Manuelli & Seshadri: Human Capital and the Wealth of Nations

Econ821

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April 8, 2016

Introduction

We write out an organized solution to Manuelli & Seshadri (2014 AER).

Then we think about how to organize the code for this model.

- My version of the code is on github.

A few notes

The model solution in the paper is correct only for the case where the schooling, child care and job training inputs are the same good

- i.e., $\beta = \theta$, $p_S = p_E = p_w$

The reported results are for the case where $\beta < \theta$

- a correction is being worked on

The paper contains about a dozen typos, mostly in the proofs.

The paper contains a tax rate in some places, which is set to 0 in all computations.

Model Elements

Small open economy (r exogenous)

Steady state

Demographics:

- OLG
- lifetime T
- $N(a,t) = \phi(a) e^{\eta t}$
- mass of persons aged a in steady state: $\phi(a) = \eta \frac{e^{-\eta a}}{1-e^{-\eta T}}$

Model Elements

Preferences:

- agents maximize lifetime earnings.

Endowments at birth:

- h_B units of human capital

Endowments each periods:

- 1 unit of time
- can be spent on working or learning

Human capital production

Phase 1: early childhood

$$h_E = h_B x_E^{\nu} \tag{1}$$

Phase 2: schooling:

$$\dot{h}(a) = z_h \left(h(a) n(a)\right)^{\gamma_1} x(a)^{\gamma_2} - \delta_h h(a) \tag{2}$$

- $n = 1$,
- starts at $a = 6$, duration s
- $h(6) = h_E$

Phase 3: job training:

- $n < 1$
- starts at $a = 6 + s$ with $h(6 + s)$ from schooling problem

Output Production

Consumption good:

- output per worker

$$y_c = zF\left(k_c, \bar{h}_c\right) \tag{3}$$

$$F\left(\kappa_c, 1\right) = \left(k_c/\bar{h}_c\right)^{\theta} \tag{4}$$

- h_c: human capital per worker devoted to that sector
- no resource constraint (open economy)
- numeraire
- also the x input during the job training phase $(p_w = 1)$

Human capital good:

- the same technology: $y_s = zF\left(k_s, \bar{h}_s\right) = \bar{x}_s + \bar{x}_e$
- used to produce x_s and x_e

Aggregation

Human capital per worker (labor supply in efficiency units):

$$\bar{h} = \frac{\int_{6+s}^{R} h(a)(1-n(a))\phi(a)\,da}{\bar{\phi}} \tag{5}$$

Mass working:

$$\bar{\phi} = \int_{6+s}^{R} \phi(a)\,da \tag{6}$$

Labor market clearing:

$$\bar{h} = \bar{h}_s + \bar{h}_c \tag{7}$$

Aggregation

$$\bar{x}_s = \int_6^{6+s} x_s(a)\,\phi(a)\,da \tag{8}$$

$$\bar{x}_e = \phi(6)\,x_E \tag{9}$$

Factor Prices

$$p_k (r + \delta_k) = zF_k (\kappa_c, 1) = z\theta \kappa_c^{\theta-1} \tag{10}$$

$$w = zF_h (\kappa_c, 1) = z (1 - \theta) \kappa_c^{\theta} \tag{11}$$

Note: the paper allows for the case where the prices of consumption, x_s, x_E differ.

- since this does not play a role (and the household problem in that case is not correct), we omit this

Household Problem

The household solves

$$\max \int_0^R e^{-r(a-6)} \{wh(a)(1-n(a)) - px(a)\} da - p_E x_E \qquad (12)$$

subject to

$$h(6) = h_E = h_B x_E^{\nu} \qquad (13)$$

$$\dot{h}(a) = z_h (h(a) n(a))^{\gamma_1} x(a)^{\gamma_2} - \delta_h h(a) \qquad (14)$$

$$n(a) \leq 1 \qquad (15)$$

Choice variables are: $x_E, h_E, x_s(a), s, h(a), n(a)$

How to solve this?

Backward induction.

1. Solve the job training problem (with n interior).
2. Solve the schooling problem with job training as continuation value.
3. Solve the childcare problem with schooling as continuation value.

