Stochastic Growth Model

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Planning solution

The history is of shocks is θ^t.

Preferences:

$$\sum_{t=0}^{\infty} \beta^t \sum_{\theta^t} \Pr(\theta^t|\theta_0)\, u(c[\theta^t]) \tag{1}$$

Technology:

$$\begin{aligned} X &= F(K,L,\theta)+(1-\delta)K-c && (2)\\ K' &= X && (3) \end{aligned}$$

Bellman equation

Define $k = K/L$.

$$V(k,\theta) = \max_{k' \in [0, f(k,\theta)+(1-\delta)k]} u\left(f(k,\theta)+(1-\delta)k-k'\right) \quad (4)$$

$$+\beta E\left[V\left(k',\theta'\right)|\theta\right] \quad (5)$$

First-order conditions

- Verify that A1-A5 hold ... Theorems 1-6 apply.
- FOC
$$u'(c) = \beta E V_k(k', \theta')$$
- Envelope
$$V_k(k, \theta) = u'(c)[f_k(k, \theta) + 1 - \delta]$$
- Euler
$$u'(c) = \beta E\left[u'(c')\left\{f_k(k', \theta') + 1 - \delta\right\}|\theta\right] \tag{6}$$
- Solution: $V(k, \theta)$ and $\pi(k, \theta)$ that "solve" the Bellman equation

Characterization

- Now for the bad news ... there really isn't much one can say about the solution analytically.
- But see Campbell (1994) for a discussion of a log-linear approximation.

Competitive Equilibrium

Competitive equilibrium

The model comes in 2 flavors.

1. Complete markets
   - for every history, there exists an asset that pays in that state of the world
   - the implication is complete risk sharing: all idiosyncratic risks are insured
   - aggregate risks remain
2. Incomplete markets
   - some securities are missing
   - there is no representative agent

Trading arrangements

- With complete markets, date 1 Arrow-Debreu trading is convenient
  - Uncertainty essentially disappears from the model.
- With incomplete markets, it is easiest to specify the set of securities available at each date.
  - Sequential trading.

Complete markets - Arrow Debreu trading

- The environment is standard.
- The history is of shocks is θ^t.
- Trading takes place at date 1.
- The point: This looks like a static model without uncertainty.

Household: Preferences

$$\sum_{t=0}^{\infty} \beta^t \sum_{\theta^t} \Pr(\theta^t | \theta_0) u(c[\theta^t]) \tag{7}$$

Household: budget constraint

- Expenditures in state θ^t:

$$x(\theta^t) = p(\theta^t)\left[c(\theta^t) + s(\theta^t)\right] \tag{8}$$

- $p(\theta^t)$ is the price of the good in state θ^t.
- c is consumption
- s is "saving:" buy goods (capital) and rent to firms.

Household: budget constraint

- Income in state θ^t:

$$y(\theta^t) = w(\theta^t) + R(\theta^t) s\left(\theta^{t-1}\right) \tag{9}$$

- $w(\theta^t)$ is the wage.
- $R(\theta^t)$ is the payoff from renting a unit of the good to the firm.
- Both are state contingent.
- Poor notation: keep in mind that θ^t follows θ^{t-1}

Household: budget constraint

- Lifetime budget constraint:

$$\sum_{t=0}^{\infty} \sum_{\theta^t} \left[y\left(\theta^t\right) - x\left(\theta^t\right) \right] + p\left(\theta_0\right) s_0 = 0 \qquad (10)$$

- s_0 is the initial endowment of goods.
- With complete markets, there is a lifetime budget constraint, even under uncertainty.
    - Because there really is no uncertainty any more.
    - At each node, the household's spending and income are fully predictable.

Firms

- Firms maximize the total value of profits.
  - There is no discounting because of Arrow-Debreu trading.
- Profits in state θ^t:

$$p(\theta^t)[F(K[\theta^t], L[\theta^t], \theta_t) + (1-\delta)K[\theta^t]]$$
$$-R(\theta^t)K(\theta^t) - w(\theta^t)L(\theta^t)$$

- Value of the firm: sum of profits over all states.
- FOCs are standard: since the firm does not own anything, it maximizes profits state-by-state.

Competitive Equilibrium

- Allocation: $c(\theta^t), s(\theta^t), K(\theta^t), L(\theta^t)$.
- Price system: $p(\theta^t), w(\theta^t), R(\theta^t)$ for all histories θ^t.
- These satisfy:
    1. Household optimality.
    2. Firm optimality.
    3. Market clearing:
        - $L(\theta^t) = 1$.
        - $K(\theta^t, \theta_{t+1}) = s(\theta^t)$.
        - Goods market.

