Outline

December 16, 2021

1

Recall that the Kronecker delta is defined by $\delta_i^j = \begin{cases} 1 & \text{if } i = j \\ 0 & \text{if } i \neq j \end{cases}$. Then if V is a vector space over $\mathbf{F}$ with basis $e_1, \ldots, e_n$ and W is a vector space over $\mathbf{F}$ with basis $\epsilon_1, \ldots, \epsilon_m$. Then $\mathcal{L}(V, W)$ is a vector space over $\mathbf{F}$ with basis T_i^j, where

$$T_i^j e_j = \delta_j^i \epsilon_i, \text{ where } 1 \leq j \leq n, 1 \leq i \leq m$$

Especially, $\dim \mathcal{L}(V, W) = \dim V \dim W$

2

Define $V^* = \mathcal{L}(V, \mathbf{F})$. Given $T \in \mathcal{L}(V, W)$, define $*_V^W(T) = T^* \in \mathcal{L}(W^*, V^*)$ by

$$T^*(\varphi) = \varphi \circ T, \forall \varphi \in W^*$$

then $*_V^W \in \mathcal{L}(\mathcal{L}(V, W), \mathcal{L}(W^*, V^*))$, and if we have $S \in \mathcal{L}(U, V), T \in \mathcal{L}(V, W)$, then

$$*_U^W(T \circ S) = *_U^V(S) \circ *_V^W(T)$$

Given a vector space V over $\mathbf{F}$, define $\Lambda_V v \in (V^*)^*$ by

$$(\Lambda_V v)(\varphi) = \varphi(v), \forall \varphi \in V^*$$

then $\Lambda_V \in \mathcal{L}(V, (V^*)^*)$ and it is bijective if $\dim V < \infty$. Given any $T \in \mathcal{L}(V, W)$, we always have

$$\Lambda_W \circ T = (T^*)^* \circ \Lambda_V$$

Even in infinite-dimensional cases, the double dual can be realized as the space itself if we have a good inner product (Riesz Representation Theorem).

3 Problem

Prove that $U \xrightarrow{S} V \xrightarrow{T} W$ exact if and only if $W^* \xrightarrow{T^*} V^* \xrightarrow{S^*} U^*$ exact.