Lecture Notes for **A Mathematical Introduction to Robotic Manipulation**

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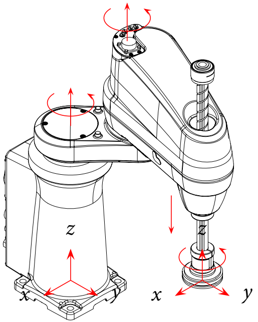
Main Concepts

3.1 Forward kinematics



Adept Cobra i600

Lower Pair Joints:

Revolute:	$S^1 \mapsto SO(2)$
Prismatic:	$\mathbb{R} \mapsto T(1)$

Forward kinematics:

Link 0: (Base)	Stationary
Link 1:	First movable link
⋮	
Link n:	End-effector attached

3.1 Forward kinematics

□ *Joint space:*

Revolute joint: $S^1, \theta_i \in S^1$ or $\theta_i \in [-\pi, \pi]$

Prismatic joint: $\mathbb{R}$

Joint Space: $Q : \underbrace{S^1 \times \cdots \times S^1}_{\text{no. of } R \text{ joint}} \times \underbrace{\mathbb{R} \times \cdots \times \mathbb{R}}_{\text{no. of } P \text{ joint}}$

Adept $\quad Q : S^1 \times S^1 \times S^1 \times \mathbb{R}$

Elbow $\quad Q = \Gamma^6 : \underbrace{S^1 \times \cdots \times S^1}_{6}$

Reference (nominal) joint config: $\theta = (0, 0, \ldots, 0) \in Q$

Reference (nominal) end-effector config: $g_{st}(0) \in SE(3)$

Arbitrary config.: $g_{st}(\theta)$, or g_{st}: $\begin{array}{l} Q \mapsto SE(3) \\ \theta \mapsto g_{st}(\theta) \end{array}$

3.1 Forward kinematics

□ *Classical Approach:*

$$g_{st}(\theta_1,\theta_2) = g_{st}(\theta_1)\cdot g_{l_1l_2}\cdot g_{l_2t}$$

Problem:

- A coordinate frame for each link

□ *The product of exponentials formula:*

Consider Fig 3.2 again.

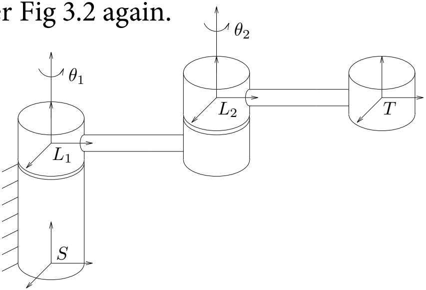


Figure 3.2: A two degree of freedom manipulator

3.1 Forward kinematics

Step 1: Rotating about ω_2 by θ_2

$$\xi_2 = \begin{bmatrix} -\omega_2 \times q_2 \\ \omega_2 \end{bmatrix}$$

$$g_{st}(\theta_2) = e^{\hat{\xi}_2 \theta_2} \cdot g_{st}(0)$$

Step 2: Rotating about ω_1 by θ_1

$$\xi_1 = \begin{bmatrix} -\omega_1 \times q_1 \\ \omega_1 \end{bmatrix}$$

$$g_{st}(\theta_1, \theta_2) = e^{\hat{\xi}_1 \theta_1} \cdot \underbrace{e^{\hat{\xi}_2 \theta_2} \cdot g_{st}(0)}_{\text{offset}}$$

$$\theta : (0, 0) \mapsto (0, \theta_2) \mapsto (\theta_1, \theta_2)$$

What if another route is taken?

$$\theta : (0,0) \mapsto (\theta_1, 0) \mapsto (\theta_1, \theta_2)$$

$$\begin{array}{rl}
\text{Step 1:} & \text{Rotating about } \omega_1 \text{ by } \theta_1 \\
 & \xi_1 = \begin{bmatrix} -\omega_1 \times q_1 \\ \omega_1 \end{bmatrix} \\
 & g_{st}(\theta_1) = e^{\hat{\xi}_1 \theta_1} \cdot g_{st}(0) \\
\text{Step 2:} & \text{Rotating about } \omega_2' \text{ by } \theta_2 \\
\text{Let} & e^{\hat{\xi}_1 \theta_1} = \begin{bmatrix} R_1 & p_1 \\ 0 & 1 \end{bmatrix} \\
 & \omega_2' = R_1 \cdot \omega_2 \\
 & q_2' = p_1 + R_1 \cdot q_2
\end{array}$$

3.1 Forward kinematics

$$\xi_2' = \begin{bmatrix} -\omega_2' \times q_2' \\ \omega_2' \end{bmatrix} = \begin{bmatrix} -R_1\hat{\omega}_2 R_1^T(p_1 + R_1 q_2) \\ R_1\omega_2 \end{bmatrix}$$

$$= \begin{bmatrix} R_1 & \hat{p}_1 R_1 \\ 0 & R_1 \end{bmatrix} \begin{bmatrix} -\omega_2 \times q_2 \\ \omega_2 \end{bmatrix} = Ad_{e^{\hat{\xi}_1\theta_1}} \cdot \xi_2 \Rightarrow \hat{\xi}_2' = e^{\hat{\xi}_1\theta_1} \cdot \hat{\xi}_2 \cdot e^{-\hat{\xi}_1\theta_1}$$

$$\begin{aligned} g_{st}(\theta_1, \theta_2) &= e^{\hat{\xi}_2'\theta_2} \cdot e^{\hat{\xi}_1\theta_1} \cdot g_{st}(0) \\ &= e^{e^{\hat{\xi}_1\theta_1} \cdot \hat{\xi}_2\theta_2 \cdot e^{-\hat{\xi}_1\theta_1}} \cdot e^{\hat{\xi}_1\theta_1} \cdot g_{st}(0) \\ &= e^{\hat{\xi}_1\theta_1} \cdot e^{\hat{\xi}_2\theta_2} \cdot e^{-\hat{\xi}_1\theta_1} \cdot e^{\hat{\xi}_1\theta_1} \cdot g_{st}(0) \\ &= \underbrace{e^{\hat{\xi}_1\theta_1} \cdot e^{\hat{\xi}_2\theta_2} \cdot g_{st}(0)}_{\text{Independent of the route taken}} \end{aligned}$$

□ *Procedure for forward kinematic map:*

Identify a nominal config.:

$$\Theta = (\theta_{10}, \ldots, \theta_{n0}) = 0, g_{st}(0) \triangleq g_{st}(\theta_{10}, \ldots, \theta_{n0})$$

Simplification of forward kinematics mapping:

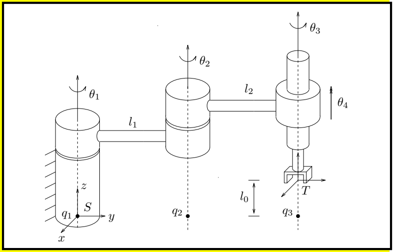
Revolute joint:	$\xi_i = \begin{bmatrix} -\omega \times q_i \\ \omega_i \end{bmatrix}$ Choose q_i s.t. ξ_i is simple.
Prismatic joint:	$\xi_i = \begin{bmatrix} v_i \\ 0 \end{bmatrix}$

Write $g_{st}(\theta) = e^{\hat{\xi}_1\theta_1} \ldots e^{\hat{\xi}_n\theta_n} \cdot g_{st}(0)$ (product of exponential mapping)

⋄ *Example: SCARA manipulator*

$$g_{st}(0) = \begin{bmatrix} I & \begin{bmatrix} 0 \\ l_1 + l_2 \\ l_0 \end{bmatrix} \\ 0 & 1 \end{bmatrix}$$

$$\omega_1 = \omega_2 = \omega_3 = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$$



$$q_1 = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}, q_2 = \begin{bmatrix} 0 \\ l_1 \\ 1 \end{bmatrix}, q_3 = \begin{bmatrix} 0 \\ l_1 + l_2 \\ 0 \end{bmatrix}$$

$$\Rightarrow \xi_1 = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 1 \end{bmatrix}, \xi_2 = \begin{bmatrix} l_1 \\ 0 \\ 0 \\ 0 \\ 0 \\ 1 \end{bmatrix}, \xi_3 = \begin{bmatrix} l_1 + l_2 \\ 0 \\ 0 \\ 0 \\ 0 \\ 1 \end{bmatrix}, \xi_4 = \begin{bmatrix} v_4 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix},$$

