21LA Linear algebra Final

Send your answer to `qiucaiyong2357@gmail.com`

Deadline: Jan.13 23:59, Open-book examination

Recall that a sequence of vector spaces and linear maps $\bullet \xrightarrow{T} \bullet \xrightarrow{S} \bullet$ is exact if and only if $\text{Range}(T) = \text{Null}(S)$.

$\prod_{i=1}^{n} V_i$ stands for $V_1 \times V_2 \times \cdots \times V_n$.

1

Let V be a finite-dimensional vector space over $\mathbf{F}$, construct an isomorphism between V and $\mathbf{F}^{\dim V}$. (10 points)

2

Let V, W be finite-dimensional vector spaces over $\mathbf{F}$, construct an isomorphism between $\mathcal{L}(V, W)$ and $\mathbf{F}^{\dim V \dim W}$. (10 points)

3

Let V be a finite-dimensional vector space over $\mathbf{F}$, and $U \leq V$ be a subspace.

3.1

Prove that there exists a subspace $W \leq V$ such that $V = U \oplus W$. (5 points)

3.2

Prove that given any $\varphi \in U^*$, there exists $\phi \in V^*$ such that $\varphi(u) = \phi(u)$ for all $u \in U$. (5 points)

4

Suppose $T \in \mathcal{L}(V, W)$ such that $\text{Null}(T), \text{Range}(T)$ are both finite-dimensional. Prove that V is finite-dimensional. (You need to modify the Theorem 3.3.1 since the Theorem 3.3.1 requires V to be finite-dimensional.) (10 points)

5

Suppose U, V, W are finite-dimensional vector space over $\mathbf{F}$, suppose we have two linear maps $S \in \mathcal{L}(V, W), T \in \mathcal{L}(U, V)$. Prove that (10 points)

$$\begin{aligned} \dim \operatorname{Null}(ST) &\leq \dim \operatorname{Null}(S) + \dim \operatorname{Null}(T) \\ \dim \operatorname{Range}(ST) &\leq \min\{\dim \operatorname{Range}(S), \dim \operatorname{Range}(T)\} \end{aligned}$$

6

Suppose V is finite-dimensional and $\mathcal{E}$ is a subspace of $\mathcal{L}(V)$ such that we have $ST, TS \in \mathcal{E}$ for all $S \in \mathcal{L}(V)$ and all $T \in \mathcal{E}$. Prove that $\mathcal{E} = \{0_V\}$ or $\mathcal{E} = \mathcal{L}(V)$. (10 points)

7

Construct an isomorphism (10 points)

$$\mathcal{L}\left(\prod_{i=1}^{n} V_i, \prod_{j=1}^{m} W_j\right) \to \prod_{i=1}^{n}\prod_{j=1}^{m} \mathcal{L}(V_i, W_j)$$

8

Given two spaces $W \leq V$, we use the following notation:

$$\begin{aligned} \iota_W^V &: W \xrightarrow{w \mapsto w} V \\ \pi_W^V &: V \xrightarrow{v \mapsto v+W} V/W \end{aligned}$$

to be the canonical inclusion and projection.

Now given $T \in \mathcal{L}(V, W)$, Let $V_0 \leq V, W_0 \leq W$ be two subspaces.

8.1

Prove that $V_0 = \operatorname{Null}(T)$ if and only if: $T \circ \iota_{V_0}^V = 0$ and for every $S \in \mathcal{L}(X, V)$ with $TS = 0$, there exists a unique $\theta \in \mathcal{L}(X, V_0)$ with $\iota_{V_0}^V \circ \theta = S$. (10 points)

8.2

Prove that $W_0 = \operatorname{Range}(T)$ if and only if: $\pi_{W_0}^W \circ T = 0$ and for every $S \in \mathcal{L}(W, Y)$ with $ST = 0$, there exists a unique $\theta \in \mathcal{L}(W_0, Y)$ with $\theta \circ \pi_{W_0}^W = S$. (10 points)

9

Suppose U, V, W are finite-dimensional. Prove that $U \xrightarrow{S} V \xrightarrow{T} W$ is exact if and only if $W^* \xrightarrow{T^*} V^* \xrightarrow{S^*} U^*$ is exact. What if U, V, W are infinite-dimensional? (10 points)

21LA Linear algebra Final

Send your answer to `qiucaiyong2357@gmail.com`

Open-book examination

Recall that a sequence of vector spaces and linear maps $\bullet \xrightarrow{T} \bullet \xrightarrow{S} \bullet$ is exact if and only if $\mathrm{Range}(T) = \mathrm{Null}(S)$.

$\prod_{i=1}^{n} V_i$ stands for $V_1 \times V_2 \times \cdots \times V_n$.

