Equipment Investment and Growth in Developing Countries
Technical Appendix

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1. Balanced Growth With External Learning

1.1 Choosing Units of Equipment

Suppose equipment comes in vintages indexed by their technical sophistication A^* and the exogenous resource cost of one "unit" is $p^*(A^*)$. Assume that more modern equipment is more cost effective in the sense that $p^*(A^*) / A^*$ is falling in A^*. Then we can redefine units by dividing by $p^*(A^*)$ such that one new unit of capital has a resource cost of one.

How much capital services does one (new) unit of A^* provide? The maximum is obviously $A(A^*) = A^* / p^*(A^*)$. This is strictly monotone and can be inverted to obtain $A^*(A)$. In general, one new unit of capital provides $q(A^*, H) / p^*(A^*) = A(A^*)\, f^*(H / A^*)$ units of capital services – it is equivalent to $1 / p^*(A^*)$ original units of capital. Thus, $q(A, H) = A\, f^*[\, p(A)\, H / A]$, where $p(A) = p^*(A^*[A])$. Since $A / p(A)$ is increasing in A, this does not affect monotonicity of the efficiency function. However, the efficiency function is, in general, *different* for each A.

A similar argument applies to the learning function. If the contribution to learning in original units is $h^*(A^*/H)$, then it is in transformed units $h^*(A/H\; p[A])/p(A)$. It remains true that more sophisticated capital contributes more to learning (per *efficiency unit* of capital), but again each grade has a different learning function. The assumption that equipment prices are the same for all grades is therefore not simply a choice of units. It is essential, however, for a balanced growth path to exist.

1.2 Concavity of the Investor's Objective Function

This section examines concavity of the investor's objective function with a degenerate maintenance cost and external learning. Note first that it is always optimal to choose θ in the range where $0 < f(\theta) < 1$. Choosing θ such that $f(\theta) = 1$ would mean that output is lower at all future dates than for the value of θ that maximizes current output. Choosing $f(\theta) = 0$ is dominated by waiting. Therefore, f is differentiable and strictly increasing over the relevant range. The value function of the investor is

$$V = \max\; k_{0,t}\left[\int_0^T e^{-rv}\left[\pi_{t+v}\, q\left(A_{0,t}, H_{t+v}\right) - m\right]dv - p^E\right].$$

Therefore, at the optimum

$$\frac{\partial V}{\partial T} = k_{0,t}\, e^{-rT} \left(\pi_{t+T}\, q(A_{0,t}, H_{t+T}) - m \right) = 0$$

$$\frac{\partial V}{\partial A_{0,t}} = k_{0,t} \int_0^T e^{-rv}\, \pi_{t+v}\, \frac{\partial q(A_{0,t}, H_{t+v})}{\partial A_{0,t}}\, dv = 0\,.$$

Assuming that at the end of equipment life capital has reached full productivity so that $\partial q / \partial H = 0$ and $\partial q / \partial A = 1$, the second order partials are

$$\begin{aligned}(1)\qquad \partial^2 V / \partial T^2 = \;& k_{0,t}\, e^{-rT} \cdot [\dot{\pi}_{t+T}\, q(A_{0,t}, H_{t+T}) \\ & + \pi_{t+T}\, \partial q(A_{0,t}, H_{t+T}) / \partial H \cdot \dot{H}_{t+T}] < 0\,.\end{aligned}$$

$$(2)\qquad \partial^2 V / \partial A^2 = k_{0,t} \int_0^T e^{-rv}\, \pi_{t+v}\, \frac{\partial^2 q(A_{0,t}, H_{t+v})}{\partial A_{0,t}^2}\, dv < 0$$

where the second term in (1) is zero. Concavity requires negative second partials and $\partial^2 V / \partial A^2 \cdot \partial^2 V / \partial T^2 - (\partial^2 V / \partial T\, \partial A)^2 > 0$, where (recall that $\partial q / \partial A = 1$)

$$\frac{\partial^2 V}{\partial T\, \partial A} = k_{0,t}\, e^{-rT}\, \pi_{t+T}\;\;.$$

A sufficient condition for strict concavity is therefore

$$(3)\qquad \left(\dot{\pi}_{t+T}\, A_{0,t}\, f(H_{t+T} / A_{0,t}) \right) \cdot \int_0^T e^{-rv}\, \pi_{t+v}\, \frac{\partial^2 q(A_{0,t}, H_{t+v})}{\partial A_{0,t}^2}\, dv - \pi_{t+T}^2 > 0$$

Since $\partial q / \partial A = f(H/A) - (H/A)\, f'(H/A)$, the second partial is

$$\partial^2 q / \partial A^2 = (H/A)^2\, f''(H/A) / A\,.$$

Also, $\pi_{t+s} = \pi_{t+T}\, e^{\gamma(T-s)}$. Equation (3) then implies

$$-\gamma\, f(H_{t+T} / A_{0,t}) \int_0^T e^{-rv}\, e^{\gamma(T-s)} \left(H_{t+v} / A_{0,t} \right)^2 f''\left(H_{t+v} / A_{0,t} \right) dv > 1$$

With $f(H_{t+T} / A_{0,t}) = 1$, we have

$$e^{-\gamma T} < -\gamma \int_0^T e^{-(r+\gamma)v}\, (H_{t+v} / A_{0,t})^2\, f''(H_{t+v} / A_{0,t})\, dv\,.$$

Since $H_{t+v} / A_{0,t} = e^{\gamma v} / \theta$, this becomes

$$\theta^2 e^{-\gamma T}/\gamma < -\int_0^T e^{-(r-\gamma)v} f''(e^{\gamma v}/\theta)\, dv .$$

Since γ and θ are bounded on any balanced growth path (as shown below) and $r = \sigma\gamma + \rho$, this is satisfied if f is sufficiently concave, except for T close to zero. However, setting T close to zero cannot be optimal because the investor pays a fixed cost (p^E) per unit of capital and a short payoff period implies that profits are negative.

1.3 Optimal Retirement and Maintenance Cost

Since the age profile of r^E is single peaked, there are only two candidate retirement dates for capital of a given technology. A fraction with very high maintenance cost $(m > p_{v_0,t+v_0})$ is retired immediately when the maintenance period starts (at age v_0). The remainder is retired at age τ where $r^E_{\tau,t+\tau} = m$.

