Quantitative Finance

ISSN: 1469-7688 (Print) 1469-7696 (Online) Journal homepage: https://www.tandfonline.com/loi/rquf20

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Quantitative Finance, 2015
Vol. 15, No. 9, 1583–1596, http://dx.doi.org/10.1080/14697688.2014.969299

Routledge
Taylor & Francis Group

Dynamic risk taking with bonus schemes

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(*Received 22 February 2013; accepted 18 September 2014*)

This paper studies dynamic risk taking by a risk-averse manager who receives a bonus; the company may default on its contractual obligations (debt and fixed compensation). We show that risk taking is time independent, and is summarized by the so-called risk aversion of derived utility. We highlight the importance of dynamic aspects and provide a foundation for common qualitative discussions that are based on characteristics of bonus functions. The paper cautions that deferral of fixed compensation may increase risk taking. Finally, we motivate a new bonus scheme that incentivizes the manager to implement the socially optimal risk level.

Keywords: Bonus; Risk; Incentives; Inside debt

JEL Classification: G11, G13, G32

1. Introduction

Bonus schemes have been blamed publicly in many financial scandals for their perceived high-powered risk-taking incentives; this has led to multiple government efforts that are targeted to control these.† The literature documented empirically and in theoretical studies states that the structure of managerial compensation matters for risk taking, see, e.g. Ross (2004), Coles *et al*. (2006), and Dittmann and Maug (2007). Theory focuses on static models, often with a discrete state-space,‡ but risk taking can be adjusted dynamically over time; in fact, it is actively managed in financial firms through value-at-risk (VaR) position limits and the allocation of risk capital across divisions; discrete state-spaces make it hard to infer general characteristics of bonus functions. Our goal is to understand whether static analyses paint an adequate picture of risk taking, deepen our understanding of the structure of bonus functions and structure them to attain desired risk taking.

We study a risk-averse manager who controls the risk level (volatility) of the continuous-time asset dynamics (linear diffusion) of a company. To prevent potentially unbounded risk taking, we specify an exogenous cap (maximum risk taking). The company has debt outstanding and may default on its obligations at a terminal date. The manager receives fixed compensation that is senior to debt in case of default; she holds a claim to a bonus payment that is junior to all other claims and determined according to a function based on terminal asset values. We analyse bonus functions that are twice continuously differentiable and strictly increasing above a threshold; we also consider in detail two common schemes (call and capped bonus schemes).

Continuous-time models are often avoided in the literature because they appear to increase technical complexity at the expense of the intuition; our paper, however, recalls the rarely used Dynkin Lemma to come up with intuitive representations. We prove that risk taking for general bonus functions is time independent and typically inverse proportional to what Ross (2004) calls the risk aversion of derived utility, i.e. that of the compounded utility and wealth functions. This allows us to determine, in closed-form, risk taking at all times and in all states for general, call and capped bonus functions. Also, we determine the socially optimal risk level, derive a partial differential equation for a bonus scheme to incentivize the manager to that level across time/states and finally analyse the resulting bonus function as well as the impact of parameter misspecification on risk taking.

A popular argument in the risk-taking literature considers the market price of a bonus claim, infers that price from option

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†Bonus schemes are usually justified on the grounds of incentive-based compensation; starting with Jensen and Meckling (1976) there is a large principal–agent literature, see, e.g. Holmström and Milgrom (1987), Holmström (1992), and Sung (1995). While standard models on moral hazard and asymmetric information also endogenize effort of the agent (manager), see, e.g. Salanié (2005), our paper focuses exclusively on the risk-taking incentives of bonus schemes. An extension should consider the impact of effort on the risk–return trade-off; however, if the effort and risk-taking decisions do not interact, see, e.g. the multiplicative model of Edmans *et al*. (2009), our main results would remain valid.

‡Notable exceptions are Carpenter (2000) and Edmans *et al*. (2012). The static risk-taking literature consists of two period models, see, e.g. Bolton *et al*. (2010), and continuous time models where the manager picks a risk level at the start and never changes it, see, e.g. Green and Talmor (1986), and Han and MacMinn (2006).

theory and argues that convexity of the bonus function leads managers to excessive risk taking, see, e.g. Green and Talmor (1986) and DeFusco *et al.* (1990). Ross (2004) stresses the importance of the manager's utility and introduces the concept of derived utility. He then uses this to introduce and discuss three separate effects that impact risk aversion of derived utility: a translation, a magnification and a convexity effect. He relates the latter to the common convexity argument and concludes with a call for an intertemporal analysis of risk taking. The empirical literature suggests the absence of large-scale manager–owner conflicts, see Fahlenbrach and Stulz (2010), and Cheng *et al.* (2012). Chaigneau (2013) argues that the regulator should set compensation to provide first-best incentives. In line with the risk-taking literature (Carpenter 2000, Ross 2004, Kadan and Swinkels 2007), our paper does not consider optimality of any particular bonus scheme within a shareholder–manager principal–agent relationship and focuses instead on managerial risk taking for a given bonus scheme.

Our paper contributes partly to a quantitative understanding of dynamic risk taking. First, whereas the literature often focuses on static risk taking, see, e.g. Bolton *et al.* (2010), our analysis highlights the importance of dynamic aspects; this is based on our observation that flat portions of the wealth function lead to bang-bang-type risk taking. Second, qualitative analyses, see, e.g. Jensen *et al.* (2004) motivate risk-level choices before the bonus payment date based on local properties of the bonus function (kinks, slopes, caps), but lack a theoretical foundation; our representation through (local) slope and curvature of the bonus (wealth) function provides a quantitative foundation for these. Third, previous studies focus on particular bonus schemes without debt claims, see, e.g. Carpenter (2000), Cadenillas *et al.* (2004), and Duan and Wei (2005), whereas we consider general as well as common bonus functions together with inside debt and fixed compensation; in addition, our characterization of risk taking is simple, intuitive; in a closely related paper† (Leisen 2013), we discuss in detail links to the risk-aversion analysis of Ross (2004).

Our paper also contributes to the inside debt literature. Jensen and Meckling (1976) noted that managerial debt holdings may reduce the agency costs of debt; recently, interest was renewed with the empirical observation of Sundaram and Yermack (2007) that debt-like managerial claims are a common form of managerial compensation. Static studies found managerial debt to reduce risk taking, see, e.g. Edmans and Liu (2011), and Bolton *et al.* (2010); empirical evidence confirmed this finding, see Wei and Yermack (2010), Tung and Wang (2011), and Cassell *et al.* (2012). On the policy sinde, the Squam Lake Working Group on Financial Regulation (2010) called for deferral of fixed compensation to counter risk-taking incentives. Our analysis is dynamic and shows that inside debt eliminates a 'gambling for resurrection' motive, consistent with the theoretical and empirical literature; however, our analysis also exhibits that inside debt coupled with deferred compensation may lead to a wealth effect that increases risk taking.

Finally, our quantitative analysis opens the door to appropriately designed bonus schemes. This contributes to a recent trend that views incentive compensation as an integral part of risk management, see, e.g. Committee of European Banking Supervisors (2010) and Federal Reserve (2011). We illustrate a potential application where the manager picks at all times and states (above our threshold) a pre-set target risk level that is built into the bonus function. We also analyse how this bonus function fares under parameter misspecification (risk aversion and wealth) and how to structure it appropriately.

The remainder of the paper is organized as follows. The next section introduces our set-up. Section 3 discusses risk taking for general bonus schemes and our particular examples. The following section considers inside debt; section 5 studies bonus schemes that incentivize the socially optimal risk level. The paper concludes with section 6. Proofs are postponed to the appendix.

2. The setup

We study a manager who is in charge of running a company over a finite time interval $[0, T]$. Time 0 is today; at time T the company is liquidated and all values paid out. The company has an outstanding (zero-coupon) debt with face value D that matures at time T. The manager does not hold company debt herself; she receives (only) at time T fixed compensation F and we denote by W her time T personal wealth.

2.1. Asset growth and risk taking

The company starts today with (balance sheet) assets in the amount A_0; at any time t the manager observes the current asset value A_t. Asset values follow the stochastic differential equation

$$dA_t = (r + \lambda\sigma(t, A_t))A_t dt + \sigma(t, A_t)A_t dB_t, \qquad (2.1)$$

where B is a standard Brownian motion on a probability space $(\Omega, \mathcal{F}, P)$, P denotes the (objective) probability measure and $\lambda > 0$ the (constant) market price of risk. This asset dynamics is a straightforward extension of the popular geometric Brownian motion asset dynamics (Black–Scholes–Merton model) to a situation where volatility can be adjusted dynamically; a similar set-up has been studied by Carpenter (2000).

The function $\sigma : [0, T] \times R^+ \to R^+$ enters multiplicatively with the Brownian motion B; so this describes what is usually called the volatility of assets, a common measure of risk. Throughout, we refer to σ as managerial *risk taking* (a.k.a. her *risk-level*). We do not discuss the process $\sigma(t, X_t)$; instead, our discussion of risk taking centres on σ as a function that gives the risk level $\sigma(t, x)$ at time $0 \le t \le T$ when the time t realized asset value $X_t = x > 0$.

Our theme in this paper is *what level* of risk the manager *chooses*, it is *not how* the manager *achieves* a (chosen) risk level. Thus, we are agnostic how the manager achieves risk taking; yet, for the interested reader, appendix 1 motivates our asset dynamics based on a portfolio theory framework.

†Both papers apply the same technique based on Dynkin's Lemma; they are complementary as Leisen (2013) studies deferral of bonuses. The first part of Leisen (2013) focuses on extending the translation, magnification and convexity effects of Ross (2004) to related deferral effects and discuss the impact of bonus deferral on risk aversion; the second part discusses the impact of bonus deferral and parameter specifications on continuous-time risk taking.

For technical reasons, we impose an upper limit $\sigma_{\max}$ on permitted risk taking†: $0 \leq \sigma \leq \sigma_{\max}$. This matches practice; usually risk taking is limited by risk management, e.g. through limits on leverage or upper bounds on the VaR.

2.2. Default

Our focus is on risk taking that comes from the manager's investment decision; we are not interested in equity risk coming from the financing decision. Therefore, we assume the debt level is exogenous and cannot be affected by the manager; in particular, we ignore how the debt level impacts the growth of assets.

We assume the company defaults at time T if assets A_T fall short of debt and fixed compensation $D + F$. Without loss of generality, we assume personal wealth W is unaffected by the company's bankruptcy; otherwise we would reinterpret the bonus to include this dependency as a form of incentive compensation. In US bankruptcy code, salaries are senior to other creditors, see, e.g. Calcagno and Renneboog (2007). Thus, in case of default, if assets fall short of the manager's fixed compensation, we assume that she will get first all remaining assets. Also we assume that bonus payments are junior to all other claims, in case of default; therefore, we assume throughout that bonus payments are zero if assets are less than $D + F$.

