Manuelli & Seshadri: Human Capital and the Wealth of Nations

Econ821

Prof. Lutz Hendricks

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Introduction

We write out an organized solution to Manuelli and Seshadri (2014).

Then we think about how to organize the code for this model.

- My version of the code is on github.

Note: almost all the derivations are straight out of MS2014.

A few notes

The model solution in the paper is correct only for the case where the schooling, child care and job training inputs are the same good

- i.e., $\beta = \theta$, $p_S = p_E = p_w$

The reported results are for the case where $\beta < \theta$

- a correction is being worked on

The paper contains about a dozen typos, mostly in the proofs.

The paper contains a tax rate in some places, which is set to 0 in all computations.

We consider a slightly more general model.

Model Elements

Small open economy (r exogenous)

Steady state

Demographics:

- OLG
- lifetime T
- $N(a,t) = \phi(a) e^{\eta t}$
- mass of persons aged a in steady state: $\phi(a) = \eta \frac{e^{-\eta a}}{1-e^{-\eta T}}$

Model Elements

Preferences:

- agents maximize lifetime earnings.

Endowments at birth:

- h_B units of human capital

Endowments each period:

- 1 unit of time
- can be spent on working or learning

Human capital production

Phase 1: early childhood

$$h_E = h_B x_E^{v} \tag{1}$$

Phase 2: schooling:

$$\dot{h}(a) = G^s\left(h(a), x(a)\right) - \delta_h h(a) \tag{2}$$

- starts at $a = a_0$, duration s (a choice)
- $h(a_0) = h_E$

Human capital production

Phase 3: job training:

$$\dot{h}(a) = G^w(h(a), n(a), x(a)) - \delta_h h(a) \tag{3}$$

- $n < 1$
- starts at $a = a_0 + s$ with $h(a_0 + s)$ from schooling problem

Output Production

Consumption good:

- output per worker

$$y_c = zF\left(k_c, \bar{h}_c\right) \tag{4}$$

$$F\left(\kappa_c, 1\right) = \left(k_c / \bar{h}_c\right)^{\theta} \tag{5}$$

- $\bar{h}_c$: human capital per worker devoted to that sector
- no resource constraint (open economy)
- numeraire
- also the x input during the job training phase $(p_w = 1)$

Human capital good:

- the same technology: $y_s = F^h\left(k_s, \bar{h}_s\right) = \bar{x}_s + \bar{x}_e$
- used to produce x_s and x_e

Aggregation

Define an aggregation function

$$M(x, a_1, a_2) = \int_{a_1}^{a_2} \phi(a) x(a) \, da \tag{6}$$

Mass working:

$$\bar{\phi} = M(1, a_0 + s, R) = \int_{6+s}^{R} \phi(a) \, da \tag{7}$$

Human capital per worker (labor supply in efficiency units):

$$\bar{h} = \frac{M(h(1-n), a_0 + s, R)}{\bar{\phi}} \tag{8}$$

Labor market clearing:

$$\bar{h} = \bar{h}_s + \bar{h}_c \tag{9}$$

Aggregation

Aggregate goods used to make x_s and x_e:

$$\bar{x}_s = M(x_s, a_0, a_0 + s) \tag{10}$$

$$\bar{x}_e = \phi(6)\, x_E \tag{11}$$

Factor Prices

Consumption goods sector:

$$p_k(r+\delta_k) = zF_k(\kappa_c, 1) = z\theta\kappa_c^{\theta-1} \tag{12}$$

$$w = zF_h(\kappa_c, 1) = z(1-\theta)\kappa_c^{\theta} \tag{13}$$

Human capital sector:

$$p_k(r+\delta_k) = p_s F_k^h(\kappa_s, 1) = z\beta\kappa_s^{\beta-1} \tag{14}$$

$$w = p_s F_h^h(\kappa_s, 1) = z(1-\beta)\kappa_s^{\beta} \tag{15}$$

Key: These determine all factor and goods prices, given the exogenous r and z.

Equilibrium

A steady state is defined by these objects:

- household: $x(a), n(a), h(a)$, x_E, h_E
- aggregates: $y, y_s, \bar{x}_s, \bar{x}_e, \bar{h}, \bar{h}_s, \bar{h}_c$
- prices: p_k, p_s, p_e, p_w

These satisfy:

- household optimization
- firm first-order conditions
- definition of $\bar{h}, \bar{x}_s, \bar{x}_e$
- $y_s = F^h\left(k_s, \bar{h}_s\right) = \bar{x}_s + \bar{x}_e$
- $\bar{h} = \bar{h}_c + \bar{h}_s$

There is no market clearing condition for final goods (small open economy).