Thus, there are three retirement phases. Up to age v_0 there is no retirement. At age v_0, a discrete amount is retired which has a negative present value of continuation. There is no additional retirement until the peak of r^E is reached (call this age v_P). Beyond this age, retirement occurs when the rental price falls below the maintenance cost.

If the cumulative distribution function of the maintenance cost is Φ, then

(4)
$$\Delta_v = \begin{cases} 1 & 0 \le v < v_0 \\ \Phi(p_{v_0,t+v_0}) & v_0 \le v < v_P \\ \min\{\Phi(r_v^E), \Phi(p_{v_0,t+v_0})\} & v_P \le v \le T_{\max} \end{cases}$$

The maintenance cost paid for surviving units is therefore equal to the conditional expectation of Φ truncated at the upper bounds given in (4). Up to age v_0, the maintenance cost is obviously 0. Between the ages of v_0 and v_P it is constant. A fraction $\Phi(0)$ has zero maintenance cost. The remainder $(\Delta_v - \Phi(0))$ costs $E(m \mid 0 \le m \le p_{v_0,t+v_0})$. The flow maintenance cost paid per unit originally invested is therefore

$$(\Delta_v - \Phi(0)) \quad E(m \mid 0 \le m \le p_{v_0,t+v_0}) .$$

To obtain the flow per suriving unit, divide by Δ_v. The revenue flow generated by a unit of capital is therefore

$$\Delta_v r_v^E - [\Delta_v - \Phi(0)] E(m \mid 0 \le m \le p_{v_0,t+v_0})$$

1.4 Choice of *A*

With external learning the optimal choice of *A* requires

$$\frac{\partial p_0}{\partial A} = \int_0^{T_{\max}} e^{-rv}\, \Delta_v \frac{\partial r_v^E}{\partial A} dv = 0 .$$

Since $r_{v,t+v}^E = \pi_{t+v}\, A_{0,t}\, f(H_{t+v} / A_{0,t})$,

$$\begin{aligned}\frac{\partial r_{v,t+v}^E}{\partial A_{0,t}} &= \pi_{t+v}\, f\left(H_{t+v} / A_{0,t}\right) - \pi_{t+v}\, A_{0,t}\, f'\left(H_{t+v} / A_{0,t}\right) H_{t+v} / A_{0,t}^2 \\ &= z / A_{0,t+v} \left\{ f\left(H_{t+v} / A_{0,t}\right) - f'\left(H_{t+v} / A_{0,t}\right) H_{t+v} / A_{0,t} \right\} \\ &= z / A_{0,t}\, e^{-\gamma v} \left\{ f\left(e^{\gamma v} / \theta\right) - f'\left(e^{\gamma v} / \theta\right) e^{\gamma v} / \theta \right\}\end{aligned}$$

Optimality of *A* thus requires

$$\frac{\partial p_0}{\partial A} = \int_0^{T_{\max}} e^{-rv}\, \Delta_v\, e^{-\gamma v} \left\{ f\left(e^{\gamma v} / \theta\right) - f'\left(e^{\gamma v} / \theta\right) e^{\gamma v} / \theta \right\} dv = 0$$

Write the efficiency function as

$$f(H/A) = \begin{cases} 1 & \text{if } H/A \geq 1 \\ 0 & \text{if } H/A \leq \eta \\ 1 - \lambda (A/H - 1)^{\varphi} & \text{otherwise} \end{cases}$$

where η is defined by $\lambda (1/\eta - 1)^{\varphi} = 1$ or $\eta = [1 + \lambda^{-1/\varphi}]^{-1}$. Its derivative in the interior region is

$$f'(H/A) = \lambda \varphi (A/H - 1)^{\varphi - 1} / (H/A)^2 .$$

1.5 Equipment Investment share

Nominal gross equipment investment is $I^E = \pi^E k_{0,t}$. The equipment share in GDP is thus $I^E / Y = \pi^E k_{0,t} / Y$. From the FOC of the firm: $K_{0,t} / (A_{0,t} L_{0,t}) = (\alpha_E / \alpha_L)(w_t / z)$. Therefore,

$$\begin{aligned}k_{0,t} &= K_{0,t} / (A_{0,t}\, f(1/\theta)) \\ &= \frac{L_{0,t} (\alpha_E / \alpha_L)(w_t / z)}{f(1/\theta)}\end{aligned}$$

Since $w = \alpha_L Y$, we have

$$\frac{k_{0,t}}{Y} = \frac{L_{0,t}\,\alpha_E}{z\,f(1/\theta)}.$$

Finally compute $L_{0,t}$ from labor market clearing together with

$$\frac{K_{v,t}}{K_{0,t}} = e^{-g_Y v}\,\Delta_v\,e^{-\gamma v}\,f\left(1/\Theta_v\right)/\,f\left(1/\theta\right).$$

1.6 Structures Investment Share

Nominal gross structures investment on the balanced growth path is $I^S = \pi^S\,(\dot{S} + \delta^S\,S) = (g_Y + \delta^S)\,\pi^S\,S$. Since the income share of structures is $r^S\,S/Y = \alpha_S$ and $r^S = (r + \delta^S)\,\pi^S$, the investment share in GDP is

$$I^S/Y = (g_Y + \delta^S)\,\alpha_S/(r + \delta^S).$$

This may be used to calibrate α_S.

1.7 Computing Average *q*

Define average q as

$$\bar{q} = \int_0^{T_{\max}} q_{v,t}\,k_{v,t}\,dv \Big/ \int_0^{T_{\max}} k_{v,t}\,dv.$$

In order to compute this, note that $k_{v,t} = k_{0,t}\,\Delta_v\,e^{-g_Y v}$.

2. Balanced Growth Without Learning Spillovers

The investor's problem is not affected by learning spillovers. Since in symmetric equilibrium $\tilde{H} = H$ for all households, I no longer explicitly distinguish between these two variables.

