# The Kelly Criterion and the Stock Market

Nikhil Khanna

June 2016

## Contents



## 1 Introduction

When considering a simple favorable bet (e.g. flipping a coin for money), a gambler must do more than make the binary decision of whether or not to play, he must also determine an appropriate amount of money to bet. The Kelly Criterion, provides a theory of optimal resource allocation when it comes to favorable bets and thus provides a criterion for bet size in a gamble to maximize long run utility. We will first explore the Kelly Criterion understanding both its derivation (A modified version of the derivation presented in Thorpe [1]) and the intuition we can gain from it about optimal bets. We will then discuss a practical application of the Kelly Criterion in determining investment amounts in the U.S. stock market to maximize gain.

## 2 Definitions

- **Random Variable**: A random variable is a function that values to each of an experiment's outcomes. For example a random variable $X$ could represent the number of heads I get in 5 flips of a coin. A random variable could be either discrete or continuous depending on the values it can take.

- **Probability distribution**: A probability distribution is some function that defines probabilities associated with all outcomes of a random variable.

- **Expectation**: For a discrete random variable, $X$ the expectation is $E[X] = \sum_{i=1}^{\infty} x_i * P(X = x_i)$, where $x_i$ is a possible value of $X$. Analogously for a continuous random variable, $Y$, the expectation is $E[Y] = \int_{-\infty}^{\infty} yf(y)dy$, where $f(y)$ is the probability density function at $y$. Additionally, one useful fact for expectation calculations is the **linearity of expectation**, the fact that expectation is a linear function (as we can see from the definitions), so for random variables $X$ and $Y$ and constants $a$ and $b$, $E[aX + bY] = aE[X] + bE[Y]$.

- **Variance**: For a random variable, $X$ the variance of $X$ is defined to be $Var(X) = E[(X - E[X])^2]$. Intuitively variance provides a measure of how spread out the values of a random variable are from its expectation (mean).

- **Standard Deviation**: For a random variable, $X$ with variance $Var(X) = \sigma^2$, we have that the standard deviation of $X$ is $\sqrt{Var(X)} = \sigma$. This measure tells us how far from the mean we should expect to see values of this random variable fall.

## 3 Motivation

Say we are playing a coin-flipping betting game against an infinitely wealthy opponent, where each time the coin is flipped you place a bet on the coin landing heads, receiving the amount you bet if the coin lands heads, and losing the amount you bet if the coin lands tails. However say the coin is biased with a probability $1 > p > \frac{1}{2}$ chance of landing heads and $q = 1 - p$ chance of landing tails. This is obviously a favorable bet for you to make as if $b$ is the amount you bet and $X$ is a **random variable** representing the total amount you gain, we can calculate the **expectation** of $X$ to see that

$$E[X] = p * b - q * b = (p - q) * b > 0$$

Because this is a positive quantity, it is definitely worth it to play this game. Let $X_i$ be a **random variable** representing the amount you have after $i$ trials, and say that at the beginning of the game, you have $X_0$ dollars. The problem we are faced with is trying to determine the optimal $B_i$, the amount you will bet on the $i$th trial. First we will examine a seemingly obvious strategy of naively trying to maximize the expectation of this game. The expected amount you will have after n trials of this game is

$$E[X_n] = X_0 + \sum_{i=1}^{n}(p - q)B_i$$

, however, we know that because $p > \frac{1}{2}$, we have that $q < p$ and so $p - q$ must be positive. Thus we maximize the expected amount of money by maximizing $B_i$; however, in the context of betting, this amounts to betting as much money as we have (which is the maximum we can bet), so $B_i = X_{i-1}$. We can trivially see that this strategy does not make much sense in the real world as if we were to lose only a single game, we would be bankrupt and the probability of losing a game after n trials is $1 - p^n$ which goes to 1 as $n \to \infty$; using this betting strategy, you are ensuring your ruin in the long run!

So maximizing expectation evidently does not yield the optimal betting amount in the long run, as it ensures ruin; additionally, if we were to instead play so as to minimize the chance of ruin, we would also minimize the expected gain so neither of these strategies seems feasible. This is where the Kelly Criterion enters the picture.

# 4 The Kelly Criterion

## 4.1 Main Idea

In the gambling game we just described, the gambling probability and payoff per bet do not change, and thus, from an intuitive standpoint, it would make sense that an optimal solution would bet the same fraction, $f$, of your money for every trial. The Kelly Criterion follows from this intuition (in order to allow for this type of fractional betting, we will assume that money is infinitely divisible). Following this strategy, after n trials if we were to call $S$ the number of successes and $F$ the number of failures (so $S + F = n$) we would have that

$$X_n = X_0(1 + f)^S(1 - f)^F$$

. We have that $e^{n\log(\frac{X_n}{X_0})^{(1/n)}} = \frac{X_n}{X_0}$ meaning that $\log(\frac{X_n}{X_0})^{(1/n)}$ measures the exponential rate of increase per trial. For the Kelly Criterion, we are concerned with maximizing the expectation of this growth.

