# Topic 9: Counting

**Read:** Chpt.6.1-6.5, 8.5-8.6, Rosen

Given two tasks A and B such that
(1) A has m distinct possible outcomes $a_1, a_2, \ldots, a_m$,
(2) B has n distinct possible outcomes $b_1, b_2, \ldots, b_n$.

**Q:** How many possible outcomes are there if we are to perform either A or B, but not both?

**A:** Possible outcomes = $\{a_1, a_2, \ldots, a_m, b_1, b_2, \ldots, b_n\}$.
\# possible outcomes = $|A| + |B| = m + n$.

**Q:** How many possible outcomes are there if we are to perform both A and B?

**A:** When performing both A and B,
Possible outcomes = $\{(a_1,b_1), (a_1,b_2), \ldots, (a_1,b_n),$
$(a_2,b_1), (a_2,b_2), \ldots, (a_2,b_n),$
$\ldots$
$(a_m,b_1), (a_m,b_2), \ldots,(a_m,b_n)\}$.
\# possible outcomes = $|A| * |B| = m * n$.

**Q:** What if we are to perform k tasks $A_1, A_2, \ldots, A_k$, where task $A_i$ has $n_i$ possible outcomes, $1 \le i \le k$?

## Basic Techniques in Counting:

1. **The Rule of Sum:**

   There are $n_1 + n_2 + \ldots + n_k$ possible outcomes if we perform exactly one of these k tasks.

2. **The Rule of Product:**

   There are $n_1 * n_2 * \ldots * n_k$ possible outcomes if we perform all of these k tasks.

## Examples:

1. In a secrete lab, an electronic lock requires a 7-digit code to operate.

Some applications of the two basic rules:

(1) #possible codes: $10*10*\ldots*10 = 10^7$.

(2) #possible codes starting with 8: $1*10^6 = 10^6$.

(3) #possible codes not starting with 4: $9*10^6$.

(It can also be computed by using

#possible codes – #possible codes starting with 4

$= 10^7 - 10^6 = 9*10^6$.)

(4) #possible codes starting with 0, or 1, or 2: $3*10^6$.

(5) #possible codes starting **and** ending with 8: $1*10^5*1$.

(6) #possible codes starting **or** ending with 8 but not both: $(1*10^5*9) + (9*10^5*1)$.

(7) #possible codes starting with 1, 3, 5 or ending with 2, 8 but not both: $(3*10^5*8) + (7*10^5*2)$.

(8) #possible codes starting with 1, 3, 5 or ending with 2, 8: $3*10^6 + 10^6*2 - 3*10^5*2$. (Why?)

2. In Fortran, an identifier is a character string consisting of 1 to 7 characters such that the first character must be a letter chosen from {a, b, …, z}, followed by up to 6 more letter(s) and/or digits.

**Q:** How many distinct identifiers can we declare in Fortran?

Let $P$ be the # of all possible identifiers,

$P_1$ be the # of all identifiers with exactly 1 char,

$P_2$ be the # of all identifiers with exactly 2 chars,

…

$P_7$ be the # of all identifiers with exactly 7 chars.

$P_1 = 26*36^0$, // 26 ways to choose the 1st char

$P_2 = 26*36^1$, // 36 ways to choose the 2nd char

$P_3 = 26*36^2$, // $36^2$ ways to choose the 2nd & 3rd chars

…

$P_7 = 26*36^6$. // $36^6$ ways to choose the 2nd to 7th chars

$$\begin{aligned} \therefore P &= P_1 + P_2 + \ldots + P_7 \\ &= 26*(36^0 + 36^1 + \ldots + 36^6) \\ &= 26*\left(\frac{36^7 - 1}{36 - 1}\right) \\ &\cong 5.82\times 10^{10}. \end{aligned}$$

**Practice HW:** Chpt.6.1, 3, 5, 17, 21, 23, 27, 29, 31.

## 3. The Pigeonhole Principle:

**Theorem:** If k+1 objects (pigeons) are to be placed in k boxes (pigeonholes), then at least one of the k boxes must contain two or more of the objects (pigeons).

