Arithmetic I Exercises

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Contents

Chapter 1

Rudiments

— —WEEK 1— —

Suppose we have a family of sets $\mathcal{C}$. If for each pair of elements $X, Y \in \mathcal{C}$, we have either $X = Y$ or $X \cap Y = \varnothing$, then we say that $\mathcal{C}$ is a disjoint family of sets or a non-intersecting family of sets. The union of all sets in $\mathcal{C}$ is denoted by

$$\bigsqcup_{X \in \mathcal{C}} X$$

We'll make the assumption that the notation $\bigsqcup$ is only used for a non-intersecting family of sets. That is

$$Y = \bigsqcup_{X \in \mathcal{C}} X$$

if and only if

$$\begin{cases} Y = \bigcup_{X \in \mathcal{C}} X \\ ((\forall X_1, X_2 \in \mathcal{C}), X_1 \cap X_2 \neq \varnothing) \Rightarrow (X_1 = X_2) \end{cases}$$

For any set X, we use the notation 2^X to denote the set of all subsets of X. That is

$$2^X = \{Y | Y \subset X\}$$

1.1

Suppose $f : X \to Y$ is a mapping. Prove that

$$X = \bigsqcup_{y \in Y} f^{-1}(\{y\})$$

1.2

Let $f : X \to Y, g : Y \to X$ be two mappings. Prove that if $gf = \mathrm{id}_X$, then f is injective and g is surjective.

1.3

Use 1.3 to prove that: f is invertible $\Leftrightarrow$ f is bijective.

1.4

Consider the mapping $f : X \to X$ where X is a finite set. Prove that the following six properties are equivalent.

f is injective	f is surjective	f is bijective
f is left-invertible	f is right-invertible	f is invertible

1.5

Suppose $X_1, X_2, \cdots, X_n$ are countable (infinite) sets, prove that their Cartesian product

$$X_1 \times X_2 \times \cdots \times X_n$$

is a countable (infinite) set.

1.6

Suppose $\sim$ is a equivalence relation on X. For every $x \in X$, define a set $[x]$ to be

$$[x] = \{y \in X | x \sim y\} (= \{y \in X | y \sim x\})$$

Prove that

1. Given $x_1, x_2 \in X$, we must have $[x_1] = [x_2]$ or $[x_1] \cap [x_2] = \varnothing$
2. $\bigcup_{x \in X} [x] = X$

(In other words, we have $X = \bigsqcup_{x \in X} [x]$)

We define the **quotient set of X under the relation $\sim$** to be

$$(X/\sim) = \{[x] | x \in X\}$$

Apparently, we have $(X/\sim) \subset 2^X$.

A **partition** $\mathcal{C}$ of a set X is defined to be a subset of 2^X such that every element $W \in \mathcal{C}$ is nonempty and

$$X = \bigsqcup_{W \in \mathcal{C}} W$$

Prove that $(X/\sim)$ is a partition of X.

1.7

Denote the set of all equivalence relations on X by $\mathrm{ER}(X)$. Denote the set of all partitions of X by $\mathrm{Par}(X)$. For any equivalence relation $\sim\in \mathrm{ER}(X)$, we define a partition $\pi_X(\sim) \in \mathrm{Par}(X)$ by

$$\pi_X(\sim) = (X/\sim) = \{[x]|x \in X\}, \text{ where } [x] = \{y \in X|x \sim y\}$$

1. Prove that $\mathrm{ER}(X) \subset 2^{(X^2)}$
2. Prove that $\mathrm{Par}(X) \subset 2^{(2^X)}$
3. Prove that π_X is a bijection

We will denote the inverse of π_X by ρ_X. Prove that if $\mathcal{C} \in \mathrm{Par}(X)$, then $(x_1, x_2) \in \rho_X(\mathcal{C})$ if and only if there exists $W \in \mathcal{C}$ such that $x_1, x_2 \in W$.

Remark.

Sets $\mathrm{ER}(X)$ and $\mathrm{Par}(X)$ have the same cardinality. When $\mathrm{Card}(X) = n$, we have $\mathrm{Card}(\mathrm{ER}(X)) = \mathrm{Card}(\mathrm{Par}(X)) = B_n$, where B_n is the n-th Bell number.

