A function $v : \mathbf{Q} \to \mathbf{R}_{\geq 0}$ is called a **size function of the rationals**, if:

- $v(0) = 0$
- if $v(x) = 0$, then $x = 0$
- $\forall a, b \in \mathbf{Q}, v(ab) = v(a)v(b)$
- $\forall a, b \in \mathbf{Q}, v(a + b) \leq v(a) + v(b)$

Example:

1. Define $v_0(x) = 0$ if $x = 0$ and $v_0(x) = 1$ if $x \neq 0$. Then v_0 is a size function.
2. Define $v_\infty(x) = |x|$. Then v_∞ is a size function.

We say a size function of the rationals v is **Archimedean**, if for every $x \in \mathbf{Q}, x \neq 0$, there exists an integer $n \in \mathbf{Z}$ such that $v(nx) > 1$.

Example:

1. v_∞ is Archimedean
2. v_0 is not Archimedean

Problem

1. Find all size functions of the rationals. (Hint: You need to know the concept of prime numbers)
2. Find all Archimedean size functions of the rationals.

Hint

Notice that $v(1) = v(1 \times 1) = v(1) \times v(1)$, so $v(1) = 1$, and from this we know that $v(-1) \times v(-1) = v(1) = 1$ so $v(-1) = 1$. Denote $\mathcal{P} = \{2, 3, 5, 7, 11, \dots\}$, since $v(p)v(\frac{1}{p}) = v(1) = 1$, we know that $v(\frac{1}{p}) = \frac{1}{v(p)}$.

Since every $x \in \mathbf{Q}$ can be written as $x = \prod_i x_i$ where $x_i \in \mathcal{P}$ or $\frac{1}{x_i} \in \mathcal{P}$, so v is completely determined by its values on $\mathcal{P}$.

Solution

This is the celebrated **Ostrowski's Theorem** (Alexander Ostrowski, 1916).

Two size functions v and v' are defined to be equivalent if there exists a real number $c > 0$ such that

$$\forall x \in \mathbf{Q}, v'(x) = (v(x))^c$$

For a prime number p, we define

$$v_p(x) = \begin{cases} 0 & x = 0 \\ p^{-n} & x \neq 0 \end{cases}, \text{ where } x = p^n \frac{a}{b} \text{ with } p \nmid a, p \nmid b, n \in \mathbf{Z}$$

The n above is uniquely determined by $x \in \mathbf{Q}$ so this is well-defined.

We now claim that:

If v is an Archimedean size function then v is equivalent to v_∞. If v is a non-Archimedean size function with $v \neq v_0$, then there exists a prime number p such that v is equivalent to v_p. And if we define $\mathrm{Pl}_{\mathbf{Q}} = \{v_\infty, v_0, v_2, v_3, v_5, v_7, v_{11} \dots\}$, then $\forall v, v' \in \mathrm{Pl}_{\mathbf{Q}}$ with $v \neq v'$ we have: v is not equivalent to v'.

Example

- $v_2(1) = 1, v_2(4) = \frac{1}{4}, v_2(6) = \frac{1}{2}, v_2(\frac{1}{3}) = 1, v_2(\frac{1}{15}) = 1, \dots$
- $v_3(1) = 1, v_3(4) = 1, v_3(6) = \frac{1}{2}, v_3(\frac{1}{3}) = 3, v_3(\frac{1}{15}) = 3, \dots$
- $v_5(1) = 1, v_5(4) = 1, v_5(6) = 1, v_5(\frac{1}{3}) = 1, v_5(\frac{1}{15}) = 5, \dots$

Some simple facts:

- If $x \in \mathbf{Q}$ with $x \neq 0$, then $v(\frac{1}{x}) = \frac{1}{v(x)}$
- If $y, x \in \mathbf{Q}$ with $x \neq 0$, then $v(\frac{y}{x}) = \frac{v(y)}{v(x)}$
- If $x \in \mathbf{Q}$, then $v(-x) = v(-1)v(x) = v(x)$
- If $n \in \mathbf{N}$, then $v(n) \leq n$

