Mathematical Analysis: An Advanced Introduction

Qiū Cáiyóng

Mathematical Analysis is the modern (and right) version of Calculus. In this course, we use this modern approach.

How did Calculus become Analysis? And for what reason? Well, mathematicians found that every proof in Calculus is essentially nonsense. Every theorem in Calculus eventually ends with ‘This is some obvious property of **real numbers**’. But how can we state and use those ‘obvious properties’ without even knowing what exactly **are** real numbers?

For engineers, Calculus=Analysis since they don’t feel the need of knowing what real numbers are. But mathematicians **do** care definitions, because we need to prove truths rather than memorize phenomena.

Nowadays, the formal definition of $\mathbf{R}$ has been found by mathematicians. And mathematicians soon realized that, all Calculus is nothing but corollaries of the definition of $\mathbf{R}$!

This approach of Calculus, is called **Mathematical Analysis**.

Contents

Chapter 1

Sequences and Series

1.1 Sets and functions

The set of all positive integers is denoted by $\mathbf{N}_+$, $\emptyset$ denotes the empty set. The union of $A_1, A_2, \dots, A_m$ is denoted by

$$A_1 \cup A_2 \cup A_3 \cup \cdots \cup A_m = \bigcup_{j=1}^{m} A_j$$

and the intersection by

$$A_1 \cap A_2 \cap A_3 \cap \cdots \cap A_m = \bigcap_{j=1}^{m} A_j$$

The union and the intersection of an infinite family of sets $A_1, A_2, \dots$ indexed by $\mathbf{N}_+$ are denoted by

$$\bigcup_{j=1}^{\infty} A_j \text{ and } \bigcap_{j=1}^{\infty} A_j$$

More generally, suppose J is a set, and suppose that for each $j \in J$ we are given a set A_j. The union and intersection of all the A_j are denoted by

$$\bigcup_{j \in J} A_j \text{ and } \bigcap_{j \in J} A_j$$

If $A \subset B$, the **complement** of A in B is the set of elements of B not in A:

$$B \setminus A = \{x | x \in B, x \notin A\}$$

Thus $C = B \setminus A$ is equivalent to the two conditions

$$A \cup C = B, A \cap C = \emptyset$$

The **product** of two sets A and B is the set of ordered pairs (x, y) where $x \in A$ and $y \in B$; this is written $A \times B$. More generally, if $A_1, A_2, \ldots, A_n$ are the sets then

$$A_1 \times A_2 \times \cdots \times A_n = \prod_{j=1}^{n} A_j$$

is the set whose elements are all the ordered n-tuples $(x_1, \ldots, x_n)$, where each $x_j \in A_j$. The product

$$A \times A \times \cdots \times A$$

of n copies of A is also written A^n.

A **function** from a set A to a set B is an assignment, to each element of A, of some unique element of B. We write

$$f : A \to B$$

for a function f from A to B. If $x \in A$, then $f(x)$ denotes the element of B assigned by f to the element x. The elements assigned by f are often called **values**. Thus a **real-valued function on** A is a function $f : A \to \mathbf{R}$, $\mathbf{R}$ the set of real numbers. A **complex-valued function on** A is a function $f : A \to \mathbf{C}$, $\mathbf{C}$ the set of complex numbers.

A function $f : A \to B$ is said to be **injective** if it assigns distinct elements of B to distinct elements of A: If $x, y \in A$ and $x \neq y$, then $f(x) \neq f(y)$. A function $f : A \to B$ is said to be **surjective** if for each element $y \in B$, there is some $x \in A$ such that $f(x) = y$. A function $f : A \to B$ which is both injective and surjective is said to be **bijective**.

If $f : A \to B$ and $g : B \to C$, the **composition** of f and g is the function denoted by $g \circ f$:

$$g \circ f : A \to C, (g \circ f)(x) = g(f(x)) \text{ for all } x \in A$$

If $f : A \to B$ is bijective, there is a unique **inverse** function $f^{-1} : B \to A$ with the properties: $(f^{-1} \circ f)(x) = (x)$, for all $x \in A$; $(f \circ f^{-1})(y) = y$, for all $y \in B$.

Example 1.1.1. Consider the functions: $f : \mathbf{Z} \to \mathbf{N}_+, g : \mathbf{Z} \to \mathbf{Z}, h : \mathbf{Z} \to \mathbf{Z}$, defined by

$$\begin{aligned} f(n) &= n^2 + 1, && n \in \mathbf{Z} \\ g(n) &= 2n, && n \in \mathbf{Z} \\ h(n) &= 1 - n, && n \in \mathbf{Z} \end{aligned}$$

Then f is neither injective nor surjective, g is injective but not surjective, h is bijective, $h^{-1}(n) = 1 - n$, and $(f \circ h)(n) = n^2 - 2n + 2$.

1.2 Cardinality

A set A is said to be **finite** if either $A = \emptyset$ or there is an $n \in \mathbf{N}_+$, and a bijective function f from A to the set $\{1, 2, \ldots, n\}$. The set A is said to be **countable** if there is a bijective $f : A \to \mathbf{N}_+$. This is equivalent to requiring that there be a bijective $g : \mathbf{N}_+ \to A$.

Proposition 1.2.1. If there is a surjective function $f : \mathbf{N}_+ \to A$, then A is either finite or countable.

Proof. Suppose A is not finite. Define $g : \mathbf{N}_+ \to \mathbf{N}_+$ as follows. Let $g(1) = 1$. Since A is not finite, $A \neq \{f(1)\}$. Let $g(2)$ be the first integer m such that $f(m) \neq f(1)$. Having defined $g(1), g(2), \ldots, g(n)$, let $g(n+1)$ be the first integer m such that $f(n+1) \notin \{f(1), f(2), \ldots, f(n)\}$. The function g defined inductively on all of $\mathbf{N}_+$ in this way has the property that $f \circ g : \mathbf{N}_+ \to A$ is bijective. In fact, it is injective by the construction. It is surjective because f is surjective and by the construction, for each n the set $\{f(1), f(2), \ldots, f(n)\}$ is a subset of $\{(f \circ g)(1), (f \circ g)(2), \ldots, (f \circ g)(n)\}$. □

Corollary 1.2.1. If B is countable and $A \subset B$, then A is finite or countable.

Proof. If $A = \emptyset$, we are done. Otherwise, choose a function $f : \mathbf{N}_+ \to A$ which is onto. Choose an element $x_0 \in A$. Define $g : \mathbf{N}_+ \to A$ by $g(n) = f(n)$ if $f(n) \in A$, $g(n) = x_0$ if $f(n) \notin A$. Then g is surjective, so A is finite or countable. □

Proposition 1.2.2. If $A_1, A_2, A_3, \ldots$ are finite or countable, then the sets

$$\bigcup_{j=1}^{n} A_j \text{ and } \bigcup_{j=1}^{\infty} A_j$$

are finite or countable.

Proof. We shall prove only the second statement. If any of the A_j are empty, we may exclude them and renumber. Consider only the second case. For each A_j we can choose a surjective function $f_j : \mathbf{N}_+ \to A_j$. Define $f : \mathbf{N}_+ \to \bigcup_{j=1}^{\infty} A_j$ be $f(1) = f_1(1), f(3) = f_1(2), f(5) = f_1(3), \ldots, f(2) = f_2(1), f(6) = f_2(2), f(10) = f_2(3), \ldots$, and in general $f(2^j(2k-1)) = f_j(k), j, k = 1, 2, 3, \ldots$. Any $x \in \bigcup_{j=1}^{\infty} A_j$ is in some A_j, and therefore there is $k \in \mathbf{N}_+$ such that $f_j(k) = x$. Then $f(2^j(2k-1)) = x$, so f is surjective. By proposition 1.2.1, $\bigcup_{j=1}^{\infty} A_j$ is finite or countable. □

Example 1.2.1. Let $\mathbf{Q}$ be the set of rational numbers: $\mathbf{Q} = \{\frac{m}{n} | m \in \mathbf{Z}, n \in \mathbf{N}_+\}$. This is countable. In fact, let $A_n = \{\frac{j}{n} | j \in \mathbf{Z}, -n^2 \leq j \leq n^2\}$. Then each A_n is finite, and $\mathbf{Q} = \bigcup_{n=1}^{\infty} A_n$.

Proposition 1.2.3. If $A_1, A_2, \ldots, A_n$ are countable sets, then the product set $A_1 \times A_2 \times \cdots \times A_n$ is countable.

Proof. Choose bijective functions $f_j : A_j \to \mathbf{N}_+, j = 1, 2, \ldots, n$. For each $m \in \mathbf{N}_+$, let B_m be the subset of the product set consisting of all n-tuples $(x_1, \ldots, x_n)$ such that each $f_j(x_j) \leq m$. Then B_m is finite (it has m^n elements) and the product set is the union of the sets B_m. Proposition 1.2.2 gives the desired conclusion. □

A **sequence** in a set A is a collection of elements of A, not necessarily distinct, indexed by some countable set J. Usually J is taken to be $\mathbf{N}_+$ or $\mathbf{N}$, and we use the notations

$$(a_n)_{n=1}^{\infty} = (a_1, a_2, a_3, \dots)$$
$$(a_n)_{n=0}^{\infty} = (a_0, a_1, a_2, \dots)$$

Proposition 1.2.4 (Cantor)**.** The set S of all sequences in the set $\{0, 1\}$ is neither finite nor countable.

