Camosun College
Department of Mathematics & Statistics

Student Guide - Math 115 (107)

- This document outlines the exact material that is to be covered in Math 115 (107). It includes required learning outcomes, formulas, definitions and proofs. This document has been created to amplify learning outcomes approved by both Camosun College and the BC Ministry of Advanced Education Provincial Level - Algebra and Trigonometry

  `https://www2.gov.bc.ca/assets/gov/education/post-secondary-education/adult-education/abe_articulation_handbook_2015_nov4.pdf`

  The textbook *Algebra and Trigonometry, Sullivan,* 10^{th} *edition* has been used as an additional reference and guide.

- No FORMULAS are to be provided on tests in this course. Learners are responsible for knowing all formulas and definitions.

- As per department policy, the only calculator permitted for use on tests and the final exam is the Sharp EL-531 scientific calculator. No other calculator or any other electronic device including cell phones, electronic translators, smart watches, iPods, etc. are allowed. Although calculators are permitted students are expected to be able to give exact answers to questions.

Chapter R. Review

This chapter is review from the previous course, learners are expected to already have a good understanding of these topics.

Section R.5 Factoring Polynomials

Formulas and Definitions
It is expected that learners will know:

(a) **Difference of Two Squares:**

$$x^2 - a^2 = (x - a)(x + a)$$

(b) **Sum or Difference of Cubes:**

$$x^3 \pm a^3 = (x \pm a)(x^2 \mp ax + a^2)$$

(c) **Steps for Factoring $\boldsymbol{Ax^2 + Bx + C}$, When $\boldsymbol{A \neq 1}$ and, $\boldsymbol{A}$, $\boldsymbol{B}$, and $\boldsymbol{C}$ Have No Common Factors:**

- Step 1: Find a pair of integers whose product is AC and that add up to B. That is find a and b such that $ab = AC$ and $a + b = B$.
- Step 2: Write $Ax^2 + Bx + C = Ax^2 + ax + bx + C$.
- Step 3: Factor the last expression by grouping.

(d) **Completing the Square of $\boldsymbol{Ax^2 + Bx + C = 0}$**

- Step 1: Identify the coefficient of the second-degree term. Factor out this coefficient. That is, factor out A from Ax^2+Bx+C.

-this method is preferable over dividing out the coefficient as it can later be used in the Conics section.

- Step 2: Identify the coefficient of the first-degree term. Multiply this coefficient by $\frac{1}{2}$ and then square the result. That is, determine the value of B and compute $\left(\frac{1}{2}B\right)^2$.
- Step 3: Add the value of $A\left(\frac{1}{2}B\right)^2$ to both sides of the equation.
- Step 4: Factor the left hand side of the equation as a perfect square.

Learning Outcomes
It is expected that learners will be able to:

(a) recognize and factor quadratics
(b) recognize and factor a difference of squares
(c) recognize and factor a sum or difference of cubes
(d) recognize and factor by grouping
(e) factor an expression by completing the square

Section R.6 Synthetic Division

This topic is visited again in section 5.5 - The Real Zeros of a Polynomial Function.

Learning Outcomes
It is expected that learners will be able to:

(a) divide polynomials using long division
(b) use synthetic division to divide a polynomial by $x - c$

Section R.7 Rational Expressions

Formulas and Definitions
It is expected that learners will know:

(a) **Multiply and Divide Rational Expressions**

If $\frac{a}{b}$ and $\frac{c}{d}$, $b \neq 0$, $d \neq 0$, are two rational expressions, then

- $\frac{a}{b}\frac{c}{d} = \frac{ac}{bd}$ if $b \neq 0$, $d \neq 0$
- $\frac{a}{b} \div \frac{c}{d} = \frac{ad}{bc}$ if $b \neq 0$, $c \neq 0$, $d \neq 0$

(b) **Add and Subtract Rational Expressions**

If $\frac{a}{b}$ and $\frac{c}{d}$ are two rational expressions, then

- $\frac{a}{b} \pm \frac{c}{d} = \frac{ad \pm cb}{bd}$, $b \neq 0$, $d \neq 0$

Learning Outcomes
It is expected that learners will be able to:

(a) simplify complex rational expressions

Section R.8 nth Roots; Rational Exponents

This section is important. The topic of factoring expressing involving rational exponents is a necessary skill for success in Math 100.

Example: Factor, write your answer using only positive exponents.

$$\frac{4}{3}(x+1)^{-1/3} - 5(x+1)^{-5/2}$$

Formulas and Definitions
It is expected the learners will know:

(a) If $n \geq 2$ is an integer and a is a real number, we have

- $\sqrt[n]{a^n} = a$ if $n \geq 3$ is odd
- $\sqrt[n]{a^n} = |a|$ if $n \geq 2$ is even

(b) **Properties of Radicals**

Let $n \geq 2$ and $m \geq 2$ denote integers, and let a and b represent real numbers. Assuming that all radicals are defined, we have the following properties:

- $\sqrt[n]{ab} = \sqrt[n]{a}\,\sqrt[n]{b}$
- $\sqrt[n]{\dfrac{a}{b}} = \dfrac{\sqrt[n]{a}}{\sqrt[n]{b}}$
- $a^{m/n} = \sqrt[n]{a^m} = (\sqrt[n]{a})^m$

Learning Outcomes
It is expected that learners will be able to:

(a) simplify rational expressions
(b) factor expressions with rational exponents

Chapter 1. Equations and Inequalities

This chapter is mostly a review from the previous course, learners are expected to already have a fairly good understanding of these topics.

Section 1.1 Linear Equations

Formulas and Definitions

It is expected that learners will know:

(a) A **linear equation in one variable** is an equation equivalent in form to

$$ax + b = 0$$

Learning Outcomes

(a) solve linear equations

(b) solve word problems involving linear equations

Section 1.2 Quadratic Equations

Formulas and Definitions

It is expected that students will know:

(a) A **quadratic equation** is an equation equivalent to one of the form

$$ax^2 + bx + c = 0, \quad a \neq 0$$

(b) Consider the quadratic equation $ax^2 + bx + c = 0$, $a \neq 0$, the solutions of this equation is (are) given by the **quadratic formula**

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

(c) **Discriminant of a Quadratic Equation:**
For a quadratic equation $ax^2 + bx + c = 0$, $a \neq 0$:

- If $b^2 - 4ac > 0$, there are two unequal real solutions.
- If $b^2 - 4ac = 0$, there is a repeated solution, a double root.
- If $b^2 - 4ac < 0$, there is no real solution.

Learning Outcomes
It is expected that students will be able to:

(a) solve quadratic equations by factoring
(b) solve quadratic equations by using the Quadratic Formula
(c) solve quadratic equations by completing the square
(d) solve quadratic word problems

Proofs
It is expected that learners will be able to prove:

(a) the Quadratic Formula by completing the square

Section 1.4 Radical Equations; Equations Quadratic in Form; Factorable Equations

Solving equations of the quadratic type is an important skill for Math 100 (108).

Formulas and Definitions
It is expected learners will know:

(a) An **equation of the quadratic type** is an equation such that an appropriate substitution u transforms an equation into one of the form

$$au^2 + bu + c = 0 \quad a \neq 0$$

Learning Outcomes
It is expected that learners will be able to:

(a) solve radical equations
(b) solve equations quadratic in form

Section 1.5 Solving Inequalities

Formulas and Definitions
It is expected that students will know:

(a) **Interval Notation**

For all real numbers a and b

- an **open interval**, denoted by (a, b), consists of all real numbers x for which $a < x < b$
- an **closed interval**, denoted by $[a, b]$, consists of all real numbers x for which $a \leq x \leq b$
- the **half-open**, or **half-closed** are $(a, b]$, consists of all real numbers x for which $a < x \leq b$, and $[a, b)$, consists of all real numbers x for which $a \leq x < b$

(b) **Multiplication Properties for Inequalities**
For all real numbers a, b, and c

- if $a < b$ and if $c > 0$, then $ac < bc$
- if $a < b$ and if $c < 0$, then $ac > bc$
- if $a > b$ and if $c > 0$, then $ac > bc$

- if $a > b$ and if $c < 0$, then $ac < bc$

(c) **Reciprocal Properties for Inequalities**

- if $a > 0$, then $\dfrac{1}{a} > 0$ if $\dfrac{1}{a} > 0$, then $a > 0$
- if $a < 0$, then $\dfrac{1}{a} < 0$ if $\dfrac{1}{a} < 0$, then $a < 0$

Learning Outcomes
It is expected that learners will be able to:

(a) solve combined inequalities
(b) use interval notation to write a set of numbers

Chapter 2. Graphs

Section 2.1 The Distance and Midpoint Formulas

Formulas an Definitions
It is expected that students will know:

(a) **Distance Formula:**
The distance between two points $P_1 = (x_1, y_1)$ and $P_2 = (x_2, y_2)$, denoted by $d(P_1, P_2)$, is

$$d(P_1, P_2) = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$$

(b) **Midpoint Formula:**
The midpoint $M = (x, y)$ of the line segment from $P_1 = (x_1, y_1)$ to $P_2 = (x_2, y_2)$ is

$$(x, y) = \left(\frac{x_1 + x_2}{2}, \frac{y_1 + y_2}{2} \right)$$

Learning Outcomes
It is expected that learners will be able to:

(a) find the distance between two points in the plane and find the midpoint of a segment
(b) apply the distance formula and mid-point formula to solve problems

Proofs
It is expected that learners will be able to prove:

(a) the distance formula using the Pythagorean Theorem
(b) the Midpoint Formula using similar triangles (Math 107 Omit.)