2.1 Wage Profile

From the household's first-order condition, the wage profile must obey

$$w_{v,H_t,t} = \mu_t - \upsilon_t\,H_t\,h\left(A_{v,t}/H_t\right).$$

Since $\mu/\lambda\,H$ is constant in steady state, the cross-section wage-profile is time-invariant:

$$\frac{w_{v,H_t,t}}{w_{0,H_t,t}} = \frac{\mu/(\upsilon H) - h(A_{v,t}/H_t)}{\mu/(\upsilon H) - h(A_{0,t}/H_t)}.$$

With external learning, the wage ratio equals one because the h terms are not valued. Internal learning adds a compensating differential which makes labor employed with old vintages relatively more expensive $[h(A_{v,t}/H_t) \le h(A_{0,t}/H_t)]$.

The intercept, $w_{0,t}$, grows at rate g_Y. The longitudinal profile facing a given vintage as it ages is therefore

$$w_{v,t+v} = w_{v,t}\, e^{g_Y v}$$

2.2 Age Profile of Rental Price

Relative factor inputs satisfy

$$\text{(5)} \qquad \frac{S_{v,H_t,t}}{L_{v,H_t,t}} = \frac{\alpha_S}{\alpha_L}\frac{w_{v,H_t,t}}{r_t^S}, \quad \frac{k_{v,H_t,t}}{L_{v,H_t,t}} = \frac{\alpha_E}{\alpha_L}\frac{w_{v,H_t,t}}{r_{v,t}^E}$$

Cost minimization implies

$$\text{(6)} \qquad \begin{aligned} r_{v,t}^E &= \alpha_E\, (S_{v,H_t,t}/k_{v,H_t,t})^{\alpha_S}\, (L_{v,H_t,t}/k_{v,H_t,t})^{\alpha_L}\, q(A_{v,t},H_t)^{\alpha_E} \\ &= \alpha_E\, (\alpha_S/\alpha_E\, r_{v,t}^E/r^S)^{\alpha_S}\, (\alpha_L/\alpha_E\, r_{v,t}^E/w_{v,H_t,t})^{\alpha_L}\, q(A_{v,t},H_t)^{\alpha_E} \\ &= \alpha_E\, q(A_{v,t},H_t)\,(\alpha_S/r^S)^{\alpha_S/\alpha_E}\,(\alpha_L/w_{v,H_t,t})^{\alpha_L/\alpha_E} \end{aligned}$$

The cross-section age profile is therefore

$$\begin{aligned} \frac{r_{v,t}^E}{r_{0,t}^E} &= \frac{q(A_{v,t},H_t)}{q(A_{0,t},H_t)}\left(\frac{w_{0,H_t,t}}{w_{v,H_t,t}}\right)^{\alpha_L/\alpha_E} \\ &= e^{-\gamma v}\,\frac{f(1/\Theta_v)}{f(1/\theta)}\left(\frac{w_{0,H_t,t}}{w_{v,H_t,t}}\right)^{\alpha_L/\alpha_E} \end{aligned}$$

Since w is increasing in v (compensating differentials), the age-profile is steeper than with external learning. Essentially, new capital receives an additional compensation for its contribution to learning. The intercept is

$$r_{0,t}^E = \alpha_E\, A_{0,t}\, f(1/\theta)\,(\alpha_S/r^S)^{\alpha_S/\alpha_E}\,(\alpha_L/w_{0,H_t,t})^{\alpha_L/\alpha_E}$$

which is time-invariant:

$$g(r_{0,t}^{E}) = \gamma - (\alpha_L / \alpha_E)\, g(w_{0,H_t,t}) = \gamma - (\alpha_L / \alpha_E)\, g_Y = 0\,.$$

2.3 Labor allocation

The relative labor allocation is determined by the firm's first-order condition

$$\frac{L_{v,H_t,t}}{L_{0,H_t,t}} = \frac{k_{v,H_t,t}}{k_{0,H_t,t}} \frac{r_{v,t}^{E}}{r_{0,t}^{E}} \frac{w_{0,H_t,t}}{w_{v,H_t,t}},$$

where
$$\frac{k_{v,t}}{k_{0,t}} = \Delta_v\, e^{-g_Y v}\,.$$

Therefore

$$\frac{L_{v,H_t,t}}{L_{0,H_t,t}} = \Delta_v\, e^{-(g_Y+\gamma)v}\, \frac{f(1/\Theta_v)}{f(1/\theta)} \left(\frac{w_{0,H_t,t}}{w_{v,H_t,t}} \right)^{1+\alpha_L/\alpha_E}$$

For a given depreciation profile, more labor is allocated to new vintages because working with new vintages conveys learning benefits which are reflected in a lower wage ratio, w_0 / w_v. Moreover, the share of earnings received by old vintage labor is also higher (again holding Δ and γ fixed). The capital-labor ratio in efficiency units

$$\frac{K_{v,H_t,t} / L_{v,H_t,t}}{K_{0,H_t,t} / L_{0,H_t,t}} = \left(\frac{w_{v,H_t,t}}{w_{0,H_t,t}} \right)^{1+\alpha_L/\alpha_E}$$

follows the wage profile.

2.4 Household with balanced growth

The household problem is described in the main text. With balanced growth, the co-state evolves according to

(7)
$$g_\upsilon = r - \int_0^{T_{\max}} L_{v,H_t,t} \left\{ \frac{\partial\, w_{v,H_t,t}}{\partial\, H_t} \frac{1}{\upsilon_t} + h\left(A_{v,t} / H_t\right) - h'\left(A_{v,t} / H_t\right) A_{v,t} / H_t \right\} dv$$

where $\dfrac{\partial\, w_{v,H_t,t}}{\partial\, H_t} \dfrac{1}{\upsilon_t}$ is constant over time (as a function of v). The derivative of the wage rate is determined by the firm's first-order condition

$$w_{v,H,t} = \alpha_L \left(S_{v,H,t} / L_{v,H,t}\right)^{\alpha_S} \left(k_{v,H_t,t}\, q(A_{v,t}, H_t) / L_{v,H,t}\right)^{\alpha_E}$$

$$= \alpha_L \left(\frac{\alpha_S}{\alpha_L} \frac{w_{v,H,t}}{r^S}\right)^{\alpha_S} \left(\frac{\alpha_E}{\alpha_L} \frac{w_{v,H,t}}{r^E_{v,t}} q(A_{v,t}, H_t)\right)^{\alpha_E}$$

$$\Rightarrow \qquad w_{v,H,t} = \alpha_L \left(\alpha_S / r^S\right)^{\alpha_S / \alpha_L} \left(\alpha_E\, q(A_{v,t}, H_t) / r^E_{v,t}\right)^{\alpha_E / \alpha_L}$$

