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OPTIMAL MEAN REVERSION TRADING WITH TRANSACTION COSTS AND STOP-LOSS EXIT

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Motivated by the industry practice of pairs trading, we study the optimal timing strategies for trading a mean-reverting price spread. An optimal double stopping problem is formulated to analyze the timing to start and subsequently liquidate the position subject to transaction costs. Modeling the price spread by an Ornstein–Uhlenbeck process, we apply a probabilistic methodology and rigorously derive the optimal price intervals for market entry and exit. As an extension, we incorporate a stop-loss constraint to limit the maximum loss. We show that the entry region is characterized by a bounded price interval that lies strictly above the stop-loss level. As for the exit timing, a higher stop-loss level always implies a lower optimal take-profit level. Both analytical and numerical results are provided to illustrate the dependence of timing strategies on model parameters such as transaction costs and stop-loss level.

Keywords: Optimal double stopping; mean reversion trading; Ornstein–Uhlenbeck process; stop-loss.

1. Introduction

It has been widely observed that many asset prices exhibit mean reversion, including commodities (see Schwartz 1997), foreign exchange rates (see Engel & Hamilton 1989, Anthony & MacDonald 1998, Larsen & Sørensen 2007), as well as US and global equities (see Poterba & Summers 1988, Malliaropulos & Priestley 1999, Balvers *et al.* 2000, Gropp 2004). Mean-reverting processes are also used to model the dynamics of interest rate, volatility, and default risk. In industry, hedge fund managers and investors often attempt to construct mean-reverting prices by simultaneously taking positions in two highly correlated or co-moving assets. The advent of exchange-traded funds (ETFs) has further facilitated this *pairs trading* approach since some ETFs are designed to track identical or similar indexes and assets.

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For instance, Triantafyllopoulos & Montana (2011) investigate the mean-reverting spreads between commodity ETFs and design model for statistical arbitrage. Dunis *et al.* (2013) also examine the mean-reverting spread between physical gold and gold equity ETFs.

Given the price dynamics of some risky asset(s), one important problem commonly faced by individual and institutional investors is to determine when to open and close a position. While observing the prevailing market prices, a speculative investor can choose to enter the market immediately or wait for a future opportunity. After completing the first trade, the investor will need to decide when is best to close the position. This motivates the investigation of the optimal sequential timing of trades.

In this paper, we study the optimal timing of trades subject to transaction costs under the Ornstein–Uhlenbeck (OU) model. Specifically, our formulation leads to an *optimal double stopping* problem that gives the optimal entry and exit decision rules. We obtain analytic solutions for both the entry and exit problems. In addition, we incorporate a stop-loss constraint to our trading problem. We find that a higher stop-loss level induces the investor to voluntarily liquidate earlier at a lower take-profit level. Moreover, the entry region is characterized by a bounded price interval that lies strictly above stop-loss level. In other words, it is optimal to wait if the current price is too high or too close to the lower stop-loss level. This is intuitive since entering the market close to stop-loss implies a high chance of exiting at a loss afterwards. As a result, the delay region (complement of the entry region) is *disconnected*. Furthermore, we show that optimal liquidation level decreases with the stop-loss level until they coincide, in which case immediate liquidation is optimal at all price levels.

A typical solution approach for optimal stopping problems driven by diffusion involves the analytical and numerical studies of the associated free boundary problems or variational inequalities (VIs); see e.g. Bensoussan & Lions (1982), Øksendal (2003), and Sun (1992). For our double optimal stopping problem, this method would determine the value functions from a pair of VIs and require regularity conditions to guarantee that the solutions to the VIs indeed correspond to the optimal stopping problems. As noted by Dayanik (2008), "the variational methods become challenging when the form of the reward function and/or the dynamics of the diffusion obscure the shape of the optimal continuation region." In our optimal entry timing problem, the reward function involves the value function from the exit timing problem, which is not monotone and can be positive and negative.

In contrast to the variational inequality approach, our proposed methodology starts with a characterization of the value functions as the smallest concave majorant of any given reward function. A key feature of this approach is that it allows us to directly construct the value function, without *a priori* finding a candidate value function or imposing conditions on the stopping and delay (continuation) regions, such as whether they are connected or not. In other words, our method will derive the structure of the stopping and delay regions as an output.

Our main results provide the analytic expressions for the value functions of the double stopping problems; see Theorems 4.1 and 4.2 (without stop-loss), and Theorems 5.1 and 5.2 (with stop-loss). In earlier studies, Dynkin & Yushkevich (1969) analyze the concave characterization of excessive functions for a standard Brownian motion, and Dayanik & Karatzas (2003) and Dayanik (2008) apply this idea to study the optimal single stopping of a one-dimensional diffusion. In this regard, we contribute to this line of work by solving a number of optimal double stopping problems with and without a stop-loss exit under the OU model.

Among other related studies, Ekstrom *et al.* (2011) analyze the optimal single liquidation timing under the OU model with zero long-run mean and no transaction cost. The current paper extends their model in a number of ways. First, we analyze the optimal entry timing as well as the optimal liquidation timing. Our model allows for a nonzero long-run mean and transaction costs, along with a stop-loss level. Song *et al.* (2009) propose a numerical stochastic approximation scheme to solve for the optimal buy-low-sell-high strategies over a finite horizon. Under a similar setting, Zhang & Zhang (2008) and Kong & Zhang (2010) also investigate the infinite sequential buying and selling/shorting problem under exponential OU price dynamics with slippage cost.

In the context of pairs trading, a number of studies have also considered market timing strategy with two price levels. For example, Gatev *et al.* (2006) study the historical returns from the buy-low-sell-high strategy where the entry/exit levels are set as ±1 standard deviation from the long-run mean. Similarly, Avellaneda & Lee (2010) consider starting and ending a pairs trade based on the spread's distance from its mean. In Elliott *et al.* (2005), the market entry timing is modeled by the first passage time of an OU process, followed by an exit at a fixed finite horizon. In comparison, rather than assigning *ad hoc* price levels or fixed trading times, our approach will generate the entry and exit thresholds as solutions of an optimal double stopping problem. Considering an exponential OU asset price with zero mean, Bertram (2010) numerically computes the optimal enter and exit levels that maximize the expected return per unit time. Gregory *et al.* (2010) also apply this approach to log-spread following the CIR and GARCH diffusion models. Other timing strategies adopted by practitioners have been discussed in Vidyamurthy (2004).

On the other hand, the related problem of constructing portfolios and hedging with mean reverting asset prices has been studied. For example, Benth & Karlsen (2005) study the utility maximization problem that involves dynamically trading an exponential OU underlying asset. Jurek & Yang (2007) analyze a finite-horizon portfolio optimization problem with an OU asset subject to the power utility and Epstein–Zin recursive utility. Chiu & Wong (2012) consider the dynamic trading of co-integrated assets with a mean–variance criterion. Tourin & Yan (2013) derive the dynamic trading strategy for two co-integrated stocks in order to maximize the expected terminal utility of wealth over a fixed horizon. They simplify the associated Hamilton–Jacobi–Bellman equation and obtain a closed-form solution.

In the stochastic control approach, incorporating transaction costs and stop-loss exit can potentially limit model tractability and is not implemented in these studies.

The rest of the paper is structured as follows. We formulate the optimal trading problem in Sec. 2, followed by a discussion on our method of solution in Sec. 3. In Sec. 4, we analytically solve the optimal double stopping problem and examine the optimal entry and exit strategies. In Sec. 5, we study the trading problem with a stop-loss constraint. The proofs of all lemmas are provided in Appendix.

2. Problem Overview

In the background, we fix the probability space $(\Omega, \mathcal{F}, \mathbb{P})$ with the historical probability measure $\mathbb{P}$. We consider an OU process driven by the SDE:

$$dX_t = \mu(\theta - X_t)\,dt + \sigma\,dB_t, \tag{2.1}$$

with constants $\mu, \sigma > 0$, $\theta \in \mathbb{R}$, and state space $\mathbb{R}$. Here, B is a standard Brownian motion under $\mathbb{P}$. Denote by $\mathbb{F} \equiv (\mathcal{F}_t)_{t \geq 0}$ the filtration generated by X.

2.1. *A pairs trading example*

Let us discuss a pairs trading example where we model the value of the resulting position by an OU process. The primary objective is to motivate our trading problem, rather than proposing new estimation methodologies or empirical studies on pairs trading. For related studies and more details, we refer to the seminal paper by Engle & Granger (1987), the books Hamilton (1994), Tsay (2005), and references therein.

We construct a portfolio by holding α shares of a risky asset $S^{(1)}$ and shorting β shares of another risky asset $S^{(2)}$, yielding a portfolio value $X_t^{\alpha,\beta} = \alpha S_t^{(1)} - \beta S_t^{(2)}$ at time $t \geq 0$. The pair of assets are selected to form a mean-reverting portfolio value. In addition, one can adjust the strategy (α, β) to enhance the level of mean reversion. For the purpose of testing mean reversion, only the ratio between α and β matters, so we can keep α constant while varying β without loss of generality. For every strategy (α, β), we observe the resulting portfolio values $(x_i^{\alpha,\beta})_{i=0,1,\dots,n}$ realized over an n-day period. We then apply the method of maximum likelihood estimation (MLE) to fit the observed portfolio values to an OU process and determine the model parameters. Under the OU model, the conditional probability density of X_{t_i} at time t_i given x_{i-1} at t_{i-1} with time increment $\Delta t = t_i - t_{i-1}$ is given by

$$f^{OU}(x_i \,|\, x_{i-1}; \theta, \mu, \sigma) = \frac{1}{\sqrt{2\pi\tilde{\sigma}^2}} \exp\left(-\frac{(x_i - x_{i-1}e^{-\mu\Delta t} - \theta(1 - e^{-\mu\Delta t}))^2}{2\tilde{\sigma}^2}\right),$$

with the constant

$$\tilde{\sigma}^2 = \sigma^2 \frac{1 - e^{-2\mu\Delta t}}{2\mu}.$$

Using the observed values $(x_i^{\alpha,\beta})_{i=0,1,\dots,n}$, we maximize the average log-likelihood defined by

$$\ell(\theta, \mu, \sigma \mid x_0^{\alpha,\beta}, x_1^{\alpha,\beta}, \dots, x_n^{\alpha,\beta}) := \frac{1}{n} \sum_{i=1}^{n} \ln f^{OU}(x_i^{\alpha,\beta} \mid x_{i-1}^{\alpha,\beta}; \theta, \mu, \sigma)$$

$$= -\frac{1}{2}\ln(2\pi) - \ln(\tilde{\sigma})$$

$$- \frac{1}{2n\tilde{\sigma}^2} \sum_{i=1}^{n} [x_i^{\alpha,\beta} - x_{i-1}^{\alpha,\beta} e^{-\mu\Delta t} - \theta(1 - e^{-\mu\Delta t})]^2,$$

and denote by $\hat{\ell}$ the maximized average log-likelihood over θ, μ, and σ for a given strategy (α, β). For any α, we choose the strategy (α, β^*), where $\beta^* = \arg\max_\beta \hat{\ell}(\theta^*, \mu^*, \sigma^* \mid x_0^{\alpha,\beta}, x_1^{\alpha,\beta}, \dots, x_n^{\alpha,\beta})$. For example, suppose we invest A dollar(s) in asset $S^{(1)}$, so $\alpha = A/S_0^{(1)}$ shares is held. At the same time, we short $\beta = B/S_0^{(2)}$ shares in $S^{(2)}$, for $B/A = 0.001, 0.002, \dots, 1$. This way, the sign of the initial portfolio value depends on the sign of the difference $A - B$, which is nonnegative. Without loss of generality, we set $A = 1$.

