

Chapter 1

Part 4

Quantifiers & Proof Technique

QUANTIFIERS

- Most of the statements in mathematics and computer science are not described properly by the propositions.
- Since most of the statements in mathematics and computer science use **variables**, the system of logic must be extended to include **statements with the variables**.

QUANTIFIERS

- Let $P(x)$ is a statement with variable x and A is a set.
- P a **propositional function** or also known as **predicate** if for each x in A , $P(x)$ is a proposition.
- Set A is the **domain of discourse** of P .
- Domain of discourse -> the particular domain of the variable in a propositional function.

QUANTIFIERS

- A **predicate** is a statement that contains variables.

Example:

$$P(x) : x > 3$$

$$Q(x,y) : x = y + 3$$

$$R(x,y,z) : x + y = z$$

Example

$x^2 + 4x$ is an odd integer

(domain of discourse is set of positive numbers).

$$x^2 - x - 6 = 0$$

(domain of discourse is set of real numbers).

x is rated as Research University in Malaysia

(domain of discourse is set of university in Malaysia).

QUANTIFIERS

A predicate becomes a proposition if the variable(s) contained is(are)

- **Assigned specific value(s)**
- **Quantified**

Example

- $P(x) : x > 3.$
What are the truth values of $P(4)$ and $P(2)$?
- $Q(x,y) : x = y + 3.$ (multiple variables)
What are the truth values of $Q(1,2)$ and $Q(3,0)$?

Propositional functions

- Let $P(x)$: “ x is a multiple of 5”
 - For what values of x is $P(x)$ true?
- Let $P(x)$: $x+1 > x$
 - For what values of x is $P(x)$ true?
- Let $P(x)$: $x + 3 = 0$
 - For what values of x is $P(x)$ true?

QUANTIFIERS

- Two types of quantifiers:
 - **Universal**
 - **Existential**

QUANTIFIERS

- Let A be a propositional function with domain of discourse B . The statement

for every x , $A(x)$

is universally quantified statement

- Symbol \forall called a **universal quantifier** is used “**for every**”.
- Can be read as “**for all**”, “**for any**”.

Universal quantifiers

Represented by an upside-down A: \forall

- It means “for all”

Example

Let $P(x) = x+1 > x$

We can state the following:

- $\forall x P(x)$
- English translation: “for all values of x, $P(x)$ is true”
- English translation: “for all values of x, $x+1>x$ is true”

QUANTIFIERS

- The statement can be written as

$$\forall x A(x)$$

- Above statement is true if $A(x)$ is true for every x in B

(**false if $A(x)$ is false for at least one x in B**).

- OR In order to prove that a universal quantification is **true**, it must be shown for **ALL cases**

In order to prove that a universal quantification is **false**, it must be shown to be false for only **ONE case**

- A value x in the domain of discourse that makes the statement $A(x)$ **false** is called a **counterexample** to the statement.

Example

- Let the universally quantified statement is

$$\forall x (x^2 \geq 0)$$

Domain of discourse is the set of real numbers.

- This statement is true** because for every real number x , it is true that the square of x is positive or zero.

Example

- Let the universally quantified statement is

$$\forall x (x^2 \leq 9)$$

Domain of discourse is a set $B = \{1, 2, 3, 4\}$

- When $x = 4$, the statement produce false value.
- Thus, the above statement is false and the **counterexample is 4.**

QUANTIFIERS

- Easy to prove a universally quantified statement is true or false if the domain of discourse is not too large.
- What happen if the domain of discourse contains a large number of elements?
- For example, a set of integer from 1 to 100, the set of positive integers, the set of real numbers or a set of students in School of Computing. It will be hard to show that every element in the set is *true*.

Use existential quantifier!!

QUANTIFIERS

- Let A be a propositional function with domain of discourse B . The statement

There exist $x, A(x)$

is **existentially quantified statement**

- Symbol \exists called an **existential quantifier** is used “**there exist**”.
- Can be read as “**for some**”, “**for at least one**”.

QUANTIFIERS

- The statement can be written as

$$\exists x A(x)$$

- Above statement is true if $A(x)$ is true for at least one x in B (**false if every x in B makes the statement $A(x)$ false**).
- **Just find one x that makes $A(x)$ true!**

Example

- Let the existentially quantified statement is

$$\exists x \left(\frac{x}{x^2 + 1} = \frac{2}{5} \right)$$

Domain of discourse is the set of real numbers.