Section 2.2 Graphs of Equations in Two Variables; Intercepts; Symmetry

Symmetry is normally omitted in this section as it is taught in terms of functions in section 3.3 - Properties of Functions.

Formulas and Definitions
It is expected that learners will know:

(a) ***x*-intercepts:** To find the x-intercept(s), if any, of the graph of an equation, let $y = 0$ in the equation and solve for x, where x is a real number.

(b) ***y*-intercept:** To find the y-intercept(s), if any, of the graph of an equation, let $x = 0$ in the equation and solve for y, where y is a real number.

Learning Outcomes:
It is expected that learners will be able to:

(a) find the coordinates of the x and y- intercepts from an equation

Section 2.3 Lines

Formulas and Definitions
It is expected that learners will know:

(a) Let $P = (x_1, y_1)$ and $Q = (x_2, y_2)$ be two distinct points. If $x_1 \neq x_2$, the **slope *m*** of the non-vertical line L containing P and Q is defined by the formula $m = \dfrac{y_2 - y_1}{x_2 - x_1}$ $x_1 \neq x_2$. If $x_1 = x_2$, then L is a **vertical line** and the slope m of L is undefined.

(b) **Equation of a vertical line:** A vertical line is given by an equation of the form $x = a$ where a is the x-intercept.

(c) **Equation of a horizontal line:** A horizontal line is given by an equation of the form $y = b$ where b is the y-intercept.

(d) **Slope-Intercept form of a line:** An equation of a line with slope m and y-intercept b is $y = mx + b$

(e) **Point-slope form of an equation of a line:** An equation of a non-vertical line with slope m that contains the point (x_1, y_1) is

$$y - y_1 = m(x - x_1)$$

(f) **Criteria for parallel lines:** Two non-vertical lines are parallel if and only if their slopes are equal and they have different y-intercepts.

(g) **Criteria for perpendicular lines:** Two non vertical lines are perpendicular if and only if the product of their slopes is -1.

Learning Outcomes
It is expected that learners will be able to:

(a) find the equation of a line given specific properties
(b) determine whether two lines are parallel, perpendicular or neither

2.4 Circles

Formulas and Definitions
It is expected that students will know:

(a) The **standard form of an equation of a circle** with radius r and center (h, k) is

$$(x - h)^2 + (y - k)^2 = r^2$$

Learning Outcomes
It is expected that learners will be able to:

(a) recognize the equation of a circle
(b) write the standard form of the equation of a circle with center at the origin and translated center (h, k)
(c) find the center and radius of a circle, given its equation, and sketch the graph

3. Functions and Their Graphs

3.1 Functions

The *difference quotient* is an important topic and in preparation for calculus the symbol Δx is used in place of h.

Formulas and Definitions
It is expected that students will know:

(a) Let X and Y be two nonempty sets. A **function** from X into Y is a relation that associates with each element of X exactly one element of Y.

(b) The **Difference Quotient** of a function f at x is given by

$$\frac{f(x+\Delta x) - f(x)}{\Delta x}$$

Note: Students in Math 115 must use Δx, not h, in the difference quotient. Students taking Math 107 may use either.

(c) If f and g are functions:

i. The **sum** $\boldsymbol{f+g}$ is the function defined by

$$\boldsymbol{(f+g)(x) = f(x) + g(x)}$$

ii. The **difference** $\boldsymbol{f-g}$ is the function defined by

$$\boldsymbol{(f-g)(x) = f(x) - g(x)}$$

iii. The **product** $\boldsymbol{f \cdot g}$ is the function defined by

$$\boldsymbol{(f \cdot g)(x) = f(x) \cdot g(x)}$$

iv. The **quotient** $\boldsymbol{\left(\frac{f}{g}\right)}$ is the function defined by

$$\boldsymbol{\left(\frac{f}{g}\right) = \frac{f(x)}{g(x)} \quad g(x) \neq 0}$$

Learning Outcomes
It is expected that learners will be able to:

(a) define the term function
(b) compose and decompose functions
(c) find the sum, difference, product and quotient of two functions and determine their domains
(d) determine if an equation represents a function
(e) evaluate the difference quotient for a given function
(f) construct algebraic functions to model simple real life problems

3.2 The Graph of a Function

Formulas and Definitions
It is expected that learners will know:

(a) **Vertical-Line Test:** A set if points in the xy-plane is the graph of a function if and only if every vertical line intersects the graph in at most one point.

Learning Outcomes
It is expected that learners will be able to:

(a) use the vertical line test to identify functions
(b) use the equation of a function to obtain information about the graph such as intercepts, domain, and range

3.3 Properties of Functions

Formulas and Definitions
It is expected that learners will know:

(a) A function is **even** if, for every number x in its domain, the number -x is also in the domain and $f(-x) = f(x)$.

(b) A function is **odd** if, for every number x in its domain, the number -x is also in the domain and $f(-x) = -f(x)$.

(c) A function is even if and only if its graph is symmetric with respect to the y-axis. A function is odd if and only if its graph is symmetric with respect to the origin.

(d) **Average rate of change:** $\dfrac{\Delta y}{\Delta x} = \dfrac{f(b) - f(a)}{b - a}, a \neq b$

(e) **Slope of the Secant Line:** The average rate of change of a function from a to b equal the slope of the secant line containing the two points $(a, f(a))$ and $(b, f(b))$ on its graph.

(f) A function f is **increasing** on an open interval I if, for any choice of x_1 and x_2 in I, with $x_1 < x_2$ we have $f(x_1) < f(x_2)$.

(g) A function f is **decreasing** on an open interval I if, for any choice of x_1 and x_2 in I, with $x_1 < x_2$ we have $f(x_1) > f(x_2)$.

(h) A function f is **constant** on an open interval I if, for all choices of x in I, the values of $f(x)$ are equal.

(i) Let f be a function defined on some interval I. If there is a number u in I for which $f(x) \leq f(u)$ for all x in I, then f has an **absolute maximum at** u, and the number $f(u)$ is the **absolute maximum of** f **on** I.

(j) Let f be a function defined on some interval I. If there is a number u in I for which $f(x) \geq f(u)$ for all x in I, then f has an **absolute minimum at** u, and the number $f(u)$ is the **absolute minimum of** f **on** I.

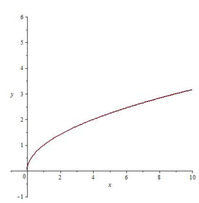
Learning Outcomes
It is expected that learners will be able to:

(a) graph functions and analyze graphs of functions, identifying: domain and range; intervals on which the function is increasing, decreasing or constant
(b) determine whether a graph is symmetric with respect to the x-axis, y-axis, and the origin
(c) identify even or odd functions and recognize their symmetries
(d) find the average rate of change of a function

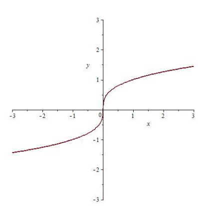
3.4 Library of Functions; Piecewise-defined Functions

Formulas and Definitions
It is expected that learners will know:

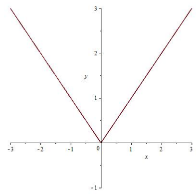
(a) the graph of $f(x) = \sqrt{x}$



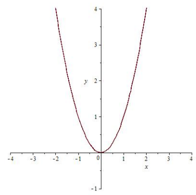
(b) the graph of $f(x) = \sqrt[3]{x}$ (Math 107 omit.)



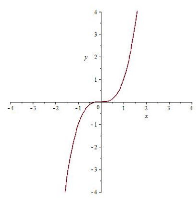
(c) the graph of $f(x) = |x|$



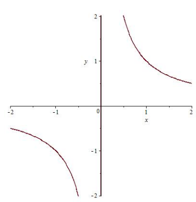
(d) the graph of $f(x) = x^2$



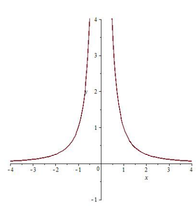
(e) the graph of $f(x) = x^3$



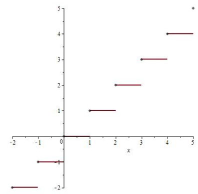
(f) the graph of $f(x) = \dfrac{1}{x}$



(g) the graph of $f(x) = \dfrac{1}{x^2}$



(h) the graph of $f(x) = \lfloor x \rfloor$ (Math 107 omit.)



