Financial Econometrics
Lecture 5: Modelling Volatility and Correlation

Dayong Zhang
Research Institute of Economics and Management

Autumn, 2011

Learning Outcomes

- Discuss the special features of financial data, motivate the use of ARCH models
- Test for ARCH effect in time series data
- Contrast various models from the GARCH family
- Maximum likelihood estimation
- Construct multivariate conditional volatility models

Financial Regularities

The linear structural (and time series) models cannot explain a number of important features common to much financial data. **Volatility clustering**: "Large changes tend to follow by large changes, ..., and small changes tend to follow by small changes...". For example:
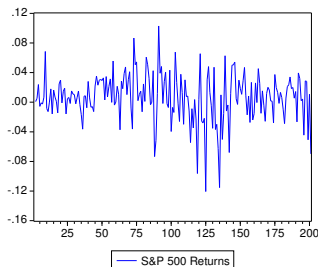
Financial Regularities Cont'd
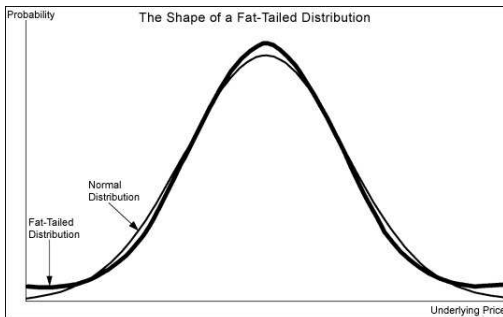
Leptokurtosis: fat tails or thick tails. Kurtosis measures the peakedness or flatness of the distribution of the series. Kurtosis is computed as

$$K = \frac{1}{N} \sum_{i=1}^{N} \left(\frac{y_i - \bar{y}}{\hat{\sigma}} \right)^4 \tag{1}$$

It equals 3 for a normal distribution and above 3 for leptokurtosis, where the distribution is peaked relative to the normal distribution.

Leptokurtosis: example

Financial Regularities Cont'd

Leverage effects: that is changes in stock prices tend to be negatively correlated with changes in volatility, first noted by Black (1976). A firm with debt and equity outstanding typically becomes more highly leveraged when the value of the firm falls. This raises equity returns volatility if the returns on the firm as a whole are constant.

Co-movements in volatilities: volatility changes are not only closely linked across asset within a market, but also across markets.

EWMA Model

Standard estimation of volatility gives equal weight to each observation:

$$\sigma_t^2 = \frac{1}{m-1} \sum_{i=1}^{m} (r_t - \bar{r})^2 \qquad (2)$$

However, it makes more sense to give more weight to the recent data. The EWMA model assumes the weights α_t decrease exponentially as we move back through time. Specifically, $\alpha_t = \lambda \alpha_{t-1}$, where λ is a constant between zero and one.

EWMA Model Cont'd

This leads to a particularly simple formula for updating volatility estimates:

$$\sigma_t^2 = \lambda \sigma_{t-1}^2 + (1 - \lambda) r_{t-1}^2 \tag{3}$$

To understand this formula, we can use backward substitution to find out that

$$\sigma_t^2 = (1 - \lambda) \sum_{i=1}^{m} \lambda^{i-1} r_{t-i}^2 + \lambda^m \sigma_{t-m}^2 \tag{4}$$

The last term is small enough to be ignored if m is large.

EWMA Model Cont'd

EWMA is attractive in that only relatively little data is required. It is designed to track changes in the volatility.

The RiskMetrices database, which was created by J.P. Morgan and made publicly available in 1994, uses EWMA with $\lambda = 0.94$ for updating daily volatility estimates.

Motivations of Using ARCH
ARCH Models
GARCH Models
Multivariate GARCH models

Definition of ARCH Models
Testing for ARCH Effects

Definition

The autoregressive conditional heteroskedastic (ARCH) class of models was introduced by Engle (1982). Arising from the use of conditional versus unconditional mean, the key insight offered by the ARCH model lies in the distinction between conditional and the unconditional second moments.

While the unconditional covariance matrix for the variables of interest may be time invariant, the conditional variances and covariances often depend non-trivially on the past states of the world.

