Review Questions: Overlapping Generations

Econ720. Fall 2015. Prof. Lutz Hendricks

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1 A Savings Function

Consider the standard two-period household problem. The household receives a wage w_t when young and a rate of return R_{t+1} on savings.

(a) Illustrate the household's intertemporal budget constraint and the optimal choice of consumption in a diagram. Label it clearly.
(b) If the wage rate rises, what happens to savings? What if the household receives an additional endowment when old? Explain and illustrate in your diagram. No math please.
(c) Derive the consumption function $c\left(w_t, R_{t+1}\right)$ and the savings function $s\left(w_t, R_{t+1}\right)$ for the utility function

$$u(c_t, x_{t+1}) = c_t^{\alpha} + x_{t+1}^{\alpha}$$

where $0 < \alpha < 1$ is a constant.
(d) Do the same for the utility function

$$u(c_t, x_{t+1}) = A \cdot \ln(c_t^{\alpha} + x_{t+1}^{\alpha}), 0 < A < 1$$

What do you find and why?

1.1 Answer: Saving Function

(a) This is the standard diagram of indifference curve and budget line.
(b) If both goods are normal, an increase in the wage rate or in the endowment when old leads to higher consumption at both ages. Therefore:

$$w \uparrow \Rightarrow s \uparrow$$

but

$$y_2 \uparrow \Rightarrow s \downarrow$$

(c) The household solves:

$$\max c^{\alpha} + x^{\alpha}$$

subject to the budget constraint $c + x/R = w$. The first-order condition is $u_c = Ru_x$ or

$$\alpha c^{\alpha-1} = R\alpha x^{\alpha-1}$$

Thus

$$x/R = cR^{\alpha/(1-\alpha)}$$

Substituting this into the budget constraint yields

$$c = w/(1 + R^{\alpha/(1-\alpha)})$$

and

$$s = w(1 - 1/[1 + R^{\alpha/(1-\alpha)}])$$

(d) The answer is exactly the same. The utility function in (c) is a monotone increasing transformation of the one in (d).

2 Log-utility Example

Consider the standard two-period OLG model with log utility: $U(c_{yt}, c_{o,t+1}) = \ln c_{yt} + \beta \ln c_{o,t+1}$.

1. Solve for the household's saving function.
2. Find a law of motion for $k_t = K_t/L_t$.
3. Show that the economy has a unique steady state (not counting $k = 0$), if (i) the old do not work ($\ell = 0$) and (ii) the production function is Cobb-Douglas: $F(K_t, L_t) = K_t^{\alpha} L_t^{1-\alpha}$.

2.1 Answer: Log-utility example

1. Saving function The Euler equation becomes

$$\frac{c_{o,t+1}}{c_{y,t}} = \beta \; R_{t+1} \tag{1}$$

A solution to the household problem is a vector $(C_{y,t}, C_{o,t+1}, A_{t+1})$ that satisfies the Euler equation and two budget constraints.

The present value budget constraint is

$$c_{y,t} + \frac{c_{o,t+1}}{R_{t+1}} = \bar{W}_t = W_t + \frac{\ell \; W_{t+1}}{R_{t+1}} \tag{2}$$

Substitute the Euler equation into the budget constraint to obtain

$$c_{y,t} = \frac{\bar{W}_t}{1+\beta} \tag{3}$$

$$c_{o,t+1} = \frac{\beta \bar{W}_t}{1+\beta} \tag{4}$$

$$A_{t+1} = s(W_t, \ell W_{t+1}, R_{t+1}) = W_t - \frac{\bar{W}_t}{1+\beta} \tag{5}$$

2. Law of motion. Capital market clearing requires

$$K_{t+1} = N_t \; A_{t+1} = N_t \; \left[\frac{\beta}{1+\beta} W_t - \frac{\ell W_{t+1}}{(1+\beta) \; R_{t+1}}\right] \tag{6}$$

From the firm's problem we know that $W_t = W(k_t)$ with $W' > 0$ and $R_t = R(k_t)$ with $R' < 0$. Dividing through by N_t therefore yields a law of motion for k:

$$\begin{aligned} k_{t+1} &= \frac{K_{t+1}}{L_{t+1}} = \frac{K_{t+1}}{L_t (1+n)} = \frac{K_{t+1}}{N_t (1+n)} \frac{N_t}{L_t} \\ &= s(W(k_t), \ell W(k_{t+1}), R(k_{t+1})) \; \frac{N_t/L_t}{1+n} \end{aligned}$$

where $L_t = N_t + \ell N_{t-1} = N_t \; (1 + \ell/[1+n])$ is just a constant.