- Statement is true** because it is **possible to find at least one real number x** to make the proposition true.
- For example, if $x = 2$, we obtain the true proposition as below

$$\left(\frac{x}{x^2 + 1} = \frac{2}{5} \right) = \left(\frac{2}{2^2 + 1} = \frac{2}{5} \right)$$

Negation of Quantifiers

- Distributing a negation operator across a quantifier changes a universal to an existential and vice versa.

$$\neg (\forall x P(x)) ; \exists x \neg P(x)$$

$$\neg (\exists x P(x)) ; \forall x \neg P(x)$$

Example

- Let $P(x) : x$ is taking Discrete Structure course with the domain of discourse is the set of all students.

$\forall x P(x)$

All students are taking Discrete Structure course.

$\exists x P(x)$

There is some students who are taking Discrete Structure course.

Example

$$\neg (\exists x P(x)) ; \forall x \neg P(x)$$

$$\neg \exists x P(x)$$

None of the students are taking Discrete Structure course.

$$\forall x \neg P(x)$$

All students are not taking Discrete Structure course.

Example

$$\neg (\forall x P(x)) ; \exists x \neg P(x)$$

$$\neg \forall x P(x)$$

Not all students are taking Discrete Structure course.

$$\exists x \neg P(x)$$

There is some students who are not taking Discrete Structure course

- Consider “**For every student in this class, that student has studied calculus**”
- Rephrased: “**For every student x in this class, x has studied calculus**”
 - Let $C(x)$ be “ **x has studied calculus**”
 - Let $S(x)$ be “ **x is a student in this class**”

$$\forall x C(x)$$

- True if the universe of discourse is **all students in this class**

- What about if the universe of discourse is **all students (or all people?)**

- $\forall x (S(x) \wedge C(x))$ X
 - This is wrong! Why? (because this statement says that all people are students in this class and have studied calculus)
- $\forall x (S(x) \rightarrow C(x))$ \checkmark

$C(x)$: “ x has studied calculus”

$S(x)$: “ x is a student in this class”

- Consider:

- “**Some students have visited Mexico**”
- Rephrasing: “**There exists a student who has visited Mexico**”

- Let:

- $S(x)$ be “**x is a student**”
- $M(x)$ be “**x has visited Mexico**”

$\exists x M(x)$

- True if the universe of discourse is all students

- What about if the universe of discourse is all people?

$$\exists x (S(x) \wedge M(x))$$

$\exists x (S(x) \rightarrow M(x))$ **This is wrong! Why?**

suppose someone is not student= F->T or F->F, both make the statement true (**refer to truth table $p \rightarrow q$**)

S(x) : “x is a student”

M(x): “x has visited Mexico”

Translating from English

- Consider “**Every student in this class has visited Canada or Mexico**”
- Let, $S(x)$ be “**x is a student in this class**”
 $M(x)$ be “**x has visited Mexico**”
 $C(x)$ be “**x has visited Canada**”

$$\forall x (M(x) \vee C(x))$$

(When the universe of discourse is all students in this class)

$$\forall x (S(x) \rightarrow (M(x) \vee C(x)))$$

(When the universe of discourse is all people or all students)

Mathematical systems consists:

- **Axioms:** assumed to be true.
- **Definitions:** used to create new concepts.
- **Undefined terms:** some terms that are not explicitly defined.
- **Theorem**
 - Statement that can be shown to be true (under certain conditions)
 - Typically stated in one of three ways:
 - As Facts
 - As Implications
 - As Bi-implications

Direct Proof (Direct Method)

- Proof of those theorems that can be expressed in the form $\forall x (P(x) \rightarrow Q(x))$, D is the domain of discourse.
- Select a particular, but arbitrarily chosen, member a of the domain D .
- Show that the statement $P(a) \rightarrow Q(a)$ is true.
(Assume that $P(a)$ is true).
- Show that $Q(a)$ is true.
- By the rule of Universal Generalization (UG),
 $\forall x (P(x) \rightarrow Q(x))$ is true.

Example

“For all integer x , if x is odd, then x^2 is odd”

Or $P(x) : x$ is an odd integer

$Q(x) : x^2$ is an odd integer

$$\forall x(P(x) \rightarrow Q(x))$$

The domain of discourse is set Z of all integer.