**Proof:** (Proof by Contradiction) Assume that k+1 objects are to be placed in the k boxes but none of the k boxes contains more than one object to obtain a contradiction. Since each box contains at most one object, the maximum number of objects in these k boxes must be $\leq$ k, which contradicts to the assumption that there are k+1 objects in the k boxes. Hence, the Pigeonhole Principle holds.

**Remark:** This is also known as Shoebox Principle, or Dirichlet Principle.

**Applications:**

1. Among 13 people in a party, two of them must have their birthday in a same month.

2. Among 102 students taking an exam (max = 100, min = 0), two of them must receive identical score.

3. Given 210 pairs of married couples (420 people). A minimum of 211 people must be selected so as to guarantee that at least a married couple will be included.

**More Applications:**

4. A chess grandmaster has 11 weeks to prepare for a tournament and he/she has decided to practice at least 1 game each day but play no more than 12 games during any 7 consecutive days. Prove that there must be a period of m consecutive days during which the grandmaster will play exactly 21 games.

**Proof:** There are 77 days in 11 weeks.

Let $s_1$ be the total #games played on the first day,
$s_2$ be the total #games played on the first two days,
…
$s_{77}$ be the total #games played on the first 77 days.

$\therefore\ 1 \le s_1 < s_2 < \ldots < s_{77} \le 11*12 = 132.$

By adding 21 to each term of the above equation, we have

$$22 \le s_1+21 < s_2+21 < \ldots < s_{77}+21 \le 132+21 = 153.$$

Among these154 integers $s_1, s_2, \ldots, s_{77}, s_1+21, s_2+21, \ldots, s_{77}+21$, they must satisfy the following inequality:

$$1 \le s_1, s_2, \ldots, s_{77}, s_1+21, s_2+21, \ldots, s_{77}+21 \le 153.$$

Since each of these 154 numbers must be between 1 and 153, by Pigeonhole Principle, at least two of them must have the same value. Hence, $\exists\ i, j \in N,\ 1 \le i < j \le 77$, such that $s_j = s_i + 21$. (Why?)

$\therefore\ s_j - s_i = 21.$

Hence, $s_j - s_i = 21$ and, from day (i+1) to day j, the master must have played exactly 21 games.

5. Given an arbitrary sequence of m positive integers $a_1, a_2, \ldots, a_m$, $m \geq 1$. Prove that there exist integers i, j, $0 \leq i < j \leq m$, such that the subsum $a_{i+1} + a_{i+2} + \ldots + a_j$ is divisible by m.

**Proof:** Define the following partial sums:

$$\begin{aligned} S_1 &= a_1, \\ S_2 &= a_1 + a_2, \\ &\ldots \\ S_m &= a_1 + a_2 + \ldots + a_m. \end{aligned}$$

Consider the following two cases:

***Case 1***: If $\exists$ k, $1 \leq k \leq m$, such that $m|S_k$, we can choose $i = 0$, $j = k$, such that $m|S_k = a_1 + a_2 + \ldots + a_k$.

***Case 2***: Assume that $\forall$ k, $1 \leq k \leq m$, $m \nmid S_k$.

By Division Theorem, we must have

$$\begin{aligned} S_1 &= mq_1 + r_1, && 1 \leq r_1 \leq m - 1, \\ S_2 &= mq_2 + r_2, && 1 \leq r_2 \leq m - 1, \\ &\ldots \\ S_m &= mq_m + r_m, && 1 \leq r_m \leq m - 1. \end{aligned}$$

Observe that there are m remainders $r_1, r_2, \ldots, r_m$ but only $m - 1$ distinct values for the reminders to choose from. Hence, by Pigeonhole Principle, at least two of the m remainders must have the same value!

WOLOG (**W**ith**O**ut **L**oss **O**f **G**enerality), let $r_p = r_t = r$, and $1 \leq p < t \leq m$.

Hence, we have

$$S_p = mq_p + r, \text{ and}$$
$$S_t = mq_t + r.$$

$\therefore\ S_t - S_p = m(q_t - q_p)$ and $m|S_t-S_p$.

By construction,

$$S_t - S_p$$
$$= (a_1 + \ldots + a_p + a_{p+1} + \ldots + a_t) - (a_1 + a_2 + \ldots + a_p)$$
$$= (a_{p+1} + a_{i+2} + \ldots + a_t).$$

Hence, $m|(a_{p+1} + a_{i+2} + \ldots + a_t)$ and we can then choose $i = p$ and $j = t$.