In Fig. 1, we illustrate an example based on two pairs of exchange-traded funds (ETFs), namely, the Market Vectors Gold Miners (GDX) and iShares Silver Trust (SLV) against the SPDR Gold Trust (GLD) respectively. These liquidly traded funds aim to track the price movements of the NYSE Arca Gold Miners Index (GDX), silver (SLV), and gold bullion (GLD) respectively. These ETF pairs are also used in Triantafyllopoulos & Montana (2011) and Dunis *et al.* (2013) for their statistical and empirical studies on ETF pairs trading.
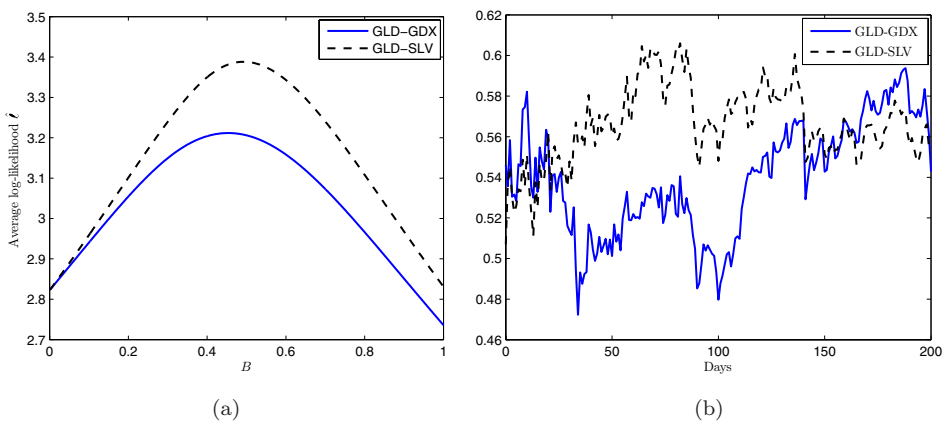

Fig. 1. (a) Average log-likelihood plotted against B. (b) Historical price paths with maximum average log-likelihood. The solid line plots the portfolio price with longing \$1 GLD and shorting \$0.454 GDX, and the dashed line plots the portfolio price with longing \$1 GLD and shorting \$0.493 SLV.

Table 1. MLE estimates of OU process parameters using historical prices of GLD, GDX, and SLV from August 2011 to May 2012. The portfolio consists of \$1 in GLD and −\$0.454 in GDX (respectively, −\$0.493 in SLV). For each pair, the second row (simulated) shows the MLE parameter estimates based on a simulated price path corresponding to the estimated parameters from the first row (empirical).

	Price	$\hat{\theta}$	$\hat{\mu}$	$\hat{\sigma}$	$\hat{\ell}$
GLD-GDX	empirical	0.5388	16.6677	0.1599	3.2117
	simulated	0.5425	14.3893	0.1727	3.1304
GLD-SLV	empirical	0.5680	33.4593	0.1384	3.3882
	simulated	0.5629	28.8548	0.1370	3.3898

Using price data from August 2011 to May 2012 ($n = 200$, $\Delta t = 1/252$), we compute and plot in Fig. 1(a) the average log-likelihood against the cash amount B, and find that $\hat{\ell}$ is maximized at $B^* = 0.454$ (respectively, 0.493) for the GLD-GDX pair (respectively, GLD-SLV pair). From this MLE-optimal B^*, we obtain the strategy (α, β^*), where $\alpha = 1/S_0^{(1)}$ and $\beta^* = B^*/S_0^{(2)}$. In this example, the average log-likelihood for the GLD-SLV pair happens to dominate that for GLD-GDX, suggesting a higher degree of fit to the OU model. Figure 1(b) depicts the historical price paths with the strategy (α, β^*).

We summarize the estimation results in Table 1. For each pair, we first estimate the parameters for the OU model from empirical price data. Then, we use the estimated parameters to simulate price paths according to the corresponding OU process. Based on these simulated OU paths, we perform another MLE and obtain another set of OU parameters as well as the maximum average log-likelihood $\hat{\ell}$. As we can see, the two sets of estimation outputs (the rows names "empirical" and "simulated") are very close, suggesting the empirical price process fits well to the OU model.

2.2. *Optimal stopping problem*

Given that a price process or portfolio value evolves according to an OU process, our main objective is to study the optimal timing to open and subsequently close the position subject to transaction costs. This leads to the analysis of an optimal double stopping problem.

First, suppose that the investor already has an existing position whose value process $(X_t)_{t \geq 0}$ follows (2.1). If the position is closed at some time τ, then the investor will receive the value X_τ and pay a constant transaction cost $c \in \mathbb{R}$. To maximize the expected discounted value, the investor solves the optimal stopping problem

$$V(x) = \sup_{\tau \in \mathcal{T}} \mathbb{E}_x\{e^{-r\tau}(X_\tau - c)\}, \tag{2.2}$$

where $\mathcal{T}$ denotes the set of all $\mathbb{F}$-stopping times, and $r > 0$ is the investor's subjective constant discount rate. We have also used the shorthand notation: $\mathbb{E}_x\{\cdot\} \equiv \mathbb{E}\{\cdot \,|\, X_0 = x\}$.

From the investor's viewpoint, $V(x)$ represents the expected liquidation value associated with X. On the other hand, the current price plus the transaction cost constitute the total cost to enter the trade. The investor can always choose the optimal timing to start the trade, or not to enter at all. This leads us to analyze the entry timing inherent in the trading problem. Precisely, we solve

$$J(x) = \sup_{\nu \in \mathcal{T}} \mathbb{E}_x \{ e^{-\hat{r}\nu} (V(X_\nu) - X_\nu - \hat{c}) \}, \tag{2.3}$$

with $\hat{r} > 0$, $\hat{c} \in \mathbb{R}$. In other words, the investor seeks to maximize the expected difference between the value function $V(X_\nu)$ and the current X_ν, minus transaction cost $\hat{c}$. The value function $J(x)$ represents the maximum expected value of the investment opportunity in the price process X, with transaction costs $\hat{c}$ and c incurred, respectively, at entry and exit. For our analysis, the pre-entry and post-entry discount rates, $\hat{r}$ and r, can be different, as long as $0 < \hat{r} \le r$. Moreover, the transaction costs $\hat{c}$ and c can also differ, as long as $c + \hat{c} > 0$. Moreover, since $\tau = +\infty$ and $\nu = +\infty$ are candidate stopping times for (2.2) and (2.3) respectively, the two value functions $V(x)$ and $J(x)$ are nonnegative.

As extension, we can incorporate a stop-loss level of the pairs trade, that caps the maximum loss. In practice, the stop-loss level may be exogenously imposed by the manager of a trading desk. In effect, if the price X ever reaches level L prior to the investor's voluntary liquidation time, then the position will be closed immediately. The stop-loss signal is given by the first passage time

$$\tau_L := \inf\{t \ge 0 : X_t \le L\}.$$

Therefore, we determine the entry and liquidation timing from the constrained optimal stopping problem:

$$J_L(x) = \sup_{\nu \in \mathcal{T}} \mathbb{E}_x \{ e^{-\hat{r}\nu} (V_L(X_\nu) - X_\nu - \hat{c}) \}, \tag{2.4}$$

$$V_L(x) = \sup_{\tau \in \mathcal{T}} \mathbb{E}_x \{ e^{-r(\tau \wedge \tau_L)} (X_{\tau \wedge \tau_L} - c) \}. \tag{2.5}$$

Due to the additional timing constraint, the investor may be forced to exit early at the stop-loss level for any given liquidation level. Hence, the stop-loss constraint reduces the value functions, and precisely we deduce that $x - c \le V_L(x) \le V(x)$ and $0 \le J_L(x) \le J(x)$. As we will show in Secs. 4 and 5, the optimal timing strategies with and without stop-loss are quite different.

3. Method of Solution

In this section, we disucss our method of solution. First, we denote the infinitesimal generator of the OU process X by

$$\mathcal{L} = \frac{\sigma^2}{2} \frac{d^2}{dx^2} + \mu(\theta - x) \frac{d}{dx}, \tag{3.1}$$

and recall the classical solutions of the differential equation

$$\mathcal{L}u(x) = ru(x), \tag{3.2}$$

for $x \in \mathbb{R}$, are (see e.g. (Borodin & Salminen 2002, p. 542) and Proposition 2.1 of Alili *et al.* 2005):

$$F(x) \equiv F(x;r) := \int_0^\infty u^{\frac{r}{\mu}-1} e^{\sqrt{\frac{2\mu}{\sigma^2}}(x-\theta)u - \frac{u^2}{2}} \, du, \tag{3.3}$$

$$G(x) \equiv G(x;r) := \int_0^\infty u^{\frac{r}{\mu}-1} e^{\sqrt{\frac{2\mu}{\sigma^2}}(\theta-x)u - \frac{u^2}{2}} \, du. \tag{3.4}$$

Direct differentiation yields that $F'(x) > 0$, $F''(x) > 0$, $G'(x) < 0$ and $G''(x) > 0$. Hence, we observe that both $F(x)$ and $G(x)$ are strictly positive and convex, and they are, respectively, strictly increasing and decreasing.

Define the first passage time of X to some level κ by $\tau_\kappa = \inf\{t \geq 0 : X_t = \kappa\}$. As is well known, F and G admit the probabilistic expressions (see Itō & McKean 1965 and Rogers & Williams 2000):

$$\mathbb{E}_x\{e^{-r\tau_\kappa}\} = \begin{cases} \dfrac{F(x)}{F(\kappa)} & \text{if } x \leq \kappa, \\[3mm] \dfrac{G(x)}{G(\kappa)} & \text{if } x \geq \kappa. \end{cases} \tag{3.5}$$

A key step of our solution method involves the transformation

$$\psi(x) := \frac{F}{G}(x). \tag{3.6}$$

Starting at any $x \in \mathbb{R}$, we denote by $\tau_a \wedge \tau_b$ the exit time from an interval $[a, b]$ with $-\infty \leq a \leq x \leq b \leq +\infty$. With the reward function $h(x) = x - c$, we compute the corresponding expected discounted reward:

$$\mathbb{E}_x\{e^{-r(\tau_a \wedge \tau_b)} h(X_{\tau_a \wedge \tau_b})\}$$

$$= h(a)\mathbb{E}_x\{e^{-r\tau_a}\mathbb{1}_{\{\tau_a < \tau_b\}}\} + h(b)\mathbb{E}_x\{e^{-r\tau_b}\mathbb{1}_{\{\tau_a > \tau_b\}}\} \tag{3.7}$$

$$= h(a)\frac{F(x)G(b) - F(b)G(x)}{F(a)G(b) - F(b)G(a)} + h(b)\frac{F(a)G(x) - F(x)G(a)}{F(a)G(b) - F(b)G(a)} \tag{3.8}$$

$$= G(x)\left[\frac{h(a)}{G(a)}\frac{\psi(b) - \psi(x)}{\psi(b) - \psi(a)} + \frac{h(b)}{G(b)}\frac{\psi(x) - \psi(a)}{\psi(b) - \psi(a)}\right] \tag{3.9}$$

$$= G(\psi^{-1}(y))\left[H(y_a)\frac{y_b - y}{y_b - y_a} + H(y_b)\frac{y - y_a}{y_b - y_a}\right], \tag{3.10}$$

where $y_a = \psi(a)$, $y_b = \psi(b)$, and

$$H(y) := \begin{cases} \dfrac{h}{G} \circ \psi^{-1}(y) & \text{if } y > 0, \\[3mm] \lim_{x \to -\infty} \dfrac{(h(x))^+}{G(x)} & \text{if } y = 0. \end{cases} \tag{3.11}$$

The second equality (3.8) follows from the fact that $f(x) := \mathbb{E}_x\{e^{-r(\tau_a \wedge \tau_b)}\mathbb{1}_{\{\tau_a < \tau_b\}}\}$ is the unique solution to (3.2) with boundary conditions $f(a) = 1$ and $f(b) = 0$. Similar reasoning applies to the function $g(x) := \mathbb{E}_x\{e^{-r(\tau_a \wedge \tau_b)}\mathbb{1}_{\{\tau_a > \tau_b\}}\}$ with $g(a) = 0$ and $g(b) = 1$. The last equality (3.10) transforms the problem from x coordinate to $y = \psi(x)$ coordinate (see (3.6)).