Can verify the theorem for certain value of x .

$$x=3, x^2 = 9 ; \text{ odd}$$

Or show that the square of an odd number is an odd number

Rephrased: **“if n is odd, then n^2 is odd”**

Example

- a is an odd integer

$$\Rightarrow a = 2n + 1 \rightarrow \text{for some integer } n$$

$$\Rightarrow a^2 = (2n + 1)^2$$

$$\Rightarrow a^2 = 4n^2 + 4n + 1$$

$$\Rightarrow a^2 = 2(2n^2 + 2n) + 1$$

$$\Rightarrow a^2 = 2m + 1 \rightarrow \text{where } m = 2n^2 + 2n \text{ is an integer}$$

$$\Rightarrow a^2 \rightarrow \text{is an odd integer}$$

Indirect Proof

- The implication $p \rightarrow q$ is **equivalent** to the implication $(\neg q \rightarrow \neg p)$ (**contrapositive**)
- Therefore, in order to show that $p \rightarrow q$ is true, one can also **show that the implication $(\neg q \rightarrow \neg p)$ is true.**
- To show that $(\neg q \rightarrow \neg p)$ is true, assume that the negation of q is true and prove that the negation of p is true.

Example

$P(n) : n^2+3$ is an odd number

$Q(n) : n$ is even number

$$\forall n(P(n) \rightarrow Q(n))$$

$$P(n) \rightarrow Q(n) \equiv \neg Q(n) \rightarrow \neg P(n)$$

$\neg Q(n)$ is true , n is not even (n is odd), so $n=2k+1$

$$\begin{aligned}n^2 + 3 &= (2k+1)^2 + 3 \\&= 4k^2 + 4k + 1 + 3 \\&= 4k^2 + 4k + 4 \\&= 2(2k^2 + 2k + 2)\end{aligned}$$

Example

$$\begin{aligned}n^2 + 3 &= (2k + 1)^2 + 3 \\&= 4k^2 + 4k + 1 + 3 \\&= 4k^2 + 4k + 4 \\&= 2(2k^2 + 2k + 2)\end{aligned}$$

$$t = 2k^2 + 2k + 2$$



t is integer

$$n^2 + 3 = 2t$$

n^2+3 is an even integer, thus $\neg P(n)$ is true

Which to use

- When do you use a direct proof versus an indirect proof?
- If it's not clear from the problem, try direct first, then indirect second
 - If indirect fails, try the other proofs

Example

Prove that “**if n is an integer and n^3+5 is odd, then n is even**”

Via direct proof

- $n^3+5 = 2k+1$ for some integer k (definition of odd numbers)
- $n^3 = 2k - 4$
- $n = \sqrt[3]{2k - 4}$
- Umm...

So direct proof didn't work out.

Next up: indirect proof

Example

Prove that “**if n is an integer and n^3+5 is odd, then n is even**”

Via indirect proof

- Contrapositive: If n is odd, then n^3+5 is even
- Assume n is odd, and show that n^3+5 is even
- $n=2k+1$ for some integer k (definition of odd numbers)
- $n^3+5 = (2k+1)^3+5 = 8k^3+12k^2+6k+6 = 2(4k^3+6k^2+3k+3)$
- As $2(4k^3+6k^2+3k+3)$ is 2 times an integer, it is even

Proof by Contradiction

Assume that the hypothesis is true and that the conclusion is false and then, arrive at a contradiction.

Proposition “**if P then Q** ”

Proof. **Suppose P and $\sim Q$**

Since we have a contradiction, it must be that Q is true

Example

Prove that “there are infinitely many prime numbers”.

Proof:

- Assume there are **not infinitely** many prime numbers, therefore they are can be listed, i.e. p_1, p_2, \dots, p_n
- Consider the number $q = p_1 \times p_2 \times \dots \times p_n + 1$.
- q is either prime or not divisible, but not listed above. Therefore, q is a prime. However, it was not listed.
- **Contradiction!** Therefore, there are infinitely many primes numbers.

Example

- For all real numbers x and y , **if $x+y \geq 2$, then either $x \geq 1$ or $y \geq 1$.**

Proof

- Suppose that the conclusion is false. Then
 - $x < 1$ and $y < 1$Add these inequalities, $x+y < 1+1 = 2$ (**$x+y < 2$**)
- **Contradiction**
- Thus we conclude that the statement is true.

Example

Suppose $a \in \mathbb{Z}$. If a^2 is even, then a is even

Proof

- Contradiction: Suppose a^2 is even and a is not even.
- Then a^2 is even, and a is odd
- Let, $a = 2c + 1$ (odd)

$$a^2 = (2c+1)^2 = 4c^2 + 4c + 1 = 2(2c^2 + 2c) + 1 \quad (\text{odd})$$

- **Contradiction**
- Thus we conclude that the statement is true.

Thank You



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