1

Let V be a finite-dimensional vector space over $\mathbf{F}$, construct an isomorphism between V and $\mathbf{F}^{\dim V}$. (10 points)

Solution

Every finite-dimensional vector space admits a basis, actually so do infinite-dimensional spaces. Choose a basis for V, say $v_1, \ldots, v_n$. By the definition of basis, every element v in V can be written as

$$v = a_1 v_1 + \cdots + a_n v_n$$

where $a_1, \ldots, a_n \in \mathbf{F}$ is uniquely determined by v. Means that if $a_1 v_1 + \cdots + a_n v_n = b_1 v_1 + \cdots + b_n v_n$, then we have $a_i = b_i$ for all $i = 1, \ldots, n$. Hence it is possible to define

$$T : V \to \mathbf{F}^n$$

by

$$T(v) = (a_1, \ldots, a_n)$$

whenever $v = a_1 v_1 + \cdots a_n v_n$. Clearly $n = \dim V$. Now we prove that T is linear and bijective.

To prove T is linear, suppose $v = a_1 v_1 + \cdots + a_n v_n$ and $w = b_1 v_1 + \cdots b_n v_n$, we prove that $Tv + Tw = T(v + w)$.

We have $Tv + Tw = (a_1, \ldots, a_n) + (b_1, \ldots, b_n) = (a_1 + b_1, \ldots, a_n + b_n)$.

Since $v + w = (a_1 + b_1)v_1 + \cdots + (a_n + b_n)v_n$, we have

$$T(v + w) = (a_1 + b_1, \ldots, a_n + b_n)$$

This proves the additivity of T. Similarly, if $\lambda \in \mathbf{F}$, then $\lambda(Tv) = \lambda(a_1, \ldots, a_n) = (\lambda a_1, \ldots, \lambda a_n)$ and by $\lambda v = \lambda a_1 v_1 + \cdots + \lambda a_n v_n$ we have

$$T(\lambda v) = (\lambda a_1, \ldots, \lambda a_n) = \lambda(Tv)$$

Next we prove that T is injective and surjective. To prove that a linear map is injective, it suffices to prove that $\mathrm{Null}(T) = 0$. Suppose $Tv = (0, \ldots, 0)$, then $v = 0v_1 + \cdots + 0v_n = 0$, so $\mathrm{Null}(T) = 0$. The surjectiveness of T is easy, since every $(a_1, \ldots, a_n) \in \mathbf{F}^n$ is the image of $a_1v_1 + \cdots + a_nv_n \in V$.

2

Let V, W be finite-dimensional vector spaces over $\mathbf{F}$, construct an isomorphism between $\mathcal{L}(V, W)$ and $\mathbf{F}^{\dim V \dim W}$. (10 points)

Solution

Choose a basis for V, say $v_1, \ldots, v_n$ where $n = \dim V$, and a basis for W, say $w_1, \ldots, w_m$ where $m = \dim W$. Given $T \in \mathcal{L}(V, W)$, Tv_i can always be represented as a linear combination of $w_1, \ldots, w_m$:

$$Tv_i = \sum_{j=1}^{m} A_{i,j} w_j$$

And the coefficient $A_{i,j}$ are uniquely determined, since $w_1, \ldots, w_m$ is a basis for W.

We now define $\Gamma(T) = (A_{i,j}) \in \mathbf{F}^{n \times m}$, so Γ is a function from $\mathcal{L}(V, W)$ to $\mathbf{F}^{n \times m}$. We prove that Γ is linear and bijective.

First, if we have $T, S \in \mathcal{L}(V, W)$, we can always assume that

$$Tv_i = \sum_{j=1}^{m} A_{i,j} w_j, \forall i = 1, \ldots, n$$

$$Sv_j = \sum_{j=1}^{m} B_{i,j} w_j, \forall i = 1, \ldots, n$$

Those coefficients $A_{i,j}, B_{i,j}$ are determined by T, S. Notice the following fact

$$(T + S)v_i = \sum_{j=1}^{m} (A_{i,j} + B_{i,j}) w_j, \forall i = 1, \ldots, n$$

$$(\lambda T)v_i = \sum_{j=1}^{m} \lambda A_{i,j} w_j, \forall i = 1, \ldots, n$$

We conclude that $\Gamma(T + S) = \Gamma(T) + \Gamma(S), \Gamma(\lambda T) = \lambda \Gamma(T)$. So Γ is linear.

I want to point out that, the fact $\Gamma(T)$ is completely determined by T is crucial. Think of this: if you know $(T+S)v_i = \sum_{j=1}^{m}(A_{i,j} + B_{i,j})w_j$, you need the fact that this is the only possible way to express $T + S$, to conclude that $\Gamma(T + S) = (A_{i,j} + B_{i,j})$, and then we can prove that $\Gamma(T + S) = \Gamma(T) + \Gamma(S)$. From this we immediately have Γ is injective.

