Optimal Control

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Topics

Optimal control is a method for solving dynamic optimization problems in continuous time.

Generic Optimal control problem

Choose functions of time $c(t)$ and $k(t)$ so as to

$$\max \int_0^T v[k(t), c(t), t] dt \tag{1}$$

Constraints:

1. Law of motion of the **state** variable $k(t)$:

$$\dot{k}(t) = g[k(t), c(t), t] \tag{2}$$

2. Feasible set for **control** variable $c(t)$:

$$c(t) \in Y(t) \tag{3}$$

3. Boundary conditions, such as:

$$k(0) = k_0, \text{given} \tag{4}$$

$$k(T) \geq k_T \tag{5}$$

Generic Optimal control problem

- c and k can be vectors.
- $Y(t)$ is a compact, nonempty set.
- T could be infinite.
  - Then the boundary conditions change
- Important: the state cannot jump; the control can.

Example

A household chooses optimal consumption to

$$\max \int_0^T u[c(t)]dt \tag{6}$$

subject to

$$\dot{k}(t) = rk(t) - c(t) \tag{7}$$

$$c(t) \in [0, \bar{c}] \tag{8}$$

$$k(0) = k_0, \text{given} \tag{9}$$

$$k(T) \geq 0 \tag{10}$$

A Recipe for Solving Optimal Control Problems

A Recipe

1. Write down the ***Hamiltonian***

$$H(t) = v(k,c,t) + \mu(t) g(k,c,t) \tag{11}$$

- μ is essentially a Lagrange multiplier (called a "**co-state** variable).

A Recipe

2. Derive the **first order conditions** which are **necessary** for an optimum:

$$\partial H/\partial c = 0 \tag{12}$$

$$\partial H/\partial k = -\dot{\mu} \tag{13}$$

A Recipe

3. Impose the **transversality** condition:

- for finite horizon:

$$\mu(T) = 0 \tag{14}$$

- for infinite horizon:

$$\lim_{t\to\infty} H(t) = 0 \tag{15}$$

- this depends on the terminal condition (see below).

A Recipe

4. A **solution** is the a set of functions $[c(t), k(t), \mu(t)]$ which satisfy
   - the FOCs
   - the law of motion for the state
   - the boundary / transversality conditions

Example: Growth Model

$$\max \int_0^\infty e^{-\rho t} u(c(t))\,dt \tag{16}$$

subject to

$$\dot{k}(t) = f(k(t)) - c(t) - \delta k(t) \tag{17}$$

$$k(0) \text{ given} \tag{18}$$

Growth Model: Hamiltonian

$$H(k, c, \mu) = e^{-\rho t} u(c(t)) + \mu(t) [f(k(t)) - c(t) - \delta k(t)] \qquad (19)$$

Necessary conditions:

$$\begin{aligned} H_c &= e^{-\rho t} u'(c) - \mu = 0 \\ H_k &= \mu \left[f'(k) - \delta \right] = -\dot{\mu} \end{aligned}$$

Growth Model

Substitute out the co-state:

$$\dot{\mu} = e^{-\rho t} u''(c)\,\dot{c} - \rho \mu \tag{20}$$

$$\begin{aligned}\dot{c} &= \frac{\dot{\mu} + \rho \mu}{e^{-\rho t} u''(c)} &(21)\\ &= -\left(f'(k) - \delta - \rho\right) \frac{u'(c)}{u''(c)} &(22)\end{aligned}$$

Solution: c_t, k_t that solve Euler equation and resource constraint, plus boundary conditions.

Details

- First order conditions are necessary, not sufficient.
- They are necessary only if we **assume** that
  1. a continuous, interior solution exists;
  2. the objective function v and the constraint function g are continuously differentiable.
- Acemoglu (2009), ch. 7, offers some insight into why the FOCs are necessary.

Details

- If there are multiple states and controls, simply write down one FOC for each separately:

$$\begin{aligned} \delta H/\delta c_i &= 0 \\ \partial H/\partial k_j &= -\dot{\mu}_j \end{aligned}$$

- There is a large variety of cases depending on the length of the horizon (finite or infinite) and the kinds of boundary conditions.
  - Each has its transversality condition (see Leonard and Van Long 1992).

Next steps

Typical useful next things to do:

1. Eliminate μ from the system. Obtain two differential equations in (c,k).
2. Find the steady state by imposing $\dot{c} = \dot{k} = 0$.