3. Unique steady state. With $\ell = 0$ the saving function simplifies greatly (b/c the interest rate does not affect lifetime earnings $\bar{W}$):

$$A_{t+1} = \frac{\beta}{1+\beta} W_t \tag{7}$$

Moreover, $N_t = L_t$. The law of motion for k becomes

$$k_{t+1} \; (1+n) = \frac{\beta}{1+\beta} W(k_t) \tag{8}$$

As long as the production function is such that $W' > 0$ and $W'' < 0$, there can only be one solution with $k_{t+1} = k_t > 0$.

With a Cobb-Douglas production function, we can solve for the steady state. $W(k) = (1-\alpha) k^{\alpha}$. Substitute this into the law of motion and set $k_{t+1} = k_t$ to obtain

$$k_{ss}(1+n) = \frac{\beta}{1+\beta}(1-\alpha) k_{ss}^{\alpha} \tag{9}$$

The solution is

$$k_{ss} = \left(\frac{\beta}{1+\beta}\frac{1-\alpha}{1+n}\right)^{1/(1-\alpha)} \tag{10}$$

This has intuitive properties. Steady state capital rises if households become more patient ($\beta \uparrow$) or if households have fewer children ($n \downarrow$).

3 Labor income taxes

Consider a two-period OLG economy with production.

Demographics: Each period $N = 1$ young households are born. They live for 2 periods.

Endowments: Young households work 1 unit of time.

Preferences:

$$\ln(c_t^y) + \beta \ln(c_{t+1}^o)$$

Technology:

$$Y_t = K_t^{\theta} L_t^{1-\theta}$$

There is no depreciation. The resource constraint is therefore

$$Y_t + K_t = c_t^y + c_t^o + K_{t+1}$$

Markets: Standard. There are no bonds.

Questions: (a) Derive the optimal level of savings of the household as a function of w. Briefly, why do savings not depend on r?

(b) Derive the FOCs for the firm.

(c) Define a competitive equilibrium. Be sure to clearly state the market clearing conditions and to ensure that the number of independent equations equals the number of endogenous variables.

(d) Write down a difference equation for the equilibrium capital-labor ratio, ($k_t = K_t/L_t$). Sketch a graph of this relationship.

(e) The government now imposes a time-invariant tax τ on labor income of the young so that after-tax earnings are $(1-\tau)w_t$. The revenues are thrown into the ocean. By how much does this tax lower the savings of the young for given w? Briefly, what is the intuition for this result?

(f) How does the tax affect the relationship graphed in (d)? What happens to the steady state capital-labor ratio? Sketch a graph.

3.1 Answer. Labor income taxes

(a) With an eye on part (d), we set up the household problem with the wage tax. The budget constraints are

$$c_t^y = (1-\tau)w_t - s_{t+1}$$

and

$$c^o_{t+1} = (1 + r_{t+1})s_{t+1}$$

The Euler equation is

$$c^o_{t+1} = \beta R_{t+1} c^y_t$$

where $R = 1 + r$. Substituting using the budget constraints yields

$$R_{t+1}s_t = \beta R_{t+1}[(1-\tau)w_t - s_t]$$

The savings function is therefore

$$s_t = (1-\tau)w_t\beta/(1+\beta)$$

It is independent of R because of log utility: income and substitution effects cancel.

(b) This is the standard answer:

$$r = \theta k^{\theta-1}$$

and

$$w = (1-\theta)k^{\theta}$$

(c) Market clearing requires

- Capital rental:

$$k_{t+1} = s_{t+1} = (1-\tau)(1-\theta)k^{\theta}_t\beta/(1+\beta)$$

- Labor rental: $L_t = 1$
- Goods: Same as feasibility.

A competitive equilibrium is a sequence

$$(c^y_t, c^o_{t+1}, k_t, L_t, s_t, w_t, r_t)$$

that satisfies (i) the household's budget constraints (2 eqn) and FOCs (2 eqn); (ii) the firm's FOCs (2 eqn); market clearing (2 eqn). We have 8 equations (per period) and 7 unknowns, which works out given Walras' law.

(d) The difference equation is the capital market clearing condition above, which is simply a positive constant times the production function. This is strictly concave and goes through the origin. The slope at zero is infinite. There is exactly one intersection with the 45-degree line (steady state).

(e) The tax reduces savings of the young, but less than one-for-one. The household spreads the pain across the two periods (see the savings function).