Household Problem

The household solves

$$\max_{s,x(a),n(a),h(a),x_e,h_e} -p_e x_e + W(h_e) + e^{-rs} V(h_s, R - s - a_0) \tag{16}$$

subject to

- $h(a_0) = h_E = h_B x_E^{v}$
- $\dot{h}(a) = G^s(h(a), x(a)) - \delta_h h(a)$; $a \leq a_0 + s$
- $\dot{h}(a) = G^w(h(a), n(a), x(a)) - \delta_h h(a)$; $a > a_0 + s$
- $W(h_e) = \int_0^s -p_s x_s(t)\, dt$
- $V(h_s, T) = \int_0^T e^{-ra} \{wh(a)(1 - n(a)) - p_w x(a)\}\, da$
- $n(a) \leq 1$

How to solve this?

Backward induction.

1. Solve the job training problem (with n interior).
2. Solve the schooling problem with job training as continuation value.
3. Solve the childcare problem with schooling as continuation value.

Computationally, it turns out to be more efficient to solve schooling and child care simultaneously.

Job training problem

Write this as starting at $a = 0$ (shifting the age range)

$$V(h(0), T) = \max \int_0^T e^{-ra} \{wh(a)(1 - n(a)) - p_w x(a)\} da \quad (17)$$

$$\begin{aligned} \dot{h}(a) &= G^w(h(a), n(a), x(a)) - \delta_h h(a) && (18) \\ &= z_h (h(a) n(a))^{\gamma_1} x(a)^{\gamma_2} - \delta_h h(a) && (19) \end{aligned}$$

$$h(0) \; given \quad (20)$$

Hamiltonian

$$\Gamma = wh(1-n) - p_w x + q\left[G^w(h,n,x) - \delta_h h\right] \tag{21}$$

FOCs:

$$whn = q\gamma_1 G^w \tag{22}$$
$$p_w x = q\gamma_2 G^w \tag{23}$$
$$\dot{q} = rq - q\left\{\gamma_1 G^w/h - \delta_h\right\} - w(1-n) \tag{24}$$
$$q_T = 0 \tag{25}$$

Implied static condition

$$p_w x = whn\gamma_2/\gamma_1 \tag{26}$$

Solution

Plug first-order conditions into the law of motion for q:

$$\dot{q} = (r + \delta_h)\, q - w \tag{27}$$

Integrate:

$$q(a) = e^{+(r+\delta_h)a} w \int_a^T e^{-(r+\delta_h)t} dt \tag{28}$$

Let

$$m(t) = 1 - \exp((r + \delta_h)\, t) \tag{29}$$

then

$$q(t) = (1 - \tau)\, w \frac{m(t)}{r + \delta_h} \tag{30}$$

Both are functions of the time remaining t.

Solution for nh

From static FOC:

$$G^{w} = z_{h} \left(\frac{\gamma_{2}}{\gamma_{1}} \frac{w}{p_{w}} \right)^{\gamma_{2}} (hn)^{\gamma} \tag{31}$$

Sub into FOC for n to obtain

$$n(a)\,h(a) = \left[Qm(T-a)\right]^{1/(1-\gamma)} \tag{32}$$

Then $x(a)$ follows from the static condition.

An interesting feature: job training investment is **purely forward looking**.

It does not depend on current h.

Solution for $h(a)$

Plug $q(a)$ and focs into $\dot{h}$ equation and integrate:

$$h(a) = e^{-\delta_h a} h(0) + C e^{-\delta_h a} \int_0^a e^{\delta_h t} m(T-t)^{\gamma/(1-\gamma)} dt \tag{33}$$

with

$$C = z_h Q^{\gamma/(1-\gamma)} \left(\frac{\gamma_2}{\gamma_1} \frac{w}{p_w} \right)^{\gamma_2} \tag{34}$$

and

$$Q = \frac{z_h \gamma_1^{1-\gamma_2} \gamma_2^{\gamma_2}}{r + \delta_h} \left(\frac{w}{p_w} \right)^{\gamma_2} \tag{35}$$

Value function

Integration of $wh(1-n)-p_w x$ yields:

$$V(h,T) = q(T)h + w\frac{1-\gamma}{\gamma_1}Q^{1/(1-\gamma)}\int_0^T e^{-rt}m(T-t)^{1/(1-\gamma)} \quad (36)$$

Summary: Job training phase

Given: s, h_s

Closed form solutions for the entire problem.

Marginal value of human capital at the start: $q(T - a_0 - s)$, also known.

So it is logical to make the solution to this into a general purpose function.

See `BenPorathContTimeLH` class.

