# Perpetual Youth Model

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# Perpetual youth

- ▶ The standard growth model is very tractable.
- ▶ But it has an important limitation: all households are identical.
- ▶ For some questions, it is important to have households of **different ages**:
  - ▶ fiscal policies that redistribute across ages
  - ▶ models with life-cycle features: job search, matching, ...
- ▶ An analytically tractable version of the OLG model is the Blanchard-Yaari model of perpetual youth.

# Demographics

- At $t = 0$, there are $L(0) = 1$ identical persons.
  - They are all newborns.
- At each instant, $nL(t)$ identical persons are born.
- Each person dies at each instant with **Poisson** probabilty $\nu$.
- The population growth rate is $n - \nu > 0$:

$$L(t) = \exp([n - \nu]t) \tag{1}$$

# Poisson Process

- The Poisson process is the continuous time analog of i.i.d.
- It is a counting process: it describes the distribution of the number of events occurring during a particular time interval.
- It is a one-parameter distribution, characterized by the arrival rate $\nu$.
- The probability of **no event** over a period of length $\tau$ is $\exp(-\nu\tau)$.
- At each instant, fraction $\nu$ of the population experiences an event.

# Demographics

- The mass of persons at $t$ aged $t - \tau$ is

$$
\begin{aligned}
L(t|\tau) &= \exp(-v(t-\tau)) \times n \exp((n-v)\tau) \\
&= \Pr(\text{live beyond } t-\tau) \; nL(\tau)
\end{aligned}
$$

- Notation: $x(t|\tau)$ means $x$ at $t$ for those born at $\tau$.

# Preferences

- Households are indexed by $i$.
- Conditional on surviving, households utility at date $t$ is $e^{-\rho t} \ln(c_i(t))$.
- The probability of being alive after $t$ "periods" is $\exp(-vt)$.
- Expected utility for date $t$ is $e^{-vt} e^{-\rho t} \ln(c_i(t))$.
- Expected lifetime utility is

$$\int_0^\infty e^{-(\rho+v)t} \ln(c_i(t)) \, dt \qquad (2)$$

- Interesting: mortality simply increases the discount factor: $\rho + v$.

# Technology

- The resource constraint is

$$\dot{K} + C = F(K, L) - \delta K$$

- In per capita terms

$$\dot{k} = f(k) - c - (n - v + \delta)k \tag{3}$$

- $k = K/L$ is capital per capita and capital per worker.

# Markets

Competitive markets for

- goods (numeraire)
- labor rental: $w$
- capital rental: $q$
- annuities...

# Annuities

- The problem: what to do with the wealth of households who die?

    - "accidental bequests"

- Assumption: households buy fair **annuities**.

- Each cohort $\tau$ household gives $a(t|\tau)$ to the insurance company.

- He gets paid:

    1. interest $r(t)a(t|\tau)$
    2. an equal share of accidental bequests of his own cohort:

$$z(a(t|\tau)|t,\tau) = va(t|\tau) \qquad (4)$$

- Effectively, the interest rate, conditional on survival, is $r(t) + v$.

# Firms

- A representative firm solves the standard problem.
- Factor prices are

$$
\begin{aligned}
q &= f'(k) \\
w &= f(k) - f'(k)k
\end{aligned}
$$

# Equilibrium

### Definition

A CE is an allocation $[K(t), L(t), C(t), c(t|\tau), a(t|\tau)]_{t=0, \tau \leq t}^{\infty}$ and a price system $[w(t), q(t), r(t)]$ such that:

1. $c(t|\tau)$ and $a(t|\tau)$ solve the household's problem for cohort $t - \tau$.
2. $w(t)$ and $q(t)$ solve the firm's problem.
3. markets clear.
4. identities: $L(t), C(t)$, $r(t) = q(t) - \delta$

# Equilibrium

Market clearing:

- labor: implicit
- capital: $K(t) = \int_0^t L(t|\tau) a(t|\tau) d\tau$.
- goods: same as resource constraint.