Since both cases lead to the conclusion that $\exists$ i, j, $0 \leq i < j \leq m$, such that $m|(a_{i+1} + a_{i+2} + \ldots + a_j)$, the assertion must be true for all $m \geq 1$.

**Extension: Generalized Pigeonhole Principle**

If m objects are to be placed in n boxes, $m \geq n$, then there exists at least one box containing at least $\lceil \frac{m}{n} \rceil$ objects.

**Observations:**

1. If $m = n+1$, we have the Pigeonhole Principle.
2. If $m \geq kn+1$, then there exists at least one box containing at least $k+1$ objects.

**Applications:**

6. If there are 37 people in a party, how many of them were born in the same month?

Since $m = 37$, $n = 12$, # people born in the same month $\geq \lceil \frac{37}{12} \rceil = 4$.

Conclusion: Among 37 people invited to a party, at least 4 of them were born in the same month.

7. How many guests you must invite in order to guarantee that 6 of them were born in the same month?

Since $\lceil \frac{m}{12} \rceil = 6$, $m = 61$.

Conclusion: At least 61 guests must be invited in order to guarantee that 6 of them were born in the same month.

**Another Approach:**

Since $k+1 = 6$, $k = 5$. Also, $n = 12$, $m = kn+1 = 5*12+1 = 61$.

8. Given infinitely that many red, white, and blue socks in a laundry basket.

**Q**: How many socks one must select to guarantee 3 pairs of socks of the same color are chosen?

**A:** Since n = 3, k+1 = 6 (or k = 5), m = kn+1 = 5∗3+1 = 16. Hence, at least 16 socks must be selected.

**Q:** How many socks one must select to guarantee 3 pairs of red socks?

**A:** Infinity! (Why?)

9. Prove that in a party of six, either 3 of them are mutual friends or 3 of them are complete strangers to each other.

**Proof:** Let A be any person in the group. Among the 5 remaining people, by Generalized Pigeonhole Principle, either 3 or more of them are friends of A, or 3 or more of them are strangers to A.

*Case 1:* Let B, C, D be friends of A.

If two of them are friends, together with A, we have 3 mutual friends. Else, they are 3 complete strangers to each other as required.

*Case 2:* Let B, C, D be strangers to A. If 2 of them are strangers to each other, together with A, we have 3 complete strangers. Else, they are 3 mutual friends as required.

**Practice HW:** Chpt.6.2, 3, 5, 7, 9, 13, 15, 17, 19.

## 4. Permutations and Combinations:

Let's consider the ordered arrangements/selctions of three distinct objects a, b, and c.

Possible Arrangements:

abc
acb
bac
bca
cab
cba

Each one of these arrangements is a ***permutation*** of $S = \{a, b, c\}$.

**Dfn:** Given a set S of n distinct objects. An ***k-permutation*** of S, $k \leq n$, is an ordered arrangement (selection/placement) of any k objects of S.

Let P(n,k), $P_k^n$, be the # of k-permutations of n objects.

**Example:** Let $S = \{a, b, c\}$.

$P(3,1) = 3$,
$P(3,2) = 6$,
$P(3,3) = 6$.

**Q:** How do we compute $P(n,k)$?

Observe that we have

n choices for the $1^{st}$ object,
$n-1$ choices for the $2^{nd}$ object,
$n-2$ choices for the $3^{rd}$ object,
…
$n-k+1$ choices for the $k^{th}$ object.

Hence,

$$P(n,k) = n(n-1)(n-2)\ldots(n-k+1)$$
$$= (n)_k, \qquad \text{(falling factorial function)}$$
$$= \frac{n!}{(n-k)!}.$$

Observe that

$$P(n,n) = n!,$$
$$P(n,0) = 1,$$
$$P(0,0) = 1.$$

**Applications:**

1. In how many different ways can a 5-character string be formed by using characters from {a, b, c, d, e, f, g}?
   **Solution:**
   $$P(7,5) = \tfrac{7!}{2!} = 7 * 6 * 5 * 4 * 3 = 2{,}520.$$

2. There are 30 members in a social club. In how many different ways can a committee with 1 chairperson, 1 vice-chairperson, 1 secretary, and 1 treasurer be formed?
**Solution:**

$$P(30,4) = \frac{30!}{26!} = 30 * 29 * 28 * 27 = 657,720.$$

3. Given S = {2, 3, 5, 7, 9}.
(a) How many distinct 3-digit integers can be formed?

$$P(5,3) = \frac{5!}{2!} = 5 * 4 * 3 = 60.$$

(b) How many distinct 3-digit integers < 500 can be formed?

