High School Analysis

Qiū Cáiyóng

This course is taught by the author in Shenzhen Middle School, 2022.

Contents

Chapter 1

Rudiments

1.1 Set Theory

To be written.

1.2 Basic Set-theoretic Constructions

Definition 1.2.1 (Cartesian product of sets)**.** Suppose A, B are sets, we define their Cartesian product $A \times B$ by

$$A \times B = \{(a, b) : a \in A, b \in B\}$$

By definition, we know that if $A_0 \subset A, B_0 \subset B$ are subsets, then $A_0 \times B_0$ is a subset of $A \times B$. Also, $A \times B \neq B \times A$ unless $A = B$.

Definition 1.2.2 (Mapping)**.** A mapping is an ordered pair $f = (S_f, \Gamma_f)$, where S_f is an ordered pair $S_f = (D_f, C_f)$ and Γ_f is a subset of $D_f \times C_f$, such that for every $x \in D_f$, there exists uniquely a $y \in C_f$ with $(x, y) \in \Gamma_f$.

Instead of $f = ((D_f, C_f), \Gamma_f)$, we say that the domain of f is D_f and the codomain of f is C_f. We also call Γ_f the graph of f. Instead of $(x, y) \in \Gamma_f$, we say that $f(x) = y$ or $fx = y$.

Example 1.2.3. Suppose $T \subset S$ is a subset, we define the embedding mapping ι_T^S by $\iota_T^S = ((T, S), \{(t, t) : t \in T\})$.

If $T = S$, we also write 1_S instead of ι_T^S.

Example 1.2.4. Suppose S is a set, then $((\varnothing, S), \varnothing)$ is a mapping.

Proposition 1.2.5 (Composition of mappings)**.** Suppose we have two mappings $f = ((D_f, C_f), \Gamma_f)$ and $g = ((D_g, C_g), \Gamma_g)$ with $C_f = D_g$. We define their composition $g \circ f$ to be $((D_f, C_g), \Gamma_{g \circ f})$ where

$$\Gamma_{g \circ f} = \{(x, z) : \exists y \in C_f = D_g, (x, y) \in \Gamma_f, (y, z) \in \Gamma_g\}$$

Example 1.2.6. It is easy to examine that $\iota_V^W \circ \iota_U^V = \iota_U^W$.

Note that $(f \circ g) \circ h = f \circ (g \circ h)$ always hold (whenever defined).

Definition 1.2.7 (Restriction and extension)**.** Suppose $f = ((D, C), \Gamma)$ is a mapping, and $E \subset D, C \subset B$ are subsets. We define

$$f|_E = f \circ \iota_E^D, f|^B = \iota_C^B \circ f, f|_E^B = \iota_C^B \circ f \circ \iota_E^D$$

Instead of $f = ((D, C), \Gamma)$, we will write $f : D \to C$. This notation is better. For example:

$$f|_E^B = \left(E \xrightarrow{\iota_E^D} D \xrightarrow{f} C \xrightarrow{\iota_C^B} B \right)$$

Definition 1.2.8 (Monomophism)**.** Suppose $f : D \to C$ is a mapping. If there exists a mapping $g : C \to D$ such that $g \circ f = 1_D$, then we say that f is a monomorphism.

Definition 1.2.9 (Epimophism)**.** Suppose $f : D \to C$ is a mapping. If there exists a mapping $g : C \to D$ such that $f \circ g = 1_C$, then we say that f is an epimorphism.

Definition 1.2.10 (Image and inverse image)**.** Suppose $f : D \to C$ is a mapping. If $D_0 \subset D$, we define $f(D_0)$ to be the set

$$\{y \in C : \exists x \in D_0, fx = y\}$$

If $C_0 \subset C$ is a subset, we define $f^{-1}(C_0)$ to be the set

$$\{x \in D : fx \in C_0\}$$

It is possible that $f^{-1}(C_0) = \varnothing$ for some nonempty C_0. We also call $f(D)$ the image of f, it may well be a proper subset of C.

We also call $f^{-1}(\{y\})$ the fibre of y under f, or the fibre of f on y.

Definition 1.2.11 (Injective and surjective mappings)**.** Suppose $f : D \to C$ is a mapping. We say that f is injective if $f^{-1}(\{y\})$ contains at most one element for all $y \in C$. We say that f is surjective if $f(D) = C$.

Definition 1.2.12. A mapping f is called an isomorphism if it is both a monomorphism and an epimorphism. In this case, there exists a mapping g such that $f \circ g = 1, g \circ f = 1$.

Proposition 1.2.13. (Assuming AC,) A mapping is a monomorphism if and only if it is injective. A mapping is an epimorphism if and only if it is surjective.

Proposition 1.2.14 (Structure of mappings)**.** Suppose $f : D \to C$ is a mapping, there exists uniquely a mapping $f_0 : D \to f(D)$ such that f_0 is surjective, and $f = \iota_{f(D)}^D \circ f_0$.

1.3 Relations and Ordering

Definition 1.3.1 (Relation)**.** Let S be a set, a subset of $S \times S$ is called a relation on S.

Example 1.3.2. $>= \{(n, m) \in \mathbf{N} \times \mathbf{N} : n > m\}$ is a relation on $\mathbf{N}$.

Example 1.3.3. $\mathrm{div} = \{(n, m) \in \mathbf{N} \times \mathbf{N} : n|m\}$ is a relation on $\mathbf{N}$.

Definition 1.3.4 (Diagonal)**.** Suppose S is a set, we define its diagonal Δ_S by $\{(s, s) : s \in S\}$. It is a relation on S. We also call Δ_S the equal relation on S.

Definition 1.3.5 (Ordered set)**.** An ordered set is an ordered pair $P = (S_P, <_P)$, where $<_P$ is a relation on S_P such that:

- for every $x, y \in S_P$, one and only one of the following is true:

$$(x, y) \in <_P, x = y, (y, x) \in <_P$$

- $(x, y) \in <_P, (y, z) \in <_P$ implies that $(x, z) \in <_P$.