1.8

Suppose $\sim\in \mathrm{ER}(X)$, we define a mapping $p_\sim : X \to \pi_X(\sim)$ by

$$p_\sim(x) = [x]$$

Suppose $f : X \to Y$ is a mapping such that $fx_1 = fx_2$ whenever $x_1 \sim x_2$. Prove that there exists exactly one mapping $f_\sim : \pi_X(\sim) \to Y$ such that

$$f = \left(X \xrightarrow{p_\sim} \pi_X(\sim) \xrightarrow{f_\sim} Y\right)$$

The mapping $f_\sim$ is called the induced mapping of f by $\sim$.

1.9

Suppose $f : X \to Y$ is a mapping, we define a relation $\sim_f$ on X by

$$x_1 \sim_f x_2 \text{ if and only if } fx_1 = fx_2$$

Prove that $\sim_f$ is an equivalence relation on X and

$$\pi_X(\sim_f) = \{f^{-1}(\{y\})|y \in \mathrm{Im}f\}$$

Prove that the induced mapping of f by $\sim_f$ is injective, and it is surjective if and only if f is surjective. Conclude that every mapping is a composition of a projection and an injection.

1.10

In this exercise, we only consider positive integers

1. Prove that $\gcd(n,m)|n, \gcd(n,m)|m$
2. Prove that $n|\mathrm{lcm}(n,m), m|\mathrm{lcm}(n,m)$
3. Suppose $d|n, d|m$, prove that $d|\gcd(n,m)$
4. Suppose $n|D, m|D$, prove that $\mathrm{lcm}(n,m)|D$

1.11

Let $a \in \mathbf{Z}, b \in \mathbf{N}$. Prove that there exists $q, r \in \mathbf{Z}$ where $0 \leq r < b$ such that $a = bq + r$. Show that q, r are **uniquely** determined by a, b.

1.12

Show that if $n, m \in \mathbf{Z}$

$$\{an + bm | a, b \in \mathbf{Z}\} = \gcd(n,m)\mathbf{Z}$$

1.13

Prove that $\{4k+1 | k \in \mathbf{N}\} \cap \mathbf{P}$ and $\{4k-1 | k \in \mathbf{N}\} \cap \mathbf{P}$ are infinite sets.

1.14

The Euler totient function $\varphi : \mathbf{N} \to \mathbf{N}$ is defined by the following:

$$\varphi(n) = \mathrm{Card}(\{m \in \mathbf{N} | 1 \leq m \leq n, \gcd(n,m) = 1\})$$

For example, $\varphi(1) = 1, \varphi(2) = 1, \varphi(3) = 2, \varphi(4) = 2, \varphi(5) = 4$. Show that

$$\frac{\varphi(n)}{n} = \prod_{v_p(n) > 0} \left(1 - \frac{1}{p}\right)$$

1.15

Suppose $(X, *)$ is a magma, where for every $a, b \in X$ we have

$$(a * b) * b = a, a * (a * b) = b$$

Prove that $a * b = b * a$ for every $a, b \in X$.

Remark 1
Suppose $r \in \mathbf{Q}$ is a non-zero rational number, then we can write is as $r = \pm\frac{q}{Q}$ where $q, Q \in \mathbf{N}$. The p-adic valuation of r is defined by

$$v_p(r) = v_p(q) - v_p(Q)$$

Notice that this definition is well-defined and is an extension of the original v_p. We define the support set of r to be

$$\text{Supp}(r) = \{p \in \mathbf{P} | v_p(r) \neq 0\}$$

This set is always finite. For example,

$$\text{Supp}\left(\frac{9}{14}\right) = \{2, 3, 7\}$$

Use this notion, we have

$$\frac{\varphi(n)}{n} = \prod_{p \in \text{Supp}(n)} \left(1 - \frac{1}{p}\right)$$

We devide the set Support(r) into two non-intersecting subsets:

$$\text{Supp}^+(r) = \{p \in \mathbf{P} | v_p(r) > 0\}$$
$$\text{Supp}^-(r) = \{p \in \mathbf{P} | v_p(r) < 0\}$$