To prove Γ is surjective, we only need to show that given any $(A_{i,j}) \in \mathbf{F}^{n\times m}$, there exists a linear map $T \in \mathcal{L}(V, W)$ with

$$Tv_i = \sum_{j=1}^{m} A_{i,j} w_j, \forall i = 1, \ldots, n$$

This is not very obvious! At least, this requires some explanation.

First of all, to prove that there exists a $T : V \to W$, we need to define the value of Tv for all $v \in V$, not just $v_1, \ldots, v_n$. Notice that $v_1, \ldots, v_n$ is a basis of V, every element $v \in V$ can be written as a linear combination of $v_1, \ldots, v_n$:

$$v = \sum_{i=1}^{n} a_i v_i$$

Now we define

$$Tv = \sum_{i=1}^{n} \sum_{j=1}^{m} a_i A_{i,j} w_j$$

The first thing to consider is: is this well-defined? Is it possible that for a same v, this formula has defined Tv more than one time, and to different values?

The answer is: since $v_1, \ldots, v_n$ is a basis, no two different linear combinations gives the same v. So the formula has defined Tv for every $v \in V$ for just once. And hence is well-defined.

Now we prove that the T defined by

$$Tv = \sum_{i=1}^{n} \sum_{j=1}^{m} a_i A_{i,j} w_j$$

is linear. Suppose we have $v, v' \in V$, written as linear combinations:

$$v = \sum_{i=1}^{n} a_i v_i$$

$$v' = \sum_{i=1}^{n} a'_i v_i$$

Then we have

$$v + v' = \sum_{i=1}^{n} (a_i + a'_i) v_i$$

$$\lambda v = \sum_{i=1}^{n} \lambda a_i v_i$$

And

$$T(v + v') = \sum_{i=1}^{n} \sum_{j=1}^{m} (a_i + a'_i) A_{i,j} w_j = Tv + Tv'$$

$$T(\lambda v) = \sum_{i=1}^{n} \sum_{j=1}^{m} \lambda a_i A_{i,j} w_j = \lambda(Tv)$$

3

Let V be a finite-dimensional vector space over $\mathbf{F}$, and $U \leq V$ be a subspace.

3.1

Prove that there exists a subspace $W \leq V$ such that $V = U \oplus W$. (5 points)

Solution

Choose a basis for W, say $v_1, \ldots, v_m$. Recall that every independent list in V can be expanded into a basis of V. Extend this into a basis for V, say $v_1, \ldots, v_m, v_{m+1}, \ldots, v_n$. Set $U = \mathrm{Span}(v_{m+1}, \ldots, v_n)$.

We now prove that $V = W + U$ and $W \cap U = \{0\}$.

Any vector in V can be written as

$$v = a_1 v_1 + \cdots + a_n v_n$$

Notice that $a_1 v_1 + \cdots + a_m v_m \in W, a_{m+1} v_{m+1} + \cdots + a_n v_n \in U$, we have $v \in W + U$. So $V \subseteq U + W$. But obviously $U + W \subseteq V$. So $V = U + W$.

To prove $U \cap W = \{0\}$. Suppose $v \in U \cap W$, so there exists $a_1, \ldots, a_n \in \mathbf{F}$ with

$$a_1 v_1 + \cdots + a_m v_m = v = a_{m+1} v_{m+1} + \cdots + a_n v_n$$

and we have

$$a_1 v_1 + \cdots + a_m v_m + (-a_{m+1}) v_{m+1} + \cdots + (-a_n) v_n = 0$$

Since $v_1, \ldots, v_n$ is linearly independent, we must have

$$a_1 = a_2 = \cdots = a_n = 0$$

so $v = 0$.

3.2

Prove that given any $\varphi \in U^*$, there exists $\phi \in V^*$ such that $\varphi(u) = \phi(u)$ for all $u \in U$. (5 points)

Solution

By **3.1**, we can find $W \leq V$ such that $V = U \oplus W$. So every vector in $v \in V$ can be **uniquely**(by the definition of direct sum) written as $v = u + w$ where $u \in U, w \in W$. Define $\phi(v) = \varphi(u)$.

We now prove that ϕ is linear. Suppose $v_1, v_2 \in V$ such that $v_1 = u_1 + w_1, v_2 = u_2 + w_2$ where $u_1, u_2 \in U, w_1, w_2 \in W$. Then we must have

$$T(v_1 + v_2) = \varphi(u_1 + u_2) = Tv_1 + Tv_2$$

$$T(\lambda v_1) = \varphi(\lambda v_1) = \lambda T v_1$$

Obviously, if $v \in U$, then $v = v + 0$ is the only possible way to express v as the sum of elements in U and W. So when $v \in U$, $\phi(v) = \varphi(v)$.

Remark

Some of you think that U^* is a subspace of V^*. But this is wrong.