The derivative of wages w.r.to H in (7) is therefore

$$\frac{\partial w_{v,H,t}}{\partial H_t} = \frac{\alpha_E}{\alpha_L} \frac{f'(H_t / A_{v,t})}{q(A_{v,t}, H_t)} w_{v,H,t} \tag{8}$$

2.5 Investor with balanced growth

The optimal choice of A still requires

$$\frac{\partial p_{0,t}}{\partial A_{0,t}} = \int_0^{T_{\max}} e^{-r v}\, \Delta_v \frac{\partial r^E_{v,t+v}}{\partial A_{0,t}} dv = 0\,,$$

but the derivative of r^E is now changed. The age profile of rental prices is

$$r^E_{v,t+v} = \alpha_E\, A_{0,t}\, f(H_{t+v} / A_{0,t}) (\alpha_S / r^S)^{\alpha_S / \alpha_E} (\alpha_L / w_{v,H_{t+v},t+v})^{\alpha_L / \alpha_E}$$

Let $C_1 = \alpha_E (\alpha_S / r^S)^{\alpha_S / \alpha_E} (\alpha_L)^{\alpha_L / \alpha_E}$. Then

$$r^E_{v,t+v} = C_1\, A_{0,t}\, f(H_{t+v} / A_{0,t})\, w_{v,H_{t+v},t+v}{}^{-\alpha_L / \alpha_E}$$

The derivative of r^E w.r.to A is therefore

$$\frac{\partial r^E_{v,t+v}}{\partial A_{0,t}} = C_1\, w_{v,H_{t+v},t+v}{}^{-\alpha_L / \alpha_E} \left(f(e^{\gamma v} / \theta) - f'(e^{\gamma v} / \theta) e^{\gamma v} / \theta\right)$$

$$- \frac{\alpha_L}{\alpha_E} \frac{r^E_{v,t+v}}{w_{v,H_{t+v},t+v}} \frac{\partial w_{v,H_{t+v},t+v}}{\partial A_{0,t}}$$

The age profile of wages facing the investor is

$$w_{v,H_{t+v},t+v} = \mu_{t+v} - \upsilon_{t+v}\, H_{t+v}\, h\left(A_{v,t+v} / H_{t+v}\right)$$

The derivative is therefore

$$\frac{\partial w_{v,H_{t+v},t+v}}{\partial A_{0,t}} = \frac{\partial w_{v,H_{t+v},t+v}}{\partial A_{v,t+v}}$$

$$= -\upsilon_{t+v}\, h'\left(A_{v,t+v} / H_{t+v}\right)$$

$$= -\frac{\upsilon_{t+v}\, H_{t+v}}{w_{v,H_{t+v},t+v}} \frac{w_{v,H_{t+v},t+v}}{H_{t+v}} h'\left(A_{v,t+v} / H_{t+v}\right)$$

where $w_{v,H} /(\upsilon H)$ is constant over time.

The derivative of the rental price may be written as

$$\frac{\partial r^E_{v,t+v}}{\partial A_{0,t}} = w_{v,H_{t+v},t+v}{}^{-\alpha_L/\alpha_E} \begin{Bmatrix} C_1 \left(f(1/\Theta_v) - f'(1/\Theta_v)/\Theta_v\right) \\ + \dfrac{\alpha_L}{\alpha_E} \dfrac{r^E_{v,t+v}\, w_{v,H_{t+v},t+v}{}^{\alpha_L/\alpha_E}}{H_{t+v}} \dfrac{\upsilon_{t+v}\, H_{t+v}}{w_{v,H_{t+v},t+v}} h'\left(A_{v,t+v} / H_{t+v}\right) \end{Bmatrix}$$

$$= w_{v,H_{t+v},t+v}{}^{-\alpha_L/\alpha_E}\, C_1 \left\{ \left(f(1/\Theta_v) - f'(1/\Theta_v)/\Theta_v\right) + \frac{\alpha_L}{\alpha_E} \frac{A_{0,t}\, f(1/\Theta_v)}{H_{t+v}} \frac{\upsilon H}{w_{v,H}} h'(\Theta_v) \right\}$$

$$= w_{v,H_{t+v},t+v}{}^{-\alpha_L/\alpha_E}\, C_1 \left\{ \left(f(1/\Theta_v) - f'(1/\Theta_v)/\Theta_v\right) + \frac{\alpha_L}{\alpha_E} \Theta_v\, f(1/\Theta_v) \frac{\upsilon H}{w_{v,H}} h'(\Theta_v) \right\}$$

To compute this, use

$$\frac{w_{v,H_{t+v},t+v}}{\upsilon_t\, H_t} = e^{g_Y v} \frac{w_{v,H_t,t}}{\upsilon_t\, H_t} = e^{g_Y v} \left(\mu/(\upsilon H) - h(\Theta_v)\right)$$

together with the fact that the derivative is needed only up to a constant.

2.6 Equipment Investment share

Nominal gross equipment investment is given by $I^E = \pi^E k_{0,t}$ or $I^E / Y = \pi^E k_{0,t} / Y$. Since equipment receives a constant income share:

$$\alpha_E Y_t = \int_0^{T_{\max}} k_{v,t}\, r^E_{v,t}\, dv$$

$$= k_{0,t} \int_0^{T_{\max}} (k_{v,t} / k_{0,t})\, r^E_v\, dv$$

$$= k_{0,t} \int_0^{T_{\max}} e^{-g_Y v}\, \Delta_v\, r^E_v\, dv$$

k_0/Y can then be computed from the knowledge of r^E and Δ.