Example 1.3.6. $(\mathbf{N}, >)$ is an ordered set. $(\mathbf{N}, \mathrm{div})$ is an ordered set.

Every subset of an ordered set is also an ordered set in a canonical way:

Proposition 1.3.7. Suppose $(S, <)$ is an ordered set, and $T \subset S$ a subset. Then $(T, < \cap (T \times T))$ is also an ordered set.

Proof. I only point out that $<$ and $T \times T$ are both subsets of $S \times S$. $\square$

Proposition 1.3.8. (Assuming AC) For every set S, there exists $< \subset S \times S$ such that $(S, <)$ is an ordered set.

1.4 Fields

Definition 1.4.1 (Operation on a set)**.** Suppose S is a set, a mapping $* : S \times S \to S$ is called an operation on S.

The notation $*(x, y)$ or $*xy$ is called the Polish notation. We usually use the infix notation $x * y$. The Polish notation is somehow better:

$$(x * y) * z = * * xyz, x * (y * z) = *x * yz$$

Definition 1.4.2 (Field)**.** A field $\mathbf{F}$ is an ordered pair $\mathbf{F} = (S_{\mathbf{F}}, \mathrm{op}_{\mathbf{F}})$, where $\mathrm{op}_{\mathbf{F}}$ is an ordered pair $(+_{\mathbf{F}}, \times_{\mathbf{F}})$, such that:

- $+_{\mathbf{F}}, \times_{\mathbf{F}}$ are operations on $S_{\mathbf{F}}$
- for all $x, y, z \in S_{\mathbf{F}}$, we have

$$(x +_{\mathbf{F}} y) +_{\mathbf{F}} z = x +_{\mathbf{F}} (y +_{\mathbf{F}} z), (x \times_{\mathbf{F}} y) \times_{\mathbf{F}} z = x \times_{\mathbf{F}} (y \times_{\mathbf{F}} z)$$

- for all $x, y \in S_{\mathbf{F}}$, we have

$$x +_{\mathbf{F}} y = y +_{\mathbf{F}} x, x \times_{\mathbf{F}} y = y \times_{\mathbf{F}} x$$

- there exists an element $0_{\mathbf{F}} \in S_{\mathbf{F}}$ such that $0_{\mathbf{F}} +_{\mathbf{F}} x = x$ for all $x \in S_{\mathbf{F}}$.
- there exists an element $1_{\mathbf{F}} \in S_{\mathbf{F}}$ such that $1_{\mathbf{F}} \neq 0_{\mathbf{F}}$ and $1_{\mathbf{F}} \times_{\mathbf{F}} x$ for all $x \in S_{\mathbf{F}}$.
- for any $x \in S_{\mathbf{F}}$, there exists an element $y \in S_{\mathbf{F}}$ such that $x +_{\mathbf{F}} y = 0_{\mathbf{F}}$
- for any $x \in S_{\mathbf{F}}$ with $x \neq 0_{\mathbf{F}}$, there exists an element $y \in S_{\mathbf{F}}$ such that $x \times_{\mathbf{F}} y = 1_{\mathbf{F}}$
- for all $x, y, z \in S_{\mathbf{F}}$, we have $x \times_{\mathbf{F}} (y +_{\mathbf{F}} z) = (x \times_{\mathbf{F}} y) +_{\mathbf{F}} (x \times_{\mathbf{F}} z)$

We call $S_{\mathbf{F}}$ the underlying set of $\mathbf{F}$.

Example 1.4.3. The set $\mathbf{Q}$ with the usual addition and usual multiplication is a field.

Example 1.4.4. The field $\mathbf{F}_2$ consists of two elements $\{0, 1\}$, in which we define $1 + 1 = 0$.

Definition 1.4.5. Suppose we have two fields $\mathbf{F} = (S_{\mathbf{F}}, (+_{\mathbf{F}}, \times_{\mathbf{F}}))$ and $\mathbf{E} = (S_{\mathbf{E}}, (+_{\mathbf{E}}, \times_{\mathbf{E}}))$. If $S_{\mathbf{F}}$ is a subset $S_{\mathbf{E}}$, and

$$+_{\mathbf{F}} = +_{\mathbf{E}}|_{S_{\mathbf{F}} \times S_{\mathbf{F}}}, \times_{\mathbf{F}} = \times_{\mathbf{E}}|_{S_{\mathbf{F}} \times S_{\mathbf{F}}}$$

then we say that $\mathbf{F}$ is a subfield of $\mathbf{E}$, and $\mathbf{E}$ is an field extension of $\mathbf{F}$.

Example 1.4.6. $\mathbf{F}_2$ is **not** a subfield of $\mathbf{Q}$.

1.5 Exercise

Exercise 1.5.1. In a field, the additive and multiplicative unit and inverse are unique in some sence. State this property in a precise way and proof it.

Exercise 1.5.2. Suppose $\mathbf{F}$ is a field. Prove that exact one of the following is true:

- if $x \neq 0$, then $\underbrace{x + \cdots + x}_{n} \neq 0$ for all $n \in \mathbf{N}_+$
- there exists (uniquely) a prime number p such that $\underbrace{x + \cdots + x}_{p} = 0$ for all $x \in \mathbf{F}$.

1.6 General Theory on Cartesian Products

You might have noticed that, although we've defined the ordered pair, we didn't define the ordered triple. In our definition of fields, we defined a field as an ordered pair consists of ordered pair.

This approach is not wrong, but feels cliché and redundant.

Chapter 2

Numbers

2.1 Ordered Fields

Definition 2.1.1 (Ordered field).

2.2 Dedekind completeness

2.3 Archimedean properties

2.4 n-th root of a positive real number

Recall that any nonempty subset of $\mathbf{R}$ bounded above has a unique least upper bound, any nonempty subset of $\mathbf{R}$ bounded below has a unique greatest lower bound.