S is **uncountable**

Proof. Suppose $f : \mathbf{N}_+ \to A$. We shall show that f is not surjective. For each $m \in \mathbf{N}_+$, $f(m)$ is a sequence $(a_{n,m})_{n=1}^{\infty} = (a_{1,m}, a_{2,m}, \dots)$, where each $a_{n,m}$ is 0 or 1. Define a sequence $(a_n)_{n=1}^{\infty}$ by setting $a_n = 0$ if $a_{n,n} = 1$, $a_n = 1$ if $a_{n,n} = 0$. Then for each $m \in \mathbf{N}_+$, $(a_n)_{n=1}^{\infty} \neq (a_{n,m})_{n=1}^{\infty} = f(m)$. Thus f is not surjective. $\square$

We introduce some more items of notation. The symbol $\Rightarrow$ means 'implies'; the symbol $\Leftarrow$ means 'is implied by'; the symbol $\Leftrightarrow$ means 'is equivalent to'.

1.3 Axioms of R

We denote by $\mathbf{R}$ the set of all real numbers. The operations of addition and multiplication can be thought of as functions from the product set $\mathbf{R} \times \mathbf{R}$ to $\mathbf{R}$. Addition assigns to the ordered pair (x, y) an element of $\mathbf{R}$ denoted by $x + y$; multiplication assigns an element of $\mathbf{R}$ denoted by xy.

Axioms of addition

A1. $(x + y) + z = x + (y + z)$, for any $x, y, z \in \mathbf{R}$.

A2. $x + y = y + x$, for any $x, y \in \mathbf{R}$.

A3. There is an element 0 in $\mathbf{R}$ such that $x + 0 = x$ for every $x \in \mathbf{R}$.

A4. For each $x \in \mathbf{R}$ there is an element $-x \in \mathbf{R}$ such that $x + (-x) = 0$.

So far, $\mathbf{R}$ is a so-called additive abelian group.

Note that the element 0 is unique. In fact, if $0'$ is an element such that $x + 0' = x$ for every x, then
$$0' = 0' + 0 = 0 + 0' = 0$$
Also, given x the element $-x$ is unique. In fact, if $x + y = 0$, then

$$\begin{aligned} y &= y + 0 = y + (x + (-x)) = (y + x) + (-x) \\ &= (x + y) + (-x) = 0 + (-x) = (-x) + 0 = -x \end{aligned}$$

This uniqueness implies $-(-x) = x$, since $(-x) + x = x + (-x) = 0$.

Axioms of multiplication

M1. $(xy)z = x(yz)$, for any $x, y, z \in \mathbf{R}$.

M2. $xy = yx$, for any $x, y \in \mathbf{R}$.

M3. There is an element $1 \neq 0$ in $\mathbf{R}$ such that $x1 = x$ for any $x \in \mathbf{R}$.

M4. For each $x \in \mathbf{R}, x \neq 0$, there is an element $x^{-1} \in \mathbf{R}$ such that $xx^{-1} = 1$.

Note that 1 and x^{-1} are unique. We leave the proofs as an exercise.

Distributive law

DL. $x(y+z) = xy + xz$, for any $x, y, z \in \mathbf{R}$.

So far, $\mathbf{R}$ is a so-called field.

Note that DL and A2 imply $(x+y)z = xz + yz$.

We can now readily deduce some other well-known facts. For example,

$$0x = (0+0)x = 0x + 0x$$

so $0x = 0$. Then

$$x + (-1)x = 1x + (-1)x = (1 + (-1))x = 0x = 0$$

so $(-1)x = -x$. Also,

$$(-x)y = ((-1)x)y = (-1)(xy) = -xy$$

The axioms A1-A4, M1-M4, and DL do **not** determine $\mathbf{R}$. In fact there is a set $\mathbf{F}_2$ consisting of two elements, together with operations of addition and multiplication, such that the axioms above are all satisfied: if we denote the elements of the set by $0, 1$, we can define addition and multiplication by

In other words, $\mathbf{F}_2$ is a field.

$$0 + 0 = 1 + 1 = 0, 0 + 1 = 1 + 0 = 1$$

$$0 \cdot 0 = 1 \cdot 0 = 0 \cdot 1 = 0, 1 \cdot 1 = 1$$

There is an additional familiar notion in $\mathbf{R}$, that of **positivity**, from which one can derive the notion of an ordering of $\mathbf{R}$. We axiomatize this by introducing a subset $P \subset \mathbf{R}$, the set of 'positive' elements.

Axioms of order

O1. If $x \in \mathbf{R}$, then exactly on of the following holds: $x \in P, x = 0$, or $-x \in P$.

O2. If $x, y \in P$, then $x + y \in P$.

O3. If $x, y \in P$, then $xy \in P$.

So far, $\mathbf{R}$ is a so-called ordered field.

It follows from these that if $x \neq 0$, then $x^2 \in P$. In fact if $x \in P$ then this follows from O3, while if $-x \in P$, then $(-x)^2 \in P$, and $(-x)^2 = -(x(-x)) = -(-x^2) = x^2$. In particular, $1 = 1^2 \in P$.

We **define** $x < y$ if $y - x \in P$, $x > y$ if $y < x$. It follows that $x \in P \Leftrightarrow x > 0$. Also, if $x < y$ and $y < z$, then

$$z - x = (z - y) + (y - x) \in P$$

so $x < z$. In terms of this order, we introduce the **Archimedean axiom**.

O4. If $x, y > 0$, then there is a positive integer n such that $nx = \underbrace{x + x + \cdots + x}_{n}$ is $> y$. (Given enough time, one can empty a large bathtub with a small spoon.)

So far, $\mathbf{R}$ is a so-called Archimedean ordered field.

The axioms given so far still do not determine $\mathbf{R}$; they are all satisfied by the subset $\mathbf{Q}$ of rational numbers. The following notions will make a distinction between these two sets.

$\mathbf{Q}$ is also an Archimedean ordered field.

1.4 Completeness axiom

A nonempty subset $A \subset \mathbf{R}$ is said to be **bounded above** if there is an $x \in \mathbf{R}$ such that every $y \in A$ satisfies $y \leq x$. Such a number x is called an **upper bound** for A. Similarly, if there is an $x \in \mathbf{R}$ such that every $y \in A$ satisfies $x \leq y$, then A is said to be **bounded below** and x is called a **lower bound** for A.

A number $x \in \mathbf{R}$ is said to be a **least upper bound** for a nonempty set $A \subset \mathbf{R}$ if x is an upper bound, and if every other upper bound x' satisfies $x' \geq x$. If such an x exists it is clearly unique, and we write

$$x = \sup A$$

Similarly, x is a **greatest lower bound** for A if it is a lower bound and if every other lower bound x' satisfies $x' \leq x$. Such an x is unique, and we write

$$x = \inf A$$

The final axiom for $\mathbf{R}$ is called the **completeness axiom**.

O5. If A is a nonempty subset of $\mathbf{R}$ which is bounded above, then A has a least upper bound.

So far, $\mathbf{R}$ is a so-called Dedekind-complete Archimedean ordered field.

Note that if $A \subset \mathbf{R}$ is bounded below, then the set $B = \{x | x \in \mathbf{R}, -x \in A\}$ is bounded above. If $x = \sup B$, then $-x = \inf A$. Therefore O5 is equivalent to: a nonempty subset of $\mathbf{R}$ which is bounded below has a greatest lower bound.

Theorem 1.4.1. $\mathbf{Q}$ does not satisfy the completeness axiom.

That is, $\mathbf{Q}$ is **not** a Dedekind-complete Archimedean ordered field.

Proof. Recall that there is no rational $\frac{p}{q}, p, q \in \mathbf{Z}$, such that $\left(\frac{p}{q}\right)^2 = 2$: in fact if there were, we could reduce to lowest terms and assume either p or q is odd. But $p^2 = 2q^2$ is even, so p is even, so $p = 2m, m \in \mathbf{Z}$. Then $4m^2 = 2q^2$, so $q^2 = 2m^2$ is even and q is also even, a contradiction.

Let $A = \{x | x \in \mathbf{Q}, x^2 < 2\}$. This is nonempty, since $0, 1 \in A$. It is bounded above, since $x \geq 2$ implies $x^2 \geq 4$, so 2 is an upper bound. We shall show that no $x \in \mathbf{Q}$ is a least upper bound for A.

If $x \leq 0$, then $x < 1 \in A$, so x is not an upper bound. Suppose $x > 0$ and $x^2 < 2$. Suppose $h \in \mathbf{Q}$ and $0 < h < 1$. Then $x + h \in \mathbf{Q}$ and $x + h > x$. Also,

$(x+h)^2 = x^2 + 2xh + h^2 < x^2 + 2xh + h = x^2 + (2x+1)h$. If we choose $h > 0$ so small that $h < 1$ and $h < \frac{2-x^2}{2x+1}$, then $(x+h)^2 < 2$. Then $x+h \in A$, and $x + h > x$, so x is not an upper bound of A.

Finally, suppose $x \in \mathbf{Q}, x > 0$, and $x^2 > 2$. Suppose $h \in \mathbf{Q}$ and $0 < h < x$. Then $x - h \in \mathbf{Q}$ and $x - h > 0$. Also, $(x-h)^2 = x^2 - 2xh + h^2 > x^2 - 2xh$. If we choose $h > 0$ so small that $h < 1$ and $h < \frac{x^2-2}{2x}$, then $(x-h)^2 > 2$. It follows that if $y \in A$, then $y < x - h$. Thus $x - h$ is an upper bound for A less than x, and x is not the least upper bound. □

We used the non-existence of a square root of 2 in $\mathbf{Q}$ to show that O5 does not hold. We may turn the argument around to show, using O5, that there is a real number $x > 0$ such that $x^2 = 2$. In fact, let $A = \{y | y \in \mathbf{R}, y^2 < 2\}$. The argument proving Theorem 1.4.1 proves the following: A is bounded above; its least upper bound x is positive; if $x^2 < 2$ then x would not be an upper bound, while if $x^2 > 2$ then x would not be the least upper bound. Thus $x^2 = 2$.

