PDE: Smoothing

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1 One way of thinking

First order Taylor expansion:

$$u(x_i) \approx u(x) + \frac{\partial u}{\partial x}(x_i - x) \tag{1}$$

Then

$$y_i \approx u(x) + \frac{\partial u}{\partial x}(x_i - x) + \epsilon_i \tag{2}$$

$$\hat{u}(x) = \sum_{i=1}^{n} y_i w_i \tag{3}$$

$$= \sum_{i=1}^{n} \left(u(x) + \frac{\partial u}{\partial x}(x_i - x) + \epsilon_i \right) w_i \tag{4}$$

$$= u(x) + \sum_{i=1}^{n} w_i \epsilon_i + \frac{\partial u}{\partial x} \cdot \sum_{i=1}^{n} w_i (x_i - x) \tag{5}$$

$$\hat{u}(x) - u(x) = \sum_{i=1}^{n} w_i \epsilon_i + \frac{\partial u}{\partial x} \cdot \sum_{i=1}^{n} w_i (x_i - x) \tag{6}$$

$$E\left[(\hat{u}(x) - u(x))^2 \right] = E\left[(w_1 \epsilon_1 + w_2 \epsilon_2 + ... + w_n \epsilon_n)^2 \right] \tag{7}$$

$$+ E\left[\left(\frac{\partial u}{\partial x} \cdot \sum_{i=1}^{n} w_i (x_i - x) \right)^2 \right] \tag{8}$$

$$E\left[(\hat{u}(x_i) - u(x_i))^2 \right] = \sigma^2 \sum_{i=1}^{n} w_i^2 + \left[\frac{\partial u}{\partial x} \cdot \sum_{i=1}^{n} w_i (x_i - x) \right]^2 \tag{9}$$

1.1 Linear

We let

$$\phi_b(x) = \frac{1}{\sqrt{2\pi b}} \exp(\frac{-x^2}{2b}) \tag{10}$$

where b is the band width parameter. Then we have our MSE at x_i

$$MSE_j = \sigma^2 \sum_{i=1}^{n} \phi_b^2(x_j - y_i) + \left[\frac{\partial u}{\partial x} \cdot \sum_{i=1}^{n} \phi_b(x_j - y_i) \cdot (x_j - y_i)\right]^2 \tag{11}$$

Now we replace summation with integration

$$\begin{aligned} MSE_j &= \sigma^2 \int_\omega \phi_b^2(x_i - y) dy + \left[\frac{\partial u}{\partial x} \cdot \int_\omega \phi_b(x_j - y_i) \cdot (x_j - y) dy\right]^2 && (12) \\ &= \sigma^2 \sqrt{2\pi \cdot b} && (13) \end{aligned}$$

1.2 Quadratic

We let

$$\phi_b(x) = \frac{1}{\sqrt{2\pi b}} \exp(\frac{-x^2}{2b}) \tag{14}$$

where b is the band width parameter. Then we have our MSE at x_i

$$(\hat{u}(x_i) - u(x_i)) = \sigma^2 \sum_{i=1}^{n} w_i^2 + \left[\frac{\partial u}{\partial x} \cdot \sum_{i=1}^{n} w_i(x_i - x)\right] + \left[\frac{\partial^2 u}{2\partial x^2} \cdot \sum_{i=1}^{n} w_i(x_i - x)^2\right] + \sum_{i=1}^{n} \epsilon_i w_i \tag{15}$$

$$E\left[(\hat{u}(x_i) - u(x_i))^2\right] = Var\left[(\hat{u}(x_i) - u(x_i))\right] + \left(E\left[(\hat{u}(x_i) - u(x_i))\right]\right)^2 \tag{16}$$

$$= \sigma^2 \sum_{i=1}^{n} w_i^2 + \left[\frac{\partial u}{\partial x} \cdot \sum_{i=1}^{n} w_i(x_i - x) + \frac{\partial^2 u}{2\partial x^2} \cdot \sum_{i=1}^{n} w_i(x_i - x)^2\right]^2 \tag{17}$$

$$MSE_j = \sigma^2 \sum_{i=1}^{n} \phi_b^2(x_j - y_i) + \left[\frac{\partial u}{\partial x} \cdot \sum_{i=1}^{n} \phi_b(x_j - y_i) \cdot (x_j - y_i) + \frac{\partial^2 u}{2\partial x^2} \cdot \sum_{i=1}^{n} \phi_b(x_j - y_i) \cdot (x_j - y_i)^2\right]^2 \tag{18}$$