2.7 Outline of Algorithm

The algorithm iterates over guesses of θ, r, $\mu/\upsilon\ H$, and r_0^E. First, it computes $w_{v,H}/\upsilon H$ from

$$w_{v,H_t,t}/(\upsilon_t H_t)=\mu_t/(\upsilon_t H_t)-h(\Theta_v)$$

Next, the age profile of r^E can be computed from (6) up to a constant

$$\frac{r_v^E}{r_0^E}=\frac{A_{v,t}\ f(1/\Theta_v)/(w_{v,H_t,t})^{\alpha_L/\alpha_E}}{A_{0,t}\ f(1/\theta)/(w_{0,H_t,t})^{\alpha_L/\alpha_E}}$$

It is then rescaled using the guess r_0^E. This allows to calculate the optimal depreciation profile Δ_v and thus k_v. The age profile of the labor allocation can be found from the firm's first-order condition (up to a constant):

$$L_{v,H}=\frac{\alpha_L}{\alpha_E}\frac{r_v^E\ k_{v,H}}{w_{v,H}}.$$

The scale of the L_v profile is determined from labor market clearing. Finally compute the derivative of w w.r.to H from (8)

$$\begin{aligned}\frac{\partial w_{v,H_t,t}}{\partial H_t}\frac{1}{\upsilon_t}&=\frac{\alpha_E}{\alpha_L}\frac{f'(1/\Theta_v)}{f(1/\Theta_v)}\frac{w_{v,H_t,t}}{A_{v,t}\,\upsilon}\\&=\frac{\alpha_E}{\alpha_L}\frac{f'(1/\Theta_v)}{f(1/\Theta_v)}\frac{w_{v,H_t,t}}{H_t\,\upsilon_t}\frac{e^{\gamma v}}{\theta}\end{aligned}$$

The resulting values are used to compute deviations from the remaining equilibrium conditions (first-order condition for A; learning equation; zero profit condition; law of motion for υ).

3. Comparative Balanced Growth Results

This section develops additional analytical results for the special case where the maintenance cost distribution is degenerate and learning is external with $\psi = 1$.

3.1 Characterization of Balanced Growth

To characterize the balanced growth path, it is convenient to substitute the optimal labor allocation into the learning equation which yields

(9)
$$\gamma = \frac{\int_{-T}^{0} e^{(g_Y+\gamma)s} f(e^{-\gamma s}/\theta)\, h(e^{\gamma s}\theta)\, ds}{\int_{-T}^{0} e^{(g_Y+\gamma)s} f(e^{-\gamma s}/\theta)\, ds}.$$

A balanced growth path is then characterized by (γ, θ, T, x) which solve the zero profit condition, the first-order condition for T $(\theta = e^{\gamma T} x)$, the first-order condition for A, and (9) where the interest rate is determined by the consumption Euler equation.

It is often useful to eliminate x from the system by substituting the first-order condition for T into the no-arbitrage condition. From

$$\pi^E = \int_0^T e^{-rs}\left(\pi_t\, e^{-\gamma s}\, H_t\, \theta\, f(H_{t+s}/A_t) - m\right) ds$$

and $\pi_t\, H_t\, \theta / m = e^{\gamma T}$ we have $\frac{\pi^E}{m} = \int_0^T e^{-rs}\left(e^{\gamma(T-s)}\, f(e^{\gamma s}/\theta) - 1\right) ds$ or

(10)
$$\frac{\pi^E}{m} = e^{\gamma T}\int_0^T e^{-(r+\gamma)s} f(e^{\gamma s}/\theta)\, ds - \frac{1-e^{-rT}}{r}$$

3.2 Existence

Existence of a balanced growth path exists can be shown as follows. Write the first-order condition for A as

$$\theta = F_\theta(\theta,\gamma,T) = \int_0^T e^{-rs}\, f'(e^{\gamma s}/\theta)\, ds \Big/ \int_0^T e^{-(r+\gamma)s}\, f(e^{\gamma s}/\theta)\, ds$$

and write the solution of (10) as $T = F_T(\theta,\gamma)$. Finally, write the learning function (9, not its solution) as $\gamma = F_\gamma(\theta,\gamma,T)$. Then the vector valued function $\mathbf{F} = (F_\theta, F_T, F_\gamma)$ maps a compact set into itself.

To see this, note first that for any $0 < T \le \infty$ and $0 \le \gamma < \infty$, optimal technology choice implied by $F\theta$ is bounded below by $\underline{\theta} = \operatorname{argmax} \zeta\, f(1/\zeta) > 1$ because choosing a θ below the level that maximizes current $q(.)$ yields lower output at all future dates as well. It is bounded above by $\overline{\theta}$ with $f(1/\overline{\theta}) = 0$ because choosing an A that yields no current output is strictly inferior to postponing the investment. For T = 0, impose the limiting value of θ as $T \to 0$. This restriction will not bind.

For any $0 < T \le \infty$, $\underline{\theta} \le \theta \le \overline{\theta}$ and $0 \le \gamma < \infty$, the learning rate implied by the left-hand side of (9) is bounded above by $\gamma \le \overline{\gamma} = \max h(\zeta)$ and below by $\gamma \ge \underline{\gamma} > 0$. Again, for T = 0, impose the limiting value of γ as $T \to 0$.

Finally, for any $\underline{\theta} \leq \theta \leq \overline{\theta}$ and $\underline{\gamma} \leq \gamma \leq \overline{\gamma}$, a strictly positive and finite T solves (10), i.e. $F_T \geq \underline{T}$. Note that the right hand side approaches zero as $T \to 0$ and positive infinity as $T \to \infty$. Therefore, we can find bounds such that $0 < \underline{T} \leq T \leq \overline{T} < \infty$.

It follows that **F** maps a compact set $[\underline{\gamma}, \overline{\gamma}] \times [\underline{T}, \overline{T}] \times [\underline{\theta}, \overline{\theta}]$ into itself. Since the mapping is continuous, an equilibrium exists by a standard fixed-point argument. ■

3.3 The Role of Service Lives

In the economics literature, depreciation is typically viewed as physical decay that is beyond the control of the firm. However, as Feldstein and Foot (1971, p. 50) observe: "Plant and equipment neither evaporate by radioactive decay nor fall apart like the legendary 'one hoss shay'; rather they are scrapped and replaced when the balance of economic forces makes that decision most profitable."[1] In the present model, allowing firms to choose when to retire old capital is essential as the age distribution of the capital stock determines productivity and growth by altering the rate of learning.

