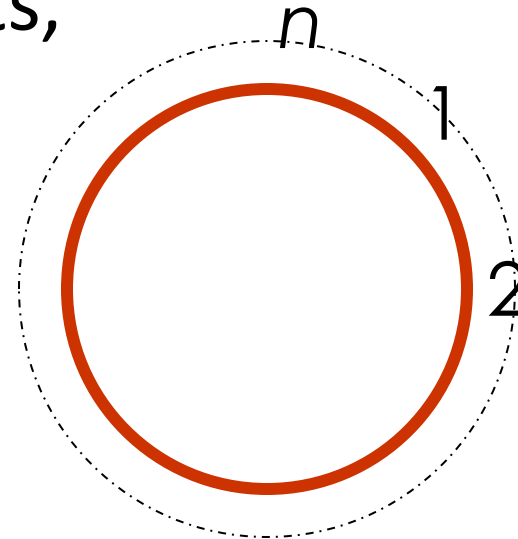


- Using our reasoning from before, we can see that the number of circular arrangements is equal to the number of linear arrangements, $10!$, divided by ten to compensate for the fact that each circular permutation corresponds to ten different linear ones.
- This gives as the number of ways to arrange ten guests around a table.
- We can generalize this to say that n elements can be arranged in $(n-1)!$ ways around a circle.

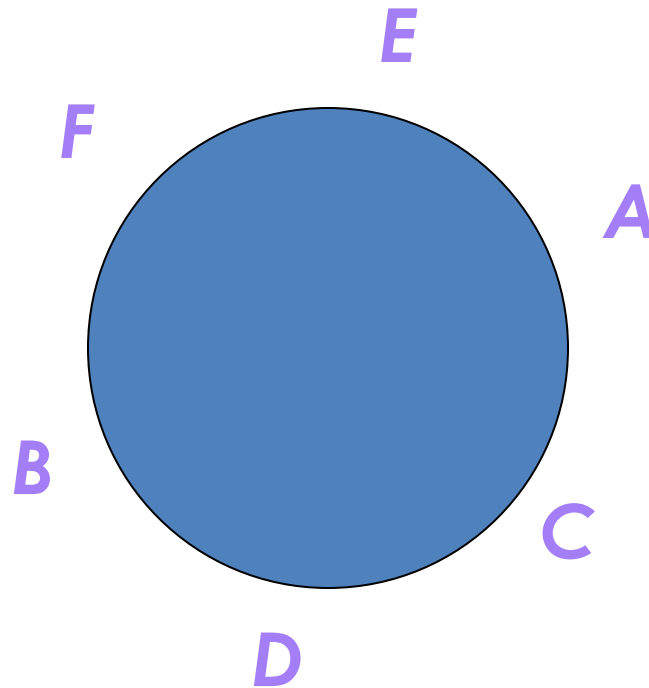
Permutations (circular arrangement)

- Permutations of n elements,

- $P(n) = (n-1)!$



Example 2



$$(n-1)! = (6-1)! = 5! = 120$$

More Examples

- How many circular permutation can you form with 10 objects

$$(n-1)! = (10-1)! = 9! = 362880$$

- How many ways can four boys and two girls be seated at a round table?

$$(n-1)! = (6-1)! = 5! = 120$$

r -Permutations

(Permutations without repetitions)

- An r -permutations of n (distinct) elements x_1, \dots, x_n is an ordering of an r -element subset of $\{x_1, \dots, x_n\}$.
- The number of r -permutations of a set of n distinct elements is,

$$P(n, r) = {}^n P_r = {}_n P_r = \frac{n!}{(n - r)!}$$

Example 1

- Back to our billiard balls example.
- If you wanted to select just 3, then you have to stop the multiplying after 14. How do you do that? There is a neat trick ... you divide by **13!** ...

$$\frac{16 \times 15 \times 14 \times 13 \times 12 \times \dots}{13 \times 12 \times \dots} = 16 \times 15 \times 14 = 3,360$$