Motivations of Using ARCH
ARCH Models
GARCH Models
Multivariate GARCH models

Definition of ARCH Models
Testing for ARCH Effects

Definition Cont'd

For a covariance-stationary AR(p) process:

$$y_t = c + \phi_1 y_{t-1} + \ldots + \phi_p y_{t-p} + u_t \qquad (5)$$

where u_t is white noise:

$$
\begin{aligned}
E(u_t) &= 0 & (6) \\
E(u_t u_\tau) &= \left\{ \begin{array}{ll} \sigma^2 & \text{for } t = \tau \\ 0 & \text{otherwise} \end{array} \right. & (7)
\end{aligned}
$$

Motivations of Using ARCH
ARCH Models
GARCH Models
Multivariate GARCH models

Definition of ARCH Models
Testing for ARCH Effects

Conditional Vs. Unconditional Mean

The unconditional mean for the series is a constant given by

$$E(y_t) = \frac{c}{1 - \phi_1 - \ldots - \phi_p} \tag{8}$$

However, the conditional mean for y_t is the linear projection:

$$E(\hat{y}_t | y_{t-1}, y_{t-2}, \ldots) = c + \phi_1 y_{t-1} + \ldots + \phi_p y_{t-p} \tag{9}$$

which is changing over time.

Motivations of Using ARCH
ARCH Models
GARCH Models
Multivariate GARCH models

Definition of ARCH Models
Testing for ARCH Effects

ARCH Process

The unconditional variance of u_t is the constant of σ^2 according to the definition, but the conditional variance of u_t could change over time. And if the square of u_t following an AR(m) process:

$$u_t^2 = \omega + \alpha_1 u_{t-1}^2 + \ldots + \alpha_m u_{t-m}^2 + v_t \tag{10}$$

where v_t is another white noise process, we can say u_t is an *autoregressive conditional heteroskedastic* process of order m, or $u_t \sim ARCH(m)$.

Motivations of Using ARCH
ARCH Models
GARCH Models
Multivariate GARCH models

Definition of ARCH Models
Testing for ARCH Effects

Another Way

It is often convenient to use an alternative expression for an ARCH(m) process such as:

$$u_t = \sqrt{h_t} \cdot v_t \tag{11}$$

where $\{v_t\}$ is an i.i.d sequence with zero means and unit variance and h_t evolves according to

$$h_t = \omega + \alpha_1 u_{t-1}^2 + \ldots + \alpha_m u_{t-m}^2 \tag{12}$$

Motivations of Using ARCH
ARCH Models
GARCH Models
Multivariate GARCH models

Definition of ARCH Models
Testing for ARCH Effects

Non-negativity Constraints

Since the variable u_t^2 cannot be negative, it normally requires all the coefficients of αs nonnegative (a sufficient but no necessary condition).

With the normal stationarity condition for an AR model, it further requires $\sum_{i=1}^{m} \alpha_i < 1$.

Motivations of Using ARCH
ARCH Models
GARCH Models
Multivariate GARCH models

Definition of ARCH Models
Testing for ARCH Effects

ARCH LM Test

The ARCH LM test statistic is computed from an auxiliary test regression. To test the null hypothesis that there is no ARCH up to order q in the residuals, we run the regression:

$$y_t = c + \phi_1 y_{t-1} + \ldots + \phi_p y_{t-p} + u_t \qquad (13)$$

Save the residual $\hat{u}_t$, then run another regression of the squared residuals on a constant and lagged squared residuals up to order q.

Motivations of Using ARCH
ARCH Models
GARCH Models
Multivariate GARCH models

Definition of ARCH Models
Testing for ARCH Effects

ARCH LM Test Cont'd

The Engle (1982) LM test statistic is defined as the number of observation multiply by the R^2 for the second regression, and $TR^2 \sim \chi^2(q)$.

The null is a joint hypothesis that all of the coefficients of q order of squared residuals are zeros.

Motivations of Using ARCH
ARCH Models
GARCH Models
Multivariate GARCH models

the GARCH Models
Estimation of GARCH Models
Extensions to GARCH Family

Limitations of ARCH Models

▶ How should the value of q, the number of lags of the squared residual in the model, be decided?

▶ The number of q could be very large to capture all of the dependence in the conditional variance.

▶ Non-negativity constraints might be violated.

Solution? the Generalized ARCH (GARCH).

Motivations of Using ARCH
ARCH Models
GARCH Models
Multivariate GARCH models

the GARCH Models
Estimation of GARCH Models
Extensions to GARCH Family

GARCH Model

Bollerslev (1986) and Taylor (1986) independently generalized Engle's model. From the ARCH model, the conditional variance equation can be written as:

$$h_t = \omega + \sum_{i=1}^{p} \alpha_i u_{t-i}^2 + \sum_{j=1}^{q} \beta_j h_{t-j} \qquad (14)$$

or in lag operator, for $\alpha(L) = \alpha_1 L + \ldots + \alpha_p L^p$ and $\beta(L) = \beta_1 L + \ldots + \beta_q L^q$:

$$h_t = \omega + \alpha(L) u_t^2 + \beta(L) h_t \qquad (15)$$

this is defined as $GARCH(p, q)$ model.