(f) The tax shifts the k_{t+1} line down and therefore reduces the steady state capital stock.

4 Fully-Funded Social Security

Fully-funded social security authority taxes households when young, invests the tax revenues, and pays benefits to the old out of the capital income accumulated on their own contributions.

(a) Explain why fully-funded social security does not affect the steady state capital stock, if public and private savings earn the same rate of return.

(b) How would this result change if the public rate of return was lower than the private one?

4.1 Answer: Fully-funded Social Security

(a) Households only care about the present value of future tax payments when deciding how much to consume. If the government earns the same rate of return as does the private sector, fully-funded social security does not alter this present value. Thus, consumption does not change and households reduce private savings by exactly the tax revenue. Therefore, total saving (public + private) remains unchanged.

(b) If the public rate of return is lower than the private one, the present value of lifetime resources available to the household declines. Given a rate of return, consumption at all ages is reduced. Therefore, private savings falls by less than the tax revenue and the capital stock increases. An easy way to see this is to note that such a policy is equivalent to a combination of case (a) plus a tax on the old.

5 Efficiency in an OLG model

[Due to Oksana Leukhina] Consider an overlapping generations endowment economy with a single non-storable good. Time is indexed by $t = 1, 2, 3...$ The representative consumer of the generation born in period t lives in periods t and $t+1$, has preferences represented by $c_t^y + \alpha \ln c_{t+1}^o$, and endowment stream $(e_t^y, e_t^o) = (1, 1)$. There is no fiat money.

a. Define an Arrow-Debreu equilibrium.

b. Calculate the unique equilibrium (do not prove it is unique).

c. Define a Pareto-efficient allocation for this economy.

d. Suppose that $\alpha = 2$. Show that the equilibrium is not Pareto-efficient.

5.1 Answer: Efficiency in an OLG model

a). An Arrow-Debreu equilibrium is given by the allocations of consumers born in $t \geq 1$, $\left\{\hat{c}_t^y, \hat{c}_{t+1}^o\right\}_{t\geq 1}$, allocation of the initial old $\hat{c}_t^o$ and prices $\{\hat{p}_t\}_{t\geq 1}$ such that

1) $\hat{c}_t^o$ solves the problem of the initial old, i.e. it solves

Given $\hat{p}_1$, max c_1^o s.t. $\hat{p}_1 c_1^o < \hat{p}_1$.

2) For all $t \geq 1$, $\left\{\hat{c}_t^y, \hat{c}_{t+1}^o\right\}_{t\geq 1}$ solves the problem of consumers born in $t:$

Given $(\hat{p}_t, \hat{p}_{t+1})$, max $c_t^y + \alpha \ln c_{t+1}^o$ s.t. $\hat{p}_t c_t^y + \hat{p}_{t+1} c_{t+1}^o = \hat{p}_t + \hat{p}_{t+1}$.

3) Equilibrium allocations satisfy $\hat{c}_t^y + \hat{c}_t^o = 2\ \forall t$.

b). The initial old eats $\hat{c}_t^o = 1$. Then the market clearing in period 1 is $\hat{c}_1^y + 1 = 2$, i.e. $\hat{c}_1^y = 1$.

The FOCs for consumers born in $t \geq 1$ are given by

$$\frac{p_t}{p_{t+1}} = \frac{c_{t+1}^o}{\alpha}. \tag{11}$$

Normalizing $p_1 = 1$, we have $p_2 = \frac{\alpha}{c_2^o}$. Using the budget constraint for the consumer born in $t = 1$ we have

$$c_1^y + \frac{\alpha}{c_2^o} c_2^o = 1 + \frac{\alpha}{c_2^o} 1,$$

which together with already determined $\hat{c}_1^y = 1$ implies $\hat{c}_2^o = 1$.

Continuing in this way, we get

$$\hat{c}_t^y = \hat{c}_t^o = 1,$$

i.e. an autarkic equilibrium.

