The Growth Model in Continuous Time (Ramsey Model)

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The Growth Model in Continuous Time

We add optimizing households to the Solow model.

We first study the planner's problem, then the CE.

Planning Problem

Planning Problem

The social planner maximizes

$$\int_{t=0}^{\infty} e^{-(\rho-n)t} u(c_t) dt \tag{1}$$

subject to the resource constraint

$$\begin{aligned} \dot{k}_t &= f(k_t) - (n+\delta)k_t - c_t && (2) \\ & k_0 \text{ given} && (3) \\ k_t &\geq 0 && (4) \end{aligned}$$

Planning Problem

The current value Hamiltonian is

The state is k and the control is c.

The optimality conditions are

Planner: TVC

The TVC is:

$$\lim_{t\to\infty} e^{-(\rho-n)t}\mu(t)k(t) = 0 \tag{5}$$

To check this:

- we need u and $g(k,c)$ to be monotone
- u is obvious.
- $g(k,c) = f(k) - c - \delta k$ is monotone in c but not k.
- However, we "know" that k never rises above the golden rule point where $f'(k) = \delta$ - unless $k(0)$ is too high.
- Then g is increasing in k.

Sufficiency

This is an example where the easiest (1st) set of sufficiency conditions applies:

u is strictly concave in c (only).

$g(k,c)$ is jointly concave in k and c.

First order conditions are sufficient.

Planner: Solution

A solution consists of functions of time

$$c_t, k_t, \mu_t$$

that satisfy:

1. The first-order conditions (2)
2. The resource constraint
3. The boundary conditions k_0 given and the TVC

$$\lim e^{-(\rho-n)t} \mu_t k_t = 0 \tag{6}$$

Planner: Euler Equation

We eliminate the multiplier.

Differentiating the FOC yields

$$\dot{\mu} = u''(c)\dot{c} \tag{7}$$

and therefore

$$\dot{\mu}/\mu = u''(c)\dot{c}/u'(c) \tag{8}$$

Substitute into the law of motion for μ:

$$\dot{c} = u'(c)/u''(c) \cdot [\rho + \delta - f'(k)] \tag{9}$$

Planner: Euler Equation

$$g(c) = [f'(k) - \delta - \rho]/\sigma \tag{10}$$

where

$$\begin{aligned}\sigma &= -u''_c c/u' &(11)\\ &= -\frac{du'(c)}{dc}\frac{c}{u'(c)} &(12)\end{aligned}$$

is the intertemporal elasticity of substitution (and the coefficient of relative risk aversion).

Note: $u(c) = c^{1-\phi}/1-\phi$ implies $\sigma = \phi$.

Planner: Euler Equation

$$g(c) = [f'(k) - \delta - \rho]/\sigma \tag{13}$$

Recall the discrete time version:

$$\frac{c_{t+1}}{c_t} = [\beta R]^{1/\sigma} \tag{14}$$

The same idea:

- consumption growth rises with the interest rate
- declines with the discount rate.

Planner: Summary

- The planner's problem solves for functions of time $c(t)$ and $k(t)$.
- These satisfy two differential equations

$$
\begin{aligned}
g(c) &= \frac{f'(k)-\delta-\rho}{\sigma} && (15)\\
\dot{k} &= f(k)-(n+\delta)k-c && (16)
\end{aligned}
$$

  and two boundary conditions

$$
\begin{aligned}
& k_0 \text{ given} \\
\lim_{t\to\infty}\beta^t u'(c(t))\ k(t) &= 0
\end{aligned}
$$

- How can we analyze the dynamics of this system?

Phase Diagram

Phase Diagram

- Phase diagrams can be used to analyze the dynamics of systems of 2 differential equations.
- Consider the example

$$\begin{aligned} \dot{x} &= A - ax + by \\ \dot{y} &= B + cx - dy \end{aligned}$$

- Assume $a, b, c, d > 0$.

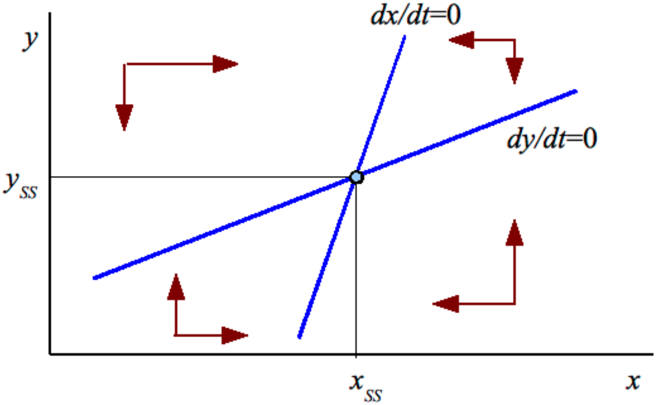
Phase Diagram: Steps

- Step 1: In an (x,y) plane, plot combinations of (x,y) that yield $\dot{x}=0$ or $\dot{y}=0$.