Competitive Equilibrium
Comments

- This looks like a static model without uncertainty.
    - Each history defines new goods: output, labor, capital rental.
- The setup is far more complicated than the recursive one.

Risk Sharing

- What if agents are heterogeneous?
- With complete markets, risk is perfectly shared.
- The simplest case: An endowment economy with Arrow-Debreu trading.
- The state is θ^t.

Risk Sharing
Households

- There are I types of households, indexed by i.
- Endowments are $y^i(\theta^t)$.
- Preferences are

$$\sum_t \sum_{\theta^t} \beta^t q(\theta^t) u^i\left(c^i[\theta^t]\right)$$

- Budget constraints:

$$\sum_t \sum_{\theta^t} p(\theta^t)\left[c^i(\theta^t) - y^i(\theta^t)\right] = 0 \tag{11}$$

Risk Sharing
Households

- First-order conditions are as usual:

$$q\left(\theta^t\right)\beta^t \frac{\partial u^i\left(c^i\left[\theta^t\right]\right)}{\partial c^i\left[\theta^t\right]} = \lambda_i p\left(\theta^t\right) \tag{12}$$

where λ_i is the Lagrange multiplier.

Risk Sharing

- Complete risk sharing: For all θ^t the MRS is equated across households:

$$MRS\left(\theta^t, \hat{\theta}^\tau\right) = -\frac{\beta^t \ \partial u^i \left(c^i \left[\theta^t\right]\right) / \partial c^i \left[\theta^t\right]}{\beta^\tau \ \partial u^i \left(c^i \left[\hat{\theta}^\tau\right]\right) / \partial c^i \left[\hat{\theta}^\tau\right]} = \frac{p\left(\theta^t\right) / q\left(\theta^t\right)}{p\left(\hat{\theta}^\tau\right) / q\left(\hat{\theta}^\tau\right)} \tag{13}$$

- Equivalently, the ratio of marginal utilities between 2 agents is the same for all θ^t:

$$\frac{\partial u^i \left(c^i \left[\theta^t\right]\right) / \partial c^i \left[\theta^t\right]}{\partial u^j \left(c^i \left[\theta^t\right]\right) / \partial c^j \left[\theta^t\right]} = \frac{\lambda_i}{\lambda_j} \tag{14}$$

- If households have identical preferences and there is no aggregate uncertainty (the aggregate endowment is the same in all states), then individual consumption is constant.

Sequential Trading

Sequential Trading

- We set up the C.E. with sequential trading.
- If we want complete markets, we need **Arrow securities**.
- Each security, $a\left(\theta^{t+1}\right)$ is indexed by the state of the world in which it pays off: θ^{t+1}.
- The asset is purchased for price $\bar{p}(\theta^t, \theta')$ in state θ^t.
- It pays one unit of consumption if $\theta^{t+1} = [\theta^t, \theta']$.

Household

- Budget constraint:

$$
\begin{aligned}
c(\theta^t) + s(\theta^t) &= w(\theta^t) + a(\theta^t) + R(\theta^t) k(\theta^t) && (15)\\
s(\theta^t) &= \sum_{\theta_{t+1}} \bar{p}(\theta^t, \theta_{t+1}) a(\theta^t, \theta_{t+1}) + x(\theta^t) && (16)\\
k(\theta^t, \theta_{t+1}) &= x(\theta^t) && (17)
\end{aligned}
$$

- Numeraire: consumption at each node θ^t.

Household

- Household problem:

$$\max \sum_{t=0}^{\infty} \beta^t \sum_{\theta^t} \Pr(\theta^t | \theta_0) u(c[\theta^t]) \tag{18}$$

s.t. budget constraints for all θ^t.

Recursive household problem

- State: $(\overrightarrow{a}, k, \theta)$.
  - $\overrightarrow{a}$: holdings of all the $a(\theta)$.
- Given prices: w and $\bar{p}(\theta, \theta')$.
- Bellman equation:

$$V(\overrightarrow{a}, k, \theta) = \max_{c, a'(\theta'), k'} u(c) + \beta \sum_{\theta'} q(\theta'|\theta) V(\overrightarrow{a}', k', \theta')$$

s.t. budget constraint

$$\sum_{\theta'} \bar{p}(\theta, \theta') a'(\theta') + k' + c = w + a(\theta) + Rk$$

First order conditions

- For $a'(\theta')$:

$$u'(c)\bar{p}\left(\theta,\theta'\right)=\beta q\left(\theta'|\theta\right)\frac{\partial V(\overrightarrow{a}'[\theta'],k',\theta')}{\partial a(\theta')} \tag{19}$$