Motivations of Using ARCH
ARCH Models
GARCH Models
Multivariate GARCH models

the GARCH Models
Estimation of GARCH Models
Extensions to GARCH Family

the Benefits of GARCH

The benefit of GARCH model over ARCH model is to provide a simple framework but richer information.

Using recursive substitution for a simple GARCH(1,1) model as for the ARMA(1,1) model, we can easily find out it is equivalent to an $ARCH(\infty)$ process.

In the empirical application, the simple GARCH(1,1) is sufficient to capture volatility clustering in financial data, thus higher order of GARCH models may not be necessary in general.

Motivations of Using ARCH
ARCH Models
GARCH Models
Multivariate GARCH models

the GARCH Models
Estimation of GARCH Models
Extensions to GARCH Family

Recursive Substitution

Suppose we have

$$h_t = \omega + \alpha u_{t-1}^2 + \beta h_{t-1} \tag{16}$$

one period lag gives:

$$h_{t-1} = \omega + \alpha u_{t-2}^2 + \beta h_{t-2} \tag{17}$$

substituting back to have

$$h_t = \omega + \omega\beta + \alpha u_{t-1}^2 + \alpha\beta u_{t-2}^2 + \beta^2 h_{t-2} \tag{18}$$

Motivations of Using ARCH
ARCH Models
GARCH Models
Multivariate GARCH models

the GARCH Models
Estimation of GARCH Models
Extensions to GARCH Family

Recursive Substitution Cont'd

Keep substituting backward to obtain

$$h_t = \gamma + \gamma_1 u_{t-1}^2 + \gamma_2 u_{t-2}^2 + \cdots \qquad (19)$$

where $\gamma = \omega(1 + \beta + \beta^2 + \cdots)$.

This is a restricted infinite order ARCH model, thus the GARCH(1,1), with only three parameters, is a very parsimonious model with rich information.

Motivations of Using ARCH
ARCH Models
GARCH Models
Multivariate GARCH models

the GARCH Models
Estimation of GARCH Models
Extensions to GARCH Family

Unconditional Variance in GARCH

The unconditional variance of a GARCH model is constant and given by:

$$var(u_t) = \frac{\omega}{1 - (\alpha + \beta)} \tag{20}$$

so long as $\beta + \alpha < 1$, otherwise, the unconditional variance is not defined, or 'nonstationary in variance'.

For $\alpha + \beta = 1$, it is known as a 'unit root in variance', and the model is called 'Integrated GARCH (IGARCH)'.

Motivations of Using ARCH
ARCH Models
GARCH Models
Multivariate GARCH models

the GARCH Models
Estimation of GARCH Models
Extensions to GARCH Family

Estimation of GARCH Models

- ▶ Since the model is no longer of the usual linear form, we cannot use OLS.
- ▶ Another technique known as maximum likelihood is employed.
- ▶ The method works by finding the most likely values of the parameters given the actual data.
- ▶ More specifically, a log-likelihood function is formed and the values of the parameters that maximise it are sought.

Motivations of Using ARCH
ARCH Models
GARCH Models
Multivariate GARCH models

the GARCH Models
Estimation of GARCH Models
Extensions to GARCH Family

Maximum Likelihood Estimation: An Example

Consider a simple regression with homoscedastic error:

$$y_t = \alpha + \beta x_t + \epsilon_t \tag{21}$$

Assuming $\epsilon_t \sim N(0, \sigma^2)$, so that $y_t \sim N(\alpha + \beta x_t, \sigma^2)$, the probability density function for a normally distributed random variable is:

$$f(y_t | \alpha + \beta x_t, \sigma^2) = \frac{1}{\sigma \sqrt{2\pi}} \exp \left[-\frac{1}{2} \frac{(y_t - \alpha - \beta x_t)^2}{\sigma^2} \right] \tag{22}$$

Motivations of Using ARCH
ARCH Models
GARCH Models
Multivariate GARCH models

the GARCH Models
Estimation of GARCH Models
Extensions to GARCH Family

Likelihood Function

If ϵ_t is iid, then y_t is also iid. The joint distribution of all y_t can be expressed as the product of individual density functions:

$$f(y_1, y_2, \ldots, y_t | \alpha + \beta x_t, \sigma^2) = \prod_{t-1}^{T} f(y_t | \alpha + \beta x_t, \sigma^2) \qquad (23)$$

Alternatively:

$$f(y_1, y_2, \ldots, y_t | \alpha + \beta x_t, \sigma^2) = \frac{1}{\sigma^T (\sqrt{2\pi})^T} \exp\left[-\frac{1}{2} \sum_{t=1}^{T} \frac{(y_t - \alpha - \beta x_t)^2}{\sigma^2} \right]$$
$$(24)$$

Motivations of Using ARCH
ARCH Models
GARCH Models
Multivariate GARCH models

the GARCH Models
Estimation of GARCH Models
Extensions to GARCH Family

Log Likelihood Function

Equation (24) is the likelihood function that can be written as $LF(\alpha, \beta, \sigma^2)$. And the MLE is to find out the value of these variables that maximize the likelihood function.