Schooling

The household solves

$$\max_{s,x(a),n(a),h(a),x_e,h_e} -p_e x_e + \int_0^s -p_s x_s(t)\,dt + e^{-rs} V(h_s, R - s - a_0) \quad (37)$$

subject to

- $h(a_0) = h_E = h_B x_E^{\nu}$
- $\dot{h}(a) = G^s(h(a), x(a)) - \delta_h h(a)$; $a \leq a_0 + s$
- $h_s = h(a_0 + s)$

Terminal value function from job training.

Hamiltonian

$$\Gamma = -p_s x_s + q\left(G^s(h, x_s) - \delta_h h\right) \tag{38}$$

First order conditions (essentially the same as job training without the condition for n):

$$-p_s + q\partial G^s/\partial x_s = 0 \tag{39}$$
$$\dot{q} = (r + \delta_h)q - q\partial G^s/\partial h \tag{40}$$
$$q(s) = \partial V/\partial h \tag{41}$$

In addition: x_e solves

$$V_0 = \max_{x_e} q_e h_B x_e^v - p_e x_e \tag{42}$$

where $q_e = q(a_0) = \partial W/\partial h$ from schooling problem.

Special case in the paper

Schooling and job training have the same technologies and input prices.

Then the terminal conditions become:

1. The terminal value of h is given by the job training problem:

$$q(a_0 + s) = (1 - \tau) w \frac{m(R - a_0 - s)}{r + \delta_h} \tag{43}$$

2. When job training starts, the agent chooses $n = 1$
   Take the FOC for nh at $a = a_0 + s$, set $n = 1$, and you get

$$h(a_0 + s) = [Qm(R - a_0 - s)]^{1/(1-\gamma)} \tag{44}$$

Schooling: Solution

(40) implies

$$x(a) = (z_h \beta_2 / p_s)^{1/(1-\beta_2)} \left(q(a) h(a)^{\beta_1} \right)^{1/(1-\beta_2)} \tag{45}$$

Therefore

$$g(x_s) = \frac{g(q) + \beta_1 g(h)}{1 - \beta_2} \tag{46}$$

From laws of motion

$$g(h) = G^s / h - \delta_h \tag{47}$$

$$\dot{q} = (r + \delta_h) q - \beta_1 G^s / h \tag{48}$$

Therefore, x grows at a constant rate:

Schooling: Solution

Therefore,

$$q(s)\,h_s^{\beta_1} = q_E h_E^{\beta_1} e^{g(x_s)(1-\beta_2)s} \tag{50}$$

Schooling: Solution

Solution is now: $x_e, h_E, q_E, h_s, s, q_s$ that solve:

1. foc for x_E and production function for h_E:

$$h_E = [h_B (vq_E/p_E)^v]^{1/(1-v)} \tag{51}$$

2. constant growth of qh^{β_1}: (50)
3. foc for x_s: (40)
4. equation for $q(a_0 + s)$ from OJT

All we need now is a first-order condition for s

Optimal s

This is a problem with free terminal time (see Leonard and Van Long (1992) p. 245).

A necessary condition is

$$e^{-rs}\Gamma(s) + \frac{d}{ds}e^{-rs}V(h_s, T - a_0 - s) = 0 \tag{52}$$

or

$$-p_s x_s(s) + q(s)\left(G^s(h(s), x_s(s)) - \delta_h h(s)\right) - rV(h(s), s) + \partial V/\partial s = 0 \tag{53}$$

Intuition...

After simplification

$$-rV + \partial V/\partial s = \frac{wh}{r+\delta_h}\left[m'(R - a_0 - s) - rm(R - a_0 - s)\right] - w\frac{1-\gamma}{\gamma_1}Q^{1/(1-\gamma)} \tag{54}$$

Schooling: Summary

We have a closed form solution the entire schooling problem

- easy to solve numerically

Key feature:

- we do not need to solve the job training problem in order to solve the schooling problem
- all we need to know is the function for $q(a_0 + s)$: (43)

Equilibrium

Equilibrium prices do not depend on household decisions

- see equations (9) and following
- this is because of the small open economy assumption
- r determines the k/h ratio in production and thus the wage

Given prices, we know how to solve the schooling / child care problem

- this yields $h_E, q_E, x_E, s, h(a_0+s)$

Given $s, h(a_0+s)$, we know how to solve the job training problem

Now we just need to aggregate to get the equilibrium

- no need to worry about market clearing b/c of the small open economy

What's next

Time to think about how to organize the code...

References I

Leonard, D. and N. Van Long (1992): *Optimal control theory and static optimization in economics*, Cambridge University Press.

Manuelli, R. E. and A. Seshadri (2014): "Human Capital and the Wealth of Nations," *The American Economic Review*, 104, 2736–2762.