3.1 Forward kinematics

$$g_{st}(\theta) = e^{\hat{\xi}_1\theta_1} \cdot e^{\hat{\xi}_2\theta_2} \cdot e^{\hat{\xi}_3\theta_3} \cdot e^{\hat{\xi}_4\theta_4} \cdot g_{st}(0) = \begin{bmatrix} R(\theta) & p(\theta) \\ 0 & 1 \end{bmatrix}$$

$$e^{\hat{\xi}_1\theta_1} = \begin{bmatrix} c_1 & -s_1 & 0 & 0 \\ s_1 & c_1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}, e^{\hat{\xi}_2\theta_2} = \begin{bmatrix} c_2 & -s_2 & 0 & l_1 s_2 \\ s_2 & c_2 & 0 & l_1 s_2 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix},$$

$$e^{\hat{\xi}_3\theta_3} = \begin{bmatrix} c_3 & -s_3 & 0 & (l_1 s_2)s_3 \\ s_3 & c_3 & 0 & (l_1 s_2)v_3 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}, e^{\hat{\xi}_4\theta_4} = \begin{bmatrix} I & \begin{bmatrix} 0 \\ 0 \\ \theta_4 \end{bmatrix} \\ 0 & 1 \end{bmatrix}$$

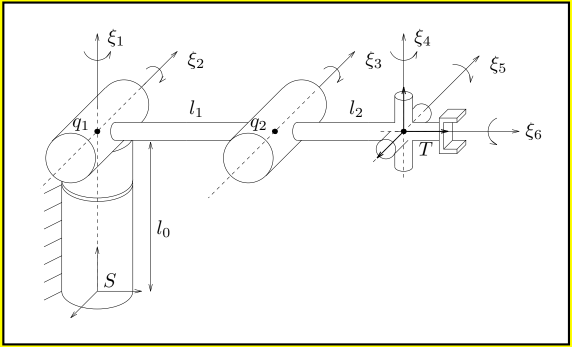
$$g_{st}(\theta) = \begin{bmatrix} c_{123} & -s_{123} & 0 & -l_1 s_1 - l_2 s_{12} \\ s_{123} & c_{123} & 0 & l_1 c_1 + l_2 c_{12} \\ 0 & 0 & 1 & l_0 + \theta_4 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

in which, $c_{123} = \cos(\theta_1 + \theta_2 + \theta_3)$

3.1 Forward kinematics

◇ *Example: Elbow manipulator*

$$g_{st}(0) = \begin{bmatrix} I & \begin{bmatrix} 0 \\ l_1 + l_2 \\ l_0 \end{bmatrix} \\ 0 & 1 \end{bmatrix}$$



3.1 Forward kinematics

$$\xi_1 = \begin{bmatrix} -\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \times \begin{bmatrix} 0 \\ 0 \\ l_0 \end{bmatrix} \\ \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 1 \end{bmatrix},$$

$$\xi_2 = \begin{bmatrix} -\begin{bmatrix} -1 \\ 0 \\ 0 \end{bmatrix} \times \begin{bmatrix} 0 \\ 0 \\ l_0 \end{bmatrix} \\ \begin{bmatrix} -1 \\ 0 \\ 0 \end{bmatrix} \end{bmatrix} = \begin{bmatrix} 0 \\ -l_0 \\ 0 \\ -1 \\ 0 \\ 0 \end{bmatrix}, \xi_3 = \begin{bmatrix} 0 \\ -l_0 \\ l_1 \\ -1 \\ 0 \\ 0 \end{bmatrix},$$

$$\xi_4 = \begin{bmatrix} l_1 + l_2 \\ 0 \\ 0 \\ 0 \\ 0 \\ 1 \end{bmatrix}, \xi_5 = \begin{bmatrix} 0 \\ -l_0 \\ l_1 + l_2 \\ -1 \\ 0 \\ 0 \end{bmatrix}, \xi_6 = \begin{bmatrix} -l_0 \\ 0 \\ 0 \\ 0 \\ 1 \\ 0 \end{bmatrix}$$

$$\rightarrow g_{st}(\theta_1, \ldots \theta_6) = e^{\hat{\xi}_1 \theta_1} \ldots e^{\hat{\xi}_6 \theta_6} \cdot g_{st}(0) = \begin{bmatrix} R(\theta) & p(\theta) \\ 0 & 1 \end{bmatrix}$$

3.1 Forward kinematics

$$p(\theta) = \begin{bmatrix} -s_1(l_2c_2 + l_2c_{23}) \\ c_1(l_1c_2 + l_2c_{23}) \\ l_0 - l_1s_2 - l_2s_{23} \end{bmatrix}, R(\theta) = [r_{ij}]$$

in which, $r_{11} = c_6(c_1c_4 - s_1c_{23}s_4) + s_6(s_1s_{23}c_5 + s_1c_{23}c_4s_5 + c_1s_4s_5)$

$$r_{12} = -c_5(s_1c_{23}c_4 + c_1s_4) + s_1s_{23}s_5$$

$$r_{13} = c_6(-c_5s_1s_{23} - (c_{23}c_4s_1 + c_1s_4)s_5) + (c_1c_4 - c_{23}s_1s_4)s_6$$

$$r_{21} = c_6(c_4s_1 + c_1c_{23}s_4) - (c_1c_5s_{23} + (c_1c_{23}c_4 - s_1s_4)s_5)s_6$$

$$r_{22} = c_5(c_1c_{23}c_4 - s_1s_4) - c_1s_{23}s_5$$

$$r_{23} = c_6(c_1c_5s_{23} + (c_1c_{23}c_4 - s_1s_4)s_5) + (c_4s_1 + c_1c_{23}s_4)s_6$$

$$r_{31} = -(c_6s_{23}s_4) - (c_{23}c_5 - c_4s_{23}s_5)s_6$$

$$r_{32} = -(c_4c_5s_{23}) - c_{23}s_5$$

$$r_{33} = c_6(c_{23}c_5 - c_4s_{23}s_5) - s_{23}s_4s_6$$

Simplify forward Kinematics Map:

Choose base frame or ref. Config. s.t. $g_{st}(0) = I$

3.1 Forward kinematics

□ *Manipulator Workspace:*

$$W = \{g_{st}(\theta) | \forall \theta \in Q\} \subset SE(3)$$

- Reachable Workspace:

$$W_R = \{p(\theta) | \forall \theta \in Q\} \subset \mathbb{R}^3$$

- Dextrous Workspace:

$$W_D = \{p \in \mathbb{R}^2 | \forall R \in SO(3), \exists \theta, g_{st}(\theta) = (p, R)\}$$

3.1 Forward kinematics

◇ *Example: A planar serial 3-bar linkage*

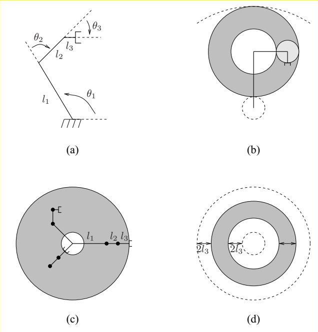
(a) Workspace calculation:

$$g = (x, y, \phi)$$
$$x = l_1 c_1 + l_2 c_{12} + l_3 c_{123}$$
$$y = l_1 s_1 + l_2 s_{12} + l_3 s_{123}$$
$$\phi = \theta_1 + \theta_2 + \theta_3$$

(b) Construction of Workspace:

(c) Reachable Workspace:

(d) Dextrous Workspace:



□ ***Manipulator's maximum workspace (Paden):***

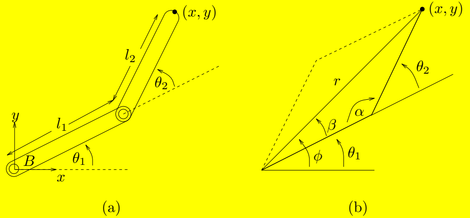
Elbow manipulator and its kinematics inverse.

