Optimal Control

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Topics

Optimal control is a method for solving dynamic optimization problems in continuous time.

Generic Optimal control problem

Choose functions of time c(t) and k(t) so as to

$$\max \int_0^T v[k(t), c(t), t] dt \tag{1}$$

Constraints:

1. Law of motion of the **state** variable k(t):

$$\dot{k}(t) = g[k(t), c(t), t] \tag{2}$$

2. Feasible set for **control** variable c(t):

$$c(t) \in Y(t) \tag{3}$$

3. Boundary conditions, such as:

$$k(0) = k_0$$
, given (4)

$$k(T) \ge k_T$$
 (5)

Generic Optimal control problem

- c and k can be vectors.
- ightharpoonup Y(t) is a compact, nonempty set.
- T could be infinite.
 - ▶ Then the boundary conditions change
- Important: the state cannot jump; the control can.

Example

A household chooses optimal consumption to

$$\max \int_0^T u[c(t)]dt \tag{6}$$

subject to

$$\dot{k}(t) = rk(t) - c(t) \tag{7}$$

$$c(t) \in [0, \bar{c}] \tag{8}$$

$$k(0) = k_0, \text{given} \tag{9}$$

$$k(T) \ge 0 \tag{10}$$

A Recipe for Solving Optimal Control Problems

A Recipe

Step1. Write down the *Hamiltonian*

$$H(t) = v(k, c, t) + \mu(t)g(k, c, t)$$
 (11)

 μ is essentially a Lagrange multiplier (called a "co-state variable).

Step 2. Derive the **first order conditions** which are **necessary** for an optimum:

$$\partial H/\partial c = 0 \tag{12}$$

$$\partial H/\partial k = -\dot{\mu} \tag{13}$$

A Recipe

Step 3. Impose the **transversality** condition:

for finite horizon:

$$\mu\left(T\right) = 0\tag{14}$$

for infinite horizon:

$$\lim_{t \to \infty} H(t) = 0 \tag{15}$$

This depends on the terminal condition (see below).

A Recipe

Step 4. A **solution** is the a set of functions $[c(t), k(t), \mu(t)]$ which satisfy

- the FOCs
- the law of motion for the state
- the boundary / transversality conditions

Example: Growth Model

$$\max \int_0^\infty e^{-\rho t} u(c(t)) dt \tag{16}$$

subject to

$$\dot{k}(t) = f(k(t)) - c(t) - \delta k(t)$$

$$k(0) \text{ given}$$
(17)

Growth Model: Hamiltonian

$$H(k, c, \mu) = e^{-\rho t} u(c(t)) + \mu(t) [f(k(t)) - c(t) - \delta k(t)]$$
 (19)

Necessary conditions:

$$H_c = e^{-\rho t} u'(c) - \mu = 0$$

$$H_k = \mu \left[f'(k) - \delta \right] = -\dot{\mu}$$

Growth Model

Substitute out the co-state:

$$\dot{\mu} = e^{-\rho t} u''(c) \dot{c} - \rho \mu \tag{20}$$

$$\dot{c} = \frac{\dot{\mu} + \rho \mu}{e^{-\rho t} u''(c)} \tag{21}$$

$$= -\left(f'(k) - \delta - \rho\right) \frac{u'(c)}{u''(c)} \tag{22}$$

Solution: c_t, k_t that solve Euler equation and resource constraint, plus boundary conditions.

Details

First order conditions are necessary, not sufficient.

They are necessary only if we assume that

- 1. a continuous, interior solution exists;
- 2. the objective function v and the constraint function g are continuously differentiable.

Acemoglu (2009), ch. 7, offers some insight into why the FOCs are necessary.

Details

If there are multiple states and controls, simply write down one FOC for each separately:

$$\begin{array}{rcl} \delta H/\delta c_i &=& 0 \\ \partial H/\partial k_j &=& -\dot{\mu}_j \end{array}$$

There is a large variety of cases depending on the length of the horizon (finite or infinite) and the kinds of boundary conditions.

Each has its transversality condition (see Leonard and Van Long 1992).

Next steps

Typical useful next things to do:

- 1. Eliminate μ from the system. Obtain two differential equations in (c,k).
- 2. Find the steady state by imposing $\dot{c} = \dot{k} = 0$.

Sufficient conditions

First-order conditions are sufficient, if the programming problem is **concave**.