Sufficient conditions

First-order conditions are sufficient, if the programming problem is **concave**.

This can be checked in various ways.

Sufficient Conditions I:

- The objective function and the constraints are concave functions of the controls and the states.
- The co-state must be positive.
- This condition is easy to check, but very stringent.

Sufficient Conditions II

First-order conditions are sufficient, if the Hamiltonian is concave in controls and states, where the co-state is evaluated at the optimal level.

- This, too is very stringent.

Sufficient Conditions III

Arrow and Kurz (1970)

First-order conditions are sufficient, if the *maximized* Hamiltonian is concave in the states.

Maximized Hamiltonian:

Substitute controls and co-states out, so that the Hamiltonian is only a function of the states.

This is less stringent and by far the most useful set of sufficient conditions.

Discounting: Current value Hamiltonian

Problems with discounting

Current utility depends on time only through an exponential discounting term $e^{-\rho t}$.

The generic discounted problem is

$$\max \int_0^T e^{-\rho t} v[k(t), c(t)] dt \tag{23}$$

subject to the same constraints as above.

Applying the Recipe

$$H(t) = e^{\rho t} v(k, c) + \hat{\mu} g(k, c) \tag{24}$$

$$\frac{\partial H}{\partial c_t} = 0 \implies e^{-\rho t} v_c(k_{t,} c_t) = -\hat{\mu}_t g_c(k_t, c_t) \tag{25}$$

$$\frac{\partial H}{\partial k_t} = e^{-\rho t} v_k(k_t, c_t) + \hat{\mu}_t g_k(k_t, c_t) = -\dot{\hat{\mu}}_t \tag{26}$$

Applying the Recipe

Let

$$\mu_t = e^{\rho t} \hat{\mu}_t \tag{27}$$

and multiply through by $e^{\rho t}$:

$$v_c(t) = -\mu_t g_k(t)$$

This is the standard FOC, but with μ instead of $\hat{\mu}$.

Applying the Recipe

$$v_k(t) + e^{\rho t}\hat{\mu}_t g_k(t) = -e^{\rho t}\dot{\hat{\mu}}_t \tag{28}$$

Substitute out $\dot{\hat{\mu}}_t$ using

$$\dot{\mu}_t = \frac{de^{\rho t}\hat{\mu}_t}{dt} = \rho\mu_t + e^{\rho t}\dot{\hat{\mu}}_t$$

we have

$$v_k(t) + \mu_t g_k(t) = -\dot{\mu}_t + \rho\mu_t$$

This is the standard condition with an additional $\rho\mu$ term.

Shortcut

We now have a shortcut for discounted problems.

Hamiltonian (drop the discounting term):

$$H = v(k,c) + \mu g(k,c) \tag{29}$$

FOCs:

$$\partial H/\partial c = 0 \tag{30}$$

$$\partial H/\partial k = \underbrace{\mu(t)\rho}_{\text{added}} - \dot{\mu}(t) \tag{31}$$

and the TVC

$$\lim_{T\to\infty} e^{-\rho T}\mu(T)k(T) = 0 \tag{32}$$

Equality constraints

Equality constraints of the form

$$h[c(t),k(t),t]=0 \tag{33}$$

are simply added to the Hamiltonian as in a Lagrangian problem:

$$H(t)=v(k,c,t)+\mu(t)g(k,c,t)+\lambda(t)h(k,c,t) \tag{34}$$

FOCs are unchanged:

$$\begin{aligned} \partial H/\partial c &= 0 \\ \partial H/\partial k &= -\dot{\mu} \end{aligned}$$

For inequality constraints:

$$h(c,k,t)\geq 0;\ \lambda h=0 \tag{35}$$

Transversality Conditions

Finite horizon: Scrap value problems

- The horizon is T.
- The objective function assigns a scrap value to the terminal state variable: $e^{-\rho T}\phi(k(T))$:

$$\max \int_0^T e^{-\rho t} v[k(t), c(t), t] dt + e^{-\rho T}\phi(k(T)) \qquad (36)$$

- Hamiltonian and FOCs: unchanged.
- Replace the TVC with

$$\mu(T) = \phi'(k(T)) \qquad (37)$$

- Intuition: μ is the marginal value of the state k.