Identities:

- $C(t) = \int_0^t L(t|\tau) c(t|\tau) d\tau$ etc

# Math Digression: Leibniz's Rule
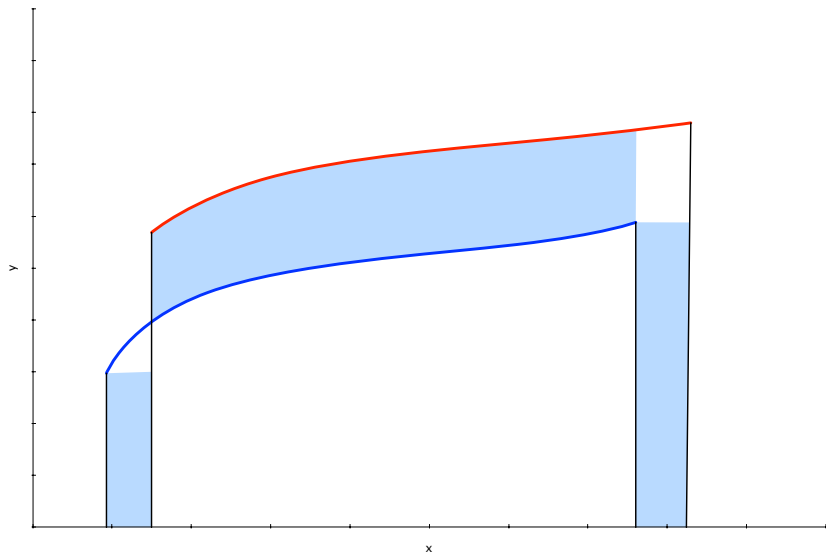
We want to differentiate an integral

Given

$$F(\theta) = \int_{a(\theta)}^{b(\theta)} f(x, \theta)\, dx \tag{5}$$

We have

$$\frac{\partial F}{\partial \theta} = f(b(\theta), \theta) b'(\theta) - f(a(\theta), \theta) a'(\theta) + \int_{a(\theta)}^{b(\theta)} f_{\theta}(x, \theta)\, dx \tag{6}$$

# Leibniz's Rule

# Households

The representative member of cohort $\tau$ solves

$$\max \int_{\tau}^{\infty} e^{-(\rho+v)(t-\tau)} \ln(c(t|\tau))\, dt$$

subject to

$$\dot{a}(t|\tau) = [r(t)+v]a(t|\tau) - c(t|\tau) + w(t) \tag{7}$$

# Lifetime Budget Constraint

$$a(t|\tau) = a(\tau|\tau) D_{\tau,t} + \int_\tau^t D_{z,t} [w(z) - c(z|\tau)] dz \qquad (8)$$

Define the discount factor

$$D_{t,\tau} = \exp\left( -\int_\tau^t [r(z) + v] dz \right) \qquad (9)$$

Note that

- $D_{t,\tau}$ discounts a date $t$ payment to $\tau$.
- $D_{\tau,t} = \exp\left(\int_\tau^t [r(z) + v] dz\right)$ discounts a date $\tau$ payment to $t$.

# Lifetime Budget Constraint: Proof

Take the derivative of $a(t|\tau)$:

$$\dot{a}(t|\tau) = a(\tau|\tau)\frac{\partial D_{\tau,t}}{\partial t} + D_{t,t}[w(t) - c(t|\tau)] + \int_\tau^t \frac{\partial D_{z,t}[w(z) - c(z|\tau)]}{\partial t}dz$$

(10)

and note that

$D_{t,t} = \exp(0) = 1$

and

$\frac{\partial D_{\tau,t}}{\partial t} = D_{\tau,t}[r(t) - v]$

# Household solution

This is a standard problem with Euler equation

$$\frac{\dot{c}(t|\tau)}{c(t|\tau)} = [r(t) + v] - [\rho + v] = r(t) - \rho \tag{11}$$

budget constraint and TVC

$$\lim_{t \to \infty} D_{t,\tau} a(t|\tau) = 0 \tag{12}$$

# Household: PIH

- Claim: because of log utility, the household consumes a constant fraction of "wealth:"

$$c(t|\tau) = (\rho + \nu)\left[a(t|\tau) + \omega(t)\right] \qquad (13)$$

- Human wealth for all alive at $t$ is the same:

$$\omega(t) = \int_t^\infty D_{s,t} w(s)\, ds \qquad (14)$$

# Proof: PIH

- Present value budget constraint:

$$\int_\tau^t D_{\tau,t} c(t|\tau) dt = a(t|\tau) + \omega(t) \qquad (15)$$

- Integrate Euler:

$$c(t|\tau) = c(\tau|\tau) \exp\left(\int_\tau^t [r(z) - \rho]\, dz\right) \qquad (16)$$

- Verify by differentiating and comparing with Euler

# Proof: PIH

- Multiply both sides by $D_{t,\tau}$:

$$D_{t,\tau} c(t|\tau) = c(\tau|\tau) \exp \left( \int_{\tau}^{t} [r(z) - \rho - r(z) + v] dz \right) \qquad (17)$$

$$= c(\tau|\tau) \exp \left( -[\rho - v][t - \tau] \right) \qquad (18)$$

- Present value of consumption

$$\int_{\tau}^{\infty} D_{t,\tau} c(t|\tau) = \frac{c(\tau|\tau)}{\rho - v} = a(\tau|\tau) + \omega(\tau) \qquad (19)$$

- Last step: show that the same holds for $t > \tau$...