Prices supporting this equilibrium are given by (11), i.e.

$$\hat{p}_t = \alpha^{t-1}.$$

c). A Pareto-efficient allocation is $c_1^o, \left\{c_t^y, c_{t+1}^o\right\}_{t\geq 1}$ such that

1. It is feasible, i.e. $c_t^y + c_t^o = 2$ and
2. There exists no other allocation $\tilde{c}_1^o, \left\{\tilde{c}_t^y, \tilde{c}_{t+1}^o\right\}_{t\geq 1}$ such that it is also feasible (i.e. $\tilde{c}_t^y + \tilde{c}_t^o = 2$ for all t) and

$$\begin{aligned} \tilde{c}_1^o &\geq c_1^o, \\ \tilde{c}_t^y + \alpha \ln \tilde{c}_{t+1}^o &\geq c_t^y + \alpha \ln c_{t+1}^o, \ \forall t, \end{aligned}$$

with at least one of these holding with a strict inequality.

d) Assume $\alpha = 2$. Then the competitive equilibrium found in b is not Pareto efficient. Consider an alternative allocation $\tilde{c}_1^o = 2$, $(\tilde{c}_t^y \tilde{c}_{t+1}^o) = (0, 2)$ $\forall t$. This allocation is feasible and everyone is strictly better off:

The initial old is better off because $2 > 1$. Consumers born in t are better off too: $0 + 2\ln 2 > 1 + 2\ln 1 = 1$.

6 Social Security

[Due to Joydeep Bhattacharya] In a fully funded pension system, the contributions made by the young at time t are invested and returned with interest at time $t+1$ to the same agent. Consider a standard Diamond (1965) model. There are N young agents born at the start of each date t. Agents work when young and retire when old. Assume that agents when young have to pay lump-sum taxes a_t in the form of contributions to a fully-funded pension program.

1. Suppose capital markets are perfect. Setup the model carefully and prove that a fully funded pension system is neutral, i.e., it does not affect capital accumulation. Assume a is less than wage income at any date.

   The Diamond model without any social security system:

   $$K_{t+1} = L_t s_t^* \quad \Leftrightarrow \quad (1+n)\, k_{t+1} = s_t^* \quad \Leftrightarrow \quad k_{t+1} = \frac{1}{1+n} s\left[w_t, R_{t+1}\right]$$

   where the optimal savings (saving function) $s_t^* \equiv s\left[w_t, R_{t+1}\right]$ is a solution to the individual problem:

   $$\begin{aligned} \max_{c_t, c_{t+1}, s_t} u_t &= u\left(c_t, c_{t+1}\right) \quad \text{s.t.} \quad c_t = w_t - s_t \quad \text{and} \quad c_{t+1} = R_{t+1} s_t \\ &\Leftrightarrow \quad \max_{s_t} u_t = u\left(w_t - s_t, R_{t+1} s_t\right) \\ &\Leftrightarrow \quad \frac{u_1\left(w_t - s_t^*, R_{t+1} s_t^*\right)}{u_2\left(w_t - s_t^*, R_{t+1} s_t^*\right)} = R_{t+1} \end{aligned}$$

   and the factor prices are given by

   $$r_t = f'\left(k_t\right) \quad \text{and} \quad w_t = f\left(k_t\right) - k_t f'\left(k_t\right).$$

   The equilibrium law of motion for k_t is

   $$k_{t+1} = \frac{1}{1+n} s\left[f\left(k_t\right) - k_t f'\left(k_t\right), f'\left(k_{t+1}\right)\right].$$

   A fully-funded social security system: the contributions of the young at time t are invested and returned with interest at time $t+1$ to the then old. Therefore the per-period budget constraints are modified into

   $$c_t = w_t - \widetilde{s}_t - a_t \quad \text{and} \quad c_{t+1} = R_{t+1}\widetilde{s}_t + R_{t+1} a_t.$$

Then the F.O.C. for utility maximization is given by

$$\frac{u_1\left[w_t - \left(\widetilde{s}_t^* + a_t\right), R_{t+1}\left(\widetilde{s}_t^* + a_t\right)\right]}{u_2\left[w_t - \left(\widetilde{s}_t^* + a_t\right), R_{t+1}\left(\widetilde{s}_t^* + a_t\right)\right]} = R_{t+1} \quad \Rightarrow \quad \widetilde{s}_t^* \equiv \widetilde{s}\left[w_t, R_{t+1}, a_t\right]$$

and the capital endowment for time $t+1$ becomes

$$\widetilde{K}_{t+1} = L_t \widetilde{s}_t^* + L_t a_t \quad \Leftrightarrow \quad (1+n)\,\widetilde{k}_{t+1} = \widetilde{s}_t^* + a_t \quad \Leftrightarrow \quad \widetilde{k}_{t+1} = \frac{1}{1+n}\left[\widetilde{s}_t^* + a_t\right].$$