4

Suppose $T \in \mathcal{L}(V, W)$ such that $\text{Null}(T), \text{Range}(T)$ are both finite-dimensional. Prove that V is finite-dimensional. (You need to modify the Theorem 3.3.1 since the Theorem 3.3.1 requires V to be finite-dimensional.) (10 points)

Solution

Choose a basis for $\text{Null}(T)$, say $v_1, \ldots, v_n$. Choose a basis for $\text{Range}(T)$, say $w_1, \ldots, w_m$. There exists $u_1, \ldots, u_m \in V$ such that $Tu_i = w_i$ for all i.

We claim that V can be spanned by $v_1, \ldots, v_n, w_1, \ldots, w_m$. Given $v \in V$, Tv can be written as a linear combination of $w_1, \ldots, w_m$, say

$$Tv = a_1 w_1 + \cdots + a_m w_m$$

Then we compute

$$T(v - a_1 u_1 - \cdots - a_m u_m) = Tv - a_1 Tu_1 - \cdots a_m Tu_m = 0$$

So $v - a_1 u_1 - \cdots - a_m u_m \in \text{Null}(T)$. So there exists $b_1, \ldots, b_n \in \mathbf{F}$ such that

$$v - a_1 u_1 - \cdots - a_m u_m = b_1 v_1 + \cdots + b_n v_n$$

Being the span of finitely many vectors, V must be finite-dimensional.

5

Suppose U, V, W are finite-dimensional vector space over $\mathbf{F}$, suppose we have two linear maps $S \in \mathcal{L}(V, W), T \in \mathcal{L}(U, V)$. Prove that (10 points)

$$\dim \text{Null}(ST) \leq \dim \text{Null}(S) + \dim \text{Null}(T)$$
$$\dim \text{Range}(ST) \leq \min\{\dim \text{Range}(S), \dim \text{Range}(T)\}$$

Solution

Let $l = \dim(\text{Null}(S) \cap \text{Range}(T)) \leq \dim \text{Null}(S)$, then there exists $u_1, \ldots, u_l \in U$ such that $Tu_1, \ldots, Tu_l$ is a basis for $\text{Null}(S) \cap \text{Range}(T)$. Choose a basis for $\text{Null}(T)$, say $t_1, \ldots, t_n$ where $n = \dim \text{Null}(T)$.

We claim that $u_1, \ldots, u_l, t_1, \ldots, t_n$ spans $\text{Null}(ST)$. If $v \in \text{Null}(ST)$, then $S(Tv) = 0$, so $Tv \in \text{Null}(S) \cap \text{Range}(T)$. We can write

$$Tv = a_1 Tu_1 + \cdots + a_l Tu_l$$

From this we have

$$T(v - a_1 u_1 - \cdots - a_l u_l) = 0$$

So we can write

$$v - a_1 u_1 - \cdots - a_l u_l = b_1 t_1 + \cdots + b_n t_n$$

and we have $v \in \text{Span}(u_1, \ldots, u_l, t_1, \ldots, t_n)$. This tells us

$$\dim \text{Null}(ST) \leq l + n \leq \text{Null}(S) + \dim \text{Null}(T)$$

Notice that $\text{Range}(ST) \subseteq \text{Range}(S)$, so $\dim \text{Range}(ST) \leq \dim \text{Range}(S)$. Let $v_1, \ldots, v_m$ be a basis of $\text{Range}(T)$, then $\text{Range}(ST) = \text{Span}(Sv_1, \ldots, Sv_m)$. So $\dim \text{Range}(ST) \leq m = \dim \text{Range}(T)$.

6

Suppose V is finite-dimensional and $\mathcal{E}$ is a subspace of $\mathcal{L}(V)$ such that we have $ST, TS \in \mathcal{E}$ for all $S \in \mathcal{L}(V)$ and all $T \in \mathcal{E}$. Prove that $\mathcal{E} = \{0_V\}$ or $\mathcal{E} = \mathcal{L}(V)$. (10 points)

Lemma If $v_1, \ldots, v_n$ is a basis of V, and $w_1, \ldots, w_n \in V$. Then there exists $T \in \mathcal{L}(V)$ such that $Tv_i = w_i$ for all $i = 1, \ldots, n$.

For a proof, see Lecture Notes Proposition 3.1.1.

Solution

If $\mathcal{E} \neq \{0_V\}$, then there exists $T \in \mathcal{E}$ such that $T \neq 0_V$. So there exists $v \in V$ such that $Tv \neq 0$.

