Asset Pricing

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Econ720

November 8, 2016

Topics

1. What determines the rates of return / prices of various assets?
2. How can risk be measured and priced?

- ▶ We use the Lucas (1978) fruit tree model.
- ▶ The implications are far more general than the simple model.
- ▶ The model forms the basis for the CAPM and the β risk measure.

The Lucas (1978) Fruit Tree Model

- **Agents**:
  - A single representative household.
- **Preferences**:

$$\max E_0 \sum_{t=0}^{\infty} \beta^t u(c_t) \tag{1}$$

- E_0 is the expectation as of time $t = 0$.

Technology

- This is an endowment economy.
- There are K identical fruit trees.
- Each tree yields d_t units of consumption goods in period t.
- d_t is random and the same for all trees.
- Trees cannot be produced.
- Fruits cannot be stored.

Technology

- The aggregate resource constraint:

$$c_t = Kd_t \qquad (2)$$

- Assume that d is a finite Markov chain with transition matrix $\pi(d', d)$.
- An important feature: All uncertainty is **aggregate**.
- There are no opportunities for households to insure each other.
- This is why we can work with a representative household.

Markets

- There are markets for fruits and for trees.
- There is also a one period bond, issued by households (in zero net supply).
  - Its purpose is to determine a risk-free interest rate.

Household problem

- The household starts out with bonds (b_0) and shares (k_0).
- At each date, he chooses c_t, b_{t+1}, k_{t+1}.
- The **budget constraint** is

$$p_t k_{t+1} + b_{t+1} = R_t b_t + (p_t + d_t) k_t - c_t \qquad (3)$$

- Notation:
    - p : the price of trees. Suppressing dependence on the state.
    - R : the real interest rate on bonds.
    - the price of bonds is normalized to 1 (how?).

Household problem

$$V(k,b,d) = \max u(c) + \beta E V(k',b',d') \tag{4}$$

subject to

$$Rb + (p+d)k - c + pk' - b' = 0 \tag{5}$$

Household problem

First-order conditions:

$$\begin{aligned}
c &: \quad u'(c) = \lambda \\
k' &: \quad \lambda p = EV_k(k', b', d') \\
b' &: \quad \lambda = EV_b(k', b', d')
\end{aligned}$$

Envelope:

$$\begin{aligned}
V_k &= \lambda(p + d) \\
V_b &= \lambda R
\end{aligned}$$

Euler equations

$$
\begin{aligned}
u'(c_t) &= \beta\, E_t\left\{ u'(c_{t+1}) R_{t+1} \right\} \\
&= \beta\, E_t\{ u'(c_{t+1}) \underbrace{\frac{p_{t+1} + d_{t+1}}{p_t}}_{R^S_{t+1}} \}
\end{aligned}
$$

This is very general - holds for any number of assets / for any type of asset.

Solution

- A solution consists of state contingent plans
  $\{c(d^t), k(d^t), b(d^t)\}$ for all histories d^t.
- These satisfy:
    - 2 Euler equations
    - 1 budget constraint.
    - b_0 and k_0 given.
    - Transversality: $\lim_{t \to \infty} E_0 \beta^t u'(c_t) \, [b_t + p_t k_t] = 0$.

Market clearing

For every history we need:

Bonds:

$$b_t = 0$$

Trees:

$$k_t = K_t$$

Goods:

$$c_t = K_t d_t$$

There is no trade in equilibrium!

Competitive Equilibrium

- A CE consists of:
  1. an allocation: $\{c(d^t), b(d^t), k(d^t)\}$.
  2. a price system: $\{p(d^t), R(d^t)\}$
- These satisfy:
  1. household: 2 Euler equations and 1 budget constraint.
  2. 3 market clearing conditions.

Recursive Competitive Equilibrium

Objects:

- Solution to the household problem: $V(k,b,d)$ and $c(k,b,d)$, $k' = \kappa(k,b,d)$, $b' = B(k,b,d)$.
- Price functions: $p(d), R(d)$.

Equilibrium conditions:

- Household: 4
- Market clearing: 2
- No need for consistency: law of motion of the aggregate state is exogenous.