The candidate optimal exit interval $[a^*, b^*]$ is determined by maximizing the expectation in (3.7). This is equivalent to maximizing (3.10) over y_a and y_b in the transformed problem. This leads to

$$W(y) := \sup_{\{y_a, y_b : y_a \le y \le y_b\}} \left[H(y_a) \frac{y_b - y}{y_b - y_a} + H(y_b) \frac{y - y_a}{y_b - y_a} \right]. \tag{3.12}$$

This is the smallest concave majorant of H. Applying the definition of W to (3.10), we can express the maximal expected discounted reward as

$$G(x)W(\psi(x)) = \sup_{\{a, b : a \le x \le b\}} \mathbb{E}_x\{e^{-r(\tau_a \wedge \tau_b)}h(X_{\tau_a \wedge \tau_b})\}.$$

Remark 3.1. If $a = -\infty$, then we have $\tau_a = +\infty$ and $\mathbb{1}_{\{\tau_a < \tau_b\}} = 0$ a.s. In effect, this removes the lower exit level, and the corresponding expected discounted reward is

$$\mathbb{E}_x\{e^{-r(\tau_a \wedge \tau_b)}h(X_{\tau_a \wedge \tau_b})\} = \mathbb{E}_x\{e^{-r\tau_a}h(X_{\tau_a})\mathbb{1}_{\{\tau_a < \tau_b\}}\} + \mathbb{E}_x\{e^{-r\tau_b}h(X_{\tau_b})\mathbb{1}_{\{\tau_a > \tau_b\}}\}$$

$$= \mathbb{E}_x\{e^{-r\tau_b}h(X_{\tau_b})\}.$$

Consequently, by considering interval-type strategies, we also include the class of stopping strategies of reaching a single upper level b (see Theorem 4.1).

Next, we prove the optimality of the proposed stopping strategy and provide an expression for the value function.

Theorem 3.1. *The value function $V(x)$ defined in (2.2) is given by*

$$V(x) = G(x)W(\psi(x)), \tag{3.13}$$

where G, ψ and W are defined in (3.4), (3.6) and (3.12), respectively.

The proof is provided in Appendix A.1. Let us emphasize that the optimal levels (a^*, b^*) may depend on the initial value x, and can potentially coincide, or take values $-\infty$ and $+\infty$. As such, the structure of the stopping and delay regions can potentially be characterized by multiple intervals, leading to *disconnected* delay regions (see Theorem 5.2).

We follow the procedure for Theorem 3.1 to derive the expression for the value function J in (2.3). First, we denote $\hat{F}(x) = F(x; \hat{r})$ and $\hat{G}(x) = G(x; \hat{r})$ (see (3.3)–(3.4)), with discount rate $\hat{r}$. In addition, we define the transformation

$$\hat{\psi}(x) := \frac{\hat{F}}{\hat{G}}(x) \quad \text{and} \quad \hat{h}(x) = V(x) - x - \hat{c}. \tag{3.14}$$

Using these functions, we consider the function analogous to H:

$$\hat{H}(y) := \begin{cases} \dfrac{\hat{h}}{\hat{G}} \circ \hat{\psi}^{-1}(y) & \text{if } y > 0, \\[2mm] \lim\limits_{x \to -\infty} \dfrac{(\hat{h}(x))^+}{\hat{G}(x)} & \text{if } y = 0. \end{cases} \tag{3.15}$$

Following the steps (3.7)–(3.12) with F, G, ψ, and H replaced by $\hat{F}$, $\hat{G}$, $\hat{\psi}$, and $\hat{H}$, respectively, we write down the smallest concave majorant $\hat{W}$ of $\hat{H}$, namely,

$$\hat{W}(y) := \sup_{\{y_{\hat{a}}, y_{\hat{b}} : y_{\hat{a}} \le y \le y_{\hat{b}}\}} \left[\hat{H}(y_{\hat{a}}) \frac{y_{\hat{b}} - y}{y_{\hat{b}} - y_{\hat{a}}} + \hat{H}(y_{\hat{b}}) \frac{y - y_{\hat{a}}}{y_{\hat{b}} - y_{\hat{a}}} \right].$$

From this, we seek to determine the candidate optimal entry interval $(y_{\hat{a}^*}, y_{\hat{b}^*})$ in the $y = \hat{\psi}(x)$ coordinate. Following the proof of Theorem 3.1 with the new functions $\hat{F}$, $\hat{G}$, $\hat{\psi}$, $\hat{H}$, and $\hat{W}$, the value function of the optimal entry timing problem admits the expression

$$J(x) = \hat{G}(x)\hat{W}(\hat{\psi}(x)). \tag{3.16}$$

An alternative way to solve for $V(x)$ and $J(x)$ is to look for the solutions to the pair of variational inequalities

$$\min\{rV(x) - \mathcal{L}V(x), V(x) - (x - c)\} = 0, \tag{3.17}$$

$$\min\{\hat{r}J(x) - \mathcal{L}J(x), J(x) - (V(x) - x - \hat{c})\} = 0, \tag{3.18}$$

for $x \in \mathbb{R}$. With sufficient regularity conditions, this approach can verify that the solutions to the VIs, $V(x)$ and $J(x)$, indeed correspond to the optimal stopping problems (see, for example, Theorem 10.4.1 of Øksendal (2003)). Nevertheless, this approach does not immediately suggest candidate optimal timing strategies or value functions, and typically begins with a conjecture on the structure of the optimal stopping times, followed by verification. In contrast, our approach allows us to directly construct the value functions, at the cost of analyzing the properties of H, W, $\hat{H}$, and $\hat{W}$.

4. Analytical Results

We will first study the optimal exit timing in Sec. 4.1, followed by the optimal entry timing problem in Sec. 4.2.

4.1. *Optimal exit timing*

We now analyze the optimal exit timing problem (2.2). In preparation for the next result, we summarize the crucial properties of H.

Lemma 4.1. *The function H is continuous on $[0, +\infty)$, twice differentiable on $(0, +\infty)$ and possesses the following properties:*

(a) $H(0) = 0$, *and*

$$H(y) \begin{cases} < 0 & \text{if } y \in (0, \psi(c)), \\ > 0 & \text{if } y \in (\psi(c), +\infty). \end{cases}$$

(b) *Let x^* be the unique solution to $G(x) - (x - c)G'(x) = 0$. Then, we have*

$$H(y) \text{ is strictly} \begin{cases} \text{decreasing} & \text{if } y \in (0, \psi(x^*)), \\ \text{increasing} & \text{if } y \in (\psi(x^*), +\infty), \end{cases}$$

and $x^* < c \wedge L^*$ with

$$L^* = \frac{\mu\theta + rc}{\mu + r}. \tag{4.1}$$

(c)

$$H(y) \text{ is} \begin{cases} \text{convex} & \text{if } y \in (0, \psi(L^*)], \\ \text{concave} & \text{if } y \in [\psi(L^*), +\infty). \end{cases}$$

Based on Lemma 4.1, we sketch H in Fig. 2. The properties of H are essential in deriving the value function and optimal liquidation level, as we show next.

Theorem 4.1. *The optimal liquidation problem* (2.2) *admits the solution*

$$V(x) = \begin{cases} (b^* - c)\dfrac{F(x)}{F(b^*)} & \text{if } x \in (-\infty, b^*), \\ x - c & \text{otherwise,} \end{cases} \tag{4.2}$$

where the optimal liquidation level b^* is found from the equation
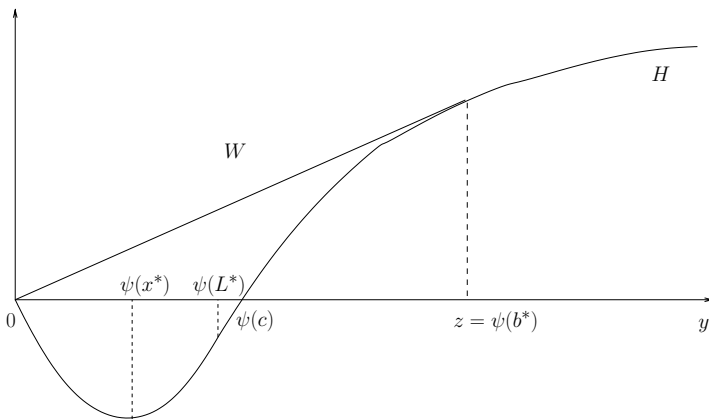
$$F(b) = (b - c)F'(b), \tag{4.3}$$

Fig. 2. Sketches of H and W. By Lemma 4.1, H is convex on the left of $\psi(L^*)$ and concave on the right. The smallest concave majorant W is a straight line tangent to H at z on $[0, z)$, and coincides with H on $[z, +\infty)$.

and is bounded below by $L^* \vee c$. The corresponding optimal liquidation time is given by

$$\tau^* = \inf\{t \geq 0 : X_t \geq b^*\}. \tag{4.4}$$

Proof. From Lemma 4.1 and the fact that $H'(y) \to 0$ as $y \to +\infty$ (see also Fig. 2), we infer that there exists a unique number $z > \psi(L^*) \vee \psi(c)$ such that

$$\frac{H(z)}{z} = H'(z). \tag{4.5}$$

In turn, the smallest concave majorant is given by

$$W(y) = \begin{cases} y\dfrac{H(z)}{z} & \text{if } y < z, \\[2mm] H(y) & \text{if } y \geq z. \end{cases} \tag{4.6}$$

Substituting $b^* = \psi^{-1}(z)$ into (4.5), we have the LHS

$$\frac{H(z)}{z} = \frac{H(\psi(b^*))}{\psi(b^*)} = \frac{b^* - c}{F(b^*)}, \tag{4.7}$$

and the RHS

$$H'(z) = \frac{G(\psi^{-1}(z)) - (\psi^{-1}(z) - c)G'(\psi^{-1}(z))}{F'(\psi^{-1}(z))G(\psi^{-1}(z)) - F(\psi^{-1}(z))G'(\psi^{-1}(z))}$$

$$= \frac{G(b^*) - (b^* - c)G'(b^*)}{F'(b^*)G(b^*) - F(b^*)G'(b^*)}.$$

Equivalently, we can express condition (4.5) in terms of b^*:

$$\frac{b^* - c}{F(b^*)} = \frac{G(b^*) - (b^* - c)G'(b^*)}{F'(b^*)G(b^*) - F(b^*)G'(b^*)},$$

which can be further simplified to

$$F(b^*) = (b^* - c)F'(b^*).$$

Applying (4.7) to (4.6), we get

$$W(\psi(x)) = \begin{cases} \psi(x)\dfrac{H(z)}{z} = \dfrac{F(x)}{G(x)}\dfrac{b^* - c}{F(b^*)} & \text{if } x < b^*, \\[3mm] H(\psi(x)) = \dfrac{x - c}{G(x)} & \text{if } x \geq b^*. \end{cases} \tag{4.8}$$

In turn, we obtain the value function $V(x)$ by substituting (4.8) into (3.13). $\qquad\square$

Next, we examine the dependence of the investor's optimal timing strategy on the transaction cost c.