3.2 Inverse Kinematics

Given $g \in SE(3)$, find $\theta \in Q$ s.t.

$$g_{st}(\theta) = g, \text{ where } g_{st} : Q \mapsto SE(3)$$

◇ *Example: A planar example*



$$x = l_1 \cos\theta_1 + l_2 \cos(\theta_1 + \theta_2)$$
$$y = l_1 \sin\theta_1 + l_2 \sin(\theta_1 + \theta_2)$$

Given (x, y), solve for (θ_1, θ_2)

3.2 Inverse Kinematics

♢ *Review:*

Polar Coordinates:

$$(r, \phi), r = \sqrt{x^2 + y^2}$$

Law of cosines:

$$\theta_2 = \pi \pm \alpha, \alpha = \cos^{-1} \frac{l_1^2 + l_2^2 - r^2}{2 l_1 l_2}$$

Flip solution: $\pi + \alpha$

$$\theta_1 = \operatorname{atan2}(y, x) \pm \beta, \beta = \cos^{-1} \frac{r^2 + l_1^2 - l_2^2}{2 l_1 r}$$

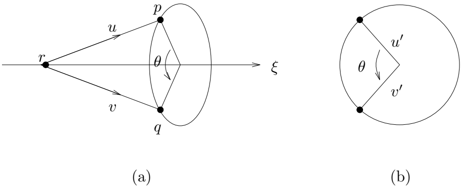
Hight Lights:

- Subproblems
- Each has zero, one or two solutions!

□ *Paden-Kahan Subproblems:*

Subproblem 1: Rotation about a single axis

Let ξ be a zero-pitch twist, with unit magnitude and two points $p, q \in \mathbb{R}^3$. Find θ s.t. $e^{\hat{\xi}\theta} p = q$



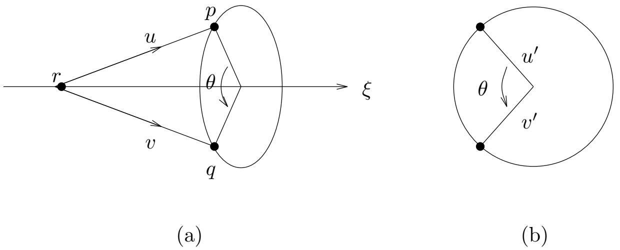
Solution: Let $r \in l_\xi$, define $u = p - r, v = q - r, e^{\hat{\xi}\theta} r = r$

3.2 Inverse Kinematics

Also,

$$\Rightarrow e^{\hat{\xi}\theta} p = q \Rightarrow e^{\hat{\xi}\theta} \underbrace{(p - r)}_{u} = \underbrace{q - r}_{v} \Rightarrow \begin{bmatrix} e^{\hat{\omega}\theta} & * \\ 0 & 1 \end{bmatrix} \begin{bmatrix} u \\ 0 \end{bmatrix} = \begin{bmatrix} v \\ 0 \end{bmatrix}$$

$$\Rightarrow e^{\hat{\omega}\theta} u = v \qquad \begin{cases} w^T u = w^T v \\ \|u\|^2 = \|v\|^2 \end{cases}$$



3.2 Inverse Kinematics

$$u' = (I - \omega\omega^T)u, v' = (I - \omega\omega^T)v$$

The solution exists only if $\begin{cases} \|u'\|^2 = \|v'\|^2 \\ \omega^T u = \omega^T v \end{cases}$

If $u' \neq 0$, then

$$u' \times v' = \omega \sin\theta \|u'\| \|v'\|$$
$$u' \cdot v' = \cos\theta \|u'\| \|v'\|$$

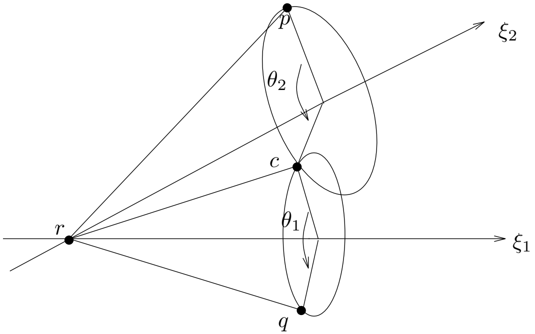
$$\Rightarrow \theta = \text{atan2}(\omega^T(u' \times v'), u'^T v')$$

If $u' = 0$, $\Rightarrow$ Infinite number of solutions!

3.2 Inverse Kinematics

Subproblem 2: Rotation about two subsequent axes

let ξ_1 and ξ_2 be two zero-pitch, unit magnitude twists, with intersecting axes, and $p, q \in R^3$. find θ_1 and θ_2 s.t. $e^{\hat{\xi}_1\theta_1}e^{\hat{\xi}_2\theta_2}p = q$.



3.2 Inverse Kinematics

Solution: If two axes of ξ_1 and ξ_2 coincide, then we get:

Subproblem 1: $\theta_1 + \theta_2 = \theta$

If the two axes are not parallel, $\omega_1 \times \omega_2 \neq 0$, then, let c satisfy:

$$e^{\hat{\xi}_2\theta_2} p = c = e^{-\hat{\xi}_1\theta_1} q$$

Set $r \in l_{\xi_1} \cap l_{\xi_2}$

$$e^{\hat{\xi}_2\theta_2} \underbrace{p - r}_{u} = \underbrace{c - r}_{z} = e^{-\hat{\xi}_1\theta_1} \underbrace{(q - r)}_{v}, \Rightarrow e^{\hat{\omega}_2\theta_2} u = z = e^{-\hat{\omega}_1\theta_1} v$$

$$\Rightarrow \begin{cases} \omega_2^T u = \omega_2^T z \\ \omega_1^T v = \omega_1^T z \end{cases}, \|u\|^2 = \|z\|^2 = \|v\|^2$$

As ω_1, ω_2 and $\omega_1 \times \omega_2$ are linearly independent,

$$z = \alpha\omega_1 + \beta\omega_2 + \gamma(\omega_1 \times \omega_2)$$

$$\Rightarrow \|z\|^2 = \alpha^2 + \beta^2 + 2\alpha\beta\omega_1^T\omega_2 + \gamma^2\|\omega_1 \times \omega_2\|^2$$

3.2 Inverse Kinematics

$$\begin{matrix} \omega_2^T u = \alpha \omega_2^T \omega_1 + \beta \\ \omega_1^T v = \alpha + \beta \omega_1^T \omega_2 \end{matrix} \Rightarrow \begin{cases} \alpha & = \frac{(\omega_1^T \omega_2)\omega_2^T u - \omega_1^T v}{(\omega_1^T \omega_2)^2 - 1} \\ \beta & = \frac{(\omega_1^T \omega_2)\omega_1^T v - \omega_2^T u}{(\omega_1^T \omega_2)^2 - 1} \end{cases}$$

$$\|z\|^2 = \|u\|^2 \Rightarrow \gamma^2 = \frac{\|u\|^2 - \alpha^2 - \beta^2 - 2\alpha\beta\omega_1^T \omega_2}{\|\omega_1 \times \omega_2\|^2} \quad (*)$$

$(*)$ has zero, one or two solution(s):

$$\text{Given } z \Rightarrow c \Rightarrow \begin{cases} e^{\hat{\xi}_2 \theta_2} p = c \\ e^{\hat{\xi}_1 \theta_1} p = c \end{cases}$$

for θ_1 and θ_2

1. Two solutions when the two circles intersect.
2. One solution when they are tangent
3. Zero solution when they do not intersect

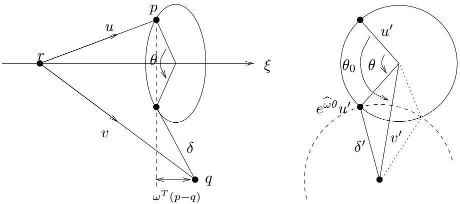
Subproblem 3: Rotation to a given point

Given a zero-pitch twist ξ, with unit magnitude and $p, q \in \mathbb{R}^3$, find θ s.t. $\|q - e^{\hat{\xi}\theta} p\| = \delta$

Define: $u = p - r, v = q - r, \|v - e^{\hat{\omega}\theta} u\|^2 = \delta^2$

$$u' = u - \omega\omega^T u$$

$$v' = v - \omega\omega^T v$$



$$\Rightarrow u = u' + \omega\omega^T u, v = v' + \omega\omega^T v, \delta'^2 = \delta^2 - |\omega^T(p-q)|^2$$

3.2 Inverse Kinematics

$$\|(v' + \omega\omega^T v) - e^{\hat{\omega}\theta}(u' + \omega\omega^T u)\|^2 = \delta^2 \Rightarrow$$

$$\|v' - e^{\hat{\omega}\theta}u' + \underbrace{\omega\omega^T(v-u)}_{\omega\omega^T(q-p)}\|^2 = \delta^2$$

$$\|v' - e^{\hat{\omega}\theta}u'\|^2 = \delta^2 - \|\omega^T(p-q)\|^2 = \delta'^2,$$

$$\theta_0 = \operatorname{atan2}(\omega^T(u' \times v'), u'^T v'),$$

$$\phi = \theta_0 - \theta \Rightarrow \|u'\|^2 + \|v'\|^2 - 2\|u'\| \cdot \|v'\| \cos\phi = \delta'^2,$$

$$\theta = \theta_0 \pm \cos^{-1}\frac{\|u'\| + \|v'\| - \delta'^2}{2\|u'\| \cdot \|v'\|} \quad (*)$$

Zero, one or two solutions!

