5-lemma: Diagram Chasing

Qiu Caiyong

November 25, 2021

Suppose we have the following commutative exact diagram of vector spaces and linear maps:

$$\begin{array}{ccccccccc} Q & \xrightarrow{A} & R & \xrightarrow{B} & S & \xrightarrow{C} & T & \xrightarrow{D} & U \\ \downarrow{\scriptstyle J} & & \downarrow{\scriptstyle K} & & \downarrow{\scriptstyle L} & & \downarrow{\scriptstyle M} & & \downarrow{\scriptstyle N} \\ V & \xrightarrow{E} & W & \xrightarrow{F} & X & \xrightarrow{G} & Y & \xrightarrow{H} & Z \end{array}$$

where K, M are isomorphisms, J is surjective, N is injective. Prove that L is an isomorphism.

Solution:

By exactness and commutativity, we know that

$$\mathrm{Range}A = \mathrm{Null}B, \mathrm{Range}B = \mathrm{Null}C, \mathrm{Range}C = \mathrm{Null}D$$

$$\mathrm{Range}E = \mathrm{Null}F, \mathrm{Range}F = \mathrm{Null}G, \mathrm{Range}G = \mathrm{Null}H$$

$$K \circ A = E \circ J, L \circ B = F \circ K, M \circ C = G \circ L, N \circ D = H \circ M$$

First we prove that L is injective. To do this, we need to show that if $Ls = 0$ then $s = 0$.

Suppose $Ls = 0$, then $M(Cs) = G(Ls) = G0 = 0$, so $Cs \in \mathrm{Null}M$. But M is an isomorphism (hence injective), so $Cs = 0$, which means that $s \in \mathrm{Null}C = \mathrm{Range}B$. So there exists $r \in R$ such that $Br = s$. Now $F(Kr) = L(Br) = Ls = 0$, so $Kr \in \mathrm{Null}F = \mathrm{Range}E$. So there exists $v \in V$ such that $Ev = Kr$. Notice that J is surjective, so there exists $q \in Q$ such that $Jq = v$, we then have $Kr = Ev = E(Jq) = K(Aq)$. But K is an isomorphism (hence injective), so $r = Aq$ and $s = Br = B(Aq) = 0$ since $\mathrm{Range}A = \mathrm{Null}B$.

Next we prove that L is surjective. To do this, we need to show that $\forall x \in X$, there exists $p \in S$ such that $Lp = x$.

First of all, M is an isomorphism so M is surjective, we can find $t \in T$ such that $Mt = Gx$. Now $N(Dt) = H(Mt) = H(Gx) = 0$ since $\mathrm{Range}G = \mathrm{Null}H$. Notice that N is injective, so $Dt = 0$, which means that $t \in \mathrm{Null}D = \mathrm{Range}C$. So there exists $s \in S$ such that $Cs = t$. So $Gx = Mt = M(Cs) = G(Ls)$, which is equivalent to $x - Ls \in \mathrm{Null}G = \mathrm{Range}F$. So we can find a $w \in W$ such that $Fw = x - Ls$. Since K is an isomorphism, there exists $r \in R$ such that $Kr = w$, and we have $x = Ls + Fw = Ls + F(Kr) = Ls + L(Br) = L(s + Br)$. Let $p = s + Br$ then $Lp = x$.