- Do you see? **$16! / 13! = 16 \times 15 \times 14$**



- Our "order of 3 out of 16 pool balls example" would be:

$$P(16,3) = \frac{16!}{(16-3)!} = \frac{16!}{13!} = \frac{20,922,789,888,000}{6,227,020,800} = 3,360$$

- which is just the same as:

$$16 \times 15 \times 14 = 3,360$$

Example 2

- How many ways can first and second place be awarded to 10 people?

$$P(10,2) = \frac{10!}{(10-2)!} = \frac{10!}{8!} = \frac{3,628,800}{40,320} = 90$$

- which is just the same as:

$$10 \times 9 = 90$$

Example 3

- 2- permutations of *a, b, c* are,
 - *ab, ac, ba, bc, ca, cb*

$$P(3,2) = \frac{3!}{(3-2)!} = \frac{3!}{1!} = 3.2 = 6$$

Example 4

- In how many ways can we select a **chairperson, vice-chairperson, secretary, and treasurer** from a group of **10 persons**?

$$P(10,4) = \frac{10!}{(10-4)!} = \frac{10!}{6!} = 10.9.8.7 = 5040$$

Example 5

- How many **dance pairs**, (dance pairs means a pair (W,M), where W stands for a women and M for man), can be formed from a group of **6 women** and **10 men**?

$$P(10,6) = \frac{10!}{(10-6)!} = \frac{10!}{4!} = 10.9.8.7.6.5 = 151200$$

Example 6

- How many numbers between 20000 and 50000 can be formed with the digits 1,2,3,4,5,6 such that no digits are repeated in any of the numbers so formed?

Example 6 - Solutions

20000 – 50000

{ 1,2,3,4,5,6 }

$$P(6,5) = 6.5.4.3.2 = 720$$

$$P(5,4) = 5.4.3.2 = 120$$

$$P(5,4) = 5.4.3.2 = 120$$

$$P(5,4) = 5.4.3.2 = 120$$

$$720-120-120-120=360$$

Or

$$3. 5. 4. 3. 2 = 360$$

— — — — —

1 — — — —

5 — — — —

6 — — — —

Example 7

- In how many ways can **six boys** and **five girls** stand in a line so that no two girls are next to each other?
 - G B₁G B₂G B₃G B₄G B₅G B₆G

$$P(7,5) = \frac{7!}{(7-5)!} = \frac{7!}{2!} = 2520 \quad P(6,6) = 6! = 720$$

$$P(7,5) \cdot P(6,6) = 1814400$$

r -Permutations

(Permutations with repetitions allowed)

- r -permutations of a set of n distinct elements if repetitions are allowed.

$$P(n, r) = n^r$$

- In other words:
 - There are ***n*** possibilities for the first choice,
 - THEN there are ***n*** possibilities for the second choice,
 - and so on, multiplying each time.
- Which is easier to write down using an exponent of ***r***:

$$n \times n \times \dots (r \text{ times}) = n^r$$

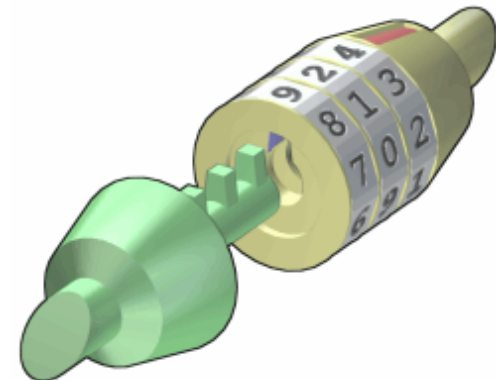
Example 1

- In the lock, there are 10 numbers to choose from (0,1,..9) and you choose 3 of them:

$$10 \times 10 \times \dots (3 \text{ times})$$

$$= 10^3$$

$$= 1,000 \text{ permutations}$$



Example 2

- How many **five-letters** word can be formed from the letters **A-Z**?

$$- P(26,5) = 26^5$$

Permutations

(n objects, k different types)