2 ways to choose the 1st digit (using 2 or 3),
4 ways to choose the 2nd digit,
3 ways to choose the 3rd digit,

∴ # distinct 3-digit integers < 500
= 2∗4∗3
= 24.

(c) How many distinct 3-digit odd integers can be formed?

3∗4∗4 = 48. (Why?)

(d) How many distinct 3-digit odd integers < 500 can be formed?

24 – 3 = 21 (Why?)

Let's now consider the unordered arrangements of distinct objects.

**Dfn:** Given a set S of n distinct objects.
An ***k-combination*** of S is an unordered arrangement (selection/placement) of any k objects of S.

Let C(n,k) = $C_k^n = \binom{n}{k}$ be the # k-combinations of n objects.

**Example:** Let S = {a,b,c}.

1-combination of S: a, b, c

2-combination of S: ab, ac, bc

3-combination of S: abc

C(3,1) = 3,
C(3,2) = 3,
C(3,3) = 1.

**Q:** How do we compute C(n,r)?

**Theorem:** P(n,k) = C(n,k)∗P(k,k).

**Proof:** In order to select k objects from S, where ordering of selected objects is critical, we may first select, without regarding to order, any k objects from S and then orderly arrange these k objects in all possible ways. Hence, by the Rule of Product, we have

P(n,k) = C(n,k)∗P(k,k).

$$\text{Corollary: } \mathrm{C}(n,k) = \frac{\mathrm{P(n,k)}}{\mathrm{P(k,k)}} = \frac{n!}{(n-k)!k!} = \mathrm{C}(n,n-k).$$

**Applications:**

4. There are 30 members in a social club. In how many different ways can a committee of 5 be formed among these 30 members?

$$\mathrm{C}(30,5) = \frac{30!}{25!5!} = 142,506.$$

5. A menu has 6 sodas, 10 sandwiches and 5 desserts. In how many different ways can we order 3 sodas, 5 sandwiches and 2 desserts?

$$\mathrm{C}(6,3) * \mathrm{C}(10,5) * \mathrm{C}(5,2) = \frac{6!}{3!3!} * \frac{10!}{5!5!} * \frac{5!}{3!2!} = 50,400.$$

6. To win the grand prize of the Multi State Powerball, one must match all five white balls labeled from 1 to 69, in any order, and the red Powerball labeled from 1 to 26. In how many ways can we purchase a ticket in order to win the Powerball drawing?

   **Solution:**

   $$C(69,5)*C(26,1) = \frac{69!}{64!5!}*\frac{26!}{25!1!} = \frac{69*68*67*66*65*26}{5!}$$

   $$= 292,201,338.$$

7. A student must answer 10 out of 13 questions in an exam.

   (a) In how many different ways can one take this exam?

   C(13,10) = 286

   (b) What if one must answer the first 2 questions?

   C(2,2)*C(11,8) = 165

   (c) What if one must answer either the first or second, but not both, questions?

   C(2,1)*C(11,9) = 110

   (d) What if one must answer exactly 3 out of the first 5 questions?

   C(5,3) *C(8,7) = 80

   (e) What if one must answer at least 3 out of the first 5 questions?

   C(5,3) *C(8,7) + C(5,4) *C(8,6) + C(5,5) *C(8,5) = 276

8. **Pascal Identity:**

$$C(n,k) = C(n-1,k) + C(n-1,k-1).$$

**Proof:** Let S be a set with n objects and x $\in$ S. Any selection of k objects from S will either include or exclude x. If x is included in a selection, the #ways to select the remaining k-1 objects from the remaining n-1 objects is C(n-1,k-1). If x is excluded, the #ways to select the k objects from the remaining n-1 objects is C(n-1,k). Hence, by Sum Rule, we have C(n,k) = C(n-1,k) + C(n-1,k-1).