In our set-up, the company can default only at time T; it is a common assumption in structural models of the credit risk literature that default occurs only at a single date; this started with Black and Scholes (1973) and is motivated by debt maturity at that date.

Default could be modelled in two alternative ways within structural models. First, so-called first-passage models assume default once asset values hit an exogenous boundary before time T; in our diffusion set-up, asset paths are continuous such that default occurs in a first-passage model once asset values hit a default boundary at the time value of $D + F$; this would mean asset values cannot fall below the time value of $D+F$ at any point in time; a major insight of our model is time independence of the risk taking strategy; this greatly simplifies our discussion but it is unlikely that time-independence will continue to hold; our model then provides an initial, educated guess of such a more general framework. A second alternative modelling of default would allow for jumps (discontinuities) in the asset value process. The risk of downward jumps will most likely lead the manager to more cautious risk taking when the company is close to default; a focus of our analysis is on risk taking (well) above the default threshold and, therefore, we expect that major insights continue to hold. Although the previously mentioned extensions are important ones to consider, we refrain from modelling these here, because they would lead to tedious path dependencies that considerably complicate our analysis and interpretations: the Dynkin Lemma would no longer apply and risk taking would have to be studied as a dynamic programming problem; unfortunately, such problems have to be solved numerically, typically.

†This also ensures the existence and uniqueness of a continuous, strong solution of our continuous-time asset dynamics, see analogously our discussion at the end of appendix 1.

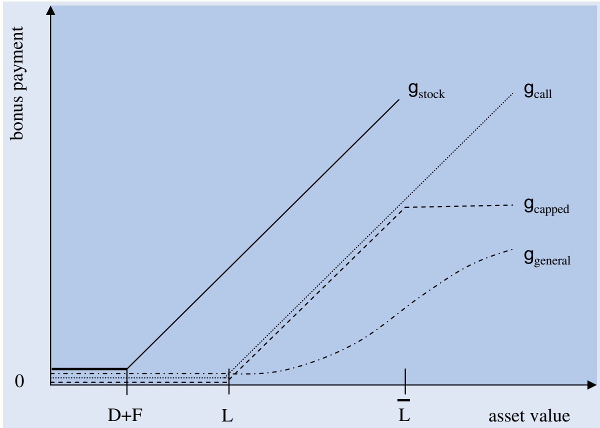

Figure 1. Stock, call and capped bonus as functions of the time T realized asset value $a > 0$.

2.3. Bonus schemes

The manager receives a bonus payment paid at time T. It is determined by the so-called *bonus function* on the positive real line, where the input parameter is the time T realized asset value. In our analysis, we distinguish general bonus schemes that fulfil convenient smoothness properties from particular bonus schemes that match those used in practice.

A *general bonus scheme* is characterized by a bonus function g_{gen} that is zero up to a threshold L, and twice continuously differentiable, strictly increasing and positive for all other asset values; this means that the bonus payment is $g_{gen}(a)$ when the realized time T asset value is $A_T = a$. Our standing assumption introduced in the previous subsection is that the bonus payment is zero if assets are less than $D+F$; this requires that the threshold L is larger than or equal to $D + F$.

In addition to general bonus schemes, we study three particular forms‡§; figure 1 illustrates their bonus functions. The first is the *call bonus scheme*: the manager receives at time T the fraction $\beta > 0$ of the time T realized asset value a that exceeds a threshold L; otherwise nothing is paid. The bonus function is then

$$g_{call}(a) = \beta \max\{a - L, 0\}. \tag{2.2}$$

The second form of bonus is the so-called *capped bonus scheme*. It pays at time T the fraction $\beta > 0$ of what exceeds a lower threshold L up to a maximum that is reached at an upper threshold $\bar{L} > L$. The bonus function is

$$g_{capped}(a) = \beta \cdot \left(\max\{a - L, 0\} - \max\{a - \bar{L}, 0\}\right), \tag{2.3}$$

‡These three bonus schemes are common in practice, see, e.g. Jensen *et al.* (2004). The Institute of International Finance (2009) reports on p. 20 of its survey that 14% of respondents use the stock bonus. Based on Towers Perrin's Incentive Plan Design Survey, 1997, Murphy (1999) reports that 63% of plans set a threshold for performance pay and that 89% cap the bonus claim; this leads us to call and capped bonus schemes.

§The manager is risk averse; thus it is 'costly' to expose here to income risk via bonus schemes; the principal–agent literature provides an extensive discussion of effort incentives and the underinsured position of the manager, see also our footnote † on the first page of this paper. However, in line with the risk-taking literature, our paper studies exclusively the risk-taking decision of the manager and, therefore, we focus on these three particular forms.

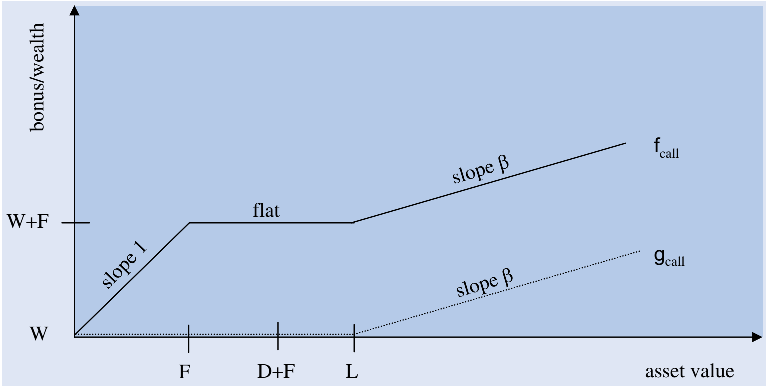

Figure 2. Wealth (bonus) function f_{call} (g_{call}) with call bonus; both are functions of the time T realized asset value $a > 0$.

where a stands for the realized asset value at time T. The capped bonus scheme can be interpreted as a call bonus for which payment is capped at $\beta(\bar{L} - L)$.

The third form is the *stock bonus scheme*. Common stock is junior to debt and fixed compensation and will pay only what exceeds $D + F$; the bonus function is therefore

$$g_{stock}(a) = \beta \max\{a - (D + F), 0\}, \tag{2.4}$$

where a stands for the realized asset value at time T. The stock bonus scheme can be interpreted as a call bonus scheme with $L = D + F$ and therefore we do not discuss it explicitly; in general, our analysis focuses on general, call and capped bonus schemes. Only our analysis of inside debt in section 4 considers explicitly stock bonus schemes.

2.4. *Wealth functions and risk preferences*

For a general bonus function, we denote f_{gen} the manager's time T total wealth function based on the company's asset values. We have

$$f_{gen}(a) = \begin{cases} W + a & ; \text{ if } 0 < a \leq F \\ W + F & ; \text{ if } F \leq a \leq L \\ W + F + g_{gen}(a) & ; \text{ otherwise} \end{cases}. \tag{2.5}$$

The wealth functions for stock, call and capped bonus schemes are denoted by f_{stock}, f_{call} and f_{capped}, respectively. They are defined analogous for the general bonus function.

Figure 2 shows the wealth function f_{call} that results from a call bonus with threshold L. The call bonus function g_{call} (dotted line) is as shown in figure 1. The wealth function f_{call} increases with slope 1 as the asset values go up to F, because the manager's fixed compensation is senior to all other claims. For asset values between F and L, the wealth function is flat; for higher asset values, the bonus function pays out a fraction of the excess (slope β). Although not depicted, the reader can infer the wealth function for stock and capped bonus functions from figure 2: Moving L leftwards to $L = D + F$, the reader gets the wealth function of the stock bonus; capping the bonus payment at $W + F + \bar{L} - L$ gives the one for the capped bonus.

We assume that the risk preferences of the manager are characterized by a strictly increasing, concave and twice differentiable function u. For a given wealth function f, the manager chooses at all times $0 \leq t \leq T$ with current asset value A_t, the risk-taking function $\sigma^*_{f,t} : [t, T] \times R^+ \to R^+$ that maximizes her (conditional) expected utility

$$E[u(f(A_T))|A_t].$$

Depending on the bonus function under consideration, the wealth function f here stands for the wealth function of general, stock, call or capped functions; we will refer to the corresponding risk-taking functions as $\sigma^*_{gen,t}$, $\sigma^*_{stock,t}$, $\sigma^*_{call,t}$ and $\sigma^*_{capped,t}$, respectively. In line with Merton (1969), we ignore time preferences because all payments are at the same point in time (T) and preferences are over wealth, only. We define the Arrow–Pratt (absolute) risk-aversion coefficient R of the manager by setting $R(x) = -u''(x)/u'(x)$ for any $x > 0$; at times we study R as a function in x and refer to this as the (absolute) risk-aversion function, then.

3. Risk taking

This section discusses optimal risk taking. The first subsection studies general bonus schemes while the second subsection looks in more detail at call and capped bonus schemes.

Wealth functions describe the manager's wealth at time T. Strictly speaking, they shall be evaluated at realized time T asset values, only. Our analysis below shows, however, that these functions are a crucial input parameter that drives risk taking at all times. For simplicity, we continue to refer to them as wealth functions, even when we apply them at time $0 \leq t < T$. Similarly, when we discuss the functional dependence of wealth on realized asset values, we distinguish them based on time T thresholds.

3.1. *General bonus scheme*

Throughout this subsection, we study a general bonus schemes g_{gen}, only. Applying Dynkin's formula, appendix 2 proves:

THEOREM 1 (Risk Taking with General Bonus) *Assume a general bonus function g_{gen} with associated wealth function f_{gen} and denote by $R_{gen} = R(f_{gen})f'_{gen} - f''_{gen}/f'_{gen}$ a function on the positive real line. We define a function $\sigma^*_{gen} : R^+ \to R^+$ by setting at realized asset values $a > 0$*

$$\sigma^*_{gen}(a) = \min\left\{\left\{\begin{array}{ll}\lambda\cdot(aR(W+a))^{-1} & ;\ \textit{if } a \le F\\ \sigma_{\max} & ;\ \textit{if } F < a < L\\ \sigma_{\max} & ;\ \textit{if } a \ge L \textit{ and } R_{gen}(a) \le 0\\ \lambda\cdot\left(aR_{gen}(a)\right)^{-1} & ;\ \textit{if } a \ge L \textit{ and } R_{gen}(a) > 0\end{array}\right\}, \sigma_{\max}\right\}.$$

The function σ^*_{gen} describes the manager's risk taking at all times; in particular, the manager's risk taking is time independent.