Choose a basis for V, say $v_1, \ldots, v_n$. By the lemma, there exists $T_1, \ldots, T_n, S_1, \ldots, S_n \in \mathcal{L}(V)$ such that

$$\begin{cases} T_1 v_1 &= v \\ T_1 v_2 &= 0 \\ & \vdots \\ T_1 v_n &= 0 \end{cases}, \begin{cases} T_2 v_1 &= 0 \\ T_2 v_2 &= v \\ & \vdots \\ T_2 v_n &= 0 \end{cases}, \ldots, \begin{cases} T_n v_1 &= 0 \\ T_n v_2 &= 0 \\ & \vdots \\ T_n v_n &= v \end{cases}, \begin{cases} S_1(Tv) &= v_1 \\ S_2(Tv) &= v_2 \\ & \vdots \\ S_n(Tv) &= v_n \end{cases}$$

By the assumption, $S_1 T T_1 + \cdots + S_n T T_n \in \mathcal{E}$. We compute:

$$(S_1 T T_1 + \cdots + S_n T T_n) v_j = S_j T v = v_j$$

So $I_V = S_1 T T_1 + \cdots + S_n T T_n \in \mathcal{E}$, and we then have $\mathcal{E} = \mathcal{L}(V)$.

7

Construct an isomorphism (10 points)

$$\mathcal{L}\left(\prod_{i=1}^{n} V_i, \prod_{j=1}^{m} W_j\right) \to \prod_{i=1}^{n} \prod_{j=1}^{m} \mathcal{L}(V_i, W_j)$$

Fast and dirty Solution

$$\dim \mathcal{L}\left(\prod_{i=1}^{n} V_i, \prod_{j=1}^{m} W_j\right) = \left(\sum_{i=1}^{n} \dim V_i\right)\left(\sum_{j=1}^{n} \dim W_j\right) = \dim \prod_{i=1}^{n} \prod_{j=1}^{m} \mathcal{L}(V_i, W_j)$$

True Solution

Define $\sharp_i^V : V_i \to V_1 \times \cdots \times V_n$ by

$$\sharp_i^V(v) = (0, \ldots, v, \ldots, 0)$$

Define $\flat_j^W : W_1 \times \cdots \times W_m \to W_j$ by

$$\flat_j^W(w_1, \ldots, w_m) = w_j$$

Obviously they are linear.

Given any $T : \prod_{i=1}^n V_i \to \prod_{j=1}^m W_j$, we can define

$$\flat_j^W T \sharp_i^V : V_i \xrightarrow{\sharp_i^V} \prod_{i=1}^n V_i \xrightarrow{T} \prod_{j=1}^m W_j \xrightarrow{\flat_j^W} W_j$$

Now we define a map $\natural_i^j : \mathcal{L}\left(\prod_{i=1}^n V_i, \prod_{j=1}^m W_j\right) \to \mathcal{L}(V_i, W_j)$ by

$$\natural_i^j(T) = \flat_j^W T \sharp_i^V$$

We claim that $\natural_i^j$ is linear, since

$$\natural_i^j(T_1 + T_2) = \flat_j^W (T_1 + T_2)\sharp_i^V = \flat_j^W T_1 \sharp_i^V + \flat_j^W T_2 \sharp_i^V = \natural_i^j(T_1) + \natural_i^j(T_2)$$

$$\natural_i^j(\lambda T) = \flat_j^W (\lambda T)\sharp_i^V = \lambda \flat_j^W T \sharp_i^V = \lambda \natural_i^j(T)$$

Now we define for every $T \in \mathcal{L}\left(\prod_{i=1}^n V_i, \prod_{j=1}^m W_j\right)$

$$\prod_{i=1}^n \prod_{j=1}^m \mathcal{L}(V_i, W_j) \ni \natural(T) = \left(\natural_i^j(T)\right)_{1 \le i \le n, 1 \le j \le m}$$

We claim that $\natural$ is linear, since

$$\natural(T_1 + T_2) = \left(\natural_i^j(T_1 + T_2)\right)_{i,j} = \left(\natural_i^j(T_1)\right)_{i,j} + \left(\natural_i^j(T_2)\right)_{i,j} = \natural(T_1) + \natural(T_2)$$

$$\natural(\lambda T) = \left(\natural_i^j(\lambda T)\right)_{i,j} = \left(\lambda \natural_i^j(T)\right)_{i,j} = \lambda \natural(T)$$

So we have a linear map

$$\natural : \mathcal{L}\left(\prod_{i=1}^n V_i, \prod_{j=1}^m W_j\right) \to \prod_{i=1}^n \prod_{j=1}^m \mathcal{L}(V_i, W_j)$$

We prove that $\natural$ is injective and surjective.

Suppose $\natural(T) = 0$, then $\flat_j^W T \sharp_i^V = \natural_i^j(T) = 0$ for all i, j. Then for all j:

$$(\flat_j^W T \sharp_1^V + \cdots + \flat_j^W T \sharp_n^V)v = \flat_j^W T(\sharp_1^V v + \cdots + \sharp_n^V v) = \flat_j^W (Tv) = 0$$

So $\flat_j^W (Tv) = 0$ for all j and all $v \in \prod_{i=1}^n V_i$, we have $T = 0$. Hence $\natural$ is injective.