Learning Outcomes
It is expected that learners will be able to:

(a) identify graphs of common algebraic functions
(b) evaluate and graph piecewise defined functions

3.5 Graphing Techniques: Transformations

Formulas and Definitions
It is expected that learners will know:

(a) If a positive real number k is added to the output of a function $y = f(x)$, the graph of the new function $y = f(x) + k$ is the graph of f **shifted vertically up** k units.

(b) If a positive real number k is subtracted from the output of a function $y = f(x)$, the graph of the new function $y = f(x) - k$ is the graph of f **shifted vertically down** k units.

(c) If the argument x of a function f is replaced by $x - h$, $h > 0$, the graph of the new function $y = f(x - h)$ is the graph of f **shifted horizontally right** h units.

(d) If the argument x of a function f is replaced by $x + h$, $h > 0$, the graph of the new function $y = f(x + h)$ is the graph of f **shifted**

horizontally left h units.

(e) When the right side of a function $y = f(x)$ is multiplied by a positive number a, the graph of the new function $y = af(x)$ is obtained by multiplying each y-coordinate on the graph of $y = f(x)$ by a. The new graph is a **vertically compressed** (if $0 < a < 1$) or a **vertically stretched** (if $a > 1$) version of the graph of $y = f(x)$.

(f) If the argument x of a function $y = f(x)$ is multiplied by a positive number a, then the graph of the new function $y = f(ax)$ is obtained by multiplying each x-coordinate of $y = f(x)$ by $\frac{1}{a}$. A **horizontal compression** results if $a > 1$, and a **horizontal stretch** results if $0 < a < 1$.

(g) When the right side of the function $y = f(x)$ is multiplied by -1, the graph of the new function $y = -f(x)$ is the **reflection about the x-axis** of the graph of the function $y = f(x)$.

(h) When the graph of the function $y = f(x)$ is known, the graph of the new function $y = f(-x)$ is the **reflection about the y-axis** of the graph of the function $y = f(x)$.

Learning Outcomes

It is expected that learners will be able to:

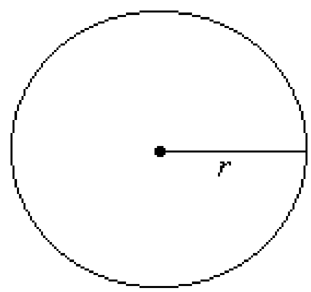
(a) use transformations to sketch functions including piecewise-defined functions

3.6 Mathematical Models: Building Functions

Formulas and Definitions
It is expected that learners will know:

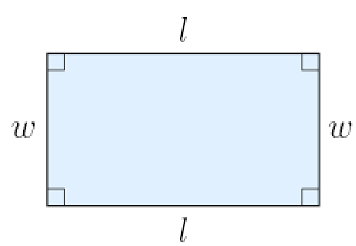
(a) **Circle:** r =Radius, A =Area, C = Circumference

$$A = \pi r^2 \quad C = 2\pi r$$

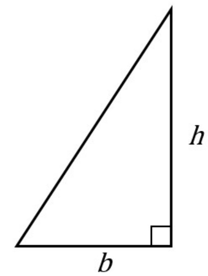


(b) **Rectangle:** l = Length, w =width, A =Area, P =Perimeter

$$A = lw \quad P = 2l + 2w$$

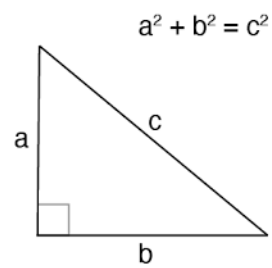


(c) **Right Triangle:** b =base, h =height, A =Area



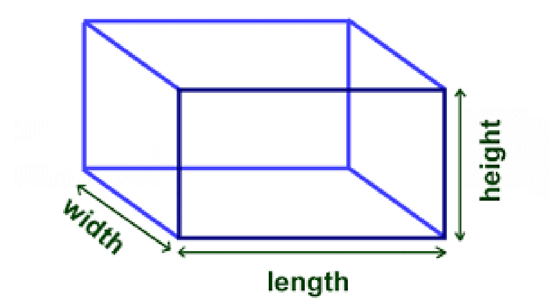
$$A = \frac{1}{2}bh$$

(d) **Pythagorean Theorem:** In a right triangle, the square of the length of the hypotenuse is equal to the sum of the squares of the lengths of the legs.



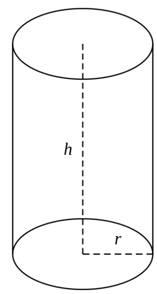
(e) **Rectangular Box:** l = Length, w = Width, h = Height, V = Volume, S = Surface Area

$$V = lwh \quad S = 2lw + 2lh + 2wh$$



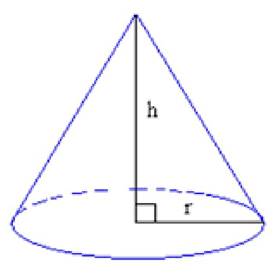
(f) **Right Circular Cylinder:** r = Radius, h = Height, V = Volume, S = Surface Area

$$V = \pi r^2 h \quad S = 2\pi r^2 + 2\pi r h$$



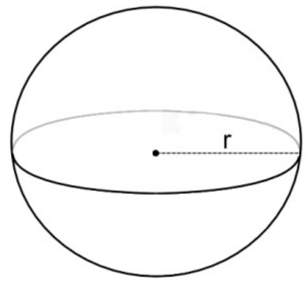
(g) **Right Circular Cone:** r = Radius, h = height, V = Volume

$$V = \frac{1}{3}\pi r^2 h$$



(h) **Sphere:** r = Radius, V = Volume

$$V = \frac{4}{3}\pi r^3$$



Learning Outcomes
It is expected that learners will be able to:

(a) write formulas or functions to model real life applications

4. Linear and Quadratic Functions

4.1 Properties of Linear Functions and Linear models

Formulas and Definitions
It is expected that learners will know:

(a) **Supply Function S(p)**: The quantity supplied of a good is the amount of a product that a company is willing to make available for sale at a given price.

(b) **Demand Function D(p)**; The quantity demanded of a good is the amount of a product that consumers are willing to purchase at a given price.

(c) **Equilibrium Point**: $S(p) = D(p)$

Learning Outcomes
It is expected that learners will be able to:

(a) solve linear and quadratic word problems

4.3 Quadratic Functions and Their Properties

Formulas and Definitions
It is expected that learners will know:

(a) A **Quadratic Function:** is a function of the form

$$f(x) = ax^2 + bx + c, a \neq 0$$

(b) The **Revenue** R from selling x items at the price p is given by: $R = xp$

(c) The graph of a quadratic function $f(x) = ax^2 + bx + c$, $a \neq 0$ is a parabola with vertex at $\left(-\frac{b}{2a}, f\left(-\frac{b}{2a}\right)\right)$.

(d) The **Equation of a Parabola** having vertex (h, k) is given by

$$f(x) = a(x - h)^2 + k$$

Parabola opens up if $a > 0$; the vertex is a minimum point.
Parabola opens down if $a < 0$; the vertex is a maximum point.

(e) **The x-intercepts of a Quadratic Function**

i. If the discriminant $b^2 - 4ac > 0$, the graph of $f(x) = ax^2 + bx + c$ has two distinct x-intercepts so it crosses the x-axis in two places.

ii. If the discriminant $b^2-4ac=0$, the graph of $f(x)=ax^2+bx+c$ has one x-intercept so it touches the x-axis at its vertex.

iii. If the discriminant $b^2-4ac<0$, the graph of $f(x)=ax^2+bx+c$ has no x-intercepts so it does not cross the x-axis.

Learning Outcomes
It is expected that learners will be able to:

(a) graph quadratic functions and analyze graphs of quadratic functions identifying the vertex, line of symmetry, maximum/minimum values, and intercepts
(b) graph quadratics using transformations
(c) find the standard form of a Parabola by completing the square
(d) solve optimization problems

4.4 Build Quadratic Models from Verbal Descriptions and from Data

Formulas and Definitions
It is expected that learner will know:

(a) **Revenue:** In economics, revenue R, in dollars, is defined as the amount of money received from the sale of an item and is equal to the unit selling price p, in dollars, of the item times the number x of units actually sold. That is,
$$R = xp$$
(b) The equation that relates p and x is called the **demand equation**.