However, defining

$$x_t \equiv \widetilde{s}_t^* + a_t = \widetilde{s}\left[w_t, R_{t+1}, a_t\right] + a_t$$

yields the F.O.C. rewritten as

$$\frac{u_1\left(w_t - x_t, R_{t+1}x_t\right)}{u_2\left(w_t - x_t, R_{t+1}x_t\right)} = R_{t+1},$$

from which x_t can be implicitly solvable in the saving function of the economy without social security:

$$x_t \equiv \widetilde{s}_t^* + a_t = s\left[w_t, R_{t+1}\right].$$

Substituting this into the expression for $\widetilde{k}_{t+1}$ yields

$$\widetilde{k}_{t+1} = \frac{1}{1+n}x_t = s\left[w_t, R_{t+1}\right].$$

Together with the factor pricing rule:

$$r_t = f'\left(\widetilde{k}_t\right) \quad \text{and} \quad w_t = f\left(\widetilde{k}_t\right) - k_t f'\left(\widetilde{k}_t\right)$$

we get the same equilibrium law of motion for k_t as before. This is possible provided that $a_t < (1+n)\,k_{t+1}(\ = s_t\ )$ in the economy without social security system, that is, social security contributions do not exceed the amount of saving that would otherwise have occurred. We may call x_t total savings, which is not affected.

2. Provide a clear intuition for this result.

   The intuitive explanation is that the increase in social security saving (a_t) is exactly offset by a decrease in private saving $(\widetilde{s}_t)$ in such a way that the total savings, $x_t = \widetilde{s}_t + a_t$, is equal to the previous level of s_t. The social security saving has the same rate of return as the one with private saving, so that the social security system perfectly substitutes for private saving in each individual's saving/investment decision. The argument assumes a perfect capital market; meaning that the young must be able to borrow against their future pension earnings.

3. Now suppose, capital markets are not perfect. In particular, suppose the young *cannot borrow*. Will the neutrality result in (1) above survive? Provide a clear intuiton for your answer. There is no need to work out all the model details.

   No; if the young cannot borrow against their future pension rights, they will save an amount different from what they did w/o the system.

7 Overlapping Generations With Human Capital

Consider the following overlapping generations model. Agents live for three periods. At each date, a unit measure of households is born. Cohort t is born at $t-1$ and middle-aged at t.

When **young**, the household can produce human capital according to $h_t = e_t^\beta$ with $0 < \beta < 1$. The household borrows the human capital investment e_t at interest rate r_t. When **adult** (middle aged), the household therefore has debt equal to $(1+r_t)e_t$. He chooses consumption (c_{1t}) and savings (s_t) subject to the budget constraint

$$c_{1t} + s_t = w_t h_t - (1 + r_t) e_t$$

When **old**, the household makes no decisions and simply consumes his wealth:

$$c_{2t+1} = (1 + r_{t+1}) s_t$$

Preferences are

$$u(c_{1t}, c_{2t}) = (c_{1t} - \gamma a_t)^{\theta} (c_{2t+1})^{1-\theta}$$

where a represents an "aspiration" level which is inherited from the parents:

$$a_t = c_{1t-1}$$

and $0 < \gamma < 1$. The idea is that growing up rich raises the standards for a "good life."

There is also a single representative firm which rents capital and labor services from the household to produce a single good. It maximizes profits in a competitive environment. The technology is $Y_t = K_t^{\alpha} h_t^{1-\alpha}$ with $0 < \alpha < 1$. Capital depreciates fully.

Questions: .

(a) Solve the problem of an **adult** (middle aged) household. You should obtain a saving function in closed form. Also derive an indirect utility function $U(e_t, a_t)$. *Hint*: You should find

$$U(e_t, a_t) = \theta^{\theta} (1 - \theta)^{1-\theta} (1 + r_{t+1})^{1-\theta} (w_t h_t - (1 + r_t) e_t - \gamma a_t) \tag{12}$$

(b) Solve for the optimal level of human capital investment by a young agent. Use the indirect utility function.

(c) Characterize optimal firm behavior.

(d) Define a competitive equilibrium. Make sure you explicitly state the market clearing conditions and that you have the same number of variables and equations.

(e) Consider the special case $\beta = 0$, so that human capital is an exogenous constant ($h_t = h$) and $e_t = 0$. Show that the steady state level of $k = K/h$ is

$$k^{SS} = \left(\frac{(1-\theta)(1-\alpha)(1-\gamma)}{1 - \gamma(1-\theta)} \right)^{1/(1-\alpha)}$$

Hint: There is no need to resolve for the saving function of the adult household (why not?).