3.2 Inverse Kinematics

□ *Solving inverse kinematics using subproblems:*

Technique 1: Eliminate the dependence on a joint

$e^{\hat{\xi}\theta}p = p$, if $p \in l_{\xi}$. Given $e^{\hat{\xi}_1\theta_1}e^{\hat{\xi}_2\theta_2}e^{\hat{\xi}_3\theta_3} = g$,
select $p \in l_{\xi_3}$, $p \notin l_{\xi_1}$ or l_{ξ_2}, then

$$gp = e^{\hat{\xi}_1\theta_1}e^{\hat{\xi}_2\theta_2}p$$

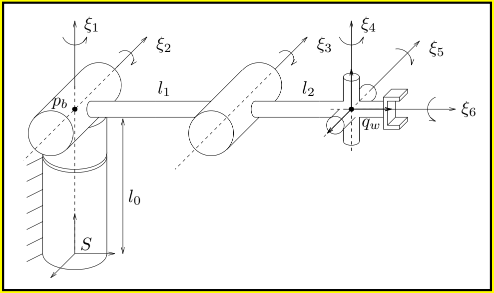
Technique 2: subtract a common point

$e^{\hat{\xi}_1\theta_1}e^{\hat{\xi}_2\theta_2}e^{\hat{\xi}_3\theta_3} = g, q \in l_{\hat{\xi}_1} \cap l_{\hat{\xi}_2} \Rightarrow$
$e^{\hat{\xi}_1\theta_1}e^{\hat{\xi}_2\theta_2}e^{\hat{\xi}_3\theta_3}p - q = gp - q \Rightarrow$

$$\|e^{\hat{\xi}_3\theta_3}p - q\| = \|gp - q\|$$

3.2 Inverse Kinematics

◇ *Example: Elbow manipulator*



$$g_{st}(\theta) = e^{\hat{\xi}_1\theta_1} e^{\hat{\xi}_2\theta_2} e^{\hat{\xi}_3\theta_3} e^{\hat{\xi}_4\theta_4} e^{\hat{\xi}_5\theta_5} e^{\hat{\xi}_6\theta_6} g_{st}(0) = g_d$$

$$\Rightarrow e^{\hat{\xi}_1\theta_1} e^{\hat{\xi}_2\theta_2} e^{\hat{\xi}_3\theta_3} e^{\hat{\xi}_4\theta_4} e^{\hat{\xi}_5\theta_5} e^{\hat{\xi}_6\theta_6} = g_d \cdot g_{st}^{-1}(0) = g_1$$

Step 1: Solve for θ_3

Let $e^{\hat{\xi}_1\theta_1} \dots e^{\hat{\xi}_6\theta_6} q_\omega = g_1 \cdot q_\omega$

$$\Rightarrow e^{\hat{\xi}_1\theta_1} e^{\hat{\xi}_2\theta_2} e^{\hat{\xi}_3\theta_3} q_\omega = g_1 \cdot q_\omega$$

Subtract p_b from $g_1 q_\omega$:

$$\left\| e^{\hat{\xi}_1\theta_1} e^{\hat{\xi}_2\theta_2} (e^{\hat{\xi}_3\theta_3} q_\omega - p_b) \right\| = \| g_1 q_\omega - p_b \|$$

$$\Rightarrow \left\| e^{\hat{\xi}_3\theta_3} q_\omega - p_b \right\| \triangleq \delta \leftarrow \text{Subproblem 3}$$

Step 2: Given θ_3, solve for θ_1, θ_2

$e^{\hat{\xi}_1\theta_1} e^{\hat{\xi}_2\theta_2} (e^{\hat{\xi}_3\theta_3} q_\omega) = g_1 q_\omega$, Subproblem 2 $\Rightarrow \theta_1, \theta_2$

3.2 Inverse Kinematics

Step 3: Given $\theta_1, \theta_2, \theta_3$, solve θ_4, θ_5

$$e^{\hat{\xi}_4\theta_4} e^{\hat{\xi}_5\theta_5} e^{\hat{\xi}_6\theta_6} = \underbrace{e^{-\hat{\xi}_3\theta_3} e^{-\hat{\xi}_2\theta_2} e^{-\hat{\xi}_1\theta_1} g_1}_{g_2}$$

let $p \in l_{\xi_6}, p \notin l_{\xi_4}$ or l_{ξ_5}, $e^{\hat{\xi}_4\theta_4} e^{\hat{\xi}_5\theta_5} p = g_2 p$,
Subproblem 2 $\Rightarrow \theta_4$ and θ_5.

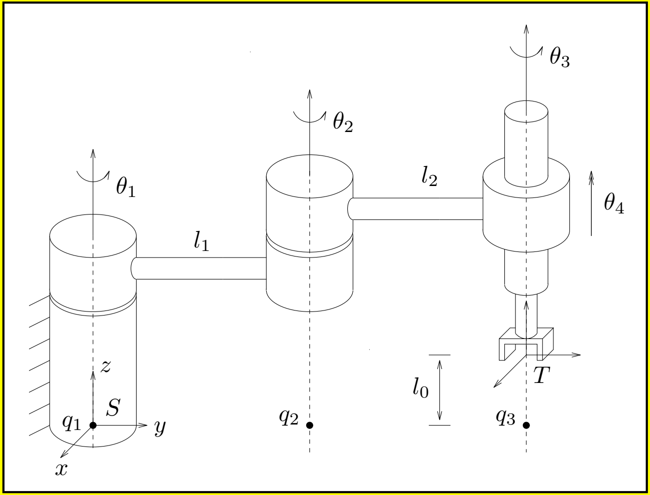
Step 4: Given $(\theta_1, \ldots, \theta_5)$, solve for θ_6

$$e^{\hat{\xi}_6\theta_6} = (e^{\hat{\xi}_1\theta_1} \ldots e^{\hat{\xi}_5\theta_5})^{-1} \cdot g_1 \triangleq g_3$$

Let $p \notin l_{\xi_6} \Rightarrow e^{\hat{\xi}_6\theta_6} p = g_3 \cdot p = q \Leftarrow$ Subproblem 1
Maximum of solutions: 8

3.2 Inverse Kinematics

⋄ *Example: Inverse Kinematics of SCARA*



3.2 Inverse Kinematics

$$g_{st}(\theta) = e^{\hat{\xi}_1\theta_1} \dots e^{\hat{\xi}_4\theta_4} g_{st}(0) = \begin{bmatrix} c\phi & -s\phi & 0 & x \\ s\phi & c\phi & 0 & y \\ 0 & 0 & 1 & z \\ 0 & 0 & 0 & 1 \end{bmatrix} \triangleq g_d,$$

$$p = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix} \Rightarrow p(\theta) = \begin{bmatrix} -l_1 s_1 - l_2 s_{12} \\ l_1 c_1 + l_2 c_{12} \\ l_0 + \theta_4 \end{bmatrix} = \begin{bmatrix} x \\ y \\ z \end{bmatrix} \Rightarrow \theta_4 = z - l_0$$

$$e^{\hat{\xi}_1\theta_1} e^{\hat{\xi}_2\theta_2} e^{\hat{\xi}_3\theta_3} = g_d g_{st}^{-1}(0) e^{-\hat{\xi}_4\theta_4} \triangleq g_1$$

3.2 Inverse Kinematics

Let $p \in l_{\xi_3}, q \in l_{\xi_1} \Rightarrow e^{\hat{\xi}_1\theta_1} e^{\hat{\xi}_2\theta_2} p = g_1 p,$

$$\|e^{\hat{\xi}_1\theta_1}(e^{\hat{\xi}_2\theta_2} p - q)\| = \|g_1 p - q\|,$$

$\|e^{\hat{\xi}_2\theta_2} p - q\| = \delta \leftarrow$ Subproblem 3 to get θ_2

$\Rightarrow e^{\hat{\xi}_1\theta_1}(e^{\hat{\xi}_2\theta_2} p) = g_1 p \Rightarrow \theta_1 \leftarrow$ Subproblem 1 to get θ_1

$\Rightarrow e^{\hat{\xi}_3\theta_3} = e^{-\hat{\xi}_2\theta_2} e^{-\hat{\xi}_1\theta_1} g_d g_{st}^{-1}(0) e^{-\hat{\xi}_4\theta_4} \triangleq g_2$

$e^{\hat{\xi}_3\theta_3} p = g_2 p, p \notin l_{\xi_3}$

There are a maximum of two solutions!

