Dynamic Programming Theorems

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Dynamic Programming Theorems

Useful theorems to characterize the solution to a DP problem.

There is no reason to remember these results.

But you need to know they exist and can be looked up when you need them.

Generic Sequence Problem (P1)

$$
\begin{aligned}
V^*(x(0)) &= \max_{\{x(t+1)\}_{t=0}^{\infty}} \sum_{t=0}^{\infty} \beta^t U(x(t), x(t+1)) \\
& \quad \textit{subject to} \\
x(t+1) &\in G(x(t)) \\
& \quad x(0) \text{ given}
\end{aligned}
$$

$x(t) \in X \subset \mathbb{R}^k$ is the set of allowed states.

The correspondence $G : X \rightrightarrows X$ defines the constraints.

A solution is a sequence $\{x(t)\}$

Mapping into the growth model

$$\max_{\{k(t+1)\}_{t=0}^{\infty}} \sum_{t=0}^{\infty} \beta^t U\left(f(k(t)) - k(t+1)\right)$$

subject to

$$k(t+1) \in G(k(t)) = [0, f(k(t))]$$
$$k(0) \text{ given}$$

Recursive Problem (P2)

$$V(x) = \max_{y \in G(x)} U(x,y) + \beta V(y), \ \forall x \in X$$

A solution is a policy function $\pi : X \longrightarrow X$ and a value function $V(x)$ such that

1. $V(x) = U(x, \pi(x)) + \beta V(\pi(x)), \ \forall x \in X$
2. When $y = \pi(x)$, now and forever, the max value is attained.

The Main Point

This is the upshot of everything that follows:

> *If it is possible to write the optimization problem in the* ***format*** *of P1*
> *and if mild* ***conditions*** *hold,*
> *then solving P1 and P2 is* ***equivalent****.*

Assumptions That Could Be Relaxed

1. Stationarity: U and G do not depend on t.
2. Utility is additively separable.
   - Time consistency
3. The control is $x(t+1)$.
   - There could be additional controls that don't affect $x(t+1)$.
   - They are "max'd out". Ex: 2 consumption goods.

Dynamic Programming Theorems

- The payoff of DP: it is easier to prove that solutions exist, are unique, monotone, etc.
- We state some assumptions and theorems using them.

Assumption 1: Non-emptiness

- Define the set of feasible paths starting at $x(0)$ by $\Phi(x(0))$.
- $G(x)$ is **nonempty** for all $x \in X$.
  - needed to prevent a currently good looking path from running into "dead ends"
- $\lim_{n\to\infty} \sum_{t=0}^{n} \beta^t U(x(t), x(t+1))$ exists and is **finite**, for all $x(0) \in X$ and feasible paths $\mathbf{x} \in \Phi(x(0))$.
  - cannot have unbounded utility

Assumption 2: Compactness

- The set X in which x lives is **compact**.
- G is compact valued and continuous.
- U is continuous.

Notes:

- Compactness avoids existence issues: without it, there could always be a slightly better $\mathbf{x}$
- Compact X creates trouble with endogenous growth, but can be relaxed.
- Think of A1 and A2 together as the "existence conditions."

Assumption 3: Convexity

- U is strictly concave.
- G is convex (for all x, $G(x)$ is a convex set).

Typical assumptions to ensure that **first order conditions** are sufficient.

Assumption 4: Monotonicity

- $U(x, y)$ is strictly increasing in x.
  - more capital is better
- G is monotone in the sense that $x \leq x'$ implies $G(x) \subset G(x')$.

This is needed for **monotonicity** of policy function.

Assumption 5: Differentiability

- U is continuously **differentiable** on the interior of its domain.

So we can work with first-order conditions.

Main Result

Principle of Optimality + Equivalence of values:

A1 and A2 $\Longrightarrow$ solving P1 and solving P2 yield the same value and policy functions.

Now for the details...

Theorem 1: Equivalence of values

- Assume A1 and A2.
- Then for any x, $V^*(x) = V(x)$.
  - The value that comes out of solving the sequence problem also solves P2.
  - Solving P2 means: Stick V^* into P2 and max. Then V^* pops out on the LHS.
- And any $V(x)$ that solves P2 and satisfies $\lim_{t\to\infty} \beta^t V(x(t)) = 0$ for all feasible $\mathbf{x}$ satisfies $V(x) = V^*(x)$.