## 4.2 Deriving The Criterion

We define the growth rate coefficient to the the expectation of the exponential rate of increase per trial

$$G(f) = E[\log(\frac{X_n}{X_0})^{(1/n)}] = E[\frac{S}{n}\log(1 + f) + \frac{F}{n}\log(1 - f)]$$

. We can compute this expectation by noticing that $S$ is a **Binomial random variable** with parameters $n$ and $p$ and thus $E[S] = np$. Similarly $E[F] = n(1 - p) = nq$. Via the linearity of expectation, we can then get that

$$G(f) = p\log(1 + f) + q\log(1 - f)$$

. We now will attempt to maximize $G(f)$ via simple calculus. We can easily calculate that

$$G'(f) = \frac{p}{1 + f} - \frac{q}{1 - f} = \frac{p - q - f}{(1 + f)(1 - f)}$$

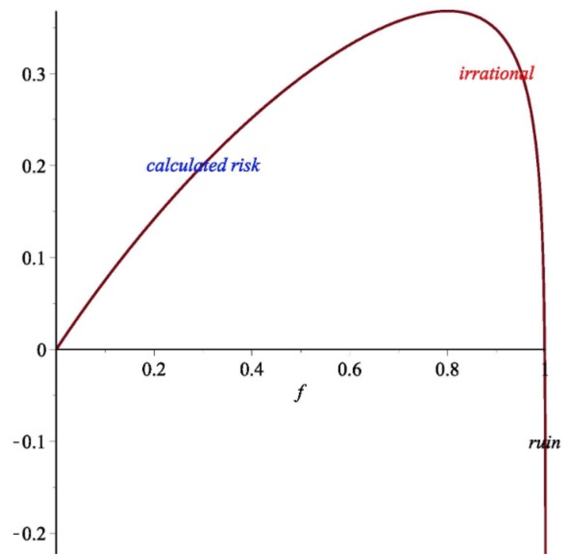
and by setting $G'(f) = 0$ and solving for $f$ we can find that $f = p - q$. We can further verify that this is indeed a maximum by taking a second derivative to find that

$$G''(f) = \frac{-f^2 + 2f(p-q) - 1}{(1-f^2)^2}$$

which is less than 0 at $f = p - q$ so we have a local maximum. We can further confirm this is a global maximum by noting that it is the only critical point and $G(0) = 0$ while as $\lim_{f \to 1^-} G(f) = -\infty$. So we have that $G$ is maximized at $f = p - q$ (which we will now call $f^*$). This means that, on each trial, you want to bet the fraction $f^* = p - q$ of your total money in order to maximize the expected exponential rate of increase.

### 4.3 Intuition and Further Analysis

So we have that $G$ is maximized at $f = f^* = p - q$. Further examining some properties of $G$, we can see that $G(0) = 0$, so, as we can intuitively guess, the exponential rate of growth will be 0 if we bet 0$ and $\lim_{f \to 1^-} G(f) = -\infty$ meaning that, as we saw earlier, betting all your money is guaranteed to lose you all your money. We can also note that because $G(f^*)$ is a maximum (which is positive), as $f \to 1$ we have $G(f) \to -\infty$, and $G$ is continuous, there must also be some crossover point, call it $f_c$, where $G(f_c) = 0$. You can clearly see both $f^*$ and $f_c$ in the graph below where $f^*$ is the peak of the curve and $f_c$ is the point at the rightmost x-intercept.




One interesting thing to note about $G$ that gives us some additional intuition is that we can rewrite $G(f) = \frac{1}{n}E(\log X_n) - \frac{1}{n}(\log X_0)$. However, $X_0$, the amount of money you start with is given, so for $n$ fixed, we can see that maximizing $G(f)$ is the same as maximizing $E(\log X_n)$. So, for a favorable trial with a probability of success of $p$ and $q = 1 - p$, the optimal fraction of your money to bet $p - q$, is exactly the quantity that maximizes $E\log(X_n)$ under this fractional betting scheme.

There are several additional useful properties of the criterion, such as the fact that when $G(f) > 0$ then $\lim_{n\to\infty} X_n = \infty$ (any favorable growth rate will cause the money to infinity in the long run), when $G(f) < 0$ then $\lim_{n\to\infty} X_n = 0$ (on the other hand, a negative growth rate will cause the money to shrink to 0). We can also prove that any strategy other than maximizing $E\log(X_n)$ is worse than ours (formally that $\lim_{n\to\infty} \frac{X_n(\Phi^*)}{X_n(\Phi)} = \infty$ where $\Phi^*$ is a strategy that maximizes $E\log(X_n)$ and $\Phi$ is essentially any other strategy), and finally, that the expected time for your "running capital" $X_n$ to reach any predefined number $\mathbf{X}$, is least with a strategy like ours that maximizes $E(\log X_n)$.