The result in this Theorem can be linked to mean–variance analysis. Based on Ross (2004, p. 213), we find that $R_{gen} = -V''_{gen}/V'_{gen}$ describes the Arrow–Pratt (absolute) risk-aversion coefficient of the derived utility function. Here, $V_{gen} = u(f_{gen})$ denotes the so-called derived utility function. Using this representation for $a \ge L$, we see that risk taking is based on a risk–return trade-off that also shows up similarly in mean–variance portfolio theory, see, e.g. Duffie (2001) and Pennacchi (2008).†

Within that line of argumentation, our set-up is characterized by a so-called constant investment opportunity set; the time independence then comes as no surprise: Mossin (1968) and Merton (1969) proved in discrete and in continuous time, respectively, that an investor is myopic if she derives utility from terminal wealth, only. In their analysis of the optimal contract, Holmström and Milgrom (1987), and Sung (2005) also find that the manager's choice is time independent. Because risk taking is time independent, in the remainder of this paper we refer to σ^*_{gen} as the risk-taking function and discuss it as a function of the realized asset value, only.

To gain further insight into Theorem 1, let us depart for a moment from our assumption of a risk-averse manager and consider a risk-neutral manager (linear u). Then $R = 0$ and so $R_{gen}(a) = R(f_{gen}(a))f'_{gen}(a) - f''_{gen}(a)/f'_{gen}(a) = -g''_{gen}(a)/g'_{gen}(a)$ for realized asset values larger $a \ge L$. Then, the risk-neutral manager picks maximum risk $\sigma_{\max}$ if $g''_{gen}(a) = f''_{gen}(a) \ge 0$ and $\max\{-\lambda/a \cdot g'_{gen}(a)/g''_{gen}(a), \sigma_{\max}\} > 0$ if $g''_{gen}(a) = f''_{gen}(a) < 0$. This means that risk taking of a risk-neutral manager is driven by the sign of $g''_{gen}(a)$ at the current realized asset value, i.e. on the *local* curvature of the bonus function: For concave bonus functions, the optimal risk level σ^*_{gen} is given by the market price of risk (λ), scaled by the relative ratio of the slope of the bonus function to its curvature. We see that a higher market price of risk λ, higher slope g'_{gen} and less concavity $|g''_{gen}|$, all increase risk taking. This provides a clear and simple answer as to risk taking based on slope and curvature, only. In line with Ross (2004), we find here that the concavity of the bonus function can lead to risk taking below the maximum level *even for risk-neutral agents*; he refers to this as a convexity effect. For a locally convex or locally linear bonus function at the realized asset value ($g''_{gen}(a) \ge 0$), we find that the manager takes maximum risk $\sigma_{\max}$.

Let us now return to our analysis of a risk-averse manager. Compared to the risk-neutral manager, she never takes more risk: for realized asset values $F < a < L$, risk taking continues to be at the maximum level; for realized asset values $0 < a \le F$, risk taking may be below the cap; for realized asset values $a > L$, a strictly positive term $R(f_{gen}(a)) \cdot f'_{gen}(a)$ is added in the denominator which never increases risk taking.

For realized asset values above L, the additional term $R(f_{gen}(a))f'_{gen}(a)$ has an interesting interpretation in the spirit of Ross (2004): It captures two effects that are due to being employed at a company and receiving wealth f_{gen} instead of having personal wealth W. First, $R(f_{gen})$ measures that the risk aversion function is evaluated at total wealth f_{gen} instead of W; it is always at least as high as her wealth W and leads to a change in managerial risk aversion; this measures what Ross (2004) refers to as the translation effect. Second, $f'_{gen}(a)$ shows that the (local) slope of the bonus function matters, the so-called pay-for-performance sensitivity; Ross (2004) refers to this as the magnification effect.

The reader may be surprised to see that a risk-averse manager takes maximum risks on the flat portion of the wealth function, i.e. for asset values between F and L. The intuition here is as follows: On the flat portion, higher risk taking increases the manager's chance on a strictly positive bonus payment (higher wealth). She does not 'fear' any risk taking, because she can react before facing a reduction in wealth as the asset value changes continuously through time; in particular, when the current realized asset value is at F, she will take less than maximum risk $\sigma_{\max}$, if she is sufficiently risk averse.‡ Essentially, this is a form of managerial 'gambling for resurrection'. While this has been noted previously with respect to risk-neutral preferences, see, e.g. Tirole (2005, p. 24), we find it here for risk-averse managers in a dynamic context.

Theorem 1 describes risk taking by the manager based on her wealth function f. Specifying the owner's wealth (and utility) functions, we could infer her desired risk taking, by analogy. A reasonable specification for the owner's wealth function seems to be the asset value less than any bonus payments. However, if the second derivative of the bonus functions is strictly positive at a realized asset value, then the second derivative of the wealth function for the manager (owner) is strictly positive (negative); an analogous statement holds when the second derivative of the bonus functions is strictly negative. Based on Theorem 1 (and its analogous statement), it can then be shown that desired risk taking of manager and owner differs, unless the bonus function g is linear. Within our framework, this confirms the finding of Bannier *et al*. (2013) that managerial risk taking may be excessive from the owner's perspective.

Holmström and Milgrom (1987), and Sung (2005) consider a closely related set-up where a manager with CARA preferences controls the drift and volatility of a Brownian motion. (The drift depends on the costly effort and volatility decision of

†Risk taking here does not come from an investment strategy, but it could be motivated from an appropriate set-up, see appendix 1.

‡For asset values below fixed compensation, the manager's fixed compensation is a claim that is senior to all others. Therefore, a sufficiently risk-averse manager will take less than maximum risk $\sigma_{\max}$.

the manager. Their set-up could be reinterpreted as a risk-neutral manager controlling a geometric Brownian motion.) Our paper focuses on the risk-taking decision for a given bonus scheme; differently to Holmström and Milgrom (1987), and Sung (2005), we do not endogenize the optimal bonus scheme for the owner (a linear contract). The manager's volatility decision drives the quadratic variation process of the asset value process and is measurable with respect to the path; therefore, no agency conflict arises with respect to risk taking, when the owner observes the asset value process. Based on Dynkin's Lemma, our paper could be extended to study the optimal effort level and risk taking of the manager and infers the optimal contract from the viewpoint of the owner. To make such an analysis interesting, it would have to consider more general drift functions than our linear risk–return trade-off; however, we focus on risk taking and such an extension would come at the cost of additional complexity that would interfere with our discussion and interpretation.

3.2. *Particular bonus scheme*

This subsection studies call and capped bonus schemes. We can apply directly the results of Theorem 1 to the call bonus; note that we have with $f_{gen} = f_{call}$ that $R_{gen} = R(f_{call})\beta > 0$ for $a \geq L$. The capped bonus function is flat above the upper threshold $\bar{L}$ and we prove this case in appendix 2:

Theorem 2 (Risk Taking with Particular Bonus) *With a call bonus,*

$$\sigma^*_{call}(a) = \min\left\{\left\{\begin{array}{ll}\lambda \cdot (aR(W+a))^{-1} & ;\ if\, a \leq F\\ \sigma_{\max} & ;\ if\, F < a < L\\ \lambda \cdot (a\beta R(W+F+\beta(a-L)))^{-1} & ;\ if\, L \leq a\end{array}\right\}, \sigma_{\max}\right\}.$$

With a capped bonus,

$$\sigma^*_{capped}(a) = \min\left\{\left\{\begin{array}{ll}\lambda \cdot (aR(W+a))^{-1} & ;\ if\, a \leq F\\ \sigma_{\max} & ;\ if\, F < a < L\\ \lambda \cdot (a\beta R(W+F+\beta(a-L)))^{-1} & ;\ if\, L \leq a < \bar{L}\\ 0 & ;\ if\, \bar{L} \leq a\end{array}\right\}, \sigma_{\max}\right\}.$$

The results for realized asset values $a < L$ are not surprising; they have been noted already for general bonus schemes and discussed in the previous subsection. More interesting is that risk taking of call and capped bonus schemes is identical for all $L < a < \bar{L}$ when their parameterization of threshold L and PPS β is identical. This means that their risk taking differs only above the threshold $\bar{L}$, i.e. on the portion where the bonus (wealth) functions actually differ. The manager takes strictly positive risks with the call bonus but risk taking vanishes with a capped bonus.

Vanishing risk taking is an entirely dynamic aspect that does not show up in static analysis of risk taking. As long as the realized asset value is less than the upper threshold, it makes sense for the manager to take non-vanishing risks because she holds a chance on a higher pay-off. However, it also makes sense that risk taking vanishes above the upper threshold with capped bonus schemes: Once the asset value has reached the upper threshold, the manager has ensured her maximum bonus payment. She will not benefit from any risk taking; actually, strictly, positive risk taking would mean that she faces the risk of falling below the threshold and faces a reduction in her bonus payment. Therefore, it is in her personal interest to eliminate all risk taking.

Increasing the PPS β impacts risk taking only for realized asset values $a > L$ ($L < a < \bar{L}$) for the call (capped) bonus. Let us focus for a moment on these values. There, increasing PPS affects risk taking through a translation effect and a magnification effect. The latter leads to a reduction in risk taking but the impact of the former is unclear: increasing PPS increases wealth, but unless we know the monotonicity of the risk aversion function, we cannot hope to infer the impact. It is usually assumed that preferences exhibit decreasing absolute risk aversion (DARA); if we adopt this assumption, however, then an increase in PPS means that the translation effect leads to larger risk taking and the overall effect remains unclear. To restrict ourselves further, a common form of risk preferences are hyperbolic absolute risk aversion (HARA) preferences with decreasing absolute risk aversion (DARA); for these preferences, it follows then that an increase in PPS decreases risk taking.†

We would prefer dynamic risk taking to be smooth in asset values but find here that it is of bang-bang type for both bonus schemes: for both bonus schemes, risk taking is at its maximum when the realized asset value is between F and L but may be lower at all other realized asset values, depending on PPS β and risk aversion. In addition, for the capped bonus, risk taking is zero for realized asset values larger than $\bar{L}$, but it is strictly positive for all other realized asset values.

4. Inside debt

This section studies risk taking when the manager's compensation is paid out partly in the form of debt.‡ Here, we denote by $0 < \alpha \leq 1$ the fraction of outstanding debt that the manager holds and by $\bar{F}$ her fixed compensation. (We continue to denote by F the fixed compensation without inside debt.) Throughout this section we assume that $\bar{F} < F$, i.e. the additional debt claim with inside debt reduces her fixed compensation; at the

†Carpenter (2000) studied risk taking with call bonus and hyperbolic absolute risk aversion (HARA) preferences with decreasing absolute risk aversion (DARA) in a set-up without default; she came to the same conclusion.

