The Growth Model: Discrete Time Competitive Equilibrium

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Econ720

September 9, 2015

Competitive Equilibrium

We show that the CE allocation coincides with the planner's solution.

Model setup:

- Preferences, endowments, and technology are the same as before.
- Markets: goods, capital rental, labor rental

Households

A single representative household owns the capital and rents it to firms at rental rate q.

It supplies one unit of labor to the firm at wage rate w.

Preferences are

$$\sum_{t=0}^{\infty} \beta^t u(c_t)$$

The budget constraint is:

$$k_{t+1} = (1-\delta)k_t + w_t + q_t k_t - c_t$$

Households: DP Representation

State variable: k.

Control: k'.

Bellman equation:

FOC

Envelope:

Euler equation:

$$u'(c) = \beta(1 + q' - \delta)u'(c')$$

Household: Solution

A pair of policy functions $c = \phi(k)$ and $k' = h(k)$ and a value function such that:

1. the policy functions solve the "max" part of the Bellman equation, given V;
2. the value function satisfies

$$V(k) = u([1-\delta] + w + qk - h(k)) + \beta\ V(h(k))$$

In terms of **sequences**: $\{c_t, k_{t+1}\}$ that solve the Euler equation and the budget constraint.

The boundary conditions are k_0 given and the transversality condition (TVC)

$$\lim_{t\to\infty} \beta^t u'(c_t) k_t = 0$$

Firms

- Firms rent capital and labor services from households, taking prices (q, w) as given.
- They maximize current period profits:

$$\max G(K, L) - wL - qK$$

- FOC

$$\begin{aligned} G_K(K, L) &= q \\ G_L(K, L) &= w \end{aligned}$$

Firms

- Assume constant returns to scale. Define

$$g(k^F)L = G(K,L)$$

- FOC's become

$$\begin{aligned} g'(k^F) &= q \\ g(k^F) - g'(k^F)k^F &= w \end{aligned}$$

- A **solution** is a pair (K,L) that satisfies the 2 FOC.

Equilibrium

An equilibrium is a sequence of
that satisfy:

Comparison with the Planner's Solution

One way of showing that the Planner's solution coincides with the CE is to appeal to the First and Second Welfare Theorems.

A more direct way is to show that the equations that characterize CE and the planner's solution are the same.

CE
$u'(c) = (1 + q' - \delta)\ \beta\ u'(c')$
$k' + c = g(k) + (1 - \delta)\ k$
$k' = (1 - \delta)k + w + qk - c$
$q = g'(k)$
$w = g(k) - g'(k)k$

Planner
$u'(c) = (g'(k') + 1 - \delta)\ \beta\ u'(c')$
$k' + c = g(k) + (1 - \delta)\ k$

Recursive Competitive Equilibrium

Recursive competitive equilibrium

Recursive CE is an alternative way of representing a CE that is more fully consistent with the DP approach.

- Everything is written as functions of the state variables.
- There are no sequences.

This is useful especially in models with

- heterogeneous agents where the distribution of households is a state variable;
- uncertainty, where we cannot assume that agents take future prices as given.

Recursive competitive equilibrium

- By definition, everything in the economy is a function of the state variables.
- All the agents need to know are the **laws of motion** for the state variables.
  - E.g., to form expectations over future interest rate, use the law of motion for k and the price function $q = f'(k)$.
- Agents' policy functions depends on the laws of motion.
- The laws of motion depend on agents' policy functions.

RCE in the example

- The economy's *state variable* is K.
- Call its law of motion $K' = \varphi(K)$.
    - This is part of the equilibrium.
- We solve the household problem for a saving function $k' = h(k, K)$.
    - It depends on the private state k and the aggregate state K.
- We solve the firm's problem for price functions $q(K), w(K)$.

Household

The household solves

$$\max \sum_{t=0}^{\infty} \beta^t u(c_t)$$

subject to

$$k_{t+1} = (1-\delta)k_t + w(K_t) + q(K_t)k_t - c_t$$

The household's problem has an individual state k and an aggregate state K.