(f) It is easy to show from the solution to (e) that the steady state level of k falls when γ rises. What is the intuition for this result?

7.1 Answer: OLG With Human Capital

(a) The household is endowed with e_t and h_t. He solves

$$\max \left(w_t h_t - (1 + r_t) e_t - s_t - \gamma a_t \right)^{\theta} \left((1 + r_{t+1}) s_t \right)^{1-\theta}$$

The FOC is

$$\theta (c_{1t} - \gamma a_t)^{\theta - 1} c_{2t+1}^{1-\theta} = (1 - \theta)(c_{1t} - \gamma a_t)^{\theta} c_{2t+1}^{-\theta} (1 + r_{t+1})$$

$\Rightarrow$

$$c_{2t+1} / (c_{1t} - \gamma a_t) = (1 + r_{t+1})(1 - \theta)/\theta$$

Using the budget constraints we find the saving function

$$\frac{w_t h_t - (1 + r_t) e_t - s_t - \gamma a_t}{(1 + r_{t+1}) s_t} = \frac{\theta/(1-\theta)}{(1 + r_{t+1})}$$

$\Rightarrow$

$$s_t = (1 - \theta)(w_t h_t - (1 + r_t) e_t - \gamma a_t)$$

This makes sense: Without "aspirations" the household would save a constant fraction of income due to log utility.

Indirect utility function: Substitute saving function into the objective function and collect terms:

$$U(e_t, a_t) = \theta^{\theta} (1 - \theta)^{1-\theta} (1 + r_{t+1})^{1-\theta} (w_t h_t - (1 + r_t) e_t - \gamma a_t)$$

(b) When young: the household maximizes the indirect utility function. In this case, this is equivalent to maximizing the present value of earnings (why?).

$$\max w_t e_t^{\beta} - (1 + r_t)e_t$$

Solution:

$$e_t = (\beta w_t/(1 + r_t))^{1/(1-\beta)}$$

(c) The firm's problem is standard. Define $k = K/h$. Then $w_t = (1-\alpha)\, k_t^{\alpha}$ and $1 + r_t = \alpha\, k_t^{\alpha-1}$. The same r as in the household problem because capital depreciates fully.

(d) A competitive equilibrium is a sequence of quantities $(c_{1t}, c_{2t}, s_t, e_t, h_t, K_t, a_t)$ and prices (w_t, r_t) that satisfy:

- Household: saving function and 2 budget constraints; optimality of e;
- Firm: two FOCs
- Law of motion for a.
- Market clearing

Capital market clearing requires

$$s_t = K_{t+1} + e_{t+1}$$

The e term appears because education requires capital. Goods market clearing requires:

$$F(K_t, h_t) = K_{t+1} + e_{t+1} + c_{1t} + c_{2t}$$

(e) The saving function does not change because the adult household takes (h, e) as given. Now they are simply fixed. Capital market clearing is now

$$s = (1-\theta)(wh - \gamma a) = K = kh$$

We can substitute out a using $a = c_1$. The adult budget constraint implies $c_1 = wh - s$. Therefore,

$$s = (1-\theta)(wh - \gamma(wh - s))$$

or

$$(1-\theta)(1-\gamma)wh = s(1 - \gamma(1-\theta))$$

Now use $w = (1-\alpha)\, k^{\alpha}$ to get

$$\frac{(1-\theta)(1-\gamma)}{1-(1-\theta)\gamma} = \frac{s}{wh} = \frac{k}{w} = \frac{k}{(1-\alpha)k^{\alpha}}$$

Rearranging leads to the equation we are supposed to prove.

(f) If children inherit aspirations from their parents, steady state capital is lower (they save less). Intuition: Ceteris paribus, a higher γ reduces saving; people require more consumption to maintain the same marginal utility.

8 Human capital

[Due to Joydeep Bhattacharya] Consider a fairly standard two-period lived OG model. The number of young agents at any date t is fixed at N. Agent preferences are described by

$$U(c_{1t}, c_{2t+1}) = \ln c_{1t} + \ln c_{2t+1}$$

where c_{1t} is young-age consumption at date t and c_{2t+1} is old-age consumption at date $t+1$. The production function is given by

$$Y_t = S_t^{\gamma} L_t^{1-\gamma}$$

where L_t is the labor of young (unskilled) workers and S_t is the labor of old (skilled) workers.