‡In a closely related paper (Leisen 2013), we focus on the deferral of *bonus* payments. A major insight of that paper is that bonus deferral leads to increased risk taking for many asset values and that risk taking may decrease only for asset values on a well-specified interval and only for judiciously chosen specifications of deferral parameters.

end of this section, we focus on deferral of fixed compensation and restrict us to $\bar{F} = F - \alpha D$.

A large part of the literature on inside debt considers debt claims jointly with the stock bonus; therefore, differently to the remainder of this paper, this section considers the stock bonus, only. Recall from subsection 2.3 that we treat the stock bonus as a call bonus with threshold $D + F$ (without inside debt), respectively $D + \bar{F}$ (with inside debt). The wealth function with inside debt is given by

$$f_{inside}(a) = \begin{cases} W + a & ;\ \text{if } a \le \bar{F} \\ W + \bar{F} + \alpha(a - \bar{F}) & ;\ \text{if } \bar{F} < a < D + \bar{F} \\ W + \bar{F} + \alpha D + \beta(a - (D + \bar{F})) & ;\ \text{if } D + \bar{F} \le a \end{cases}. \tag{4.1}$$

Figure 3 illustrates the wealth function f_{stock} without inside debt (solid line); it also presents the wealth function f_{inside} for three cases with inside debt that differs in their choice of α in relation to β: $0 < \alpha < \beta$ (dashed line), $0 < \alpha = \beta$ (dash-dot line), and $0 < \beta < \alpha \le 1$ (dotted line). Without inside debt (solid line), the wealth function grows linearly with slope 1 up to F, is flat with value $W + F$ for realized asset values between F and $D + F$; it grows with slope β above that. With inside debt (all other lines), the wealth function grows linearly with slope 1 up to $\bar{F} < F$; then it grows linearly with slope α up to $D + \bar{F}$ due to the inside debt claim; for larger realized asset values, it grows with slope β.

Static analysis may draw misleading conclusions about risk-taking incentives; to see why, we restrict for a moment our attention to the domain of asset values larger than $\bar{F}$. We look at figure 3 and compare the wealth function with inside debt when (1) $\alpha < \beta$, when (2) $\alpha = \beta$ and when (3) $\alpha > \beta$: we note that for cases (1) and (3), there is a kink and that this kink leads the wealth function in case (1) to be convex on the domain $a > \bar{F}$ and to be concave in case (3); only in case (2) there is no kink and the wealth function is linear (on the domain $a > \bar{F}$). Such a static and qualitative analysis of convexity/concavity would be along the lines of Jensen *et al.* (2004) and is common in the literature. This kind of analysis would conclude that inside debt with $\alpha < \beta$ leads to 'excessive' risk taking $\sigma_{\max}$ and suggests setting $\alpha \ge \beta$. Our continuous-time analysis, however, clarifies that the local properties of the wealth function (slope, curvature) together with those of the risk-aversion function matter. Extending Theorem 1, appendix 2 proves:

THEOREM 3 *Assume the manager holds a fraction $0 < \alpha \le 1$ of debt. The manager picks for realized asset values $a > 0$ the risk level*

$$\sigma^*_{inside}(t, a) = \left\{ \sigma_{\max}, \lambda(aR(f_{inside}(a)))^{-1} \begin{Bmatrix} 1 & ;\ \text{if } a \le \bar{F} \\ \alpha^{-1}; & ;\ \text{if } \bar{F} < a < D + \bar{F} \\ \beta^{-1} & ;\ \text{if } D + \bar{F} \le a \end{Bmatrix} \right\},$$

where the wealth function f_{inside} is given through equation (4.1).

A key role of inside debt is to eliminate the 'gambling for resurrection' motive; we noted in subsection 3.1 that this motive is due to a flat wealth function on a portion of asset values. The introduction of inside debt leads to a strictly positive slope $\alpha > 0$ for realized asset values between F to $D + F$. Theorem 3 shows that risk taking *with inside debt* is less than $\sigma_{\max}$ at all asset values, if the manager is sufficiently risk averse. Thus, we can confirm that inside debt eliminates such a 'gambling for resurrection' motive. This is a *positive aspect* of inside debt.

Note in figure 3 that the wealth function with inside debt may be larger than that without for asset values $a > D + \bar{F}$. In the previous section, we studied the impact of introducing a bonus on risk taking; we related a shift in the wealth function to the translation effect of Ross (2004). At a minimum, further analysis requires assumptions about the monotonicity of the risk-aversion function R.

For further analysis, in this section, we adopt the usual assumption that preferences are DARA, i.e. that the absolute risk-aversion function R is decreasing. A straightforward calculation reveals for all asset values $a \ge D + F$, that the wealth function with inside debt is larger than the wealth function without, if and only if $\bar{F} > F - \alpha D/(1 - \beta)$. Since the slope of the wealth function is the same, this translation effect (a wealth effect) means that the introduction of inside debt increases risk

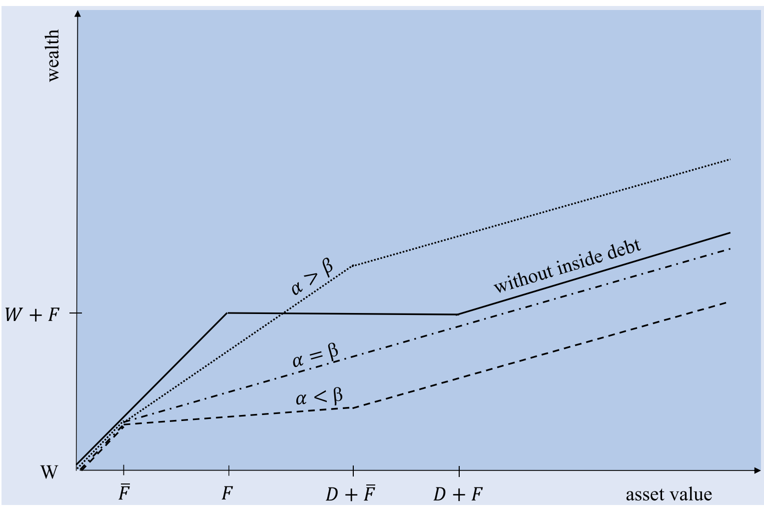

Figure 3. Wealth function based on realized time T asset value without inside debt (solid line) as well as three cases with inside debt: $0 < \alpha < \beta$ (dashed line), $0 < \alpha = \beta$ (dash-dot line), and $0 < \beta < \alpha \le 1$ (dotted line).

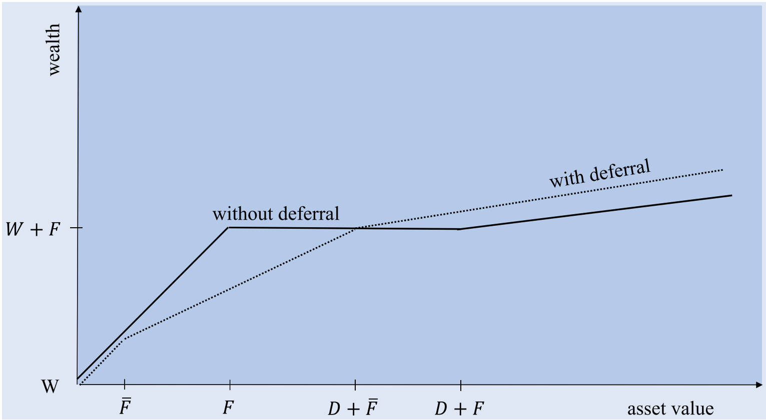

Figure 4. Wealth function based on realized time T asset value without deferral (solid line) and with deferral of fixed compensation (dotted line); depicted is the case $0 < \alpha < \beta$.

taking for asset values above the threshold $D + F$. This is a *negative aspect* of inside debt. The previous literature found that inside debt reduces risk taking, see, e.g. Edmans and Liu (2011), and Wei and Yermack (2010); we can confirm this for a subset of realized asset values but also show that risk taking increases for many other situations.

In response to the financial crisis, the Squam Lake Working Group on Financial Regulation (2010) called for deferral of fixed compensation to counter risk-taking incentives. To study this, we assume the debt holding is part of the overall managerial compensation arrangement, i.e. deferral reduces her fixed compensation by αD to $\bar{F} = F - \alpha D$ in exchange for the claim on a fraction α of debt. With this choice of $\bar{F}$, we continue to use our descriptions of the wealth function f_{inside} in equation (4.1) and of risk taking in Theorem 3.

Note that we have $f_{inside}(D+\bar{F}) = W+F = f_{stock}(D+\bar{F})$ since $D + \bar{F} < D + F$. This implies that the wealth function with deferral of fixed compensation is larger than that without for all asset values $a > D + \bar{F}$. (Note that for deferral of fixed compensation, this result holds whatever α in relation to β.) Figure 4 illustrates the wealth function without (solid line) and with deferral of fixed compensation (dotted line). It shows that the wealth function with deferral is larger than that without for asset values $a > D+\bar{F}$. (Differently to figure 3 we illustrate in figure 4 only the case $0 < \alpha < \beta$, but the equality of the wealth functions with/without inside debt holds for all α in relation to β.) Figure 4 shows also that the wealth function with deferral is smaller than that without for asset values $\bar{F} < a < D + \bar{F}$.

Let us assume that the manager is sufficiently risk averse at an asset value $a > D+\bar{F}$ to take risk below the maximum $\sigma_{\max}$. Then, we conclude from the DARA property that deferral of fixed compensation increases risk taking. Based on Theorems 2 and 3 we find:

THEOREM 4 *Assume preferences with DARA. Subject to risk taking being below the maximum $\sigma_{\max}$, introducing deferral of fixed compensation with $\bar{F} = F - \alpha D$ leads to the following changes in risk taking:*

$$\begin{aligned} \textit{unchanged}\ &;\ \textit{for } a < \bar{F},\\ \textit{increases}\ &;\ \textit{for } \bar{F} < a \le F,\\ \textit{decreases}\ &;\ \textit{for } F < a < D+\bar{F},\\ \textit{decreases}\ &;\ \textit{for } D+\bar{F} < a < D+F\\ \textit{increases}\ &;\ \textit{for } a > D+F. \end{aligned}$$

We could provide sufficient conditions for risk taking being below the maximum; doing so would make it harder to see the message of Theorem 4; therefore, we refrain from presenting these here and present the statement only for risk taking subject to the maximum constraint not being bound.

Theorem 4 comes to the same results that we noted above for inside debt, in general: deferral of fixed compensation decreases risk taking for realized asset values $F < a < D+F$, i.e. it eliminates partly a 'gambling for resurrection' motive; however, it increases risk taking for realized asset values $\bar{F} < a < F$, i.e. it increases risk taking near bankruptcy. Our main interest lies in asset values well above bankruptcy, i.e. $a > D + F$. Unfortunately, as noted before, the deferral of fixed compensation increases risk taking for these values.