**Computing C(n,k):**

1. Using the definition of C(n,k) and factorial functions:

   Inefficient and imprecise! (Why?)

2. Using Pascal Identity and Forward Evaluation:

   Let's try to compute $C(n,k)$ in terms of some previous computed values. Initially, we have $C(n,1) = C(n,n) = 1, \forall n.$ Starting at n = 2, using Pascal Identity, we can then compute $C(n,1), C(n,2), \ldots, C(n,n-1)$ in increasing order of n.

   Observe that we are computing $C(n,k)$ row-by-row as the following table illustrated.

| | 0 | 1 | 2 | 3 | 4 | 5 | 6 |
|---|---|---|---|---|---|---|---|
| 0 | 1 | | | | | | |
| 1 | 1 | 1 | | | | | |
| 2 | 1 | 2 | 1 | | | | |
| 3 | 1 | 3 | 3 | 1 | | | |
| 4 | 1 | 4 | 6 | 4 | 1 | | |
| 5 | 1 | 5 | 10 | 10 | 5 | 1 | |
| 6 | 1 | 6 | 15 | 20 | 15 | 6 | 1 |

**Algorithm:**

```
int binomial(int n, int k)
    if (n < k)                          // initialization
        then return 0;
    for m = 0 to n do
        C(m,0) = 1; endfor;
    for  m = 1 to n do
        C(m,m) = 1; endfor;
    for m = 2 to n do              // computing C(n,k)
        i = min{m-1,k};
        for j = 1 to i do
            C(m,j) = C(m-1,j-1) + C(m-1,j); endfor;
    endfor;
    return C(n,k);
```

**Complexity Analysis:**

$$T(n,k) = \sum_{m=2}^{n} \sum_{j=1}^{k} C = \Theta(nk).$$

**Practice HW:** Chpt.6.3, 3, 5, 13, 15, 17, 19, 21, 25, 27.

## 5. Binomial Theorem:

Consider the binomial expansion $(x + y)^n$, where x, y are real numbers, n is a non-negative integer.

$(x + y)^0 = 1,$
$(x + y)^1 = 1x^1 + 1y^1,$
$(x + y)^2 = 1x^2 + 2x^1y^1 + 1y^2,$
$(x + y)^3 = 1x^3 + 3x^2y^1 + 3x^1y^2 + 1y^3,$
…
$(x + y)^n = ?x^{n-0}y^0 + ?x^{n-1}y^1 + ?x^{n-2}y^2 + \ldots + ?x^1y^{n-1} + ?x^0y^{n-0}.$

**Q**: How do we determine the coefficients of $x^{n-k}y^k$, $0 \le k \le n$?

**A Combinatorial Argument:**

Consider

$$(x + y)^n = (x + y)(x + y)\ldots(x + y) \qquad \text{(n factors).}$$

Each factor $(x + y)$ will contribute either an x, or a y, in the computation of $x^{n-k}y^k$, $0 \le k \le n$. Hence, in order to obtain the product $x^{n-k}y^k$, exactly n-k factors of $(x + y)$ must contribute an x to this product (the remaining k factors of $(x + y)$ must contribute a y to the same product).

**Q**: In how many different ways can we select n-k factors of $(x + y)$ to contribute an x to $x^{n-k}y^k$?

**C(n,n-k) (or C(n,k)).**

Hence, the coefficient of $x^{n-k}y^k$, $0 \le k \le n$, must be C(n,k).

**Binomial Theorem:**
Let x and y be real numbers, n be a non-negative integer.

$$(x + y)^n = C(n,0)x^{n-0}y^0 + C(n,1)x^{n-1}y^1 + C(n,2)x^{n-2}y^2 + \ldots + C(n,k)x^{n-k}y^k + \ldots + C(n,n)x^0y^{n-0}$$

$$= \sum_{k=0}^{n} \binom{n}{k} x^{n-k} y^k$$

**Remark**: $C(n,k) = \binom{n}{k}$ is also known as binomial coefficient.

**Another Proof:** Use induction on n.
**Basis step:** When n = 1, $(x + y)^1 = 1x^1 + 1y^1$, and

$$\sum_{k=0}^{1} \binom{1}{k} x^{1-k} y^k = \binom{1}{0} x^1 + \binom{1}{1} y^1 = 1x^1 + 1y^1$$

Hence, the theorem holds when n = 1.