3.3 Manipulator Jacobian

Given $g_{st} : Q \to SE(3)$, $\theta(t) = (\theta_1(t) \dots \theta_n(t))^T \to e^{\hat{\xi}_1\theta_1} \dots e^{\hat{\xi}_n\theta_n} g_{st}(0)$ and $\dot{\theta}(t) = (\dot{\theta}_1(t) \dots \dot{\theta}_n(t))^T$,

What is the velocity of the tool frame?

$$\hat{V}^s_{st} = \dot{g}_{st}(\theta)g^{-1}_{st}(\theta) = \sum_{i=1}^{n}(\frac{\partial g_{st}}{\partial \theta_i}\dot{\theta}_i)g^{-1}_{st}(\theta)$$

$$= \sum_{i=1}^{n}(\frac{\partial g_{st}}{\partial \theta_i}g^{-1}_{st}(\theta))\dot{\theta}_i \Rightarrow V^s_{st} = \sum_{i=1}^{n}(\frac{\partial g_{st}}{\partial \theta_i}g^{-1}_{st}(\theta))^\vee\dot{\theta}_i$$

$$= \underbrace{[(\frac{\partial g_{st}}{\partial \theta_1}g^{-1}_{st}(\theta))^\vee, \dots, (\frac{\partial g_{st}}{\partial \theta_n}g^{-1}_{st}(\theta))^\vee]}_{J^s_{st}(\theta)\in\mathbb{R}^{6\times n}} \begin{bmatrix} \dot{\theta}_1 \\ \vdots \\ \dot{\theta}_n \end{bmatrix}$$

3.3 Manipulator Jacobian

Chapter 3 Manipulator Kinematics
Forward kinematics
Inverse Kinematics
Manipulator Jacobian
Redundant and Parallel Manipulators

$$g_{st}(\theta) = e^{\hat{\xi}_1\theta_1} \dots e^{\hat{\xi}_n\theta_n} g_{st}(0)$$

$$\frac{\partial g_{st}}{\partial \theta_1} g_{st}^{-1}(\theta) = (\hat{\xi}_1 e^{\hat{\xi}_1\theta_1} \dots e^{\hat{\xi}_n\theta_n} g_{st}(0))(g_{st}(\theta))^{-1} = \hat{\xi}_1 \Rightarrow$$

$$(\frac{\partial g_{st}}{\partial \theta_1} g_{st}^{-1}(\theta))^{\vee} = \xi_1$$

$$\frac{\partial g_{st}}{\partial \theta_2} g_{st}^{-1}(\theta) = (e^{\hat{\xi}_1\theta_1} \hat{\xi}_2 e^{\hat{\xi}_2\theta_2} \dots e^{\hat{\xi}_n\theta_n} g_{st}(0))(g_{st}(\theta))^{-1}$$

$$= e^{\hat{\xi}_1\theta_1} \hat{\xi}_2 e^{\hat{\xi}_2\theta_2} \dots e^{\hat{\xi}_n\theta_n} g_{st}(0) g_{st}^{-1}(\theta) = e^{\hat{\xi}_1\theta_1} \hat{\xi}_2 e^{-\hat{\xi}_1\theta_1} \triangleq \hat{\xi}'_2$$

3.3 Manipulator Jacobian

$$\left(\frac{\partial g_{st}}{\partial \theta_2} g_{st}^{-1}(\theta)\right)^\vee = \mathrm{Ad}_{e^{\hat{\xi}_1 \theta_1}} \xi_2 = \xi_2'$$

$$\vdots$$

$$\left(\frac{\partial g_{st}}{\partial \theta_i} g_{st}^{-1}(\theta)\right)^\vee = \mathrm{Ad}_{e^{\hat{\xi}_1 \theta_1} \ldots e^{\hat{\xi}_{i-1} \theta_{i-1}}} \xi_i \triangleq \xi_i'$$

$$\Rightarrow J_{st}^s(\theta) = [\xi_1, \xi_2', \ldots, \xi_n']$$

Chapter 3 Manipulator Kinematics

Forward kinematics

Inverse Kinematics

Manipulator Jacobian

Redundant and Parallel Manipulators

□ *Interpretation of ξ_i':*

- ξ_i' is only affected by $\theta_1 \ldots \theta_{i-1}$
- The twist associated with joint i, at the present configuration.

□ *Body jacobian:*

$$
\begin{aligned}
V_{st}^b &= J_{st}^b(\theta) \cdot \dot{\theta} \\
J_{st}^b(\theta) &= [\xi_1^\dagger \ldots \xi_{n-1}^\dagger, \xi_n^\dagger] \\
\xi_i^\dagger &= \mathrm{Ad}^{-1}_{e^{\hat{\xi}_{i+1}\theta_{i+1}} \ldots e^{\hat{\xi}_n\theta_n} g_{st}(0)} \xi_i
\end{aligned}
$$

Joint twist written with respect to the body frame at the current configuration!

$$J_{st}^s(\theta) = \mathrm{Ad}_{g_{st}(\theta)} \cdot J_{st}^b(\theta)$$

If J_{st}^s is invertible, $\dot{\theta}(t) = (J_{st}^s(\theta))^{-1} \cdot V_{st}^s(t)$

3.3 Manipulator Jacobian

Given $g(t)$, how to find $\theta(t)$?

$$\left.\begin{array}{ll} 1) & \hat{V}^s_{st} = \dot{g}(t)g^{-1}(t) \\ 2) & \left\{ \begin{array}{l} \dot{\theta}(t) = (J^s_{st}(\theta))^{-1}V^s_{st}(t) \\ \theta(0) = \theta_0 \end{array} \right. \end{array}\right\} \Rightarrow \theta(t)$$

◇ *Example: Jacobian for a SCARA manipulator*

$$q_1 = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}, q_2' = \begin{bmatrix} -l_1 s_1 \\ l_1 c_1 \\ 0 \end{bmatrix}, q_3' = \begin{bmatrix} -l_1 s_1 - l_2 s_{12} \\ l_1 c_1 + l_2 c_{12} \\ 0 \end{bmatrix},$$

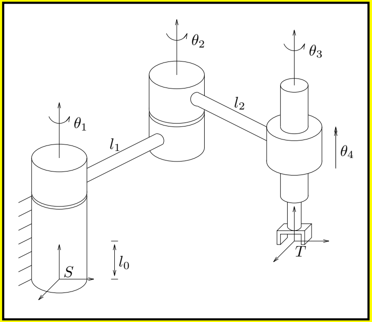
$$\omega_1 = \omega_2' = \omega_3' = [0\ 0\ 1]^T$$

$$\xi_1 = \begin{bmatrix} -\omega_1 \times q_1 \\ \omega_1 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}^T$$

$$\xi_2' = \begin{bmatrix} -\omega_2' \times q_2' \\ \omega_2' \end{bmatrix} = \begin{bmatrix} l_1 c_1 & l_1 s_1 & 0 & 0 & 0 & 0 \end{bmatrix}^T$$

$$\xi_3' = \begin{bmatrix} -\omega_3' \times q_3' \\ \omega_3' \end{bmatrix} = \begin{bmatrix} l_1 c_1 + l_2 c_{12} & l_1 s_1 + l_2 s_{12} & 0 & 0 & 0 & 1 \end{bmatrix}^T$$

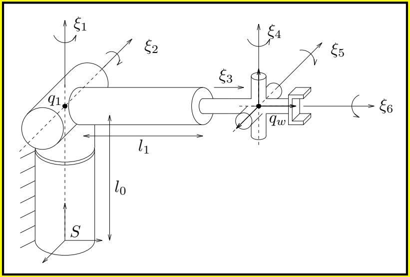
3.3 Manipulator Jacobian



$$\xi_4' = \begin{bmatrix} v_4' \\ 0 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 1 & 0 & 0 & 0 \end{bmatrix}^T$$

$$J_{st}^{s}(\theta) = \begin{bmatrix} 0 & l_1 c_1 & l_1 c_1 + l_1 c_{12} & 0 \\ 0 & l_1 s_1 & l_1 s_1 + l_1 s_{12} & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 1 & 1 & 0 \end{bmatrix}$$

