AK Model:
Phase Diagram

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Econ720

October 21, 2015

Introduction

We study an endogenous growth model with transitional dynamics.
The model is asymptotically AK.

As an example of a phase diagram with endogenous growth.

The Model

▶ We modify the Ak model's production function:

$$H(K,L) = AK + F(K,L) \qquad (1)$$

▶ In intensive form

$$h(k) = Ak + f(k)$$

where $F(K,L) = Lf(k)$ satisfies Inada conditions and has constant returns to scale in K and L jointly.

▶ For simplicity, assume $f(k) = k^\alpha$ with $\alpha < 1$.

Equilibrium

- The only change to the equilibrium conditions of the Ak model: the marginal product of capital is not A but

$$H_K(K,L) = A + F_K(K,L) = A + f'(k) \tag{2}$$

- Laws of motion:

$$\dot{k} = (A + f(k)/k - \delta - n)k - c \tag{3}$$
$$g(c) = (A + f'(k) - \delta - \rho)/\sigma \tag{4}$$

- Asymptotically, $f'(k) \to 0$ and the model becomes Ak.

Phase Diagram with Endogenous Growth

How to draw a phase diagram when c and k grow at endogenous rates?

Answer: Find ratios that are constant asymptotically:

- $z = h(k)/k$
- $x = c/k$.

Equivalently, **detrend** the model and then draw the phase diagram.

Laws of motion

$$g(z) = g(h(k)) - g(k) \qquad (5)$$
$$g(x) = g(c) - g(k) \qquad (6)$$

▶ We therefore need to find expressions for $g(h(k))$, $g(k)$, and $g(c)$ in terms of z and x only.

▶ First rewrite (3) as

$$g(k) = [h(k)/k - \delta - n] - c/k \qquad (7)$$
$$= z - \delta - n - x \qquad (8)$$

Laws of motion

- Next, $g(c) = [h'(k) - \delta - \rho]/\sigma$.
  - We need to replace $h'(k)$.
- Note that

$$h'(k) = A + \alpha f(k)/k = A + \alpha (z - A) = \alpha z + (1 - \alpha)A$$

- Use this to rewrite (4) as

$$g(c) = \frac{\alpha z + (1 - \alpha)A - \delta - \rho}{\sigma}$$

Laws of motion

Finally,

$$g(h(k)) = \frac{h'(k)\,k}{h(k)}\,g(k) = \frac{\alpha z + (1-\alpha)A}{z}\,g(k)$$

Laws of motion

$$
\begin{aligned}
g(z) &= g(h(k)) - g(k) \\
&= [\frac{\alpha z + (1-\alpha)A}{z} - 1][z - x - n - \delta] \\
&= (1-\alpha)(A/z - 1)[z - x - n - \delta]
\end{aligned}
$$

and

$$
\begin{aligned}
g(x) &= g(c) - g(k) \\
&= \frac{\alpha z + (1-\alpha)A - \rho - \delta}{\sigma} - z + x + n + \delta \\
&= \varphi + x + z(\alpha/\sigma - 1)
\end{aligned}
$$

where $\varphi = n + \delta + (1-\alpha)A/\sigma - (\rho + \delta)/\sigma$.

Steady state

In steady state,

$$z_{ss} = A \tag{9}$$

or

$$z = x + n + \delta \tag{10}$$

and

$$x_{ss} = (1 - \alpha/\sigma) z_{ss} - \varphi \tag{11}$$

We will see in a minute that the relevant condition is $z_{ss} = A$.

$\dot{x} = 0$ Locus

The $\dot{x} = 0$ locus:

$$x = -\varphi + (1 - \alpha/\sigma)z \tag{12}$$

For realistic parameter values (e.g. $\alpha \simeq 0.3$ and $\sigma \geq 1$), we have $0 < 1 - \alpha/\sigma < 1$.

Negative intercept.

Slope < 1.

$\dot{z} = 0$ Locus

$\dot{z} = 0$ has two solutions:

- $z = A$ or
- $x = z - n - \delta$.

The first solution is irrelevant because $z = A/k + f(k)/k > A$.

Negative intercept

Slope $= 1$

Summary
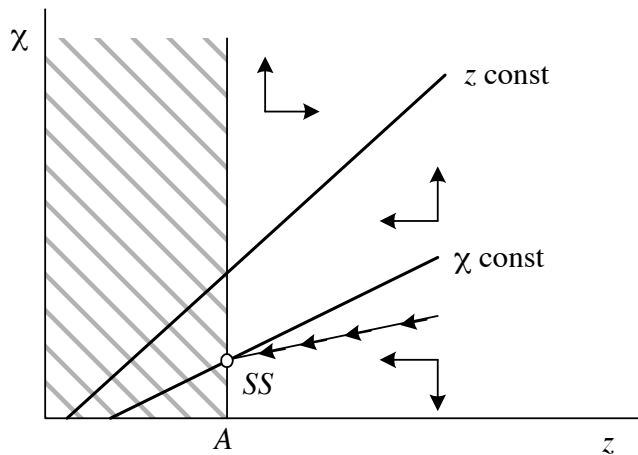
Laws of motion:

$$\dot{z} = (1-\alpha)(A-z)(z-x-n-\delta)$$
$$g(x) = \varphi + x + z(\alpha/\sigma - 1)$$

$$\dot{z} = 0 : x = z - n - \delta$$
$$\dot{x} = 0 : x = -\varphi + (1-\alpha/\sigma)z$$

Steady state: $z = A$.

Phase Diagram with Endogenous Growth

Phase Diagram with Endogenous Growth

- This system is **saddle-path stable**.
- If x_0 is too small, then the trajectory crosses into the $c < 0$ quadrant.
- If x_0 is too large, then the trajectory takes off to the north-east.
    - This violates feasibility: $x = c/k$ would grow without bounds.
- Both x and z converge monotonically to the steady state.

Summary

The important point is the general approach for dealing with the dynamics of growing economies:

1. Write out the equilibrium conditions as usual.
2. Find conditions characterizing the balanced growth path.
3. Find ratios that are constant on the balanced growth path (x and z).
4. Express the laws of motion of the economy in terms of these ratios.

An alternative approach is to transform the economy into stationary form before characterizing its equilibrium.

Reading

- Acemoglu (2009), ch. 11.
- Krusell (2014), ch. 8.

References I

Acemoglu, D. (2009): *Introduction to modern economic growth*, MIT Press.

Krusell, P. (2014): "Real Macroeconomic Theory," Unpublished.