Nothing in Theorem 4 provides a foundation for setting α in relation to β. In particular, nothing motivates setting $\alpha = \beta$; this means we cannot provide a foundation for the recommendation of Jensen and Meckling (1976) that $\alpha = \beta$ aligns managerial risk-taking incentives with bondholders' interests; our result matches Edmans and Liu (2011).

5. Incentivizing the socially optimal risk level

This section discusses bonus schemes that incentivize the manager to pick the first-/second-best risk level across time/realized asset values. The typical application would be a regulator that is interested in the risk management role of bonus schemes and wants to ensure the socially optimal risk level through appropriately chosen bonus functions.

5.1. *First-best (socially optimal) and second-best risk taking*

The corporate finance literature discusses extensively the concept of financial distress costs; we take from that literature that risk taking is costly for the company. Usually, adjusting risk taking is also costly; for example, to reduce firm risk, the manager may need to liquidate some risky projects, which may impose a loss. The instantaneous cost of risk taking is denoted $c(a, \sigma)$, at current asset value a and risk level σ; similar to Sung (2005), the function c is assumed to be an increasing, concave function in the risk level and the cumulative cost up to a time t is $\int_0^t c(A_s, \sigma(s, A_s))ds$ without discounting.

In financial crises, banks are often bailed out with associated costs to the taxpayer. While manager and owner are concerned about their respective shares of the asset value, society as a whole will be concerned about the aggregate (the entire company). Similarly, Merton (1977) determines the optimal cost of deposit insurance in such a way that only a bank's asset value matters and no subsidy from deposit insurance (society) is involved. Therefore, we determine the socially optimal risk level as that risk level which maximizes the net expected terminal value of the entire company, irrespective of how it is distributed among manager and owner,† i.e. the expected terminal asset value less cumulative costs. Overall, this is the risk level that maximizes

$$E\left[A_T - \frac{1}{2}\int_0^T c(A_s, \sigma(s, A_s))ds\right].$$

We denote the first-best (socially optimal) risk level by σ_F^* and study it as a function of current time and asset value. Appendix 2 proves:

THEOREM 5 *At all times $0 \leq t \leq T$ and for all realized asset values $a > 0$, the socially optimal risk level $\sigma_F^*(t, a)$ solves*

$$\frac{1}{2}\frac{\partial c}{\partial \sigma}(a, \sigma_F^*(t, a)) = \lambda a.$$

The Theorem has an interesting interpretation: the instantaneous benefit from a marginal change in risk taking is given by the marginal contribution to the drift rate λa of our asset dynamics; the instantaneous marginal cost of risk taking is $\partial c/\partial \sigma$. Theorem 5 states that at the continuous-time, socially optimal risk-taking strategy equates marginal costs and benefits at all times and for all realized asset values. Note that marginal benefit is time independent; so the socially optimal risk level is time independent, too.

For further illustration, we assume throughout the remainder of this paper a functional form for the cost function: Similar to Bolton *et al*. (2010), we adopt a cost function that is linear in size and quadratic in risk taking, i.e. we take the cost function $c(a, \sigma) = \kappa\sigma^2 a$ for a suitable constant $\kappa > 0$. Then, we calculate based on Theorem 5 the *socially optimal (first-best) risk level* at all times $0 \leq t \leq T$ and for all realized asset values $a > 0$:

$$\sigma_F^*(t, a) = \frac{\lambda}{\kappa}.$$

Because bonus payments are junior in case of default, bonus functions need to be zero below a threshold $L \geq D + F$, see our discussion in subsection 2.2. For such realized asset values, Theorem 1 shows that managerial risk taking is independent of the functional form of the bonus function, that it is always given by $\sigma^*(t, a) = \lambda/\gamma$ for $0 < a \leq F$ and that it is at its cap for realized asset values $F < a < L$. Typically, $\gamma \neq \kappa$ and so risk taking will differ from the socially optimal risk level for realized asset values $0 < a \leq F$; also the cap $\sigma_{\max}$ is typically different from the socially optimal risk level. In our set-up, this means that we cannot incentivize the first-best risk level for all realized asset values; thus, we draw our attention to realized asset values $a \geq D + F$.

This suggests taking the minimum permitted threshold $L = D + F$ and we adopt that choice. In addition, for asset values smaller than the threshold $D + F$, the company's assets are less than fixed compensation such that the company would typically be liquidated before time T; in our set-up, however, we excluded such liquidation for simplicity; therefore, we are not concerned about risk levels for asset values below $D + F$.

Throughout the remainder of this paper, we focus on the *second-best risk level* that fulfils at all times $0 \leq t \leq T$ and for all realized asset values $a \geq D + F$:

$$\sigma_S^*(t, a) = \frac{\lambda}{\kappa}.$$

5.2. *Second-best bonus functions*

We assume that the social planner cannot enforce the socially optimal risk level; she can only *incentivize* the manager. This means the social planner can decide on the bonus scheme and does it in such a way that the manager, acting in her own interest, picks the risk level that is desired by the social planner. We discussed at the end of the previous subsection that the social planner can only hope for the second-best risk level. We call *second-best bonus function* a bonus function with the property that the manager picks the risk level λ/κ at all times $0 \leq t \leq T$ and for all realized asset values $a \geq D + F$.

Without functional form of the utility function u, i.e. of the risk-aversion function R, we cannot characterize target bonus schemes. To illustrate our approach, therefore, in the remainder of this section, we restrict ourselves to specific preferences with constant relative risk aversion (CRRA, a.k.a. power utility); they are characterized by

$$u(x) = \frac{x^{1-\gamma}}{1-\gamma} \text{ and } 1 \neq \gamma > 0.$$

The case $\gamma = 1$ would correspond to log utility; to simplify our presentation, we restrict ourselves to risk-aversion parameters $\gamma \neq 1$. It is well known that then $R(a) = \gamma/a$.

Throughout the remainder of this paper, we assume that the socially optimal risk level is less than the cap, $0 < \lambda/\kappa < \sigma_{\max}$. It is important to note that below the threshold $L = D + F$, risk taking is not affected by bonus schemes. Therefore, unless stated otherwise, our analysis in this section focuses exclusively on asset values above the threshold $D + F$. Subject to suitable differentiability assumptions, Theorem 1 guides us in finding a second-best bonus function: to incentivize the target risk level λ/κ, this requires the following *risk targeting differential equation* to hold for the wealth function (above $D + F$):

$$a\left(R(f_S)f_S'(a) - \frac{f_S''(a)}{f_S'(a)}\right) = a\left(\gamma\frac{f_S'(a)}{f_S(a)} - \frac{f_S''(a)}{f_S'(a)}\right) = \kappa. \tag{5.1}$$

The wealth function f_S that we search is related to the bonus function through $f_S = g_S + W + F$ above $D + F$; in a second step, this allows us to infer the bonus function g_S.

†Essentially, we consider here the economic concept of standard of living; it should be distinguished from the economic concept of social welfare (aggregate utility). We focus on the concept of standard of living, since the current policy debate centres on the total gains to society less costs of risk taking; the costs of risk taking include the costs of bank bailouts, which may be interpreted as a form of financial distress costs. It would be interesting to study aggregate risk-sharing aspects and analyse social welfare (aggregate utility), but this is beyond the focus of this paper.

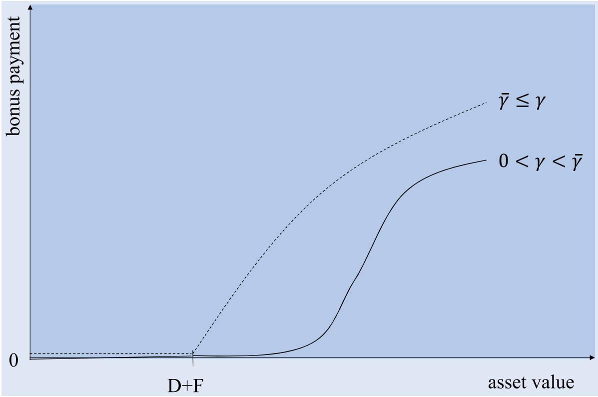

Figure 5. Shapes of the power bonus function.

Appendix 3 provides a characterization of the wealth functions and corresponding bonus functions that fulfil this partial differential equation. Throughout the remainder of this section, we consider only the case $\kappa > 1$ and the *power bonus* function g_{power}, defined for realized asset values $a > 0$ by setting

$$g_{power}(a) = \begin{cases} 0 & ; \text{ if } a \le D+F \\ (W+F)\left(\left(1+\xi\cdot\left(1-\left(\frac{D+F}{a}\right)^{\kappa-1}\right)\right)^{\frac{1}{1-\gamma}}-1\right) & ; \text{ if } a \ge D+F \end{cases}. \tag{5.2}$$

Here, ξ is a constant with the property that $\xi > 0$ ($\xi < 0$) for $\gamma < 1$ (for $\gamma > 1$). Theorem 1 confirms that the power bonus function g_{power} defined above is a second-best bonus function, i.e. the manager picks the second-best risk level λ/κ for all $a \ge D+F$.

Figure 5 illustrates the shapes of power bonus functions, in general. The function is a continuous function in the realized asset value, in particular it is continuous at $L = D+F$. Below $D+F$ it is zero, while above $D+F$ it is strictly positive and strictly increasing. Depending on the parameterization and $\gamma > 0$, the bonus function is either locally concave at all realized asset values above $D+F$ or it is first locally convex and then becomes locally concave; the threshold† $\bar{\gamma}$ depends on $\kappa - 1$ in relation to ξ.

For $0 < \gamma \le \bar{\gamma}$, the shape of the bonus function resembles that of a so-called S-curve bonus. The Institute of International Finance (2009) reports on p. 20 of its survey that 31% of respondents use a formal payout function to determine bonus payments and that 17% use the so-called S-curve bonus, i.e. more than half of the respondents that use a formal payout function use one that resembles the power bonus scheme that we introduced in this section.

5.3. *Model misspecification*

Managerial risk aversion and personal wealth are usually unknown in practice. This subsection studies the implications of a misspecified risk-aversion coefficient with CRRA preferences and of a misspecification of the manager's wealth. Throughout this subsection, we consider the power bonus function introduced in equation (5.2) that incentivizes the second-best risk level for a manager who has wealth W, fixed compensation F and CRRA preferences with risk-aversion coefficient γ; also, we continue to focus on asset values above the threshold $L = D+F$.