In fact, endogenous service lives are *necessary* for equipment prices to have any growth effects at all. To see this, consider an economy where $T(t)$ is fixed at T. The definition of balanced growth equilibrium is unchanged, except that the first order condition for T is dropped.

Proposition A1. Assume that $T > 0$ is fixed at for all t. Then the balanced growth rate of the economy is independent of the price of imported capital goods.

Proof. If T is exogenous, the system becomes block recursive. The growth rate is determined by the learning function (9) and the first-order condition for A, which do not depend on π^E. ■

The intuition again relies on the fact that the choice of technology does not *directly* depend on equipment prices due to the additive separability of equipment prices in the investor's problem (recall the discussion of the investor's problem).

3.4 Growth Effects of Distortionary Policies

The balanced growth path can be depicted graphically (Figure 2) as the intersection of the learning equation ("LBD," (9)), the combined no arbitrage condition and optimal choice of

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[1] Boddy and Gort (1975) provide evidence that obsolescence, not physical decay, accounts for most of depreciation.

T ("NA", (10)), and the first order condition for technology choice ("FOC-A"). The latter represents the combinations of (γ, T) for which the investor leaves θ unchanged. From the investor's problem we know that θ depends positively on T and negatively on γ, which implies that this curve is upward sloping. Moreover, shifting the curve up requires a lower θ. All three curves are drawn for a fixed θ.

The slope of LBD represents is ambiguous a priori, but negative in the empirically relevant case. The direct effect of reducing T is to increase the rate of learning as a larger fraction of capital contributes to learning. However, there may be a positive feedback: Faster learning implies that new vintages are relatively more productive (lower A/H) and they are relatively larger (higher k). It is possible that these feedbacks are so strong that the slope of (9) becomes positive. It can be shown that is the case only if the learning period is sufficiently long. Since the learning period in the data is only a small fraction of total equipment life, we focus on the case of a negative slope.[2]

The graph of the no-arbitrage condition (NA) is downward sloping "on average" in the sense that, along NA, $\gamma \to \infty$ as $T \to 0$ and $\gamma \to 0$ as $T \to \infty$. Given parameter values from the data, it is likely that NA is downward sloping everywhere (section 3.5). The intuition for the negative slope is as follows. Capital goods are retired when their payoff flows fall to zero: $\pi_{t+T}\, q(A_{0,t}, H_{t+T}) = m$. If γ rises, it is optimal to reduce T because r_E falls at a faster rate over the lifetime of equipment. This suggests that the slope of NA should be negative. However, there is an indirect effect: π_t is adjusted so as to maintain zero profits and the investor responds to higher π by raising T. For NA to be positively sloped, it has to be the case that increasing π_t so as to maintain π_{t+T} (and therefore T) unchanged is not sufficient to restore zero profits. This means that the present value of profits must fall (due to a higher discount rate r) despite the fact that the payoff π_{t+s} is higher at every date.

Exogenous Technology Choice

Consider first the case where $\theta > 1$ is exogenously given such that $f(1/\theta) > 0$ (the investor would always choose that). Existence of a balanced growth path follows as a special case of the previous argument. Uniqueness requires not only that NA be negatively sloped, but that it intersects LBD from above. The reason for potential multiplicity is that

[2] It should be emphasized again that the objective of this section is not to characterize all possible equilibria, but to provide intuition for the numerical results, where the possibility of a positive slope is never an issue.

investors respond to faster learning by reducing service lives, which reinforces the increase in the learning rate. In the calibrated version of the model, multiplicity is never an issue because the effect of T on the learning rate is quite small (see the numerical results presented below). The reason is that a very small fraction of labor is employed with old vintages.[3] If the balanced growth path is unique, lower equipment prices unambiguously lead to faster growth:

Proposition 2. Given a fixed $\theta > 1$ with $f(1/\theta) > 0$. If the balanced growth path is unique, a lower price of equipment causes faster growth and shorter service lives.

Proof. This result can be seen directly from Figure 2. Assuming that θ is fixed amounts to dropping the FOC-A schedule. The initial equilibrium occurs at *A*. Reducing p^E then shifts NA to the left, while the position of LBD does not change. The new equilibrium at *B* has shorter service lives and faster growth. ■

Thus, allowing investors to choose the technology embodied in new capital is not essential for equipment prices to affect growth (in contrast to endogenous obsolescence). The intuition is that reducing service lives frees up labor allocated to capital that does not contribute to learning. The lower tail of the labor allocation schedule is truncated resulting in faster learning.

Endogenous Technology Choice

The effect of reducing the price of equipment when both θ and T are endogenous is illustrated again in Figure 2.[4] The initial equilibrium occurs at *A*. The impact effect of a lower p^E is to shift the no-arbitrage curve (NA) to the left. If θ did not respond, the new equilibrium would be *B* with shorter service lives and faster growth as shown above. However, the investor responds by changing θ which shifts all three curves. Since a change in θ moves NA and LBD in the same direction, the new equilibrium must lie in either region *I* or *II* so that θ and T always move together. The change in the growth rate is ambiguous because the rate of learning depends positively on θ and negatively on T.

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[3] This need not be the case here because the assumption that capital does not depreciate before T overstates the amount of labor allocated to old vintages.

[4] The requirements for uniqueness of the balanced growth path are now even more severe. If NA intersects LBD from above, the balanced growth path is unique for constant θ. In addition, it is required that all three curves intersect only for a unique θ.

Since the investor chooses to lower θ at *B*, so that NA and LBD shift left, the leading case is an equilibrium in region *I*. Lower equipment prices lead to shorter service lives and investment in *inferior* technologies (lower θ). There are two ways to climb up the technological ladder: by making infrequent large steps or by making many small steps – the model suggests that fast growing countries follow the latter strategy.

