On the center of the cohomology algebra of closed orientable 3-manifolds and a theorem of Sullivan

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Closed orientable manifolds

Definition

*A Hausdorff topological space M is an n-***manifold** *or* **manifold of dimension** *n, if for all $x \in M$, there is a neighbourhood of x homeomorphic to $\mathbb{R}^n$.*

- 2-manifolds are also known as **surfaces**.
- Connected manifolds.
- Compact manifolds without boundary are said to be **closed**.
- Orientable manifolds.

Cohomology algebra

Definition

*Let k be a field. A k-**algebra** is a k-vector space equipped with a bilinear product.*

Definition

*An algebra A is a **graded algebra** if it can be decomposed into a direct sum $A = \bigoplus_{n=0}^{\infty} A_n$ such that for $a_1 \in A_p$ and $a_2 \in A_q$, $a_1 a_2 \in A_{p+q}$. The highest n for which $A_n \neq 0$ is the **degree** of the graded algebra.*

- $H^*(X; k) \equiv \bigoplus_{n=0}^{\infty} H^n(X; k)$ is the **cohomology algebra** of X. The cup product $\smile: H^p(X; k) \times H^q(X; k) \to H^{p+q}(X; k)$ makes it into an associative graded algebra.
- For simplicity denote $H^n(X; k) \equiv H^n$ and $H^*(X; k) \equiv H^*$.
- The cohomology algebra is also **graded commutative**: for $x \in H^p$ and $y \in H^q$, $xy = (-1)^{pq} yx$.

Closed orientable manifolds and Frobenius algebras

Definition

A bilinear form f on a finite-dimensional vector space V is said to be **nondegenerate** *if $f(x, y) = 0$ for all $y \in V$ implies that $x = 0$.*

Definition

An associative finite-dimensional k-algebra with unit A equipped with a nondegenerate bilinear form $\sigma : A \times A \to k$ satisfying $\sigma(xy, z) = \sigma(x, yz)$ for all $x, y, z \in A$ is said to be a **Frobenius algebra***.*

The following is equivalent to Poincaré duality.

Theorem

The cohomology algebra of a closed orientable n-manifold is a graded Frobenius algebra of degree n.

3-manifolds, cup product 3-form and Sullivan's theorem

Let M be a closed orientable manifold of dimension 3.

Definition

Define the **cup product 3-form** $\mu_M : H^1 \times H^1 \times H^1 \to k$ *by* $\mu_M(x, y, z) = \sigma(xy, z)$*, where* σ *is the Frobenius form of* $H^*(M; k)$*.*

A calculation shows that μ_M is indeed an antisymmetric 3-form. In fact, μ_M determines the graded Frobenius algebra structure on $H^*(M; k)$.

Theorem (Sullivan)

Let μ *be an antisymmetric* 3*-form on a finite dimensional* k*-vector space* V*. Then there exists a closed orientable* 3*-manifold* M *such that* $\mu = \mu_M$ *and* $H^1(M; k) = V$*.*

Choice of bases

Let n be the degree of the graded Frobenius algebra.

Proposition

Given a basis $\{x_1, \cdots, x_b\}$ for H^p, there is a basis $\{\overline{x_1}, \cdots, \overline{x_b}\}$ for H^{n-p} dual to it in the sense that $\sigma(x_i, \overline{x_j}) = \delta_{ij}$.

For a 3-manifold M such that $\dim_k(H^1(M;k)) = \beta$,

- $H^0 \cong H^3 \cong k$.
- $H^1 \cong H^2 \cong k^\beta$.

Choose a basis $\{1, x_1, \cdots, x_\beta, \overline{x_1}, \cdots, \overline{x_\beta}, e\}$ for $H^* = H^0 \oplus H^1 \oplus H^2 \oplus H^3$.

Center of the cohomology algebra

- Center $Z(A)$ of an algebra A.
- H^0, H^2 and H^3 are subsets of $Z(H^*)$ since the algebra is graded commutative.

Lemma

Let $x \in H^1$. Then $x \in Z(H^*)$ if and only if $\mu_M(x, y, z) = 0$ for all $y, z \in H^1$.

Theorem

Let M be a closed orientable 3-manifold. Define the map $\iota_\mu : H^1 \to \Lambda^2(H^1)$ by $\iota_\mu(x) = \mu_M(x, \cdot, \cdot)$. Then

$$Z(H^*(M; k)) \cong H^0 \oplus \ker(\iota_\mu) \oplus H^2 \oplus H^3 \cong k^{2+\dim H^1} \oplus \ker(\iota_\mu)$$

Hochschild cohomology

Let A be a k-algebra. Define the Hochschild cochain complex

$$CC^n(A) = \mathrm{Hom}_k(A^{\otimes n}, A)$$

Where $A^{\otimes n}$ is the tensor product of A with itself n times and $A^{\otimes 0} = k$.

- Hochschild codifferential $d : CC^n(A) \to CC^{n+1}(A)$.
- For $n = 0$, $df(a_1) = a_1 f(1) - f(1)a_1$.

Define the Hochschild cohomology vector spaces
$HH^n(A) = \frac{\ker(d:CC^n(A)\to CC^{n+1}(A))}{\mathrm{im}(d:CC^{n-1}(A)\to CC^n(A))}$.

Proposition

$$HH^0(A) \cong Z(A)$$

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