Case 1: $\exists n \in \mathbf{N}, v(n) > 1$ Suppose $v(b) > 1$ where $b \in \mathbf{N}$, we know that $b \geq 2$. Let $a, n \in \mathbf{N}_+$ where $a > 1$, express b^n in base a we can write

$$b^n = \sum_{i<m} c_i a^i$$

where $c_i \in \{0, 1, \dots, a-1\}$, and we can estimate that $m \leq n \log_a b + 1$

Notice that $v(b^n) = v(b)^n$ and

$$v\left(\sum_{i<m} c_i a^i\right) \leq \sum_{i<m} v(c_i a^i) \leq \sum_{i<m} c_i v(a)^i \leq m \cdot a \cdot \max\{v(a)^{m-1}, 1\}$$

So we have,

$$\forall n \in \mathbf{N}_+, v(b) \leq (a(n \log_a b + 1))^{\frac{1}{n}} \max\{1, v(a)^{\log_a b}\}$$

We need the following lemma:

Lemma If $\alpha \in \mathbf{R}$ such that $\alpha \leq (a(n \log_a b + 1))^{\frac{1}{n}}$ for all $n \in \mathbf{N}_+$, then we have $\alpha \leq 1$.

From this we know that $\max\{1, v(a)^{\log_a b}\} \geq v(b) > 1$, so $v(a) > 1$ for all $a \in \mathbf{N}_{\geq 2}$.

Here comes the smart part: since now we have $\forall b \in \mathbf{N}_{\geq 2}, v(b) > 1$ and $\forall a \in \mathbf{N}_{\geq 2}$ we have $v(a)^{\log_a b} = \max\{1, v(a)^{\log_a b}\} \geq v(b)$. We can rewrite this as

$$\forall a, b \in \mathbf{N}_{\geq 2}, \log_b v(b) \leq \log_a v(a)$$

So we must have

$$\forall a, b \in \mathbf{N}_{\geq 2}, \log_b v(b) = \log_a v(a) = \lambda \in \mathbf{R}$$

which says that v is equivalent to v_∞

Case 2: $\forall n \in \mathbf{N}, v(n) \leq 1$ Since we've assumed that $v \neq v_0$, there exists $n \in \mathbf{N}$ such that $v(n) < 1$. Write $\mathcal{P}_n = \{p \text{ prime with } p|n\}$, then there exists at least one $p \in \mathcal{P}$ such that $v(p) < 1$. That is,

$$\mathrm{Card}(\{p \text{ prime with } v(p) < 1\}) \geq 1$$

We now claim that

$$\mathrm{Card}(\{p \text{ prime with } v(p) < 1\}) = 1$$

Suppose p, q are distinct primes with $v(p), v(q) < 1$. We can choose $e \in \mathbf{N}$ so large such that $v(p^e) < \frac{1}{2}$ as well as $v(q^e) < \frac{1}{2}$. According to the Bezout's Theorem, there exists $k, l \in \mathbf{Z}$ such that $kp^e + lq^e = 1$. But this is impossible since $v(kp^e + lq^e) \leq v(k)v(p^e) + v(l)v(q^e) < \frac{v(k)+v(l)}{2} \leq 1$.

So we know that there exists exactly one prime p with $v(p) < 1$ and we have $v(q) = 1$ for other primes. Which means that v is equivalent to v_p.

Now we've finished the classification. And all Archimedean size functions are of the Case 1.

Bibliography

[Wikipedia] Ostrowski's theorem, `https://en.wikipedia.org/wiki/Ostrowski\%27s_theorem`

[Ostrowski 1916] Über einige Lösungen der Funktionalgleichung $\phi(x)\phi(y) = \phi(xy)$. Acta Mathematica. 41: 271-284