Two important questions arise concerning the above axioms. Are the axioms consistent, and satisfied by some set $\mathbf{R}$? Is the set of real numbers the only set satisfying these axioms?

The consistency of the axioms and the existence of $\mathbf{R}$ can be demonstrated (to the satisfaction of most mathematicians) by constructing $\mathbf{R}$, starting with the rationals.

In one sense the axioms do not determine $\mathbf{R}$ uniquely. For example, let $\mathbf{R}^\circ$ be the set of all symbols x°, where x is (the symbol for) a real number. Define addition and multiplication of elements of $\mathbf{R}^\circ$ by

$$x^\circ + y^\circ = (x+y)^\circ, x^\circ y^\circ = (xy)^\circ$$

Define P° by $x^\circ \Leftrightarrow x \in P$. Then $\mathbf{R}^\circ$ satisfies the axioms above. This is clearly fraudulent: $\mathbf{R}^\circ$ is just a copy of $\mathbf{R}$. It can be shown that any set with addition, multiplication, and a subset of positive elements, which satisfies all the axioms above, is just a copy of $\mathbf{R}$.

Remark: Nowadays mathematicians use this definition: (real) numbers is (up to an isomorphism) **the** Dedekind-complete Archimedean ordered field.

1.5 Complex numbers

Starting from $\mathbf{R}$ we can construct the set $\mathbf{C}$ of complex numbers, without simply postulating the existence of a 'quantity' i such that $i^2 = -1$. Let $\mathbf{C}^\circ$ be the product set $\mathbf{R}^2 = \mathbf{R} \times \mathbf{R}$, whose elements are ordered pairs (x, y) of real numbers. Define addition and multiplication by

$$\begin{aligned}(x,y) + (x',y') &= (x+x', y+y')\\ (x,x')(y,y') &= (xx' - yy', xy' + x'y)\end{aligned}$$

It can be shown by straightforward calculations that $\mathbf{C}^\circ$ together with these operations satisfies A1, A2, M1, M2, and DL. To verify the remaining algebraic axioms, note that

$$\begin{aligned}
(x,y)+(0,0) &= (x,y)\\
(x,y)+(-x,-y) &= (0,0)\\
(x,y)(1,0) &= (x,y)\\
(x,y)\left(\frac{x}{x^2+y^2},\frac{-y}{x^2+y^2}\right) &= (1,0) \text{ if } (x,y)\neq(0,0)
\end{aligned}$$

If $x \in \mathbf{R}$, let x° denote the element $(x,0) \in \mathbf{C}^\circ$. Let i° denote the element $(0,1)$. Then we have

$$(x,y) = (x,0)+(0,y) = (x,0)+(0,1)(y,0) = x^\circ + i^\circ y^\circ$$

Also $(i^\circ)^2 = (0,1)(0,1) = (-1,0) = (-1)^\circ$. Thus we can write any element of $\mathbf{C}^\circ$ uniquely as $x^\circ + i^\circ y^\circ, x, y \in \mathbf{R}$, where $(i^\circ)^2 = (-1)^\circ$. We now drop the superscripts and write $x+iy$ for $x^\circ + i^\circ y^\circ$ and $\mathbf{C}$ for $\mathbf{C}^\circ$.

This renaming $x^\circ + i^\circ y^\circ \to x+iy$ has some issues: if $y=0$, then $x+i0$ looks like just a real number. But remember that $x+i0$ is actually an ordered pair of real numbers $(x,0)$ by the definition of complex numbers. A real number is, strictly speaking, not an ordered pair of real numbers. If this renaming is legitimate, we have to verify that for elements of $\mathbf{R}$ the new operations coincide with the old:

$$x^\circ + y^\circ = (x+y)^\circ, x^\circ y^\circ = (xy)^\circ$$

Remark: In other words, $\mathbf{C}$ is a field extension of $\mathbf{R}$, since the field $\mathbf{R}$ can be embedded into $\mathbf{C}^\circ$ and $\mathbf{C}^\circ$ is isomorphic to $\mathbf{C}$.

Often we shall denote elements of $\mathbf{C}$ by z or w. When we write $z = x+iy$, we shall understand that x, y are real. They are called the **real part** and the **imaginary part** of z, respectively:

$$z = x+iy, x = \Re(z), y = \Im(z)$$

There is a very useful operation in $\mathbf{C}$, called **complex conjugation**, defined by:

$$\bar{z} = \overline{x+iy} = x-iy$$

Then $\bar{z}$ is called the **complex conjugate** of z. It is readily checked that

$$\overline{z+w} = \bar{z}+\bar{w}, \overline{zw} = \bar{z}\bar{w}, \bar{\bar{z}} = z, \bar{z}z = x^2+y^2$$

Thus $\bar{z}z \neq 0$ if $z \neq 0$. Define the **modulus** of z, $|z|$ by

$$|z| = (\bar{z}z)^{\frac{1}{2}} = (x^2+y^2)^{\frac{1}{2}}, z = x+iy$$

Then if $z \neq 0$,

$$1 = \frac{\bar{z}z}{|z|^2} = z \cdot \frac{\bar{z}}{|z|^2} \Rightarrow z^{-1} = \frac{\bar{z}}{|z|^2}$$

Adding and subtracting gives

$$z + \bar{z} = 2x, z - \bar{z} = 2iy, \text{ if } z = x + iy$$

Thus

$$\Re(z) = \frac{z + \bar{z}}{2}, \Im(z) = \frac{1}{2} i^{-1}(z - \bar{z})$$

The usual geometric representation of **C** is by a coordinatized plane: $z = x + iy$ is represented by the point with coordinates (x, y). Then by the Pythagorean theorem, $|z|$ is the distance from (the point representing) z to the origin. More generally, $|z - w|$ is the distance from z to w.

1.6 Exercises

1. There is a unique real number $x > 0$ such that $x^3 = 2$.
2. Show that $\Re(z + w) = \Re(z) + \Re(w), \Im(z + w) = \Im(z) + \Im(w)$.
3. Suppose $z = x + iy, x, y \in \mathbf{R}$. Then

$$|x| \leq |z|, |y| \leq |z|, |z| \leq |x| + |y|$$

4. For any $z, w \in \mathbf{C}$,

$$|z\bar{w}| = |z||w|$$

5. For any $z, w \in \mathbf{C}$,

$$|z + w| \leq |z| + |w|$$

6. The Archimedean axiom O4 can be deduced from the other axioms for the real numbers. (Hint: O5.)
7. If $a > 0$ and n is a positive integer, there is a unique $b > 0$ such that $b^n = a$.

1.7 Sequence limits

Remember that every real number is a complex number.

A sequence $(z_n)_{n=1}^{\infty}$ of complex numbers is said to **converge to** $z \in \mathbf{C}$ if for each $\varepsilon > 0$, there is an integer N such that $|z_n - z| < \varepsilon$ whenever $n \geq N$. Geometrically, this says that for **any** circle with center z, the numbers z_n all lie inside the circle, **except for possibly finitely many** values of n. If this is the case we write

$$z_n \to z, \text{ or } \lim_{n\to\infty} z_n = z, \text{ or } \lim z_n = z$$

The number z is called the **limit** of the sequence $(z_n)_{n=1}^{\infty}$. Note that the limit is unique: suppose $z_n \to z$ and also $z_n \to w$. Given **any** $\varepsilon > 0$ we can take n so large that $|z_n - z| < \varepsilon$ and also $|z_n - w| < \varepsilon$. Then

$$|z - w| \leq |z_n - z| + |z_n - w| < \varepsilon + \varepsilon = 2\varepsilon$$

Since this is true for all $\varepsilon > 0$, necessarily $z = w$.

The following proposition collects some convenient facts about convergence.

Proposition 1.7.1. Suppose $(z_n)_{n=1}^{\infty}$ and $(w_n)_{n=1}^{\infty}$ are sequences in $\mathbf{C}$.

1. $z_n \to z$ if and only if $z_n - z \to 0$.
2. Let $z_n = x_n + iy_n$, x_n, y_n real. Then $z_n \to z = x + iy$ if and only if $x_n \to x$ and $y_n \to y$.
3. If $z_n \to z$ and $w_n \to w$, then $z_n + w_n \to z + w$.
4. If $z_n \to z$ and $w_n \to w$, then $z_n w_n \to zw$.
5. If $z_n \to z \neq 0$, then there is an integer M such that $z_n \neq 0$ if $n \geq M$. Moreover $(z^{-1})_{n=M}^{\infty}$ converges to z^{-1}.

Proof. 1. This follows directly from the definition of convergence.

2. By Exercise 1.6.3,
$$|x_n - x| + |y_n - y| \leq |z_n - z| \leq 2|x_n - x| + 2|y_n - y|$$
It follows easily that $z_n - z \to 0$ if and only if $x_n - x \to 0$ and $y_n - y \to 0$.