If $r \in \mathbf{Z}_{\neq 0}$, then $\text{Supp}(r) = \text{Supp}^+(r)$. We define

$$\mathbf{Z}_{(p)} = \{r \in \mathbf{Q} | p \notin \text{Supp}^-(r)\}$$

Remark 2
This may be boring, but if $r \in \mathbf{Q}_{\neq 0}$, then we have

$$|r| = \prod_{p \in \text{Supp}(r)} p^{v_p(r)}$$

Or, equivalently,

$$\ln |r| = \sum_{p \in \text{Supp}(r)} v_p(r) \ln p$$

We define a function $|\cdot|_p : \mathbf{Q} \to \mathbf{R}_{\geq 0}$ by

$$|r|_p = \begin{cases} p^{-v_p(r)}, & r \neq 0 \\ 0, & r = 0 \end{cases}$$

Then we have

- $|r_1 - r_2|_p = 0$ if and only if $r_1 = r_2$
- $|r_1 r_2|_p = |r_1|_p |r_2|_p$
- $|r_1 - r_2|_p + |r_2 - r_3|_p \geq |r_1 - r_3|_p$

— —WEEK 3— —

1.16

Let m be an odd natural number, prove that

$$\frac{\sin mx}{\sin x} = (-4)^{\frac{m-1}{2}} \prod_{j=1}^{\frac{m-1}{2}} \left(\sin^2 x - \sin^2 \frac{2\pi j}{m} \right)$$

1.17

Suppose we have a system of sets and mappings:

$$A_1 \xleftarrow{\phi_2} A_2 \xleftarrow{\phi_3} A_3 \xleftarrow{\phi_4} A_4 \leftarrow \cdots$$

where every A_n is a non-empty finite set. Prove that we can find a sequence of elements $x_1 \in A_1, x_2 \in A_2, \ldots, x_n \in A_n, \ldots$, such that

$$\phi_2 x_2 = x_1, \phi_3 x_3 = x_2, \ldots$$

1.18

Show that 1.17 is wrong if we do not acquire every A_n to be finite.

1.19

Let p be a prime number, and $n \in \mathbf{Z}$ such that $\gcd(p, n) = 1$. Prove that

$$p | (n^{p-1} - 1)$$

1.20

Let p be a prime number, and $0 < n < p$ is an integer. Prove that

$$p | \mathrm{C}_p^n$$

where $\mathrm{C}_p^n = \frac{p!}{n!(p-n)!}$

1.21

Let B_n be the n-th Bell number. Let p be a prime number. Show that

$$p | (B_{n+p} - B_{n+1} - B_n)$$

Chapter 2

Algebraic Structures

— —WEEK 4— —

Groups

2.1

Let $(M, *)$ be a semigroup, if $N \subset M$ is a subset such that for all $a, b \in N$ we have $a * b \in N$, then we say that $N \subset_* M$, or N is a sub-semigroup of M. Prove or disprove:

- If M is a monoid, then N is a monoid
- If M does not have an identity, then N does not have an identity
- If M and N are monoids, then their identities are the same one

2.2

Let M be a monoid (which is by definition a semigroup), and denote the set of all invertible elements of M by $\mathrm{U}(M)$, show that $\mathrm{U}(M)$ is a sub-semigroup of M and itself is even a group. We call it the group of units of M.

2.3

Let Ω be a set, and $M(\Omega) = \{f : \Omega \to \Omega\}$ be the set of mappings, together with the composition operation $\circ$.

- Show that $\mathrm{U}(M(\Omega))$ is the set of all bijective mappings.
- If $\Omega = \{1, 2, \ldots, n\}$, we denote $\mathrm{U}(M(\Omega))$ by S_n. Show that $\mathrm{Card}(\mathrm{S}_n) = n!$

2.4

Let $(G, *)$ be a group (which is by definition a semigroup), and $H \subset_* G$ is a sub-semigroup of G. Show that if

1. the identity $e \in H$
2. for all $h \in H$ we have $h^{-1} \in H$

Then H is not only a semigroup, it is a group.

2.5

Let $(G, *)$ be a group (which is by definition a semigroup), and $H \subset_* G$ is a sub-semigroup of G. Show that if H is a group, then

1. the identity $e \in H$
2. for all $h \in H$ we have $h^{-1} \in H$

2.6

Show that H is a subgroup of G if and only if H is a nonempty subset of G and for all $h_1, h_2 \in H$ we have $h_1^{-1} h_2 \in H$.