**Inductive step:** Assume that

$$(x + y)^n = \sum_{k=0}^{n} \binom{n}{k} x^{n-k} y^k .$$

We need to prove that

$$(x + y)^{n+1} = \sum_{k=0}^{n+1} \binom{n+1}{k} x^{n+1-k} y^k .$$

$$(x+y)^{n+1}$$
$$=(x+y)(x+y)^n$$

$$= (x+y)\sum_{k=0}^{n}\binom{n}{k}x^{n-k}y^{k} \qquad \text{(Inductive Hypothesis)}$$

$$= \sum_{k=0}^{n}\binom{n}{k}x^{n+1-k}y^{k} + \sum_{k=0}^{n}\binom{n}{k}x^{n-k}y^{k+1}$$

$$= \binom{n}{0}x^{n+1} + \sum_{k=1}^{n}\binom{n}{k}x^{n+1-k}y^{k} + \sum_{k=0}^{n-1}\binom{n}{k}x^{n-k}y^{k+1} + \binom{n}{n}y^{n+1}$$

$$= x^{n+1} + \sum_{k=1}^{n}\binom{n}{k}x^{n+1-k}y^{k} + \sum_{k=1}^{n}\binom{n}{k-1}x^{n+1-k}y^{k} + \binom{n}{n}y^{n+1}$$

$$\text{(since } \sum_{k=0}^{n-1}\binom{n}{k}x^{n-k}y^{k+1} = \sum_{k=1}^{n}\binom{n}{k-1}x^{n+1-k}y^{k}\text{)}$$

$$= x^{n+1} + \sum_{k=1}^{n}\left[\binom{n}{k}+\binom{n}{k-1}\right]x^{n+1-k}y^{k} + y^{n+1}$$

$$= x^{n+1} + \sum_{k=1}^{n}\binom{n+1}{k}x^{n+1-k}y^{k} + y^{n+1} \qquad \text{(Pascal Identity)}$$

$$= \binom{n+1}{0}x^{n+1} + \sum_{k=1}^{n}\binom{n+1}{k}x^{n+1-k}y^{k} + \binom{n+1}{n+1}y^{n+1}$$

$$= \sum_{k=0}^{n+1}\binom{n+1}{k}x^{n+1-k}y^{k}$$

Hence, if the theorem holds for n, it must also be true for n+1. By induction, the theorem must hold for all n $\geq$ 1.

## Pascal Triangle for Binomial Coefficients:

| | |
|---|---|
| 1 | $(x+y)^0$ |
| 1 1 | $(x+y)^1$ |
| 1 2 1 | $(x+y)^2$ |
| 1 3 3 1 | $(x+y)^3$ |
| 1 4 6 4 1 | $(x+y)^4$ |
| 1 5 10 10 5 1 | $(x+y)^5$ |
| 1 6 15 20 15 6 1 | $(x+y)^6$ |
| … | |

## Applications:

1. $(2x-y)^3$

$$= \sum_{k=0}^{3} \binom{3}{k} (2x)^{3-k}(-y)^k$$

$$= \binom{3}{0}(2x)^3 + \binom{3}{1}(2x)^2(-y) + \binom{3}{2}(2x)(-y)^2 + \binom{3}{3}(-y)^3$$

$$= 8x^3 - 12x^2y + 6xy^2 - y^3$$

2. Compute the coefficient of $x^{12}y^{13}$ in $(2x - 3y)^{25}$.

$$(2x-3y)^{25}$$

$$= \sum_{k=0}^{25} \binom{25}{k} (2x)^{25-k} (-3y)^k$$

Hence, by choosing k = 13, the coefficient of $x^{12}y^{13}$ in $(2x - 3y)^{25}$ is given by

$$\binom{25}{13} 2^{12} (-3)^{13} = (-1)\frac{25!}{12!13!} 2^{12} 3^{13}.$$

3. In how many ways can one select 0, 1, …, n objects, without regarding to the order of selections, from a set of n distinct objects?

