

# AK Model: Phase Diagram

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Econ720

October 24, 2016

# Introduction

We study an endogenous growth model with transitional dynamics.

The model is asymptotically AK.

As an example of a phase diagram with endogenous growth.

# The Model

- ▶ We modify the  $Ak$  model's production function:

$$H(K, L) = AK + F(K, L) \quad (1)$$

- ▶ In intensive form

$$h(k) = Ak + f(k)$$

where  $F(K, L) = Lf(k)$  satisfies Inada conditions and has constant returns to scale in  $K$  and  $L$  jointly.

- ▶ For simplicity, assume  $f(k) = k^\alpha$  with  $\alpha < 1$ .

# Equilibrium

- ▶ The only change to the equilibrium conditions of the  $Ak$  model: the marginal product of capital is not  $A$  but

$$H_K(K, L) = A + F_K(K, L) = A + f'(k) \quad (2)$$

- ▶ Laws of motion:

$$\dot{k} = (A + f(k)/k - \delta - n)k - c \quad (3)$$

$$g(c) = (A + f'(k) - \delta - \rho)/\sigma \quad (4)$$

- ▶ Asymptotically,  $f'(k) \rightarrow 0$  and the model becomes  $Ak$ .

# Phase Diagram with Endogenous Growth

How to draw a phase diagram when  $c$  and  $k$  grow at endogenous rates?

Answer: Find ratios that are constant asymptotically:

- ▶  $z = h(k)/k$

- ▶  $x = c/k$ .

Equivalently, **detrend** the model and then draw the phase diagram.

## Laws of motion

$$g(z) = g(h(k)) - g(k) \quad (5)$$

$$g(x) = g(c) - g(k) \quad (6)$$

- ▶ We therefore need to find expressions for  $g(h(k))$ ,  $g(k)$ , and  $g(c)$  in terms of  $z$  and  $x$  only.
- ▶ First rewrite (3) as

$$g(k) = [h(k)/k - \delta - n] - c/k \quad (7)$$

$$= z - \delta - n - x \quad (8)$$

# Laws of motion

► Next,  $g(c) = [h'(k) - \delta - \rho] / \sigma$ .

► We need to replace  $h'(k)$ .

► Note that

$$h'(k) = A + \alpha f(k)/k = A + \alpha (z - A) = \alpha z + (1 - \alpha)A$$

► Use this to rewrite (4) as

$$g(c) = \frac{\alpha z + (1 - \alpha)A - \delta - \rho}{\sigma}$$

# Laws of motion

Finally,

$$g(h(k)) = \frac{h'(k)k}{h(k)} g(k) = \frac{\alpha z + (1 - \alpha)A}{z} g(k)$$



## Laws of motion

$$\begin{aligned}g(z) &= g(h(k)) - g(k) \\&= \left[ \frac{\alpha z + (1 - \alpha)A}{z} - 1 \right] [z - x - n - \delta] \\&= (1 - \alpha)(A/z - 1)[z - x - n - \delta]\end{aligned}$$

and

$$\begin{aligned}g(x) &= g(c) - g(k) \\&= \frac{\alpha z + (1 - \alpha)A - \rho - \delta}{\sigma} - z + x + n + \delta \\&= \varphi + x + z(\alpha/\sigma - 1)\end{aligned}$$

where  $\varphi = n + \delta + (1 - \alpha)A/\sigma - (\rho + \delta)/\sigma$ .

## Steady state

In steady state,

$$z_{ss} = A \quad (9)$$

or

$$z = x + n + \delta \quad (10)$$

and

$$x_{ss} = (1 - \alpha/\sigma)z_{ss} - \varphi \quad (11)$$

We will see in a minute that the relevant condition is  $z_{ss} = A$ .

## $\dot{x} = 0$ Locus

The  $\dot{x} = 0$  locus:

$$x = -\varphi + (1 - \alpha/\sigma)z \quad (12)$$

For realistic parameter values (e.g.  $\alpha \simeq 0.3$  and  $\sigma \geq 1$ ), we have  $0 < 1 - \alpha/\sigma < 1$ .

Negative intercept.

Slope  $< 1$ .

## $\dot{z} = 0$ Locus

$\dot{z} = 0$  has two solutions:

- ▶  $z = A$  or
- ▶  $x = z - n - \delta$ .

The first solution is irrelevant because  $z = A/k + f(k)/k > A$ .

Negative intercept

Slope = 1

# Summary

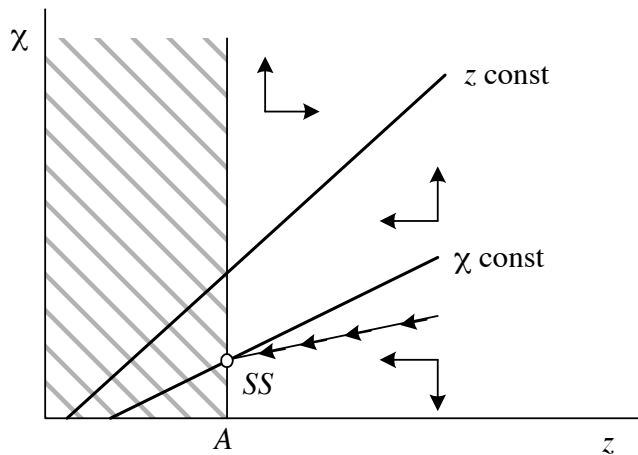
Laws of motion:

$$\begin{aligned}\dot{z} &= (1 - \alpha)(A - z)(z - x - n - \delta) \\ g(x) &= \varphi + x + z(\alpha/\sigma - 1)\end{aligned}$$

$$\begin{aligned}\dot{z} &= 0 : x = z - n - \delta \\ \dot{x} &= 0 : x = -\varphi + (1 - \alpha/\sigma)z\end{aligned}$$

Steady state:  $z = A$ .

# Phase Diagram with Endogenous Growth



# Phase Diagram with Endogenous Growth

- ▶ This system is **saddle-path stable**.
- ▶ If  $x_0$  is too small, then the trajectory crosses into the  $c < 0$  quadrant.
- ▶ If  $x_0$  is too large, then the trajectory takes off to the north-east.
  - ▶ This violates feasibility:  $x = c/k$  would grow without bounds.
- ▶ Both  $x$  and  $z$  converge monotonically to the steady state.

# Summary

The important point is the general approach for dealing with the dynamics of growing economies:

1. Write out the equilibrium conditions as usual.
2. Find conditions characterizing the balanced growth path.
3. Find ratios that are constant on the balanced growth path ( $x$  and  $z$ ).
4. Express the laws of motion of the economy in terms of these ratios.

An alternative approach is to transform the economy into stationary form before characterizing its equilibrium.



# Reading

- ▶ Acemoglu (2009), ch. 11.
- ▶ Krusell (2014), ch. 8.

# References I

Acemoglu, D. (2009): *Introduction to modern economic growth*, MIT Press.

Krusell, P. (2014): "Real Macroeconomic Theory," Unpublished.