Consumption smoothing

- The Euler equation implies (for any asset):

$$E_t \left\{ \frac{\beta u'(c_{t+1})}{u'(c_t)} R_{t+1} \right\} = 1 \qquad (6)$$

- Define: Marginal rate of substitution:

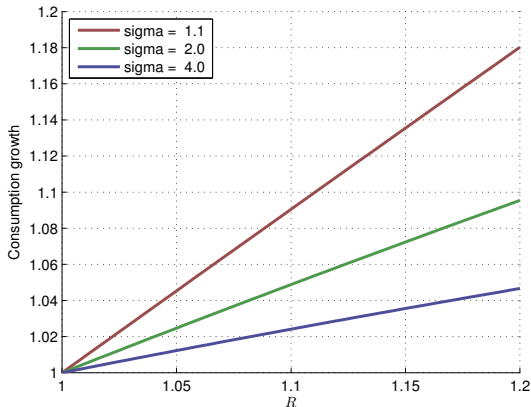$$MRS_{t+1} = \beta\, u'(c_{t+1})/u'(c_t) \qquad (7)$$

- MRS_{t+1} is inversely related to consumption growth.
- With $u(c) = \frac{c^{1-\sigma}}{1-\sigma}$:

$$
\begin{aligned}
u'(c) &= c^{-\sigma} & (8) \\
MRS_{t+1} &= \beta\, (c_{t+1}/c_t)^{-\sigma} & (9)
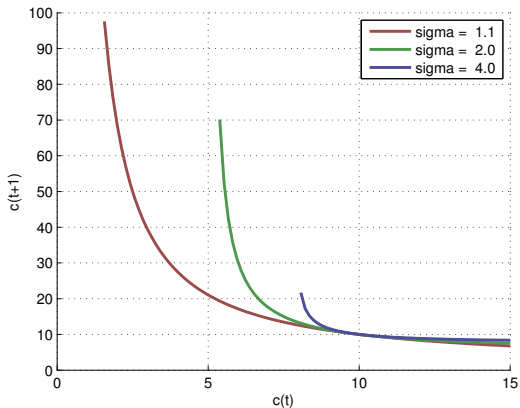\end{aligned}
$$

Consumption smoothing

- The coefficient of relative risk aversion (σ) determines how much *MRS* fluctuates with c.
- High σ implies that the household chooses smooth consumption.
- Illustration for the deterministic case:

Consumption smoothing

- With high σ, marginal utility changes a lot when c changes.
- The household then keeps c smooth.

Asset Prices

Asset pricing implications

- We will now derive the famous **Lucas asset pricing equation**.
- Define: Rate of return on trees: $R_{t+1}^S = (p_{t+1} + d_{t+1})/p_t$.
- Directly from the 2 Euler equations:

$$E_t \left\{ \frac{\beta u'(c_{t+1})}{u'(c_t)} R_{t+1} \right\} = E_t \left\{ \frac{\beta u'(c_{t+1})}{u'(c_t)} R_{t+1}^S \right\} = 1$$

- Or

$$E \left\{ MRS_{t+1} R_{t+1} \right\} = E \left\{ MRS_{t+1} R_{t+1}^S \right\} = 1 \qquad (10)$$

When does an asset pay a high expected return?

Re-write asset pricing equation using

$$Cov(x,y) = E(xy) - E(x)E(y)$$

as

$$
\begin{aligned}
1 &= E\{MRS\} E\{R\} + Cov(MRS, R) & (11)\\
E(R) &= \frac{1 - Cov(MRS, R)}{E(MRS)} & (12)
\end{aligned}
$$

When do assets pay high returns?

$$\mathbb{E}(R) = \frac{1 - Cov(MRS, R)}{\mathbb{E}(MRS)} \tag{13}$$

- Take a "safe" asset with fixed R.
  - $Cov(MRS, R) = 0$
  - $\mathbb{E}(R) = 1/\mathbb{E}(MRS)$.
- If $Cov(MRS, R) < 0$: the asset pays higher return than the safe asset
  - a **risk premium**
- If $Cov(MRS, R) > 0$: the asset pays **lower** return than the safe asset
  - important point: an asset return can have lots of volatility, but pay a lower return than a t-bill
  - examples?