This can be checked in various ways.

Sufficient conditions I

The objective function and the constraints are concave functions of the controls and the states.

▶ The co-state must be positive.

This condition is easy to check, but very stringent.

In the growth model:

- $\boldsymbol{v}(c)$ is concave in c (and, trivially, k)
- ► $f(k) \delta k c$ is concave in c and k
- $\mu = u'(c) > 0$

Sufficient Conditions II

First-order conditions are sufficient, if the Hamiltonian is concave in controls and states, where the co-state is evaluated at the optimal level.

This, too is very stringent.

Sufficient Conditions III

Arrow and Kurz (1970)

First-order conditions are sufficient, if the *maximized* Hamiltonian is concave in the states.

Maximized Hamiltonian:

Substitute controls and co-states out, so that the Hamiltonian is only a function of the states.

This is less stringent and by far the most useful set of sufficient conditions.

Discounting: Current value Hamiltonian

Problems with discounting

Current utility depends on time only through an exponential discounting term $e^{-\rho t}$.

The generic discounted problem is

$$\max \int_0^T e^{-\rho t} v[k(t), c(t)] dt \tag{23}$$

subject to the same constraints as above.

Applying the Recipe

$$H(t) = e^{\rho t} v(k,c) + \hat{\mu} g(k,c)$$
 (24)

$$\frac{\partial H}{\partial c_t} = 0 \implies e^{-\rho t} v_c(k_t, c_t) = -\hat{\mu}_t g_c(k_t, c_t)$$
 (25)

$$\frac{\partial H}{\partial k_t} = e^{-\rho t} v_k(k_t, c_t) + \hat{\mu}_t g_k(k_t, c_t) = -\dot{\hat{\mu}}_t \tag{26}$$

Applying the Recipe

Let

$$\mu_t = e^{\rho t} \hat{\mu}_t \tag{27}$$

and multiply through by $e^{\rho t}$:

$$v_c(t) = -\mu_t g_k(t)$$

This is the standard FOC, but with μ instead of $\hat{\mu}$.

Applying the Recipe

$$v_k(t) + e^{\rho t} \hat{\mu}_t g_k(t) = -e^{\rho t} \hat{\mu}_t \tag{28}$$

Substitute out $\hat{\mu}_t$ using

$$\dot{\mu}_t = rac{de^{
ho t}\hat{\mu}_t}{dt} =
ho \mu_t + e^{
ho t}\dot{\hat{\mu}}_t$$

we have

$$v_k(t) + \mu_t g_k(t) = -\dot{\mu}_t + \rho \mu_t$$

This is the standard condition with an additional $\rho\mu$ term.

Shortcut

We now have a shortcut for discounted problems.

Hamiltonian (drop the discounting term):

$$H = v(k,c) + \mu g(k,c) \tag{29}$$

FOCs:

$$\partial H/\partial c = 0 \tag{30}$$

$$\partial H/\partial k = \underbrace{\mu(t)\rho}_{\text{added}} - \dot{\mu}(t) \tag{31}$$

and the TVC

$$\lim_{T \to \infty} e^{-\rho T} \mu(T) k(T) = 0 \tag{32}$$

Equality constraints

Equality constraints of the form

$$h[c(t), k(t), t] = 0$$
 (33)

are simply added to the Hamiltonian as in a Lagrangian problem:

$$H(t) = v(k, c, t) + \mu(t)g(k, c, t) + \lambda(t)h(k, c, t)$$
 (34)

FOCs are unchanged:

$$\frac{\partial H}{\partial c} = 0$$
$$\frac{\partial H}{\partial k} = -\dot{\mu}$$

For inequality constraints:

$$h(c,k,t) \ge 0; \lambda h = 0 \tag{35}$$

Transversality Conditions

Finite horizon: Scrap value problems

The horizon is T.

The objective function assigns a scrap value to the terminal state variable: $e^{-\rho T}\phi(k(T))$:

$$\max \int_{0}^{T} e^{-\rho t} v[k(t), c(t), t] dt + e^{-\rho T} \phi(k(T))$$
 (36)

Hamiltonian and FOCs: unchanged.

The TVC is

$$\mu(T) = \phi'(k(T)) \tag{37}$$

Intuition: μ is the marginal value of the state k.