However, an equilibrium in region *II* cannot be ruled out: θ might rise by so much that both the NA and the LBD schedule shift *right*, in which case reducing p^E leads to slower growth, although this case is (unsurprisingly) never found in the numerical results.[5]

3.5 Properties of the No Arbitrage Condition

Write the no arbitrage condition (10), which also incorporates optimal choice of *T* as in implicit function

(11) $$F(\theta,\gamma,T;\pi^E) = \int_0^T e^{-rs}\left[e^{\gamma(T-s)} f(e^{\gamma s}/\theta) - 1\right] ds - \frac{\pi^E}{m} = 0 .$$

Since the bracketed term equals zero at $s = T$ (optimal choice of *T*),

(12) $$\begin{aligned}\frac{\partial F}{\partial T} &= \gamma \int_0^T e^{-rs}\, e^{\gamma(T-s)} f(e^{\gamma s}/\theta)\, ds \\ &= \gamma\, \tilde{f}\, e^{\gamma T}\, \frac{1-e^{-(r+\gamma)T}}{r+\gamma} > 0\end{aligned}$$

where $f(1/\theta) \le \tilde{f} \le 1$. Since $\partial F/\partial\theta < 0$, a higher θ shifts the NA curve to the right in Figure 2. To sign the slope, we need to sign

$$\frac{\partial F}{\partial \gamma} = \int_0^T e^{-rs}\, e^{\gamma(T-s)} \left\{ -s\frac{\sigma\alpha_E}{\alpha_L}[\,f(\zeta(s)) - e^{-\gamma(T-s)}] + (T-s)\, f(\zeta(s)) + s\, f'(\zeta(s)) e^{\gamma s}/\theta \right\} ds$$

which uses $\partial r/\partial\gamma = \sigma\alpha_E/\alpha_L$. A lower bound can be established by dropping the positive term involving *f'*

$$\frac{\partial F}{\partial \gamma} = \frac{\sigma\alpha_E}{\alpha_L}\int_0^T s\, e^{-rs}\, ds + e^{\gamma T}\int_0^T e^{-rs}\, e^{-\gamma s}\; f(\zeta(s))\, \{T - s[1+\sigma\alpha_E/\alpha_L]\} ds$$

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[5] One reason is that the shift of NA in response to changing θ is likely small for sensible parameter values; see Appendix 3.6. Another reason, born out by the numerical results, is that investors do not change θ much in response to *T* and γ. Thus, the new equilibrium occurs close to *B* and therefore in region *I*.

Note that the first term integrates by parts:

$$\int_0^T s e^{-rs} = \frac{e^{-rs}}{-r}(s+1/r)\Bigg|_0^T = -\frac{e^{-rT}}{r}(T+1/r)+\frac{1}{r^2}$$
$$=\left\{1-e^{-rT}(1+rT)\right\}/r^2 > 0$$

The second integral can be written as

$$e^{\gamma T} T \bar{f} \int_0^T e^{-(r+\gamma)s}\{1-(s/T)[1+\sigma\alpha_E/\alpha_L]\}ds$$

This can also be integrated by parts using

$$\int s e^{-(r+\gamma)s} = \frac{e^{-(r+\gamma)s}}{-(r+\gamma)}(s+1/(r+\gamma)).$$

Applying the bounds [0, *T*] yields

$$\int_0^T s e^{-(r+\gamma)s} = -\frac{e^{-(r+\gamma)T}}{(r+\gamma)}(T+1/(r+\gamma))+\frac{1}{(r+\gamma)^2}.$$

The derivative is therefore

$$\frac{\partial F}{\partial\gamma} \geq \frac{\sigma\alpha_E}{\alpha_L}\frac{1-e^{-rT}(1+rT)}{r^2}+e^{\gamma T} T \bar{f}\left\{\frac{1-e^{-(r+\gamma)T}}{r+\gamma}-\frac{1}{T}[1+\frac{\sigma\alpha_E}{\alpha_L}]\left(\frac{1}{(r+\gamma)^2}-e^{-(r+\gamma)T}\frac{T+1/(r+\gamma)}{r+\gamma}\right)\right\}$$

This simplifies to

$$\frac{\partial F}{\partial\gamma} \geq \frac{\sigma\alpha_E}{\alpha_L}\frac{1-e^{-rT}(1+rT)}{r^2}+\frac{e^{\gamma T} T \bar{f}}{r+\gamma}e^{-(r+\gamma)T}\left\{\left[e^{(r+\gamma)T}-1\right]\left[1-\frac{1+\sigma\alpha_E/\alpha_L}{(r+\gamma)T}\right]+1+\frac{\sigma\alpha_E}{\alpha_L}\right\}$$

A sufficient (but too strong) condition for the second term to be positive (and the slope of NA to be negative) is then

$$\frac{1+\sigma\alpha_E/\alpha_L}{(r+\gamma)T} < 1$$

or $(r + \gamma)\, T$ close to zero.

3.6 Shift of NA when θ changes

The amount by which NA shifts right when θ changes can be derived by implicitly differentiating (11).

$$\frac{\partial F}{\partial \theta} = -e^{\gamma T} \int_0^T e^{-(r+\gamma)s} f'(e^{\gamma s}/\theta)\, e^{\gamma s}/\theta^2\, ds$$
$$= -\frac{e^{\gamma T}}{\theta^2} \int_0^{T_L} e^{-rs} f'(e^{\gamma s}/\theta)\, ds$$

Together with (12) this implies

$$\frac{\partial T}{\partial \theta} = \frac{\bar{f}'/\theta^2 (1-e^{-rT_L})/r}{\bar{f}\,\gamma(1-e^{-(r+\gamma)T})/(r+\gamma)}$$
$$= \frac{\bar{f}'}{\bar{f}\,\gamma\theta^2} \frac{1-e^{-rT_L}}{1-e^{-(r+\gamma)T}} \frac{r+\gamma}{r}$$

where $\bar{f}$ and $\bar{f}'$ are values of *f* and *f'* that leaves the respective integrals unchanged. The elasticity of *T* with respect to θ generated by a shift in NA (holding γ fixed) is therefore

$$\eta_{T,\theta} = \frac{\bar{f}'}{\bar{f}\,\gamma\theta T} \frac{1-e^{-rT_L}}{1-e^{-(r+\gamma)T}} \frac{r+\gamma}{r}$$

This is obviously small for short T_L or long *T*.