We refer to the parameters that are used to determine the bonus function as the 'believed' managerial characteristics (risk-aversion parameter γ and personal wealth W). To consider misspecification, we distinguish them from the 'actual' (true) managerial characteristics: we denote by $\bar{W}$, $\bar{\gamma}$ and $\bar{f}_{power}$ her actual personal wealth, risk-aversion parameter and wealth function. This allows us to relate the believed and actual wealth function by $\bar{f}_{power} = f_{power} + (\bar{W} - W)$. Based on this, we denote δ the relative error in her wealth specification and f_{rel} the (believed) relative wealth function, i.e. for $a \ge D+F$:

$$\delta = \frac{\bar{W}-W}{F+W}, \text{ and } f_{\text{rel}}(a) = \frac{\bar{f}_{power}(a)}{W+F} = \left(1+\xi\cdot\left(1-\left(\frac{D+F}{a}\right)^{\kappa-1}\right)\right)^{\frac{1}{1-\gamma}} - 1.$$

According to Theorem 1, risk taking above $D+F$ is fully characterized by $R(\bar{f}_{power})\bar{f}'_{power} - \bar{f}''_{power}/\bar{f}'_{power} = \bar{\gamma} f'_{power}/(f_{power} + \bar{W} - W) - f''_{power}/f'_{power}$. Because the wealth function f_{power} fulfils the risk-targeting differential equation (5.1), risk taking can be expressed by

$$\lambda\left(\frac{\lambda}{\bar{\sigma}} + a\frac{f_{\text{rel}}(a)\cdot(\bar{\gamma}-\gamma) - \gamma\delta}{f_{\text{rel}}(a)\cdot(f_{\text{rel}}(a)+\delta)}\right)^{-1}, \tag{5.3}$$

as long as the term in the denominator is strictly positive. If this is not the case, then Theorem 1 states that the manager takes maximum risk $\sigma_{\max}$.

Based on equation (5.3), we can determine the impact of changes in γ on risk taking. To see this, it is important to note that f_{rel} does not depend on $\bar{W}$, W, or F. Assume $\bar{W} \le W$, i.e. we believed personal wealth to be higher than it actually is (or correctly specified). Then $\delta \le 0$ and we see that higher risk-aversion coefficient $\bar{\gamma} > \gamma$ leads to less risk taking. When $\bar{W} > W$, i.e. we believed personal wealth to be less than it actually is, then $\delta > 0$ and risk taking increases if $\bar{\gamma} < \gamma$, subject to the cap. In all other cases, we need to consider the functional form of f_{rel}; to analyse this quantitatively, we calculate out (5.3):

THEOREM 6 (Misspecified Preferences and Wealth) *Assume the manager receives a bonus according to the power bonus function of equation (5.2), her personal wealth is $\bar{W}$ and she has CRRA preferences with risk-aversion parameter $\bar{\gamma}$. Define a function η by setting for $a > L$:*

$$\eta(a) = \left(1+\xi-\xi\left(\frac{a}{D+F}\right)^{1-\kappa}\right)^{1/(1-\gamma)}.$$

For realized asset value, $0 < a \le F$, the manager picks the risk level $\lambda/\bar{\gamma}$; she takes maximum risk for realized asset value

†The description of the different cases is tedious and does not provide insights. We refer the interested reader to appendix 3.

Table 1. Risk taking for different realized asset values above the threshold $D+F$. The power bonus scheme was structured to incentivize the second-best risk level $\lambda/\kappa = 0.1$ for a manager with CRRA risk-aversion parameter γ. The wealth misspecification in relation to wealth and fixed compensation is δ and the actual CRRA risk-aversion parameter of the manager is $\bar{\gamma}$.

	Panel A: $\gamma = 2$									
	$\delta = 0$					$\delta = 0.1$				
	$a/(D+F)$					$a/(D+F)$				
$\bar{\gamma}$	1.1	1.2	1.3	1.4	1.5	1.1	1.2	1.3	1.4	1.5
2	0.100	0.100	0.100	0.100	0.100	$\sigma_{\max}$	0.181	0.128	0.115	0.109
3	0.033	0.050	0.061	0.069	0.075	0.109	0.075	0.076	0.080	0.083
4	0.020	0.033	0.044	0.053	0.060	0.044	0.047	0.054	0.061	0.067
	Panel B: $\gamma = 4$									
	$\delta = 0$					$\delta = 0.1$				
	$a/(D+F)$					$a/(D+F)$				
$\bar{\gamma}$	1.1	1.2	1.3	1.4	1.5	1.1	1.2	1.3	1.4	1.5
2	$\sigma_{\max}$	$\sigma_{\max}$	$\sigma_{\max}$	0.439	0.227	$\sigma_{\max}$	$\sigma_{\max}$	$\sigma_{\max}$	$\sigma_{\max}$	0.409
3	$\sigma_{\max}$	1.148	0.229	0.163	0.139	$\sigma_{\max}$	$\sigma_{\max}$	$\sigma_{\max}$	0.558	0.234
4	0.100	0.100	0.100	0.100	0.100	$\sigma_{\max}$	$\sigma_{\max}$	1.640	0.238	0.164

$F < a < L$. For realized asset value, $a \geq D+F$, she takes the risk level

$$\lambda \cdot \left(\kappa + (\kappa-1)\frac{\eta}{\eta-1} \cdot \frac{\delta\gamma + (\eta-1)(\gamma-\bar{\gamma})}{\delta(\gamma-1)+(\eta-1)(\gamma-1)} \cdot \frac{\xi}{(1+\xi)\left(\frac{a}{D+F}\right)^{\kappa-1} - \xi}\right)^{-1},$$

as long as this is strictly positive; otherwise, she takes maximum risk $\sigma_{\max}$.

Table 1 uses Theorem 6 to determine risk taking quantitatively. We assume there that $\lambda = 0.5$ (a typical value for the market price of risk), and choose $\xi = -0.5$; we set $\kappa = 5$ such that the second-best risk level is $\lambda/\kappa = 0.1$. We do not specify quantitatively the cap $\sigma_{\max}$; unless the risk level is known to be $\sigma_{\max}$ we always show the value that comes out of the formula.

It is usually assumed that the risk-aversion parameter γ is between 2 and 4. Therefore, Panel A studies the smallest value within that range ($\gamma = 2$) and Panel B the largest ($\gamma = 4$). In both Panels, the left-hand part of each panel considers $\delta = 0$ (no wealth misspecification) and the right-hand part considers $\delta = 0.1$ (actual wealth is higher by 10% of wealth and fixed compensation). Each of the four parts is structured similarly: it considers variation in $a/(D+F) = 1.1, 1.2, 1.3, 1.4$ and 1.5; it also considers variation within the typical range of the *actual* risk-aversion parameters $\bar{\gamma} = 2$, 3 and 4.

It may be surprising that risk taking may become quite large in some cases; this happens, for example, if the believed and actual risk-aversion parameters are $\gamma = 4$ and $\bar{\gamma} = 2$, respectively, both with and without misspecified wealth ($\delta = 0$ and $\delta = 0.1$). To understand this, we recall that for sufficiently small risk-aversion parameter the bonus function is initially locally convex and eventually becomes locally concave (see figure 5 for an illustration). We also recall that we stressed in subsection 3.1 that the curvature of the derived utility drives risk taking. If we have a manager with sufficiently small actual risk-aversion parameter $\bar{\gamma} < \gamma$, the derived utility function will be locally convex and risk taking is $\sigma_{\max}$, initially; eventually, as we look at larger realized asset values, the derived utility function will be concave but risk taking may be large.

We see with $\delta = 0$ that for $\gamma = \bar{\gamma} = 2$ and $\gamma = \bar{\gamma} = 4$ risk taking is equal to our second-best level 0.1, exactly as it was meant to be. We also find that decreases in $\bar{\gamma}$ increases risk taking for both $\delta = 0$ and $\delta = 0.1$. Overall, this recommends using the risk-aversion parameter $\gamma = 2$ and using values for personal wealth (W) that are upward biased to ensure that actual wealth is always less than or equal to the believed one ($\bar{W} \leq W$).

6. Conclusion

This paper studied how a risk-averse manager sets volatility (risk taking) of a linear diffusion across time and states in the presence of bonus schemes when debt and fixed compensation are subject to default risk. Our analysis provided a quantitative foundation to prior qualitative arguments of dynamic risk taking based on bonus functions and extended the characterization of Ross (2004) to continuous-time models. We found that static risk taking does not provide an accurate picture of dynamic risk taking as it does not capture bang-bang-type behaviours. In addition, we found in our dynamic analysis that inside debt eliminates the 'gambling for resurrection' motive in bonus schemes and thereby controls risk taking; however, with deferral of fixed compensation, the introduction of inside debt may increase risk taking for all realized asset values above a threshold. Finally, we determined first- and second-best optimal risk taking and introduced a new, so-called power bonus which incentivizes the second-best risk level; this feeds into a current regulatory trend to view on bonus schemes as an integral component within the risk management function of a company.

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Appendix 1. Asset dynamics in a portfolio framework

This appendix motivates the asset dynamics in equation (2.1) from a portfolio framework. We start with the assumption that the manager can invest (continuously) into a risk-free bond with constant return r over time and into a risky asset with price process

$$\frac{dS_t}{S_t} = (r + \lambda v)dt + v\,dB_t,$$

where $(\Omega, \mathcal{F}, P)$ is the same probability space we studied in section 2. The so-called volatility parameter v is assumed constant, i.e. S follows a so-called geometric Brownian motion.

We describe her investment strategy as a function of time t and asset value $a > 0$: for such a combination the manager invests the fraction $\pi(t, a) > 0$ of the company's (balance sheet) assets into the *risky* and the remainder $1 - \pi(t, a)$ into the *riskfree* bond. (As usual, negative values are interpreted as borrowing.) The manager does not take out money from the company before terminal time T and so the investment strategy is self-financing. Risk taking translates directly into an investment strategy through the relationship $\sigma(t, a) = v\pi(t, a)$. This leads to the stated asset dynamics:

$$\begin{aligned} dA_t &= (r + \lambda v\pi(t, a))A_t + v\pi(t, a)A_t dB_t \\ &= (r + \lambda\sigma(t, A_t))A_t dt + \sigma(t, A_t)A_t dB_t, \end{aligned}$$

Note that the upper limit $\sigma_{\max}$ on permitted risk taking $0 \le \sigma \le \sigma_{\max}$ imposes a limit on the leverage $|\pi(t, a)| \le \pi_{\max}$. This implies that μ and σ fulfil Lipschitz conditions, i.e. there exists a constant $C < \infty$ such that for all $0 \le t \le T$ and $a, b > 0$:

$$|\mu(t, a) - \mu(t, b)| + |\sigma(t, a) - \sigma(t, b)| \le C|a - b|. \quad \text{(A1)}$$

These inequalities ensure the existence and uniqueness of a continuous, strong solution of the asset dynamics in (2.1), see, e.g. chapter 6 of Arnold (1992).

