- Probability Concepts
    - 1. Conditional & Joint Probabilities
    - 2. Conditional Expectations & Expected Value
    - 3. Portfolio Variance, Bayes, and Counting Problems

Probability Concepts

```
Key Words: mutually exclusive, exhaustive events, odds,
unconditional/conditional probability, joint probability,union,
dependent/independent events, total probability rule, RV expected value and
variance, portfolio mean and variance, covariance matrix, correlation
matrix, Byes, Counting, factorial, combination, permutation
```

1. Conditional & Joint Probabilities

- Some concepts:
    - Event: outcome
    - Mutually exclusive events:cannot happen both simultaneously
    - Exhaustive events: include all possible outcomes
- Probability Property:
    - $0 \leq P(E_i) \leq 1$
    - $\sum P(E_i) = 1$
- Types of Probabilities:
    - Empirical:get by analyzing data(outcome)
    - Prior:get by using formal reasoning or inspection process推理
    - Subjective: the least formal method with personal judgment带有主观性
    - Objective: includes empirical or prior probabilities
- Odds: the event will or will not occur
    - format: the odd **for** event occur is $P(occur)/P(not\ occur)$, on the other hand, the odd **against** the event occur take the reciprocal.

- With the given odd for/against, we can also get the **probability** of occurrence or not occurrence.

- Conditional Probability:

  - **Unconditional Probability**(Marginal Probability): the event probabilities without any past or future occurrence of other events.不受未来或过去任何其他事件的影响。
  - **Conditional Probability**(likelihood): occurrence of one event affects that of another.
    - $P(A|B)$, the occurrence of event A, with the given occurrence of event B

- Multiplication and Addition Rules

  - Addition:
    - **Union**: at least one of two events will occur
      - $P(A \cup B) = P(A) + P(B) - P(AB)$
      - For mutually exclusive events: $P(A \cup B) = P(A) + P(B)$
  - Multiplication:
    - **Joint Probability**: two events will both occur
      - $P(AB) = P(A|B) * P(B)$
      - $P(A|B) = \frac{P(AB)}{P(B)}$, with 1 unconditional probability*1 conditional probability
      - $P(A \cap B) = P(A) * P(B)$, for independent events

2. Conditional Expectations & Expected Value

- Dependent & Independent Events

  - Independent event:
    - Definition: the occurrence of one has *no* effects on the occurrence of the others
    - Function: $P(A|B) = P(A)$ or $P(B|A) = P(B)$
  - Else, they are dependent events.

- Total Probability Rule

- Function: $P(A) = P(A|B_1)P(B_1) + P(A|B_2)P(B_2) + ... + P(A|B_N)P(B_N)$ $\leftarrow P(A) = P(AB_1)(AB_2)...(AB_N)$
  - $P(A)$: unconditional probability
  - $\{S_1, S_2, ..., S_N\}$: set of events which are <u>mutually exclusive and exhaustive</u>.
- Application:
  - Conditional Expected Values can be used in Financial investment.
  - Probability Tree: show the probabilities of various outcomes

- Expected Value, Variance, and STD of Random Variables

  - **Expected Value**: weighted average of possible outcomes of the variable
    - Function: $E(X) = \sum_{i=1}^{n} P(x_i)x_i = P(x_1)x_1 + P(x_2)x_2 + ... + P(x_n)x_n$
      - X: event
      - x_i: possible outcomes
  - **Volatility**: Variance & Standard Deviation
    - Function:
      - $Var(X) = \sum_{i=1}^{n} P[(x_i - E(X))^2]$
        - Probability-weighted sum of the squared deviations from the mean
      - $std(X) = \sqrt[2]{\sum_{i=1}^{n} P[(x_i - E(X))^2]}$

3. Portfolio Variance, Bayes, and Counting Problems

- Portfolio expected return:

  - $R_p = \sum_{i=1}^{n} w_i E(R_i)$
    - w_i=market value of asset i/that of entire portfolio

