Stochastic Two Period OLG Model

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Introduction

We build up towards solving a stochastic, heterogeneous agent OLG model like Huggett (1996).

Building on the deterministic model, we change:

- Households draw random labor endowments at each age: $e^a \in \left\{e_1^a, ..., e_{n_w}^a\right\}$.
- Endowment draws are independent over time: $\Pr(e^a = e_j^a) = P_e^a(j)$.
- Incomes received are then: $y_j^a = e_j^a \, w^a$.

Young wage rate is determined by firms' marginal product of labor.

Old "wage rate" is set by government as a transfer.

Household

$$\max\ u(c_t^y) + E_t\, \beta\, u(c_{t+1}^o)$$

subject to

$$c_{t+1}^o - y_{t+1}^o = (1 + r_{t+1})\,(y_t^y - c_t^y) \tag{1}$$

where

$$E_t\, u\left(c_{t+1}^o\right) = \sum_{j=1}^{n_w} P_e^o(j)\, u\left(c_{t+1}^o\left(y_j^o\right)\right) \tag{2}$$

Timing: Realizations of y are known before consumption is chosen.

Lagrangean

$$\Gamma(y_t^y) = \max\, u(c_t^y) + \beta\, E_t\, u\left([1 + r_{t+1}]\,[y_t^y - c_t^y] + y_{t+1}^o\right)$$

Euler equation

$$u'(c_t^y) = \beta\,(1 + r_{t+1})\, E_t\, u'\left([1 + r_{t+1}]\,[y_t^y - c_t^y] + y_{t+1}^o\right) \tag{3}$$

Example: Certainty Equivalence: $u(c) = a\,c - b\,c^2$

Euler equation becomes

$$a - bc_t^y = \beta\,(1 + r_{t+1})\,E_t\left\{a - bc_{t+1}^o\right\} \tag{4}$$

$$a - bc_t^y = \beta\,(1 + r_{t+1})\;[a - b\,(1 + r_{t+1})\,(y_t^y - c_t^y) - bE_t\,\{y^o\}] \tag{5}$$

Rearrange

$$c_t^y = \frac{[1 - \beta\;(1 + r)]\,a/b + \beta\;(1 + r)^2\,E_t\,\{W_t\}}{1 + \beta\;(1 + r)^2}$$

Implications:

- *Certainty equivalence*: Consumption only depends on the expected value of lifetime income, $E_t\,\{W_t\}$.
- Marginal propensity to consume is the same out of present income and out of expected future income.
- Consumption itself is a martingale.

Computing the Household Problem

For this model, the method we used for the deterministic case still works.

Search over values of s to find the zero of the Euler equation

$$u'\left(y^{y}-s\right)=\beta\,[1+r]\sum_{j}P_{e}^{o}(j)\,u'\left(y_{j}^{o}+[1+r]\,s\right)$$

Result: Saving function $s\left(y^{y}\right)$. This is a $(1\times n_{w})$ vector.

For multi-period models there are two complications:

- the consumption function for $t+1$ is not known (but can be found by backward induction).
- wealth is an additional, continuous state variable (this is handled by interpolation).

Household Dynamic Program

For more complicated models it is convenient to write the household problem recursively:

$$V^o(y^o, s) = u\left(y^o + [1+r]\ s\right) \tag{6}$$

$$V^y\left(y^y\right) = \max\ u(y^y - s) + \beta\, E\left\{V^o(y^o, s)\right\} \tag{7}$$

First-order conditions

$$V^o_s(y^o, s) = [1+r]\ u'\left(y^o + [1+r]\ s\right) \tag{8}$$

$$u'\left(y^y - s\right) = \beta\, E\left\{V^o_s(y^o, s)\right\} \tag{9}$$

The end result is the same.

Exercise: Pseudo code for Household Problem

Algorithm for Household Problem

Take as given: parameters, prices.

1. Calculate marginal utility when old from the budget constraint:
   $MU^o(y^o, s) = u'(y^o + [1 + r]\ s)$
   To compute this, approximate on a grid for wealth: $s \in \{s_1, ..., s_{n_s}\}$.
   Then MU^o is a $(n_w \times n_s)$ matrix.

2. Compute expected marginal utility for every wealth level:
   $EMU^o(s_i) = \sum_{j=1}^{n_w^o} P_e^o(j)\, MU^o(y^o, s_i)$.
   EMU^o is a $(1 \times n_s)$ vector.

3. Search for a zero of the Euler equation deviation

$$y^y - s = (u')^{-1}(\beta [1 + r]\, EMU^o(s)) \tag{10}$$

   `hh_solve_olg2s.m`

Note: The only change from the deterministic case is that marginal utility is replaced by EMU^o.

Stationary Equilibrium

Objects:

- Scalars: G, K, L, r, w.
- Policy functions: $c^y(y^y), s(y^y), c^o(s, y^o)$.
- A distribution of households over types; density $\Lambda(y^y, y^o)$.

