Extra Problems

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Week 1 Suppose $n = p_1^{a_1} p_2^{a_2} \cdot \dots \cdot p_k^{a_k}$ is the unique prime factorization for n, find formulae for $d(n)$ and $\varphi(n)$. Where $d(n)$ is the divisor-counting function and $\varphi(n)$ is the Euler-totient function.

Answers 1

- $d(n) = \prod_{i=1}^{k} (a_i + 1)$
- $\frac{\varphi(n)}{n} = \prod_{i=1}^{k} \left(1 - \frac{1}{p_i}\right) = \prod_{p|n} \left(1 - \frac{1}{p}\right)$

Remark 1 Here we need the **inclusion-exclusion principle**:

$$\left|\bigcup_{i=1}^{n} A_i\right| = \sum_{k=1}^{n} (-1)^{k+1} \left(\sum_{1 \le i_1 < \cdots < i_k \le n} \left| \bigcap_{m=1}^{k} A_{i_m} \right| \right)$$

Remark 2 If n, m are co-prime, then $d(nm) = d(n)d(m), \varphi(nm) = \varphi(n)\varphi(m)$.

Remark 3 Suppose p is a prime number, then $d(p^k) = k + 1$ and $\varphi(p^k) = p^k - p^{k-1}$.

Fact 1 We have $\sum_{i=0}^{k} \varphi(p^i) = p^k$.

Week 2 Prove that $\sum_{d|n} \varphi(d) = n$

Solution 2

- Method 1: Suppose $n = p_1^{a_1} p_2^{a_2} \cdot \dots \cdot p_k^{a_k}$, we know that $d|n$ if and only if $d = p_1^{b_1} p_2^{b_2} \cdot \dots \cdot p_k^{b_k}$ where $0 \le b_i \le a_i$ for all $i = 1, \dots, k$.
  We now compute:

$$\begin{aligned}
\sum_{d|n} \varphi(d) &= \sum_{0 \le b_1 \le a_1} \cdots \sum_{0 \le b_k \le a_k} \varphi(p_1^{b_1} p_2^{b_2} \cdot \dots \cdot p_k^{b_k}) \\
&\overset{\text{remark 2}}{=} \sum_{0 \le b_1 \le a_1} \cdots \sum_{0 \le b_k \le a_k} \prod_{i=1}^{k} \varphi(p_i^{b_i}) \\
&= \prod_{i=1}^{k} \sum_{t=0}^{a_i} \varphi(p_i^t) \overset{\text{Fact 1}}{=} \prod_{i=1}^{k} p^{a_i} = n
\end{aligned}$$

- Method 2: Define $\xi_n = \cos\frac{2\pi}{n} + i\sin\frac{2\pi}{n}$, let $\mu_n = \{\xi_n^m | m \in \mathbf{Z}\}$. Now for any element $x = \xi_n^k \in \mu_n$, we define its **order** by

  $$\mathrm{ord}(x) = \min\{m \in \mathbf{N}^* | x^m = 1\}$$

  Then we have $\mu_n^{[d]} = \{x \in \mu_n | \mathrm{ord}(x) = d\}$ has exactly $\varphi(d)$ elements. Combined with the fact that the set μ_n is the non-intersected union of $\mu_n^{[d]}$ for all $d|n$, this gives us $\sum_{d|n}\varphi(d) = n$.

Remark 4 The definition of $\mu_n^{[d]}$ is irrelevant to n. Actually we have

$$\mu_n^{[d]} = \mu^{[d]} = \left\{ \cos\frac{2\pi}{m} + i\sin\frac{2\pi}{m} \middle| m, d \text{ are co-prime} \right\}$$

Example 1 $\mu^{[1]} = \{1\}, \mu^{[2]} = \{-1\}, \mu^{[3]} = \{\frac{-1+i\sqrt{3}}{2}, \frac{-1-i\sqrt{3}}{2}\}, \mu^{[4]} = \{i, -i\}$.

Week 3 Suppose $\{u_n\}$ is a sequence, such that $u_{n+2} = au_{n+1} + bu_n$.

Let α_1, α_2 be the roots of $X^2 - aX - b = 0$. Show that

- If $\alpha_1 \neq \alpha_2$, then $u_n = C_1\alpha_1^n + C_2\alpha_2^n$
- If $\alpha_1 = \alpha_2 = \alpha \neq 0$, then $u_n = C_1\alpha^n + C_2 n\alpha^n$

Week 4 We define the divisor function $\sigma(n)$ to be the summation of all positive divisors of n. That is,

$$\sigma(n) = \sum_{d|n} d$$

Suppose $n = p_1^{a_1} \cdot \cdots \cdot p_k^{a_k}$ is the unique prime factorization for n.

- Prove that $\sigma(n) = \prod_{i=1}^{k}(1 + p_i + \cdots + p_i^{a_i})$
- Prove that $L_i = 1 + p_i + \cdots + p_i^{a_i} \leq 2p_i^{a_i} - 1$
- Prove that $L_1 + L_2 + \cdots + L_k \leq 2n - k$, conclude that

  $$\frac{(2n-k)!}{\sigma(n)} = \frac{(2n-k)!}{L_1 L_2 \cdot \cdots \cdot L_k} \in \mathbf{Z}$$

Week 5 The Möbius function $\mu : \mathbf{N}^* \to \mathbf{Z}$ satisfies the following for any $n \in \mathbf{N}^*$

$$\sum_{d|n}\mu(d) = \begin{cases} 1, & n = 1 \\ 0, & n > 1 \end{cases}$$

For example, $\mu(1) = 1, \mu(2) = 0 - \mu(1) = -1, \mu(3) = 0 - \mu(1) = -1$.

Find a formula for $\mu(n)$, and prove that

$$\sum_{d|n}\frac{\mu(d)}{d} = \frac{\varphi(n)}{n}$$

Week 6 Two Dynamic Systems

- Suppose $z_{n+2} = Az_{n+1} + Bz_n$, here $B \neq 0$. Find all ordered triple (z_1, z_2, N) such that $z_{n+N} = z_n$ for all n.
- Define $\mathbf{C}_\infty = \mathbf{C} \cup \{\infty\}$, and let $A, B, D \in \mathbf{C}$ be complex numbers. Suppose

$$z_{n+1} = \begin{cases} \infty, & z_n \in \mathbf{C}, z_n + D = 0 \\ \frac{Az_n+B}{z_n+D}, & z_n \in \mathbf{C}, z_n + D \neq 0 \\ A, & z = \infty \end{cases}$$

  Find all ordered pair (z_1, N) such that $z_{n+N} = z_n$ for all n.

Solution We have $z_{n+2} + \beta_i z_{n+1} = (A + \beta_i)(z_{n+1} + \beta_i z_n)$ where β_1, β_2 is the roots of the equation $(A + X)X = B$. If $\beta_1 \neq \beta_2$, then $A + \beta_i$ must be N-th root of unity. From now on we only consider the case $\beta_1 = \beta_2$.

In this case, we have $z_n = C_1\alpha^n + C_2 n\alpha^n$