$$\begin{aligned}\dot{x} &= 0 \Rightarrow y = \frac{ax-A}{b} \\ \dot{y} &= 0 \Rightarrow y = \frac{B+cx}{d}\end{aligned}$$

- Step 2: Find out in which direction the system moves when off the $\dot{x}=0$ or $\dot{y}=0$ lines.
  - raise x: $\dot{x}$ falls - move left
  - raise y: $\dot{y}$ falls - move down
- Step 3: Divide phase diagram into 4 quadrants.
  - draw arrows of movement and think...

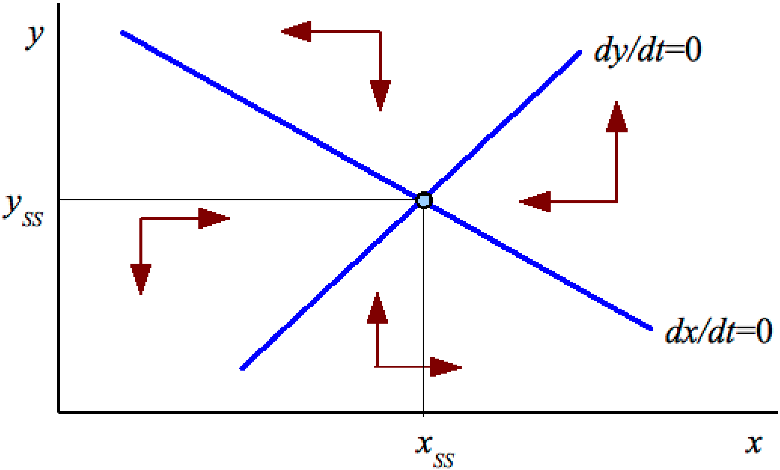
Phase Diagram



Recall: $\dot{x} = A - ax + by$. $\dot{y} = B + cx - dy$.

The steady state is stable.

Phase Diagram



With other coefficients: there are oscillations.

Applications

Galor (2000)

- studies transition from Malthusian stagnation to industrialization using a sequence of phase diagrams

Models of human capital accumulation over the life-cycle:

- Heckman (1976)

Phase Diagram: Growth Model

The $\dot{c} = 0$ locus is characterized by

$$f'(k^*) = \rho + \delta \tag{17}$$

The $\dot{k} = 0$ locus is hump-shaped:

$$c = f(k) - (n + \delta)k \tag{18}$$

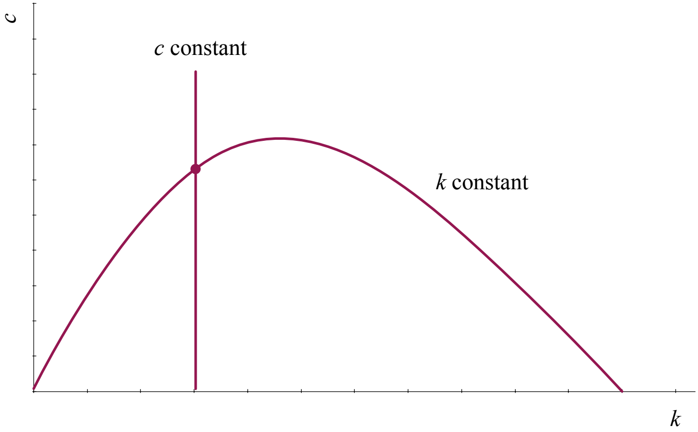
with a maximum at

$$f'(k^*) = n + \delta \tag{19}$$

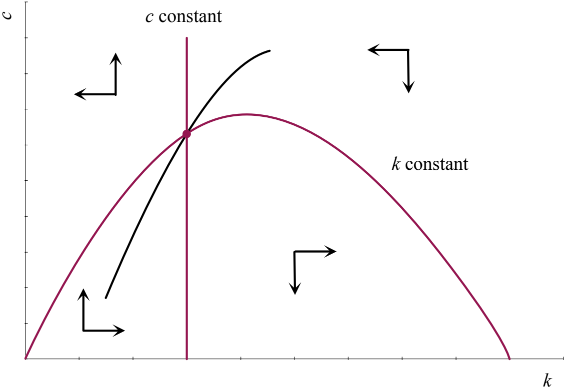
Since $\rho - n > 0$, the $\dot{c} = 0$ locus lies to the left of the peak of the $\dot{k} = 0$ locus.