When young, agents spend a fraction m_t of their time working and the remaining time $1-m_t$ in getting an education (acquiring human capital). Their human capital accumulation is described by

$$h_{t+1} = h_t + (1 - m_t)\,\theta h_t$$

where h_t is human capital of the current old, h_{t+1} is human capital acquired by the current young, and $\theta > 1$ is a parameter. The wage per unit of time worked is w_t. When young, agents cannot use the human capital they have acquired in that period.

When old, agents just work and earn a wage of v_t per unit of human capital they acquired when young. All markets are perfectly competitive.

1. Point out the *two* most important features of the human capital accumulation equation.
2. Solve for the agent's optimal consumption, saving, and human capital investment profile.
3. What is the growth rate for output in this economy?
4. Define a competitive equilibrium for this economy.
5. Can the model, as it stands, generate poverty traps? Explain briefly. [*Hint:* does an agent's human capital investment profile depend on whether they are in a high or low human capital economy?]
6. Suppose the government initiates a mandatory PAYG social security program, taxing the young a lump-sum amount T_1 and returning the old a lump-sum amount T_2. Will all countries that run such a program invest more in education than before? Show all steps. Provide a clear intuition for your answer.

8.1 Answer: Human capital

1. Point out two important features of the human capital accumulation equation.

   There is a human capital externality (the current young directly benefit from the human capital accumulated by the current old generation (their parents). Alternatively, one an point out that the production function for h is linear in h which generates a potential for endogenous growth.

2. Solve for the agent's optimal consumption, saving, and human capital investment profile.

   The budget constraints are

$$\begin{aligned} c_{1t} &= w_t m_t \\ c_{2t+1} &= v_{t+1} h_{t+1} = v_{t+1}\left[h_t + (1 - m_t)\,\theta h_t\right] \end{aligned}$$

   so agent's problem

$$\max \quad \ln c_{1t} + \ln c_{2t+1} \Leftrightarrow \max_{m_t} \quad \ln w_t m_t + \ln\left\{v_{t+1}\left[h_t + (1 - m_t)\,\theta h_t\right]\right\}$$

   Solution:

$$m_t = (1+\theta)/2\theta$$

   a constant.

$$\begin{aligned} c_{1t} &= \frac{(1+\theta)}{2\theta} w_t \\ c_{2t+1} &= v_{t+1} h_t \frac{(1+\theta)}{2} \end{aligned}$$

   since $\theta > 1$.

3. What is the growth rate for output in this economy?

$$\begin{aligned} Y_t &= S_t^{\gamma} L_t^{1-\gamma} \\ S_t &= N h_t \\ L_t &= N m_t \end{aligned}$$

$$h_{t+1} = h_t + (1 - m_t)\,\theta h_t = h_t + \left(1 - \frac{(1+\theta)}{2\theta}\right)\theta h_t = \frac{1}{2} h_t\,(\theta + 1)$$

then

$$\frac{Y_{t+1}}{Y_t} - 1 = \left(\frac{h_{t+1}}{h_t}\right)^{\gamma} - 1 = \left[\frac{1}{2}(\theta+1)\right]^{\gamma} - 1$$

4. A competitive equilibrium is a set of allocation sequences $\{c_{1t}\}, \{c_{2t}\}, \{h_t\}$, $\{m_t\}$ and price sequences $\{w_t\}, \{v_t\}$ such that a) $\{c_{1t}\}, \{c_{2t}\}, \{m_t\}$ solve the agent's problem given $\{w_t\}, \{v_t\}$ and the human capital accumulation function, b) firms maximize profits so that $\{w_t\}, \{v_t\}$ satisfy the standard marginal product relationships, and c) all markets clear

5. Can this model, as it stands, generate poverty traps? Explain. [*Hint:* does an agent's human capital investment profile depend on whether they are in a high or low human capital economy?]

   No. Here m is a constant and does not depend on the level of h. This means the young in rich or poor nations will invest the same in human capital.

6. Suppose the government initiates a mandatory PAYG social security program, taxing the young a lump-sum amount T_1 and returning the old a lump-sum amount T_2. Will all countries that run such a program invest more in education?