◇ *Example: Jacobian of Stanford arm*



$$q_1 = q_2 = \begin{bmatrix} 0 \\ 0 \\ l_0 \end{bmatrix}, \omega_1 = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \omega_2' = \begin{bmatrix} -c_1 \\ -s_1 \\ 0 \end{bmatrix}$$

$$\xi_1 = \begin{bmatrix} -\omega_1 \times q_1 \\ \omega_1 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}$$

$$\xi_2' = \begin{bmatrix} -\omega_2' \times q_2 \\ \omega_2' \end{bmatrix} = \begin{bmatrix} l_0 s_1 & = l_0 c_1 & 0 & -c_1 & -s_1 & 0 \end{bmatrix}$$

3.3 Manipulator Jacobian

$$\xi_3' = \begin{bmatrix} e^{\hat{z}\theta_1} \cdot e^{-\hat{x}\theta_2} \cdot \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \\ 0 \end{bmatrix} = [-s_1 c_2 \; c_1 c_2 \; - s_2 \; 0 \; 0 \; 0]^T = \begin{bmatrix} v_3 \\ 0 \end{bmatrix}$$

$$q_\omega' = \begin{bmatrix} 0 \\ 0 \\ l_0 \end{bmatrix} + e^{\hat{z}\theta_1} \cdot e^{-\hat{x}\theta_2} \cdot \begin{bmatrix} 0 \\ l_1 + \theta_3 \\ 0 \end{bmatrix} = \begin{bmatrix} -(l_1 + \theta_3) s_1 c_2 \\ (l_1 + \theta_3) c_1 c_2 \\ l_0 - (l_1 + \theta_3) s_2 \end{bmatrix}$$

$$\omega_4' = e^{\hat{z}\theta_1} \cdot e^{-\hat{x}\theta_2} \cdot \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} = \begin{bmatrix} -s_1 s_2 \\ c_1 s_2 \\ c_2 \end{bmatrix}$$

$$\omega_5' = e^{\hat{z}\theta_1} \cdot e^{-\hat{x}\theta_2} \cdot e^{\hat{z}\theta_4} \cdot \begin{bmatrix} -1 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} -c_1 c_4 + s_1 c_2 s_4 \\ -s_1 c_4 - c_1 c_2 s_4 \\ s_2 s_4 \end{bmatrix}$$

$$\omega_6' = e^{\hat{z}\theta_1} \cdot e^{-\hat{x}\theta_2} \cdot e^{\hat{z}\theta_4} \cdot e^{-\hat{x}\theta_5} \cdot \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} = \begin{bmatrix} -c_5(s_1 c_2 c_4) + s_1 s_2 s_5 \\ c_5(c_1 c_2 c_4 - s_1 s_4) - c_1 s_2 s_5 \\ -s_2 c_4 c_5 - c_2 s_5 \end{bmatrix}$$

$$J_{st}^s = \begin{bmatrix} 0 & -\omega_2' \times q_1 & v_3' & -\omega_5' \times q_\omega' & -\omega_5' \times q_\omega' & -\omega_6' \times q_\omega' \\ \omega_1 & \omega_2' & 0 & \omega_4' & \omega_5' & \omega_6' \end{bmatrix}$$

3.3 Manipulator Jacobian

□ *End-effector force:*

$$F_t = \begin{bmatrix} \text{force} \\ \text{torque} \end{bmatrix}$$

$$W = \int_{t_1}^{t_2} V_{st}^b \cdot F_t \mathrm{d}t = \int_{t_1}^{t_2} \dot{\theta} \cdot \tau \mathrm{d}t = \int_{t_1}^{t_2} \dot{\theta}^T (J_{st}^b(\theta))^T \cdot F_t \mathrm{d}t$$

$\Rightarrow \tau = (J_{st}^b)^T F_t = (J_{st}^s)^T F_s$

- Given F_t, what τ is required to balance that force?
- If we apply a set of joint torques, what is the resulting end-effector wrench?

□ *Structural force:*

Structural force: that produces no work on admissible velocity space V^b

$$F^b \cdot V^b = 0, \forall V^b \in \mathrm{Im} J_{st}^b(\theta) \Rightarrow F^b \in (\mathrm{Im} J_{st}^b)^\perp$$

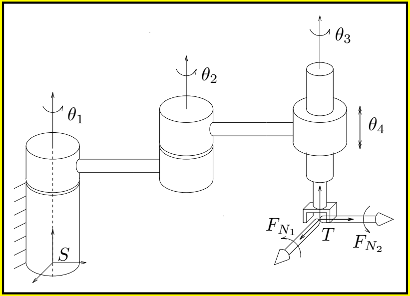
♢ *Review:*

$$\forall A \in \mathbb{R}^{m\times n}, \begin{cases} (\mathrm{Im} A)^\perp = \ker A^T \\ (\ker A)^\perp = \mathrm{Im} A^T \end{cases}$$

$$(\mathrm{Im} J_{st}^b)^\perp = \ker (J_{st}^b)^T, \tau = (J_{st}^b)^T F^b \equiv 0, \forall F^b \in \ker (J_{st}^b)^T$$

3.3 Manipulator Jacobian

◇ *Example: SCARA manipulator*



$$(J_{st}^s(\theta))^T = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 1 \\ l_1c_1 & l_1s_1 & 0 & 0 & 0 & 1 \\ l_1c_1 + l_1c_{12} & l_1s_1 + l_1s_{12} & 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 & 0 & 0 \end{bmatrix}$$

$\ker((J_{st}^s(\theta))^T)$: spanned by

$$F_{N_1} = \begin{bmatrix} 0 & 0 & 0 & 1 & 0 & 0 \end{bmatrix}$$
$$F_{N_2} = \begin{bmatrix} 0 & 0 & 0 & 0 & 1 & 0 \end{bmatrix}$$

3.3 Manipulator Jacobian

□ *Singularities:*

θ is called a singular configuration if there $\exists\dot{\theta} \neq 0$ s.t. $V_{st}^s = J_{st}^s(\theta)\dot{\theta} = 0$

or...A singularity config. is a point θ at which J_{st}^s drops rank.

Consequence: $(n = 6)$

1. Can't move in certain directions.
2. Large joint motion is required.
3. Large structural force.
4. Can't apply end-effector force in certain direction force!

Singularities for 6R-manipulators:

$$J_{st}^s = \begin{bmatrix} -\omega_1 \times q_1 & -\omega_2 \times q_2 & \cdots & -\omega_6 \times q_6 \\ \omega_1 & \omega_2 & \cdots & \omega_6 \end{bmatrix}$$

3.3 Manipulator Jacobian

Case 1: Two collinear revolute joints

$J(\theta)$ is singular if there exists two joints

$$\xi_1 = \begin{bmatrix} -\omega_1 \times q_1 \\ \omega_1 \end{bmatrix}, \xi_2 = \begin{bmatrix} -\omega_2 \times q_2 \\ \omega_2 \end{bmatrix}$$

s.t.

1. The axes are parallel, $\omega_1 = \pm\omega_2$
2. The axes are collinear, $\omega_i \times (q_1 - q_2) = 0, i = 1, 2$

Proof :

Elementary row or column operation do not change rank of $J(\theta)$

$$J(\theta) = \begin{bmatrix} -\omega_1 \times q_1 & -\omega_2 \times q_2 & \cdots \\ \omega_1 & \omega_2 & \cdots \end{bmatrix} \in \mathbb{R}^{6\times n} \xrightarrow{\omega_1=\omega_2}$$

$$J(\theta) \sim \begin{bmatrix} -\omega_1 \times q_1 & -\omega_2 \times (q_2 - q_1) & \cdots \\ \omega_1 & 0 & \cdots \end{bmatrix}$$

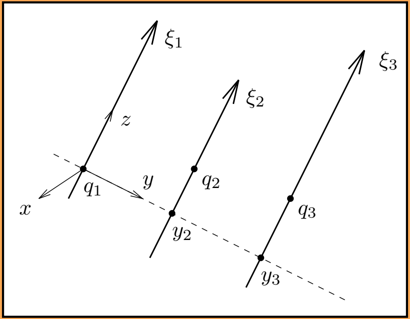
$\square$

Case 2: Parallel coplanar revolute joint axes

$J(\theta)$ is singular if there exists two joints s.t.