Learning Outcomes
It is expected that learners will be able to:

(a) solve optimization problems modeled with quadratic functions

4.5 Inequalities Involving Quadratic Functions

Learning Outcomes
It is expected that learners will be able to:

(a) solve polynomial and rational inequalities

5. Polynomial and Rational Functions

5.1 Polynomial Functions and Models

Formulas and Definitions
It is expected that learners will know:

(a) A **polynomial function** in one variable is a function of the form

$$f(x) = a_n x^n + a_{n-1}x^{n-1} + \cdots a_1 x + a_0$$

where $a_i \in \mathbb{R}$ and $n \in \mathbb{Z}$.
If $a_n \neq 0$, it is called the **leading coefficient**, and n is the **degree** of the polynomial.

(b) A **power function of degree n** is a monomial function of the form

$$y = a_n x^n$$

(c) **Properties of Power Functions, $f(x) = x^n$ where n is a Positive Even Integer**

- f is an even function, so its graph is symmetric with respect to the y-axis (the shape is similar to a parabola).
- The graph always contains the points $(-1, 1)$, $(0, 0)$, and $(1, 1)$.

(d) **Properties of Power Functions, $f(x) = x^n$ where n is a Positive Odd Integer**

- f is an odd function, so its graph is symmetric with respect to the origin (the shape is similar to a cubic).

- The graph always contains the points $(-1, -1)$, $(0, 0)$, and $(1, 1)$.

(e) **End Behavior (Leading Coefficient Test):** For large values of x, either positive or negative, the graph of $f(x) = a_n x^n + a_{n-1}x^{n-1} + \cdots a_1 x + a_0$ resembles the graph of the power function $y = a_n x^n$.

(f) **Real Zero**: If f is a function and r is a real number for which $f(r) = 0$, then r is called a real zero of f.
As a consequence of this definition, the following statements are equivalent.

- r is a real zero of a polynomial f.
- r is an x-intercept of the graph of f.
- $x - r$ is a factor of f.
- r is a solution to the equation $f(x) = 0$ that is $f(r) = 0$.

(g) **Multiplicity:** If $(x - r)^m$ is a factor of a polynomial f and $(x - r)^{m+1}$ is not a factor of f, then r is called a zero of multiplicity m

- if r is a zero of even multiplicity the graph of f touches the x-axis at r.

- if r is a zero of odd multiplicity the graph of f crosses the x-axis at r

(h) **Turning Points:** If f is a polynomial function of degree n, then the graph of f has at most $n - 1$ turning points.

Learning Outcomes
It is expected that learners will be able to:

(a) determine the behavior of the graphs of polynomial functions of higher degree using the leading coefficient test
(b) recognize characteristics of the graphs of polynomial functions including real zeros, y-intercept, relative maximal and minima, domain and range
(c) sketch polynomial functions

5.2 Properties of a Rational Functions

Formulas and Definitions
It is expected that learners will know:

(a) **Rational Function:** A rational function is a function of the from
$$R(x) = \frac{p(x)}{q(x)}$$
where p and q are polynomial functions and q is not zero.

(b) **Locating Vertical Asymptotes:**
A rational function $R(x) = \dfrac{p(x)}{q(x)}$, in lowest terms, will have a vertical asymptote $x = r$ if r is a real zero of the denominator. That is, if $x - r$ is a factor of the denominator q (and not a factor of the numerator p), R will have the vertical asymptote $x = r$.

(c) **Locating a Horizontal Asymptote:**
Consider the function
$$R(x) = \frac{p(x)}{q(x)} = \frac{a_n x^n + a_{n-1}x^{n-1} + \cdots + a_1 x + a_0}{b_m x^m + b_{m-1}x^{m-1} + \cdots + b_1 x + b_0}$$
in which the degree of the numerator is n and the degree of the denominator is m.

- If $n < m$, then the line $y = 0$ is the equation of the horizontal asymptote.
- If $n = m$, then the line $y = \frac{a_n}{b_m}$ is the equation of the horizontal asymptote.
- If $n > m$, then the graph has no horizontal asymptote.

(d) **Locating an Oblique Asymptote:**
Consider the function

$$R(x) = \frac{p(x)}{q(x)} = \frac{a_n x^n + a_{n-1}x^{n-1} + \cdots + a_1 x + a_0}{b_m x^m + b_{m-1}x^{m-1} + \cdots + b_1 x + b_0}$$

in which the degree of the numerator is n and the degree of the denominator is m.

If $n = m + 1$ (the degree of the numerator is one more than the degree of the denominator), the line $y = mx + b$ is the equation of the oblique asymptote, which is found using long division.

Learning Outcomes
It is expected that learners will be able to:

(a) determine the behavior of the graphs of rational functions near x-intercepts using multiplicity
(b) recognize characteristics of the graphs of rational functions including real zeros, x and$y$-intercept(s), relative maximal and minima, asymptotes, holes, domain and range
(c) sketch rational functions

5.3 The Graph of a Rational Function

Learning Outcomes
It is expected that learners will be able to:

(a) analyze and graph rational functions

5.4 Polynomial and Rational Inequalities

Learning Outcomes
It is expected that learners will be able to:

(a) solve polynomial and rational inequalities

5.5 The Real Zeros of a Polynomial Function

Formulas and Definitions
It is expected that learners will know:

(a) **Remainder Theorem:** Let f be a polynomial function. If $f(x)$ is divided by $x - c$, then the remainder is $f(c)$.

(b) **Factor Theorem:**
Let f be a polynomial function. Then $x - c$ is a factor of $f(x)$ if and only if $f(c) = 0$.

(c) **Rational Zeros Theorem:** Let f be a polynomial function of degree 1 or higher of the form
$f(x) = a_n x^n + a_{n-1}x^{n-1} + \cdots + a_1 x + a_0 \quad a_n \neq 0 \quad a_0 \neq 0$
where each coefficient is an integer. If $\dfrac{p}{q}$, in lowest terms, is a rational zero of f, then p must be a factor of a_0, and q must be a factor of a_n.

Learning Outcomes
It is expected that learners will be able to:

(a) determine whether a function has a real zero between two real numbers
(b) state the Remainder and Factor Theorems
(c) use the Remainder and Factor Theorems to find function values and factors of a polynomial

(d) list the possible rational zeros for a polynomial function with integer coefficients
(e) factor polynomial functions and find the zeros
(f) find a polynomial with specified zeros

6. Exponential and Logarithmic Functions

6.1 Composite Functions

Formulas and Definitions
It is expected that learners will know:

(a) Given two functions f and g, the **composite function**, denoted by $f \circ g$ is defined by

$$(f \circ g)(x) = f(g(x))$$

The domain of $f \circ g$ is the set of all numbers x in the domain of g such that $g(x)$ is in the domain of f.

Learning Outcomes
It is expected that learners will be able to:

(a) find the composition of two functions f and g, find formulas for $f(g(x))$ and $g(f(x))$, identifying the domain of the composition and evaluate the composite function
(b) decompose a function as a composition of two or more simpler functions

6.2 One-to-One Functions; Inverse Functions

Definitions and Formulas
It is expected that learners will know.

(a) A function is **one-to-one** if $x_1 \neq x_2$ then $f(x_1) \neq f(x_2)$.

(b) **Horizontal Line Test:** If every horizontal line intersects the graph of a function f in at most one point, then f is one-to-one.

(c) Suppose that f is a one-to-one function. Then, corresponding to each x in the domain of f, there is exactly one y in the range; and corresponding to each y in the range of f, there is exactly one x in the domain. The correspondence from the range of f back to the domain of f is called the **inverse function of** $\boldsymbol{f}$. The symbol f^{-1} is used to denote the inverse function of f.

Domain of f = Range of f^{-1}
Domain of f^{-1} = Range of f

(d) **Verifying Inverse Functions:** To verify two functions f and f^{-1} are inverses of each other one must show the two following conditions:

- $f(f^{-1}(x)) = x$ where x is in the domain of f
- $f^{-1}(f(x)) = x$ where x is in the domain of f^{-1}

(e) **Theorem** The graph of a one-to-one function f and the graph of its inverse f^{-1} are symmetric with respect to the line $y = x$.