Lemma 2.4.1. Let n be a natural number strictly greater than 1. a, L be positive real numbers with $a^n < L$. Then there exists a positive real number b such that $b > a, b^n < L$.

Proof. □

Lemma 2.4.2. Let n be a natural number strictly greater than 1. a, L be positive real numbers with $a^n > L$. Then there exists a positive real number b such that $b < a, b^n > L$.

Proof. □

Theorem 2.4.3. Let $\beta > 1$ be a real number, n be a natural number strictly greater than 1. Then there exists a positive real number γ such that $\gamma^n = \beta$.

Proof. Let $E = \{r > 0 : r^n < \beta\}$, we claim that E is an nonempty set and bounded above. Actually $\min\{1, \beta\}$ is an element of E and $\max\{1, \beta\}$ is an upper bound of E. Set $\gamma = \sup E$, we prove that $\gamma^n = \beta$.

Suppose $\gamma^n < \beta$, by Lemma 2.4.1 there exists $\gamma' > \gamma$ such that $\gamma' \in E$. So γ cannot be an upper bound of E, a contradiction. Suppose $\gamma^n > \beta$, by Lemma 2.4.2 there exists $\gamma' < \gamma$ such that γ' is an upper bound of E. So γ cannot the the least upper bound of E, a contraduction. $\square$

We denote this number by $\gamma = \sqrt[n]{\beta}$.

Chapter 3

Cardinality

Recall that a mapping $f : X \to Y$ is said to be

- injective, if given any $y \in Y$, there is at most one $x \in X$ such that $fx = y$
- surjective, if given any $y \in Y$, there is at least one $x \in X$ such that $fx = y$
- bijective, if f is injective and surjective

If $f : X \to Y$ is bijective, then there exists uniquely a mapping $g : Y \to X$ such that $fx = y$ if and only if $gy = x$. The mapping g is called the inverse of f. Inverse is defined only when the mapping is bijective.

3.1 Cardinality

Definition 3.1.1. We say that two sets X and Y are of the same cardinality, if there exists a bijective mapping f from X to Y. Denoted by $|X| = |Y|$.

Recall that a natrual number n is a set of n elements by definition.

Definition 3.1.2. We say a set X is

- finite, if $|X| = |n|$ for some $n \in \mathbf{N}$
- countable, if $|X| = |\mathbf{N}|$
- uncountable, if X is neither finite nor countable

We say X is at most countable if it is finite or countable. We say X is infinite if it is not finite.

Example 3.1.3. There exists a bijection from $\mathbf{N}$ to $\mathbf{Z}$. So $\mathbf{Z}$ is countable.

Theorem 3.1.4. A set X is infinite if and only if there exists a proper subset $X_0 \subset X$ such that $|X| = |X_0|$.

Theorem 3.1.5. Every subset of a countable set is at most countable.

Theorem 3.1.6. Suppose A is at most countable, and for every $\alpha \in A$, X_α is at most countable. Then $\bigcup_{\alpha \in A} X_\alpha$ is at most countable.

Example 3.1.7. The set of rationals $\mathbf{Q}$ is countable.

Example 3.1.8. The set of algebraic numbers $\mathbf{A}$ is countable.

If X is a set, we define its power set 2^X to be the set of all subsets of X. That is,

$$2^X = \{X_0 : X_0 \subset X\}$$

Theorem 3.1.9 (Cantor)**.** $|X| \neq |2^X|$

Proof. Suppose there exists a bijective mapping $f : X \to 2^X$. Define

$$Y = \{x \in X : x \notin fx\}$$

Suppose $y \in X$ such that $fy = Y$. Then $y \in Y \iff y \notin Y$. $\square$

Chapter 4

Topology

4.1 Metric Spaces

Definition 4.1.1 (Metric)**.** Suppose X is a set, a function $d : X \times X \to \mathbf{R}$ is called a metric on X, if

- $d(x, x) = 0$ for all $x \in X$
- $d(x, y) > 0$ if $x \neq y$
- $d(x, y) = d(y, x)$ for all $x, y \in X$
- $d(x, z) \leq d(x, y) + d(y, z)$ for all $x, y, z \in X$

Definition 4.1.2 (Metric Space)**.** A metric space is an ordered pair (X, d) such that d is a metric on X.

Example 4.1.3. Let $p \geq 1$ be a real number. Suppose $\mathbf{x} = (x_1, \ldots, x_n), \mathbf{y} = (y_1, \ldots, y_n) \in \mathbf{R}^n$ are two points, we define

$$d_p(\mathbf{x}, \mathbf{y}) = \left(\sum_{i=1}^{n} |x_i - y_i|^p \right)^{\frac{1}{p}}$$

Then d_p is a metric on $\mathbf{R}^n$.

$(\mathbf{R}^n, d_2)$ is called the n-dimensional Euclidean space. When we use $\mathbf{R}^n$ as a metric space, we always mean $(\mathbf{R}^n, d_2)$. When we use $\mathbf{C}$ as a metric space, we always mean $d(z_1, z_2) = |z_1 - z_2|$.

Example 4.1.4 (Discrete metric)**.** Let X be a set, we define

$$d(x, y) = \begin{cases} 0, & x = y \\ 1, & x \neq y \end{cases}$$

Then d is a metric on X, called the discrete metric on X.

Definition 4.1.5. Suppose (X, d) and (X', d') are metric spaces. If X' is a subset of X, and

$$d'(x, y) = d(x, y)$$

for all $x, y \in X'$. Then we say that (X', d') is a subspace of (X, d).

Theorem 4.1.6 (Every subset of a metric space admits a canonical metric)**.** Suppose (X, d) is a metric space, and X' is a subset of X. Then there exists uniquely a metric d' on X' such that (X', d') is a subspace of (X, d).

Example 4.1.7. Suppose (X, d) is a metric space. Define

$$\delta(x, y) = \frac{d(x, y)}{1 + d(x, y)}$$

Then (X, δ) is also a metric space.

Definition 4.1.8 (Topological pair)**.** A topological pair is an ordered triple (E, X, d) where (X, d) is a metric space and E is a subset of X.