Household

Bellman's equation is

$$
\begin{aligned}
V(k,K) &= \max u([1-\delta]k + w(K) + q(K)k - k') + \beta V(k',K') \\
K' &= \varphi(K)
\end{aligned}
$$

Solution: $k' = h(k,K)$.

Firm

Nothing changes in the firm's problem.

Solution:

$$\begin{aligned} q(K) &= g'(K) \\ w(K) &= g(K) - g'(K)K \end{aligned}$$

RCE:

Objects:

- price functions $[q(K), w(K)]$,
- a law of motion for the aggregate state: $K' = \varphi(K)$,
- a policy function $k' = h(k,K)$ and a value function $V(k,K)$.

Equilibrium conditions:

- Given $\varphi(K), q(K), w(K)$: the policy function solves the household's DP.
- The price functions satisfy firm FOCs.
- Markets clear (same as before).
- Household expectations are consistent with household behavior:
$$h(K,K) = \varphi(K)$$

Recursive CE: Example

Households

- There are N_j households of type j.
- The representative type j household solves:

$$
\begin{aligned}
& \max \sum_{t=0}^{\infty} \beta^t u(c_t, l_t) \\
s.t.\ k_{t+1} &= R_t k_t + w_t l_t - c_t
\end{aligned}
$$

Aggregate State

- The aggregate state vector is the distribution of wealth:

$$\kappa = (\kappa_1, ..., \kappa_N) \tag{1}$$

- κ_j is wealth of household j in equilibrium.
- The household knows the law of motion

$$\kappa' = \phi(\kappa) \tag{2}$$

with jth element

$$\kappa_j' = \phi_j(\kappa) \tag{3}$$

Household Dynamic Program

$$
\begin{aligned}
V_j(k_j, \kappa) &= \max u(c_j, l_j) + \beta V_j\left(k_j', \phi(\kappa)\right) \\
k_j' &= R(\kappa) k_j + w(\kappa) l_j - c_j
\end{aligned}
$$

First-order conditions:

$$
\begin{aligned}
u_c(c_j, l_j) &= \beta V_{j,1}\left(k_j', \phi(\kappa)\right) && (4) \\
u_l(c_j, l_j) &= \beta V_{j,1}\left(k_j', \phi(\kappa)\right)\ w(\kappa) && (5)
\end{aligned}
$$

Envelope:

$$
V_{j,1}(k_j, \kappa) = u_c(c_j, l_j)\ R(\kappa) \qquad (6)
$$

Household solution

A solution to the type j household problem consists of

- a value function V_j
- policy functions $k_j' = h_j(k_j, \kappa)$, $l_j = \ell_j(k_j, \kappa)$, and $c_j = g_j(k_j, \kappa)$.

These satisfy:

1. V_j is a fixed point of the Bellman equation, given h, ℓ and g.
2. h, ℓ and g "max" the Bellman equation.

Firm

- This is standard:

$$\max F(K(\kappa), L(\kappa)) - w(\kappa)L(\kappa) - q(\kappa)K(\kappa)$$

- FOC: Factor prices equal marginal products.
- Solution: $K(\kappa)$ and $L(\kappa)$.

Market clearing

- Goods:

$$F\left(K\left(\kappa\right),L\left(\kappa\right)\right)+\left(1-\delta\right)K\left(\kappa\right)=\sum_j N_j\left[g_j\left(\kappa_j,\kappa\right)+h_j\left(\kappa_j,\kappa\right)\right] \tag{7}$$

- Labor:

$$L\left(\kappa\right)=\sum_j N_j\ell\left(\kappa_j,\kappa\right) \tag{8}$$

- Capital:

$$K\left(\kappa\right)=\sum_j N_j\kappa_j \tag{9}$$

Recursive CE

- Objects: Functions V_j, h_j, ℓ_j, g_j and $K(\kappa), L(\kappa), w(\kappa), q(\kappa), R(\kappa)$ and ϕ.
- These satisfy:
  1. Household solution (4)
  2. Firm first order conditions (2)
  3. Market clearing (3 - 1 redundant)
  4. Identity: $R(\kappa) = q(\kappa) + 1 - \delta$.
  5. Consistency:
  $$\phi_j(\kappa) = h_j(\kappa_j, \kappa) \;\; \forall j \tag{10}$$

Notes on RCE

All the objects to be found are functions, not sequences.