The steady state is located at the intersection of the two curves.

Phase Diagram



Dynamics



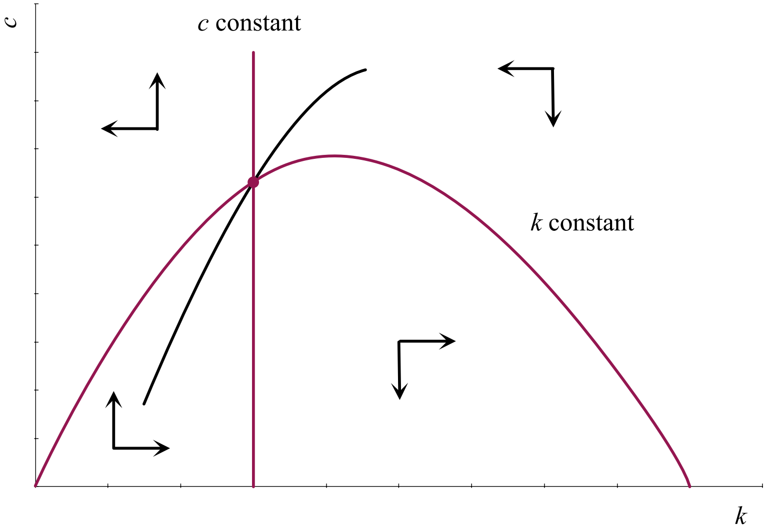
$$\dot{k} = f(k) - (n+\delta)k - c$$

- $c \uparrow \implies \dot{k} \downarrow$
- $k \uparrow \implies \dot{k} \downarrow$

$$g(c) = \frac{f'(k) - \delta - \rho}{\sigma}$$

- $k \uparrow \implies \dot{c} \downarrow$

Dynamics: Possible Paths



Dynamics: Saddle-path Stability

Only one value of c avoids moving into “forbidden” regions for given k.

For this c, the economy converges to the steady state.

Such a system is called "saddle-path stable."

Competitive Equilibrium

Competitive Equilibrium

- Firms solve the same problem as in the Solow model.
- We add a government that imposes lump-sum taxes to finance government spending.
- The budget constraint is $\tau_t = G_t$.

Households

$$\max \int_{t=0}^{\infty} e^{-(\rho-n)t} u(c_t) dt \tag{20}$$

subject to: k_0 given, the TVC, and the budget constraint

$$\dot{k}_t = w_t + (r_t - \delta - n)k_t - c_t - \tau_t \tag{21}$$

Households

Hamiltonian:

$$H = u(c) + \lambda[w + (r - \delta - n)k - c - \tau] \tag{22}$$

First-order conditions

$$\partial H/\partial c = 0 \Rightarrow u'(c) = \lambda \tag{23}$$

$$\begin{aligned}
\dot{\lambda} &= (\rho - n)\lambda - \partial H/\partial k \\
&= \lambda[\rho - n - (r - \delta - n)] \\
&= \lambda(\rho - r + \delta)
\end{aligned}$$

Transversality:

$$\lim_{t \to \infty} e^{-(\rho - n)t} \lambda_t k_t = 0 \tag{24}$$

Households

Eliminate λ:

$$u''(c)\dot{c} = \dot{\lambda} \tag{25}$$

Substitute into the law of motion for λ:

$$\dot{c} = u'(c)/u''(c) \cdot [\rho + \delta - r]$$

or

$$g_c = (r - \delta - \rho)/\sigma \tag{26}$$

Solution: Functions c_t, k_t that solve the Euler equation, the budget constraint, and the boundary conditions.

Competitive Equilibrium

Objects: Functions $c_t, k_t, \tau_t, w_t, r_t$.

Equilibrium conditions:

- Household (2)
- Firm (2)
- Government (1)
- Market clearing (1)

Dynamics

Simplify to obtain two differential equations:

$$\dot{c} = u'(c)/u''(c) \cdot [\rho + \delta - f'(k)] \tag{27}$$

$$\dot{k} = f(k) - (n + \delta)k - c - G \tag{28}$$

The planning solution and the CE coincide (with $G = 0$).

Applications

Models of consumption-saving over the life-cycle

- Carroll, C. D., Overland, J., & Weil, D. N. (2000). Saving and growth with habit formation. American Economic Review, 341–355.

Growth models (we study those later).

Detrending the Model

Detrending a model

- Consider the Cass Koopmans model with productivity growth:

$$\max \int_0^\infty e^{-(\rho-n)t} u(c_t) dt \tag{29}$$

$$\dot{k}_t = F(k_t, A_t) - (n+\delta)k_t - c_t \tag{30}$$

with

$$A_t = e^{gt} \tag{31}$$

- What does the Planner's solution look like?
- The problem: the model has no steady state.
- How can we analyze its dynamics?