3. This follows easily from the inequality
$$|(z_n + w_n) - (z + w)| = |(z_n - z) + (w_n - w)| \leq |z_n - z| + |w_n - w|$$

4. Choose M so large that if $n \geq M$, then $|z_n - z| < 1$. Then for $n \geq M$,
$$|z_n| = |(z_n - z) + z| < 1 + |z|$$
Let $K = 1 + |w| + |z|$. Then for all $n \geq M$,
$$\begin{aligned} |z_n w_n - zw| &= |z_n(w_n - w) + (z_n - z)w| \\ &\leq |z_n||w_n - w| + |z_n - z||w| \\ &\leq K(|w_n - w| + |z_n - z|) \end{aligned}$$
Since $w_n - w \to 0$ and $z_n - z \to 0$, it follows that $z_n w_n - zw \to 0$.

5. Take M so large that $|z_n - z| \leq \frac{|z|}{2}$ when $n \geq M$. Then for $n \geq M$,
$$\begin{aligned} |z_n| &= |z_n| + \frac{|z|}{2} - \frac{|z|}{2} \\ &\geq |z_n| + |z_n - z| - \frac{|z|}{2} \geq |z_n + (z - z_n)| - \frac{|z|}{2} = \frac{|z|}{2} \end{aligned}$$
Therefore, $z_n \neq 0$. Also for $n \geq M$.
$$\begin{aligned} |z_n^{-1} - z^{-1}| &= \left| \frac{(z - z_n)}{zz_n} \right| \\ &\leq |z - z_n| \cdot |z|^{-1} \cdot \left(\frac{|z|}{2} \right)^{-1} = 2|z|^{-2} \cdot |z - z_n| \end{aligned}$$
Since $z - z_n \to 0$ we have $z_n^{-1} - z^{-1} \to 0$.

$\square$

A sequence $(z_n)_{n=1}^{\infty}$ in $\mathbf{C}$ is said to be **bounded** if there is an $M \geq 0$ such that $|z_n| \leq M$ for all n; in other words, there is a fixed circle around the origin which encloses all the z_n's.

A sequence $(x_n)_{n=1}^{\infty}$ in $\mathbf{R}$ is said to be **increasing** if for each n, $x_n \leq x_{n+1}$; it is said to be **decreasing** if for each n, $x_n \geq x_{n+1}$.

It makes **no** sense to say a **complex** sequence $(z_n)_{n=1}^{\infty}$ is increasing or decreasing: $\mathbf{C}$ is not an ordered field.

Proposition 1.7.2 (Monotone convergence theorem)**.** A bounded, increasing sequence in $\mathbf{R}$ converges. A bounded, decreasing sequence in $\mathbf{R}$ converges.

Proof. Suppose $(x_n)_{n=1}^{\infty}$ is a bounded, increasing sequence. Then the set $\{x_n | n = 1, 2, \dots\}$ is bounded above. Let x be its least upper bound. Given $\varepsilon > 0, x - \varepsilon$ is not an upper bound, so there is an N such that $x_N \geq x - \varepsilon$. If $x \geq N$, then

$$x - \varepsilon \leq x_N \leq x$$

so $|x_n - x| \leq \varepsilon$. Thus $x_n \to x$. The proof for a decreasing sequence is similar. $\square$

Suppose $(x_n)_{n=1}^{\infty}$ is a bounded sequence of reals. We shall associate with this given sequence two order sequences, one increasing and the other decreasing. For each n, let $A_n = \{x_n, x_{n+1}, x_{n+2}, \dots\}$, and set

$$x'_n = \inf A_n, x''_n = \sup A_n$$

Now $A_n \supset A_{n+1}$, so any lower or upper bound for A_n is a lower or upper bound for A_{n+1}. Thus

$$x'_n \leq x'_{n+1}, x''_n \geq x''_{n+1}$$

Choose M so that $|x_n| \leq M$, all n. Then $-M$ is a lower bound and M an upper bound for each A_n. Thus

$$-M \leq x'_n \leq x''_n \leq M, \text{ all } n$$

We may apply Proposition 1.7.2 to the bounded increasing sequence $(x'_n)_{n=1}^{\infty}$ and the bounded decreasing sequence $(x''_n)_{n=1}^{\infty}$ and conclude that both converge. We define

$$\liminf x_n = \lim x'_n$$
$$\limsup x_n = \lim x''_n$$

These numbers are called the **lower limit** and the **upper limit** of the sequence $(x_n)_{n=1}^{\infty}$, respectively. It follows from

$$-M \leq x'_n \leq x''_n \leq M, \text{ all } n$$

that

$$-M \leq \liminf x_n \leq \limsup x_n \leq M$$

1.8 Cauchy sequences

Cauchy found a way to decide whether a sequence is convergent or not without knowing the limit that it convergences to. Remember that every real number is a complex number.

A sequence $(z_n)_{n=1}^{\infty}$ in $\mathbf{C}$ is said to be a **Cauchy sequence** if for each $\varepsilon > 0$ there is an integer N such that $|z_n - z_m| < \varepsilon$ whenever $n \geq N$ and $m \geq N$. The following theorem is of fundamental importance.

Theorem 1.8.1 (Cauchy's convergence test)**.** A sequence in $\mathbf{C}$ (or $\mathbf{R}$) converges if and only if it is a Cauchy sequence.

Proof. Suppose first that $z_n \to z$. Given $\varepsilon > 0$, we can choose N so that $|z_n - z| < \frac{1}{2}$ if $n \geq N$. Then if $n, m \geq N$ we have

$$|z_n - z_m| \leq |z_n - z| + |z - z_m| < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon$$

Conversely, suppose $(z_n)_{n=1}^{\infty}$ is a Cauchy sequence. We consider first the case of a real sequence $(x_n)_{n=1}^{\infty}$ which is a Cauchy sequence. The sequence $(x_n)_{n=1}^{\infty}$ is bounded: in fact, choose M so that $|x_n - x_m| < 1$ if $n, m \geq M$. Then if $n \geq M$,

$$|x_n| \leq |x_n - x_M| + |x_M| < 1 + |x_M|$$

Let $K = \max\{|x_1|, |x_2|, \ldots, |x_{M-1}|, |x_M| + 1\}$. Then for any n, $|x_n| \leq K$. Now since the sequence is bounded, we can associate the sequences $(x'_n)_{n=1}^{\infty}$ and $(x''_n)_{n=1}^{\infty}$ as above. Given $\varepsilon > 0$, choose N so that $|x_n - x_m| < \varepsilon$ if $n, m \geq N$. Now suppose $n \geq m \geq N$. It follows that

$$x_n - \varepsilon \leq x_n \leq x_m + \varepsilon, n \geq m \geq N$$

By definition of x'_n we also have, therefore,

$$x_m - \varepsilon \leq x'_n \leq x_m + \varepsilon, n \geq m \geq N$$

Letting $x = \liminf x_n = \lim x'_n$, we have

$$x_m - \varepsilon \leq x \leq x_m + \varepsilon, m \geq N$$

or $|x_m - x| \leq \varepsilon, m \geq N$. Thus $x_n \to x$.

Now consider the case of a complex Cauchy sequence $(z_n)_{n=1}^{\infty}$. Let $z_n = x_n + iy_n, x_n, y_n \in \mathbf{R}$. Since $|x_n - x_m| \leq |z_n - z_m|$, $(x_n)_{n=1}^{\infty}$ is a Cauchy sequence. Therefore $x_n \to x \in \mathbf{R}$. Similarly, $y_n \to y \in \mathbf{R}$. By Proposition 1.7.1.2, $z_n \to x + iy$. $\square$

The importance of this theorem lies partly in the fact that it gives a criterion for the **existence** of a limit **in terms of the sequence itself**.

We conclude this section with a useful characterization of the upper and lower limits of a bounded sequence.

Proposition 1.8.1. Suppose $(x_n)_{n=1}^{\infty}$ is a bounded sequence in $\mathbf{R}$. Then $\liminf x_n$ is the unique number x' such that

1. for any $\varepsilon > 0$, there is an N such that $x_n > x' - \varepsilon$ whenever $n \geq N$,
2. for any $\varepsilon > 0$ and any N, there is an $n \geq N$ such that $x_n < x' + \varepsilon$.

Similarly, $\limsup x_n$ is the unique number x'' such that

1. for any $\varepsilon > 0$, there is an N such that $x_n < x'' + \varepsilon$ whenever $n \geq N$,
2. for any $\varepsilon > 0$ and any N, there is an $n \geq N$ such that $x_n > x'' - \varepsilon$.

Proof. We shall prove only the assertion about $\liminf x_n$. First, let $x'_n = \inf\{x_n, x_{n+1}, \dots\} = \inf A_n$ as above, and let $x' = \lim x'_n = \liminf x_n$. Suppose $\varepsilon > 0$. Choose N so that $x'_N > x' - \varepsilon$. Then $n \geq N$ implies $x_n \geq x'_N > x' - \varepsilon$, so 1. holds. Given $\varepsilon > 0$ and N, we have $x'_N \leq x' < x' + \frac{\varepsilon}{2}$. Therefore $x' + \frac{\varepsilon}{2}$ is not a lower bound for A_N, so there is an $n \geq N$ such that $x_n \leq x' + \frac{\varepsilon}{2} < x' + \varepsilon$. Thus 2. holds.

