

PDE: Smoothing

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1 One way of thinking

First order Taylor expansion:

$$u(x_i) \approx u(x) + \frac{\partial u}{\partial x}(x_i - x) \quad (1)$$

Then

$$y_i \approx u(x) + \frac{\partial u}{\partial x}(x_i - x) + \epsilon_i \quad (2)$$

$$\hat{u}(x) = \sum_{i=1}^n y_i w_i \quad (3)$$

$$= \sum_{i=1}^n \left(u(x) + \frac{\partial u}{\partial x}(x_i - x) + \epsilon_i \right) w_i \quad (4)$$

$$= u(x) + \sum_{i=1}^n w_i \epsilon_i + \frac{\partial u}{\partial x} \cdot \sum_{i=1}^n w_i (x_i - x) \quad (5)$$

$$\hat{u}(x) - u(x) = \sum_{i=1}^n w_i \epsilon_i + \frac{\partial u}{\partial x} \cdot \sum_{i=1}^n w_i (x_i - x) \quad (6)$$

$$E \left[(\hat{u}(x) - u(x))^2 \right] = E \left[(w_1 \epsilon_1 + w_2 \epsilon_2 + \dots + w_n \epsilon_n)^2 \right] \quad (7)$$

$$+ E \left[\left(\frac{\partial u}{\partial x} \cdot \sum_{i=1}^n w_i (x_i - x) \right)^2 \right] \quad (8)$$

$$E \left[(\hat{u}(x) - u(x))^2 \right] = \sigma^2 \sum_{i=1}^n w_i^2 + \left[\frac{\partial u}{\partial x} \cdot \sum_{i=1}^n w_i (x_i - x) \right]^2 \quad (9)$$

$$(10)$$

2 Another way of thinking

Gaussian Kernel:

$$w_{ij}(x_j, g) = \exp\left[-\frac{(x_i - x_j)^2}{2g^2}\right]$$

We have

$$MSE(x_j, g) = E[\hat{\mu}(x_j) - \mu(x_j)]^2$$

We will need to minimize

$$MSE(g) = \sum_{j=1}^n MSE(x_j, g)$$

We have

$$\hat{\mu}(x_j, i) = \mu(x_j) + \sum_{i=1}^n w_{ij}(\mu'(x_j)(x_i - x_j) + \epsilon_i)$$

Where ϵ_i follows $N(0, \sigma^2)$

Hence

$$\begin{aligned} MSE(x_j, g) &= E\left[\sum_{i=1}^n w_{ij}(\mu'(x_j)(x_i - x_j) + \epsilon_i)\right]^2 \\ &= Var\left(\sum_{i=1}^n w_{ij}(\mu'(x_j)(x_i - x_j) + \epsilon_i)\right) + \left(E\left[\sum_{i=1}^n w_{ij}(\mu'(x_j)(x_i - x_j) + \epsilon_i)\right]\right)^2 \\ &= \sigma^2 \sum_{i=1}^n w_{ij}^2 + \left[\sum_{i=1}^n w_{ij}\mu'(x_j)(x_i - x_j)\right]^2 \end{aligned}$$