Theorem 1: Equivalence of values

In words:

- For any initial x, P1 and P2 yield the same values.
- This says nothing about the policies.

Theorem 2: Principle of Optimality

- Assume A1.
- In P1, for any **optimal** plan $\mathbf{x}^*$ [that attains $V^*(x(t))$ in P1] starting at $x(0)$ the Bellman equation holds:

$$V^*(x^*(t)) = U(x^*(t), x^*(t+1)) + \beta V^*(x^*(t+1)) \qquad (1)$$

- Any feasible plan $\mathbf{x}^*$ starting at $x(0)$ that satisfies (1) attains the max value in P1.

Theorem 2: Principle of Optimality

In words:

- Solve the sequence problem to get V^* and $\mathbf{x}^*$. Both satisfy the Bellman equation (without the max part).
- Part 2 gives us the max part: If (1) holds for $\mathbf{x}^*$, then $\mathbf{x}^*$ solves the max part.
- If we solve the sequence problem, we solve the recursive one.

Theorem 2: Principle of Optimality

- Part 2 says: we can go the other way.
- Solve the Bellman equation to get $V(x)$ and optimal sequences $\mathbf{x}^*$.
- They satisfy the Bellman equation
- By theorem 1, they also satisfy the Bellman equation with value $V^*(x)$.
- Part 2 says: $\mathbf{x}^*$ then also solves the sequence problem.

Theorem 2: Principle of Optimality

- To sum up: If A1 and A2 hold, then solving the sequence problem and solving the recursive problem yield the same values and policies.

Theorem 3: Uniqueness of V

- Assumptions: A1 and A2.
- Then there exists a unique, **continuous, bounded** value function that solves P1 or P2 (they are the same).
- An optimal plan $\mathbf{x}^*$ exists. But it may not be unique.

Theorem 4: Concavity of V

- Assumptions: A1-A3 (convexity).
- Then the value function is strictly concave.

Recall: A3 says that U is strictly concave and $G(x)$ is convex.

So we are solving a concave / convex programming problem.

Corollary 1

- Assumptions A1-A3.
- Then there exists a **unique optimal plan** $\mathbf{x}^*$ for all $x(0)$.
- It can be written as $x^*(t+1) = \pi(x^*(t))$.
- π is continuous.

Reason: The Bellman equation is a concave optimization problem with convex choice set.

Theorem 5: Monotonicity of V

- Assumptions: A1, A2, A4.
- Recall A4: U and G are monotone.
- V is strictly increasing in all arguments (states).

Theorem 6: Differentiability of V

- Assumptions A1, A2, A3, A5.
- A5: U is differentiable.
- Then $V(x)$ is continuously differentiable at all interior points x' with $\pi(x') \in IntG(x')$.
- The derivative is given by:

$$DV\left(x'\right) = D_x U\left(x', \pi\left(x'\right)\right) \tag{2}$$

This is an envelope condition: we can ignore the response of π when x' changes.

Contraction mapping theorem

- How could one show that V is increasing? Or concave? Etc.
- Thinking of the Bellman equation as a functional equation helps...
- Think of the Bellman equation as mapping V on the RHS into $\hat{V}$ on the LHS:

$$\hat{V}(x) = \max_{y \in G(x)} U(x,y) + \beta V(y) \tag{3}$$

- The RHS is a function of V.
- The Bellman equation maps the space of functions V lives in into itself.

$$\hat{V} = T(V) \tag{4}$$

- The solution is the function V that is a fixed point of T:

$$V = T(V) \tag{5}$$

Notation

- If $T : X \to X$, we write:
    1. Tx instead of the usual $T(x)$
    2. $T(\hat{X})$ as the image of the set $\hat{X} \subset X$.

Contraction mapping theorem

- The Bellman equation is $\hat{V} = TV$.
- Suppose we could show:
  1. If V is increasing, then $\hat{V}$ is increasing.
  2. There is a fixed point in the set of increasing functions.
  3. The fixed point is unique.
- Then we would have shown that the solution V is increasing.
- The contraction mapping theorem allows us to make arguments like this.