We need to compute

$$\binom{n}{0} + \binom{n}{1} + \ldots + \binom{n}{n} = \sum_{k=0}^{n} \binom{n}{k}.$$

Consider

$$(x+y)^n = \sum_{k=0}^{n} \binom{n}{k} x^{n-k} y^k.$$

By taking x = y = 1, we have

$$(x+y)^n = 2^n = \sum_{k=0}^{n} \binom{n}{k}.$$

Hence, there are $2^n$ ways to make the unordered selections.

**Corollary:** The # of subsets in a set with n elements = $2^n$.

4. In how many ways can one select an even number of objects, without regarding to the order of selections, from a set of n distinct objects?

We need to compute

$$\binom{n}{0}+\binom{n}{2}+\ldots+.$$

Consider

$$(x+y)^n = \sum_{k=0}^{n}\binom{n}{k}x^{n-k}y^k.$$

By taking x = 1 and y = -1, we have

$$(1-1)^n = 0 = \sum_{k=0}^{n}\binom{n}{k}(-1)^k.$$

Hence,

$$\binom{n}{0}-\binom{n}{1}+\binom{n}{2}-\binom{n}{3}+\ldots+(-1)^n\binom{n}{n}=0.$$

Or,

$$\binom{n}{0}+\binom{n}{2}+\ldots$$

$$=\binom{n}{1}+\binom{n}{3}+\ldots$$

$$=\frac{2^n}{2}$$

$$=2^{n-1}.$$

5. Compute $1\binom{n}{1}+2\binom{n}{2}+\dots+n\binom{n}{n}=\sum_{k=0}^{n}k\binom{n}{k}$.

Consider

$$(1+x)^n = \sum_{k=0}^{n}\binom{n}{k}x^k.$$

$$\frac{d}{dx}(1+x)^n$$

$$= n(1+x)^{n-1}$$

$$= \frac{d}{dx}\sum_{k=0}^{n}\binom{n}{k}x^k$$

$$= \sum_{k=0}^{n}\binom{n}{k}\frac{d}{dx}x^k$$

$$= \sum_{k=0}^{n}k\binom{n}{k}x^{k-1}.$$

By taking x = 1, we have

$$1\binom{n}{1}+2\binom{n}{2}+\dots+n\binom{n}{n}=n2^{n-1}.$$

**Q**: What if n ≠ non-negative integer?

What is $\binom{\alpha}{k}$ if α ≠ non-negative integer?