To prove $\natural$ is surjective, we need to show that: given a family of linear maps $\left(T_i^j \in \mathcal{L}(V_i, W_j)\right)_{i,j}$, there exists T such that for all i, j

$$\natural_i^j(T) = T_i^j$$

Define $\sharp_j^W : W_j \to W_1 \times \cdots \times W_m$ by

$$\sharp_j^W(w) = (0, \ldots, w, \ldots, 0)$$

Define $\flat_i^V : V_1 \times \cdots \times V_n \to V_i$ by

$$\flat_i^V(v_1, \ldots, v_n) = v_i$$

Obviously they are linear. And we have

$$\flat_j^W \sharp_b^W = \begin{cases} I, & j = b \\ 0, & j \neq b \end{cases}$$

$$\flat_a^V \sharp_i^V = \begin{cases} I, & a = i \\ 0, & a \neq i \end{cases}$$

Then we have for any i, j,

$$\sharp_j^W T_i^j \flat_i^V : \prod_{i=1}^n V_i \xrightarrow{\flat_i^V} V_i \xrightarrow{T_i^j} W_j \xrightarrow{\sharp_j^W} \prod_{j=1}^m W_j$$

If we define

$$T = \sum_{i=1}^n \sum_{j=1}^m \sharp_j^W T_i^j \flat_i^V$$

we claim that $\natural_i^j(T) = T_i^j$. This is just a simple calculation:

$$\natural_i^j(T) = \flat_j^W \left(\sum_{a=1}^n \sum_{b=1}^m \sharp_b^W T_a^b \flat_a^V \right) \sharp_i^V = \sum_{a=1}^n \sum_{b=1}^m \flat_j^W \sharp_b^W T_a^b \flat_a^V \sharp_i^V = T_i^j$$

Remark

Some of you got the idea right, but don't know how to write it down.

What I've done can be summarized by the following diagram:

$$\begin{tikzcd} \prod V_i \arrow[r, "T"] & \prod W_j \arrow[d, "\flat"] \\ V_i \arrow[u, "\sharp"] \arrow[r, dashed, "\natural_i^j T"'] & W_j \end{tikzcd} \qquad \begin{tikzcd} V_i \arrow[r, "T_i^j"] & W_j \arrow[d, "\sharp"] \\ \prod V_i \arrow[u, "\flat"] \arrow[r, dashed, "\sum \sharp_j^W T_i^j \flat_i^V"'] & \prod W_j \end{tikzcd}$$

Remark

It is possible to generalize Problem 7 to the case of infinite product and infinite direct sum:

$$\mathcal{L}\left(\bigoplus_{i \in \mathcal{I}} V_i, \prod_{j \in \mathcal{J}} W_j \right) \simeq \prod_{i \in \mathcal{I}} \prod_{j \in \mathcal{J}} \mathcal{L}(V_i, W_j)$$

8

Given two spaces $W \leq V$, we use the following notation:

$$\iota_W^V : W \xrightarrow{w \mapsto w} V$$
$$\pi_W^V : V \xrightarrow{v \mapsto v+W} V/W$$

to be the canonical inclusion and projection.

Now given $T \in \mathcal{L}(V, W)$, Let $V_0 \leq V, W_0 \leq W$ be two subspaces.

8.1

Prove that $V_0 = \mathrm{Null}(T)$ if and only if: $T \circ \iota_{V_0}^V = 0$ and for every $S \in \mathcal{L}(X, V)$ with $TS = 0$, there exists a unique $\theta \in \mathcal{L}(X, V_0)$ with $\iota_{V_0}^V \circ \theta = S$. (10 points)

Solution

If $V_0 = \mathrm{Null}(T)$, then

$$\mathrm{Null}(T) = V_0 \xrightarrow{\iota_{V_0}^V} V \xrightarrow{T} W$$

is clearly 0. And if we have $X \xrightarrow{S} V \xrightarrow{T} W$ with $TS = 0$. Then $\mathrm{Range}(S) \subseteq \mathrm{Null}(T) = V_0$. So we can define $\theta : X \to V_0$ by $\theta(x) = Sx$. This is well-defined.

Since S is linear, θ is also linear. And θ is unique since we must have $\iota_{V_0}^V(\theta(x)) = Sx$.

Now suppose $T \circ \iota_{V_0}^V = 0$ and for every $S \in \mathcal{L}(X, V)$ with $TS = 0$, there exists a unique $\theta \in \mathcal{L}(X, V_0)$ with $\iota_{V_0}^V \circ \theta = S$. We have $V_0 = \mathrm{Range}(\iota_{V_0}^V) \subseteq \mathrm{Null}(T)$.

To prove $\mathrm{Null}(T) \subseteq V_0$, consider $X = \mathrm{Null}(T)$ and $S = \iota_{\mathrm{Null}(T)}^V$, by assumption there exists a unique $\theta : \mathrm{Null}(T) \to V_0$ such that $\theta(v) = Sv = v$ for all $v \in X = \mathrm{Null}(T)$. So we have $\theta(v) = v \in V_0$ for all $v \in \mathrm{Null}(T)$, and consequently $\mathrm{Null}(T) \subseteq V_0$.

