The Growth Model In Continuous Time

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Topics

- We study the standard growth model in continuous time.
- To solve it: Optimal Control
- To characterize it: phase diagrams

Continuous Time vs. Discrete Time

[Some of you will find the next several slides obvious.]

Continuous time

- So far, time was divided into discrete "periods."
- It is often more convenient to shrink the length of periods to 0.
- Difference equations then become differential equations.

Continuous time

Example: Law of motion for capital

- Discrete time:

$$K_{t+1} - K_t = I_t - \delta K_t \tag{1}$$

- More generally:

$$K_{t+\Delta t} - K_t = [I_t - \delta K_t]\ \Delta t \tag{2}$$

- Continuous time $(\Delta t \to 0)$:

$$\lim_{\Delta t \to 0} \frac{K_{t+\Delta t} - K_t}{\Delta t} = \dot{K}_t = I_t - \delta K_t \tag{3}$$

Notation: $\dot{K} = dK/dt$.

Growth rates in continuous time

The growth rate of a variable is defined as

$$g(x) = \frac{\dot{x}}{x} = \frac{d \ln x}{dt} \tag{4}$$

Growth rate rules (easy to prove):

1. $g(xy) = g(x) + g(y)$.
2. $g(x/y) = g(x) - g(y)$.
3. $g(x^{\alpha}) = \alpha g(x)$.
4. $x(t) = e^{\gamma t} \Longrightarrow g(x) = \gamma$.

Differential equations

Differential equations

Take a function of time:

$$x(t) = a + bt \tag{5}$$

There is another way of describing this function:

- Take the derivative:

$$\dot{x}(t) = dx(t)/dt = b \tag{6}$$

- Fix $x(0) = a$.
- The two pieces of information (the derivative and $x(0)$) completely describe $x(t)$.
- Only one function $x(t)$ satisfies both pieces.
  - But note that infinitely many functions satisfy the derivative!

Definition: Differential equation

- A differential equation (DE) is a function of the form

$$\dot{x}(t) = f(x(t), t) \tag{7}$$

  - This is actually a "first-order" DE.
- **Higher order** DEs contain higher order derivatives of time.
  - E.g.: A second order DE

$$d^2x(t)/dt^2 + dx(t)/dt = a + bt \tag{8}$$

- Together with a boundary condition, the DE can be solved for $x(t)$.

Solving DEs

- The bad news: There is no algorithm for solving DEs.
- But one look up solutions in tables.
- It is also easy to **verify** a solution one may guess.

Guess + Verify

Consider again

$$\dot{x}(t) = b \tag{9}$$
$$x(0) = a \tag{10}$$

Guess

$$x(t) = a + bt \tag{11}$$

Verify:

- Take the time derivative and find that it matches $\dot{x} = b$.
- Verify that $x(0) = a$.

Example

$$\begin{aligned} \dot{x}(t) &= b\, x(t) && (12) \\ x(0) &= a && (13) \end{aligned}$$

Guess:

$$x(t) = a\, e^{bt} \tag{14}$$

Verify: Take the derivative

$$\begin{aligned} \dot{x}(t) &= b\, a\, e^{bt} = b\, x(t) && (15) \\ x(0) &= a\, e^{0} = a && (16) \end{aligned}$$

Example: Boundary conditions

$$\begin{aligned} \dot{x}(t) &= b\, x(t) \qquad (17) \\ x(T) &= a \qquad (18) \end{aligned}$$

Guess:

$$x(t) = D\ e^{bt} \qquad (19)$$

Verify:

$$\dot{x}(t) = b\ D\ e^{bt} = b\ x(t) \qquad (20)$$

Find D from

$$x(T) = a = D\ e^{bT} \qquad (21)$$

Boundary conditions

Boundary conditions can take many forms:

- $\int_a^b x(s)\,ds = 5$.
- $\dot{x}(T) = 5$.
- $x(T) - x(T-2) = 5$.
- etc.

The Solow Model

The Solow Model - Structure

- Modify the discrete time growth model in two ways:
  1. Continuous time.
  2. Fixed saving rate.
- This is not an equilibrium model, but can be interpreted as one.