- A collection of n objects of k different types.
- The total number of different arrangements of these n objects is,

$$P(n) = \frac{n!}{(n_1!n_2!\dots n_k!)}$$

Example 1

- Find the number of different ways the letters of the word **ASSESSMENT** can be arranged?
- 10 letters, A (1), S(4), E(2), M(1), N(1), T(1)

$$P(10) = \frac{10!}{(1! 4! 2! 1! 1! 1!)} = 75600$$

Combinations

- There are also two types of combinations (remember the order does **not** matter now):
- **Repetition is Allowed:** such as coins in your pocket (5,5,5,10,10)
- **No Repetition:** such as combination of subjects to enroll (Maths., English, Music)

Combinations (without Repetition)

Given a set $X = \{x_1, \dots, x_n\}$ containing n (distinct) elements.

An r -combination of X is an unordered selection of r -elements of X .

The number of r -combinations of a set of n distinct elements is,

$$C(n, r) = {}^nC_r = {}_nC_r = \binom{n}{r} = \frac{n!}{r!(n-r)!}$$

Example 1

- Back to our pool ball example, let us say that you just want to know which 3 pool balls were chosen, not the order.
- We already know that 3 out of 16 gave us 3,360 permutations.
- But many of those will be the same to us now, because we don't care what order!



- For example, let us say balls 1, 2 and 3 were chosen. These are the possibilities:

Order does matter	Order doesn't matter
1 2 3 1 3 2 2 1 3 2 3 1 3 1 2 3 2 1	1 2 3

- So, the permutations will have 6 times as many possibilities.

- In fact there is an easy way to work out how many ways "1 2 3" could be placed in order, and we have already talked about it.
- The answer is:

$$3! = 3 \times 2 \times 1 = 6$$

- So, all we need to do is adjust our permutations formula to **reduce it** by how many ways the objects could be in order (because we aren't interested in the order anymore):

$$\frac{n!}{(n-r)!} \times \frac{1}{r!} = \frac{n!}{r!(n-r)!}$$

- So, our pool ball example (now without order) is:

$$\frac{16!}{3!(16-3)!} = \frac{16!}{3! \times 13!} = \frac{20,922,789,888,000}{6 \times 6,227,020,800} = 560$$

- Or you could do it this way:

$$\frac{16 \times 15 \times 14}{3 \times 2 \times 1} = \frac{3360}{6} = 560$$



Example 2

- In how many ways can we select a committee of **three** from a group of **10 distinct persons**?

$$C(10,3) = \frac{10!}{3!(10-3)!} = \frac{10!}{3!7!} = 120$$

Example 3

- In how many ways can we select a committee of 2 women and 3 men from a group of 5 distinct women and 6 distinct men?

$$C(5,2) = \frac{5!}{2!3!} = 10 \qquad C(6,3) = \frac{6!}{3!3!} = 20$$

$$10 \cdot 20 = 200$$

Example 4

- How many 8-bit strings contain exactly four 1's ?

$$C(8,4) = \frac{8!}{4!4!} = 70$$

Example 5

- A committee of six is to be made from 4 students and 8 lecturers. In how many ways can this be done:
 - If the committee contains exactly 3 students?
 - If the committee contains at least 3 students?

Example 5 - Solutions

- If the committee contains exactly 3 students?

$$C(4,3) = \frac{4!}{3! 1!} = 4$$

- Select 3 students

$$C(8,3) = \frac{8!}{3! 5!} = 56$$

- Select 3 lecturers

- $4.56 = 224$

Example 5 - Solutions

- If the committee contains at least 3 students?
- We have to consider 2 cases
 - Case 1: 3 students and 3 lecturers
 - Case 2: 4 students and 2 lecturers

- Case 1: 3 students and 3 lecturers
 - 224 ways

- Case 2: 4 students and 2 lecturers

$$C(4,4) = 1 \qquad C(8,2) = \frac{8!}{2!6!} = 28$$

– 1.28 = 28

- Case 1+ case 2 = 224 + 28 = 252

Example 6

- A student is required to answer 7 out of 12 questions, which are divided into two groups, each containing 6 questions.
- The student is not permitted to answer more than 5 questions from either group.
- In how many different ways can the student choose the 7 questions?

