Problem 1 Suppose W and V are vector spaces, and $W \subset V$ is a subset of V. Prove that the inclusion map

$$\iota_W^V : W \to V = (w \mapsto w)$$

is a linear map if and only if W is a subspace of V.

Problem 2 Suppose $W \leq V$ is a subspace, define the quotient set

$$V/W = \{v + W | v \in V\}$$

where

$$v + W = \{v + w | w \in W\}$$

define the quotient map

$$\pi_W^V : V \to V/W = (v \mapsto v + W)$$

Prove that there exists precisely one vector space structure on V/W such that π_W^V is a linear map.

Problem 3 Suppose $T \in \mathcal{L}(V, W)$. Prove that there exists precisely one linear map

$$\tilde{T} : V/\mathrm{Null}(T) \to \mathrm{Range}(T)$$

such that

$$V \xrightarrow{\pi_{\mathrm{Null}(T)}^V} V/\mathrm{Null}(T) \xrightarrow{\tilde{T}} \mathrm{Range}(T) \xrightarrow{\iota_{\mathrm{Range}(T)}^W} W$$

equals to T.

Problem 4 Recall that a sequence

$$\cdots \to V_{n-1} \xrightarrow{T_{n-1}} V_n \xrightarrow{T_n} V_{n+1} \to \cdots$$

is exact if

$$\mathrm{Range}(T_{n-1}) = \mathrm{Null}(T_n), \text{ for all } n$$

Prove that

$$0 \to \mathrm{Null}(T) \xrightarrow{\iota_{\mathrm{Null}(T)}^V} V \xrightarrow{T} W \xrightarrow{\pi_{\mathrm{Range}(T)}^W} W/\mathrm{Range}(T) \to 0$$

is exact.

Problem 5 Suppose $V_1, V_2 \leq V$ are subspaces.

Construct linear maps such that

$$0 \to V_1 \cap V_2 \xrightarrow{?} V_1 \xrightarrow{?} (V_1 + V_2)/V_2 \to 0$$

is exact.

Problem 6 Suppose $V_0 \leq V_1 \leq V_2 \leq V$ are subspaces.

Construct linear maps such that

$$0 \to V_1/V_0 \xrightarrow{?} V_2/V_0 \xrightarrow{?} V_2/V_1 \to 0$$

is exact.