Appendix 2. Proofs on dynamic risk taking

For smooth pasting purposes, we use the well-known Black and Scholes (1973) call and put option pricing formula (Call, Put) defined by

$$\begin{aligned}\texttt{Call}(a,K,r,v,\bar{t}) &= a\mathcal{N}(d_1) - Ke^{-r\bar{t}}\mathcal{N}(d_2),\ \texttt{Put}(a,K,r,v,\bar{t})\\ &= Ke^{-r\bar{t}}\mathcal{N}(-d_2) - a\mathcal{N}(-d_1),\end{aligned} \tag{B1}$$

as well as its $d_{1/2}$ terms:

$$d_{1/2} = \frac{\ln(a/K) + \left(r \pm \frac{v^2}{2}\right)\bar{t}}{v\sqrt{\bar{t}}}.$$

(Here, $\mathcal{N}(\cdot)$ denotes the standard normal cumulative distribution function.) In our application, we use these functions only as a tool to describe a sequence of (smooth pasting) functions that converges to a function under consideration; it is immaterial for us what are drift and volatility of the underlying Black–Scholes diffusion set-up.

Theorem 7 *Assume a wealth function f that is strictly increasing and twice continuously differentiable on the* entire *positive line and denote by $R_f = R(f)f' - f''/f'$ a function on the positive real line. The manager's risk taking function is time independent, i.e. $\sigma_f^* : R^+ \to R^+$ describes the entire risk taking function at all times. At realized asset value $a > 0$ the function is defined by*

$$\sigma_f^*(a) = \min\left\{\begin{Bmatrix}\sigma_{\max} & ;\ \text{if } R_f(a) \le 0\\ \lambda\cdot\left(aR_f(a)\right)^{-1} & ;\ \text{if } R_f(a) > 0\end{Bmatrix}, \sigma_{\max}\right\}.$$

Proof of Theorem 7 In line with Ross (2004) we denote $V = u(f)$ the derived utility function. For a moment, let us fix a function $\sigma : R^+ \times R \to R$. Then we define for all time points $0 \le t \le T$ and all asset values $a > 0$ the generator G of the process $V(A)$ by setting

$$G(t,a) = rV(a) + (r + \lambda\sigma(t,a))aV'(a) + \frac{1}{2}(\sigma(t,a))^2 a^2 V''(a).$$

Using the Dynkin formula, see Dynkin (1965, p. 133), and Oksendal (1995) we find that managerial utility is

$$E[V(A_T)] = V(A_0) + E\left[\int_t^T G(s,A_s)ds\right]. \tag{B2}$$

Therefore, risk taking $\sigma^*(t,a)$ is given at all $0 \le t \le T$ and all $a > 0$ as the value $\sigma(t,a)$ that maximizes $G(t,a)$.

For given $0 \le t \le T$ and $a > 0$, we determine the optimal risk level through the first- and second-order derivatives of the manager's generator G w.r.t. σ:

$$\frac{\partial G}{\partial\sigma} = \lambda aV'(a) + \sigma a^2 V''(a), \qquad \frac{\partial^2 G}{\partial\sigma^2} = a^2 V''(a). \tag{B3}$$

We have $V' = u'(f)f' > 0$. If $V''(a) \ge 0$, then $\frac{\partial G}{\partial\sigma} > 0$ for all $0 \le \sigma \le \sigma_{\max}$; therefore, the manager will pick $\sigma_{\max}$. If $V''(a) < 0$, then $\sigma = -\lambda\frac{V'(a)}{aV''(a)}$ characterizes the (unique) solution to the first-order condition $\frac{\partial G}{\partial\sigma} = 0$; because $\frac{\partial^2 G}{\partial\sigma^2} < 0$ for all $\sigma \ge 0$, this is a global maximum. The statement follows from the representation $-V''(x)/V'(x) = R(f)f' - f''/f'$, see Ross (2004, p. 213). □

Proposition 8 *Assume $\tilde{F} < \tilde{L}$ and consider the wealth function $f(a) = \tilde{F} - (\tilde{F} - a)^+ + (a - \tilde{L})^+$. With risk averse manager, risk taking is $\sigma_{\max}$ for $\tilde{F} < a < \tilde{L}$.*

Proof of Proposition 8 For any $\epsilon > 0$ we do a local smooth pasting replacing the wealth function f by the wealth function f_ϵ, where $f_\epsilon(a) = W - \texttt{Put}(a,\tilde{F},r,\sigma_{\max},\epsilon) + \beta\texttt{Call}(a,\tilde{L},r,\sigma_{\max},\epsilon)$, where the functions Call, Put are given by equation (B1). (The idea is to smooth the kinks through the Black–Scholes call/put values with time to maturity ϵ at asset value a.) We then have

$$f'_\epsilon(a) = \mathcal{N}\Big(-d_1(a,\tilde{F},r,\sigma_{\max},\epsilon)\Big) + \beta\mathcal{N}\Big(d_1(a,\tilde{L},r,\sigma_{\max},\epsilon)\Big),$$

$$f''_\epsilon(a) = \frac{-\mathcal{N}'\Big(d_1(a,\tilde{F},r,\sigma_{\max},\epsilon)\Big) + \beta\mathcal{N}'\Big(d_1(a,\tilde{L},r,\sigma_{\max},\epsilon)\Big)}{a\sigma_{\max}\sqrt{\epsilon}}.$$

We calculate $\ln^2 a/\tilde{L} - \ln^2 a/\tilde{F} = \ln\tilde{F}/\tilde{L}\ln(a^2/(\tilde{L}\tilde{F}))$ such that

$$-d_1^2(\tilde{F}) + d_1^2(\tilde{L}) = \ln(\tilde{F}/\tilde{L})\frac{\ln(a^2/(\tilde{L}\tilde{F})) + (2r + \sigma_{\max})\epsilon}{\sigma_{\max}^2\epsilon},$$

and so

$$\begin{aligned} &f''_\epsilon(a)a\sigma_{\max}\sqrt{\epsilon}\sqrt{2\pi}\\ &= -\mathcal{N}'\Big(d_1(a,\tilde{F},r,\sigma_{\max},\epsilon)\Big) + \beta\mathcal{N}'\Big(d_1(a,\tilde{L},r,\sigma_{\max},\epsilon)\Big)\\ &= \exp\left(-\frac{d_1^2(\tilde{L})}{2}\right)\\ &\quad\times\left(-\exp\left(\ln(\tilde{F}/\tilde{L})\frac{\ln(a^2/(\tilde{L}\tilde{F})) + (2r+\sigma_{\max})\epsilon}{\sigma_{\max}^2\epsilon}\right) + \beta\right).\end{aligned}$$

First, let us consider the case $a^2 > \tilde{L}\tilde{F}$, then this implies

$$\lim_{\epsilon\to 0}\frac{f''_\epsilon(a)a\sigma_{\max}\sqrt{\epsilon}}{\beta\mathcal{N}'\Big(d_1(a,\tilde{L},r,\sigma_{\max},\epsilon)\Big)} = 1,$$

and, therefore,

$$\begin{aligned}\lim_{\epsilon\to 0}\frac{f'_\epsilon(a)}{f''_\epsilon(a)} &= \lim_{\epsilon\to 0}\\ &\times\frac{\mathcal{N}\Big(-d_1(a,\tilde{F},r,\sigma_{\max},\epsilon)\Big) + \beta\mathcal{N}\Big(d_1(a,\tilde{L},r,\sigma_{\max},\epsilon)\Big)}{\beta\mathcal{N}'\Big(d_1(a,\tilde{L},r,\sigma_{\max},\epsilon)\Big)}\\ &\times a\sigma_{\max}\sqrt{\epsilon} = \infty.\end{aligned}$$

This shows $\lim_{\epsilon\to 0} R(f_\epsilon(a))f'_\epsilon(a) - f''_\epsilon(a)/f'_\epsilon(a) = 0$. Taking limits and using Theorem 7 proves the statement on risk taking in this case. Next, let us consider the case $a^2 < \tilde{L}\tilde{F}$. Then, the above decomposition shows that

$$\lim_{\epsilon\to 0}\frac{f''_\epsilon(a)a\sigma_{\max}\sqrt{\epsilon}}{\beta\mathcal{N}'\Big(d_1(a,\tilde{L},r,\sigma_{\max},\epsilon)\Big)} = -\infty,$$

and it follows analogously that

$$\begin{aligned}\lim_{\epsilon\to 0}\frac{f'_\epsilon(a)}{f''_\epsilon(a)} &= \lim_{\epsilon\to 0}\\ &\times\frac{\mathcal{N}\Big(-d_1(a,\tilde{F},r,\sigma_{\max},\epsilon)\Big) + \beta\mathcal{N}\Big(d_1(a,\tilde{L},r,\sigma_{\max},\epsilon)\Big)}{\beta\mathcal{N}'\Big(d_1(a,\tilde{L},r,\sigma_{\max},\epsilon)\Big)}\\ &\times a\sigma_{\max}\sqrt{\epsilon} = 0.\end{aligned}$$

The convergence is from above. This shows $\lim_{\epsilon\to 0} R(f_\epsilon(a)) f'_\epsilon(a) - f''_\epsilon(a)/f'_\epsilon(a) = -\infty$. The negative sign means that for sufficiently small $\epsilon > 0 : R(f_\epsilon(a))f'_\epsilon(a) - f''_\epsilon(a)/f'_\epsilon(a) < 0$. Taking limits and using Theorem 7 proves the statement on risk taking in this case. □

Proof of Theorem 1 We apply a smooth pasting procedure and define the function f_ϵ as follows: we set $f_\epsilon = a$ for $a < F - \epsilon$; we define f_ϵ equal to the wealth function in Proposition 8 with $\tilde{F} = F$ and $\tilde{L} = L$ for asset values $F + \epsilon < a < L - \epsilon$; we set $f_\epsilon = W + F + f$ for $a > L + \epsilon$; for asset values in between we do a smooth pasting to the left and right boundaries such that the function f_ϵ is twice continuously differentiable for all $a > 0$. Applying Theorem 7 together with Proposition 8 and taking limits ends the proof. □

Proof of Theorem 2 Based on Theorem 1 we only need to consider the impact of the cap on risk taking at asset values above L. We carry out a smooth pasting procedure and define for all $\epsilon > 0$ a bonus function f_ϵ as follows: For $a < (L + \bar{L})/2 - \epsilon$ we define f_ϵ to be the Black–Scholes value of the call option with time to maturity ϵ and strike L, i.e. $f_\epsilon(a) = \beta\texttt{Call}(a,L,r,\sigma_{\max},\epsilon)$; for $a > (L+\bar{L})/2+\epsilon$ we define the function f_ϵ by setting $f_\epsilon(a) = \beta(\bar{L} - L) - \beta\texttt{Put}(a,\bar{L},r,\sigma_{\max},\epsilon)$; for $(L+\bar{L})/2-\epsilon \le a \le (L+\bar{L})/2+\epsilon$ we do a smooth pasting to the left and right boundaries such that the function f_ϵ is twice continuously differentiable for all $a > 0$. (Here, the functions Call, Put are given by equation (B1).)