Now suppose x' is a number satisfying 1. and 2. From 1. it follows that $\inf A_n > x' - \varepsilon$ whenever $n \geq N$. Thus $\liminf x_n \geq x' - \varepsilon$, all ε, so $\liminf x_n \geq x'$. From 2. it follows that for any N and any ε, $\inf A_N < x' + \varepsilon$. Thus for any N, $\inf A_n \leq x'$. We have $\liminf x_n = x'$. □

1.9 Exercises

1. The sequence $\left(\frac{1}{n}\right)_{n=1}^{\infty}$ has limit 0. (Hint: Archimedean axiom.)
2. If $x_n > 0$ and $x_n \to 0$, then $\sqrt{x_n} \to 0$.
3. If $a > 0$, then $a^{\frac{1}{n}} \to 1$ as $n \to \infty$. (Hint: if $a \geq 1$, let $a^{\frac{1}{n}} = 1 + x_n$. By the binomial expansion, or by induction, $a = (1 + x_n)^n \leq 1 + nx_n$. Thus $x_n < \frac{a}{n} \to 0$. If $a < 1$, then $a^{\frac{1}{n}} = (b^{\frac{1}{n}})^{-1}$ where $b = a^{-1} > 1$.)
4. $\lim n^{\frac{1}{n}} = 1$. (Hint: Let $n^{\frac{1}{n}} = 1 + y_n$. For $n \geq 2$, $n = (1 + y_n)^n \geq 1 + ny_n + \frac{n(n-1)y_n^2}{2} > \frac{n(n-1)y_n^2}{2}$, so $y_n^2 \leq \frac{2}{n-1} \to 0$. Thus $y_n \to 0$.)
5. If $z \in \mathbf{C}$ and $|z| < 1$, then $z^n \to 0$ as $n \to \infty$.
6. Suppose $(x_n)_{n=1}^{\infty}$ is a bounded real sequence. Show that $x_n \to x$ if and only if $\liminf x_n = x = \limsup x_n$.
7. Prove the second part of Proposition 1.8.1.
8. Suppose $(x_n)_{n=1}^{\infty}$ and $(a_n)_{n=1}^{\infty}$ are two bounded real sequences such that $a_n \to a > 0$. Then

$$\liminf a_n x_n = a \cdot \liminf x_n, \limsup a_n x_n = a \cdot \limsup x_n$$

1.10 Series

Suppose $(z_n)_{n=1}^{\infty}$ is a sequence in $\mathbf{C}$. We associate to it a second sequence $(s_n)_{n=1}^{\infty}$ where

$$s_n = \sum_{m=1}^{n} z_n = z_1 + z_2 + \cdots + z_n$$

If $(s_n)_{n=1}^{\infty}$ converges to s, it is reasonable to consider s as the infinite sum $\sum_{n=1}^{\infty} z_n$. Whether $(s_n)_{n=1}^{\infty}$ converges or not, the formal symbol $\sum_{n=1}^{\infty} z_n$ or $\sum z_n$ is called an **infinite series**, or simply **series**. The number z_n is called the nth **term** of the series, s_n is called the nth **partial sum**. If $s_n \to s$ we say that the series $\sum z_n$ **converges** and that its **sum** is s. This is written

$$s = \sum_{n=1}^{\infty} z_n$$

If the sequence $(s_n)_{n=1}^{\infty}$ does not converge, the series $\sum z_n$ is said to **diverge**.

In particular, suppose $(x_n)_{n=1}^{\infty}$ is a real sequence, and suppose each $x_n \geq 0$. Then the sequence $(s_n)_{n=1}^{\infty}$ of partial sums is clearly an increasing sequence. Either it is bounded, so by Proposition 1.7.2 convergent; or for each $M > 0$ there is an N such that

$$s_n = \sum_{m=1}^{n} x_m > M \text{ whenever } n \geq N$$

In the first case we write

$$\sum_{n=1}^{\infty} x_n < \infty \Leftrightarrow \sum x_n \text{ converges}$$

and in the second case we write

$$\sum_{n=1}^{\infty} x_n = \infty \Leftrightarrow \sum x_n \text{ diverges}$$

Example 1.10.1. 1. Consider the series $\sum_{n=1}^{\infty} \frac{1}{n}$. We claim $\sum_{n=1}^{\infty} \frac{1}{n} = \infty$. In fact,

$$\begin{aligned}
\sum \frac{1}{n} &= 1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \frac{1}{5} + \frac{1}{6} + \frac{1}{7} + \frac{1}{8} + \cdots \\
&\geq \frac{1}{2} + \frac{1}{2} + \left(\frac{1}{4} + \frac{1}{4}\right) + \left(\frac{1}{8} + \frac{1}{8} + \frac{1}{8} + \frac{1}{8}\right) + \cdots \\
&= \frac{1}{2} + \frac{1}{2} + \frac{1}{2} + \cdots = \infty
\end{aligned}$$

2. $\sum_{n=1}^{\infty} \frac{1}{n^2} < \infty$. In fact,

$$\begin{aligned}\sum \frac{1}{n^2} &= 1 + \left(\frac{1}{2}\right)^2 + \left(\frac{1}{3}\right)^2 + \cdots + \left(\frac{1}{7}\right)^2 + \cdots \\ &\le 1 + \left(\frac{1}{2}\right)^2 + \left(\frac{1}{2}\right)^2 + \left(\frac{1}{4}\right)^2 + \left(\frac{1}{4}\right)^2 + \left(\frac{1}{4}\right)^2 + \left(\frac{1}{4}\right)^2 + \cdots \\ &= 1 + \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \cdots = 2\end{aligned}$$

We leave it to the reader to make the above rigorous by considering the respective partial sums.

How does one tell whether a series converges? The question is whether the sequence $(s_n)_{n=1}^{\infty}$ of partial sums converges. Theorem 1.8.1 gives a necessary and sufficient condition for convergence of this sequence: that it be a Cauchy sequence. However this only refines our original question to: how does one tell whether a series has a sequence of partial sums which is a Cauchy sequence? The five propositions below give some answers.

Proposition 1.10.1. If $\sum_{n=1}^{\infty} z_n$ converges, then $z_n \to 0$.

Proof. If $\sum z_n$ converges, then the sequence $(s_n)_{n=1}^{\infty}$ of partial sums is a Cauchy sequence, so $s_n - s_{n-1} \to 0$. But $s_n - s_{n-1} = z_n$. $\square$

Note that the converse is false: $\frac{1}{n} \to 0$ but $\sum \frac{1}{n}$ diverges.

Proposition 1.10.2. If $|z| < 1$, then $\sum_{n=0}^{\infty} z^n$ converges; the sum is $\frac{1}{1-z}$. If $|z| \ge 1$, then $\sum_{n=0}^{\infty} z^n$ diverges.

Proof. The nth partial sum is

$$s_n = 1 + z + z^2 + \cdots + z^{n-1}$$

Then $s_n(1-z) = 1 - z^n$, so $s_n = \frac{1-z^n}{1-z}$. If $|z| < 1$, then as $n \to \infty, z^n \to 0$ (Exercise 1.9.5). Therefore $s_n \to \frac{1}{1-z}$. If $|z| \ge 1$, then $|z^n| \ge 1$, and Proposition 1.10.1 shows divergence. $\square$

The series $\sum_{n=0}^{\infty}$ is called a **geometric series**.

Proposition 1.10.3 (Comparison test)**.** Suppose $(z_n)_{n=1}^{\infty}$ is a sequence in $\mathbf{C}$ and $(a_n)_{n=1}^{\infty}$ is a sequence in $\mathbf{R}$ with each $a_n \ge 0$. If there are constants M, N such that

$$|z_n| \le M a_n \text{ whenever } n \ge N$$

and if $\sum a_n$ converges, then $\sum z_n$ converges.

Proof. Let $s_n = \sum_{m=1}^{n} z_m, b_n = \sum_{m=1}^{n} a_m$. If $n, m \geq N$ then

$$\begin{aligned} |s_n - s_m| &= \left| \sum_{j=m+1}^{n} z_n \right| \leq \sum_{j=m+1}^{n} |z_n| \\ &\leq M \sum_{j=m+1}^{n} a_n = M(b_n - b_m) \end{aligned}$$

But $(b_n)_{n=1}^{\infty}$ is a Cauchy sequence, so this inequality implies that $(s_n)_{n=1}^{\infty}$ is also a Cauchy sequence. □

Proposition 1.10.4 (Ratio test)**.** Suppose $(z_n)_{n=1}^{\infty}$ is a sequence in $\mathbf{C}$ and suppose $z_n \neq 0$, all n.

1. If
$$\limsup \left| \frac{z_{n+1}}{z_n} \right| < 1$$
then $\sum z_n$ converges.
2. If
$$\liminf \left| \frac{z_{n+1}}{z_n} \right| > 1$$
then $\sum z_n$ diverges.

Proof.

1. In this case, take r so that $\limsup |z_n|^{\frac{1}{n}} < r < 1$. By Proposition 1.8.1, there is an N so that $\left| \frac{z_{n+1}}{z_n} \right| \leq r$ whenever $n \geq N$. Thus if $n > N$,
$$|z_n| \leq r|z_{n-1}| \leq r^2 |z_{n-2}| \leq \cdots \leq r^{n-N} |z_N| = M r^n$$
where $M = \frac{|z_N|}{r^N}$. Proposition 1.10.2 and 1.10.3 imply convergence.
2. In this case, Proposition 1.8.1 implies that for some N, $\left| \frac{z_{n+1}}{z_n} \right| \geq 1$ if $n \geq N$. Thus for $n > N$.
$$|z_n| \geq |z_{n-1}| \geq \cdots \geq |z_N| > 0$$
We cannot have $z_n \to 0$, so Proposition 1.10.1 implies divergence.

□

Corollary 1.10.1. If $z_n \neq 0$ for $n = 1, 2, \ldots$ and if $\lim \left| \frac{z_{n+1}}{z_n} \right|$ exists, then the series $\sum z_n$ converges if the limit is < 1 and diverges if the limit is > 1.