Maximizing the multiplicative function with respect to these parameters is complicated, therefore its logrithm is taken to obtain the log likelihood function (LLF):

$$LLF = -Tlog(\sigma) - \frac{T}{2}log(2\pi) - \frac{1}{2}\sum_{t=1}^{T}\frac{(y_t - \alpha - \beta x_t)^2}{\sigma^2} \quad (25)$$

Motivations of Using ARCH
ARCH Models
GARCH Models
Multivariate GARCH models

the GARCH Models
Estimation of GARCH Models
Extensions to GARCH Family

Maximum Likelihood Estimation: solution

Equation (25) can also be written as:

$$LLF = -\frac{T}{2}log(\sigma^2) - \frac{T}{2}log(2\pi) - \frac{1}{2}\sum_{t=1}^{T}\frac{(y_t - \alpha - \beta x_t)^2}{\sigma^2} \quad (26)$$
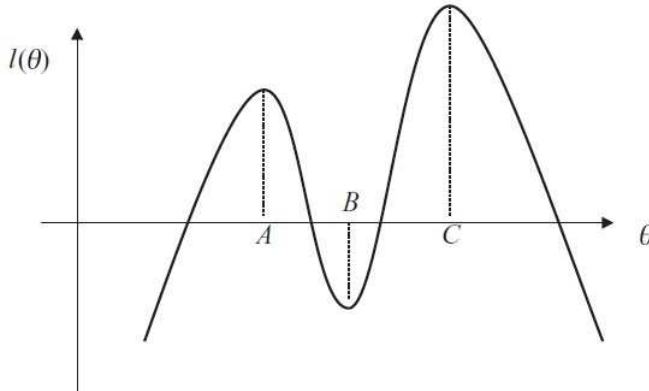
Solving this function with the First Order Conditions for each parameters to get identical estimator to the OLS, however, the variance is different: $\sigma^2 = \frac{1}{T}\sum \hat{u}_t^2$, whereas the OLS variance estimator is $\sigma^2 = \frac{1}{T-k}\sum \hat{u}_t^2$. Therefore, the MLE variance is consistant but biased.

Motivations of Using ARCH
ARCH Models
GARCH Models
Multivariate GARCH models

the GARCH Models
Estimation of GARCH Models
Extensions to GARCH Family

Estimate GARCH Model with MLE

For the GARCH model, variance is time varying, therefore we have to replace $-\frac{T}{2}log(\sigma^2)$ by $-\frac{1}{2}\sum_{t=1}^{T}log(\sigma_t^2)$. The optimization cannot be reached analytically except for the simplest case. It is complicated, so a numerical procedure is used, for example:

1. Set up LLF
2. Use regression to get initial guesses for the mean parameters.
3. Choose some initial guesses for the conditional variance parameters.
4. Specify a convergence criterion - either by criterion or by value.

Motivations of Using ARCH
ARCH Models
GARCH Models
Multivariate GARCH models

the GARCH Models
Estimation of GARCH Models
Extensions to GARCH Family

Local Optima or Multimodalities

Motivations of Using ARCH
ARCH Models
GARCH Models
Multivariate GARCH models

the GARCH Models
Estimation of GARCH Models
Extensions to GARCH Family

the 'Trinity' of tests

There are three classical tests: Likelihood Ratio (LR), Lagrange Multiplier (LM) and Wald tests, they are trying to answer the following question:

▶ (LR) Did the likelihood change much under the null hypotheses versus the alternative?

▶ (Wald) Are the estimated parameters very far away from what they would be under the null hypothesis?

▶ (LM) If I had a less restrictive likelihood function, would its derivative be close to zero here at the restricted ML estimate?

Motivations of Using ARCH
ARCH Models
GARCH Models
Multivariate GARCH models

the GARCH Models
Estimation of GARCH Models
Extensions to GARCH Family

Likelihood Ratio Test

The LR test requires estimation of both restricted model and unrestricted model. The LR statistic is given by:

$$LR = -2(L_r - L_u) \sim \chi^2(m) \tag{27}$$

where L_u is the LLF for the unrestricted model and L_r is for the restricted model; m refers to the number of restrictions.