8.2

Prove that $W_0 = \mathrm{Range}(T)$ if and only if: $\pi_{W_0}^W \circ T = 0$ and for every $S \in \mathcal{L}(W, Y)$ with $ST = 0$, there exists a unique $\theta \in \mathcal{L}(W/W_0, Y)$ with $\theta \circ \pi_{W_0}^W = S$. (10 points)

Solution

If $W_0 = \mathrm{Range}(T)$, then

$$V \xrightarrow{T} W \xrightarrow{\pi_{W_0}^W} W/W_0 = W/\mathrm{Range}(T)$$

is clearly 0. And if we have $V \xrightarrow{T} W \xrightarrow{S} Y$ with $ST = 0$. Then $W_0 = \mathrm{Range}(T) \subseteq \mathrm{Null}(S)$. So we can define $\theta : W/W_0 \to Y$ by $\theta(w + W_0) = Sw$. This is well-defined.

Since S is linear, θ is also linear. And θ is unique since we must have $\theta(\pi_{W_0}^W(w)) = Sw$.

Now suppose $\pi^W_{W_0} \circ T = 0$ and for every $S \in \mathcal{L}(W, Y)$ with $ST = 0$, there exists a unique $\theta \in \mathcal{L}(W/W_0, Y)$ with $\theta \circ \pi^W_{W_0} = S$. We have $\text{Range}(T) \subseteq \text{Null}(\pi^W_{W_0}) = W_0$.

To prove $W_0 \subseteq \text{Range}(T)$, consider $Y = W/\text{Range}(T)$ and $S = \pi^W_{\text{Range}(T)}$, by assumption there exists a unique $\theta : W/W_0 \to W/\text{Range}(T)$ such that $\theta(w + W_0) = Sw = w + \text{Range}(T)$ for all $w \in W$. If we let $w \in W_0$, then $w + W_0 = 0 + W_0$, and linear maps send 0 to 0, we have $w + \text{Range}(T) = 0 + \text{Range}(T)$, so $w \in \text{Range}(T)$ and consequently we have $W_0 \subseteq \text{Range}(T)$.

Remark

Did you notice that the proofs of 8.1 and 8.2 are the same, word for word?

9

Suppose U, V, W are finite-dimensional. Prove that $U \xrightarrow{S} V \xrightarrow{T} W$ is exact if and only if $W^* \xrightarrow{T^*} V^* \xrightarrow{S^*} U^*$ is exact. What if U, V, W are infinite-dimensional? (10 points)

Solution

Suppose $\text{Range}(S) = \text{Null}(T)$, then $S^*(T^*\varphi) = \varphi TS = 0$ for all $\varphi \in W^*$. So $\text{Range}(T^*) \subseteq \text{Null}(S^*)$. We now prove that $\text{Range}(T^*) \supseteq \text{Null}(S^*)$.

Suppose $\psi \in V^*$ such that $S^*\psi = \psi S = 0$. Then $\forall v \in \text{Null}(T) = \text{Range}(S)$, we have $\psi(v) = 0$. So $\psi(\text{Null}(T)) = 0$

Define

$$\tilde{T} : V/\text{Null}(T) \to \text{Range}(T)$$

by

$$\tilde{T}(v + \text{Null}(T)) = Tv$$

Define

$$\tilde{\psi} : V/\text{Null}(T) \to \mathbf{F}$$

by

$$\tilde{\psi}(v + \text{Null}(T)) = \psi(v)$$

These two are well-defined, meaning that if $v_1 + \text{Null}(T) = v_2 + \text{Null}(T)$, then

$$\tilde{T}(v_1 + \text{Null}(T)) = \tilde{T}(v_2 + \text{Null}(T))$$

$$\tilde{\psi}(v_1 + \text{Null}(T)) = \tilde{\psi}(v_2 + \text{Null}(T))$$

Obviously they are linear and $\tilde{T}$ is injective and surjective. So it has an linear inverse

$$\tilde{T}^{-1} : \text{Range}(T) \to V/\text{Null}(T)$$

such that $\tilde{T}^{-1}(Tv) = v + \text{Null}(T)$

Consider the linear functional

$$\phi : \text{Range}(T) \xrightarrow{\tilde{T}^{-1}} V/\text{Null}(T) \xrightarrow{\tilde{\psi}} \mathbf{F}$$

By problem 3.2, there exists $\theta \in W^*$ such that

$$\theta(w) = \phi(w), \forall w \in \text{Range}(T)$$

Notice that we have

$$T = V \xrightarrow{\pi^V_{\text{Null}(T)}} V/\text{Null}(T) \xrightarrow{\tilde{T}} W$$

$$\psi = V \xrightarrow{\pi^V_{\text{Null}(T)}} V/\text{Null}(T) \xrightarrow{\tilde{\psi}} \mathbf{F}$$

We compute $T^*\theta = V \xrightarrow{T} W \xrightarrow{\theta} \mathbf{F}$, we use the fact that $Tv \in \text{Range}(T)$

$$(T^*\theta)(v) = \theta(Tv) = \phi(Tv) = \tilde{\psi}\left(\tilde{T}^{-1}(Tv)\right) = \tilde{\psi}(v + \text{Null}(T)) = \psi(v)$$

Which means that $\psi = T^*\theta \in \text{Range}(T^*)$.