Proposition 4.1. *The value function $V(x)$ of (2.2) is decreasing in the transaction cost c for every $x \in \mathbb{R}$, and the optimal liquidation level b^* is increasing in c.*

Proof. For any $x \in \mathbb{R}$ and $\tau \in \mathcal{T}$, the corresponding expected discounted reward, $\mathbb{E}_x\{e^{-r\tau}(X_\tau - c)\} = \mathbb{E}_x\{e^{-r\tau}X_\tau\} - c\,\mathbb{E}_x\{e^{-r\tau}\}$, is decreasing in c. This implies that $V(x)$ is also decreasing in c. Next, we treat the optimal threshold $b^*(c)$ as a function of c, and differentiate (4.3) w.r.t. c to get

$$b^{*\prime}(c) = \frac{F'(b^*)}{(b^* - c)F''(b^*)} > 0.$$

Since $F'(x) > 0$, $F''(x) > 0$ (see (3.3)), and $b^* > c$ according to Theorem 4.1, we conclude that b^* is increasing in c. $\qquad\square$

In other words, if the transaction cost is high, the investor would tend to liquidate at a higher level, in order to compensate the loss on transaction cost. For other parameters, such as μ and σ, the dependence of b^* is generally not monotone.

4.2. *Optimal entry timing*

Having solved for the optimal exit timing, we now turn to the optimal entry timing problem. In this case, the value function is

$$J(x) = \sup_{\nu \in \mathcal{T}} \mathbb{E}_x\{e^{-\hat{r}\nu}(V(X_\nu) - X_\nu - \hat{c})\}, \quad x \in \mathbb{R}, \tag{4.9}$$

where $V(x)$ is given by Theorem 4.1.

To solve for the optimal entry threshold(s), we will need several properties of $\hat{H}$, as we summarize below.

Lemma 4.2. *The function $\hat{H}$ is continuous on $[0, +\infty)$, differentiable on $(0, +\infty)$, and twice differentiable on $(0, \hat{\psi}(b^*)) \cup (\hat{\psi}(b^*), +\infty)$, and possesses the following properties*:

(a) $\hat{H}(0) = 0$. *Let $\bar{d}$ denote the unique solution to $\hat{h}(x) = 0$, then $\bar{d} < b^*$ and*

$$\hat{H}(y) \begin{cases} > 0 & \text{if } y \in (0, \hat{\psi}(\bar{d})), \\ < 0 & \text{if } y \in (\hat{\psi}(\bar{d}), +\infty). \end{cases}$$

(b) $\hat{H}(y)$ *is strictly decreasing if $y \in (\hat{\psi}(b^*), +\infty)$.*
(c) *Let $\underline{b}$ denote the unique solution to $(\mathcal{L} - \hat{r})\hat{h}(x) = 0$, then $\underline{b} < L^*$ and*

$$\hat{H}(y) \text{ is } \begin{cases} concave & \text{if } y \in (0, \hat{\psi}(\underline{b})), \\ convex & \text{if } y \in (\hat{\psi}(\underline{b}), +\infty). \end{cases}$$

In Fig. 3, we give a sketch of $\hat{H}$ according to Lemma 4.2. This will be useful for deriving the optimal entry level.

Theorem 4.2. *The optimal entry timing problem* (2.3) *admits the solution*

$$J(x) = \begin{cases} V(x) - x - \hat{c} & \text{if } x \in (-\infty, d^*], \\ \dfrac{V(d^*) - d^* - \hat{c}}{\hat{G}(d^*)}\hat{G}(x) & \text{if } x \in (d^*, +\infty), \end{cases} \tag{4.10}$$
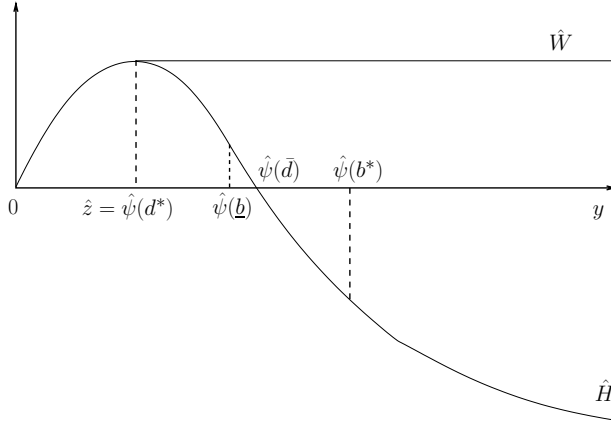
Fig. 3. Sketches of $\hat{H}$ and $\hat{W}$. The function $\hat{W}$ coincides with $\hat{H}$ on $[0, \hat{z}]$ and is equal to the constant $\hat{H}(\hat{z})$ on $(\hat{z}, +\infty)$.

where the optimal entry level d^* is found from the equation

$$\hat{G}(d)(V'(d) - 1) = \hat{G}'(d)(V(d) - d - \hat{c}). \tag{4.11}$$

Proof. We look for the value function of the form: $J(x) = \hat{G}(x)\hat{W}(\hat{\psi}(x))$, where $\hat{W}$ is the the smallest concave majorant of $\hat{H}$. From Lemma 4.2 and Fig. 3, we infer that there exists a unique number $\hat{z} < \hat{\psi}(b^*)$ such that

$$\hat{H}'(\hat{z}) = 0. \tag{4.12}$$

This implies that

$$\hat{W}(y) = \begin{cases} \hat{H}(y) & \text{if } y \leq \hat{z}, \\ \hat{H}(\hat{z}) & \text{if } y > \hat{z}. \end{cases} \tag{4.13}$$

Substituting $d^* = \hat{\psi}^{-1}(\hat{z})$ into (4.12), we have

$$\hat{H}'(\hat{z}) = \frac{\hat{G}(d^*)(V'(d^*) - 1) - \hat{G}'(d^*)(V(d^*) - d^* - \hat{c})}{\hat{F}'(d^*)\hat{G}(d^*) - \hat{F}(d^*)\hat{G}'(d^*)} = 0,$$

which is equivalent to condition (4.11). Furthermore, using (3.14) and (3.15), we get

$$\hat{H}(\hat{z}) = \frac{V(d^*) - d^* - \hat{c}}{\hat{G}(d^*)}. \tag{4.14}$$

To conclude, we substitute $\hat{H}(\hat{z})$ of (4.14) and $\hat{H}(y)$ of (3.15) into $\hat{W}$ of (4.13), which by (3.16) yields the value function $J(x)$ in (4.10). □

With the analytic solutions for V and J, we can verify by direct substitution that $V(x)$ in (4.2) and $J(x)$ in (4.10) satisfy both (3.17) and (3.18).

Since the optimal entry timing problem is nested with another optimal stopping problem, the parameter dependence of the optimal entry level is complicated. Below, we illustrate the impact of transaction cost.

Proposition 4.2. *The optimal entry level d^* of* (2.3) *is decreasing in the transaction cost $\hat{c}$.*

Proof. Considering the optimal entry level d^* as a function of $\hat{c}$, we differentiate (4.11) w.r.t. $\hat{c}$ to get

$$d^{*\prime}(\hat{c}) = \frac{-\hat{G}'(d^*)}{\hat{G}(d^*)} \left[V''(d^*) - \frac{V(d^*) - d^* - \hat{c}}{\hat{G}(d^*)} \hat{G}''(d^*) \right]^{-1}. \tag{4.15}$$

Since $\hat{G}(d^*) > 0$ and $\hat{G}'(d^*) < 0$, the sign of $d^{*\prime}(\hat{c})$ is determined by $V''(d^*) - \frac{V(d^*)-d^*-\hat{c}}{\hat{G}(d^*)}\hat{G}''(d^*)$. Denote $\hat{f}(x) = \frac{V(d^*)-d^*-\hat{c}}{\hat{G}(d^*)}\hat{G}(x)$. Recall that $\hat{h}(x) = V(x) - x - \hat{c}$,

$$J(x) = \begin{cases} \hat{h}(x) & \text{if } x \in (-\infty, d^*], \\ \hat{f}(x) > \hat{h}(x) & \text{if } x \in (d^*, +\infty), \end{cases}$$

and $\hat{f}(x)$ smooth pastes $\hat{h}(x)$ at d^*. Since both $\hat{h}(x)$ and $\hat{f}(x)$ are positive decreasing convex functions, it follows that $\hat{h}''(d^*) \leq \hat{f}''(d^*)$. Observing that $\hat{h}''(d^*) = V''(d^*)$ and $\hat{f}''(d^*) = \frac{V(d^*)-d^*-\hat{c}}{\hat{G}(d^*)}\hat{G}''(d^*)$, we have $V''(d^*) - \frac{V(d^*)-d^*-\hat{c}}{\hat{G}(d^*)}\hat{G}''(d^*) < 0$. Applying this to (4.15), we conclude that $d^{*\prime}(\hat{c}) \leq 0$. □

We end this section with a special example in the OU model with no mean reversion.

Remark 4.1. If we set $\mu = 0$ in (2.1), with r and $\hat{r}$ fixed, it follows that X reduces to a Brownian motion: $X_t = \sigma B_t$, $t \geq 0$. In this case, the optimal liquidation level b^* for problem (2.2) is

$$b^* = c + \frac{\sigma}{\sqrt{2r}},$$

and the optimal entry level d^* for problem (2.3) is the root to the equation

$$\left(1 + \sqrt{\frac{\hat{r}}{r}}\right) e^{\frac{\sqrt{2r}}{\sigma}(d-c-\frac{\sigma}{\sqrt{2r}})} = \frac{\sqrt{2\hat{r}}}{\sigma}(d+\hat{c}) + 1, \quad d \in (-\infty, b^*).$$

5. Incorporating Stop-Loss Exit

Now we consider the optimal entry and exit problems with a stop-loss constraint. For convenience, we restate the value functions from (2.4) and (2.5):

$$J_L(x) = \sup_{\nu \in \mathcal{T}} \mathbb{E}_x\{e^{-\hat{r}\nu}(V_L(X_\nu) - X_\nu - \hat{c})\}, \tag{5.1}$$

$$V_L(x) = \sup_{\tau \in \mathcal{T}} \mathbb{E}_x\{e^{-r(\tau \wedge \tau_L)}(X_{\tau \wedge \tau_L} - c)\}. \tag{5.2}$$

After solving for the optimal timing strategies, we will also examine the dependence of the optimal liquidation threshold on the stop-loss level L.

5.1. *Optimal exit timing*

We first give an analytic solution to the optimal exit timing problem.

Theorem 5.1. *The optimal liquidation problem* (5.2) *with stop-loss level L admits the solution*

$$V_L(x) = \begin{cases} CF(x) + DG(x) & \text{if } x \in (L, b_L^*), \\ x - c & \text{otherwise,} \end{cases} \tag{5.3}$$

where

$$C = \frac{(b_L^* - c)G(L) - (L - c)G(b_L^*)}{F(b_L^*)G(L) - F(L)G(b_L^*)}, \quad D = \frac{(L - c)F(b_L^*) - (b_L^* - c)F(L)}{F(b_L^*)G(L) - F(L)G(b_L^*)}. \tag{5.4}$$

The optimal liquidation level b_L^* is found from the equation

$$[(L - c)G(b) - (b - c)G(L)]F'(b) + [(b - c)F(L) - (L - c)F(b)]G'(b)$$
$$= G(b)F(L) - G(L)F(b). \tag{5.5}$$

Proof. Due to the stop-loss level L, we consider the smallest concave majorant of $H(y)$, denoted by $W_L(y)$, over the restricted domain $[\psi(L), +\infty)$ and set $W_L(y) = H(y)$ for $y \in [0, \psi(L)]$.