4. Evidence on Plant Learning

This section provides a brief summary of the evidence on plant learning provided in the literature. Evidence of learning-by-doing is found in data on productivity growth of new plants and on productivity growth of new workers carrying out a given task. The parameters of interest for this study are the duration of learning and the total amount of productivity improvement due to learning. In the model, the duration of learning is represented by time until a new technology reaches full efficiency ($f = 1$). On a balanced growth path, this duration (T_L) is implicitly defined by $e^{\gamma T_L} = \theta$. The amount of learning is given by $f(1/\theta)^{-\alpha_E}$.

4.1 Duration of Learning

Data on plant learning are very diverse. Using data for U.S. manufacturing firms Power (1998, p. 308) finds that learning continues for the entire length of her sample (8 years) and yields a productivity improvement of around 16%. This controls for plant fixed effects. Interestingly, she does not find a significant effect of previous investment on plant productivity levels or growth rates. Baloff (1966) finds much shorter learning periods of 2 to 43 months after new plants start production with total output gains between 1.6 and 11%. David (1975) finds that output per hour grew by 2.3% per year over a 20 year period for U.S. cotton mills in the mid 19th century.

Bahk and Gort (1993) and Gort et al. (1993) find that "capital learning," which "refers to increases in knowledge about the characteristics of given physical capital" (p. 575) continues for five to six years after plant birth. This is measured by the change in the output elasticity with respect to the capital stock. However, capital learning in the model appears as a shift of the intercept of the production function. In Bahk and Gort's data, this type of learning peaks somewhat earlier in the life of a plant.[6]

4.2 Amount of Learning

Hulten (1992) finds that embodied efficiency in best practice firms is about 23 percent above sample average. In the model, this corresponds to the difference between the maximum and average values of $q(A, H)$ in the cross section.

Annual productivity growth rates between 1 and 2 percent are implied by Hulten's (1992) estimate of the rate of embodied technical change and by Baily et al.'s (1992) estimates of vintage effects for young plants.

I conclude that most estimates suggest a rate of productivity growth between 1 and 2% per year with a duration of learning between 5 and 15 years. The baseline case therefore has 6 years of learning leading to a total productivity gain of 12%.

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[6] Interestingly, Greenwood and Jovanovic (1998) interpret Bahk and Gort's data to show a total productivity growth of 15% over 14 years. My interpretation is closer to Parente's (1998).

5. Figures

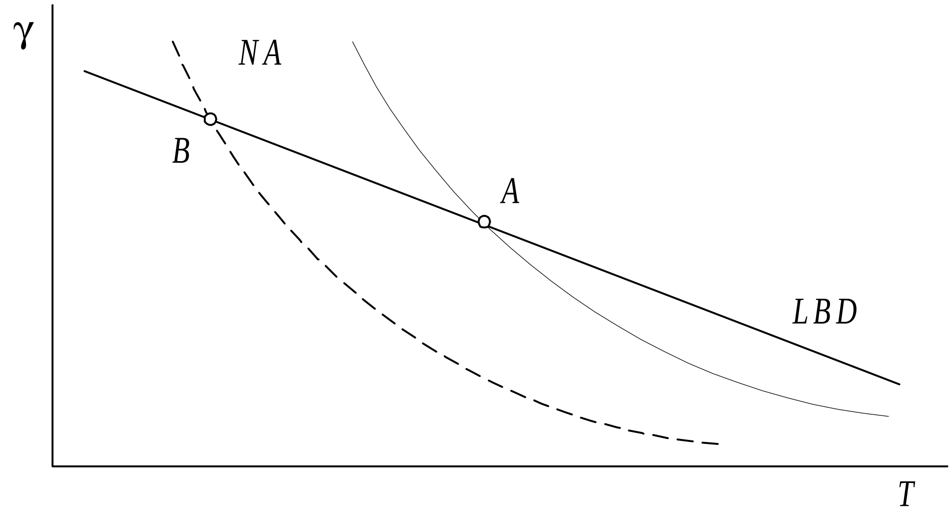

Figure 1. Balanced Growth with Exogenous θ

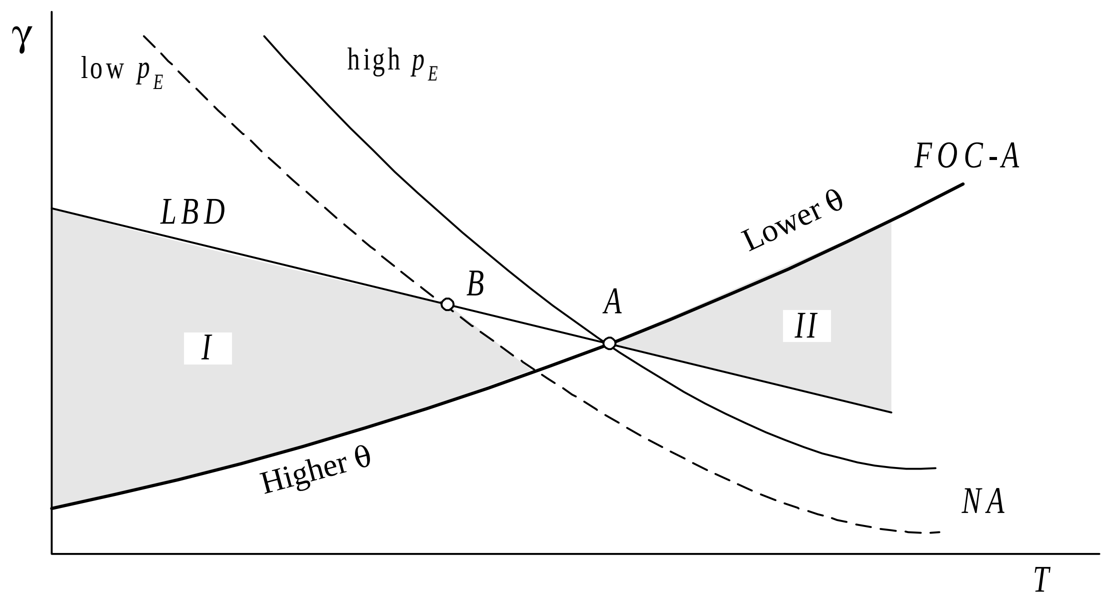

Figure 2. Endogenous Technology and Service Lives