Approach 1: Solve and detrend

- Unchanged: the Planner's optimality conditions in terms of original variables:

$$\dot{c}/c = \frac{\frac{\partial F(k,A)}{\partial k} - n - \delta - (\rho - n)}{\sigma(c)} \tag{32}$$

- But we cannot draw the phase diagram without a steady state.
- Solution: detrend the variables to make them stationary.
  1. Find the balanced growth rate for each variable.
  2. Divide each variable by a scale factor that grows at its balanced growth rate.

Balanced growth rates

- The same as in the Solow model with growth:

$$g(c) = g(k) = g \tag{33}$$

- Define the detrended variables:

$$\tilde{c}_t = c_t / A_t \tag{34}$$

$$\tilde{k}_t = k_t / A_t \tag{35}$$

- Law of motion:

$$\begin{aligned} g\left(\tilde{k}\right) &= g(k) - g \\ &= \frac{F\left(\tilde{k}, 1\right) A - (n+\delta)\tilde{k}A - \tilde{c}A}{k} - g \\ d\tilde{k}/dt &= f\left(\tilde{k}\right) - (n+\delta+g)\tilde{k} - \tilde{c} \qquad (36) \end{aligned}$$

Detrended first-order conditions

- Optimality conditions in terms of detrended variables:

$$
\begin{aligned}
\frac{d\tilde{c}/dt}{\tilde{c}} &= \frac{\dot{c}}{c} - g \\
&= \frac{f'\left(\tilde{k}\right) - \delta - \rho}{\sigma\left(c\right)} - g
\end{aligned}
\tag{37}
$$

- This is true because

$$
\frac{\partial F\left(k,A\right)}{\partial k} = \frac{\partial F\left(\tilde{k}A,A\right)}{\partial \tilde{k}} \frac{\partial \tilde{k}}{\partial k} = Af'\left(\tilde{k}\right)\frac{1}{A} \tag{38}
$$

Detrended first-order conditions

- Assume CRRA preferences:

$$u(c) = c^{1-\sigma}/(1-\sigma) \tag{39}$$

- Then $\sigma(c) = \sigma$ is constant.
- **CRRA is required for balanced growth** - an important result.
  - Otherwise $\sigma(c)$ is not constant.

Approach 2: Detrend and solve

- Steps:
  1. Find balanced growth rates - as before.
  2. Write the economy in detrended variables.
  3. Take the first-order conditions.
  4. Define the solution.
  5. Convert back into (undetrended) variables.
- This is useful for solution methods that only work on stationary problems (such as DP).
- Exercise: show that this yields the same answer for the growth model.

Detrending the Model

Summary

In the growth model, optimality conditions change only by adding the 2 occurrences of g:

$$
\begin{aligned}
g(\tilde{c}) &= \frac{f'\left(\tilde{k}\right) - \delta - \rho}{\sigma} - g && (40)\\
d\tilde{k}/dt &= f\left(\tilde{k}\right) - (n + \delta + g)\tilde{k} - \tilde{c} && (41)
\end{aligned}
$$

Detrending the Model

Why do we care?

1. The balanced growth $\tilde{k}$ now depends on preferences:

$$g(\tilde{c}) = 0 \Rightarrow f(\tilde{k}) = \delta + \rho + \sigma g \tag{42}$$

2. We see that preferences must be CRRA for a steady state to exist.
3. Quantitative differences.

Reading

- Acemoglu (2009), ch. 8. Ch. 8.6 covers the detrended model. Ch. 7 covers Optimal Control.
- Barro and Martin (1995), ch. 2, explains the Cass-Koopmans/Ramsey model in great detail.
- Blanchard and Fischer (1989), ch. 2
- Romer (2011), ch. 2A
- Phase diagram: Barro and Martin (1995), ch. 2.6

References I

Acemoglu, D. (2009): *Introduction to modern economic growth*, MIT Press.

Barro, R. and S.-i. Martin (1995): "X., 1995. Economic growth," *Boston, MA*.

Blanchard, O. J. and S. Fischer (1989): *Lectures on macroeconomics*, MIT press.

Galor, O. (2000): "Ability Biased Technological Transition, Wage Inequality, and Economic Growth," *Quarterly Journal of Economics*, 115, 469–498.

Heckman, J. J. (1976): "A Life-Cycle Model of Earnings, Learning, and Consumption," *Journal of Political Economy*, 84, pp. S11–S44.

Romer, D. (2011): *Advanced macroeconomics*, McGraw-Hill/Irwin.