From Lemma 4.1 and Fig. 4, we see that $H(y)$ is convex over $(0, \psi(L^*)]$ and concave in $[\psi(L^*), +\infty)$. If $L \geq L^*$, then $H(y)$ is concave over $[\psi(L), +\infty)$, which implies that $W_L(y) = H(y)$ for $y \geq 0$, and thus $V_L(x) = x - c$ for $x \in \mathbb{R}$. On the
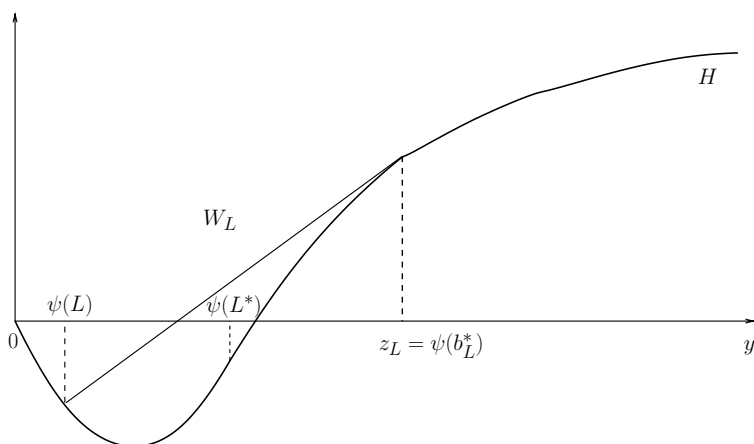

Fig. 4. Sketch of W_L. On $[0, \psi(L)] \cup [z_L, +\infty)$, W_L coincides with H, and over $(\psi(L), z_L)$, W_L is a straight line tangent to H at z_L.

other hand, if $L < L^*$, then $H(y)$ is convex on $[\psi(L), \psi(L^*)]$, and concave strictly increasing on $[\psi(L^*), +\infty)$. There exists a unique number $z_L > \psi(L^*)$ such that

$$\frac{H(z_L) - H(\psi(L))}{z_L - \psi(L)} = H'(z_L). \tag{5.6}$$

In turn, the smallest concave majorant admits the form:

$$W_L(y) = \begin{cases} H(\psi(L)) + (y - \psi(L))H'(z_L) & \text{if } y \in (\psi(L), z_L), \\ H(y) & \text{otherwise.} \end{cases} \tag{5.7}$$

Substituting $b_L^* = \psi^{-1}(z_L)$ into (5.6), we have from the LHS

$$\frac{H(z_L) - H(\psi(L))}{z_L - \psi(L)} = \frac{H(\psi(b_L^*)) - H(\psi(L))}{\psi(b_L^*) - \psi(L)} = \frac{\frac{b_L^*-c}{G(b_L^*)} - \frac{L-c}{G(L)}}{\frac{F(b_L^*)}{G(b_L^*)} - \frac{F(L)}{G(L)}} = C,$$

and the RHS

$$H'(z_L) = \frac{G(\psi^{-1}(z_L)) - (\psi^{-1}(z_L) - c)G'(\psi^{-1}(z_L))}{F'(\psi^{-1}(z_L))G(\psi^{-1}(z_L)) - F(\psi^{-1}(z_L))G'(\psi^{-1}(z_L))}$$

$$= \frac{G(b_L^*) - (b^* - c)G'(b_L^*)}{F'(b_L^*)G(b_L^*) - F(b_L^*)G'(b_L^*)}.$$

Therefore, we can equivalently express (5.6) in terms of b_L^*:

$$\frac{(b_L^* - c)G(L) - (L - c)G(b_L^*)}{F(b_L^*)G(L) - F(L)G(b_L^*)} = \frac{G(b_L^*) - (b_L^* - c)G'(b_L^*)}{F'(b_L^*)G(b_L^*) - F(b_L^*)G'(b_L^*)},$$

which by rearrangement immediately simplifies to (5.5).

Furthermore, for $x \in (L, b_L^*)$, $H'(z_L) = C$ implies that

$$W_L(\psi(x)) = H(\psi(L)) + (\psi(x) - \psi(L))C.$$

Substituting this to $V_L(x) = G(x)W_L(\psi(x))$, the value function becomes

$$V_L(x) = G(x)[H(\psi(L)) + (\psi(x) - \psi(L))C] = CF(x) + G(x)[H(\psi(L)) - \psi(L)C],$$

which resembles (5.3) after the observation that

$$H(\psi(L)) - \psi(L)C = \frac{L - c}{G(L)} - \frac{F(L)}{G(L)} \frac{(b_L^* - c)G(L) - (L - c)G(b_L^*)}{F(b_L^*)G(L) - F(L)G(b_L^*)}$$

$$= \frac{(L - c)F(b_L^*) - (b_L^* - c)F(L)}{F(b_L^*)G(L) - F(L)G(b_L^*)} = D. \qquad \square$$

We can interpret the investor's timing strategy in terms of three price intervals, namely, the liquidation region $[b_L^*, +\infty)$, the delay region (L, b_L^*), and the stop-loss region $(-\infty, L]$. In both liquidation and stop-loss regions, the value function $V_L(x) = x - c$, and therefore, the investor will immediately close out the position. From the proof of Theorem 5.1, if $L \geq L^* = \frac{\mu\theta + rc}{\mu + r}$ (see (4.1)), then $V_L(x) = x - c, \forall x \in \mathbb{R}$. In other words, if the stop-loss level is too high, then the delay region

completely disappears, and the investor will liquidate immediately for every initial value $x \in \mathbb{R}$.

Corollary 5.1. *If $L < L^*$, then there exists a unique solution $b_L^* \in (L^*, +\infty)$ that solves (5.5). If $L \geq L^*$, then $V_L(x) = x - c$, for $x \in \mathbb{R}$.*

The direct effect of a stop-loss exit constraint is forced liquidation whenever the price process reaches L before the upper liquidation level b_L^*. Interestingly, there is an additional indirect effect: a higher stop-loss level will induce the investor to *voluntarily* liquidate earlier at a lower take-profit level.

Proposition 5.1. *The optimal liquidation level b_L^* of (5.2) strictly decreases as the stop-loss level L increases.*

Proof. Recall that $z_L = \psi(b_L^*)$ and ψ is a strictly increasing function. Therefore, it is sufficient to show that z_L strictly decreases as $\tilde{L} := \psi(L)$ increases. As such, we denote $z_L(\tilde{L})$ to highlight its dependence on $\tilde{L}$. Differentiating (5.6) w.r.t. $\tilde{L}$ gives

$$z_L'(\tilde{L}) = \frac{H'(z_L) - H'(\tilde{L})}{H''(z_L)(z_L - \tilde{L})}. \tag{5.8}$$

It follows from the definitions of W_L and z_L that $H'(z_L) > H'(\tilde{L})$ and $z_L > \tilde{L}$. Also, we have $H''(z) < 0$ since H is concave at z_L. Applying these to (5.8), we conclude that $z_L'(\tilde{L}) < 0$. □

Figure 5 illustrates the optimal exit price level b_L^* as a function of the stop-loss levels L, for different long-run means θ. When b_L^* is strictly greater than L (on the left of the straight line), the delay region is nonempty. As L increases, b_L^* strictly decreases and the two meet at L^* (on the straight line), and the delay region vanishes.

Also, there is an interesting connection between cases with different long-run means and transaction costs. To this end, let us denote the value function by
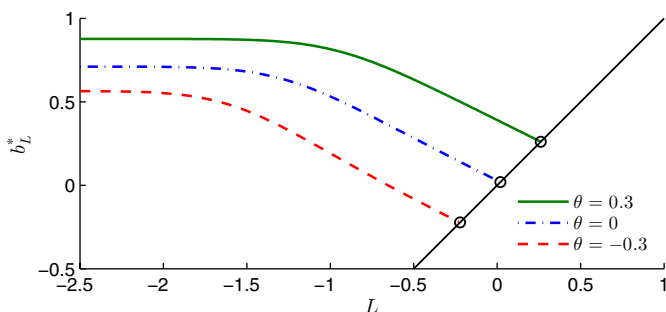

Fig. 5. The optimal exit threshold b_L^* is strictly decreasing with respect to the stop-loss level L. The straight line is where $b_L^* = L$, and each of the three circles locates the critical stop-loss level L^*.

$V_L(x; \theta, c)$ to highlight the dependence on θ and c, and the corresponding optimal liquidation level by $b_L^*(\theta, c)$. We find that, for any $\theta_1, \theta_2 \in \mathbb{R}$, $c_1, c_2 > 0$, $L_1 \leq \frac{\mu\theta_1 + rc_1}{\mu + r}$, and $L_2 \leq \frac{\mu\theta_2 + rc_2}{\mu + r}$, the associated value functions and optimal liquidation levels satisfy the relationships:

$$V_{L_1}(x + \theta_1; \theta_1, c_1) = V_{L_2}(x + \theta_2; \theta_2, c_2), \tag{5.9}$$

$$b_{L_1}^*(\theta_1, c_1) - \theta_1 = b_{L_2}^*(\theta_2, c_2) - \theta_2, \tag{5.10}$$

as long as $\theta_1 - \theta_2 = c_1 - c_2 = L_1 - L_2$. These results (5.9) and (5.10) also hold in the case without stop-loss.

5.2. *Optimal entry timing*

We now discuss the optimal entry timing problem $J_L(x)$ defined in (5.1). Since $\sup_{x \in \mathbb{R}}(V_L(x) - x - \hat{c}) \leq 0$ implies that $J_L(x) = 0$ for $x \in \mathbb{R}$, we can focus on the case with

$$\sup_{x \in \mathbb{R}}(V_L(x) - x - \hat{c}) > 0, \tag{5.11}$$

and look for nontrivial optimal timing strategies.

Associated with reward function $\hat{h}_L(x) := V_L(x) - x - \hat{c}$ from entering the market, we define the function $\hat{H}_L$ as in (3.11) whose properties are summarized in the following lemma.

Lemma 5.1. *The function $\hat{H}_L$ is continuous on $[0, +\infty)$, differentiable on $(0, \hat{\psi}(L)) \cup (\hat{\psi}(L), +\infty)$, twice differentiable on $(0, \hat{\psi}(L)) \cup (\hat{\psi}(L), \hat{\psi}(b_L^*)) \cup (\hat{\psi}(b_L^*), +\infty)$, and possesses the following properties:*

(a) $\hat{H}_L(0) = 0$. $\hat{H}_L(y) < 0$ *for $y \in (0, \hat{\psi}(L)] \cup [\hat{\psi}(b_L^*), +\infty)$.*
(b) $\hat{H}_L(y)$ *is strictly decreasing for $y \in (0, \hat{\psi}(L)) \cup (\hat{\psi}(b_L^*), +\infty)$.*
(c) *There exists some constant $\bar{d}_L \in (L, b^*)$ such that $(\mathcal{L} - \hat{r})\hat{h}_L(\bar{d}_L) = 0$, and*

$$\hat{H}_L(y) \text{ is } \begin{cases} convex & \text{if } y \in (0, \hat{\psi}(L)) \cup (\hat{\psi}(\bar{d}_L), +\infty), \\ concave & \text{if } y \in (\hat{\psi}(L), \hat{\psi}(\bar{d}_L)). \end{cases}$$

In addition, $\hat{z}_1 \in (\hat{\psi}(L), \hat{\psi}(\bar{d}_L))$, where $\hat{z}_1 := \arg\max_{y \in [0, +\infty)} \hat{H}_L(y)$.