- For k':

$$u'(c)=\beta\sum_{\theta'}q\left(\theta'|\theta\right)\frac{\partial V(\overrightarrow{a}',k',\theta')}{\partial k'} \tag{20}$$

First order conditions

- Envelope:

$$\begin{aligned} \partial V(\overrightarrow{a},k,\theta)/\partial a(\theta) &= u'(c) && (21)\\ \partial V\left(\overrightarrow{a},k,\hat{\theta}\right)/\partial a(\theta) &= 0 && (22)\\ \partial V(\overrightarrow{a},k,\theta)/\partial k &= u'(c)R && (23) \end{aligned}$$

- Euler equation holds state by state for state contingent claims:

$$u'(c)\bar{p}\left(\theta,\theta'\right) = \beta q\left(\theta'|\theta\right)\ u'\left(c\left[a'\left(\theta'\right),\theta'\right]\right) \qquad (24)$$

- Euler equation for capital:

$$\begin{aligned} u'(c) &= \beta\sum_{\theta'} q\left(\theta'|\theta\right)R\left(\theta,\theta'\right)u'\left(c\left[a'\left(\theta'\right),k',\theta'\right]\right) && (25)\\ &= \beta E\ R'\ u'\left(c'\right) \end{aligned}$$

No arbitrage

- Since capital can be replicated by buying a set of Arrow securities:

$$\sum_{\theta'} \bar{p}\left(\theta, \theta'\right) R\left(\theta, \theta'\right) = 1 \tag{26}$$

- Proof: Solve (24) for $q(\theta'|\theta)$ and substitute into (25).

Equilibrium

- We can write down a sequential equilibrium definition, similar to the Arrow-Debreu.
  - Everything is indexed by θ^t.
- More powerful: Recursive Competitive Equilibrium.
  - Everything is a function of the current state.

Recursive CE

- Define an aggregate state vector: $S = (\theta, K)$.
  - In general: we need to keep track of the distribution of (θ_i, k_i) across households.
  - Here: all households are identical.
- The law of motion for the aggregate state:

$$\begin{aligned} \Pr\left(\theta'|\theta\right) &= q\left(\theta'|\theta\right) \\ K' &= G(\theta, K) \end{aligned}$$

where G is endogenous.

Recursive CE

Household

- Given:
  - aggregate state and its law of motion.
  - price functions: $w(S), R(S)$ and $\bar{p}(S, \theta')$.
- Bellman equation:

$$V(\overrightarrow{a}, k, S) = \max_{c, a'(\theta'), k'} u(c) + \beta \sum_{\theta'} q(\theta'|\theta) V\left(\overrightarrow{a}'[\theta'], k', S'\right)$$

s.t. budget constraint

$$\sum_{\theta'} \bar{p}(\theta, \theta') a'(\theta') + k' + c = w(S) + a(\theta) + R(S) k$$

and aggregate law of motion

$$S' = G(S)$$

Recursive CE

- First-order conditions: unchanged.
- Solution: $V(a,k,S)$ and policy functions $c(a,k,S)$, $k' = \kappa(a,k,S)$.

Recursive CE

Firm

- Always the same because the firm has a static problem:
- Solution: $R(S), w(S)$.

Recursive CE

- Equilibrium objects:
  1. Household: Value function and policy functions.
  2. Firm: Price functions.
  3. Aggregate law of motion: $K' = G(\theta, K)$.
- Equilibrium conditions:
  1. Household optimality.
  2. Firm optimality.
  3. Market clearing.
  4. Consistency:

$$G(\theta, K) = \kappa(K, \theta, K) \tag{27}$$

where the household's policy function is $k' = \kappa(k, \theta, K)$.

Recursive CE

- Note: We could toss out all the Arrow securities without changing anything.
- The model boils down to:
  1. Euler equation for K: $u'(c) = \beta E[R'u'(c')]$
  2. Law of motion for K: $K' = F(K,L) + (1-\delta)K - c$.
  3. FOC: $R = F_K(K,L) + 1 - \delta$.
- This changes when individuals are not identical.

Recursive CE

What do we gain?

- Avoid having to carry around infinite histories.
- Equilibrium contains few objects.
  - Especially when the economy is **stationary**.
- All endogenous objects are functions.
  - Results from functional analysis can be used to determine their properties.
- Recursive CE is easy to compute.

Reading

- Acemoglu (2009) ch. 16-17.
- Krusell (2014) ch. 6
- Stokey et al. (1989) discuss the technical details of stochastic Dynamic Programming.
- Ljungqvist and Sargent (2004), ch. 2 talk about Markov chains. Ch. 7 covers complete market economies (Arrow-Debreu and sequential trading). Ch. 6: Recursive CE.
- Campbell (1994) discusses an analytical solution (approximate)

References I

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