Next, suppose $W^* \xrightarrow{T^*} V^* \xrightarrow{S^*} U^*$ is exact. We need the following lemma:

[←**Lemma**
If V is finite-dimensional, define

$$\Lambda_V : V \to V^{**}$$

by

$$(\Lambda_V v)\varphi = \varphi(v), \forall \varphi \in V^*$$

Then Λ_V is linear and bijective. →]

Since $W^* \xrightarrow{T^*} V^* \xrightarrow{S^*} U^*$ is exact, we have $U^{**} \xrightarrow{S^{**}} V^{**} \xrightarrow{T^{**}} W^{**}$ is exact.
Consider the diagram

$$\begin{tikzcd}
U \arrow[r, "S"] \arrow[d, "\Lambda_U"] & V \arrow[r, "T"] \arrow[d, "\Lambda_V"] & W \arrow[d, "\Lambda_W"] \\
U^{**} \arrow[r, "S^{**}"] & V^{**} \arrow[r, "T^{**}"] & W^{**}
\end{tikzcd}$$

It is commutative. For example, we prove that $\Lambda_V \circ S = S^{**} \circ \Lambda_U$: $\forall u \in U$ and $\forall \theta \in V^*$, we have

$$\left(\Lambda_V(Su)\right)(\theta) = \theta(Su)$$

$$\left(S^{**}(\Lambda_U u)\right)(\theta) = (\Lambda_U u \circ S^*)(\theta) = (\Lambda_U u)\,(\theta \circ S) = (\theta \circ S)u = \theta(Su)$$

Which tells us that $\Lambda_V(Su) = S^{**}(\Lambda_U u)$ for all $u \in U$.

OK. Now we know that

$$\begin{tikzcd}
U \arrow[r, "S"] \arrow[d, "\Lambda_U"] & V \arrow[r, "T"] \arrow[d, "\Lambda_V"] & W \arrow[d, "\Lambda_W"] \\
U^{**} \arrow[r, "S^{**}"] & V^{**} \arrow[r, "T^{**}"] & W^{**}
\end{tikzcd}$$

is a commutative diagram. The lower row is exact, and all three Λ's are isomorphisms. So

$$TS = {\Lambda_W}^{-1}\Lambda_W TS = {\Lambda_W}^{-1}T^{**}S^{**}\Lambda_U = 0$$

To prove that $\mathrm{Null}(T) \subseteq \mathrm{Range}(S)$, if $Tv = 0$, then

$$0 = \Lambda_W Tv = T^{**}\Lambda_V v$$

so $\Lambda_V v \in \mathrm{Null}(T^{**}) = \mathrm{Range}(S^{**})$, so there exists $t \in U^{**}$ such that $S^{**}t = \Lambda_V v$. We compute

$$\Lambda_V S {\Lambda_U}^{-1} t = S^{**}\Lambda_U {\Lambda_U}^{-1} t = S^{**}t = \Lambda_V v$$

Since Λ_V is injective, we have $v = S({\Lambda_U}^{-1}t) \in \mathrm{Range}(S)$.

Afterthoughts of teaching linear algebra

You may have noticed that, my solution is very abstract. That is because I want to show you as more as possible techniques in linear algebra.

The main feature of linear algebra is its abstractness. But to be honest, linear algebra is actually the simplest abstractness. It is possible to teach linear algebra in a more concrete way, by focusing on linear equations and matrix computations. But this approach is somehow not basis-free, meaning that we always consider the space $\mathbf{R}^n$ with the canonical basis. A linear space may has many different basis, so mathematically this version is not so linear algebra. Maybe a better name is “Matrix theory with applications”.

Concrete computation has its benefit: it may be easier to accept. That is the reason why in today universities, linear algebra is still taught in a concrete style. Since you will learn linear algebra in that form in university, I want to offer you the chance to learn about linear spaces, linear maps. And focus on how to use definitions, theorems. I hope that you think our approach is elegant.

Our course also highlights some methods from something called homological algebra. The prove of 5-lemma, is an elementary technique in homological algebra, called the diagram chasing. Homological algebra is sometimes called “abstract nonsense”, and homological algebra is a direct generalization of linear algebra in a sense.