Motivations of Using ARCH
ARCH Models
GARCH Models
Multivariate GARCH models

the GARCH Models
Estimation of GARCH Models
Extensions to GARCH Family

Lagrange Multiplier Test

The LM test requires only estimation of the restricted model and it involves the first and second derivatives of the log-likelihood function with respect to the parameters at the constrained estimate.
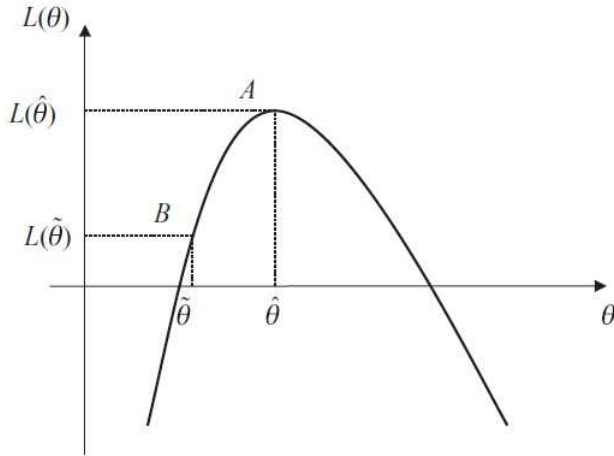
The first derivatives of the log-likelihood function are collectively known as the score vector, measuring the slope of the LLF for each possible value of the parameters. The expected values of the second derivatives is called Hessian and its negative value is the information matrix (variance and covariance matrix).

Motivations of Using ARCH
ARCH Models
GARCH Models
Multivariate GARCH models

the GARCH Models
Estimation of GARCH Models
Extensions to GARCH Family

Wald Test

The Wald test involves estimating only an unrestricted regression, and the usual OLS t-tests and F-tests are examples of Wald tests.

All these three tests assume nomality of estimators, and the test statistics are asymptotically χ^2. They are asymptotically eqivalent.

Motivations of Using ARCH
ARCH Models
GARCH Models
Multivariate GARCH models

the GARCH Models
Estimation of GARCH Models
Extensions to GARCH Family

Graphic Demonstration

Motivations of Using ARCH
ARCH Models
GARCH Models
Multivariate GARCH models

the GARCH Models
Estimation of GARCH Models
Extensions to GARCH Family

EGARCH

A GARCH model successfully captures several stylized facts, however, it is symmetric and not able to show the "leverage effects" mentioned above. Nelson (1991) proposed an exponential GARCH model (EGARCH) that allows asymmetric effects. The conditional variance equation is:

$$ln(h_t) = \omega + \beta ln(h_{t-1}) + \gamma \frac{u_{t-1}}{\sqrt{h_{t-1}}} + \alpha \left[\frac{|u_{t-1}|}{\sqrt{h_{t-1}}} - \sqrt{\frac{2}{\pi}} \right] \quad (28)$$

Motivations of Using ARCH
ARCH Models
GARCH Models
Multivariate GARCH models

the GARCH Models
Estimation of GARCH Models
Extensions to GARCH Family

EGARCH cont'd

The benefits of such parameterizations is no need for further nonnegative restrictions for the parameters since nature logarithm is been applied, furthermore, if there is indeed asymmetric effects where negative shock cause larger volatility, γ will be negative.

An additional notes about EGARCH is that Nelson (1991) proposed using the Generalized error distribution (GED) normalized to have zero mean and unit variance rather than normal distribution for the ML estimation.

Motivations of Using ARCH
ARCH Models
GARCH Models
Multivariate GARCH models

the GARCH Models
Estimation of GARCH Models
Extensions to GARCH Family

GJR-GARCH

An alternative way of showing asymmetric effect is to use the GJR-GARCH proposed by Glosten, Jagannathan and Runkel (1993). It is also called Threshold GARCH (TGARCH) model. It adds an additional term in the GARCH conditional variance equation that:

$$h_t = \omega + \alpha u_{t-1}^2 + \beta h_{t-1} + \gamma u_{t-1}^2 \cdot I_{t-1} \qquad (29)$$

where $I_{t-1} = \begin{cases} 1 & \text{if } u_{t-1} < 0 \\ 0 & \text{otherwise} \end{cases}$.

The coefficient $\gamma > 0$ captures the leverage effects. Remember, here we still require the nonnegative restrictions.

Motivations of Using ARCH
ARCH Models
GARCH Models
Multivariate GARCH models

the GARCH Models
Estimation of GARCH Models
Extensions to GARCH Family

Tests for Asymmtries

Engle and Ng (1993) have proposed a set of tests for asymmetry in volatility, known as sign and size bias tests. In practice, the Engle-Ng tests are usually applied to the residuals of a GARCH fit to the data.