Definition 4.1.9 (Neighborhood)**.** Suppose (X, d) is a metric space, $r \in \mathbf{R}$, $x \in X$. We define

$$\begin{aligned}
\mathbf{B}_r^{(X,d)}(x) &= \{y \in X : d(x, y) < r\} \\
\bar{\mathbf{B}}_r^{(X,d)}(x) &= \{y \in X : d(x, y) \leq r\} \\
\mathring{\mathbf{B}}_r^{(X,d)}(x) &= \{y \in X : 0 < d(x, y) < r\}
\end{aligned}$$

4.2 Topology

Definition 4.2.1 (Limit point and derived set)**.** Suppose (E, X, d) is a topological pair, a point $p \in X$ is called a limit point of (E, X, d) if for all $r > 0$, such that $\mathring{\mathbf{B}}_r^{(X,d)}(p) \cap E \neq \varnothing$. The set of all limit points of (E, X, d) is called the derived set of (E, X, d) and denoted by $\mathsf{D}(E, X, d)$.

Definition 4.2.2 (Interior point and interior)**.** Suppose (E, X, d) is a topological pair, a point $p \in X$ is called an interior point of (E, X, d) if there exists $r > 0$, such that $\mathbf{B}_r^{(X,d)}(x) \subseteq E$. The set of all interior points of (E, X, d) is called the interior of (E, X, d) and denoted by $\mathsf{In}(E, X, d)$.

Definition 4.2.3 (Open and closed subsets)**.** Suppose (X, d) is a metric space and E a subset of X. We say that

- E is open in (X, d), if $E \subseteq \mathsf{In}(E, X, d)$
- E is closed in (X, d), if $\mathsf{D}(E, X, d) \subseteq E$
- E is perfect in (X, d), if $\mathsf{D}(E, X, d) = E$

Theorem 4.2.4 (Open neighborhood and closed neighborhood)**.** Suppose (X, d) is a metric space and $p \in X$ is a point. Then $\mathbf{B}_r^{(X,d)}(x)$ is open in (X, d) for all $r \in \mathbf{R}$ and $\bar{\mathbf{B}}_r^{(X,d)}(x)$ is closed in (X, d) for all $r \in \mathbf{R}$.

Theorem 4.2.5. Suppose (E, X, d) is a topological pair, and $p \in \mathsf{D}(E, X, d)$. Then every neighborhood of p with positive radius contains infinitely many points of E.

Proof. $\mathring{\mathbf{B}}_{1/n}^{(X,d)}(p) \cap E$ is nonempty for all $n \in \mathbf{N}_+$. □

Theorem 4.2.6. Suppose (E, X, d) is a topological pair. Then E is open in (X, d) if and only if $\complement_X E$ is closed in (X, d). Hence E is closed in (X, d) if and only if $\complement_X E$ is open in (X, d).

Theorem 4.2.7. Suppose (X, d) is a topological space, and X_α is open in (X, d) for all α. Then $\bigcup_\alpha X_\alpha$ is open in (X, d).

Theorem 4.2.8. Suppose (X, d) is a topological space, and X_α is closed in (X, d) for all α. Then $\bigcap_\alpha X_\alpha$ is closed in (X, d).

Theorem 4.2.9. Suppose (X, d) is a topological space, and X_i is open in (X, d) for all i. Then $\bigcap_{i=1}^n X_i$ is open in (X, d).

Theorem 4.2.10. Suppose (X, d) is a topological space, and X_i is closed in (X, d) for all i. Then $\bigcup_{i=1}^n X_i$ is closed in (X, d).

Definition 4.2.11. Suppose (E, X, d) is a topological pair. We define the closure of (E, X, d) to be

$$\mathsf{Cl}(E, X, d) = \mathsf{D}(E, X, d) \cup E$$

If $\mathsf{Cl}(E, X, d) = X$, we say that E is dense in X.

Theorem 4.2.12. Suppose (E, X, d) is a topological pair. Then $\mathsf{Cl}(E, X, d)$ and $\mathsf{D}(E, X, d)$ is closed in (X, d).

Proof. □

Theorem 4.2.13. Suppose (E, X, d) is a topological pair. Then $E = \mathsf{Cl}(E, X, d)$ if and only if E is closed in (X, d).

Theorem 4.2.14. Suppose (E, X, d) is a topological pair. Then $\mathsf{In}(E, X, d)$ is open in (X, d). And $E = \mathsf{In}(E, X, d)$ if and only if E is open in (X, d).

Theorem 4.2.15. Suppose (X, d) is a metric space, $E \subseteq F$ are subsets of X. If F is closed in (X, d) then $\mathsf{Cl}(E, X, d) \subseteq F$.

Theorem 4.2.16. Suppose (X, d) is a metric space, $E \supseteq G$ are subsets of X. If G is open in (X, d) then $\mathsf{In}(E, X, d) \supseteq G$.

Theorem 4.2.17. Suppose (E, X, d) is a topological pair. Then

$$\complement_X \mathsf{In}(E, X, d) = \mathsf{Cl}(\complement_X E, X, d)$$

Theorem 4.2.18. Suppose (E, X, d) is a topological pair. Then

$$\complement_X \mathsf{Cl}(E, X, d) = \mathsf{In}(\complement_X E, X, d)$$

4.3 More on Topology

Definition 4.3.1 (Topology)**.** Suppose X is a set. A subset $\mathcal{T} \subseteq 2^X$ is called a topology on X, if:

- $\varnothing \in \mathcal{T}, X \in \mathcal{T}$
- if $X_\alpha \in \mathcal{T}$ for all $\alpha \in \Lambda$, then $\bigcup_{\alpha\in\Lambda} X_\alpha \in \mathcal{T}$
- if $X_i \in \mathcal{T}$ for all $i = 1, \dots, n$, then $\bigcap_{i=1}^n X_i \in \mathcal{T}$

Example 4.3.2. Suppose (X, d) is a metric space, then

$$\mathcal{T}_{(X,d)} = \{E : E \text{ is open in } (X, d)\}$$

is a topology on X. It is called the induced topology of (X, d).