Theorem 5.2. *The optimal entry timing problem (5.1) admits the solution*

$$J_L(x) = \begin{cases} P\hat{F}(x) & \text{if } x \in (-\infty, a_L^*), \\ V_L(x) - x - \hat{c} & \text{if } x \in [a_L^*, d_L^*], \\ Q\hat{G}(x) & \text{if } x \in (d_L^*, +\infty), \end{cases} \tag{5.12}$$

where

$$P = \frac{V_L(a_L^*) - a_L^* - \hat{c}}{\hat{F}(a_L^*)}, \quad Q = \frac{V_L(d_L^*) - d_L^* - \hat{c}}{\hat{G}(d_L^*)}. \tag{5.13}$$

T. Leung & X. Li

The optimal entry time is given by

$$\nu_{a_L^*, d_L^*} = \inf\{t \geq 0 : X_t \in [a_L^*, d_L^*]\}, \tag{5.14}$$

where the critical levels a_L^* and d_L^* satisfy, respectively,

$$\hat{F}(a)(V_L'(a) - 1) = \hat{F}'(a)(V_L(a) - a - \hat{c}), \tag{5.15}$$

and

$$\hat{G}(d)(V_L'(d) - 1) = \hat{G}'(d)(V_L(d) - d - \hat{c}). \tag{5.16}$$

Proof. We look for the value function of the form: $J_L(x) = \hat{G}(x)\hat{W}_L(\hat{\psi}(x))$, where $\hat{W}_L$ is the smallest nonnegative concave majorant of $\hat{H}_L$. From Lemma 5.1 and the sketch of $\hat{H}_L$ in Fig. 6, the maximizer of $\hat{H}_L$, $\hat{z}_1$, satisfies

$$\hat{H}_L'(\hat{z}_1) = 0. \tag{5.17}$$

Also there exists a unique number $\hat{z}_0 \in (\hat{\psi}(L), \hat{z}_1)$ such that

$$\frac{\hat{H}_L(\hat{z}_0)}{\hat{z}_0} = \hat{H}_L'(\hat{z}_0). \tag{5.18}$$

In turn, the smallest nonnegative concave majorant admits the form:

$$\hat{W}_L(y) = \begin{cases} y\hat{H}_L'(\hat{z}_0) & \text{if } y \in [0, \hat{z}_0), \\ \hat{H}_L(y) & \text{if } y \in [\hat{z}_0, \hat{z}_1], \\ \hat{H}_L(\hat{z}_1) & \text{if } y \in (\hat{z}_1, +\infty). \end{cases}$$
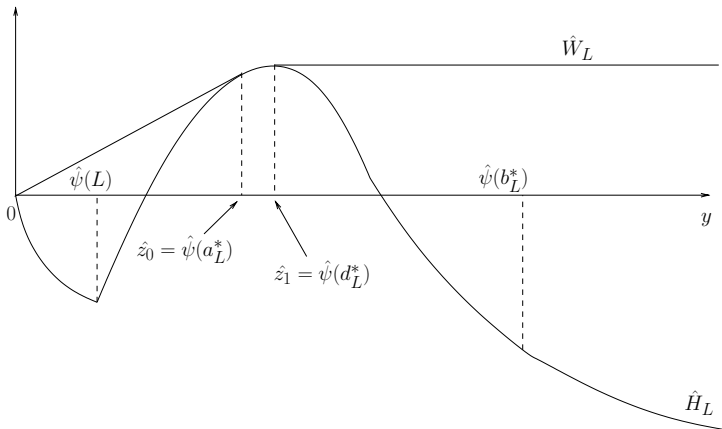

Fig. 6. Sketches of $\hat{H}_L$ and $\hat{W}_L$. $\hat{W}_L$ is a straight line tangent to $\hat{H}_L$ at $\hat{z}_0$ on $[0, \hat{z}_0)$, coincides with $\hat{H}_L$ on $[\hat{z}_0, \hat{z}_1]$, and is equal to the constant $\hat{H}_L(\hat{z}_1)$ on $(\hat{z}_1, +\infty)$. Note that $\hat{H}_L$ is not differentiable at $\hat{\psi}(L)$.

Substituting $a_L^* = \hat{\psi}^{-1}(\hat{z}_0)$ into (5.18), we have

$$\frac{\hat{H}_L(\hat{z}_0)}{\hat{z}_0} = \frac{V_L(a_L^*) - a_L^* - \hat{c}}{\hat{F}(a_L^*)},$$

$$\hat{H}_L'(\hat{z}_0) = \frac{\hat{G}(a_L^*)(V_L'(a_L^*) - 1) - \hat{G}'(a_L^*)(V_L(a_L^*) - a_L^* - \hat{c})}{\hat{F}'(a_L^*)\hat{G}(a_L^*) - \hat{F}(a_L^*)\hat{G}'(a_L^*)}.$$

Equivalently, we can express condition (5.18) in terms of a_L^*:

$$\frac{V_L(a_L^*) - a_L^* - \hat{c}}{\hat{F}(a_L^*)} = \frac{\hat{G}(a_L^*)(V_L'(a_L^*) - 1) - \hat{G}'(a_L^*)(V_L(a_L^*) - a_L^* - \hat{c})}{\hat{F}'(a_L^*)\hat{G}(a_L^*) - \hat{F}(a_L^*)\hat{G}'(a_L^*)}.$$

Simplifying this shows that a_L^* solves (5.15). Also, we can express $\hat{H}_L'(\hat{z}_0)$ in terms of a_L^*:

$$\hat{H}_L'(\hat{z}_0) = \frac{\hat{H}_L(\hat{z}_0)}{\hat{z}_0} = \frac{V_L(a_L^*) - a_L^* - \hat{c}}{\hat{F}(a_L^*)} = P.$$

In addition, substituting $d_L^* = \hat{\psi}^{-1}(\hat{z}_1)$ into (5.17), we have

$$\hat{H}_L'(\hat{z}_1) = \frac{\hat{G}(d_L^*)(V_L'(d_L^*) - 1) - \hat{G}'(d_L^*)(V_L(d_L^*) - d_L^* - \hat{c})}{\hat{F}'(d_L^*)\hat{G}(d_L^*) - \hat{F}(d_L^*)\hat{G}'(d_L^*)} = 0,$$

which, after a straightforward simplification, is identical to (5.16). Also, $\hat{H}_L(\hat{z}_1)$ can be written as

$$\hat{H}_L(\hat{z}_1) = \frac{V_L(d_L^*) - d_L^* - \hat{c}}{\hat{G}(d_L^*)} = Q.$$

Substituting these to $J_L(x) = \hat{G}(x)\hat{W}_L(\hat{\psi}(x))$, we arrive at (5.12). $\qquad\square$

Theorem 5.2 reveals that the optimal entry region is characterized by a price interval $[a_L^*, d_L^*]$ strictly above the stop-loss level L and strictly below the optimal exit level b_L^*. In particular, if the current asset price is between L and a_L^*, then it is optimal for the investor to wait even though the price is low. This is intuitive because if the entry price is too close to L, then the investor is very likely to be forced to exit at a loss afterwards. As a consequence, the investor's delay region, where she would wait to enter the market, is disconnected.

Figure 7 illustrates two simulated paths and the associated exercise times. We have chosen L to be 2 standard deviations below the long-run mean θ, with other parameters from our pairs trading example. By Theorem 5.2, the investor will enter the market at $\nu_{a_L^*, d_L^*}$ (see (5.14)). Since both paths start with $X_0 > d_L^*$, the investor waits to enter until the OU path reaches d_L^* from above, as indicated by ν_d^* in panels (a) and (b). After entry, Fig. 7(a) describes the scenario where the investor exits voluntarily at the optimal level b_L^*, whereas in Fig. 7(b) the investor is forced to exit at the stop-loss level L. These optimal levels are calculated from Theorem 5.2 and Theorem 5.1 based on the given estimated parameters.
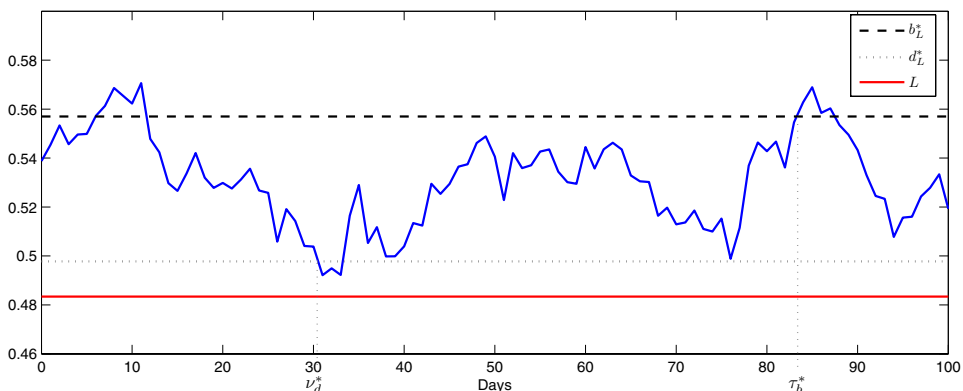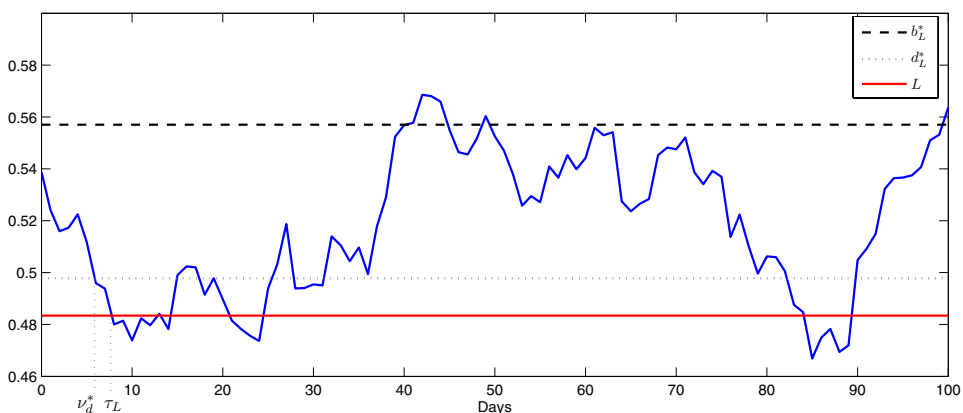
Fig. 7. Simulated OU paths and exercise times. (a) The investor enters at $\nu_d^* = \inf\{t \geq 0 : X_t \leq d_L^*\}$ with $d_L^* = 0.4978$, and exit at $\tau_b^* = \inf\{t \geq \nu_d^* : X_t \geq b_L^*\}$ with $b_L^* = 0.5570$. (b) The investor enters at $\nu_d^* = \inf\{t \geq 0 : X_t \leq d_L^*\}$ but exits at stop-loss level $L = 0.4834$. Parameters: $\theta = 0.5388$, $\mu = 16.6677$, $\sigma = 0.1599$, $r = \hat{r} = 0.05$, and $c = \hat{c} = 0.05$.

Remark 5.1. We remark that the optimal levels a_L^*, d_L^* and b_L^* are outputs of the models, depending on the parameters (μ, θ, σ) and the choice of stop-loss level L. Recall that our model parameters are estimated based on the likelihood maximizing portfolio discussed in Sec. 2.1. Other estimation methodologies and price data can be used, and may lead to different portfolio strategies (α, β) and estimated parameters values (μ, θ, σ). In turn, the resulting optimal entry and exit thresholds may also change accordingly.