Define S_t^{-1} as an indicator dummy that takes the value 1 if $\hat{u}_{t-1} < 0$ and zero otherwise. The test for sign bias is based on the significance or otherwise of ϕ_1 in

$$\hat{u}_t^2 = \phi_0 + \phi_1 S_t^{-1} + v_t \tag{30}$$

Motivations of Using ARCH
ARCH Models
GARCH Models
Multivariate GARCH models

the GARCH Models
Estimation of GARCH Models
Extensions to GARCH Family

Tests for Asymmtries Cont'd

It could also be the case that the magnitude or size of the shock will affect whether the response of volatility to shocks is symmetric or not.

In this case, a negative size bias test would be conducted, negative size bias is argued to be present if ϕ_1 is statistically significant in the regression:

$$\hat{u}_t^2 = \phi_0 + \phi_1 S_t^{-1} \hat{u}_{t-1} + v_t \tag{31}$$

Motivations of Using ARCH
ARCH Models
GARCH Models
Multivariate GARCH models

the GARCH Models
Estimation of GARCH Models
Extensions to GARCH Family

News Impact Curves (NIC)

In the asymmetric volatility models good news and bad news have different impact on future volatility. The news impact curve characterizes the impact of past return shocks on the return volatility which is implicit in a volatility model.

The NIC for a GARCH(1,1) model is centered around $u_{t-1} = 0$ and has the following expression:

$$
\begin{align}
h_t &= A + \alpha u_{t-1}^2 \tag{32} \\
A &\equiv \omega + \beta \sigma^2 \tag{33}
\end{align}
$$

Motivations of Using ARCH
ARCH Models
GARCH Models
Multivariate GARCH models

the GARCH Models
Estimation of GARCH Models
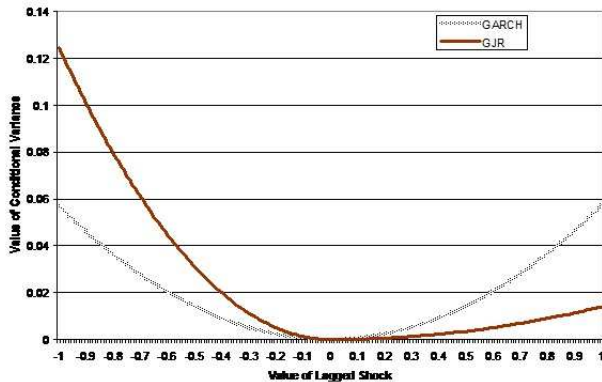**Extensions to GARCH Family**

NIC Cont'd

For a GJR-GARCH model, the NIC can be shown to be asymmetric:

$$h_t = \begin{cases} A + \alpha u_{t-1}^2 & \text{if } u_{t-1} \geq 0 \\ A + (\alpha + \gamma) u_{t-1}^2 & \text{otherwise} \end{cases} \tag{34}$$

$$A \equiv \omega + \beta \sigma^2 \tag{35}$$

Motivations of Using ARCH
ARCH Models
GARCH Models
Multivariate GARCH models

the GARCH Models
Estimation of GARCH Models
Extensions to GARCH Family

NIC Cont'd

Motivations of Using ARCH
ARCH Models
GARCH Models
Multivariate GARCH models

the GARCH Models
Estimation of GARCH Models
Extensions to GARCH Family

GARCH in Mean

It is normally agreed that an asset with a higher risk would pay a higher return. The conditional covariance with an appropriately defined benchmark portfolio often serves to price the assets.

For example, according to the traditional capital asset pricing model (CAPM) the excess returns on all risky assets are proportional to the non-diversifiable risk as measured by the covariances with the market portfolio.

Motivations of Using ARCH
ARCH Models
GARCH Models
Multivariate GARCH models

the GARCH Models
Estimation of GARCH Models
Extensions to GARCH Family

GARCH in Mean Cont'd

The GARCH in mean (GARCH-M) model proposed by Engle, Lilien and Robins (1987) are defined as:

$$
\begin{align}
y_t &= \mu + \delta g(h_{t-1}) + u_t \tag{36} \\
h_t &= \omega + \alpha u_{t-1}^2 + \beta h_{t-1} \tag{37}
\end{align}
$$

where $g(\cdot)$ is a function of conditional variance, which can be the level, the square roots or natural logarithm.

Motivations of Using ARCH
ARCH Models
GARCH Models
Multivariate GARCH models

Covariance and Correlation
Application: Dynamic Hedge Ratio

Estimate Covariance and Correlation

Covariance or correlation between two series can be calculated in the standard way using a set of historical data.