Note that for both the series $\sum \frac{1}{n}$ and $\sum \frac{1}{n^2}$, the limit in Corollary 1.10.1 equals 1. Thus either convergence or divergence is possible in this case.

Proposition 1.10.5 (Root test)**.** Suppose $(z_n)_{n=1}^{\infty}$ is a sequence in $\mathbf{C}$.

1. If
$$\limsup |z_n|^{\frac{1}{n}} < 1$$
then $\sum z_n$ converges.
2. If
$$\limsup |z_n|^{\frac{1}{n}} > 1$$
then $\sum z_n$ diverges.

Proof. 1. In this case, take r so that $\limsup |z_n|^{\frac{1}{n}} < r < 1$. By Proposition 1.8.1, there is an N so that $|z_n|^{\frac{1}{n}} \leq r$ whenever $n \geq N$. Thus if $n \geq N$, then $|z_n|^{\frac{1}{n}} \leq r^n$. Proposition 1.10.2 and 1.10.3 imply convergence.

2. In this case, Proposition 1.8.1 implies that $|z_n|^{\frac{1}{n}} \geq 1$ for infinitely many values of n. Thus Proposition 1.10.1 implies divergence.

Note the tacit assumption in the statement and proof that $(|z_n|^{\frac{1}{n}})_{n=1}^{\infty}$ is a **bounded** sequence, so that the upper and lower limits exist. However, if this sequence is not bounded, then in particular $|z_n| \geq 1$ for infinitely many values of n, and Proposition 1.10.1 implies divergence. □

Corollary 1.10.2. If $\lim |z_n|^{\frac{1}{n}}$ exists, then the series $\sum z_n$ converges if the limit is < 1 and diverges if the limit is > 1.

Note that for both the series $\sum \frac{1}{n}$ and $\sum \frac{1}{n^2}$, the limit in Corollary 1.10.2 equals 1 (see Exercise 1.9.4). Thus either convergence or divergence is possible in this case.

1.11 Power series

A particularly important class of series are the **power series**. If $(a_n)_{n=0}^{\infty}$ is a sequence in $\mathbf{C}$ and z_0 a fixed element of $\mathbf{C}$, then the series
$$\sum_{n=0}^{\infty} a_n (z - z_0)^n$$
is the **power series around** z_0 with coefficients $(a_n)_{n=0}^{\infty}$. Here we use the convention that $w^0 = 1$ for all $w \in \mathbf{C}$, including $w = 0$. Thus $\sum_{n=0}^{\infty} a_n (z - z_0)^n$ is defined, as a series, for each $z \in C$. For $z = z_0$ it converges (with sum a_0), but for other values of z it may or may not converge.

Theorem 1.11.1. Consider the power series $\sum_{n=0}^{\infty} a_n (z - z_0)^n$. Define R by
$$R = 0 \text{ if } \left(|a_n|^{\frac{1}{n}}\right)_{n=1}^{\infty} \text{ is not a bounded sequence}$$
$$R = \frac{1}{\limsup |a_n|^{\frac{1}{n}}} \text{ if } \limsup |a_n|^{\frac{1}{n}} > 0$$
$$R = \infty \text{ if } \limsup |a_n|^{\frac{1}{n}} = 0$$
Then the power series $\sum_{n=0}^{\infty} a_n (z - z_0)^n$ converges if $|z - z_0| < R$, and diverges if $|z - z_0| > R$.

Proof. We have

$$|a_n(z-z_0)^n|^{\frac{1}{n}} = |a_n|^{\frac{1}{n}}|z-z_0|$$

Suppose $z \neq z_0$. If $\left(|a_n|^{\frac{1}{n}}\right)_{n=1}^{\infty}$ is not a bounded sequence, then neither is $|a_n|^{\frac{1}{n}}|z-z_0|$, and we have divergence. Otherwise the conclusions follow from $|a_n(z-z_0)^n|^{\frac{1}{n}} = |a_n|^{\frac{1}{n}}|z-z_0|$ and the root test, Proposition 1.10.5. □

The number R defined in the statement of Theorem 1.11.1 is called the **radius of convergence** of the power series $\sum_{n=0}^{\infty} a_n(z-z_0)^n$. It is the radius of the **largest** circle in the complex plane **inside** which $\sum_{n=0}^{\infty} a_n(z-z_0)^n$ converges.

Theorem 1.11.1 is quite satisfying from a theoretical point of view: the radius of convergence is shown to exist and is (in principle) determined in all cases. However, recognizing $\limsup |a_n|^{\frac{1}{n}}$ may be very difficult in practice. The following is often helpful.

Theorem 1.11.2. Suppose $a_n \neq 0$ for $n \geq N$, and suppose $\lim \left|\frac{a_{n+1}}{a_n}\right|$ exists. Then the radius of convergence R of the power series $\sum_{n=0}^{\infty} a_n(z-z_0)^n$ is given by

$$R = \frac{1}{\lim \left|\frac{a_{n+1}}{a_n}\right|} \text{ if } \lim \left|\frac{a_{n+1}}{a_n}\right| > 0$$

$$R = \infty \text{ if } \lim \left|\frac{a_{n+1}}{a_n}\right| = 0$$

Proof. Apply Corollary 1.10.1 to the series $\sum_{n=0}^{\infty} a_n(z-z_0)^n$, noting that if $z \neq z_0$ then

$$\left|\frac{a_{n+1}(z-z_0)^{n+1}}{a_n(z-z_0)^n}\right| = \left|\frac{a_{n+1}}{a_n}\right| \cdot |z-z_0|$$

□

1.12 Exercises

1. If $\sum_{n=1}^{\infty} z_n$ converges with sum s and $\sum_{n=1}^{\infty} w_n$ converges with sum t, then $\sum_{n=1}^{\infty}(z_n+w_n)$ converges with sum $s+t$.
2. Suppose $\sum a_n$ and $\sum b_n$ each have all non-negative terms. If there are constants $M > 0$ and N such that $b_n \geq Ma_n$ whenever $n \geq N$, and if $\sum a_n = \infty$, then $\sum b_n = \infty$.
3. Show that $\sum_{n=1}^{\infty} \frac{n+1}{2n^2+1}$ diverges and $\sum_{n=1}^{\infty} \frac{n+1}{2n^3+1}$ converges. (Hint: use Proposition 1.10.3 and Exercise 1.12.2, and compare these to $\sum \frac{1}{n}, \sum \frac{1}{n^2}$.)
4. (2^k-Test). Suppose $a_1 \geq a_2 \geq \cdots \geq a_n \geq 0$, all n. Then $\sum_{n=1}^{\infty} a_n < \infty \Leftrightarrow \sum_{k=1}^{\infty} 2^k a_{2^k} < \infty$. (Hint: use the methods used to show divergence of $\sum \frac{1}{n}$ and convergence of $\sum \frac{1}{n^2}$.)

5. Suppose $p > 0$. The series $\sum_{n=1}^{\infty} \frac{1}{n^p}$ converges if $p > 1$ and diverges if $p \leq 1$. (Hint: use Exercise 1.12.4.)
6. The series $\sum_{n=2}^{\infty} \frac{1}{n(\log n)^2}$ converges; the series $\sum_{n=2}^{\infty} \frac{1}{n \log n}$ diverges.
7. The series $\sum_{n=0}^{\infty} \frac{z^n}{n!}$ converges for any $z \in \mathbf{C}$. Here $0! = 1$.
8. Determine the radius of convergence of

$$\sum_{n=0}^{\infty} \frac{2^n z^n}{n}, \sum_{n=1}^{\infty} \frac{n^n z^n}{n!}, \sum_{n=0}^{\infty} n! z^n, \sum_{n=0}^{\infty} \frac{n! z^n}{(2n)!}$$

9. (Alternating series). Suppose $|x_1| \geq |x_2| \geq \cdots \geq |x_n|$, all n, $x_n \geq 0$ if n odd, $x_n \leq 0$ if n even, and $x_n \to 0$. Then $\sum x_n$ converges. (Hint: the partial sums satisfy $s_2 \leq s_4 \leq s_6 \leq \cdots \leq s_5 \leq s_3 \leq s_1$.)
10. $\sum_{n=1}^{\infty} \frac{(-1)^n}{n}$ converges.

Chapter 2

Topology of Metric Spaces and Continuity

2.1 Metric spaces

A **metric** on a set S is a function d from the product set $S \times S$ to $\mathbf{R}$, with the properties

D1. $d(x, x) = 0, d(x, y) > 0$, if $x, y \in S, x \neq y$.

D2. $d(x, y) = d(y, x)$, all $x, y \in S$.

D3. $d(x, z) \leq d(x, y) + d(y, z)$, all $x, y, z \in S$.

We shall refer to $d(x, y)$ as the **distance** from x to y. A **metric space** is a set S together with a given metric d. The inequality D3 is called the **triangle inequality**. The elements of S are often called **points**.

As an example, take $S = \mathbf{R}^2 = \mathbf{R} \times \mathbf{R}$, with

$$d\left((x, y), (x', y')\right) = \left[(x - x')^2 + (y - y')^2\right]^{\frac{1}{2}}$$

If we coordinatize the Euclidean plane in the usual way, and if $(x, y), (x', y')$ are the coordinates of points P and P' respectively, then d gives the length of the line segment PP' (Pythagorean theorem). In this example, D3 is the analytic expression of the fact that the length of one side of a triangle is at most the sum of the lengths of the other two sides. The same example in different guise is obtained by letting $S = \mathbf{C}$ and taking

$$d(z, w) = |z - w|$$

as the metric. Then D3 is a consequence of Exercise 1.6.5.