When do assets pay high returns?

- High returns require low / negative $Cov(MRS, R)$.
- Example: log utility
  - $u'(c) = 1/c$
  - $MRS = \beta u'(c_{t+1})/u'(c_t) = \beta c_t / c_{t+1}$.
- High MRS means low consumption growth.
- Therefore: Assets pay high returns if their returns are **positively correlated with consumption growth**.

Intuition

- Imagine there are good times (high c) and bad times (low c).
- There are 2 assets: A pays dividends in good times, B pays in bad times.
- The value of the dividend is $u'(c)$.
- Assets that pay in good times are not valuable: $u'(c)$ is low.
- Assets that pay in bad times provide insurance - they are valuable (have low expected returns).

Risk (premia)

- The "risk free" assets has expected return

$$E(R_f) = \frac{1}{E(MRS)} \tag{14}$$

- A "risky" asset has expected return

$$E(R) = \frac{1 - Cov(MRS, R)}{E(MRS)} \tag{15}$$

- The **risk premium** is

$$E(R) - E(R_f) = -\frac{Cov(MRS, R)}{E(MRS)} \tag{16}$$

- This defines what **risk** means: **covariance with consumption growth**.
- Note that risk can be **negative** (insurance).

The Equity Premium Puzzle

- Mehra and Prescott (1985): Asset return data pose a puzzle for the theory.
- The equity premium is "high" (6-7% p.a.)
- The cov of c growth and R_s is low.
  - The reason: Consumption is very smooth.

The Equity Premium Puzzle

TABLE 1
Summary Statistics
United States Annual Data, 1889–1978

	Sample Means		
R_t^s	0.070		
R_t^b	0.010		
C_t/C_{t-1}	0.018		

	Sample Variance-Covariance		
	R_t^s	R_t^b	C_t/C_{t-1}
R_t^s	0.0274	0.00104	0.00219
R_t^b	0.00104	0.00308	−0.000193
C_t/C_{t-1}	0.00219	−0.000193	0.00127

The Equity Premium Puzzle

A back-of-the envelope calculation with CRRA utility:

$$EP = -\frac{Cov\left(\beta\left[c_{t+1}/c_t\right]^{-\sigma},\ R_s\right)}{E\left\{\beta\left[c_{t+1}/c_t\right]^{-\sigma}\right\}} \tag{17}$$

Take log utility: $\sigma = 1$.

- $Cov(MRS, R_s) \simeq -0.0022$.
- $E(MRS) \simeq 1$.
- $EP \simeq 0.2\%$.
- Replicating the observed equity premium requires **very high risk aversion** ($\sigma = 40$).

How severe is the puzzle?

Investors forego very large returns.

<table>
<tr>
<td colspan="5">**Table 3**

Terminal value of $1 invested in Stocks and Bonds</td>
</tr>
<tr>
<td>**Investment Period**</td>
<td colspan="2">**Stocks**</td>
<td colspan="2">**T-bills**</td>
</tr>
<tr>
<td></td>
<td>Real</td>
<td>Nominal</td>
<td>Real</td>
<td>Nominal</td>
</tr>
<tr>
<td>1802-1997</td>
<td>$558,945</td>
<td>$7,470,000</td>
<td>$276</td>
<td>$3,679</td>
</tr>
<tr>
<td>1926-2000</td>
<td>$266.47</td>
<td>$2,586.52</td>
<td>$1.71</td>
<td>$16.56</td>
</tr>
</table>

Source: Mehra and Prescott (2003)

Long holding periods

Over 20 year holding periods: stocks dominate bonds.