5.3. *Relative stop-loss exit*

For some investors, it may be more desirable to set the stop-loss contingent on the entry level. In other words, if the value of X at the entry time is x, then the investor
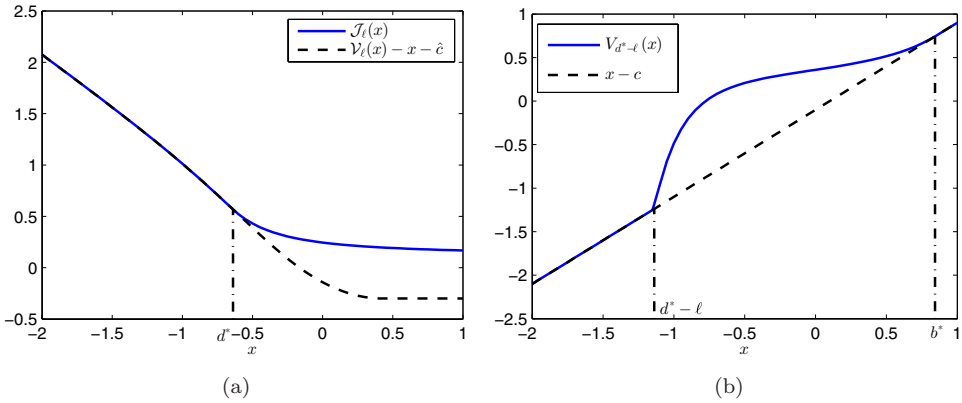
Fig. 8. (a) The optimal entry value function $\mathcal{J}_\ell(x)$ dominates the reward function $\mathcal{V}_\ell(x) - x - \hat{c}$, and they coincide for $x \leq d^*$. (b) For the exit problem, the stop-loss level is $d^* - \ell$ and the optimal liquidation level is b^*.

would assign a lower stop-loss level $x - \ell$, for some constant $\ell > 0$. Therefore, the investor faces the optimal entry timing problem

$$\mathcal{J}_\ell(x) = \sup_{\nu \in \mathcal{T}} \mathbb{E}_x\{e^{-\hat{r}\nu}(\mathcal{V}_\ell(X_\nu) - X_\nu - \hat{c})\}, \tag{5.19}$$

where $\mathcal{V}_\ell(x) := V_{x-\ell}(x)$ (see (5.2)) is the optimal exit timing problem with stop-loss level $x - \ell$. The dependence of $V_{x-\ell}(x)$ on x is significantly more complicated than $V(x)$ or $V_L(x)$, making the problem much less tractable. In Fig. 8, we illustrate numerically the optimal timing strategies. The investor will still enter at a lower level d^*. After entry, the investor will wait to exit at either the stop-loss level $d^* - \ell$ or an upper level b^*.

5.4. *Concluding remarks*

Other extensions include adapting our double optimal stopping problem to the exponential OU, Cox–Ingersoll–Ross (CIR), or other underlying dynamics, and to countable number of trades (Zervos *et al.* 2013, Zhang & Zhang 2008). Alternatively, one can model asset prices by specifying the dynamics of the dividend stream. For instance, Scheinkman & Xiong (2003) study the optimal timing to trade between two speculative traders with different beliefs on the mean-reverting (OU) dividend dynamics. Other than trading of risky assets, it is also useful to study the timing to buy/sell derivatives written on a mean-reverting underlying (see e.g. Leung & Liu 2012 and Leung & Shirai 2013). For all these applications, it is natural to examine the optimal stopping problems over a finite horizon although explicit solutions are less available.

Acknowledgments

Appendix A

A.1. *Proof of Theorem* **3.1** (*Optimality of V*)

Since $\tau_a \wedge \tau_b \in \mathcal{T}$, we have $V(x) \geq \sup_{\{a,b:a \leq x \leq b\}} \mathbb{E}_x\{e^{-r(\tau_a \wedge \tau_b)}h(X_{\tau_a \wedge \tau_b})\} = G(x)W(\psi(x))$.

To show the reverse inequality, we first show that $G(x)W(\psi(x)) \geq \mathbb{E}_x\{e^{-r(t \wedge \tau)}G(X_{t \wedge \tau})W(\psi(X_{t \wedge \tau}))\}$, for $\tau \in \mathcal{T}$ and $t \geq 0$. The concavity of W implies that, for any fixed y, there exists an affine function $L_y(z) := m_y z + c_y$ such that $L_y(z) \geq W(z)$ and $L_y(y) = W(y)$ at $z = y$, where m_y and c_y are both constants depending on y. This leads to the inequality

$$
\mathbb{E}_x\{e^{-r(t \wedge \tau)}G(X_{t \wedge \tau})W(\psi(X_{t \wedge \tau}))\}
$$

$$
\leq \mathbb{E}_x\{e^{-r(t \wedge \tau)}G(X_{t \wedge \tau})L_{\psi(x)}(\psi(X_{t \wedge \tau}))\}
$$

$$
= m_{\psi(x)}\mathbb{E}_x\{e^{-r(t \wedge \tau)}G(X_{t \wedge \tau})\psi(X_{t \wedge \tau})\} + c_{\psi(x)}\mathbb{E}_x\{e^{-r(t \wedge \tau)}G(X_{t \wedge \tau})\}
$$

$$
= m_{\psi(x)}\mathbb{E}_x\{e^{-r(t \wedge \tau)}F(X_{t \wedge \tau})\} + c_{\psi(x)}\mathbb{E}_x\{e^{-r(t \wedge \tau)}G(X_{t \wedge \tau})\}
$$

$$
= m_{\psi(x)}F(x) + c_{\psi(x)}G(x) \tag{A.1}
$$

$$
= G(x)L_{\psi(x)}(\psi(x))
$$

$$
= G(x)W(\psi(x)), \tag{A.2}
$$

where (A.1) follows from the martingale property of $(e^{-rt}F(X_t))_{t \geq 0}$ and $(e^{-rt}G(X_t))_{t \geq 0}$.

By (A.2) and the fact that W majorizes H, it follows that

$$
G(x)W(\psi(x)) \geq \mathbb{E}_x\{e^{-r(t \wedge \tau)}G(X_{t \wedge \tau})W(\psi(X_{t \wedge \tau}))\}
$$

$$
\geq \mathbb{E}_x\{e^{-r(t \wedge \tau)}G(X_{t \wedge \tau})H(\psi(X_{t \wedge \tau}))\} = \mathbb{E}_x\{e^{-r(t \wedge \tau)}h(X_{t \wedge \tau})\}. \tag{A.3}
$$

Maximizing (A.3) over all $\tau \in \mathcal{T}$ and $t \geq 0$ yields that $G(x)W(\psi(x)) \geq V(x)$. $\qquad\square$

A.2. *Proof of Lemma* **4.1** (*Properties of H*)

The continuity and twice differentiability of H on $(0, +\infty)$ follow directly from those of h, G and ψ. To show the continuity of H at 0, since $H(0) = \lim_{x \to -\infty} \frac{(x-c)^+}{G(x)} = 0$,

we only need to show that $\lim_{y \to 0} H(y) = 0$. Note that $y = \psi(x) \to 0$, as $x \to -\infty$. Therefore,

$$\lim_{y \to 0} H(y) = \lim_{x \to -\infty} \frac{h(x)}{G(x)} = \lim_{x \to -\infty} \frac{x - c}{G(x)} = \lim_{x \to -\infty} \frac{1}{G'(x)} = 0.$$

We conclude that H is also continuous at 0.

(a) One can show that $\psi(x) \in (0, +\infty)$ for $x \in \mathbb{R}$ and is a strictly increasing function. Then property (i) follows directly from the fact that $G(x) > 0$.

(b) By the definition of H,

$$H'(y) = \frac{1}{\psi'(x)} \left(\frac{h}{G} \right)'(x) = \frac{h'(x)G(x) - h(x)G'(x)}{\psi'(x)G^2(x)}, \quad y = \psi(x).$$

Since both $\psi'(x)$ and $G^2(x)$ are positive, we only need to determine the sign of $h'(x)G(x) - h(x)G'(x) = G(x) - (x - c)G'(x)$.

Define $u(x) := (x - c) - \frac{G(x)}{G'(x)}$. $u(x) + c$ is the intersecting point at x axis of the tangent line of $G(x)$. Since $G(\cdot)$ is a positive, strictly decreasing and convex function, $u(x)$ is strictly increasing and $u(x) < 0$ as $x \to -\infty$. Also, note that

$$u(c) = -\frac{G(c)}{G'(c)} > 0,$$

$$u(L^*) = (L^* - c) - \frac{G(x)}{G'(x)} = \frac{\mu}{r}(\theta - L^*) - \frac{G(L^*)}{G'(L^*)} = -\frac{\sigma^2}{2r} \frac{G''(L^*)}{G'(L^*)} > 0.$$

Therefore, there exists a unique root x^* that solves $u(x) = 0$, and $x^* < c \wedge L^*$, such that

$$G(x) - (x - c)G'(x) \begin{cases} < 0 & \text{if } x \in (-\infty, x^*), \\ > 0 & \text{if } x \in (x^*, +\infty). \end{cases}$$

Thus $H(y)$ is strictly decreasing if $y \in (0, \psi(x^*))$, and increasing otherwise.

(c) By the definition of H,

$$H''(y) = \frac{2}{\sigma^2 G(x)(\psi'(x))^2}(\mathcal{L} - r)h(x), \quad y = \psi(x).$$

Since $\sigma^2, G(x)$ and $(\psi'(x))^2$ are all positive, we only need to determine the sign of $(\mathcal{L} - r)h(x)$:

$$(\mathcal{L} - r)h(x) = \mu(\theta - x) - r(x - c)$$

$$= (\mu\theta + rc) - (\mu + r)x \begin{cases} \geq 0 & \text{if } x \in (-\infty, L^*], \\ \leq 0 & \text{if } x \in [L^*, +\infty). \end{cases}$$

Therefore, $H(y)$ is convex if $y \in (0, \psi(L^*)]$, and concave otherwise. $\qquad \square$

A.3. *Proof of Lemma* 4.2 (*Properties of* $\hat{H}$)

We first show that $V(x)$ and $\hat{h}(x)$ are twice differentiable everywhere, except for $x = b^*$. Recall that

$$V(x) = \begin{cases} (b^* - c)\dfrac{F(x)}{F(b^*)} & \text{if } x \in (-\infty, b^*), \\ x - c & \text{otherwise,} \end{cases} \quad \text{and} \quad \hat{h}(x) = V(x) - x - \hat{c}.$$

Therefore, it follows from (4.3) that

$$V'(x) = \begin{cases} (b^* - c)\dfrac{F'(x)}{F(b^*)} = \dfrac{F'(x)}{F'(b^*)} & \text{if } x \in (-\infty, b^*), \\ 1 & \text{if } x \in (b^*, +\infty), \end{cases}$$

which implies that $V'(b^*-) = 1 = V'(b^*+)$. Therefore, $V(x)$ is differentiable everywhere and so is $\hat{h}$. However, $V(x)$ is not twice differentiable since

$$V''(x) = \begin{cases} \dfrac{F''(x)}{F'(b^*)} & \text{if } x \in (-\infty, b^*), \\ 0 & \text{if } x \in (b^*, +\infty), \end{cases}$$

and $V''(b^*-) \neq V''(b^*+)$. Consequently, $\hat{h}(x) = V(x) - x - \hat{c}$ is not twice differentiable at b^*.

The twice differentiability of $\hat{G}$ and $\hat{\psi}$ are straightforward. The continuity and differentiability of $\hat{H}$ on $(0, +\infty)$ and twice differentiability on $(0, \hat{\psi}(b^*)) \cup (\hat{\psi}(b^*), +\infty)$ follow directly. Observing that $\hat{h}(x) > 0$ as $x \to -\infty$, $\hat{H}$ is also continuous at 0 by definition. We now establish the properties of $\hat{H}$.