Some other possible metrics on $\mathbf{R}^2$ are:

$$d_1\left((x, y), (x', y')\right) = |x - x'| + |y - y'|$$
$$d_2\left((x, y), (x', y')\right) = \max\{|x - x'|, |y - y'|\}$$
$$d_3\left((x, y), (x', y')\right) = 0 \text{ if } (x, y) = (x', y'), \text{ and } 1 \text{ otherwise}$$

Verification that the functions d_1, d_2 and d_3 satisfy the conditions D1, D2, D3 is left as an exercise. Note that d_3 words for **any** set S: if $x, y \in S$ we set $d(x, y) = 1$ if $x \neq y$ and 0 if $x = y$.

This is called **the discrete metric** on S

A still simpler example of a metric space is $\mathbf{R}$, with distance function d given by

$$d(x, y) = |x - y|$$

Again this coincides with the usual notion of the distance between two points on the (coordinatized) line.

Another important example is $\mathbf{R}^n$, the space of ordered n-tuples $x = (x_1, \ldots, x_n)$ of elements of $\mathbf{R}$. There are various possible metrics on $\mathbf{R}^n$ like the metrics d_1, d_2, d_3 defined above for $\mathbf{R}^n$, but we shall consider here only the generalization of the Euclidean distance in $\mathbf{R}^2$ and $\mathbf{R}^3$. If $x = (x_1, \ldots, x_n)$ and $y = (y_1, \ldots, y_n)$ we set

$$d(x, y) = \left(\sum_{j=1}^{n} (x_j - y_j)^2 \right)^{\frac{1}{2}}$$

When $n = 1$ we obtain $\mathbf{R}$ with the metric

$$d(x, y) = |x - y|$$

when $n = 2$ we obtain $\mathbf{R}^2$ with the metric

$$d\left((x, y), (x', y')\right) = \left[(x - x')^2 + (y - y')^2\right]^{\frac{1}{2}}$$

in somewhat different notation. It is easy to verify that d given by

$$d(x, y) = \left(\sum_{j=1}^{n} (x_j - y_j)^2 \right)^{\frac{1}{2}}$$

satisfies D1 and D2, but condition D3 is not so easy to verify.

Often when the metric d is understood, one refers to a set S alone as a metric space. For example, when we refer to $\mathbf{R}$, $\mathbf{C}$, or $\mathbf{R}^n$ as a metric space with no indication what metric is taken, we mean the metric to be given by

$$d(x, y) = |x - y|, d(z, w) = |z - w|, d(x, y) = \left(\sum_{j=1}^{n} (x_j - y_j)^2 \right)^{\frac{1}{2}}$$

Suppose (S, d) is a metric space and T is a subset of S. We can consider T as a metric space by taking the distance function on $T \times T$ to be the restriction of d to $T \times T$.

The concept of metric space has been introduced to provide a uniform treatment of such notions as distance, convergence, and limit which occur in many contexts in analysis.

Suppose (S, d) is a metric, x is a point of S, and r is a positive real number. The **ball of radius** r **about** x is defined to be the subset of S consisting of all points in S whose distance from x is less than r:

$$\mathbf{B}_r(x) = \{y | y \in S, d(x, y) < r\}$$

Clearly $x \in \mathbf{B}_r(x)$. If $0 < r < s$, then $\mathbf{B}_r(x) \subset \mathbf{B}_s(x)$.

Example 2.1.1. When $S = \mathbf{R}$ (metric understood), $\mathbf{B}_r(x)$ is the open interval $(x-r, x+r)$. When $S = \mathbf{R}^2$ or $\mathbf{C}$, $\mathbf{B}_r(z)$ is the open disc of radius r centered at z. Here we take the adjective 'open' understood; we shall see that the interval and the disc in question are also open in the sense defined below.

2.2 Topology of metric spaces

Let (S, d) be a metric space.

A subset $A \subset S$ is said to be a **neighborhood** of the point $x \in S$ if A contains $\mathbf{B}_r(x)$ for some $r > 0$. Roughly speaking, this says that A contains all points **sufficiently closed to** x. In particular, if A is a neighborhood of x it contains x itself.

A **topology** on a set S is just a collection $\mathcal{T}$ of subsets of S, which we call **open subsets**. The most common way to give a topology to a set is by using a metric on this set. Remember that (S, d) is now already a metric space. A subset $A \subset S$ is said to be **open** if it is a neighborhood of **each of its points**. Note that the empty set **is** an open subset of S since it has no points.

Actually, $\mathcal{T}$ has to satisfy some axioms to be a topology: T1: $\mathcal{T}$ is closed under arbitrary union, T2: $\mathcal{T}$ is closed under finite intersection.

Now S is a topology space since we have defined those subsets of S which we call them open subsets. A topology is the minimal structure on a set in order that the word **continuous** makes sense. Metric structure is too much: every metric induces a topology and different metrics may induce the same topology. Actually d_1, d_2, d_3 on $\mathbf{R}^2$ all give the same topology. We call these metrics **equivalent**. We will not need the notion of topological space in general. Metric spaces are sufficient for analysis. But you should be aware that not every topology is induced by some metric. Actually, every metric space is a so-called Hausdorff topological space, but there are topological spaces that are not Hausdorff. Voila, now forget about topology. Just remember what is an open subset.

Hausdorff spaces and T_2 spaces are the same.

Example 2.2.1. Consider the interval $A = (0, 1] \subset \mathbf{R}$. This is a neighborhood of each of its points except $x = 1$. In fact, if $0 < x < 1$, let $r = \min\{x, 1 - x\}$. Then $A \supset \mathbf{B}_r(x) = (x-r, x+r)$. However, for any $r > 0$, $\mathbf{B}_r(1)$ contains $1 + \frac{r}{2}$, which is not in A. So A is not open.

Now we explain what do we mean by 'closed under arbitrary union and finite intersection':

Proposition 2.2.1. Suppose (S, d) is a metric space.

1. For any $x \in S$ and any r, $\mathbf{B}_r(x)$ is open.

2. If $A_1, A_2, \ldots, A_n$ are open subsets of S, then $\bigcap_{m=1}^{n} A_m$
3. If $(A_\beta)_{\beta \in B}$ is any collection of open subsets of S, then $\bigcup_{\beta \in B} A_\beta$ is also open.

Proof. 1. Suppose $y \in \mathbf{B}_r(x)$. We want to show that for some $s > 0$, $\mathbf{B}_s(y) \subset \mathbf{B}_r(x)$. The triangle inequality makes this easy, for we can choose $s = r - d(y,x)$. Since $y \in \mathbf{B}_r(x)$. If $z \in \mathbf{B}_s(y)$, then

$$d(z,x) \leq d(z,y) + d(y,x) < s + d(y,x) = r$$

Thus $z \in \mathbf{B}_r(x)$.

2. Suppose $x \in \bigcap_{m=1}^{n} A_m$. Since each A_m is open, there is $r(m) > 0$ so that $\mathbf{B}_{r(m)}(x) \subset A_m$. Let $r = \min\{r(1), r(2), \ldots, r(n)\}$. Then $r > 0$ (If we are considering infinite intersections, then we do not necessarily have $r > 0$.) and $\mathbf{B}_r(x) \subset \mathbf{B}_{r(m)}(x) \subset A_m$ for all m. So $\mathbf{B}_r(x) \subset \bigcap_{m=1}^{n} A_m$.
3. Suppose $x \in A = \bigcup_{\beta \in B} A_\beta$. Then for some particular β, $x \in A_\beta$. Since A_β is open, there is an $r > 0$ so that $\mathbf{B}_r(x) \subset A_\beta \subset A$. Thus A is open.

□

Again suppose (S,d) is a metric space and suppose $A \subset S$. A point $x \in S$ is said to be a **limit point of** A if for every $r > 0$ there is a point of A with distance from x less than r:

$$\mathbf{B}_r(x) \cap A \neq \emptyset \text{ if } r > 0$$

In particular, if $x \in A$ then x is a limit point of A. The set is said to be closed if it contains each of its limit points. Note that the empty set is closed since is has no limit points so it contains each of its limit points.

You can think that a limit point x of A is a point which is infinitely close to A.

Example 2.2.2. The interval $(0,1] \subset \mathbf{R}$ has as its set of limit points the closed interval $[0,1]$. In fact if $0 < x \leq 1$, then x is certainly a limit point. If $x = 0$ and $r > 0$, then $\mathbf{B}_r(0) \cap (0,1] = (-r,r) \cap (0,1] \neq \emptyset$. If $x < 0$ and $r = |x|$, then $\mathbf{B}_r(x) \cap (0,1] = \emptyset$, while if $x > 1$ and $r = x - 1$, then $\mathbf{B}_r(x) \cap (0,1] = \emptyset$. Thus the interval $(0,1]$ is **neither open nor closed**. The exact relationship between open sets and closed sets is given in Proposition 2.2.3 below.

The following is the analogue for closed sets of Proposition 2.2.1.

Proposition 2.2.2. Suppose (S,d) is a metric space.

1. For any $x \in S$ and any $r > 0$, the closed ball $C = \{y | y \in S, d(x,y) \leq r\}$ is a closed set.
2. If $A_1, A_2, \ldots, A_n$ are closed subsets of S, then $\bigcup_{m=1}^{n} A_m$ is closed.
3. If $(A_\beta)_{\beta \in B}$ is any collection of closed subsets of S, then $\bigcap_{\beta \in B} A_\beta$ is closed.