Equity Risk Premium Over 20-Year Periods
1926–2000

Average Equity Risk Premium (in percent)

20-Year Period Ending
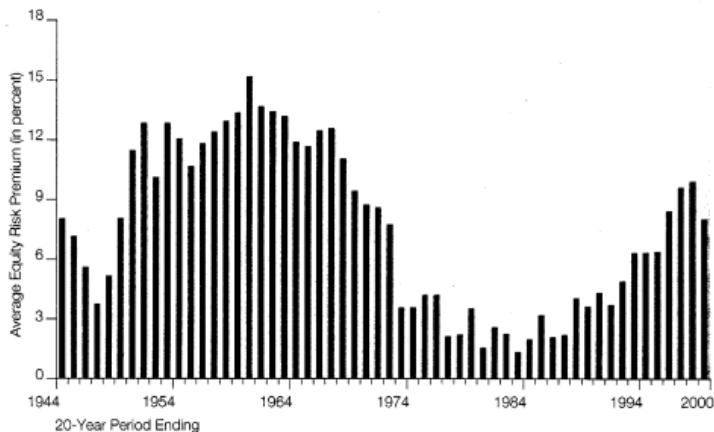
Source: Mehra and Prescott (2003)

Why do we care?

▶ The EP puzzle shows that we do not understand
1. what households view as "risky"
2. why households place a high value on smooth consumption

▶ This has implications for:
1. The welfare costs of business cycles
   ▶ They are very low in standard models.
2. Stock price volatility.
   ▶ Standard models fail to explain it (see below).

How to resolve the puzzle

Proposed explanations include:

1. Habit formation: $u(c_t, c_{t-1}) = \frac{[c_t - \gamma c_{t-1}]^{1-\sigma}}{1-\sigma}$.
   - Implies high risk aversion when c_t is close to c_{t-1}.

2. Heterogeneous agents
   - Implicit in the standard model: all idiosyncratic risk is perfectly insured.

3. Borrowing constraints
   - The young should hold stocks (long horizon), but cannot.
   - The old receive mostly capital income and find stocks risky.

4. Taxes / regulations (McGrattan and Prescott, 2000)
   - The runup in stock prices since the 1960s stems from lower dividend taxes & laws permitting institutional investors to hold equity.

Beta

- Now we derive the famous "beta" measure of risk.
- Suppose asset m (the market) is perfectly correlated with marginal utility:

$$u'(c_{t+1}) = -\gamma \, R_{m,t+1} \qquad (18)$$

- The market's expected return is

$$E \, R_m - R = -\frac{Cov(MRS, R_m)}{E(MRS)} \qquad (19)$$

Beta

▶ Now we relate the covariance term to marginal utility:

$$Cov(MRS, R_m) = Cov\left(\frac{u'(c_{t+1})}{u'(c_t)}, R_{m,t+1}\right) = \frac{Cov(u'(c_{t+1}), R_{m,t+1})}{u'(c_t)}$$

$$E(MRS) = \frac{E(u'(c_{t+1}))}{u'(c_t)}$$

▶ Therefore:

$$E(R_m) - R = -\frac{Cov(u'(c_{t+1}), R_{m,t+1})}{E\, u'(c_{t+1})} = -\frac{\gamma\, Var(R_{m,t+1})}{E\, u'(c_{t+1})}$$

Beta

- For any asset i:

$$E\,R_i - R = -\frac{Cov\left(u'\left(c_{t+1}\right), R_i\right)}{E\,u'\left(c_{t+1}\right)} = \frac{\gamma\,Cov\left(R_m, R_i\right)}{E\,u'\left(c_{t+1}\right)}$$

- Take the ratio for assets i and m:

$$
\begin{aligned}
E\,R_i - R &= \beta_i\,[E\,R_m - R] \\
\beta_i &= \frac{Cov\left(R_m, R_i\right)}{Var\left(R_m\right)}
\end{aligned}
$$

- This is the famous CAPM asset pricing equation.

Beta

- The risk premium for asset i depends on:
  - it's **beta** (essentially the correlation with the market)
  - the market price of risk: $E\,R_m - R$.
- A stock's beta can be estimated from data on past returns of the stock (R_i) and the market (using a broad stock index).
- Betas are used to
  - Measure the risk of an asset.
  - Calculate the required rate of return for investment projects.
  - Evaluation of mutual fund managers.