(a) First we prove the value of $\hat{H}$ at 0:

$$\hat{H}(0) = \lim_{x \to -\infty} \frac{(\hat{h}(x))^+}{\hat{G}(x)}$$

$$= \limsup_{x \to -\infty} \frac{\frac{(b^*-c)}{F(b^*)}F(x) - x - \hat{c}}{\hat{G}(x)} = \limsup_{x \to -\infty} \frac{\frac{(b^*-c)}{F(b^*)}F'(x) - 1}{\hat{G}'(x)} = 0.$$

Next, observe that $\lim_{x \to -\infty} \hat{h}(x) = +\infty$ and $\hat{h}(x) = -(c + \hat{c})$, for $x \in [b^*, +\infty)$. Since $F'(x)$ is strictly increasing and $F'(x) > 0$ for $x \in \mathbb{R}$, we have, for $x < b^*$,

$$\hat{h}'(x) = V'(x) - 1 = \frac{F'(x)}{F'(b^*)} - 1 < \frac{F'(b^*)}{F'(b^*)} - 1 = 0,$$

which implies that $\hat{h}(x)$ is strictly decreasing for $x \in (-\infty, b^*)$. Therefore, there exists a unique solution $\bar{d}$ to $\hat{h}(x) = 0$, and $\bar{d} < b^*$, such that $\hat{h}(x) > 0$ if $x \in (-\infty, \bar{d})$ and $\hat{h}(x) < 0$ if $x \in (\bar{d}, +\infty)$. It is trivial that $\hat{\psi}(x) \in (0, +\infty)$ for $x \in \mathbb{R}$ and is a strictly increasing function. Therefore, along with the fact that $\hat{G}(x) > 0$, property (a) follows directly.

(b) With $y = \hat{\psi}(x)$, for $x > b^*$,

$$\hat{H}'(y) = \frac{1}{\hat{\psi}'(x)} \left(\frac{\hat{h}}{\hat{G}} \right)'(x) = \frac{1}{\hat{\psi}'(x)} \left(\frac{-(c+\hat{c})}{\hat{G}(x)} \right)' = \frac{1}{\hat{\psi}'(x)} \frac{(c+\hat{c})\hat{G}'(x)}{\hat{G}^2(x)} < 0,$$

since $\hat{\psi}'(x) > 0$, $\hat{G}'(x) < 0$, and $\hat{G}^2(x) > 0$. Therefore, $\hat{H}(y)$ is strictly decreasing for $y > \hat{\psi}(b^*)$.

(c) By the definition of $\hat{H}$,

$$\hat{H}''(y) = \frac{2}{\sigma^2 \hat{G}(x)(\hat{\psi}'(x))^2} (\mathcal{L} - \hat{r})\hat{h}(x), \quad y = \hat{\psi}(x).$$

Since σ^2, $\hat{G}(x)$ and $(\hat{\psi}'(x))^2$ are all positive, we only need to determine the sign of $(\mathcal{L} - \hat{r})\hat{h}(x)$:

$$(\mathcal{L} - \hat{r})\hat{h}(x) = \frac{1}{2}\sigma^2 V''(x) + \mu(\theta - x)V'(x) - \mu(\theta - x) - \hat{r}(V(x) - x - \hat{c})$$

$$= \begin{cases} (r - \hat{r})V(x) + (\mu + \hat{r})x - \mu\theta + \hat{r}\hat{c} & \text{if } x < b^*, \\ \hat{r}(c + \hat{c}) > 0 & \text{if } x > b^*. \end{cases}$$

To determine the sign of $(\mathcal{L} - \hat{r})\hat{h}(x)$ in $(-\infty, b^*)$, first note that $[(\mathcal{L} - \hat{r})\hat{h}](x)$ is a strictly increasing function in $(-\infty, b^*)$, since $V(x)$ is a strictly increasing function and $r \geq \hat{r}$ by assumption. Next note that for $x \in [L^*, b^*)$,

$$(\mathcal{L} - \hat{r})\hat{h}(x) = (r - \hat{r})V(x) + (\mu + \hat{r})x - \mu\theta + \hat{r}\hat{c}$$

$$\geq (r - \hat{r})(x - c) + (\mu + \hat{r})x - \mu\theta + \hat{r}\hat{c}$$

$$= (r + \mu)x - (\mu\theta + rc) + \hat{r}(c + \hat{c})$$

$$\geq (r + \mu)L^* - (\mu\theta + rc) + \hat{r}(c + \hat{c}) = \hat{r}(c + \hat{c}) > 0.$$

Also, note that $(\mathcal{L} - \hat{r})\hat{h}(x) \to -\infty$ as $x \to -\infty$. Therefore, $(\mathcal{L} - \hat{r})\hat{h}(x) < 0$ if $x \in (-\infty, \underline{b})$ and $(\mathcal{L} - \hat{r})\hat{h}(x) > 0$ if $x \in (\underline{b}, +\infty)$ with $\underline{b} < L^*$ being the break-even point. From this, we conclude property (c). $\qquad \square$

A.4. *Proof of Lemma* **5.1** (*Properties of* $\hat{H}_L$)

(a) The continuity of $\hat{H}_L(y)$ on $(0, +\infty)$ is implied by the continuities of $\hat{h}_L$, $\hat{G}$ and $\hat{\psi}$. The continuity of $\hat{H}_L(y)$ at 0 follows from

$$\hat{H}_L(0) = \lim_{x \to -\infty} \frac{(\hat{h}_L(x))^+}{\hat{G}(x)} = \lim_{x \to -\infty} \frac{0}{\hat{G}(x)} = 0,$$

$$\lim_{y \to 0} \hat{H}_L(y) = \lim_{x \to -\infty} \frac{\hat{h}_L}{\hat{G}}(x) = \lim_{x \to -\infty} \frac{-(c + \hat{c})}{\hat{G}(x)} = 0,$$

where we have used that $y = \hat{\psi}(x) \to 0$ as $x \to -\infty$.

Furthermore, for $x \in (-\infty, L] \cup [b_L^*, +\infty)$, we have $V_L(x) = x - c$, and thus, $\hat{h}_L(x) = -(c + \hat{c})$. Also, with the facts that $\hat{\psi}(x)$ is a strictly increasing function and $\hat{G}(x) > 0$, property (a) follows.

(b) By the definition of $\hat{H}_L$, since $\hat{G}$ and $\hat{\psi}$ are differentiable everywhere, we only need to show the differentiability of $V_L(x)$. To this end, $V_L(x)$ is differentiable at b_L^* by (5.3)–(5.5), but not at L. Therefore, $\hat{H}_L$ is differentiable for $y \in (0, \hat{\psi}(L)) \cup (\hat{\psi}(L), +\infty)$.

In view of the facts that $\hat{G}'(x) < 0$, $\hat{\psi}'(x) > 0$, and $\hat{G}^2(x) > 0$, we have for $x \in (-\infty, L) \cup [b_L^*, +\infty)$,

$$\hat{H}_L'(y) = \frac{1}{\hat{\psi}'(x)} \left(\frac{\hat{h}_L}{\hat{G}} \right)'(x) = \frac{1}{\hat{\psi}'(x)} \left(\frac{-(c + \hat{c})}{\hat{G}(x)} \right)' = \frac{(c + \hat{c})\hat{G}'(x)}{\hat{\psi}'(x)\hat{G}^2(x)} < 0.$$

Therefore, $\hat{H}_L(y)$ is strictly decreasing for $y \in (0, \hat{\psi}(L)) \cup [\hat{\psi}(b_L^*), +\infty)$.

(c) Both $\hat{G}$ and $\hat{\psi}$ are twice differentiable everywhere, while $V_L(x)$ is twice differentiable everywhere except at $x = L$ and b^*, and so is $\hat{h}_L(x)$. Therefore, $\hat{H}_L(y)$ is twice differentiable on $(0, \hat{\psi}(L)) \cup (\hat{\psi}(L), \hat{\psi}(b^*)) \cup (\hat{\psi}(b^*), +\infty)$.

To determine the convexity/concavity of $\hat{H}_L$, we look at the second-order derivative:

$$\hat{H}_L''(y) = \frac{2}{\sigma^2 \hat{G}(x)(\hat{\psi}'(x))^2}(\mathcal{L} - \hat{r})\hat{h}_L(x),$$

whose sign is determined by

$$(\mathcal{L} - \hat{r})\hat{h}_L(x) = \frac{1}{2}\sigma^2 V_L''(x) + \mu(\theta - x)V_L'(x) - \mu(\theta - x) - \hat{r}(V_L(x) - x - \hat{c})$$

$$= \begin{cases} (r - \hat{r})V_L(x) + (\mu + \hat{r})x - \mu\theta + \hat{r}\hat{c} & \text{if } x \in (L, b_L^*), \\ \hat{r}(c + \hat{c}) > 0 & \text{if } x \in (-\infty, L) \cup (b_L^*, +\infty). \end{cases}$$

This implies that $\hat{H}_L$ is convex for $y \in (0, \hat{\psi}(L)) \cup (\hat{\psi}(b_L^*), +\infty)$.

On the other hand, the condition $\sup_{x \in \mathbb{R}} \hat{h}_L(x) > 0$ implies that $\sup_{y \in [0, +\infty)} \hat{H}_L(y) > 0$. By property (a) and twice differentiability of $\hat{H}_L(y)$ for $y \in (\hat{\psi}(L), \hat{\psi}(b_L^*))$, there must exist an interval $(\hat{\psi}(\underline{a}_L), \hat{\psi}(\bar{d}_L)) \subseteq (\hat{\psi}(L), \hat{\psi}(b_L^*))$ such that $\hat{H}_L(y)$ is concave, maximized at $\hat{z}_1 \in (\hat{\psi}(\underline{a}_L), \hat{\psi}(\bar{d}_L))$.

Furthermore, if $V_L(x)$ is strictly increasing on (L, b_L^*), then $(\mathcal{L} - \hat{r})\hat{h}_L(x)$ is also strictly increasing. To prove this, we first recall from Lemma 4.1 that $H(y)$ is strictly increasing and concave on $(\psi(L^*), +\infty)$. By Proposition 5.1, we have $b_L^* < b^*$, which implies $z_L < z$, and thus, $H'(z_L) > H'(z)$.

Then, it follows from (4.5), (4.6) and (5.7) that $W_L'(y) = H'(z_L) > H'(z) = W'(y)$ for $y \in (\psi(L), z_L)$. Next, since $W_L(y) = \frac{V_L}{G} \circ \psi^{-1}(y)$, we have

$$W_L'(y) = \frac{1}{\psi'(x)} \left(\frac{V_L}{G} \right)'(x) = \frac{1}{\psi'(x)} \left(\frac{V_L'(x)G(x) - V_L(x)G'(x)}{G^2(x)} \right).$$

The same holds for $W'(y)$ with $V(x)$ replacing $V_L(x)$. As both $\psi'(x)$ and $G^2(x)$ are positive, $W'_L(y) > W'(y)$ is equivalent to $V'_L(x)G(x) - V_L(x)G'(x) > V'(x)G(x) - V(x)G'(x)$. This implies that

$$V'_L(x) - V'(x) = -\frac{G'(x)}{G(x)}(V(x) - V_L(x)) > 0,$$

since $G(x) > 0$, $G'(x) < 0$, and $V(x) > V_L(x)$. Recalling that $V'(x) > 0$, we have established that $V_L(x)$ is a strictly increasing function, and so is $(\mathcal{L} - \hat{r})\hat{h}_L(x)$. As we have shown the existence of an interval $(\hat{\psi}(\underline{a}_L), \hat{\psi}(\bar{d}_L)) \subseteq (\hat{\psi}(L), \hat{\psi}(b^*_L))$ over which $\hat{H}(y)$ is concave, or equivalently $(\mathcal{L} - \hat{r})\hat{h}_L(x) < 0$ with $x = \hat{\psi}^{-1}(y)$. Then by the strictly increasing property of $(\mathcal{L} - \hat{r})\hat{h}_L(x)$, we conclude $\underline{a}_L = L$ and $\bar{d}_L \in (L, b^*_L)$ is the unique solution to $(\mathcal{L} - \hat{r})\hat{h}_L(x) = 0$, and

$$(\mathcal{L} - \hat{r})\hat{h}_L(x) \begin{cases} < 0 & \text{if } x \in (L, \bar{d}_L), \\ > 0 & \text{if } x \in (-\infty, L) \cup (\bar{d}_L, b^*_L) \cup (b^*_L, +\infty). \end{cases}$$

Hence, we conclude the convexity and concavity of the function $\hat{H}_L$. □

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