2.7

Show that if $\varphi : G_1 \to G_2$ is an isomorphism, then $f(e_1)$ is the identity of G_2, where e_1 is the identity of G_1.

2.8

Show that if $\varphi : G_1 \to G_2$ is an isomorphism, then $f(a^{-1}) = (f(a))^{-1}$.

2.9

Let G be a group, we define a new magma $(G^{\text{op}}, *)$ by $x * y = yx$. Show that $(G^{\text{op}}, *)$ is actually a group, called the opposite group of G.

2.10

Show that G and G^{op} are isomorphic (=find an isomorphism between them).

2.11

Show that if $\varphi : G_1 \to G_2$ is an isomorphism, then the inverse mapping φ^{-1} is an isomorphism. Show that if $\varphi : G \to H$ and $\psi : H \to K$ are isomorphisms, then their composition

$$\psi \circ \varphi : G \to K$$

is an isomorphism.

2.12

An automorphism of a group G is an isomorphism from G to G. Denote the set of all automorphisms of G by $\mathrm{Aut}(G)$. Show that $\mathrm{Aut}(G)$ is a subgroup of $\mathrm{Perm}(G) = \mathrm{U}(M(G))$. (Hint: Use **2.7**)

2.13

Give an example of two non-isomorphic groups of cardinality 4.

2.14

Write down the Cayley table for S_3 and for $\mathrm{Aut}(\mathrm{S}_3)$. Are these two groups isomorphic?

2.15

Let G be a group, we define $\mathrm{SG}(G)$ to be the sets of all subgroups of G. Suppose $S \subset G$ is a **subset**, we define

$$\langle S \rangle = \bigcap_{S \subset H \in \mathrm{SG}(G)} H$$

1. Prove that $\mathrm{SG}(G)$ is closed under arbitrary intersection.
2. Deduce that $\langle S \rangle \in \mathrm{SG}(G)$, which is called the subgroup generated by S. If $G = \langle S \rangle$, we say that G is generated by S.

2.16

Show that S_4 can be generated by $\{(12), (13), (14)\}$.

Show that S_4 can be generated by $\{(12), \theta\}$, here θ is the mapping $\theta(1) = 2, \theta(2) = 3, \theta(3) = 4, \theta(4) = 1$.

2.17

Find all subgroups of S_4.

2.18

S_n is called the n-th symmetric group. We write down some of its elements:

$$(12) = \begin{cases} 1 \mapsto 2 \\ 2 \mapsto 1 \\ 3 \mapsto 3 \\ \vdots \\ n \mapsto n \end{cases}, (13) = \begin{cases} 1 \mapsto 3 \\ 2 \mapsto 2 \\ 3 \mapsto 1 \\ \vdots \\ n \mapsto n \end{cases}, \ldots, (1n) = \begin{cases} 1 \mapsto n \\ 2 \mapsto 2 \\ 3 \mapsto 3 \\ \vdots \\ n \mapsto 1 \end{cases}, \theta = \begin{cases} 1 \mapsto 2 \\ 2 \mapsto 3 \\ 3 \mapsto 4 \\ \vdots \\ n-1 \mapsto n \\ n \mapsto 1 \end{cases}$$

1. Prove that S_n is generated by $\{(12), (13), \ldots, (1n)\}$.
2. Prove that S_n is generated by $\{(12), \theta\}$.
3. Prove the Cayley's theorem: every finite group of cardinality n is isomorphic to some subgroup of S_n.

2.19

Construct a non-abelian group of cardinality 8.

2.20

For $n = 1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11$, classify all groups of cardinality n (=of order n).

2.21

Suppose $\varphi : G_1 \to G_2$ is a group homomorphism, show that the image and the kernel of φ

$$\mathrm{im}\varphi = \{\gamma \in G_2 : \exists g \in G_1, \varphi g = \gamma\}$$
$$\ker\varphi = \{g \in G_1 : \varphi g = e\}$$

are subgroups of G_1 and G_2.

2.22

Suppose $\varphi : G_1 \to G_2$ is a group homomorphism, show that φ is injective if and only if $\ker\varphi = 1$, and φ is surjective if and only if $\mathrm{im}\varphi = G_2$.