- Covariance with Assets:

  - How two assets move together.
  - $Cov(R_i, R_j) = E\{[R_i - E(R_i)][R_j - E(R_j)]\}$
  - Covariance Properties:
    - $Cov(R_i, R_i) = Var(R_i)$
    - $Cov(R_i, R_j) = Cov(R_j, R_i)$, the order is not important
    - range: $[-\infty, \infty]$

        - Cov>0, means one random variable above the mean, the others trend to **above** the mean as well.
        - Cov<0, means one random variable above the mean, the others trend to **below** the mean.
    - Sample Covariance for sample return data:
        - $s_{X,Y} = \frac{\sum_{i=1}^{n}[(R_{1,i}-\bar{R}_1)(R_{2,i}-\bar{R}_2)]}{n-1}$
            - $R_{1,i}$:an observation of returns on asset 1
            - $\bar{R}_1$: mean return of asset 1
    - Covariance Matrix:
        - shows the covariances between returns on group of assets(n*n)
        - elements on <u>diagonal</u> are the **variance** of each asset's return
        - For n assets, we have n variances, and $n(n-1)/2$ unique covariance terms in matrix.

- Portfolio Variance:

    - $Var(R_p) = \sum_{i=1}^{N}\sum_{j=1}^{N} Cov(R_i, R_j)$
    - if N=2, $Var(R_p) = w_A w_B Cov(R_A, R_A) + w_A w_B Cov(R_A, R_B) + w_B w_A Cov(R_B, R_A) + w_B w_B Cov(R_B, R_B)$ or
        - $\sigma_p^2 = w_A^2\sigma_A^2 + w_B^2\sigma_B^2 + 2w_A w_B Cov_{AB}$
    - if N=3, $\sigma_p^2 = w_A^2\sigma_A^2 + w_B^2\sigma_B^2 + w_C^2\sigma_C^2 + 2w_A w_B Cov_{AB} + 2w_A w_C Cov_{AC} + 2w_B w_C Cov_{BC}$
    - the units of Variance and Covariance is %^2
    - The lower covariance between assets, the lower portfolio variance

- Correlation:

    - use *correlation matrix* to solve
    - Function: $\rho_{AB} = \frac{Cov_{AB}}{\rho_A \rho_B}$
    - the diagonal items=1

- Bayes:

    - Logic:

```
Updated Probability=(probability of new information for a given event/unconditional probability of new information) * prior probability of event
```

- Function:
  - $P(B|A) * P(A) = P(B \cap A)$
  - $P(A|B) * P(B) = P(A \cap B)$
  - since, $P(AB) = P(BA)$, we get:
    - $P(B|A) * P(A) = P(A|B) * P(B)$
    - $\frac{P(B|A)P(A)}{P(B)} = \frac{P(B \cap A)}{P(B)}$

- Counting
  - Labeling: there are n items that each one receives one of K different labels.
    - Total numbers of ways that the label can be $\frac{n!}{(n_1!)(n_2!)*...*(n_k!)}$
      - n_1: the number of items receiving label 1
      - $\sum_{i=1}^{k} n_i = n$
      - k: number of labels
    - Financial calculator: to compute n!, `[n] [2nd] [x!]`
  - Combination:
    - Intuition: the number of ways selecting r items from a set of n items when the order is not important.
    - when $k = 2$, $r = n_1$, and $n_2 = n - r$, the special case of original counting rule
    - Function: $C_k^n = \frac{n!}{(n-r)!r!}$
    - Financial Calculator: `[n] [2nd] [nCr] [r] [=]`
  - Permutation:
    - Intuition: the number of ways selecting r items in special order from n items
    - Function: $P_k^n = \frac{n!}{(n-r)!}$
    - Relationship: $P_k^n = r! * C_k^n$
    - Financial Calculator: `[n] [2nd] [nPr] [r] [=]`