Learning Outcomes
It is expected that learners will be able to:

(a) use the horizontal line test to determine if a function is one-to-one and therefore has an inverse that is a function
(b) find $f^{-1}(f(x))$ and $f(f^{-1}(x))$ for any number x in the domains of the functions when the inverse of a function is also a function
(c) find inverse functions algebraically and graphically

6.3 Exponential Functions

Formulas and Definitions
It is expected that students will know:

(a) **Laws of Exponents**
If s, t, a and b are real numbers with $a > 0$ and $b > 0$, then

- $a^s \cdot a^t = a^{s+t}$
- $(a^s)^t = a^{st}$
- $(ab)^s = a^s b^s$
- $1^s = 1$
- $a^{-s} = \dfrac{1}{a^s} = \left(\dfrac{1}{a}\right)^s$
- $a^0 = 1$

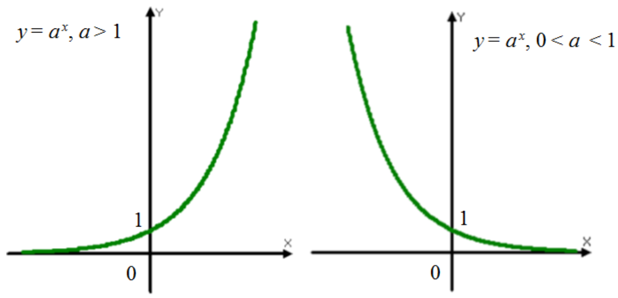
(b) An **exponential function** is a function of the form

$$f(x) = Ca^x$$

where a is a positive real number ($a > 0$), $a \neq 1$

(c) **Properties of the Exponential Function $\boldsymbol{f(x) = a^x}$, $\boldsymbol{a > 1}$**

- The domain is the set of all real numbers, $(-\infty, \infty)$. The range is the set of all positive real numbers, $(0, \infty)$.
- There are no x-intercepts; the y-intercept is at 1.
- The x-axis ($y = 0$) is the horizontal asymptote.



(d) **Properties of the Exponential Function** $\boldsymbol{f(x) = a^x, 0 < a < 1}$

- The domain is the set of all real numbers, $(-\infty, \infty)$. The range is the set of all positive real numbers, $(0, \infty)$.
- There are no x-intercepts; the y-intercept is at 1.
- The x-axis ($y = 0$) is the horizontal asymptote.

(e) The **number** $\boldsymbol{e}$ is defined using limit notation as:

$$e = \lim_{n \to \infty} \left(1 + \frac{1}{n}\right)^n$$

Learning Outcomes
It is expected that learners will be able to:

(a) evaluate exponential functions including functions with base e
(b) graph exponential functions including transformations and analyze the graphs in terms of: x- or y-intercepts, asymptotes, increasing or decreasing, domain and range
(c) solve exponential equations

6.4 Logarithmic Functions

Formulas and Definitions
It is expected that learners will know:

(a) The **logarithmic function with base a** is defined by

$$y = \log_a x \text{ if and only if } a^y = x$$

(b) **Properties of the Logarithmic Function**
$\boldsymbol{f(x) = \log_a x, \quad a > 0, a \neq 1}$

- The domain is the set of positive real numbers $(0, \infty)$. The range is the set of all real numbers $(-\infty, \infty)$.
- The x-intercept of the graph is 1. There is no y-intercept.

(c) **The Natural Logarithm Function:**

$$\ln x = \log_e x$$

Learning Outcomes
It is expected that learners will be able to:

(a) recognize the inverse relationship between exponential and logarithmic functions
(b) graph logarithmic functions including transformations and analyze the graphs in terms of: x- or y-intercepts, asymptotes, increasing or decreasing, domain and range
(c) convert between exponential and logarithmic equations
(d) find common and natural logarithms using a calculator

6.5 Properties of Logarithms

Formulas and Definitions
It is expected that learners will know:

(a) **Properties of Logarithms:**
In the properties given, M, N and a are positive real numbers, $a \neq 1$, and r is any real number.

i. $a^{\log_a M} = M$

ii. $\log_a a^r = r$

iii. $\log_a(MN) = \log_a M + \log_a N$

iv. $\log_a \left(\dfrac{M}{N}\right) = \log_a M - \log_a N$

v. $\log_a M^r = r \log_a M$

vi. Change-of-Base Formula: $\log_a M = \dfrac{\log_b M}{\log_b a}$

Learning Outcomes
It is expected that learners will be able to:

(a) use the properties of logarithms to expand or condense logarithmic expressions
(b) use the change of base property to find a logarithm with base other than 10 or e
(c) use the properties of logarithms to simplify expressions

Proofs
It is expected that learners will be able to prove:

(a) the properties of logarithms (Math 107 omit proofs of properties *i* and *ii*.)

6.6 Logarithmic and Exponential Equations

Learning Outcomes
It is expected that learners will be able to:

(a) use the properties of logarithms and exponentials to solve equations

6.7 Financial Models

Formulas and Definitions
It is expected that learners will know:

(a) **Compound Interest Formula:**
The amount A after t years due to a principal P invested at an annual interest rate r, expressed as a decimal, compounded n times per year is

$$A = P\left(1 + \frac{r}{n}\right)^{nt}$$

(b) **Continuous Compounding:** The amount A after t years due to a principal P invested at an annual interest rate r compounded continuously is

$$A = Pe^{rt}$$

Learning Outcomes
It is expected that learners will be able to:

(a) solve applied financial problems

6.8 Exponential Growth and Decay Models; Logistic Growth and Decay Models

Formulas and Definitions
It is expected that learners will know:

(a) **Uninhibited Growth and Decay:** A model that gives the number A of cells in a culture (or amount of radioactive material present) at time t is given by

$$A = Pe^{rt}$$

where P is the initial number of cells (or initial amount of radioactive material) and r is the growth/decay constant.

Learning Outcomes
It is expected that learners will be able to:

(a) solve applied problems involving exponential growth, exponential decay and logistic growth

7. Trigonometric Functions

Trigonometry is the most important topic taught in Math 115.

7.1 Angles and Their Measure

Formulas and Definitions
It is expected that learners will know:

(a) **Radian:**If the radius of the circle is r and the length of the arc subtended by the central angle is also r, then the measure of the angle is **1 radian**. One radian is equivalent to 180 degrees.

$$180^\circ = \pi \text{ radians}$$

(b) **Arc Length:** For a circle of radius r, a central angle of θ radians subtends an arc whose length s is

$$s = r\theta$$

(c) **Area of a Sector:** The area A of the sector of a circle of radius r formed by a central angle of θ radians is

$$A = \frac{1}{2}r^2\theta$$

Learning Outcomes
It is expected that learners will be able to:

(a) identify angles in standard position, positive and negative angles
(b) convert between degree and radian measures of angles
(c) find the length of an arc, radian measure of central angle, or radius of a circle
(d) define a radian and work with radian measure

7.2 Right Triangle Trigonometry

Formulas and Definitions
It is expected that students will know:

(a) **Trigonometric Definitions** The six ratios of the lengths of the sides of a right triangle are called **trigonometric functions of acute angles** and are defined as follows:

- $\sin\theta = \dfrac{opposite}{hypotenuse}$

- $\cos\theta = \dfrac{adjacent}{hypotenuse}$

- $\tan\theta = \dfrac{opposite}{adjacent}$

- $\csc\theta = \dfrac{hypotenuse}{opposite}$

- $\sec\theta = \dfrac{hypotenuse}{adjacent}$

- $\cot\theta = \dfrac{adjacent}{opposite}$

(b) **Reciprocal Identities**

- $\csc\theta = \dfrac{1}{\sin\theta}$
- $\sec\theta = \dfrac{1}{\cos\theta}$
- $\cot\theta = \dfrac{1}{\tan\theta}$

Quotient Identities

- $\tan\theta = \dfrac{\sin\theta}{\cos\theta}$
- $\cot\theta = \dfrac{\cos\theta}{\sin\theta}$

(c) **Pythagorean Identities**

- $\sin^2\theta + \cos^2\theta = 1$
- $\tan^2\theta + 1 = \sec^2\theta$
- $1 + \cot^2\theta = \csc^2\theta$

(d) Two acute angles are called **complementary** if their sum is $\dfrac{\pi}{2}$.