Infinite horizon TVC

- The finite horizon TVC with the boundary condition $k(T) \geq k_T$ is $\mu(T) = 0$.
  - Intuition: capital has no value at the end of time.
- But the infinite horizon boundary condition is NOT $\lim_{t \to \infty} \mu(t) = 0$.
- The next example illustrates why.

Infinite horizon TVC: Example

$$
\begin{aligned}
& \max \int_0^{\infty} \left[\ln(c(t)) - \ln(c^*)\right] dt \\
& \textit{subject to} \\
\dot{k}(t) &= k(t)^{\alpha} - c(t) - \delta k(t) \\
k(0) &= 1 \\
\lim_{t\to\infty} k(t) &\geq 0
\end{aligned}
$$

- c^* is the max steady state (golden rule) consumption.
- No discounting - subtracting c^* makes utility finite.

Infinite horizon TVC

- Hamiltonian

$$H(k,c,\lambda) = \ln c - \ln c^* + \lambda \left[k^\alpha - c - \delta k\right] \tag{38}$$

- Necessary FOCs

$$\begin{aligned} H_c &= 1/c - \lambda = 0 && (39) \\ H_k &= \lambda \left[\alpha k^{\alpha-1} - \delta\right] = -\dot{\lambda} && (40) \end{aligned}$$

Infinite horizon TVC

- We show: $\lim_{t\to\infty} c(t) = c^*$ [why?]
- Limiting steady state solves

$$\begin{aligned} \dot{\lambda}/\lambda &= \alpha k^{\alpha-1} - \delta = 0 \\ \dot{k} &= k^{\alpha} - 1/\lambda - \delta k = 0 \end{aligned}$$

- Solution is the golden rule:

$$k^* = (\alpha/\delta)^{1/(1-\alpha)} \tag{41}$$

- Verify that this max's steady state consumption.

Infinite horizon TVC

- Implications for the TVC...
- $\lambda(t) = 1/c(t)$ implies $\lim_{t\to\infty} \lambda(t) = 1/c^*$.
- Therefore, neither $\lambda(t)$ nor $\lambda(t)k(t)$ converge to 0.
- The correct TVC: $\lim_{t\to\infty} H(t) = 0$.
- The only reason why the standard TVC does not work: there is no discounting in the example.

Infinite horizon TVC: Discounting

- With discounting, the TVC is easier to check.
- Assume:
  - the objective function is $e^{-\rho t} v[k(t), c(t)]$
  - it only depends on t through the discount factor
  - v and g are weakly monotone
- Then the TVC becomes

$$\lim_{t \to \infty} e^{-\rho t} \mu(t) k(t) = 0 \tag{42}$$

where μ is the costate of the current value Hamiltonian.

- This is exactly analogous to the discrete time version

$$\lim_{t \to \infty} \beta^t u'(c_t) k_t = 0 \tag{43}$$

Example: renewable resource

Example: Renewable resource

$$\max \int_0^\infty e^{-\rho t} u(y(t))\, dt \tag{44}$$

$$\textit{subject to} \tag{45}$$

$$\dot{x}(t) = -y(t) \tag{46}$$

$$x(0) = 1 \tag{47}$$

$$x(t) \geq 0 \tag{48}$$

Example: Renewable resource

Current value Hamiltonian

Necessary FOCs

Example: Renewable resource

FOC

Therefore:

$$\mu(t) = \mu(0)\, e^{\rho t} \tag{49}$$

$$y(t) = u'^{-1}\left[\mu(0)\, e^{\rho t}\right] \tag{50}$$

Solution

The optimal path has $\lim x(t) = 0$ or

$$\int_0^\infty y(t)\,dt = \int_0^\infty u'^{-1}\left[\mu(0)\,e^{\rho t}\right]dt = 1 \tag{51}$$

This solves for $\mu(0)$.

Example: Renewable resource

- TVC for infinite horizon case:

$$\lim e^{-\rho t}\mu(0)e^{\rho t}x(t) = 0 \tag{52}$$

- Equivalent to

$$\lim x(t) = 0 \tag{53}$$

Reading

- Acemoglu (2009), ch. 7. Proves the Theorems of Optimal Control.
- Barro and Martin (1995), appendix.
- Leonard and Van Long (1992): A fairly comprehensive treatment. Contains many variations on boundary conditions.

References I

Acemoglu, D. (2009): *Introduction to modern economic growth*. MIT Press.

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