# Aggregation

$$
\begin{aligned}
c(t) &= \int_{-\infty}^{t} L(t,\tau)c(t|\tau)d\tau/L(t) & (20) \\
&= (\rho + \nu)[a(t) + \omega(t)] & (21)
\end{aligned}
$$

▶ This is a form of **aggregation**: Aggregate consumption
  behaves like individual consumption.
  ▸ As if a single individual made the choice.
▶ The budget constraint aggregates in the same way.
▶ How general is this?

# Equilibrium Dynamics

It would be tempting to say:

- Euler is unchanged relative to growth model
- Resource constraint is unchanged
- Everything behaves like the growth model

Why does this not work?

# Equilibrium Dynamics

- We have a system in $c, a, \omega$.
- Equations: consumption function, budget constraint, def of lifetime wealth:

$$
\begin{aligned}
c(t) &= (\rho + v)[a(t) + \omega(t)] \\
\dot{a}(t) &= (r(t) - (n - v))a(t) + w(t) - c(t) \\
\omega(t) &= \int_t^\infty \exp\left(-\int_t^s [r(\iota) + v]\,d\iota\right) w(s)\,ds
\end{aligned}
$$

- The strategy: Derive an Euler equation for aggregate consumption by differentiating the $c(t)$ equation.

# Equilibrium Dynamics

- Differentiate the consumption function:

$$\dot{c} = (\rho + v)[\dot{a} + \dot{\omega}] \tag{22}$$

- Sub in budget constraint for $\dot{a}$.
- Differentiate def of $\omega$ (Leibniz's rule - next slide):

$$\dot{\omega}(t) = (r(t) + v)\omega(t) - w(t) \tag{23}$$

- Sub that into $\dot{c}$ and collect terms:

$$\dot{c}(t) = [r(t) - \rho]c(t) - (\rho + v)na(t) \tag{24}$$

- Sub in $k(t) = a(t)$ and the firm foc for $r(t)$:

$$\frac{\dot{c}(t)}{c(t)} = f'(k(t)) - \delta - \rho - (\rho + v)n\frac{k(t)}{c(t)} \tag{25}$$

# Note: Differentiating $\omega(t)$

$$\omega(t) = \int_t^\infty \exp\left(-\int_t^s [r(\iota) + v]\, d\iota\right) w(s)\, ds \qquad (26)$$

$\dot{\omega}(t)$ has 2 pieces:

1. Effect of changing lower bound of integral is integrand evaluated at $s = t$: $w(t)$.

2. Derivative of integrand w.r.to $t$:
   $-[r(t) + v]\,\omega(t) = \int_t^\infty w(s) \frac{d}{dt} \exp\left(-\int_t^s [r(\iota) + v]\, d\iota\right) ds$.

Now note that
$\frac{d}{dt} \exp\left(-\int_t^s [r(\iota) + v]\, d\iota\right) = \exp\left(-\int_t^s [r(\iota) + v]\, d\iota\right) \times [-(r(t) + v)]$.

# Intuition for $\omega(t)$

- Think of human wealth as an asset with price $\omega(t)$.
- Its instantaneous payoff consists of:
  1. "dividend" $w(t)$
  2. capital gain $\dot{\omega}(t)$
- The asset price equals [required rate of return] $\times$ [dividend + capital gain]
- Required rate of return is $r(t) + \nu$.

# Phase diagram

$$\frac{\dot{c}(t)}{c(t)} = f'(k(t)) - \delta - \rho - (\rho + v)n\frac{k(t)}{c(t)} \tag{27}$$

$$\dot{k} = f(k) - c - (n - \delta - v)k \tag{28}$$

with boundary conditions $k(0)$ given and TVC (which is not so obvious...)

This looks a lot like a standard growth model...