It can also be estimated using EWMA specificaiton:

$$\sigma(x, y)_t = (1 - \lambda) \sum_{i=0}^{m} \lambda^i x_{t-i} y_{t-i} \tag{38}$$

Multivariate GARCH models are used to estimate and to forecast covariances and correlations to allow for the time-varying nature.

Motivations of Using ARCH
ARCH Models
GARCH Models
Multivariate GARCH models

Covariance and Correlation
Application: Dynamic Hedge Ratio

Consider an $(n \times 1)$ vector $\mathbf{y}_t$:

$$\mathbf{y}_t = \Pi' \cdot \mathbf{x}_t + \mathbf{u}_t \qquad (39)$$

Where $\mathbf{u}_t$ is a vector of residuals. Let $\mathbf{H}_t$ denotes the $n \times n$ conditional variance-covariance matrix of the residuals.

$$\mathbf{H}_t = E(\mathbf{u}_t \mathbf{u}_t' | \mathbf{y}_{t-1}, \ldots, \mathbf{x}_{t-1}, \ldots) \qquad (40)$$

There are three methods to model $\mathbf{H}_t$: the VECH, the diagonal VECH and the BEKK models.

Motivations of Using ARCH
ARCH Models
GARCH Models
Multivariate GARCH models

Covariance and Correlation
Application: Dynamic Hedge Ratio

VECH model

A common specification of the VECH model, initially due to Bollerslev, Engle and Wooldridge (1988):

$$VECH(\mathbf{H}_t) = \mathbf{W} + \mathbf{A}\,VECH(\mathbf{u}_{t-1}\mathbf{u}_{t-1}\prime) + \mathbf{B}\,VECH(\mathbf{H}_{t-1}) \quad (41)$$

where $VECH(\cdot)$ denote the column stacking operator, which stacks the lower triangular elements of an $n \times n$ matrix as an $[n(n+1)/2] \times 1$ vector.

Motivations of Using ARCH
ARCH Models
GARCH Models
Multivariate GARCH models

Covariance and Correlation
Application: Dynamic Hedge Ratio

VECH Model Cont'd

For example, in a (2×2) case, we have:

$$\mathbf{H}_t = \begin{pmatrix} h_{11t} & h_{12t} \\ h_{21t} & h_{22t} \end{pmatrix}, \quad \mathbf{W} = \begin{pmatrix} c_{11} \\ c_{21} \\ c_{31} \end{pmatrix}, \quad \mathbf{A} = \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix}$$

$$\mathbf{B} = \begin{pmatrix} b_{11} & b_{12} & b_{13} \\ b_{21} & b_{22} & b_{23} \\ b_{31} & b_{32} & b_{33} \end{pmatrix}, \quad \mathbf{u}_t = \begin{pmatrix} u_{1t} \\ u_{2t} \end{pmatrix}$$

Motivations of Using ARCH
ARCH Models
GARCH Models
Multivariate GARCH models

Covariance and Correlation
Application: Dynamic Hedge Ratio

VECH Cont'd

The VECH representation can is written as:

$$VECH(\mathbf{H}_t) = \begin{pmatrix} h_{11t} \\ h_{22t} \\ h_{12t} \end{pmatrix}, VECH(\mathbf{u}_t) = \begin{pmatrix} u_{1t}^2 \\ u_{2t}^2 \\ u_{1t}u_{2t} \end{pmatrix} \qquad (42)$$

Then Equation (41) can be written as:

$$h_{11t} = c_{11} + a_{11}u_{1t-1}^2 + a_{12}u_{2t-1}^2 + a_{13}u_{1t-1}u_{2t-1} + b_{11}h_{11t-1} + b_{12}h_{22t-1} + b_{13}h_{12t-1} \qquad (43)$$

$$h_{22t} = c_{21} + a_{21}u_{1t-1}^2 + a_{22}u_{2t-1}^2 + a_{23}u_{1t-1}u_{2t-1} + b_{21}h_{11t-1} + b_{22}h_{22t-1} + b_{23}h_{12t-1} \qquad (44)$$

$$h_{12t} = c_{31} + a_{31}u_{1t-1}^2 + a_{32}u_{2t-1}^2 + a_{33}u_{1t-1}u_{2t-1} + b_{31}h_{11t-1} + b_{32}h_{22t-1} + b_{33}h_{12t-1} \qquad (45)$$

Motivations of Using ARCH
ARCH Models
GARCH Models
Multivariate GARCH models

Covariance and Correlation
Application: Dynamic Hedge Ratio

Diagonal VECH

Bollerslev et al. (1988) suggest that the coefficient matrix **A** and **B** to be diagonal. Which means the $(i, j)^{th}$ elements of $\mathbf{H}_t$ only depend on the corresponding $(i, j)^{th}$ elements of $\mathbf{H}_{t-1}$ and $\mathbf{u}_{t-1}\mathbf{u}'_{t-1}$.