1. The axes are parallel, $\omega_i = \pm\omega_j, i, j = 1, 2, 3$
2. The axes are coplanar, i.e. there exists a plane with normal n

s.t. $n^T\omega_i = 0, n^T(q_i - q_j) = 0, i, j = 1, 2, 3$



3.3 Manipulator Jacobian

Proof :

Using change of frame $J \sim \mathrm{Ad}_g J$ and assume

$$J(\theta) = \begin{bmatrix} -\omega_1 \times q_1 & -\omega_2 \times (q_2 - q_1) & \cdots \\ \omega_1 & \omega_2 & \cdots \end{bmatrix},$$

$$\mathrm{Ad}_g J(\theta) = \underbrace{\begin{bmatrix} 0 & \pm y_2 & \pm y_3 & \cdots \\ 0 & 0 & 0 & \cdots \\ 0 & 0 & 0 & \cdots \\ 0 & 0 & 0 & \cdots \\ 0 & 0 & 0 & \cdots \\ 1 & \pm 1 & \pm 1 & \cdots \end{bmatrix}}_{\text{Linearly independent}}$$

Examples are such as the Elbow manipulator in its reference config. $\square$

3.3 Manipulator Jacobian

Case 3: Four intersecting revolute joints axes

$J(\theta)$ is singular if there exists four concurrent revolute joints with intersection point q s.t.:

$$\omega_i \times (q_i - q) = 0, i = 1, \ldots, 4$$

Proof :

Choose the frame origin at q,

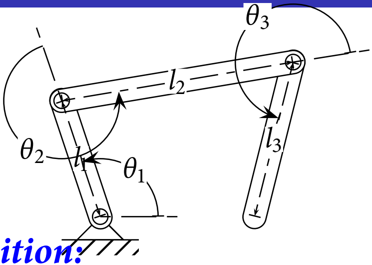
$$p = q_i, i = 1, \ldots, 4$$

$$J(\theta) = \begin{bmatrix} 0 & 0 & 0 & 0 & \cdots \\ \omega_1 & \omega_2 & \omega_3 & \omega_4 & \cdots \end{bmatrix}$$

$\square$

3.3 Manipulator Jacobian

□ *Manipulability:*



$$\begin{bmatrix} \dot{x} \\ \dot{y} \\ \dot{\phi} \end{bmatrix} = J(\theta)\dot{\theta}$$

□ *Singular value decomposition:*

Given $A : \mathbb{R}_n \mapsto \mathbb{R}_m$, assume: $n \leq m$ and $\text{rank}(A) = r \leq n$, then

$$A = U\Sigma V^T = [u_1 \ \cdots \ u_n] \begin{bmatrix} \sigma_1 & & & \\ & \sigma_2 & & \\ & & \ddots & \\ & & & \sigma_n \end{bmatrix} \begin{bmatrix} v_1^T \\ \vdots \\ v_n^T \end{bmatrix}$$

where $U \in \mathbb{R}^{m\times n}$, $V \in \mathbb{R}^{n\times n}$, $\Sigma \in \mathbb{R}^{n\times n}$, $\sigma_1 \geq \cdots \geq \sigma_r > \sigma_{r+1} = \cdots = \sigma_n = 0$ and $U^TU = V^TV = VV^T = I_{n\times n}$.
Moreover, $\text{Im}(A) = \{u_1, \ldots, u_r\}$ and $\ker(A) = \{v_{r+1}, \ldots, v_n\}$.

3.3 Manipulator Jacobian

◇ *Review: Basic facts of a linear transformation*

$$\begin{aligned}
\mathbb{R}^n &= \mathrm{Im}(A^T) \oplus \ker(A) \\
\mathbb{R}^m &= \mathrm{Im}(A) \oplus \ker(A^T) \\
\mathrm{rank}(A) &= \dim(\mathrm{Im}(A)) \\
\mathrm{Im}(A) &= \{u_1, \dots, u_r\} \\
\ker(A) &= \{v_{r+1}, \dots, v_n\} \\
\mathrm{Im}(A^T) &= (\ker A)^{\perp} = \{v_1, \dots, v_r\} \\
\ker(A^T) &= (\mathrm{Im}(A))^{\perp} = \{u_{r+1}, \dots, u_m\}
\end{aligned}$$

(need to span $\{u_1, \dots, u_n\}$ to a full orhogonal set $\{u_1, \dots, u_m\}$)

□ *Redundant manipulator:*

No. of joints > Dimension of Task Space.

E.g.

No. of joints = 3, Task space $\mathbb{R}^2(\dim = 2)$
No. of joints = 7, Task space $SE(3)(\dim = 6)$.

Purpose of Redundancy:

1. Avoid obstacles
2. Optimize some cost criteria.

$$g_{st}(\theta) = e^{\hat{\xi}_1\theta_1} \dots e^{\hat{\xi}_6\theta_6} g_{st}(0),$$

$$J_{st}(\theta) = \begin{bmatrix} \xi_1 & \xi_2' & \dots & \xi_n' \end{bmatrix}, n > p, p = 3, 6$$

Inverse Kinematis:

$g_{st}(\theta(t)) = g_d \; Q_s = \{\theta \in Q | g_{st}(\theta) = g_d\}$: self motion manifold
$T_\theta Q_s \triangleq \{\dot{\theta} \in T_\theta Q | J_{st}^s(\theta)\dot{\theta} = 0\}$: Internal motion

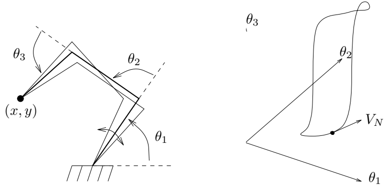
3.4 Redundant and Parallel Manipulators

Given $V_{st} = J_{st}(\theta)\dot{\theta} \Rightarrow \dot{\theta} = \underbrace{J_{st}^{\dagger}(\theta)}_{J^T(JJ^T)^{-1}\text{:Moore-Penrose generalized inverse}} V_{st}$

$$\begin{cases} l_1 c_1 + l_2 c_{12} + l_3 c_{123} = x \\ l_1 s_1 + l_2 s_{12} + l_3 s_{123} = y \end{cases}$$

$$J = \frac{\partial p}{\partial \theta} = \left[\begin{array}{c|c|c} -l_1 s_1 - l_2 s_{12} - l_3 s_{123} & -l_2 s_{12} - l_3 s_{123} & -l_3 s_{123} \\ \hline l_1 c_1 + l_2 c_{12} + l_3 c_{123} & l_2 c_{12} + l_3 c_{123} & l_3 c_{123} \end{array}\right],$$

$$v_N = \begin{bmatrix} l_2 l_3 s_3 \\ -l_2 l_3 s_3 - l_1 l_3 s_{23} \\ l_1 l_2 s_3 + l_1 l_3 s_{23} \end{bmatrix}$$



(a) (b)

3.4 Redundant and Parallel Manipulators

Robot	d.o.f	mass	load	Repeatability	$\frac{\text{load}}{\text{mass}}$
Adept 3	4	205	25	±0.025	0.122
Epson H803N-MZ	4	96	8	±0.02	0.0833
GMF A-600	4	120	6	±0.02	0.05
Pana Robo HR50	4	52	5	±0.02	0.096
Puma RS84	4	130	5	±0.02	0.0384
Sankyo Skilam SR-3C	4	120	6	±0.02	0.05
Sony SRX-3CH	4	64	2	±0.02	0.0625

TABLE 1.1.Characteristics of industrial manipulators (SCARA type, mass of the robot and load capacity in kg, repeatability in mm, according to the manufacturers notice).

3.4 Redundant and Parallel Manipulators

Robot	mass	load	Repeatability	$\frac{\text{load}}{\text{mass}}$
Acma SR 400	430	10	±0.1	0.0232
Acma YR500	590	30	±0.2	0.0508
ABB IRB L6	145	6	±0.2	0.0414
ABB IRB 2000	370	10	±0.1	0.027
CM $T^3$646	1500	22	±0.25	0.0147
CM $T^3$786	2885	90	±0.25	0.0312
GMF Arc Mate	120	5	±0.2	0.0417
GMF S 10	200	10	±0.1	0.05
Hitachi M6100	405	6	±0.2	0.0148
Hitachi M6100	410	10	±0.1	0.0243
Kuka IR 163/65	1700	60	±0.5	0.0353
Kuka IR 363/15	290	8	±0.1	0.0276
Puma 550	63	4	±0.1	0.0634
Puma 762	590	20	±0.2	0.0338

TABLE 1.2.Characteristics of industrial manipulators (spherical type, mass of the robot and load capacity in kg, repeatability in mm, according to the manufacturers notice).