Scrap value examples

1. Household with bequest motive

$$U = \int_0^T e^{\rho t} u(c(t)) + V(k_T)$$
 (38)

with $\dot{k} = w + rk - c$.

2. Maximizing the present value of earnings

$$Y = \int_0^T e^{-rt} wh(t) [1 - l(t)]$$
 (39)

subject to $\dot{h}(t) = Ah(t)^{\alpha} l(t)^{\beta} - \delta h(t)$ Scrap value is 0. TVC: $\mu(T) = 0$.

The finite horizon TVC with the boundary condition $k(T) \ge k_T$ is $\mu(T) = 0$.

Intuition: capital has no value at the end of time.

But the infinite horizon boundary condition is NOT $\lim_{t\to\infty} \mu(t) = 0$.

The next example illustrates why.

Infinite horizon TVC: Example

$$\max \int_{0}^{\infty} \left[\ln \left(c(t) \right) - \ln \left(c^{*} \right) \right] dt$$

$$subject \ to$$

$$\dot{k}(t) = k(t)^{\alpha} - c(t) - \delta k(t)$$

$$k(0) = 1$$

$$\lim_{t \to \infty} k(t) \ge 0$$

 c^{st} is the max steady state (golden rule) consumption. No discounting - subtracting c^{st} makes utility finite.

Hamiltonian

$$H(k,c,\lambda) = \ln c - \ln c^* + \lambda \left[k^{\alpha} - c - \delta k \right]$$
 (40)

Necessary FOCs

$$H_c = 1/c - \lambda = 0 \tag{41}$$

$$H_k = \lambda \left[\alpha k^{\alpha - 1} - \delta \right] = -\dot{\lambda} \tag{42}$$

We show: $\lim_{t\to\infty} c(t) = c^*$ [why?] Limiting steady state solves

$$\dot{\lambda}/\lambda = \alpha k^{\alpha - 1} - \delta = 0$$
$$\dot{k} = k^{\alpha} - 1/\lambda - \delta k = 0$$

Solution is the golden rule:

$$k^* = (\alpha/\delta)^{1/(1-\alpha)} \tag{43}$$

Verify that this max's steady state consumption.

Implications for the TVC...

$$\lambda(t) = 1/c(t)$$
 implies $\lim_{t\to\infty} \lambda(t) = 1/c^*$.

Therefore, neither $\lambda(t)$ nor $\lambda(t)k(t)$ converge to 0.

The correct TVC: $\lim_{t\to\infty} H(t) = 0$.

The only reason why the standard TVC does not work: there is no discounting in the example.

Infinite horizon TVC: Discounting

With discounting, the TVC is easier to check.

Assume:

the objective function is $e^{-\rho t}v[k(t),c(t)]$ it only depends on t through the discount factor v and g are weakly monotone Then the TVC becomes

$$\lim_{t \to \infty} e^{-\rho t} \mu(t) k(t) = 0 \tag{44}$$

where μ is the costate of the current value Hamiltonian.

This is exactly analogous to the discrete time version

$$\lim_{t \to \infty} \beta^t u'(c_t) k_t = 0 \tag{45}$$

$$\max \int_{0}^{\infty} e^{-\rho t} u(y(t)) dt \qquad (46)$$

$$subject \ to \qquad (47)$$

$$\dot{x}(t) = -y(t) \qquad (48)$$

$$x(0) = 1 \qquad (49)$$

$$x(t) \ge 0 \qquad (50)$$

Current value Hamiltonian Necessary FOCs

FOC

Therefore:

$$\mu(t) = \mu(0)e^{\rho t} \tag{51}$$

$$y(t) = u'^{-1} [\mu(0) e^{\rho t}]$$
 (52)

Solution

The optimal path has $\lim x(t) = 0$ or

$$\int_0^\infty y(t) dt = \int_0^\infty u'^{-1} \left[\mu(0) e^{\rho t} \right] dt = 1$$
 (53)

This solves for $\mu(0)$.

TVC for infinite horizon case:

$$\lim e^{-\rho t} \mu(0) e^{\rho t} x(t) = 0$$
 (54)

Equivalent to

$$\lim x(t) = 0 \tag{55}$$

Reading

- ► Acemoglu (2009), ch. 7. Proves the Theorems of Optimal Control.
- Barro and Martin (1995), appendix.
- ► Leonard and Van Long (1992): A fairly comprehensive treatment. Contains many variations on boundary conditions.

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