Diagonal VECH model does not allow for causality in variance, co-persistence in variance, or asymmetries. Furthermore, the model may not guarantee a positive definite conditional covariance matrix.

Motivations of Using ARCH
ARCH Models
GARCH Models
Multivariate GARCH models

Covariance and Correlation
Application: Dynamic Hedge Ratio

BEKK Model

An alternative model proposed by Engle and Kroner (1995) is the BEKK model that have:

$$\mathbf{H}_t = \mathbf{V}'\mathbf{V} + \mathbf{A}'\mathbf{H}_{t-1}\mathbf{A} + \mathbf{B}'\mathbf{u}_{t-1}\mathbf{u}_{t-1}\mathbf{B} \qquad (46)$$

which guaranteed for the conditional covariance matrix to be positive definite due to the quadratic nature of the terms on the RHS of equations.

Motivations of Using ARCH
ARCH Models
GARCH Models
Multivariate GARCH models

Covariance and Correlation
Application: Dynamic Hedge Ratio

Hedging

The simplest and perhaps the most widely used method for reducing and managing risk is hedging with futures contracts.

A hedge is achieved by taking opposite positions in spot and futures markets simultaneously, so that any loss sustained from an adverse price movement in one market should to some degree be offset by a favourable price movement in the other.

Motivations of Using ARCH
ARCH Models
GARCH Models
Multivariate GARCH models

Covariance and Correlation
Application: Dynamic Hedge Ratio

Hedge Ratio

The ratio of the number of units of the futures asset that are purchased relative to the number of units of the spot asset is known as the **hedge ratio**.

The *optimal hedge ratio* is to choose a ratio which minimises the variance of the returns of a portfolio containing the spot and futures position.

Motivations of Using ARCH
ARCH Models
GARCH Models
Multivariate GARCH models

Covariance and Correlation
Application: Dynamic Hedge Ratio

Definitions

- ▶ Change in spot price during the life of the hedge ΔS
- ▶ Change in futures price ΔF
- ▶ Standard deviation of the spot prices σ_s
- ▶ Standard deviation of the future prices σ_F
- ▶ Correlation coefficient between changes in spot prices and future prices ρ
- ▶ Hedge ratio h

Motivations of Using ARCH
ARCH Models
GARCH Models
Multivariate GARCH models

Covariance and Correlation
Application: Dynamic Hedge Ratio

the Variance of Hedge Portfolio

The variance of long or short portfolio is the same and can be obtained from

$$var(h\Delta F - \Delta S) = var(h\Delta F) + var(\Delta S) - 2cov(\Delta S, h\Delta F) \quad (47)$$

or

$$v = \sigma_s^2 + h^2 \sigma_F^2 - 2h\rho\sigma_s\sigma_F \quad (48)$$

Motivations of Using ARCH
ARCH Models
GARCH Models
Multivariate GARCH models

Covariance and Correlation
Application: Dynamic Hedge Ratio

Optimal Hedge Ratio

The optimal hedge ratio can be found by minimizing Equation (48) with respect to h:

$$h = \rho \frac{\sigma_s}{\sigma_F} \tag{49}$$

According to the intuition of ARCH models, the RHS variable is not constants, but time-varying. It is then necessary to model the optimal hedge ratio with an Multivariate GARCH Model:

$$h_t = \rho_t \frac{\sigma_{s,t}}{\sigma_{F,t}} \tag{50}$$

Motivations of Using ARCH
ARCH Models
GARCH Models
Multivariate GARCH models

Covariance and Correlation
Application: Dynamic Hedge Ratio

Effectiveness of hedge

Brooks, Henry and Persand (2002) compared the effectiveness of hedging on the basis of hedge ratios derived from various multivariate GARCH specifications and other, simpler techniques.

	Out-of-sample			
	Unhedged $\beta = 0$	Naive hedge $\beta = -1$	Symmetric time-varying hedge $\beta_t = \frac{h_{FS,t}}{h_{F,t}}$	Asymmetric time-varying hedge $\beta_t = \frac{h_{FS,t}}{h_{F,t}}$
Return	0.0819 {1.4958}	−0.0004 {0.0216}	0.0120 {0.7761}	0.0140 {0.9083}
Variance	1.4972	0.1696	0.1186	0.1188