(e) **Complementary Angle Theorem**

- $\sin\theta = \cos\left(\dfrac{\pi}{2} - \theta\right)$
- $\cos\theta = \sin\left(\dfrac{\pi}{2} - \theta\right)$
- $\tan\theta = \cot\left(\dfrac{\pi}{2} - \theta\right)$
- $\cot\theta = \tan\left(\dfrac{\pi}{2} - \theta\right)$

- $\csc\theta = \sec\left(\frac{\pi}{2} - \theta\right)$
- $\sec\theta = \csc\left(\frac{\pi}{2} - \theta\right)$

Learning Outcomes
It is expected that learners will be able to:

(a) determine the six trigonometric functions of an angle in standard position given a point on its terminal side
(b) recognize and use the reciprocal, quotient and Pythagorean identities
(c) recognize and use the Complementary Angle Theorem
(d) state the right triangle definitions for the trigonometric functions

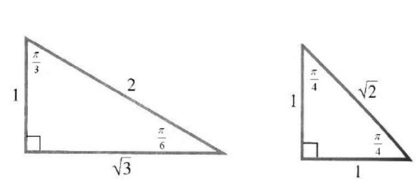
Proofs
It is expected that learners will be able to prove:

(a) the Pythagorean Identities using a unit circle argument

7.3 Computing the Values of Trigonometric Functions of Acute Angles

Formulas and Definitions
It is expected that learners will know:

(a) Special Triangles



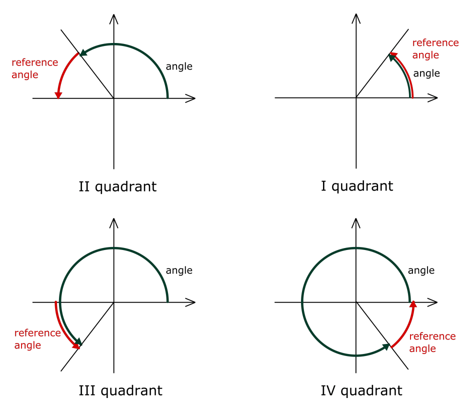
Learning Outcomes
It is expected that learners will be able to:

(a) find the exact values of the trigonometric functions of special acute angles $\frac{\pi}{4}$, $\frac{\pi}{6}$ and $\frac{\pi}{3}$

7.4 Trigonometric Functions of Any Angle

Formulas and Definitions
It is expected that learners will know:

(a) **Reference Angle:** Let θ denote an angle that lies in a quadrant. The acute angle formed by the terminal side of θ and the x-axis is called the **reference angle** for θ.



Learning Outcomes
It is expected that learners will be able to:

(a) identify special angles on a unit circle and reference angles

(b) use reference triangles to find exact values of trigonometric functions of special angles

7.5 Unit Circle Approach; Properties of the Trigonometric Functions

Formulas and Definitions
It is expected that learners will know:

(a) **Periodic Properties**

- $\sin(\theta + 2\pi) = \sin\theta$
- $\csc(\theta + 2\pi) = \csc\theta$
- $\cos(\theta + 2\pi) = \cos\theta$
- $\sec(\theta + 2\pi) = \sec\theta$
- $\tan(\theta + \pi) = \tan\theta$
- $\cot(\theta + \pi) = \cot\theta$

(b) **Even-Odd Properties**

- $\sin(-\theta) = -\sin(\theta)$
- $\csc(-\theta) = -\csc(\theta)$
- $\cos(-\theta) = \cos(\theta)$
- $\sec(-\theta) = \sec(\theta)$
- $\tan(-\theta) = -\tan(\theta)$
- $\cot(-\theta) = -\cot(\theta)$

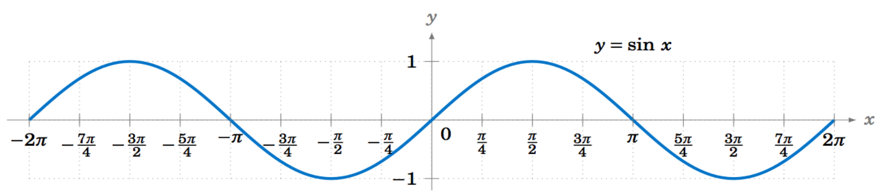
Learning Outcomes
It is expected that learners will be able to:

(a) Use the Even-Odd and Periodic Properties to simplify expressions.

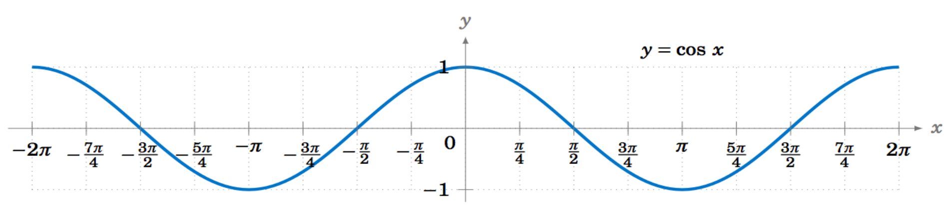
7.6 Graphs of the Sine and Cosine Functions

Formulas and Definitions
It is expected that learners will know:

(a) the graph of the Sine function $y = \sin x$



(b) the graph of the Cosine function $y = \cos x$



(c) the graphs of $y = A\sin(\omega x)$ and $y = A\cos(\omega x)$

- Amplitude $= |A|$
- Period $= \dfrac{2\pi}{\omega}$

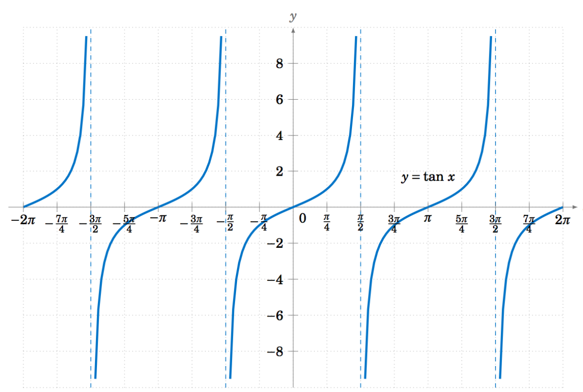
Learning Outcomes
It is expected that learners will be able to:

(a) graph the sine and cosine functions using transformations and state their properties
(b) analyze sinusoidal graphs and construct possible equations

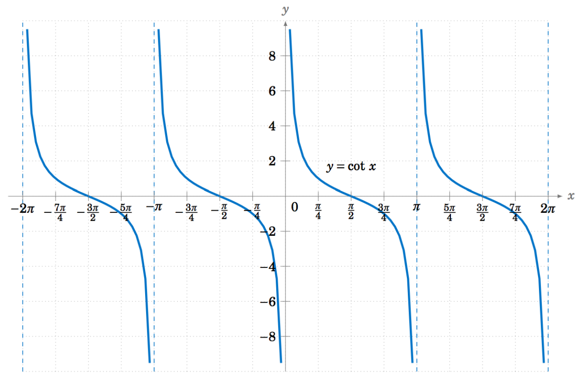
7.7 Graphs of the Tangent, Cotangent, Cosecant, and Secant Functions

Formulas and Definitions
It is expected that learners will know:

(a) the graph of the Tangent function $y = \tan x$



(b) the graph of the Cotangent function $y = \cot(x)$
(Math 107 responsible for basic graph only.)

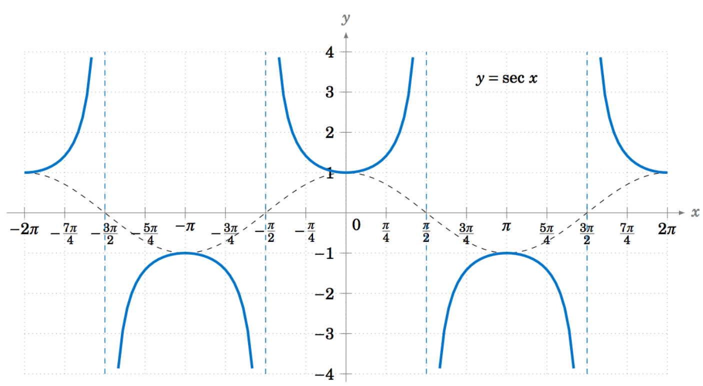


(c) the graphs of $y = A\tan(\omega x) + B$ and $y = A\cot(\omega x) + B$

- magnitude of the vertical stretch $= A$
- Period $= \dfrac{\pi}{\omega}$
- Vertical shift $= B$

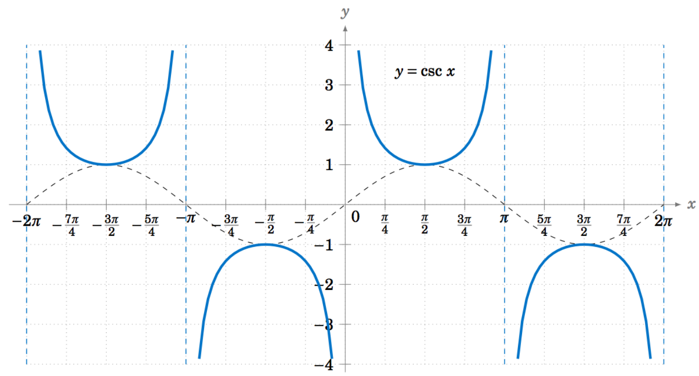
(d) the graph of the Secant function $y = \sec x$

(Math 107 responsible for basic graph only.)



(e) the graph of the Cosecant function $y = \csc(x)$

(Math 107 responsible for basic graph only.)



(f) the graphs of $y = A\sec(\omega x) + B$ and $y = A\csc(\omega x) + B$

- range: $|y| \geq A$
- Period $= \dfrac{2\pi}{\omega}$
- Vertical shift $= B$

Learning Outcomes
It is expected that learners will be able to:

(a) graph tangent, cotangent, cosecant, secant using transformations and state their properties

(Math 107 responsible for basic graph only of cotangent, cosecant and secant.)