Steady state conditions:
In this simple model the density is exogenous: $\Lambda\left(y_j^y, y_i^o\right) = P_e^y(j) \cdot P_e^o(i)$.
Policy functions solve the household problem.
Firm:

$$r = (1 - \tau_r)\ F_K(K, L) - \delta \tag{11}$$

$$w = (1 - \tau_w)\ F_L(K, L) \tag{12}$$

Market clearing:

$$L = N \sum_j P_e^y(j)\ e_j^y \tag{13}$$

$$K\ (1 + n) = N \sum_j P_e^y(j)\ s\left(y_j^y\right) \tag{14}$$

$$G + \sum_j P_e^o(j)\ y_j^o N/(1 + n) = \tau_w F_L(K, L)L + \tau_r F_K(K, L)K \tag{15}$$

Steady State in Per Capita Terms

Scalars: $g = G/N, k = K/L, l = L/N, r, w$.
Steady state conditions:

$$r = (1 - \tau_r)\ f'(k) - \delta \tag{16}$$

$$w = (1 - \tau_w)\ [f(k) - f'(k)\, k] \tag{17}$$

$$l = \sum_j P_e^y(j)\ e_j^y \tag{18}$$

$$s(y^y) = y^y - c^y(y^y) \tag{19}$$

$$c^o(s, y^o) = y^o + (1 + r)\, s \tag{20}$$

$$k\, l\ (1 + n) = \sum_{j=1}^{n_e^y} P_e^y(j)\ s\left(y_j^y\right) \tag{21}$$

$$g + (1 + n)^{-1} \sum_{j=1}^{n_e^o} P_e^o(j)\ y_j^o = \tau_w\ [f(k) - f'(k)\, k]\ \ l + \tau_r\, f'(k)\ k\ l \tag{22}$$

Note: l is exogenous.

Exercise: Pseudo-code for Algorithm

Algorithm

Guess k.

Compute r, w from steady state conditions.

Solve household problem for policy functions for every e_j^y.

Compute aggregate k from household savings.

Iterate until deviation from capital market clearing is close to zero.

Program: `bg_comp_olg2s.m`.

Computing Aggregate Capital

Direct method: In this simple model, the distribution of savings is known. Aggregate saving can be computed directly as

$$k\ l\,(1+n) = \sum_{j=1}^{n_e^y} P_e^y\,(j)\ s\left(y_j^y\right)$$

Simulation method: In multi-period models, we don't know the distribution of savings.

Then we simulate a large number of households, starting from age 1 forward.

Compute aggregate wealth as: [mean of savings over all simulated households] $\times$ [mass of households].

Calibration Algorithm

Fix k at the value implied by the target level for K/Y: $k = (AK/Y)^{1/(1-\alpha)}$

Compute r, w from marginal products.

Guess β.

Solve household problem for $s(y^y)$.

Iterate until deviation from capital market clearing is small.

Program: `cal_comp_olg2s.m`.

Parameters

By default, parameters are the same as in the deterministic model.

Labor endowments: Should be calibrated to match earnings processes. For now: set to arbitrary values.

$e^a \in \{0.5, 1, 1.5\}$.

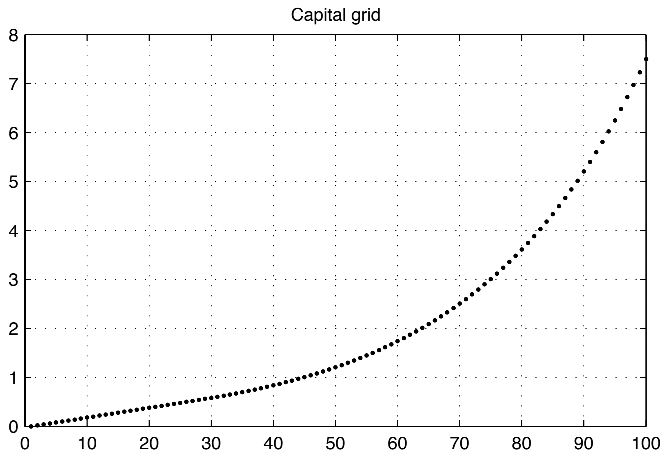
$P_e^a = [0.3, 0.4, 0.3]$.

Capital grid: Grid must be tight at low s where marginal utility may be highly nonlinear.

Number of grid points must be large enough so that approximation errors are small: $n_k = 100$.

`set_kgrid_olg2s.m` sets a capital grid with constants steps for low s and constant exponential steps for high s.

Capital Grid



Calibration Results

```
>> cal_comp_olg2sS(1, 0, 111);

Deviation from calibration targets: 0.000000
beta = 2.284. Annual beta = 1.028
k = 0.151.
```

Verify that steady state computation returns desired K/Y:

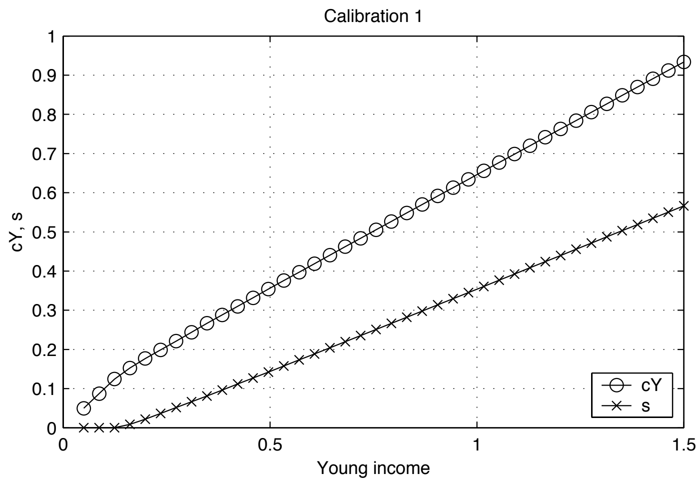
```
>> bg_comp_olg2sS(1, 0, 111);

Steady state. calNo = 1. expNo = 0.
k = 0.151. y = 1.563.
r = 3.322. w = 1.000.
cY/wY = 0.645.    s/y = 0.227
Deviation = 0.000000
Steady state K/Y = 0.097. Target: 0.097.
```

Verify that the model replicates the certainty case: `calNo = 50`.

Consumption Function

Solve household problem for various levels of young income.

`show_cons_fct_olg2s.m`



Consumption function is nearly linear.

References

Huggett, M. (1996): "Wealth distribution in life-cycle economies," *Journal of Monetary Economics*, 38, 469–494.