Number of question
from group A

5

$$C(6,5).C(6,2)=90$$

4

$$C(6,4).C(6,3)=300$$

3

$$C(6,3).C(6,4)=300$$

2

$$C(6,2).C(6,5)=90$$

Number of question
from group B

2

3

4

5

$$90+300+300+90 = 780$$

Combinations

(Repetition Allowed)

The number of r -combinations of n objects with repetitions allowed is,

$$C(n + r - 1, r) = \binom{n + r - 1}{r} = \frac{(n + r - 1)!}{r!(n - 1)!}$$

n is the number of things/objects to choose from, and you choose r of them (repetition allowed, order doesn't matter)

Combinations with Repetition

- Let us say there are five flavors of ice cream: **banana, chocolate, lemon, strawberry and vanilla.**
- You can have three scoops.
- How many variations will there be?



Combinations with Repetition

- Let's use letters for the flavors:
 $\{b, c, l, s, v\}$
- Example selections would be
 - $\{c, c, c\}$ (3 scoops of chocolate)
 - $\{b, l, v\}$ (one each of banana, lemon and vanilla)
 - $\{b, v, v\}$ (one of banana, two of vanilla)

(And just to be clear: There are $n=5$ things to choose from, and you choose $r=3$ of them. **Order does not matter, and you can repeat!**)

Combinations with Repetition

$$C(n + r - 1, r) = \binom{n + r - 1}{r} = \frac{(n + r - 1)!}{r!(n - 1)!}$$

- where n is the number of things to choose from, and you choose r of them
(Repetition allowed, order doesn't matter)

Combinations with Repetition

- So, what about our ice-cream example, what is the answer?

$$C(n + r - 1, r) = \binom{n + r - 1}{r} = \frac{(n + r - 1)!}{r!(n - 1)!}$$

$$C(5 + 3 - 1, 3) = \binom{5 + 3 - 1}{3} = \frac{(5 + 3 - 1)!}{3!(5 - 1)!} = \frac{7!}{3! \times 4!} = \frac{5040}{6 \times 24} = 35$$

Example 1

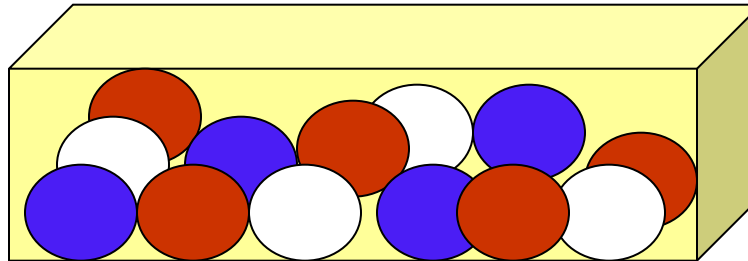
- A bakery makes 4 different varieties of donuts. Khairin wants to buy 6 donuts. How many different ways can he do it?

$$C(4 + 6 - 1, 6) = \frac{(4 + 6 - 1)!}{6!(4 - 1)!} = \frac{9!}{6!3!} = 84$$



Example 2

- There is a box containing identical white, red and blue balls.



- In how many ways can we select 4 balls?

$$C(3 + 4 - 1, 4) = \frac{(3 + 4 - 1)!}{4!(3 - 1)!} = \frac{6!}{4!2!} = 15$$

Summary

Which formula to use?

	Order Matters (Permutations)	Order Does Not Matter (Combinations)
Repetition is allowed	$P_n = n^r$	$C(n + r - 1, r) = \frac{(n + r - 1)!}{r!(n - 1)!}$
Repetition is not allowed	$P(n, r) = \frac{n!}{(n - r)!}$	$C(n, r) = \frac{n!}{r!(n - r)!}$