2.23

Show that if $\varphi : G_1 \to G_2$ is a group homomorphism, then

$$\varphi(x) = \varphi(y) \Leftrightarrow xy^{-1} \in \ker \varphi$$

2.24

The kernel of a group homomorphism is not only a subgroup: it is a **normal** subgroup. To be precise, a subgroup H of G is normal, if and only if for all $h \in H, g \in G$ we have

$$ghg^{-1} \in H$$

Show that the kernel of a group homomorphism $G_1 \to G_2$ is a normal subgroup of G_1.

If a group only has trivial normal subgroups, then it is called a simple group.

2.25

Suppose H is a subgroup of G. For any $g \in G$, we define

$$gH = \{gh : h \in H\}$$

Show that $\{gH : g \in G\}$ is a partition of G, and g_1H, g_2H have the same cardinality for any two g_1, g_2.

Conclude that if G is finite, then $\mathrm{Card}(H)$ is a divisor of $\mathrm{Card}(G)$.

2.26

Prove the Lagrange's theorem:

Suppose G is a finite group and $g \in G$. Then there exists a natural number $\mathrm{ord}_G(g)$ such that

$$\{n \in \mathbf{Z} | g^n = e\} = \mathrm{ord}_G(g)\mathbf{Z}$$

And we have $\mathrm{ord}_G(g)$ divides $\mathrm{Card}(G)$.

2.27

Let m be a natural number. Consider the set

$$\mathrm{U}[m] = \{1 \leq n \leq m : \gcd(m, n) = 1\}$$

Show that if $a, b \in \mathrm{U}[m]$, then there exists $c \in \mathrm{U}[m]$ such that $m|(ab - c)$. Show that if we define $a * b = c$ then $\mathrm{U}[m]$ is a group of $\varphi(m)$.

Show that if m is a prime, then $\mathrm{U}[m]$ is cyclic.

2.28

Let m be a natural number and $n \in \mathbf{N}$ such that $\gcd(m, n) = 1$. Prove that

$$m|(n^{\varphi(m)} - 1)$$

2.29

Let G be a group, we define $\mathrm{SG}(G)$ to be the sets of all subgroups of G. Suppose $S \subset G$ is a **subset**, we define

$$\langle S \rangle = \bigcap_{S \subset H \in \mathrm{SG}(G)} H$$

1. Elements of the form $s_1^{\epsilon_1} s_2^{\epsilon_2} \cdot \cdots \cdot s_n^{\epsilon_n}$ where $s_i \in S, \epsilon_i \in \mathbf{Z}$ are called S-words. Prove that every element of $\langle S \rangle$ is a S-word.

2. Suppose $S = \{a\}$ contains one element, we also write $\langle a \rangle$ for $\langle \{a\} \rangle$. This group is automatically a cyclic subgroup of G.

   Suppose $a, b \in G$ with $ab = ba$, and that $\langle a \rangle$ is a finite group of order n, $\langle b \rangle$ is a finite group of order m where $\gcd(n, m) = 1$.

   Prove that $\langle \{a, b\} \rangle$ is a cyclic group of order nm.

3. Consider $a = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, b = \begin{pmatrix} 0 & 1 \\ -1 & -1 \end{pmatrix}$ (as elements of $\mathrm{SL}_2(\mathbf{Z})$ if you want). Prove that these two are elements of finite order, such that ab is an element of infinite order. Also, calculate $\langle \{a, b\} \rangle$.

2.30

1. Prove that every finite group is a subgroup of some bi-generated group (=group that can be generated by only two elements).

2. Recall that there is a group homomorphism $\sigma_G : G \to \mathrm{Aut}(G)$ for every group G, defined by

   $$\left(G \xrightarrow{\sigma_G(a)} G \right) = I_a = \left(g \mapsto aga^{-1} \right)$$

   Prove that if $G = \mathrm{S}_n$ where $n \neq 2, 6$, then σ_G is an isomorphism.

3. Prove that there is only one epimorphism from S_n to S_2 (where $n \geq 2$).

4. Prove that S_n has only one subgroup, of order $\frac{1}{2}\mathrm{Card}(\mathrm{S}_n)$. This subgroup is called the n-th alternating group, denoted by A_n.

2.31

In this exercise, we study the arithmetics of cyclic groups.