## 5 Stock Market

### 5.1 Idea

We can view investing in the stock market as a continuous gambling game, and, as such, we will now examine how we can apply the Kelly Criterion to the stock market. Suppose we have initial capital $X_0$ and we want to determine the optimal betting fraction $f^*$ to invest each year in S&P 500 stocks. However, unlike in the previous situation we examined the Kelly Criterion for, there is not a finite number of outcomes of a bet on a security, so, we will use a continuous probability distribution as opposed to the discrete one we were using previously. Additionally to simplify this analysis a bit, we will ignore inflation, broker's fees, tax considerations, and other ancillary factors.

### 5.2 Derivation

Here we will present a modified version of the proof given in Thorpe [2]. We will let $X$ be the random variable representing the return per unit. We will also assume that

$$P(X = \mu + \sigma) = P(X = \mu - \sigma) = .5$$

where $\mu = E[X]$, the mean, and $\sigma^2 = Var(X)$ which means that $\sigma$ is the standard deviation. In plain English, this assumption is simplifying the outcomes of investing in the stock market to two separate equally likely outcomes. Either your investment is successful in which case your return per unit is a standard deviation above the mean, otherwise it is unsuccessful in which case your investment is a standard deviation below the mean.
Now, following analogous steps to before, if we say that the initial capital you have is $V_0$, we can see that the capital is given by

$$V(f) = V_0(1 + (1 - f)r + fX)$$

where $r$ is the rate of return of capital invested elsewhere (for instance in treasury bonds or a similar stable investment). We can then say $G(f) = E[\log(\frac{V(f)}{V_0})]$ and via the linearity of expectation we find that

$$G(f) = 0.5\log(1 + r + f(\mu - r - \sigma))$$

. Now we will divide the time interval into $n$ equal steps in order to get a situation with $n$ trials analogous to what we had in the previous section.Now instead of having a single $X$, we will have $n$ separate independent $X_i$'s with mean $\frac{\mu}{n}$ and variance $\frac{\sigma^2}{n}$ and similar to before we have that

$$P(X_i = \frac{\mu}{n} + \frac{\sigma}{\sqrt{n}}) = P(X_i = \frac{\mu}{n} - \frac{\sigma}{\sqrt{n}}) = .5$$

We can compute the ratio between our final capital and our initial capital as

$$\frac{V_n}{V_0} = \prod_{i=1}^{n}(1 + (1-f)r + fX_i)$$

. We now have that the expected exponential rate of growth is $G_n(f) = E[\log(\frac{V_n}{V_0})]$ and we will attempt to find the limit of this function as $n \to \infty$ to find the instantaneous rate of growth. We can expand this expression for $G_n(f)$ into the Taylor series around $f = 0$ to get

$$G_n(f) = r + f(\mu - r) - \frac{\sigma^2 f^2}{2} + O(n^{-(1/2)})$$

. Now we can the limit as $n \to \infty$ and see that the $O(n^{-(1/2)})$ terms vanish so we can say that the instantaneous rate of growth is

$$G_\infty(f) = r + f(\mu - r) - \frac{\sigma^2 f^2}{2}$$

. Now, as before, we can use simple calculus to maximize this quantity and can find that $f^* = \frac{\mu - r}{\sigma^2}$. Thus, under our simplifying assumptions we are able to apply the Kelly Criterion to the stock market and were able to find the optimal amount to bet.

### 5.3 Example

Now examining actual market data, as given in Rotando [1], the mean return on a blue chip stock from the S&P 500 for a 59 year period were $\mu = .058$ and the standard deviation of the returns were $\sigma = .216$. Additionally, we will assume that the alternative investment is a treasury bond, which has an average rate of return of .029 over that time period. Thus as per our equation we have that $f^* = \frac{.058 - .029}{.216^2} \approx .62$. Thus as per the Kelly criterion, we should be willing to invest 62% of our money in a blue chip stock over the period of time and put the rest in Treasury bonds. Obviously, there were several simplifying assumptions made in this calculation and care should be taken to examine how closely the stock market actually resembles how we modeled it here before immediately investing in the market based off this criterion.

## 6 Conclusion

The Kelly criterion provides a theory backing optimal bet amounts. We explored the Kelly Criterion in the context of a discrete game whose outcome was simple win or lose coin flip and then expanded the discussion to the context of a continuous gambling game based on the stock market. Overall the Kelly criterion provides a useful guide to how to structure bets in this sort of game, but, care should be taken to fully understand how the real world (particularly the stock market) actually meets the assumptions we made when deriving the Kelly criterion.

## 7 References

[1] L. M. Rotando, and E. O. Thorpe, The Kelly Criterion and the Stock Market, The American Mathematical Monthly, Vol. 99, No. 10 (Dec., 1992), 922-931

[2] E. O. Thorpe, The Kelly Criterion in Blackjack, Sport Betting, and The Stock Market. Paper presented at the 10th International Conference on Gambling and Risk Taking, Montreal, 2007