Job training problem

Write this as starting at $a = 0$ (shifting the age range)

$$V(h,a) = \max \int_0^{R-(6+s)} e^{-ra} \{wh(a)(1-n(a)) - p_w x(a)\} da \quad (16)$$

$$\dot{h}(a) = z_h (h(a) n(a))^{\gamma_1} x(a)^{\gamma_2} - \delta_h h(a) \quad (17)$$

Let F_h be the production function for h

Hamiltonian

$$\Gamma = wh(1-n) - p_w x + q\left[F_h(h,n,x) - \delta_h h\right] \tag{18}$$

FOCs:

$$
\begin{aligned}
whn &= q\gamma_1 F_h && (19)\\
p_w x &= q\gamma_2 F_h && (20)\\
\dot{q} &= rq - q\left\{\gamma_1 F_h/h - \delta_h\right\} - w(1-n) && (21)\\
q_T &= 0 && (22)
\end{aligned}
$$

Implied static condition

$$p_w x = whn\gamma_2/\gamma_1 \tag{23}$$

Solution

Plug first-order conditions into the law of motion for q:

$$\dot{q} = (r + \delta_h)\, q - w \tag{24}$$

Integrate:

$$q(a) = e^{+(r+\delta_h)a} w \int_a^T e^{-(r+\delta_h)t} dt \tag{25}$$

Let $m(a) = 1 - \exp(-(r + \delta_h)(R - a))$, then

$$q(a) = (1 - \tau)\, w \frac{m(a)}{r + \delta_h} \tag{26}$$

Solution

Plug that and focs into $\dot{h}$ equation and integrate:

$$h(a) = e^{-\delta_h a} h(0) + C e^{-\delta_h a} \int_0^a e^{\delta_h t} m(t)^{\gamma/(1-\gamma)} \, dt \tag{27}$$

with

$$C = z_h Q^{\gamma/(1-\gamma)} \left(\frac{\gamma_2}{\gamma_1} \frac{w}{p_w} \right)^{\gamma_2} \tag{28}$$

and

$$Q = \frac{z_h \gamma_1^{1-\gamma_2} \gamma_2^{\gamma_2}}{r + \delta_h} \left(\frac{w}{p_w} \right)^{\gamma_2} \tag{29}$$

From this, we get closed form solutions for $n(a)h(a)$ and $x(a)$.

Solution for nh

$$n(a)h(a) = [Qm(a)]^{1/(1-\gamma)} \tag{30}$$

An interesting feature: job training investment is purely forward looking.

It does not depend on current h.

Summary: Job training phase

Given: $s, h(6+s)$

Closed form solutions for the entire problem.

Marginal value of human capital at the start: $q(6+s)$, also known.

So it is logical to make the solution to this into a general purpose function.

See `BenPorathContTimeLH` class.

Schooling

Exactly the same problem as job training, except:

- initial $h(6) = h_E$
- $n = 1$, so we lose the FOC for n and the static condition
- terminal condition:

$$q(6+s) = (1-\tau)w\frac{m(6+s)}{r+\delta_h} \tag{31}$$

Plus, there is one more terminal condition, which says in words: "when job training starts, the agent chooses $n = 1$"

Take the FOC for nh at $a = 6+s$, set $n = 1$, and you get

$$h(6+s) = [Qm(6+s)]^{1/(1-\gamma)} \tag{32}$$

Schooling: Solution

Lemma 2 in the paper (correct when $p_S = p_E = p_w$)

The trick:

The FOC for x implies

$$x(a) = (z_h \gamma_2 / p_s)^{1/(1-\gamma_2)} \left(q(a) h(a)^{\gamma_1}\right)^{1/(1-\gamma_2)} \tag{33}$$

The rest is luck: derive the growth rates of q and h from the laws of motion.

A bunch of terms cancel and yield a constant growth rate for x (not really luck; it's again the forward looking nature of investment).

$$g(x) = \frac{g(qh^{\gamma_1})}{1-\gamma_2} = \frac{r + \delta_h (1-\gamma_1)}{1-\gamma_2} \tag{34}$$

Plug that into the h equation and integrate...

Schooling: Summary

We have a closed form solution (Lemma 2) for $x_E, h_E, s, h(6+s)$ and $q_E = q(6)$.

- 5 equations in 5 unknowns
- easy to solve numerically

Key feature:

- we do not need to solve the job training problem in order to solve the schooling problem
- all we need to know is the function for $q(6+s)$: (31)

But note the typo in (26) and in the 1st equation of the proof of Lemma 2:

- the term in the exp should enter with plus, not minus

Equilibrium

Equilibrium prices do not depend on household decisions

- see equations (9) and following
- this is because of the small open economy assumption
- r determines the k/h ratio in production and thus the wage

Given prices, we know how to solve the schooling / child care problem

- this yields $h_E, q_E, x_E, s, h(6+s)$

Given $s, h(6+s)$, we know how to solve the job training problem

Now we just need to aggregate to get the equilibrium

- no need to worry about market clearing b/c of the small open economy

What's next

Time to think about how to organize the code...