(b) analyze sinusoidal graphs and construct possible equations

7.8 Phase Shift; Sinusoidal Curve Fitting

Formulas and Definitions
It is expected that learners will know:

(a) the graphs of $y = A\sin(\omega x - \phi) + B$ and $y = A\cos(\omega x - \phi) + B$
- Amplitude $= |A|$
- Period $= \dfrac{2\pi}{\omega}$
- Phase Shift $= \dfrac{\phi}{\omega}$

(b) the graphs of $y = A\tan(\omega x - \phi) + B$
- magnitude of the vertical stretch $= A$
- Period $= \dfrac{\pi}{\omega}$
- Vertical shift $= B$
- Phase Shift $= \dfrac{\phi}{\omega}$

Formulas and Definitions (Math 107 basic graph only.)

(a) the graphs of $y = A\csc(\omega x - \phi) + B$ and $y = A\sec(\omega x - \phi) + B$
- range: $|y| \geq A$
- Period $= \dfrac{2\pi}{\omega}$
- Vertical shift $= B$
- Phase Shift $= \dfrac{\phi}{\omega}$

(b) the graphs of $y = A\cot(\omega x - \phi) + B$

- magnitude of the vertical stretch $= A$
- Period $= \dfrac{\pi}{\omega}$
- Vertical shift $= B$
- Phase Shift $= \dfrac{\phi}{\omega}$

Learning Outcomes
It is expected that learners will be able to:

(a) graph transformations of the sine, cosine and tangent functions and determine period, amplitude, and phase shift

(b) graph transformations of the cotangent, cosecant, secant functions and determine period, amplitude, and phase shift (Math 107 basic graph only.)

8. Analytic Trigonometry

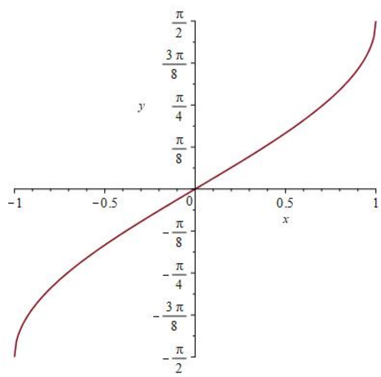
Trigonometry is the most important topic taught in Math 115.

8.1 The Inverse Sine, Cosine, and Tangent Functions

Formulas and Definitions

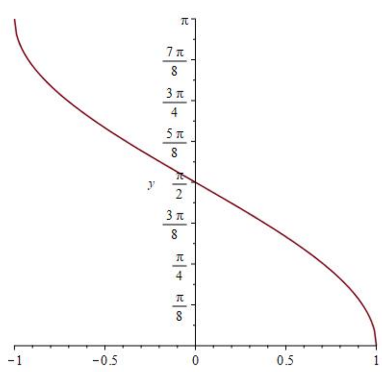
It is expected that learners will know:

(a) the graph of the inverse Sine function



- $y = \sin^{-1}(x)$ means $x = \sin(y)$
- $\sin^{-1}(\sin(x)) = x$ where $-\dfrac{\pi}{2} \leq x \leq \dfrac{\pi}{2}$
- $\sin(\sin^{-1}(x)) = x$ where $-1 \leq x \leq 1$

(b) the graph of the inverse Cosine function

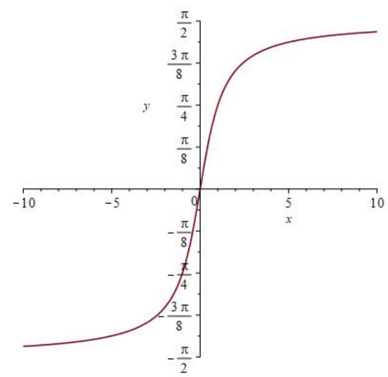


- $y = \cos^{-1}(x)$ means $x = \cos(y)$

- $\cos^{-1}(\cos(x)) = x$ where $0 \leq x \leq \pi$

- $\cos(\cos^{-1}(x)) = x$ where $-1 \leq x \leq 1$

(c) the graph of the inverse Tangent function



- $y = \tan^{-1}(x)$ means $x = \tan(y)$

- $\tan^{-1}(\tan(x)) = x$ where $-\dfrac{\pi}{2} < x < \dfrac{\pi}{2}$

- $\tan(\tan^{-1}(x)) = x$ where $-\infty < x < \infty$

Learning Outcomes
It is expected that learners will be able to:

(a) recognize and use inverse trigonometric function notation
(b) us a calculator to evaluate inverse trigonometric functions
(c) find exact values of composite functions with inverse trigonometric functions
(d) graph the inverse sine, cosine and tangent functions
(e) find exact values for compositions of trigonometric and inverse trigonometric functions

8.2 The Inverse Trigonometric Functions

Formulas and Definitions

(a) $y = \sec^{-1}(x)$ means $x = \sec(y)$

(b) $y = \csc^{-1}(x)$ means $x = \csc(y)$

(c) $y = \cot^{-1}(x)$ means $x = \cot(y)$

Learning Outcomes

It is expected that learners will be able to:

(a) find exact values of composite functions with inverse trigonometric functions
(b) find exact values for compositions of trigonometric and inverse trigonometric functions
(c) write compositions as algebraic expressions

8.3 Trigonometric Equations

Learning Outcomes

It is expected that learners will be able to:

(a) solve trigonometric equations over the interval $(0, 2\pi)$
(b) solve trigonometric equations over the interval $(-\infty, \infty)$
(c) find exact and approximate solutions of trigonometric equation, including equations involving identities

8.4 Trigonometric Identities

In order to establish an identity students must start with one side and show how to get to the other side. **Students may not work both sides of an identity at the same time.**

Learning Outcomes
It is expected that learners will be able to:

(a) use identities to simplify expressions and verify other identities

8.5 Sum and Difference Formulas

Formulas and Definitions
It is expected that learners will know:

(a) **Sum and Difference Formulas**

- $\cos(\alpha + \beta) = \cos\alpha\cos\beta - \sin\alpha\sin\beta$
- $\cos(\alpha - \beta) = \cos\alpha\cos\beta + \sin\alpha\sin\beta$
- $\sin(\alpha + \beta) = \sin\alpha\cos\beta + \cos\alpha\sin\beta$
- $\sin(\alpha - \beta) = \sin\alpha\cos\beta - \cos\alpha\sin\beta$
- $\tan(\alpha + \beta) = \dfrac{\tan\alpha + \tan\beta}{1 - \tan\alpha\tan\beta}$
- $\tan(\alpha - \beta) = \dfrac{\tan\alpha - \tan\beta}{1 + \tan\alpha\tan\beta}$

Learning Outcomes
It is expected that the learners will be able to:

(a) apply the sum or difference formulas to find exact values and to verify trigonometric identities
(b) use the sum or difference formulas to simplify expressions and verify other identities
(c) apply the sum or difference formulas to find exact values and to verify trigonometric identities

Proofs
It is expected that learners will be able to prove:

(a) the sum and difference identities
(Math 107 omit proof of $\cos(\alpha - \beta)$.)

8.6 Double-angle and Half-angle Formulas

Formulas and Definitions
It is expected that learners will know:

(a) **Double-angle Formulas**
- $\sin(2\theta) = 2\sin\theta\cos\theta$
- $\cos(2\theta) = \cos^2\theta - \sin^2\theta$
- $\cos(2\theta) = 1 - 2\sin^2\theta$
- $\cos(2\theta) = 2\cos^2\theta - 1$
- $\tan(2\theta) = \dfrac{2\tan\theta}{1 - \tan^2\theta}$

(b) **Power Reducing Formulas**

- $\sin^2\theta = \dfrac{1-\cos(2\theta)}{2}$
- $\cos^2\theta = \dfrac{1+\cos(2\theta)}{2}$
- $\tan^2\theta = \dfrac{1-\cos(2\theta)}{1+\cos(2\theta)}$

(c) **Half-angle Formulas** (Math 107 omit.)

- $\sin\left(\dfrac{\alpha}{2}\right) = \pm\sqrt{\dfrac{1-\cos\alpha}{2}}$
- $\cos\left(\dfrac{\alpha}{2}\right) = \pm\sqrt{\dfrac{1+\cos\alpha}{2}}$
- $\tan\left(\dfrac{\alpha}{2}\right) = \pm\sqrt{\dfrac{1-\cos\alpha}{1+\cos\alpha}}$

where the + or - sign is determined by the quadrant of the angle $\dfrac{\alpha}{2}$

Learning Outcomes
It is expected that the learners will be able to:

(a) apply the double-angle and half-angle (Math 107 omit.) formulas to find exact values
(b) use the double angle, half angle (Math 107 omit.) and power reducing formulas to simplify expressions and verify other identities

Proofs
It is expected that learners will be able to prove:

(a) the double angle identities
(b) the power reducing identities
(c) the half angle identities (Math 107 omit.)