Securities market line

In equilibrium, all asset returns should line up on the SML.
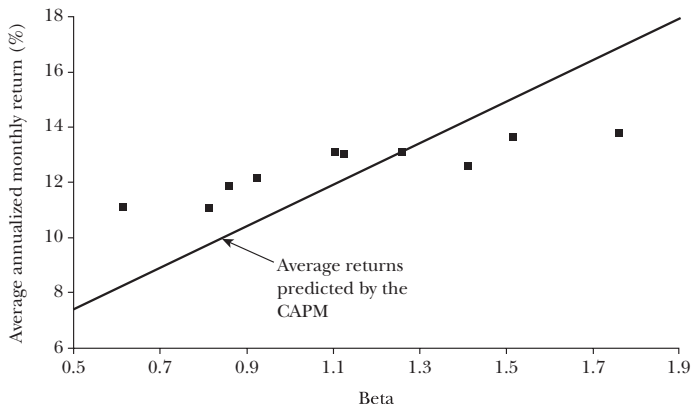
The Securities Market Line (SML)
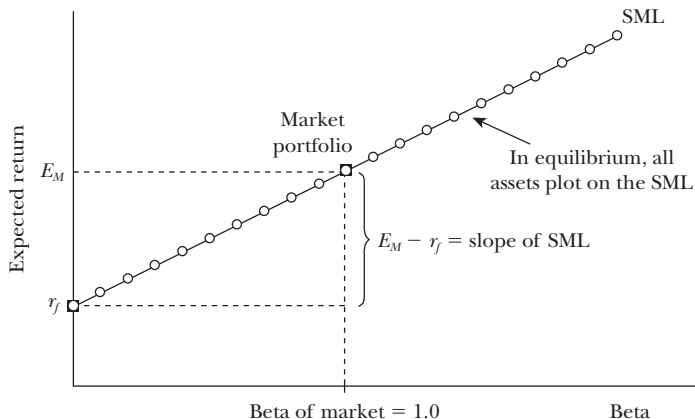
Source: Perold (2004)

Securities market line: Evidence

Stocks with higher βs have higher expected returns, but not enough.

Average Annualized Monthly Return versus Beta for Value Weight Portfolios Formed on Prior Beta, 1928–2003

Source: Fama (2004)

Solving for the asset price

- ▶ We show that the asset price equals the present discounted value of dividends.

- ▶ Start from the Euler equation:

$$u'(c_t) = \beta E_t \left\{ u'(c_{t+1}) \frac{p_{t+1} + d_{t+1}}{p_t} \right\} \qquad (20)$$

- ▶ Solve for the price:

$$p_t = E_t \left\{ \frac{\beta u'(c_{t+1})}{u'(c_t)} (p_{t+1} + d_{t+1}) \right\} \qquad (21)$$

- ▶ Replace p_{t+1} with (21) shifted to $t+1$:

$$
\begin{aligned}
p_t \;=\; & E_t \left\{ \frac{\beta u'(c_{t+1})}{u'(c_t)} d_{t+1} \right\} + \\
& E_t \left\{ \frac{\beta u'(c_{t+1})}{u'(c_t)} E_{t+1} \left[\frac{\beta u'(c_{t+2})}{u'(c_{t+1})} \right] (p_{t+2} + d_{t+2}) \right\} \quad (22)
\end{aligned}
$$

Solving for the asset price

The law of iterated expectations:

$$E_t \{E_{t+1}(x)\} = E_t(x) \tag{23}$$

Eliminate the E_{t+1}:

$$p_t = E_t \left\{ \frac{\beta \, u'(c_{t+1})}{u'(c_t)} \, d_{t+1} \right\} + E_t \left\{ \frac{\beta^2 \, u'(c_{t+2})}{u'(c_t)} \, (p_{t+2} + d_{t+2}) \right\} \tag{24}$$

Iterate forward for T periods:

$$p_t = E_t \left\{ \sum_{j=1}^{T} \frac{\beta^j \, u'(c_{t+j})}{u'(c_t)} \, d_{t+j} \right\} \tag{25}$$

$$+ E_t \left\{ \frac{\beta^{T+1} \, u'(c_{t+T+1})}{u'(c_{t+T})} \, (p_{t+T+1} + d_{t+T+1}) \right\} \tag{26}$$

Solving for the asset price

- Impose that the last term vanishes in the limit:

$$p_t = E_t \left\{ \sum_{j=1}^{\infty} \frac{\beta^j \, u'(c_{t+j})}{u'(c_t)} \, d_{t+j} \right\} \tag{27}$$

- There is no good reason for this assumption!
- The asset price equals the **discounted present value of dividends**.
- The stochastic **discount factor** is the marginal rate of substitution.