Contraction mapping theorem

Definition

Let (S,d) be a metric space and $T:S\rightarrow S$. T is a contraction mapping with modulus β, if for some $\beta\in(0,1)$,

$$d(Tz_1,Tz_2)\leq\beta d(z_1,z_2),\ \forall z_1,z_2\in S \tag{6}$$

A contraction pulls points closer together.

Contraction mapping theorem

Theorem 7: Let (S, d) be a complete metric space and let T be a contraction mapping. Then T has a unique fixed point in S.

Recall:

1. Cauchy sequence: For any ε, $\exists n$ such that $d(x_n, x_m) < \varepsilon$ for $m > n$.
2. Complete metric space: Every Cauchy sequence converges to a point in S.

Contraction mapping theorem

A helpful result for showing properties of V :

> *Theorem 8: Let (S,d) be a complete metric space and let $T : S \to S$ be a contraction mapping with fixed point $T\hat{z} = \hat{z}$.*
>
> *If S' is a closed subset of S and $T(S') \subset S'$, then $\hat{z} \in S'$.*
>
> *If $T(S') \subset S'' \subset S'$, then $\hat{z} \in S''$.*

The point: When looking for the fixed point, one can restrict the search to sub-spaces with nice properties.

Example:

- We try to show that V is strictly concave, but the set of strictly concave functions (S) is not closed.
- If we can show that T maps strictly concave functions into a closed subset S' of S, then V must be strictly concave.

Blackwell's Sufficient Conditions

This is helpful for showing that a Bellman operator is a contraction:

Theorem 9: Let $X \subseteq \mathbb{R}^K$*, and* $\mathbf{B}(X)$ *be the space of bounded functions* $f : X \to \mathbb{R}$*. Suppose that* $T : \mathbf{B}(X) \to \mathbf{B}(X)$ *satisfies:*

(1) monotonicity: $f(x) \leq g(x)$ *for all* $x \in X$ *implies* $Tf(x) \leq Tg(x)$ *for all* $x \in X$*.*

(2) discounting: there exists $\beta \in (0,1)$ *such that*

$$T[f(x) + c] \leq Tf(x) + \beta c \text{ for all } f \in \mathbf{B}(X) \text{ and } c \geq 0.$$

Then T *is a contraction with modulus* β*.*

Example: Growth Model

$$TV = \max_{k' \in [0, f(k)]} U\left(f(k) - k'\right) + \beta V(k') \tag{7}$$

Metric space:

- S: set of bounded functions on $(0, \infty)$.
- d: sup norm: $d(f, g) = \sup |f(k) - g(k)|$.

Step 1: $T : S \to S$

- need tricks if U is not bounded (argue that k is bounded along any feasible path)
- otherwise TV is the sum of bounded functions

Example: Growth Model

Step 2: Monotonicity

- Assume $W(k) \geq V(k) \forall k$.
- Let $g(k)$ be the optimal policy for $V(k)$.
- Then

$$
\begin{aligned}
TV(k) &= U(f(k) - g(k)) + \beta V(g(k)) && (8)\\
&\leq U(f(k) - g(k)) + \beta W(g(k)) && (9)\\
&\leq TW(k) && (10)
\end{aligned}
$$

Example: Growth Model

Step 3: Discounting

$$
\begin{aligned}
T(V + a(k)) &= \max U\left(f(k) - k'\right) + \beta[V(k') + a] &\quad (11)\\
&= V(k) + \beta a &\quad (12)
\end{aligned}
$$

Therefore: T is a contraction mapping with modulus β.

Summary: Contraction mapping theorem

Suppose you want to show that the value function is increasing.

1. Show that the Bellman equation is a contraction mapping - using Blackwell.
2. Show that it maps increasing functions into increasing functions.

Done.

First order conditions

Consider again Problem P2:

$$V(x) = \max_{y \in G(x)} U(x, y) + \beta V(y), \ \forall x \in X$$

If we make assumptions that ensure:

- V is differentiable and concave.
- U is concave.
- G is convex. [A1-A5 ensure all that.]

Then the RHS is just a standard concave optimization problem.

We can take the usual FOCs to characterize the solution.