Example 4.3.3. The induced topology for $(\mathbf{R}^n, d_p)$ is the same for all $p \geq 1$.

Theorem 4.3.4. Suppose (X', d') is a subspace of (X, d). Then

$$\mathcal{T}_{(X',d')} = \{E \cap X' : E \in \mathcal{T}_{(X,d)}\}$$

In general, we don't have

$$\mathcal{T}_{(X',d')} = \mathcal{T}_{(X,d)} \cap 2^{X'}$$

Corollary 4.3.5. Suppose (X, d) is a metric space and (X', d') is a subspace. A subset $E \subseteq X'$ is open in (X', d') if and only if there exists a subset $G \subseteq X$ such that G is open in (X, d) and $G \cap X' = E$.

Corollary 4.3.6. Suppose (X, d) is a metric space and (X', d') is a subspace. A subset $E \subseteq X'$ is closed in (X', d') if and only if there exists a subset $F \subseteq X$ such that F is closed in (X, d) and $F \cap X' = E$.

Definition 4.3.7 (Open cover)**.** Suppose (X, d) is a metric space, denote its induced topology by $\mathcal{T}_{(X,d)}$. Let E be a subset of X.

A subset $\mathcal{C}$ of $\mathcal{T}_{(X,d)}$ is called a (X, d)-open covering of E, if

$$\bigcup_{G\in\mathcal{C}} G \supseteq E$$

By definition, a (X, d)-open cover is a family of subsets of (X, d), each one is open in (X, d).

Definition 4.3.8 (Open subcover)**.** Suppose (X, d) is a metric space, E is a subset of X, $\mathcal{C}$ is a (X, d)-open cover of E. A finite subset $\mathcal{C}_0$ of $\mathcal{C}$ is called a (X, d)-open subcover of E if $\mathcal{C}_0$ itself is a (X, d)-open cover of E.

The following definition can be generalized to any topological space.

Definition 4.3.9 (Compact subset)**.** Suppose (X, d) is a metric space. A subset $K \subseteq X$ is said to be compact in (X, d), if every (X, d)-open covering of K admits a finite (X, d)-open subcover of K. By a finite subcover, we mean that the subcover is a finite set as a covering.

Surprisingly, the notion of campactness is not relative, it is an absolute property:

Theorem 4.3.10. Suppose (X, d) is a metric space and (X', d') is a subspace. A subset $K \subseteq X'$ is compact in (X, d) if and only if (X', d').

Proof. This is just Theorem 4.3.4. □

We say a metric space (X, d) is compact, if X is compact in (X, d). So if (X, d) is compact and (X, d) is a subspace of (Y, δ), then X is compact in (Y, δ).

Theorem 4.3.11 (Properties of compact spaces)**.** All spaces are assumed to be metric spaces.

1. Suppose (X', d') is a subspace of (X, d), (X', d') is a compact space. Then $\complement_X X'$ is open in (X, d).
2. Suppose (X', d') is a subspace of (X, d), (X, d) is a compact space and X' is closed in (X, d). Then (X', d') is a compact space.
3. Suppose (X, d) is a compact space and K_α is closed in (X, d) for all $\alpha \in \Lambda$. If the intersection of every finite subcollection of $\{K_\alpha\}$ is nonempty, then $\bigcap K_\alpha$ is nonempty.
4. Suppose (X, d) is a metric space and F is closed in (X, d), K is compact in (X, d). Then $F \cap K$ is closed and compact in (X, d).

Proof.

1. Fix a $p \in \complement_X X'$. We construct an open covering of X':
$$X' \subseteq \bigcup_{x \in X'} \mathbf{B}^{(X,d)}_{\frac{d(x,p)}{2}}(x)$$
Since (X', d') is compact, there exists a finite subset $X_0 \subseteq X'$ such that
$$X' \subseteq \bigcup_{x \in X_0} \mathbf{B}^{(X,d)}_{\frac{d(x,p)}{2}}(x)$$
And we have $\mathbf{B}^{(X,d)}_r(p) \subseteq \complement_X X'$ where
$$d = \min_{x \in X_0} \frac{d(x,p)}{2}$$
2. Notice that if $\mathcal{C}$ is an (X, d)-open covering of X', then $\mathcal{C} \cup \{\complement_X X'\}$ is an (X, d)-open covering of X.
3. Notice that $\complement_X K_\alpha$ is an open covering of X if $\bigcap K_\alpha$ is empty.

□

Currently we can only show that a finite space is compact. To increase our example, we need to study some hard theorems.

4.4 Hard Theorems

We want to study topology, not tautology. When we say $\mathbf{R}^n$, we mean the set $\mathbf{R}^n$ together with the 2-metric d_2 on it. (So $\mathbf{R}^n$ is a metric space, as well as every subset of $\mathbf{R}^n$.)

Theorem 4.4.1. If E is a subset of $\mathbf{R}$ which has a least upper bound γ. Then $\gamma \in \mathsf{Cl}(E, \mathbf{R}, d_2)$.

Proof. Recall that $\mathsf{Cl}(E, \mathbf{R}, d_2) = \mathsf{D}(E, \mathbf{R}, d_2) \cup E$. If $\gamma \in E$ then we're done. If $\gamma \notin E$ we will prove that $\gamma \in \mathsf{D}(E, \mathbf{R}, d_2)$. This follows from the definition of least upper bound. $\square$

Theorem 4.4.2. If E is a subset of $\mathbf{R}$ which has a greatest lower bound γ. Then $\gamma \in \mathsf{Cl}(E, \mathbf{R}, d_2)$.

Theorem 4.4.3. Suppose F is a closed subset of $\mathbf{R}$. If F has a least upper bound γ, then $\gamma \in F$. If F has a greatest lower bound β, then $\beta \in F$.