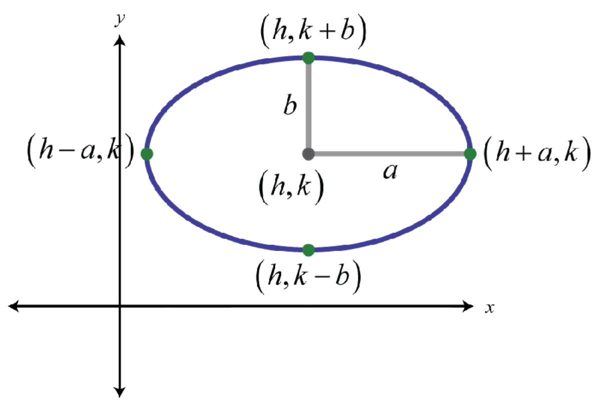
11. Analytic Geometry

11.3 The Ellipse

Formulas and Definitions
It is expected that learners will know:

(a) **Equations of an Ellipse: Center at (h, k):**
Parallel to the x-axis.

$$\frac{(x-h)^2}{a^2} + \frac{(y-k)^2}{b^2} = 1,\ a > b$$



(b) **Equation of an Ellipse: Center at (h, k):**
Parallel to the y-axis.

$$\frac{(x-h)^2}{a^2} + \frac{(y-k)^2}{b^2} = 1,\ a < b$$

Learning Outcomes
It is expected that learners will be able to:

(a) recognize the equation of an ellipse
(b) write the standard form of the equation of an ellipse with center at the origin and translated center (h, k)

(c) find the center, and vertices of an ellipse, given its equation, and sketch the graph

11.4 The Hyperbola

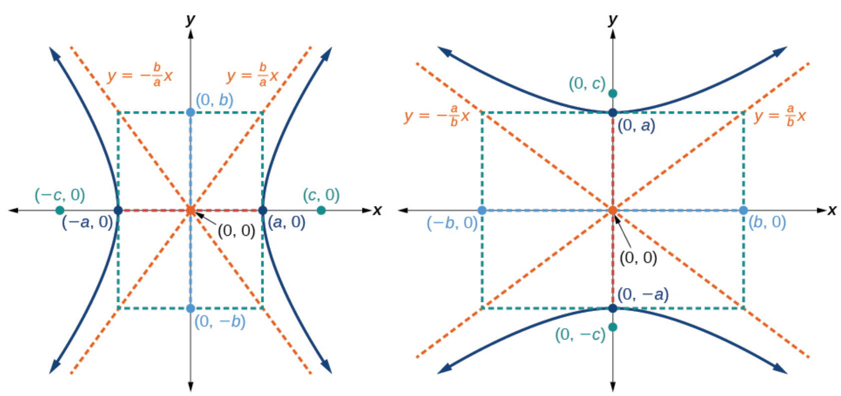
Formulas and Definitions
It is expected that learners will know:

(a) **Equations of a Hyperbola: Center at (h, k):**
Parallel to the x-axis.

$$\frac{(x-h)^2}{a^2} - \frac{(y-k)^2}{b^2} = 1$$

(b) **Equation of an Hyperbola: Center at (h, k):**
Parallel to the y-axis.

$$\frac{(y-k)^2}{a^2} - \frac{(x-h)^2}{b^2} = 1$$



Learning Outcomes
It is expected that learners will be able to:

(a) recognize the equation of a hyperbola
(b) write the standard form of the equation of a hyperbola with center at the origin and translated center (h, k)

(c) find the center, vertices and asymptotes of a hyperbola, given its equation, and sketch the graph

13 Sequences; Introduction; the Binomial Theorem

13.1 Sequences

Formulas and Definitions
It is expected that learners will know:

(a) A **Sequence** is a function whose domain is the set of positive integers.

(b) **Factorial** If $n \geq 0$ is an integer, the factorial symbol $n!$ is defined as follows.

$$0! = 1 \quad 1! = 1$$
$$n! = n \cdot (n-1) \cdots 3 \cdot 2 \cdot 1$$

(c) **Summation Notation:**

$$a_1 + a_2 + a_3 + \cdots + a_n = \sum_{k=1}^{n} a_k$$

(d) **Properties of Sequences**

- $$\sum_{k=1}^{n} (ca_k) = c \sum_{k=1}^{n} a_k$$
- $$\sum_{k=1}^{n} (a_k \pm b_k) = \sum_{k=1}^{n} a_k \pm \sum_{k=1}^{n} b_k$$

(e) **Formulas for Sums of Sequences:**

- $$\sum_{k=1}^{n} c = cn$$
- $$\sum_{k=1}^{n} k = \frac{n(n+1)}{2}$$
- $$\sum_{k=1}^{n} k^2 = \frac{n(n+1)(2n+1)}{6}$$
- $$\sum_{k=1}^{n} k^3 = \left[\frac{n(n+1)}{2}\right]^2$$

Learning Outcomes
It is expected that learners will be able to:

(a) find terms of sequences given the general or n^{th} term
(b) use summation notation to write a series and evaluate a series designated in summation notation
(c) construct the terms of a sequence defined by a recursive formula
(d) identify patterns in sequences and write formulas for general terms
(e) simplify and evaluate basic sums of sequences
(a) properties of sequences
(b) formulas for sums of sequences

13.2 Arithmetic Sequences

Formulas and Definitions:
It is expected that learners will know:

(a) An **arithmetic sequence** may be defined recursively as

$$a_1 = a \qquad a_n = a_{n-1} + d$$

where $a_1 = a$ and d are real numbers. The number a is the first term, and the number d is called the **common difference**.

(b) **n^{th} Term of an Arithmetic Sequence:**
For an arithmetic sequence $\{a_n\}$ whose first term is a_1 and whose common difference is d, the n^{th} term is determined by the formula

$$a_n = a_1 + (n-1)d$$

(c) **Sum of the First n Terms of an Arithmetic Sequence:**
Let $\{a_n\}$ be an arithmetic sequence with first term a_1 and common difference d. The sum S_n of the first n terms of $\{a_n\}$ is given by:

$$S_n = \frac{n}{2}(a_1 + a_2)$$

Learning Outcomes

It is expected that learners will be able to:

(a) recognize and write terms of arithmetic sequences

(b) find a formula for the general or n^{th} term of a given arithmetic sequence, use the n^{th} term formulas for arithmetic sequences to find a specified term, or to find n when an n^{th} term is given

(c) find the sum of the first n terms of an arithmetic sequence

(d) solve word problems involving arithmetic sequences

Proofs:

It is expected that learners will be able to prove:

(a) derive formulas for the n^{th} terms of arithmetic sequences and for the sum of the first n terms of the sequence

13.3 Geometric Sequences

Formulas and Definitions:
It is expected that learners will know:

(a) A **geometric sequence** may be defined recursively as

$$a_1 = a \qquad a_n = r a_{n-1}$$

where $a_1 = a$ and $r \neq 0$ are real numbers. The number a_1 is the first term, and the nonzero number r is called the **common ratio**.

(b) ***n**^th^* **Term of a Geometric Sequence**

For a geometric sequence $\{a_n\}$ whose first term is a_1 and whose common ratio is r, the n^{th} term is determined by the formula

$$a_n = a_1 r^{n-1} \qquad r \neq 0$$

(c) **Sum of the First *n* Terms of a Geometric Sequence**

Let $\{a_n\}$ be a geometric sequence with first term a_1 and common ratio r, where $r \neq 0$, $r \neq 1$. The sum S_n of the first n terms of $\{a_n\}$ is

$$S_n = \sum_{k=1}^{n} a_1 r^{k-1} = a_1 \cdot \frac{1 - r^n}{1 - r}$$

(d) **Convergence of an Infinite Geometric Series**

If $|r| < 1$, the infinite geometric series $\displaystyle\sum_{k=1}^{\infty} a_1 r^{k-1}$ converges. Its sum is

$$\sum_{k=1}^{\infty} a_1 r^{k-1} = \frac{a_1}{1 - r}$$

Learning Outcomes

It is expected that learners will be able to:

(a) recognize and write terms of geometric sequences

(b) find terms of sequences given the general or n^{th} term

(c) find a formula for the general or n^{th} term of a given geometric sequence

(d) find a formula for the general or n^{th} term of a given geometric sequence use n^{th} term formulas for arithmetic sequences to find a specified term, or to find n when an n^{th} term is given

(e) find the sum of the first n terms of a geometric sequence

(f) find the sum of an infinite geometric series, if it exists

(g) derive formulas for the n^{th} terms of geometric sequences and for the sum of the first n terms of the sequence

(h) solve word problems involving geometric sequences and series