First order conditions

- For y:

$$D_y U(x, \pi(x)) + \beta DV(\pi(x)) = 0 \tag{13}$$

- To find $DV(x)$ differentiate the Bellman equation:

$$DV(x) = D_x U(x, \pi(x)) + D_y U(x, \pi(x))\ D\pi(x) + \beta DV(\pi(x)) D\pi(x) = \tag{14}$$

- Apply the FOC to find the Envelope condition:

$$\begin{aligned} DV(x) &= D_x U(x, \pi(x)) && (15) \\ DV(\pi(x)) &= D_x\ U(\pi(x), \pi(\pi(x))) && (16) \end{aligned}$$

- Sub back into the FOC:

$$D_y U(x, \pi(x)) + \beta D_x U(\pi(x), \pi(\pi(x))) = 0 \tag{17}$$

First order conditions

- In the usual prime notation:

$$D_2 U\left(x, x'\right) + \beta D_1 U\left(x', x''\right) = 0 \tag{18}$$

- Think about a feasible perturbation:
  1. Raise x' a little and gain $D_2 U(x, x')$ today.
  2. Tomorrow lose the marginal value of the state x': $D_1(x', x'')$.
- Why isn't there a term as in the growth model's resource constraint: $f'(k) + 1 - \delta$?
  - By writing $U(x, x')$, the resource constraint is built into U.
  - In the growth model: $U(k, k') = u(f(k) + (1-\delta)k - k')$.
  - $D_1 U = u'(c)[f'(k) + 1 - \delta]$.

Transversality

- Even though the programming problem is concave, the first-order condition is not sufficient!
- A mechanical reason: it is a first-order difference equation - it has infinitely many solutions.
- A boundary condition is needed.

Theorem 10: Let $X \subset \mathbb{R}^K$ and assume A1-A5. Then a sequence $\{x(t+1)\}$ with $x(t+1) \in IntG(x(t))$ is optimal in P1, if it satisfies the Euler equation and the transversality condition

$$\lim_{t\to\infty} \beta^t D_x U(x(t), x(t+1))\; x(t) = 0 \tag{19}$$

Example: The growth model

$$
\begin{aligned}
& \max \sum_{t=0}^{\infty} \beta^t \ln\left(c\left(t\right)\right) \\
& \textit{subject to} \\
0 \quad \leq \quad & k\left(t+1\right) \leq k\left(t\right)^{\alpha} - c\left(t\right) \\
k\left(0\right) \quad = \quad & k_0
\end{aligned}
$$

Example: The growth model

- Step 1: Show that A1 to A5 hold.
- Define $U(k,k') = \ln(k^{\alpha} - k')$.
- A1 is obvious: $G(x)$ is non-empty. The sum of discounted utilities is bounded for all feasible paths.
- A2:
  - X is compact - no, but we can restrict k to a compact set w.l.o.g.
  - G is compact valued and continuous: check
  - U is continuous: check
- A3: U is strictly concave. $G(x)$ is convex: check.
- A4: U is strictly increasing in x. G is monotone: check.
- A5: U is continuously differentiable: check

Example: The growth model

- Step 2: Theorems 1-6 and 10 apply.
- We can characterize the solution by first-order conditions and TVC.
- FOC:
$$\frac{1}{k^{\alpha} - \pi(k)} = \beta V'(\pi(k)) \tag{20}$$
- Envelope:
$$V'(k) = \frac{\alpha k^{\alpha-1}}{k^{\alpha} - \pi(k)} \tag{21}$$
- Combine:
$$\frac{1}{k^{\alpha} - \pi(k)} = \beta \frac{\alpha \pi(k)^{\alpha-1}}{\pi(k)^{\alpha} - \pi(\pi(k))} \tag{22}$$
- Or:
$$u'(c) = \beta f'(k') u'(c') \tag{23}$$

Example: The growth model

Other things we know:

1. V is continuously differentiable, bounded, unique, strictly concave.
2. $V'(k) > 0$.
3. The optimal policy function $c = \phi(k)$ is unique, continuous.

Reading

- Acemoglu, *Introduction to Modern Economic Growth*, ch. 6
- Stokey, Lucas, with Prescott, *Recursive Methods*. A book length treatment. The standard reference.
- Krusell, "Real Macroeconomic Theory," ch. 4.