This helps when there are shocks:

- We cannot find the sequence of prices without knowing the realizations of the shocks.
- But we can find how prices evolve for each possible sequence of shocks.
- The price functions describe this together with the laws of motion for the states.

Notes on RCE

Functional analysis helps determine the properties of the policy functions and the laws of motion.

- E.g., we strictly concave utility we know that savings are increasing in k, continuous, differentiable, etc.

RCE helps compute equilibria.

- Find the household's optimal choices for every possible set of states.
- Then simulate household histories to find the laws of motion.

Example: Heterogeneous Preferences

Model

Demographics:

- There are $j = 1, ..., J$ types of households.
- The mass of type j households is μ_j.

Preferences:

- $\max \sum_{t=0}^{\infty} \beta^t u_j(c_{jt})$.
- u_j is increasing and strictly concave and obeys Inada conditions.

Model

Technology: $F(K_t, L_t) + (1 - \delta) L_t = C_t + K_{t+1}$

Endowments:

- Each household is endowed with one unit of labor in each period.
- At $t = 0$ household j is endowed with k_{j0} units of capital and with $b_{j0} = 0$ units of one period bonds.

Market arrangements are standard.

Household Problem

- Nothing new here, except everything is indexed by j.
- Define wealth as $a_{jt} = k_{jt} + b_{jt}$.
- Impose no-arbitrage: $R = q + 1 - \delta$
- Bellman equation:
- Euler Equation:

$$u_j'(c) = \beta R' u_j'(c') \tag{11}$$

- Solution (sequence language): $\{c_{jt}, a_{jt}\}$ that solve the Euler equation and budget constraint.
- Boundary conditions: a_{j0} given and TVC $\lim_{t\to\infty} \beta^t u'(c_{jt}) a_{jt} = 0$.

Competitive Equilibrium

A CE consists of sequences

which satisfy:

- 2 household conditions
- 2 firm first-order conditions (standard)
- Market clearing:

We need to distinguish k_{jt} from k_t in the equilibrium definition.

Steady State

- Similar to CE without time subscripts.
- Euler equation becomes:

$$\beta R = 1$$

- Interesting: we can find R without knowing preferences or wealth distribution.

Are there steady states with persistent inequality?

- Let's solve for steady state c_j as a function of prices and endowments (k_{j0}, b_{j0}).
- With constant prices, the household's present value budget constraint implies
- Endowing households with any k_{j0}'s that sum to the steady state k yields a steady state with persistent inequality.
- It would be harder to show that persistent inequality follows from *any* initial asset distribution which features capital inequality.

Redistribution

How does the steady state allocation change when a unit of capital is taken from household j and given to household j'?

Lump-sum Taxes

Impose a lump-sum tax τ on type j households. The revenues are given to type j' households.

How does the steady state change?

Lump-sum Taxes

What if revenues are thrown into the ocean instead?

Differences in β

- Now imagine households differ in their β's, but not in their u functions.
- For simplicity, assume that $u(c) = c^{1-\sigma}/(1-\sigma)$.
- What would the asset distribution look like in the limit as $t \to \infty$?

Reading

- Acemoglu (2009), ch. 6. Also ch. 5 for background material we will discuss in detail later on.
- Ljungqvist and Sargent (2004), ch. 3 (Dynamic Programming), ch. 7 (Recursive CE).
- Stokey, Lucas, and Prescott (1989), ch. 1 is a nice introduction.

References I

Acemoglu, D. (2009): *Introduction to modern economic growth*. MIT Press.

Ljungqvist, L., and T. J. Sargent (2004): *Recursive macroeconomic theory*.

Stokey, N., R. Lucas, and E. C. Prescott (1989): "Recursive Methods in Economic Dynamics," .