Now we replace summation with integration

$$MSE_j = \sigma^2 \int_\omega \phi_b^2(x_i - y)dy + \left[\frac{\partial u}{\partial x} \cdot \int_\omega \phi_b(x_j - y) \cdot (x_j - y)dy + \frac{\partial^2 u}{2\partial x^2} \cdot \int_\omega \phi_b(x_j - y) \cdot (x_j - y)^2 dy\right]^2 \tag{19}$$

$$= \frac{\sigma^2}{2\sqrt{\pi \cdot b}} + 0 + \left[\frac{\partial^2 u}{2\partial x^2} \cdot \int_\omega \phi_b(x_j - y) \cdot (x_j^2 - 2x_j \cdot y + y^2)dy\right]^2 \tag{20}$$

$$= \frac{\sigma^2}{2\sqrt{\pi \cdot b}} + \left[\frac{\partial^2 u}{2\partial x^2} \cdot \int_\omega \phi_b(x_j - y)(x_j^2)dy - 2\int_\omega \phi_b(x_j - y) \cdot (x_j \cdot y)dy + \int_\omega \phi_b(x_j - y) \cdot y^2 dy\right]^2 \tag{21}$$

$$= \frac{\sigma^2}{2\sqrt{\pi \cdot b}} + \left[\frac{\partial^2 u}{2\partial x^2} \cdot (x_j^2 - 2x_j^2 + b + x_j^2)\right]^2 \tag{22}$$

$$= \frac{\sigma^2}{2\sqrt{\pi \cdot b}} + \left[\frac{\partial^2 u}{2\partial x^2} \cdot b\right]^2 \tag{23}$$

Set $h(b) = \frac{\sigma^2}{2\sqrt{\pi}} \cdot \frac{1}{\sqrt{b}} + \frac{1}{4}u''^2 \cdot b^2$, and take derivative, then set $h'(b) = 0$

$$h'(b) = -\frac{\sigma^2}{4\sqrt{\pi}} \cdot b^{-\frac{3}{2}} + \frac{1}{2}u''\cdot b = 0 \tag{24}$$

$$b = \left(\frac{\sigma^2}{2\sqrt{\pi}u''^2}\right)^{\frac{2}{5}} \tag{25}$$

2 Another way of thinking

Gaussian Kernel:

$$w_{ij}(x_j, g) = \exp[-\frac{(x_i - x_j)^2}{2g^2}]$$

We have

$$MSE(x_j, g) = E[\hat{\mu}(x_j) - \mu(x_j)]^2$$

We will need to minimize

$$MSE(g) = \sum_{j=1}^{n} MSE(x_j, g)$$

We have

$$\hat{\mu}(x_j, i) = \mu(x_j) + \sum_{i=1}^{n} w_{ij}(\mu'(x_j)(x_i - x_j) + \epsilon_i)$$

Where ϵ_i follows $N(0, \sigma^2)$

Hence

$$MSE(x_j, g) = E[\sum_{i=1}^{n} w_{ij}(\mu'(x_j)(x_i - x_j) + \epsilon_i)]^2$$

$$= Var(\sum_{i=1}^{n} w_{ij}(\mu'(x_j)(x_i - x_j) + \epsilon_i)) + (E[\sum_{i=1}^{n} w_{ij}(\mu'(x_j)(x_i - x_j) + \epsilon_i)])^2$$

$$= \sigma^2 \sum_{i=1}^{n} w_{ij}^2 + [\sum_{i=1}^{n} w_{ij}\mu'(x_j)(x_i - x_j)]^2$$

3 Question

$$Y_i = g(x_i) + \epsilon_i$$

$$MSE = \sum_{i=1}^{N} (\hat{Y}_i - Y_i)^2 = \sum_{i=1}^{N} (\hat{Y}_i - g(x_i))^2$$

$$MISE = \int_1^5 [\hat{g}(x) - g(x)]^2 dx = \int_1^5 [\hat{Y}_i - g(x)]^2 dx = \sum_{i=1}^{N} [\hat{Y}_i - g(x)]^2 \cdot dx$$