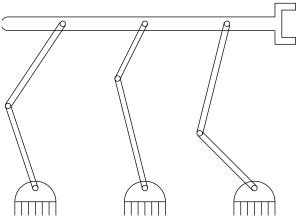
□ *Parallel Manipulator:*

Delta manipulator   Stewart manipulator

3.4 Redundant and Parallel Manipulators

□ *Structure equation:*

$$
\begin{aligned}
g_{st}(\theta) &= e^{\hat{\xi}_{11}\theta_{11}} \dots e^{\hat{\xi}_{1n_1}\theta_{1n_1}} g_{st}(0) \\
&= e^{\hat{\xi}_{21}\theta_{21}} \dots e^{\hat{\xi}_{2n_2}\theta_{2n_2}} g_{st}(0) \\
&= e^{\hat{\xi}_{k1}\theta_{k1}} \dots e^{\hat{\xi}_{kn_k}\theta_{kn_k}} g_{st}(0)
\end{aligned}
$$



$$E = \{(\theta_{11}, \dots, \theta_{1n_1}, \dots, \theta_{k1}, \dots, \theta_{kn_k})\} \text{ (free cfg. space)}$$

$$Q = \{\theta \in E | g_{st}(\theta) = g^1_{st}(\theta_1) = g^k_{st}(\theta_k)\} \text{ (actual cfg. space)}$$

Computation of dimension of Q: (DOF)

□ *Greubler's formula:*

$$F = 6N - \sum_{i=1}^{g}(6 - f_i) = 6(N - g) + \sum_{i=1}^{g} f_i$$

g =No. of joints, f_i =DOF of the i^{th} joint, N = No. of links

3.4 Redundant and Parallel Manipulators

Chapter 3 Manipulator Kinematics
Forward kinematics
Inverse Kinematics
Manipulator Jacobian
Redundant and Parallel Manipulators

Using the above equation to the manipulator shown in the figure:

$$g = 9 \quad f_t = 1 \quad N = 7 \Rightarrow F = 3(7-9) + \sum_{i=1}^{9} 1 = 3(-2) + 9 = 3$$

□ ***Parametrization of Q (the process of designating active joints):***

$$\text{Velocity Relation: } V_{st}^s = J_s^1 \cdot \dot{\theta}_1 = J_s^2 \cdot \dot{\theta}_2 = \cdots = J_s^k \cdot \dot{\theta}_k$$

$$\begin{bmatrix} J_1 & & \\ & \ddots & \\ & & J_k \end{bmatrix} \begin{bmatrix} \dot{\theta}_1 \\ \vdots \\ \dot{\theta}_k \end{bmatrix} = \begin{bmatrix} I \\ \vdots \\ I \end{bmatrix} V_{st}^s \Rightarrow \begin{cases} J \cdot \dot{\theta} = I' \cdot V_{st}^s \\ J_a \cdot \dot{\theta}_a + J_p \cdot \dot{\theta}_p = I' \cdot V_{st}^s \end{cases}$$

3.4 Redundant and Parallel Manipulators

$$\begin{aligned} \mathrm{Im}(J_1^s): &\quad \text{Allowed velocity by chain 1} \\ \mathrm{Im}(J_1^s) \cap \cdots \cap \mathrm{Im}(J_k^s): &\quad \text{Allowed velocity} \\ g: &\quad \text{Rank of structure equation} \end{aligned}$$

$$g = \dim \cap_{i=1}^{k} \mathrm{Im}(J_i(\theta))$$

Definition: Configuration Space Singularity

A point $\theta \in Q$ s.t.the structure equation drops rank.

$$\begin{aligned} J_1(\theta_1)\dot{\theta}_1 - J_2(\theta_2)\dot{\theta}_2 &= 0 \\ J_1(\theta_1)\dot{\theta}_1 - J_3(\theta_3)\dot{\theta}_3 &= 0 \\ &\vdots \\ J_1(\theta_1)\dot{\theta}_k - J_k(\theta_k)\dot{\theta}_k &= 0 \end{aligned} \Rightarrow \begin{bmatrix} J_1(\theta_1) & -J_2(\theta_2) & & & \\ J_1(\theta_1) & 0 & -J_3(\theta_3) & & \\ \vdots & & & \ddots & \\ J_1(\theta_1) & \dots & & & -J_k(\theta_k) \end{bmatrix} \begin{bmatrix} \dot{\theta}_1 \\ \dot{\theta}_2 \\ \vdots \\ \dot{\theta}_k \end{bmatrix} = 0$$

$$J(\theta)\dot{\theta} = 0$$

How many constraint equations?: $p(k-1)$

E.g. $k = 1, p = 3 \Rightarrow 3$
$k = 3, p = 3 \Rightarrow 6$
$k = 6, p = 6 \Rightarrow 6 \times 5 = 30$

$n = DOF, \dim Q = \underbrace{\dim Q}_{n}$ –No. of independent constrains

Definition: Configuration Space Singularity
A point $\theta \in Q$ s.t. $\mathrm{rank}(J(\theta))$ drops below its full rank.

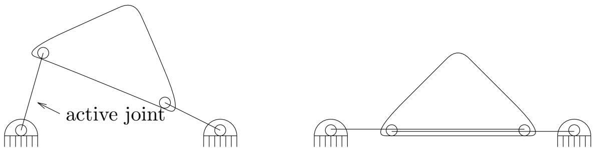
◇ *Example: 4-bar mechanism*

$$-r - l_1 \sin\theta_{11} + r\cos(\theta_{11}+\theta_{12}) = x = r - l_2 \sin\theta_{21} - r\cos\theta_{21} + \theta_{22}$$
$$l_1 \cos\theta_{11} + r\sin(\theta_{11}+\theta_{12}) = y = h + l_2\cos\theta_{21} - r\sin(\theta_{21}+\theta_{22})$$

3.4 Redundant and Parallel Manipulators

$$\theta_{11} + \theta_{12} = \phi = \theta_{21} + \theta_{22}$$

$$\begin{bmatrix} -l_1 \cos\theta_{11} - r\sin\theta_{11} & -r\sin\theta_{12} & -l_2\cos\theta_{21} + r\sin\theta_{21} & r\sin\theta_{22} \\ l_1\sin\theta_{11} + r\cos\theta_{12} & r\cos\theta_{12} & -l_2\sin\theta_{21} + r\cos\theta_{22} & r\cos\theta_{22} \\ 1 & 1 & -1 & -1 \end{bmatrix} \dot{\theta} = 0$$



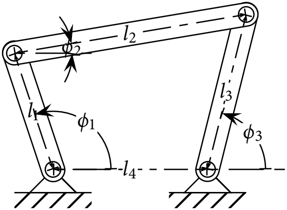
(a) Singular configuration

(b) Uncertainty configuration

3.4 Redundant and Parallel Manipulators

□ *An alternative method:*

$$
\begin{aligned}
l_1 \cos\phi_1 + l_2 \cos\phi_2 - l_3 \cos\phi_3 - l_4 &= 0 \\
l_1 \sin\phi_1 + l_2 \sin\phi_2 - l_3 \sin\phi_3 &= 0 \\
Q &= (\phi_1, \phi_2, \phi_3)
\end{aligned}
$$



$$
\underbrace{\begin{bmatrix} -l_1 \sin\phi_1 & -l_2 \sin\phi_2 & l_3 \sin\phi_3 \\ l_1 \cos\phi_1 & l_2 \cos\phi_2 & -l_3 \cos\phi_3 \end{bmatrix}}_{J(\theta)} \begin{bmatrix} \dot\phi_1 \\ \dot\phi_2 \\ \dot\phi_3 \end{bmatrix} = 0
$$

$$
J(\phi) \text{ drop rank!} \begin{cases} \phi_1 - \phi_2 = \pi, \text{ or } 0 \\ \phi_2 - \phi_3 = \pi, \text{ or } 0 \\ \phi_1 - \phi_3 = \pi, \text{ or } 0 \end{cases}
$$

3.4 Redundant and Parallel Manipulators

If $J(\phi)$ is of normal rank, the configuration space of the system is:

$$Q = f^{-1}(0), \text{ where } f = 0 \text{ is the close loop constraint}$$

How to parameterize Q by active joints: θ_a, θ_p

$$J_1(\theta)\dot{\theta}_a + J_2(\theta)\dot{\theta}_p = 0$$

Actuator singularity if rank $J_2(\theta) <$ normal rank of $J_2(\theta)$.