   The budget constraints are

$$\begin{aligned} c_{1t} &= w_t m_t - T_1 \\ c_{2t+1} &= v_{t+1} h_{t+1} + T_2 \end{aligned}$$

   so agent's problem

$$\max_{m_t} \quad \ln(w_t m_t - T_1) + \ln\{v_{t+1}[h_t + (1 - m_t)\theta h_t] + T_2\}$$

   Solution:

$$\begin{aligned} \frac{w_t}{w_t m_t - T_1} &= \frac{v_{t+1}\theta h_t}{v_{t+1}[h_t + (1-m_t)\theta h_t] + T_2} \\ \Rightarrow m_t &= \frac{1}{2\theta h_t w_t v_{t+1}}\left(T_2 w_t + h_t w_t v_{t+1} + \theta T_1 h_t v_{t+1} + \theta h_t w_t v_{t+1}\right) \end{aligned}$$

   since the govt's budget constraint is $T_2 = T_1 = T$ (with no population growth),

$$\begin{aligned} m_t &= \frac{1}{2\theta h_t w_t v_{t+1}}\left(T w_t + h_t w_t v_{t+1} + \theta T h_t v_{t+1} + \theta h_t w_t v_{t+1}\right) \\ &= \frac{(1+\theta)}{2\theta} + T\left(\frac{w_t + v_{t+1}\theta h_t}{2 v_{t+1}\theta h_t w_t}\right) > \frac{(1+\theta)}{2\theta} \end{aligned}$$

   it follows that people work more (higher m) invest less in education than in the absence of the social security program. The intuition is as follows: notice that the only asset here is human capital and human capital adds only to old age income. The SS program adds to old age income and hence reduces the need to accumulate human capital; the tax used to finance the pension reduces young age income and further raises the opportunity cost of getting an education.

9 OLG Model with Human Capital

Consider the following two period OLG model.

Households live for two periods. $N_t = (1+n)^t$ young households are born at date t. Each is endowed with human capital h_t and no assets. A young household divides his time between work (l_t) and education ($1-l_t$). Work pays $l_t h_t w_t$. Income is either consumed or saved. The young budget constraint is

$$l_t h_t w_t = c_t^y + k_{t+1}$$

Old households have human capital $h_{t+1}^o = g(1-l_t, h_t)$ and earn $h_{t+1}^o w_{t+1}$. Their budget constraint is

$$R_{t+1} k_{t+1} + h_{t+1}^o w_{t+1} = c_{t+1}^o$$

The utility function is $u\left(c_t^y, c_{t+1}^o\right)$. New agents "inherit" human capital from their parents:

$$h_{t+1} = \varphi h_t$$

Firms rent capital and labor from households at rental prices q_t and w_t, respectively. Firms produce with a constant returns to scale production function, $Y_t = F(K_t, L_t)$, and maximize period profits. Capital depreciates at rate δ.

(a) Derive and interpret the first order conditions for the **household problem**. Define a solution to the household problem.

(b) Define a solution to the **firm**'s problem.

(c) State the **market clearing** conditions.

(d) Define a **competitive equilibrium**. Make sure the number of equations matches the number of objects.

9.1 Answer: OLG model with human capital

(a) The problem is

$$\max \ u\left(c_t^y, c_{t+1}^o\right)$$

subject to the present value budget constraint

$$c_t^y + c_{t+1}^o / R_{t+1} = l_t h_t w_t + \frac{g(1-l_t, h_t) w_{t+1}}{R_{t+1}}$$

First order conditions are:

$$\begin{aligned} u_1 &= u_2 R_{t+1} \\ \frac{g_1 w_{t+1}}{R_{t+1}} &= w_t h_t \end{aligned}$$

The first is a standard Euler equation. The second requires the marginal return to human capital to equal the interest rate.

A solution is a vector $\left(c_t^y, c_{t+1}^o, l_t, k_{t+1}, h_{t+1}^o\right)$ that solves the 2 focs, the 2 budget constraints, and the law of motion for h.

(b) The firm is standard: (K_t, L_t) that satisfy

$$\begin{aligned} q_t &= f'(K_t/L_t) \\ w_t &= f(K_t/L_t) - f'(K_t/L_t) \ K_t/L_t \end{aligned}$$

(c) Market clearing:

$$\begin{aligned} K_{t+1} &= N_t k_{t+1} \\ L_t &= N_t h_t l_t + N_{t-1} g(1-l_{t-1}, h_{t-1}) \\ F(K_t, L_t) + (1-\delta) K_t &= N_t c_t^y + N_{t-1} c_t^o + K_{t+1} \end{aligned}$$

(d) A CE consists of sequences

$$\{c_t^y, c_t^o, k_t, l_t, K_t, L_t, h_t, h_t^o, q_t, w_t, R_t\}$$

[10 objects] that satisfy

- 5 household conditions;
- 2 firm conditions;
- 3 market clearing conditions;
- $R_{t+1} = 1 - \delta + q_{t+1}$.
- $h_{t+1} = \varphi h_{t+1}^o$.