Proof. 1. Suppose z is a limit point of the set C. Given $\varepsilon > 0$, there is a point $y \in \mathbf{B}_\varepsilon(z) \cap C$. Then

$$d(z,x) \leq d(z,y) + d(y,x) < \varepsilon + r$$

Since this is true for every $\varepsilon > 0$, we must have $d(z,x) \leq r$. Thus $z \in C$.

2. Suppose $x \notin A = \bigcup_{m=1}^{n} A_m$. For each m, x is not a limit point of A_m, so there is $r(m) > 0$ such that $\mathbf{B}_{r(m)}(x) \cap A_m = \emptyset$. Let

$$r = \min\{r(1), r(2), \ldots, r(n)\}$$

Then $\mathbf{B}_r(x) \cap A_m = \emptyset$, all m, so $\mathbf{B}_r(x) \cap A = \emptyset$. Thus x is not a limit point of A.

3. Suppose x is a limit point of $A = \bigcap_{\beta \in B} A_\beta$. For any $r > 0$, $\mathbf{B}_r(x) \cap A \neq \emptyset$. But $A \subset A_\beta$, so $\mathbf{B}_r(x) \cap A_\beta \neq \emptyset$. Thus x is a limit point of A_β, so it is in A_β. This is true for each β, so $x \in A$.

□

Proposition 2.2.3. Suppose (S, d) is a metric space. A subset $A \subset S$ is open if and only if its complement is closed.

Proof. Let B be the complement of A. Suppose B is closed, and suppose $x \in A$. Then x is not a limit point of B, so for some $r > 0$ we have $\mathbf{B}_r(x) \cap B = \emptyset$. Thus $\mathbf{B}_r(x) \subset A$, and A is a neighborhood of x.

Conversely, suppose A is open and suppose $x \notin B$. Then $x \in A$, so for some $r > 0$ we have $\mathbf{B}_r(x) \subset A$. Then $\mathbf{B}_r(x) \cap B = \emptyset$, and x is not a limit point of B. It follows that every limit point of B is in B. □

The set of limit points of a subset $A \subset S$ is called the **closure** of A; we shall denote it by A^-. We have $A \subset A^-$ and A is closed if and only if $A = A^-$. In the example above, we saw that the closure of $(0, 1] \subset \mathbf{R}$ is $[0, 1]$.

Suppose A, B are subsets of S and $A \subset B$. We say that A is **dense in** B if $B \subset A^-$. In particular, A is dense in S if $A^- = S$. As an example, $\mathbf{Q}$ is dense in $\mathbf{R}$. In fact, suppose $x \in \mathbf{R}$ and $r > 0$. Choose a positive integer n so large that $\frac{1}{n} < r$. There is a unique integer m so that $\frac{m}{n} < x < \frac{m+1}{n}$. Then $d(x, \frac{m}{n}) = x - \frac{m}{n} < \frac{m+1}{n} - \frac{m}{n} = \frac{1}{n} < r$, so $\frac{m}{n} \in \mathbf{B}_r(x)$. Thus $x \in \mathbf{Q}^-$.

We've defined sequence limit of real numbers and complex numbers, actually, sequence limit can be defined in any metric space:

A sequence $(x_n)_{n=1}^{\infty}$ in S is said to **converge to** $x \in S$ if for each $\varepsilon > 0$ there is an N so that $d(x_n, x) < \varepsilon$ if $n \geq N$. The point x is called the **limit** of the sequence, and we write

$$\lim_{n \to \infty} x_n = x \text{ or } x_n \to x$$

When $S = \mathbf{R}$ or $\mathbf{C}$ with the usual metric, this coincides with the definition in Chapter 1. Again the limit, if any, is unique.

A sequence $(x_n)_{n=1}^{\infty}$ in S is said to be a **Cauchy sequence** if for each $\varepsilon > 0$ there is an N so that $d(x_n, x_m) < \varepsilon$ if $n, m \geq N$. Again when $S = \mathbf{R}$ or $\mathbf{C}$, this coincides with the definition in Chapter 1.

The metric space (S, d) is said to be **complete** if every Cauchy sequence in S converges to a point of S. As an example, Theorem 1.8.1 says precisely that $\mathbf{R}$ and $\mathbf{C}$ are complete metric spaces with respect to the usual metrics.

Many processes in analysis produce sequences of numbers, functions, etc., in various metric spaces. It is important to know when such sequences converge. Knowing that the metric space in question is complete is a powerful tool, since the condition that the sequence be a Cauchy sequence is then a necessary and sufficient condition for convergence. We have already seen this in our discussion of series, for example.

Note that $\mathbf{R}^n$ is complete. To see this note that in $\mathbf{R}^n$,

$$\max\{|x_j - y_j|, j = 1, \ldots, n\} \leq d(x, y) \leq n \max\{|x_j - y_j|, j = 1, \ldots, n\}$$

It follows that a sequence of points in $\mathbf{R}^n$ converges if and only if each of the n corresponding sequences of coordinates converges in $\mathbf{R}$. Similarly, a sequence of points in $\mathbf{R}^n$ is a Cauchy sequence if and only if each of the n corresponding sequences of coordinates is a Cauchy sequence in $\mathbf{R}$. Thus completeness of $\mathbf{R}^n$ follows from completeness of $\mathbf{R}$. (This is simply a generalization of the argument showing $\mathbf{C}$ is complete.)

2.3 Exercises

1. If (S, d) is a metric space, $x \in S$, and $r \geq 0$, then

   $$\{y | y \in S, d(y, x) > r\}$$

   is an open subset of S.

2. The point x is a limit point of a set $A \subset S$ if and only if there is a sequence $(x_n)_{n=1}^{\infty}$ in A such that $x_n \to x$.

3. If a sequence $(x_n)_{n=1}^{\infty}$ in a metric space converges to $x \in S$ and also converges to $y \in S$, then $x = y$.

4. If a sequence converges, then it is a Cauchy sequence.

5. If (S, d) is a complete metric space and $A \subset S$ is closed, then (A, d) is complete. Conversely, if $B \subset S$ and (B, d) is complete, then B is a closed subset of S.

6. When we say A is an open subset of a metric space S. The condition that we're viewing A as a subset of S is important. Saying A itself is open does not make sense: The interval $(0, 1)$ is open as a subset of $\mathbf{R}$, but **not** as a subset of $\mathbf{C}$.

7. Let $S = \mathbf{Q}$ and let $d(x, y) = |x - y|, x, y \in \mathbf{Q}$. Show that (S, d) is not complete.

8. The set of all elements $x = (x_1, \ldots, x_n)$ in $\mathbf{R}^n$ such that each x_j is rational is a dense subset of $\mathbf{R}^n$.

9. Verify that $\mathbf{R}^n$ is complete.

2.4 Compact spaces

Suppose that (S, d) is a metric space, and suppose A is a subset of S. The subset A is said to be **compact** if it has the following property: suppose that for each $x \in A$ there is given a neighborhood of x, denoted $N(x)$; then there are finitely many points $x_1, \dots, x_n$ in A such that A is contained in the union of $N(x_1), N(x_2), \dots, N(x_n)$. Note that we are saying that this is true for any choice of neighbothoods of points of A, though the selection of points $x_1, x_2, \dots$ **may** depend on the selection of neighborhoods. It is bovious that any finite subset A is compact.

Example 2.4.1.

1. The infinite interval $(0, \infty) \subset \mathbf{R}$ is not compact. For example, let $N(x) = (x-1, x+1), x \in (0, \infty)$. Clearly no finite colloection of these intervals of finite length can cover all of $(0, \infty)$.
2. Even the finite interval $(0, 1] \subset \mathbf{R}$ is not compact. To see this, let $N(x) = (\frac{1}{x}, 2), x \in (0, 1]$. For any $x_1, x_2, \dots, x_n \in (0, 1]$, the union of the intervals $N(x_i)$ will not contain y if $y \le \frac{1}{2} \min\{x_1, \dots, x_n\}$.
3. The infinite set $A = \{0\} \cup \{1, \frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \dots\} \subset \mathbf{R}$ is compact. In fact, suppose for each $x \in A$ we are given a neighborhood $N(x)$. In particular, the neighborhood $N(0)$ of contains an interval $(-\varepsilon, \varepsilon)$. Let M be a positive integer larger that $\frac{1}{\varepsilon}$. Then $\frac{1}{n} \in N(0)$ for $n \ge M$, and it follows that $A \subset N(0) \cup N(1) \cup N(\frac{1}{2}) \cup N(\frac{1}{3}) \cup \dots \cup N(\frac{1}{M})$.

The first two examples illustrate general requirements which compact sets must satisfy. A subset A of S, when (S, d) is a metric space, is said to be **bounded** if there is a ball $\mathbf{B}_R(x)$ containing A.

Proposition 2.4.1. Suppose (S, d) is a metric space, $S \neq \emptyset$, and suppose $A \subset S$ is compact. Then A is closed and bounded.

Proof. □

The converse of Proposition 2.4.1, that a closed, bounded subset of a metric space is compact, is not true in general. It is a subtle but extremely important fact that it is true in $\mathbf{R}^n$, however.

Theorem 2.4.1 (Heine-Borel)**.** A subset of $\mathbf{R}^n$ or of $\mathbf{C}$ is compact if and only if it is closed and bounded.

Proof. □

Chapter 3

Integration and Differentiation

Chapter 4

Sequences and Series of Functions

Chapter 5

Differential Equations

Chapter 6

Multi-variable Analysis

Chapter 7

Bonus Chapter: Partition of Unity