Model Elements

- Demographics: households live forever;

$$L_t = e^{nt} \tag{22}$$

- Preferences:

$$\int_{t=0}^{\infty} e^{-(\rho-n)t} u(c_t) dt \tag{23}$$

- Endowments:
  - at each moment, the household has 1 unit of work time
  - at $t = 0$ he has K_0 goods

Model Elements

Technology:

$$F(K_t, L_t) = \dot{K}_t + \delta K_t + L_t c_t \tag{24}$$

- F: constant returns to scale

Markets:

- competitive markets for goods (numeraire), labor rental, capital rental

Firms

The firm solves a static problem.

The same as in discrete time.

$$\max F(K,L) - wL - qK \tag{25}$$

FOC

$$q_t = F_K \tag{26}$$

$$w_t = F_L \tag{27}$$

Firms: Intensive Form

Define $k^F = K/L$ and

$$f\left(k^F\right) = F(K,L)/L = F\left(k^F,1\right) \tag{28}$$

The first order conditions are then

$$q = f'(k^F) \tag{29}$$

and

$$w = f(k^F) - f'(k^F)k^F \tag{30}$$

Households

Budget constraint

$$\dot{K}_t = w_t L_t + (q_t - \delta) K_t - L_t c_t$$

It is convenient to have everything per capita.

Define $k = K/L$.

Law of motion for k:

$$\begin{aligned} \dot{k}/k &= \dot{K}/K - n \\ &= w/k + (q - \delta) - c/k \end{aligned}$$

Or

$$\dot{k}_t = w_t + (q_t - \delta - n) k_t - c_t \tag{31}$$

Constant saving rate

- The modern way: Set up an optimization problem and derive the saving function.
- The Solow way: Assume that the saving rate is fixed:

$$c = (1-s)(w+qk) \tag{32}$$

- Therefore:

$$\dot{k} = s(w+qk) - (n+\delta)k \tag{33}$$

Market Clearing

Capital rental:

$$k = k^F \tag{34}$$

Goods market:

$$F(K_t, L_t) = C_t + \delta K_t + \dot{K}_t$$

or in per capita terms

$$\dot{k} = f(k) - (n+\delta)k - c \tag{35}$$

Equilibrium

An equilibrium is a collection of *functions* (of time)

$$c_t, k_t, k_t^F, w_t, q_t$$

that satisfy

1. the firm's first order conditions (2)
2. the household's budget constraint and the behavioral equation

$$\dot{k} = s(w + qk) - (n + \delta)k$$

3. market clearing (2)

Law of Motion

- The entire model boils down to to one key equation:

$$\dot{k}_t = sf(k_t) - (n+\delta)k_t \tag{36}$$

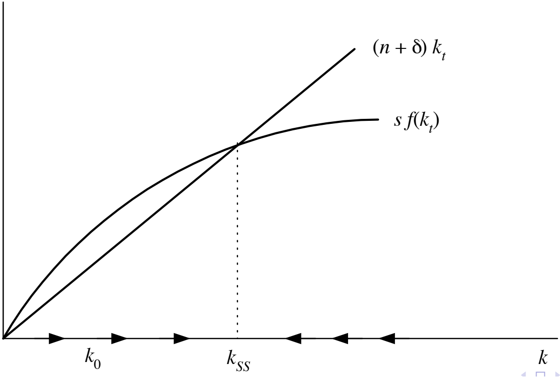
- This is simply the household's behavioral equation after applying $f(k) = w + qk$.

Steady state

The steady state requires $\dot{k} = 0$ or

$$sf(k) = (n+\delta)k \tag{37}$$

With strictly concave f, there is a unique steady state with $k > 0$.



Steady state: Golden Rule

- Which k maximizes steady state consumption?
- Steady state consumption is

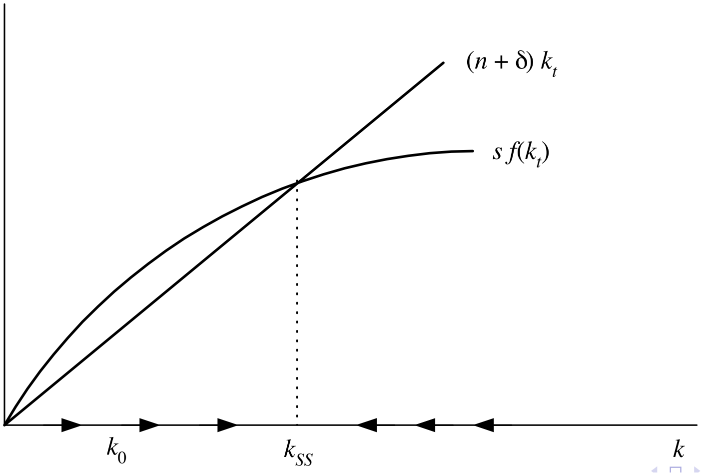
$$c = f(k) - (n + \delta)k \tag{38}$$

- Maximizing this yields the first-order condition:

$$f'(k^{GR}) - \delta = n \tag{39}$$

- This is the exact analogue to the discrete time case: The interest rate must equal the population growth rate.

Dynamics



The steady state is *stable*.

Convergence is monotone.

Adding Technical Change

- The model does not have sustained growth in per capita income.
- This requires technical change (A grows).
- Assume exogenous growth in A:

$$A(t) = A(0)e^{\gamma t} \tag{40}$$

Adding Technical Change

- Assume that technical change takes the following form:

$$Y(t) = F(K(t), A(t)L(t)) \tag{41}$$

- This type of technical change is called "labor-augmenting" or "Harrod-neutral."
- This is the *only* form of technical change that is consistent with *balanced growth*.

Definition

A balanced growth path is a path along which all growth rates are constant.

How to analyze a growing model?

- Construct a **stationary transformation**.
- Divide each variable by its balanced growth factor:

$$\tilde{x}(t) = x(t) e^{-g_x t} \tag{42}$$

  where g_x is the **balanced growth rate** of x.
- Or take ratios of variables that grow at the same rate.
- The economy in transformed variables $(\tilde{x})$ has a steady state.

How to find the balanced growth rates?

- For equations that involve sums:

$$Y(t) = C(t) + I(t) + G(t) \tag{43}$$

  Constant growth (usually) requires that all summands grow at the same rate.

- For other equations: Try taking the growth rate of the whole equation.
- Example:

$$Y(t) = K(t)^{\alpha} \; [A(t) L(t)]^{1-\alpha} \tag{44}$$

  implies

$$g(Y) = \alpha g(K) + (1-\alpha)[g(A) + n] \tag{45}$$

Balanced growth path: Solow Model

- Start from

$$\begin{aligned} \dot{K}(t) &= sF(K(t), A(t)L(t)) - \delta K(t) && (46) \\ g(K(t)) &= sF\left(1, \frac{A(t)L(t)}{K(t)}\right) - \delta && (47) \end{aligned}$$

- Constant growth requires that

$$\bar{k}(t) = \frac{K(t)}{A(t)L(t)} \tag{48}$$

be constant over time. Thus, on a balanced growth path:

$$g(K) = \gamma + n \tag{49}$$

Balanced growth path

- Production function:

$$\bar{y}(t) = \frac{Y(t)}{A(t)L(t)} = F\left(\bar{k}(t), 1\right) \tag{50}$$

  must be constant on a balanced growth path.
- Thus: The model has a steady state in $\left(\bar{k}, \bar{y}\right)$.

Law of motion

$$
\begin{aligned}
g\left(\bar{k}\right) &= g\left(K\right) - \gamma - n \\
&= sF\left(1, \frac{A\left(t\right)L\left(t\right)}{K\left(t\right)}\right) - \delta - \gamma - n \\
&= s\,f\left(\bar{k}\right)/\bar{k} - \delta - \gamma - n
\end{aligned}
$$

Or

$$\dot{\bar{k}}(t) = sf(\bar{k}(t)) - (n + \delta + \gamma)\bar{k}(t) \tag{51}$$

Nothing changes, except the constant term in the law of motion.

Reading

- Acemoglu (2009), ch. 2 covers the Solow model and stationary transformations of growing economies.
- Barro and Martin (1995), ch. 1
- Romer (2011), ch. 1

References I

Acemoglu, D. (2009): *Introduction to modern economic growth*. MIT Press.

Barro, R., and S.-i. Martin (1995): "X., 1995. Economic growth," *Boston, MA*.

Romer, D. (2011): *Advanced macroeconomics*. McGraw-Hill/Irwin.