Example: Log Utility

- In the Lucas model, assume: $u(c) = \ln(c)$. $K = 1$.
- In equilibrium: $c_t = d_t$.
- $MRS_{t+1} = \frac{\beta \; u'(c_{t+1})}{u'(c_t)} = \frac{\beta \; d_t}{d_{t+1}}$.
- The asset pricing equation becomes

$$
\begin{aligned}
p_t &= E_t \left\{ \sum_{j=1}^{\infty} \frac{\beta^j \, d_t}{d_{t+j}} \, d_{t+j} \right\} \\
&= d_t \, \frac{\beta}{1 - \beta}
\end{aligned}
$$

Example: Periodic dividends

- In the Lucas model, assume:
  - Utility is $u(c) = c^{1-\sigma}/(1-\sigma)$.
  - d_t alternates between d^H and d^L.
  - Asset pricing equation:

$$
\begin{aligned}
p_t &= \sum \beta^j (d_t/d_{t+j})^\sigma \, d_{t+j} \\
&= d_t^\sigma \sum \beta^j d_{t+j}^{1-\sigma}
\end{aligned}
\tag{28}
$$

- On good days, p_t is pulled up by low $u'(c')$, but is pushed down by low d_{t+1}.

The Excess Volatility Puzzle

- Consider a stock with dividend process d_t.
- Its price is given by

$$p_t = E_t \left\{ \sum_{j=1}^{\infty} \frac{\beta^j u'(c_{t+j})}{u'(c_t)} d_{t+j} \right\} \qquad (29)$$

- In the data:
  - Dividends are very smooth (a goal of company policy).
  - Stock prices are much more volatile than dividends.
- But in the theory: stock prices should be the **average** of future dividends and thus **smoother** than dividends.
- This is the flip-side of the Equity Premium Puzzle.

Excess Volatility
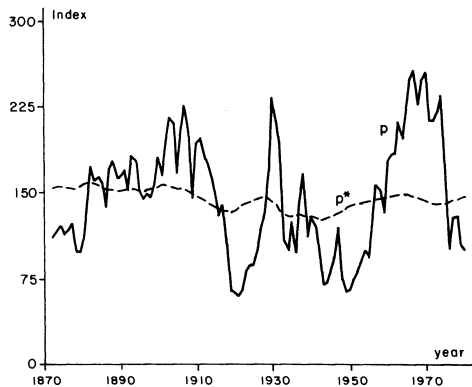
Source: Shiller (1981), figure 1

Bubbles

- Recall how the asset pricing formula is derived:
- We iterate forward on the asset pricing Euler equation

$$p_t = E_t \left\{ \frac{\beta\, u'(c_{t+1})}{u'(c_t)} \left(p_{t+1} + d_{t+1} \right) \right\} \tag{30}$$

- We assume that the p_{t+1} term vanishes in the limit.
- What if it does not vanish?
- Then **any** (current) **asset price** can satisfy the asset pricing equation.
- The deviation between p_t and the **fundamental price** from (30) is called a **bubble**.
- It is purely a self-fulfilling expectation.

Bubbles: Example

- Consider an asset that pays no dividends.
- Its **fundamental price** is 0.
- Assume that the MRS is constant at $\frac{\beta \ u'(c_{t+1})}{u'(c_1)} = 1$.
- The the asset pricing equation is

$$p_t = E_t p_{t+1} \tag{31}$$

- One price process that satisfies this: p doubles with probability 1/2 and drops to 0 otherwise.
- This satsifies (31) for **any** p_t.
- Bubbles are a possible explanation for asset price volatility.
- Note that the bubble does not offer any excess return opportunities.

Reading

- Romer (2011), ch. 7.5
- Ljungqvist and Sargent (2004), ch. 7.
- On the equity premium puzzle: Mehra and Prescott (1985, 2003)

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