1. Suppose G is a cyclic group, and H is a subgroup of G. Prove that H is also a cyclic group.

2. Suppose G is a cyclic group of infinite order, and H is a non-trivial subgroup of G. Prove that H is also a cyclic group of infinite order.

3. Suppose G is a cyclic group of order n, and H is a subgroup of G. Prove that the order of H divides n.

4. Suppose G is a cyclic group of order n, and m is a natural number dividing n. Prove that G has a unique subgroup of order m.

5. Let G be a cyclic group of order n. An element $g \in G$ is called a generator of G if $G = \langle g \rangle$. Prove that the number of generators of G is $\varphi(n)$.

6. Prove that $\sum_{d|n} \varphi(d) = n$. (Hint: How many elements of C_n, the cyclic group of order n, generates a (cyclic) group of order d?)

7. For any group G, we define

$$\mathbf{u}_d(G) = \{g \in G | g^d = 1\}, u_d(G) = \mathrm{Card}(\mathbf{u}_d(G))$$

   Let G be a cyclic group of order n, and let $d|n$. Prove that $u_d(G) = d$.

8. Prove the following cyclic-forcing theorem:

   Suppose G is finite, and $u_d(G) \leq d$ for all $d \in \mathbf{N}$. Prove that G is cyclic.

2.32

Let n be a natural number with $\gcd(n, \varphi(n)) = 1$. Prove that every group of order n is isomorphic to C_n.

2.33

Prove that the only normal subgroups of A_n are the trivial groups. Here $n \geq 5$.

This result seems to be trivial, but it explains why equations of degree 5 are not solvable.

2.34

Let p be a prime number, and G a group of order p^n. (These groups are called p-groups.) Prove that σ_G is not injective.

Rings and Fields

Now we study algebraic structures with two operations on it. It is possible to define an algebraic structure Shenzhong as follows:

A shenzhong is a triple $(S, \circ, \bullet)$, such that $(S, \circ)$ is a magma/semigroup/monoid/group and $(S, \bullet)$ is a magma/semigroup/monoid/group...

But this is pointless, since the two operations are not linked by any formulae. Basically we are just studying two algebraic structures $(S, \circ)$ and $(S, \bullet)$, and we should study them separately.

It turns out that, we should add a property, called the distributivity, to get a reasonable theory, the theory of rngs:

Definition A rg is a triple $(R, +, \times)$ such that

- $(R, +)$ is a semigroup
- $(R, \times)$ is a semigroup
- $x \times (a + b) = (x \times a) + (x \times b)$ for all $a, b, x \in R$
- $(a + b) \times y = (a \times y) + (b \times y)$ for all $a, b, y \in R$

If $(R, +, \times)$ is a rg, then $(R, \times, +)$ is in general not a rg.

Definition A rg $(R, +, \times)$ is a rng, if $(R, +)$ is an abelian group.

Definition A rg $(R, +, \times)$ is a ring, if $(R, +)$ is a group and $(R, \times)$ is a monoid.

It may not be obvious, but actually every ring is a rng.

Definition A ring $(R, +, \times)$ is commutative, if $(R, \times)$ is an abelian monoid.

Methods used in the studying of commutative rings are radically different from general rng theory, and became a new theory called commutative algebra, which then become the scheme theory. If you want to learn more about this, I recommend Einsenbud's textbook for commutative algebra and of course the EGA.

OK, let me explain something. Every commutative ring A can be realized as an ordered pair $(\mathrm{Spec}A, \mathcal{O}_{\mathrm{Spec}A})$, which is a **topological space** and a **sheaf of rings**, and we called it a ringed space. A ringed space is actually a geometric object, rather than an algebraic one. So now we can do analysis and geometry on commutative rings.

This approach is called scheme theory, and it solved many geometry problems and algebra problems. Here's my recommend for textbooks in commutative algebra, algebraic geometry and scheme theory:

- David Eisenbud *Commutative Algebra*
- Robin Hartshorne *Algebraic Geometry*
- A. Grothendieck, J. Dieudonné *Eléments de Géométrie Algébrique* (EGA)
- Stacks Project `https://stacks.math.columbia.edu`

— —WEEK 5— —

Chapter 3

Ring-theoretic Constructions

Chapter 4

Linear Algebra

Chapter 5

Finite Fields and Reciprocity

Chapter 6

p-adic Numbers

Chapter 7

Hilbert Symbol