Theorem 4.4.4. Suppose G is an open subset of $\mathbf{R}$. If G has a least upper bound γ, then $\gamma \notin G$. If G has a greatest lower bound β, then $\beta \notin G$.

Theorem 4.4.5. The only clopen subset of $\mathbf{R}$ are $\varnothing$ and $\mathbf{R}$.

Proof. Suppose A is a clopen nonempty proper subset of $\mathbf{R}$. Let $b \notin A$. We define

$$A_* = \{x \in A : x < b\}$$
$$A^* = \{x \in A : x > b\}$$

Since $A = A_* \cup A^*$, at least one of it is nonempty, say A_*. A_* is bounded above so it has a least upper bound α. Notice that $A_* = A \cap (-\infty, b) = A \cap (-\infty, b]$, so A_* is also clopen. We must have $\alpha \notin A_*$ and $\alpha \in A_*$. $\square$

Definition 4.4.6 (Connected space)**.** We say a metric space (X, d) is connected, if its only clopen subsets are $\varnothing$ and X.

Example 4.4.7. Intervals are connected.

Example 4.4.8. $\mathbf{R}^n$ is connected.

Actually, we can describe all open subsets of $\mathbf{R}$:

Theorem 4.4.9. A subset of $\mathbf{R}$ is open if and only if it is a countable union of a disjoint open intervals.

Proof. Suppose G is open, for every $t \in \mathbf{R}$ we define

$$G_t = \bigcup_{t \in (a,b) \subseteq G} (a, b)$$

Then $G = \bigcup_{t \in \mathbf{Q}} G_t$. $\square$

The following result is simple yet powerful:

Theorem 4.4.10. Suppose X, Y are two nonempty subsets of $\mathbf{R}$, such that $x \leq y$ for all $x \in X, y \in Y$. Then X is bounded above and Y is bounded below, and

$$\sup X \leq \inf Y$$

We now study the Heine-Borel-Bolzano-Weierstrass Theorems.

Theorem 4.4.11 (Heine-Borel)**.** A subset K of $\mathbf{R}^n$ is compact if and only if it is closed and bounded.

Proof. Consider the case $n = 1$, we only need to show that $[-a, a]$ is compact.

Suppose $[-a, a]$ is not compact, so there exists $\mathcal{C} = \{G_\alpha | \alpha \in \Lambda\}$, which is a family of open subsets of $\mathbf{R}$ such that no finite subfamily of $\mathcal{C}$ covers $[-a, a]$.

$\rightarrow$ Get a nested closed intervals $[a_n, b_n]$, and $\sup a_n \leq \inf b_n$

$\rightarrow$ Choose $\sup a_n \leq r \leq \inf b_n$ and $r \in [*, *] \subseteq G_{\alpha_0}$ □

The general Heine-Borel is true in any metric spaces, it is irrelevant to Dedekind completeness:

Theorem 4.4.12 (General Heine-Borel)**.** If (K_n, d_n) are compact spaces for all $n \in \mathbf{N}_+$. Suppose (K_{n+1}, d_{n+1}) is a subspace of (K_n, d_n) for all $n \in \mathbf{N}_+$. Then there exists a point x such that $x \in K_n$ for all $n \in \mathbf{N}_+$.

Theorem 4.4.13 (Bolzano-Weierstrass)**.** A subset K of $\mathbf{R}^n$ is compact if and only if $\mathsf{D}(K_0, \mathbf{R}^n, d_2) \cap K \neq \varnothing$ for every infinite subset $K_0 \subseteq K$.

Remark. Is it true that $\mathsf{D}(K_0, \mathbf{R}^n, d_2) \cap K = \mathsf{D}(K_0, K, d_2)$?

Theorem 4.4.14 (General Bolzano-Weierstrass)**.** If (X, d) is a compact space, then $\mathsf{D}(X_0, X, d) \neq \varnothing$ for every infinite subset $X_0 \subseteq X$.

The most useful form of Heine-Borel-Bolzano-Weierstrass is the following:

Corollary 4.4.15. Every bounded (infinite) sequence of real numbers has a convergent (infinite) subsequence.

Chapter 5

Limits

The word "space" means metric space throughout this chapter. The theory of limits in a general topological space is not well-behaved. Sometimes we will use a single capital to denote a metric space.

5.1 Sequencial Limit

Definition 5.1.1 (Sequence, bounded sequence)**.** A sequence in a space X is a nothing but a mapping from $\mathbf{N}$ or $\mathbf{N}_+$ to X. If the range(=image) of this mapping is a bounded subset of X, then we say that the sequence is bounded.

For example, if a is a sequence in X, then $a(n) \in X$ for every $n \in \mathbf{N}$. But we will write a_n instead of $a(n)$.

If a sequence takes its values in $\mathbf{C}$, the space of complex numbers with the usual metric, we say that the sequence is a numerical sequence.

Definition 5.1.2. Suppose (X, d) is a space and $\{p_n\}$ is a sequence in X. Let $p \in X$ be a point, we say that $\{p_n\}$ converges to p under the metric d, if there is a function

$$\delta : \mathbf{R}_{>0} \to \mathbf{N}$$

such that for all $\epsilon > 0$ we have

$$p_n \in \mathbf{B}_\epsilon^{(X,d)}(p), \forall n \geq \delta(\epsilon)$$

If a sequence converges to two points p, p' then these two points are the same. So we can talk about **the** limit of a sequence if such limit exists. Denoted by

$$\lim_{n\to\infty}^{(X,d)} p_n = p$$

If a sequence converges to no point in (X, d), we say that it diverges.

Example 5.1.3. Consider the sequence $1/n$, it converge to 0 in $\mathbf{R}$ with the usual metric. It diverges in $\mathbf{R}_{>0}$ with the usual metric (=subspace topology). It also diverges in $\mathbf{R}$ with the discrete metric.

Example 5.1.4. Any converge sequence is bounded, so unbounded sequence cannot be convergent.