# Steady state

$\dot{c} = 0 \implies$

$$c = \frac{(\rho + \nu)n}{f'(k) - \delta - \rho}k \tag{29}$$

Properties:

1. $k \longrightarrow 0 \implies c \longrightarrow 0$ [as $f' \longrightarrow \infty$]
2. $k \longrightarrow k^{MGR}$ where $f'(k^{MGR}) = \delta + \rho \implies c \longrightarrow \infty$
3. $c''(k) > 0$ [verify]

# Steady state

$\dot{k} = 0 \implies$

$$c = f(k) - (n + \delta - \nu)k \tag{30}$$

Properties: as the standard growth model.

# Steady state

Solution for steady state $k^*$

$$\frac{f(k^*)}{k^*} - (n - v + \delta) - \frac{(\rho + v)n}{f'(k^*) - \delta - \rho} = 0 \tag{31}$$

Unique steady state $k^*$: $f(k)/k \searrow$ in $k$. $-1/f'(k) \searrow$ in $k$.

# Dynamic efficiency

- **Golden Rule** maximizes

$$c^* = f(k^*) - (n + \delta - \nu)k^* \qquad (32)$$
$$f'(k_{GR}) - \delta = n - \nu \qquad (33)$$

- Steady state:

$$f'(k^*) - \delta > \rho \qquad (34)$$

[otherwise $c/k < 0$]

- There can be overaccumulation relative to the Golden Rule.
- This happens when households are sufficiently impatient (high $\rho$).
- Similar to the finite lifetime OLG model.

# Dynamic efficiency

- **Modified Golden Rule** for planner with discount factor $\rho$ [effects of mortality and "annuities" cancel]:

$$f'(k_{MGR}) - \delta = \rho \qquad (35)$$

- Equilibrium avoids overaccumulation relative to MGR.
- This is not a robust feature of the model.
- Giving households a stronger motive to save for "old age" can lead to overaccumulation.
- Example: labor efficiency declines with age.

# Dynamic efficiency

- ▶ Finite lifetimes are not necessary to generate overaccumulation.
- ▶ In this model, it is the presence of overlapping generations that destroys the welfare theorems.

# Phase diagram
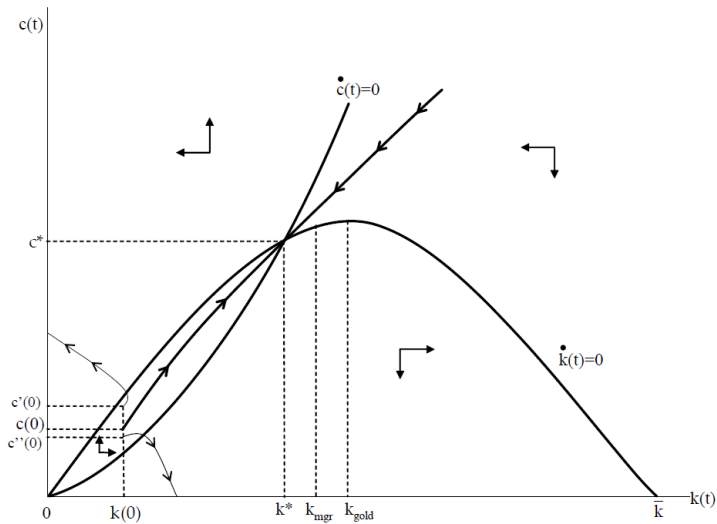
# Phase diagram

- The dynamics closely resemble the growth model.
- A unique, globally saddle path stable steady state exists.
- Convergence is monotone.
- An analytically tractable model with OLG.

## Where Is This Used?

Models of human capital

- ▶ combine the convenience of an infinitely lived decision maker
- ▶ capture that only young invest in education
- ▶ Akyol and Athreya (2005)

Models of income / wealth distribution

- ▶ a version of perpetual youth: agents age stochastically
- ▶ Castaneda et al. (2003)

# Reading

- Acemoglu (2009), ch. 9.7-9.8.
- Blanchard and Fischer (1989), ch. 3.3

# References I

Acemoglu, D. (2009): *Introduction to modern economic growth*, MIT Press.

Akyol, A. and K. Athreya (2005): "Risky higher education and subsidies," *Journal of Economic Dynamics and Control*, 29, 979 – 1023.

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Castaneda, A., J. Diaz-Gimenez, and J. V. Rios-Rull (2003): "Accounting for the US earnings and wealth inequality," *Journal of political economy*, 111, 818–857.