PDE: Smoothing

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1 One way of thinking

First order Taylor expansion:

$$u(x_i) \approx u(x) + \frac{\partial u}{\partial x}(x_i - x) \tag{1}$$

Then

$$y_i \approx u(x) + \frac{\partial u}{\partial x}(x_i - x) + \epsilon_i \tag{2}$$

$$\hat{u}(x) = \sum_{i=1}^{n} y_i w_i \tag{3}$$

$$= \sum_{i=1}^{n} \left(u(x) + \frac{\partial u}{\partial x}(x_i - x) + \epsilon_i \right) w_i \tag{4}$$

$$= u(x) + \sum_{i=1}^{n} w_i \epsilon_i + \frac{\partial u}{\partial x} \cdot \sum_{i=1}^{n} w_i (x_i - x) \tag{5}$$

$$\hat{u}(x) - u(x) = \sum_{i=1}^{n} w_i \epsilon_i + \frac{\partial u}{\partial x} \cdot \sum_{i=1}^{n} w_i (x_i - x) \tag{6}$$

$$E\left[(\hat{u}(x) - u(x))^2\right] = E\left[(w_1\epsilon_1 + w_2\epsilon_2 + ... + w_n\epsilon_n)^2\right] \tag{7}$$

$$+ E\left[\left(\frac{\partial u}{\partial x} \cdot \sum_{i=1}^{n} w_i (x_i - x)\right)^2\right] \tag{8}$$

$$E\left[(\hat{u}(x) - u(x))^2\right] = \sigma^2 \sum_{i=1}^{n} w_i^2 + E\left[\frac{\partial u}{\partial x} \cdot \sum_{i=1}^{n} w_i (x_i - x)\right]^2 \tag{9}$$

$$\tag{10}$$

2 Another way of thinking

Gaussian Kernel:

$$w_{ij}(x_j, g) = \exp[-\frac{(x_i - x_j)^2}{2g^2}]$$

We have

$$MSE(x_j, g) = E[\hat{\mu}(x_j) - \mu(x_j)]^2$$

We will need to minimize

$$MSE(g) = \sum_{j=1}^{n} MSE(x_j, g)$$

We have

$$\hat{\mu}(x_j, i) = \mu(x_j) + \sum_{i=1}^{n} w_{ij}(\mu'(x_j)(x_i - x_j) + \epsilon_i)$$

Where ϵ_i follows $N(0, \sigma^2)$
Hence

$$MSE(x_j, g) = E[\sum_{i=1}^{n} w_{ij}(\mu'(x_j)(x_i - x_j) + \epsilon_i)]^2$$

$$= Var(\sum_{i=1}^{n} w_{ij}(\mu'(x_j)(x_i - x_j) + \epsilon_i)) + (E[\sum_{i=1}^{n} w_{ij}(\mu'(x_j)(x_i - x_j) + \epsilon_i)])^2$$

$$= \sigma^2 \sum_{i=1}^{n} w_{ij}^2 + [\sum_{i=1}^{n} w_{ij}\mu'(x_j)(x_i - x_j)]^2$$