We can give another definition of "convergence":

Proposition 5.1.5. Suppose (p_n) is a sequence in X, then (p_n) converges to $p \in X$ under d, if and only if every nonempty open neighborhood of p contains almost all (=all but finitely many) p_n.

Proposition 5.1.6. Suppose (E, X, d) is a topological pair and $p \in \mathsf{D}(E, X, d)$. Then there is a sequence $\{p_n\}$ in E such that

$$\overset{(X,d)}{\lim_{n\to\infty}} p_n = p$$

Theorem 5.1.7. A real sequence converges to a real number, if and only if it converges (as a numerical sequence) to a complex number which is real.

Theorem 5.1.8. We consider numerical sequences (that is, sequence in $\mathbf{C}$, where the metric is the usual one). Suppose

$$\lim_{n\to\infty} s_n = s, \lim_{n\to\infty} t_n = t$$

then:

- $\lim_{n\to\infty}(s_n + t_n) = s + t$
- $\lim_{n\to\infty} cs_n = cs$
- $\lim_{n\to\infty} c + s_n = c + s$
- $\lim_{n\to\infty}(s_n t_n) = st$
- Suppose $s_n \neq 0$ for all n and $s \neq 0$, then $\lim_{n\to\infty} \frac{1}{s_n} = \frac{1}{s}$

We automatically have the next theorem:

Theorem 5.1.9. We consider real sequences (that is, sequence in $\mathbf{R}$, where the metric is the usual one). Suppose

$$\lim_{n\to\infty} s_n = s, \lim_{n\to\infty} t_n = t$$

then:

- $\lim_{n\to\infty}(s_n + t_n) = s + t$
- $\lim_{n\to\infty} cs_n = cs$
- $\lim_{n\to\infty} c + s_n = c + s$
- $\lim_{n\to\infty}(s_n t_n) = st$
- Suppose $s_n \neq 0$ for all n and $s \neq 0$, then $\lim_{n\to\infty} \frac{1}{s_n} = \frac{1}{s}$

5.2 Cauchy Theory

Until now, we can only verify that a sequence is convergent by finding its limit. Is there a way to decide the convergence of a sequence without finding its limit (if exists)? There is. This is called the Cauchy theory.

Definition 5.2.1 (Subsequence)**.** Suppose $\{p_n\}$ is a sequence, and

$$n_1 < n_2 < \cdots < n_i < \cdots$$

is a sequence of natural numbers. Then the sequence $\{p_{n_i}\}$ is called a subsequence.

Theorem 5.2.2. A sequence converges to p if and only if its every subsequence converges to the same limit.

Definition 5.2.3 (Subsequential limit)**.** Suppose $\{p_n\}$ is a sequence in a metric space (X, d). If its some subsequence is convergent, the limit is called a subsequential limit of $\{p_n\}$. The set of all subsequential limit of $\{p_n\}$ is denoted by $\mathsf{Sl}(p, X, d)$ or $\mathsf{Sl}(\{p_n\}, X, d)$. If $(X, d) = \mathbf{R}$, then we set

$$\limsup_{n\to\infty} p_n = \sup \mathsf{Sl}(p, X, d), \quad \liminf_{n\to\infty} p_n = \inf \mathsf{Sl}(p, X, d)$$

We have

$$\limsup_{n\to\infty} p_n \leq \sup\{p_n\}, \quad \liminf_{n\to\infty} p_n \geq \inf\{p_n\}$$

Theorem 5.2.4. $\mathsf{Sl}(p, X, d)$ is closed in (X, d), and if (X, d) is compact, then $\mathsf{Sl}(p, X, d)$ is nonempty.

Proof. To be written. $\square$

Definition 5.2.5. Suppose (X, d) is a metric, we define the diameter of X by

$$\operatorname{diam}(X, d) = \sup_{p,q\in X} d(p, q)$$

In general, if (E, X, d) is a topological pair, we can define the diameter of E by

$$\operatorname{diam}(E, X, d) = \sup_{p,q\in E} d(p, q)$$

We have $\operatorname{diam}(\mathsf{Cl}(E, X, d), X, d) = \operatorname{diam}(E, X, d)$

Proposition 5.2.6.

Suppose $\{p_n\}$ is a sequence in a metric space (X, d), then for any $n \in \mathbf{N}$, the set

$$\{p_m : m \geq n\}$$

is a subspace of (X, d). So we can talk about its diameter.

Definition 5.2.7 (Cauchy sequence)**.** Suppose $\{p_n\}$ is a sequence in a metric space (X, d). We say that $\{p_n\}$ is Cauchy sequence in (X, d), if

$$\lim_{n\to\infty} \operatorname{diam}\{p_m : m \geq n\} = 0$$

Theorem 5.2.8 (Cauchy's criterion of convergence)**.** When we say spaces, we mean metric spaces.

- Any convergent sequence is a Cauchy sequence.
- A metric space in which every Cauchy sequence converges is called a complete space.
- Any closed subspace of a complete space is complete.
- A space (X, d) is totally bounded if and only if for every $\epsilon > 0$ there exists finitely many points $x_1, \dots$ such that $\bigcup_i \mathbf{B}_\epsilon^{(X,d)}(x_i) = X$.
- A space is compact if and only if it is complete and totally bounded.
- Any discrete space is complete.
- $\mathbf{R}^n$ and $\mathbf{C}^n$ are complete, but $\mathbf{Q}$ is not complete.

Theorem 5.2.9. Suppose (X, d) is a metric space, then there exists a complete metric space $(X^\natural, d^\natural)$ and a mapping $\iota : X \to X^\natural$ such that

- $d^\natural(\iota x, \iota y) = d(x, y)$
- the image of ι is dense in $(X^\natural, d^\natural)$

Theorem 5.2.10. Suppose $(X, d), (X_1^\natural, d_1^\natural), (X_2^\natural, d_2^\natural)$ are metric spaces, and $\iota_1 : X \to X_1^\natural, \iota_2 : X \to X_2^\natural$ are mappings such that

- $(X_1^\natural, d_1^\natural), (X_2^\natural, d_2^\natural)$ are complete
- $d_1^\natural(\iota_1 x, \iota_1 y) = d_2^\natural(\iota_2 x, \iota_2 y) = d(x, y)$
- the image of ι_1 is dense in $(X_1^\natural, d_1^\natural)$, the image of ι_2 is dense in $(X_2^\natural, d_2^\natural)$,

Then there exists a bijection $\phi : X_1 \to X_2$ such that $d_2(\phi x, \phi y) = d_1(x, y)$.

