# The Growth Model: Discrete Time

Prof. Lutz Hendricks

Econ720

September 9, 2015

# The standard growth model

- The neoclassical growth model, aka the standard growth model, is the most important model in macro.
- It underlies entire branches of the literature (parts of growth theory and business cycle theory, for example).
- Here, we study this model in discrete time.
- The **main issues** of this section are:
  - Tools: Dynamic programming
  - The neoclassical growth model

# Model structure

There are many versions of the growth model. This is a basic version.

1. Households are identical and live forever.
2. Firms produce a single good using capital and labor.
3. All agents are price takers.
4. All prices are perfectly flexible. All markets clear at all times.

# Infinite horizons

- So far we have assumed that agents are finitely lived.
- Analytically more convenient: infinite lifetimes.
- How to justify this?
  - Reduced form of an OLG model with **altruism**.
  - Stochastic deaths (perpetual youth models).
  - But really: convenience + show it does not matter.

# Demographics

There is a continuum of households (uncountably infinite number).

All households are identical.

- This is stronger than needed (see notes on aggregation later on).

We can think of a single, price-taking household.

The measure of households is 1.

Therefore, per capita and aggregate variables are the same.

**Exercise**: Redo everything when the number of households is $N_t = (1+n)^t$.

# Preferences

The household values discounted utility from consumption:

$$\sum_{t=0}^{\infty} \beta^t u(c_t), \quad 0 < \beta < 1 \tag{1}$$

Utility is time separable (for tractability).

Discounting is exponential (to avoid time consistency problems).

Time consistency means: If $\{c_t\}_{t=0}^{\infty}$ solves the problem with start date 0, then $\{c_t\}_{t=\tau}^{\infty}$ solves the problem with start date $\tau$.

The household does not want to change past plans.

# Endowments

The household has

- $k_0$ units of the good at $t = 0$
- 1 unit of time in each period

# Technology

- Resource constraint:

$$k_{t+1} = f(k_t) - c_t \tag{2}$$

- We assume Inada conditions for $f$.
- Capital cannot be negative: $k_t \geq 0$.

# Markets

Goods: numeraire.

Labor: $w_t$

Capital rental: $q_t$

All markets are competitive.

# Planning Problem

- The planner maximizes discounted utility of the representative household

$$\sum_{t=0}^{\infty} \beta^t u(c_t), \quad 0 < \beta < 1$$

- Constraints:

$$\begin{aligned} k_{t+1} &= f(k_t) - c_t \\ k_{t+1} &\geq 0 \\ & k_0 \text{ given} \end{aligned}$$

# Lagrangian

$$
\begin{aligned}
\Gamma &= \sum_{t=0}^{\infty} \beta^t u(c_t) \\
&\quad + \sum_{t=0}^{\infty} \lambda_t [f(k_t) - c_t - k_{t+1}]
\end{aligned}
$$

FOCs for an interior solution:

$$
\begin{aligned}
\beta^t u'(c_t) &= \lambda_t \\
\lambda_{t+1} f'(k_{t+1}) &= \lambda_t
\end{aligned}
$$

# Euler equation

$$\beta u'(c_{t+1})f'(k_{t+1}) = u'(c_t) \tag{3}$$

- This is exactly the same Euler equation we saw many times before.
- The Euler equation implicitly defines a law of motion for the capital stock:

$$\beta u'(f(k_{t+1}) - k_{t+2})f'(k_{t+1}) = u'(f(k_t) - k_{t+1}) \tag{4}$$

- This is a second order difference equation.

# Planner: Solution

- A solution is a sequence $\{c_t, k_{t+1}\}_{t=0}^{\infty}$.
- These satisfy:
  1. Euler equation
  2. Resource constraint
- We have two difference equations - we need two **boundary conditions**:
  1. $k_0$ given
  2. **Transversality**:

$$\lim_{t \to \infty} \beta^t \; u'(c_t) \; k_{t+1} = 0 \tag{5}$$

# Digression: Transversality Conditions

# Digression: Transversality Conditions

Consider the following example:

$$\begin{aligned} & \max \sum_{t=0}^{T} \beta^t u(c_t) \\ s.t. \;\; k_{t+1} &= e_t + (1+r_t)k_t - c_t \end{aligned}$$

# Digression: Transversality Conditions

Lagrangian

$$\begin{aligned}\Gamma &= \sum_{t=0}^{T} \beta^t u(c_t) \\ &\quad + \sum_{t=0}^{T} \lambda_t \{e_t + (1+r_t)k_t - c_t - k_{t+1}\}\end{aligned}$$

FOCs (necessary):

$$u'(c_t) = \beta \; u'(c_{t+1}) \; (1 + r_{t+1})$$

# Solution

Sequences $\{c_t, k_{t+1}\}$ that satisfy:

- Euler equation
- budget constraint
- $k_0$ given

# Problems

**Problem 1**:

- We allowed the household to choose $c_t \to \infty$ and $k_{t+1} \to -\infty$.
- The household problem has no solution.

**Problem 2**:

- We have 2 difference equations, but only one boundary condition.
- The solution is not uniquely determined by those.

We need one more boundary condition to ensure that utility is finite.

# Where to Find a Boundary Condition?

- The economics of the problem must suggest the right condition.
- It needs to be imposed as part of the original problem with some economic justification.
- A natural candidate in this example: $k_{T+1} = 0$.
  - The household cannot die in debt.

# Infinite horizon case

- What if $T \to \infty$?
- We could impose $\lim_{T\to\infty} k_{T+1} = 0$, but it does not make economic sense.
  - This would prevent the household from perpetually growing its capital stock.
- We need to find a weak condition that makes utility finite.

# Infinite horizon case

One solution:

- Write the present value budget constraint as

$$\sum_{t=0}^{T} \frac{c_t}{R_t} = \sum_{t=0}^{T} \frac{e_t}{R_t} + k_0 - \frac{k_{T+1}}{R_{T+1}}$$

  where $R_t = (1+r_1) \times ... \times (1+r_t)$ is a cumulative discount factor.
- Require that $\lim_{T\to\infty} k_{T+1}/R_{T+1} = 0$.
- That ensures finite consumption and picks out a unique solution.

# Infinite horizon case

An equivalent solution:

Impose

$$\lim_{T\to\infty} \beta^T \; u'(c_T) \; k_{T+1} = 0$$

This is the same because, by the Euler equation:

$$\beta^T u'(c_T) \; R_T = u'(c_0)$$

# Transversality Conditions

The general point:

- Each dynamic optimization problem requires as many boundary conditions as there are difference equations.
- In economic problems, we are usually short one boundary condition (only $k_0$ is given).
- We need to find a second boundary condition that is economically justifiable and keeps utility finite.

# Reading

- Acemoglu (2009), ch. 6. Also ch. 5 for background material we will discuss in detail later on.
- Stokey, Lucas, and Prescott (1989), ch. 1 is a nice introduction.
- Blanchard and Fischer (1989) is a good introduction to the standard growth model.

# References I


Acemoglu, D. (2009): *Introduction to modern economic growth*. MIT Press.

Blanchard, O. J., and S. Fischer (1989): *Lectures on macroeconomics*. MIT press.

Stokey, N., R. Lucas, and E. C. Prescott (1989): "Recursive Methods in Economic Dynamics," .