Problem Set 5: Growth Model in Continuous Time

Econ720. Fall 2015. Prof. Lutz Hendricks

---

1 Relative Wealth Preferences

Consider the following version of the growth model in continuous time.

Notation: c: consumption, k: capital, $\bar{k}$: average capital in the economy.

Demographics: There is one representative household who lives forever.

Preferences:

$$\int_0^\infty e^{-\rho t} \left[U(c_t) + V\left(k_t/\bar{k}_t\right)\right] dt \tag{1}$$

Endowments: The household starts with k_0.

Technology:

$$\dot{k}_t = f(k_t) - c_t \tag{2}$$

Government budget constraint: The government taxes consumption at rate τ_c and lump-sum rebates the revenues R_t to the household.

$$R_t = \tau_c c_t \tag{3}$$

Markets: Goods (numeraire).

Household budget constraint:

$$\dot{k}_t = f(k_t) - (1+\tau_c) c_t + R_t \tag{4}$$

Assumptions: U, V, f are strictly increasing and strictly concave. $f'(0) = \infty$. $f'(\infty) = 0$.

Questions:

1. State the household's current value Hamiltonian and derive the first-order conditions. Do not yet substitute out the co-state. Define a solution to the household problem.
2. Define a competitive equilibrium. Make sure that the number of objects equals the number of equations.
3. Derive an equation that implicitly solves for the steady state capital stock.
4. Draw the phase diagram. Start with $\dot{k} = 0$ and discuss its shape.
5. Derive $\dot{c} = 0$ and discuss its slope / intercept. For which values of k does $\dot{c} = 0$ have a solution? Hint: It is easier to write down $\dot{\lambda} = 0$, where λ is the co-state. Then use the fact that $\dot{\lambda} > 0$ implies $\dot{c} < 0$.
6. Assume that $\dot{c} = 0$ is concave,
$$\partial^2 c / \partial k^2 |_{\dot{c}=0} < 0 \tag{5}$$
and that it intersects $\dot{k} = 0$ twice. Discuss the stability properties of the two steady states.

2 Money in the Utility Function

Demographics: Time is continuous. There is a single representative household who lives forever.

Preferences:

$$\int_{t=0}^{\infty} e^{-\rho t} u(c_t, m_t) dt \tag{6}$$

where c is consumption and m denotes real money balances.

Endowments: Households work 1 unit of time at each instant. Households are initially endowed with k_0 units of capital and m_0 units of real money.

Technology:

$$f(k_t) - \delta k_t = c_t + \dot{k}_t \tag{7}$$

Money: nominal money grows at exogenous rate $g(M)$. New money is handed to households as a lump-sum transfer: $\dot{M}_t = p_t x_t$.

Markets: money (numeraire), goods, capital rental (price r), labor (w).

Questions:

1. The household's budget constraint is given by

$$\dot{k}_t + \dot{m}_t = w + r_t k_t + x_t - c_t - \pi_t m_t - g(\dot{m}_t) \tag{8}$$

where $g(\dot{m}_t)$ is the cost of adjusting the money stock. $g'(0) = 0$ and $g''(\dot{m}_t) > 0$. State the Hamiltonian. If you cannot figure this out, assume $g(\dot{m}) = 0$ and proceed (for less than full credit).

2. State the first-order conditions.
3. Define a competitive equilibrium.
4. Characterize the steady state to the extent possible. What is the effect of a permanent change in $g(M)$?
5. What is the optimal rate of inflation? Explain.