For $a < (L+\bar{L})/2 - \epsilon$ we use the results for the call bonus (Theorem 1) to conclude the stated risk taking result. For $a > (L+\bar{L})/2 + \epsilon$ we have that

$$f'_\epsilon(a) = \beta\mathcal{N}\Big(-d_1(a,\bar{L},r,\sigma_{\max},\epsilon)\Big),\ f''_\epsilon(a)$$
$$= -\frac{\beta\mathcal{N}'\Big(-d_1(a,\bar{L},r,\sigma_{\max},\epsilon)\Big)}{a\sigma_{\max}\sqrt{\epsilon}}.$$

Therefore, $\lim_{\epsilon\to 0} R(f_\epsilon)f'_\epsilon - f''_\epsilon/f'_\epsilon = \lim_{\epsilon\to 0} R(f_\epsilon)$ and so Theorem 1 implies the stated result.

For further analysis of the capped bonus we recall that the first derivative has the properties that $\lim_{\epsilon\to 0} f'_\epsilon(a) = 1$ for $(L+\bar{L})/2 + \epsilon < a < \bar{L}$, that $\lim_{\epsilon\to 0} f'_\epsilon(a) = \beta/2$ for $a = \bar{L}$ and that $\lim_{\epsilon\to 0} f'_\epsilon(a) = 0$ for $a > \bar{L}$; in addition we recall that $\lim_{\epsilon\to 0} f''_\epsilon(a) = 0$ for $a \neq \bar{L}$ and $(L+\bar{L})/2 + \epsilon < a$, that $\lim_{\epsilon\to 0} f''_\epsilon(a) = \infty$ for $a = \bar{L}$; we also note that $\lim_{\epsilon\to 0} d_{1/2}\sigma\sqrt{\epsilon} = \ln(a/\bar{L})$.

For $(L+\bar{L})/2 + \epsilon < a < \bar{L}$, when ϵ tends to zero, because $f''_\epsilon(a)$ tends to zero and $f'_\epsilon(a)$ tends to one, the term $-f'_\epsilon(a)/(af''_\epsilon(a))$ tends to infinity. For $a = \bar{L}$, when ϵ tends to zero, because $f''_\epsilon(a)$ tends to infinity and $f'_\epsilon(a)$ tends to $1/2$, the term $-f'_\epsilon(a)/(af''_\epsilon(a))$ tends to zero. For $a > \bar{L}$, when ϵ tends to zero, both $f'_\epsilon(a)$ and $f''_\epsilon(a)$ tend to zero, so that further analysis is required. For this we recall well-known results about the greeks (Charm and Color) for the put option:

$$\frac{df'_\epsilon}{d\epsilon} = \beta\mathcal{N}'(d_1)\frac{2r\epsilon - d_2\sigma\sqrt{\epsilon}}{2\epsilon\sigma\sqrt{\epsilon}}, \text{ and}$$
$$\frac{df''_\epsilon}{d\epsilon} = \beta\frac{\mathcal{N}'(d_1)}{2a\sigma\sqrt{\epsilon}}\left(1 + \frac{2r\epsilon - d_2\sigma\sqrt{\epsilon}}{2\epsilon\sigma\sqrt{\epsilon}}d_1\right).$$

We then apply l'Hôpital's rule to get for $a > \bar{L}$ that

$$\lim_{\epsilon\to 0} -\frac{f'_\epsilon(a)}{af''_\epsilon(a)} = \lim_{\epsilon\to 0}\frac{\frac{df'_\epsilon}{d\epsilon}}{a\cdot\frac{df''_\epsilon}{d\epsilon}} = \lim_{\epsilon\to 0} -\frac{2r\epsilon - d_2\sigma\sqrt{\epsilon}}{1 + \frac{2r\epsilon - d_2\sigma\sqrt{\epsilon}}{2\epsilon\sigma\sqrt{\epsilon}}d_1} = 0.$$

Because $\lim_{\epsilon\to 0} R(f_\epsilon)f'_\epsilon = 0$, Theorem 1 proves zero risk taking above $\bar{L}$. □

Proof of Theorem 3 We carry out a smooth pasting procedure where we define a sequence of bonus functions f_ϵ. Each is equal to f for $a < \bar{F} - \epsilon$, $\bar{F} + \epsilon < a < L - \epsilon$, $a > L + \epsilon$; for asset values in between we do a smooth pasting to the left and right boundaries such that the function f_ϵ is twice continuously differentiable for all $a > 0$. Applying Theorem 7 and taking limits ends the proof. □

Proof of Theorem 4 The wealth functions without/with deferral differ only for asset values $a > \bar{F}$, so we focus our discussion on these values. We note that the wealth function with deferral is strictly below the one without deferral for $\bar{F} < a < D + \bar{F}$; they intersect exactly at $D + \bar{F}$; for larger realized asset values, wealth with deferral is larger than the one without; for realized asset values larger than $D + F$ both wealth functions differ by αD.

The case $a > D + F$ has been discussed in the main text. We compare theorems 2 and 3 for the other two cases of realized asset values $a > \bar{F}$. First, for $\bar{F} < a \leq F$, risk taking with deferral is larger than that without if and only if $\alpha < R(f_{inside}(a))/R(f_{stock}(a))$; since the risk aversion function is of DARA type and $f_{inside}(a) < f_{stock}(a)$ for asset values $\bar{F} < a \leq F$, this holds for all $\alpha \leq 1$. Second, for $F < a < D + F$, risk taking without deferral is $\sigma_{\max}$, whereas risk taking with deferral will be smaller for sufficiently large risk aversion. □

Proof of Theorem 5 The proof is analogous that of Theorem 7. First, we define for all time points $0 \leq t \leq T$ and all asset values $a > 0$ the generator G of the process

$$A_t - \frac{1}{2}\int_0^t c(\sigma(s, A_s), A_s)ds$$

by setting $G(t,a) = ra + (r + \lambda\sigma(t,a))a - \frac{1}{2}c(\sigma(t,a),a)$. Using the Dynkin formula, see Dynkin (1965, p. 133), and Oksendal (1995) we find that the expected terminal asset value less cumulative costs fulfills

$$E\left[A_T - \frac{1}{2}\int_0^T c(\sigma(s,A_s),A_s)ds\right] = A_0 + E\left[\int_t^T G(s,A_s)ds\right].$$

Therefore, the socially optimal risk level $\sigma^*_F(t,a)$ is given at all $0 \leq t \leq T$ and all $a > 0$ as the value $\sigma(t,a)$ that maximizes $G(t,a)$.

For given $0 \leq t \leq T$ and $a > 0$, we determine the optimal risk level through the first- and second order derivatives of the generator G w.r.t. σ:

$$\frac{\partial G}{\partial\sigma} = \lambda a - \frac{\partial c}{\partial\sigma},\quad \frac{\partial^2 G}{\partial\sigma^2} = -\frac{\partial^2 c}{\partial\sigma^2}.$$

Noting that $\frac{\partial^2 c}{\partial\sigma^2} > 0$, this ends the proof. □

Appendix 3. Risk targeting

The solution to the differential equation (5.1) characterizes the wealth function above L; it is for suitably chosen constants α_1, α_2 known to be

$$f(a) = \alpha_1\left(\alpha_2(\kappa - 1) + (\gamma - 1)a^{1-\kappa}\right)^{-\frac{1}{\gamma-1}}.$$

We need to choose the constants α_1, α_2 appropriately such that the resulting function $g = f - (W + F)$ fulfils the properties of a bonus function. Incentive theory tells us that the bonus function shall be non-negative and increasing; the same properties must then hold for the wealth function. It might be interesting to study the case $\alpha_1 < 0$ and special parameter choices for γ where this could be fulfilled, but we refrain from this. For simplicity, we focus on $\alpha_1 > 0$; let us define the inner term as a function ζ by setting $\zeta(a) = \alpha_2(\kappa - 1) + (\gamma - 1)a^{1-\kappa}$. If $\gamma < 1$, then ζ must be increasing to have f increasing; this requires $\kappa > 1$; if $\gamma > 1$, then ζ must be decreasing to have f increasing; this also requires $\kappa > 1$.

Murphy (1999) argues against 'jumps' in the bonus function on the grounds that it will induce excessive risk taking. In our earlier analysis, we could not confirm this in our dynamic analysis; nevertheless we study only a continuous bonus function here. For given threshold L, there is then a unique α_2 that ensures continuity, i.e. to have $f(L) = W + F$ for the above function f; we adopt this as the α_2 constant.

Next, we turn our attention to the curvature of the power bonus function. We have for $a > D + F$:

$$g''_{power}(a) = \frac{(\kappa-1)(W+F)\xi\left(1 + \xi - \left(\frac{a}{D+F}\right)^{1-\kappa}\right)^{1/(1-\gamma)}}{a^2(\gamma-1)^2\left(\xi - \left(\frac{a}{D+F}\right)^{\kappa-1}(1+\xi)\right)^2}$$
$$\times\left(-\gamma\xi + \kappa\left(\xi + (\gamma-1)\left(\frac{a}{D+F}\right)^{\kappa-1}(1+\xi)\right)\right).$$

The sign of $g''_{power}(a)$ is equal to that of

$$-\gamma\xi + \kappa\left(\xi + (\gamma-1)\left(\frac{a}{D+F}\right)^{\kappa-1}(1+\xi)\right).$$

This term is continuous, strictly monotonous decreasing and tends to minus infinity when a tends to infinity. Also, the sign at $a = L$ is equal to that of $-\gamma\xi + \lambda/\bar{\sigma}\,(\xi + (\gamma-1)(1+\xi))$. We calculate that this term has a zero in γ at $\kappa/(\kappa(1+\xi) - \xi)$. The last term is strictly positive for $\xi > \kappa/(\kappa-1)$ and strictly negative for $\xi < \kappa/(\kappa-1)$. This implies that there is a non-negative $\bar{\gamma}$ that depends on κ, ξ with the following properties: the bonus function g_{power} is strictly concave for all $\gamma \geq \bar{\gamma}$; for $0 < \gamma < \bar{\gamma}$ there is a threshold $\bar{a} > D + F$ such that the bonus function is locally convex on the interval $(D + F, \bar{a})$ and locally concave for asset values $a > \bar{a}$.