These two theorems can be summerized as "every metric space has completions, which are unique up to an isometry".

Example 5.2.11. $\mathbf{R}$ is **the** completion of $\mathbf{Q}$ (with the usual metric).

Example 5.2.12. The p-adic numbers $\mathbf{Q}_p$ is the completion of $\mathbf{Q}$ (with the p-adic metric).

There is no easy way to introduce p-adic numbers. So I'll just tell you what is a p-adic metric. Suppose we have two rational numbers x, y, such that

$$x - y = p^a \frac{m}{n}$$

where m, n are not multiples of p. Then

$$d_p(x, y) = p^{-a}$$

Using the 2-adic metric, the sequence $1, 3, 7, 15, 31, 63, 127, \dots$ converges to -1. In $\mathbf{Q}_p$, every open ball is closed.

5.3 More on Sequences

Theorem 5.3.1. Suppose $\{p_n\}$ is a monotonically increasing sequence of real numbers. If it is unbounded, then it is divergent. If it is bounded, then it has limit $\lim p_n = \limsup p_n$.

Theorem 5.3.2. Suppose $\{p_n\}$ is a monotonically decreasing sequence of real numbers. If it is unbounded, then it is divergent. If it is bounded, then it has limit $\lim p_n = \liminf p_n$.

Theorem 5.3.3. Suppose $\{p_n\}$ is a sequence of real numbers. Let I be an interval of finite length with $I \cap \mathsf{Sl}(\{p_n\}) = \varnothing$. Then $I \cap \{p_n : n \in \mathbf{N}\}$ is finite.

We include the important by easy theorem of Sandwich:

Theorem 5.3.4. Suppose $\{a_n\}, \{b_n\}$ are two sequences of real numbers. If $a_n \leq b_n$ for almost all n, then

$$\liminf a_n \leq \liminf a_n, \quad \limsup a_n \leq \limsup b_n$$

5.4 Important Limits

Theorem 5.4.1. Let $p > 0$ be a real number, α be a real number, z be a complex number with $|z| < 1$.

- $\lim_{n\to\infty} n^{-p} = 0$
- $\lim_{n\to\infty} p^{1/n} = 1$
- $\lim_{n\to\infty} n^{1/n} = 1$
- $\lim_{n\to\infty} \frac{n^\alpha}{(1+p)^n} = 0$
- $\lim_{n\to\infty} z^n = 0$

Theorem 5.4.2 (Euler constant)**.**

$$\lim_{n\to\infty} \sum_{i=0}^{n} \frac{1}{i!} = \lim_{n\to\infty} \left(1 + \frac{1}{n}\right)^n = 2.718281828\dots$$

This number is called the Euler constant, denoted by e. It is not an algebraic number. The Euler constant can be computed in the following sense:

$$0 < e - \left(\frac{1}{0!} + \frac{1}{1!} + \cdots + \frac{1}{n!}\right) < \frac{1}{n^2(n-1)!}$$

Chapter 6

Continuity

Appendix A

Some Inequalities

$p, q > 1, \frac{1}{p} + \frac{1}{q} = 1$, prove that

$$xy \leq \frac{x^p}{p} + \frac{y^q}{q}$$

$p, q > 1, \frac{1}{p} + \frac{1}{q} = 1$, prove that

$$\sum_{i=1}^{n} |x_i y_i| \leq \left(\sum_{i=1}^{n} |x_i|^p \right)^{\frac{1}{p}} \left(\sum_{i=1}^{n} |y_i|^q \right)^{\frac{1}{q}}$$

$p \geq 1$, prove that

$$\left(\sum_{i=1}^{n} |x_i - y_i|^p \right)^{\frac{1}{p}} + \left(\sum_{i=1}^{n} |y_i - z_i|^p \right)^{\frac{1}{p}} \geq \left(\sum_{i=1}^{n} |x_i - z_i|^p \right)^{\frac{1}{p}}$$

$p \leq 1$, prove that

$$\left(\sum_{i=1}^{n} |x_i - y_i|^p \right)^{\frac{1}{p}} + \left(\sum_{i=1}^{n} |y_i - z_i|^p \right)^{\frac{1}{p}} \leq \left(\sum_{i=1}^{n} |x_i - z_i|^p \right)^{\frac{1}{p}}$$

$p_1, \ldots, p_m > 0, \sum_{j=1}^{m} \frac{1}{p_j} = \frac{1}{r}$. Prove that

$$\left(\sum_{i=1}^{n} \left| \prod_{j=1}^{m} x_{i,j} \right|^r \right)^{\frac{1}{r}} \leq \prod_{j=1}^{m} \left(\sum_{i=1}^{n} |x_{i,j}|^{p_i} \right)^{\frac{1}{p_i}}$$

Appendix B

Problems

1. HW1: Prove Lemma 3.1.1, Lemma 3.1.2
2. HW2: Prove that if A is at most countable, and each X_α is at most countable for $\alpha \in A$, then $\bigcup_{\alpha \in A} X_\alpha$ is at most countable.
3. HW3: Prove that the set of all algebraic numbers is countable
4. HW4: Prove Exercise 1.5.2
5. HW5: Prove Theorem 5.2.4, 5.2.5, 5.2.6, 5.2.7, 5.2.8, 5.2.9, 5.2.10, 5.2.12, 5.2.13, 5.2.14, 5.2.15, 5.2.16, 5.2.17, 5.2.18
6. HW6: Prove Theorem 5.3.4
7. HW7: Prove Theorem 5.3.10