**Extension:**

Recall that, for non-negative integers n and k, we have

$$\binom{n}{k} = \frac{n!}{(n-k)!k!} = \frac{n(n-1)...(n-k+1)}{k!}.$$

## Generalized Binomial Coefficients:

**Dfn:** Given any real number α and non-negative integer k.

$$\binom{\alpha}{k} = \frac{\alpha(\alpha-1)...(\alpha-k+1)}{k!},$$

$$\binom{\alpha}{0} = 1.$$

**Generalized Binomial Theorem:**

For any real numbers x, y, α,

$$(x+y)^\alpha = \sum_{k=0}^{\infty} \binom{\alpha}{k} x^{\alpha-k} y^k.$$

**Applications:**

6. Compute $\dfrac{1}{1+x}$.

$$\frac{1}{1+x}$$

$$= (1+x)^{-1}$$

$$= \sum_{k=0}^{\infty} \binom{-1}{k} x^k$$

$$= \sum_{k=0}^{\infty} \frac{(-1)(-1-1)(-1-2)\ldots(-1-k+1)}{k!} x^k$$

$$= \sum_{k=0}^{\infty} \frac{(-1)^k (1)(2)\ldots(k)}{k!} x^k$$

$$= \sum_{k=0}^{\infty} (-1)^k x^k$$

$$= 1 - x + x^2 - x^3 + \ldots$$

7. Compute $\frac{1}{(1-x)^n}$.

$$\frac{1}{(1-x)^n}$$

$$= (1-x)^{-n}$$

$$= \sum_{k=0}^{\infty} \binom{-n}{k}(-x)^k$$

$$= \sum_{k=0}^{\infty} \frac{(-n)(-n-1)(-n-2)\ldots(-n-k+1)}{k!}(-x)^k$$

$$= \sum_{k=0}^{\infty} \frac{(n)(n+1)(n+2)\ldots(n+k-1)}{k!}\, x^k$$

$$= \sum_{k=0}^{\infty} \binom{n+k-1}{k} x^k.$$

**Practice HW**: Chpt.6.4, 7, 9, 13, 15, 17, 21, 25.

## 6. Generalized Permutation and Combination:
### (Permutation and Combination of Multisets)

Let S be a multiset with n elements.
Recall that a multi set $S = \{m_1 \bullet x_1, m_2 \bullet x_2, \ldots, m_k \bullet x_k\}$ denotes that S has k distinct types of objects $x_1, x_2, \ldots, x_k$ and there are $m_1$ copies of $x_1$, $m_2$ copies of $x_2$, …, and $m_k$ copies of $x_k$, where $m_i$, $1 \le i \le k$, is the ***repetition number*** of $x_i$ and $|S| = \sum_{i=1}^{k} m_i = n$.

**Theorem:** Let S be a multiset with k distinct types of objects each of which appears at least r times in S. Then the number of r-permutations of S is $k^r$.

**Proof:** There are k choices for the $1^{st}$ type of objects,
k choices for the $2^{nd}$ type of objects,
…
k choices for the $r^{th}$ type of objects.
Hence, by Product rule, # r-permutations = $k^r$.

**Applications:**

1. How many distinct 3-digit binary integers are there?

Observe that $S = \{\infty \bullet 0, \infty \bullet 1\}$. We have k = 2 and r =3. Hence, there are $k^r = 2^3 = 8$ different 3-digit binary integers.

2. Let S = {3•a, 2•b, 2•c}.

(a) # **1-permutations** = 3.

(b) # **2-permutations** = $3^2$ = 9.

(c) # **3-permutations**:

3-permutations can be formed by:

{3•a}, {2•a, 1•b}, {2•a, 1•c}, {1•a, 2•b},
{1•a, 2•c}, {1•a, 1•b, 1•c}, {2•b, 1•c}, {1•b, 2•c}.

Hence,

$$\binom{3}{3}+6\binom{3}{2}\binom{1}{1}+\binom{3}{1}\binom{2}{1}\binom{1}{1}$$
$$=1+18+6$$
$$=25.$$

(d) # **4-permutations**:

4-permutations can be formed by:

{3•a, 1•b}, {3•a, 1•c}, {2•a, 2•b}, {2•a, 2•c},
{2•a, 1•b, 1•c}, {1•a, 2•b, 1•c}, {1•a, 1•b, 2•c},
{2•b, 2•c}.

Hence,

$$2\binom{4}{3}\binom{1}{1}+3\binom{4}{2}\binom{2}{2}+3\binom{4}{2}\binom{2}{1}\binom{1}{1}$$
$$=8+18+36$$
$$=62.$$

(e) # **5-permutations**:

5-permutations can be formed by:

{3•a, 2•b}, {3•a, 2•c}, {2•a, 2•b, 1•c},
{2•a, 1•b, 2•c}, {1•a, 2•b, 2•c}.

Hence,

$$2\binom{5}{3}\binom{2}{2}+2\binom{5}{2}\binom{3}{2}\binom{1}{1}+\binom{5}{1}\binom{4}{2}\binom{2}{2}$$
$$=20+60+30$$
$$=110.$$

(f) # **6-permutations**:

6-permutations can be formed by:

{3•a, 2•b, 1•c }, {3•a, 1•b, 2•c}, {2•a, 2•b, 2•c}.

Hence,

$$2\binom{6}{3}\binom{3}{2}\binom{1}{1}+\binom{6}{2}\binom{4}{2}\binom{2}{2}$$
$$=120+90$$
$$=210.$$

(g) # **7-permutations**:

$$\binom{7}{3}\binom{4}{2}\binom{2}{2}=\left(\frac{7!}{3!4!}\right)\left(\frac{4!}{2!2!}\right)\left(\frac{2!}{2!0!}\right)$$
$$=\frac{7!}{3!2!2!}$$
$$=210.$$

**Theorem:** Let S be a multiset with k types of distinct objects $x_1, x_2, \ldots, x_k$ such that $x_i$ has repetition number $m_i \geq 1$, $1 \leq i \leq k$, and $\sum_{i=1}^{k} m_i = n.$ Then the number of permutations of S (n-permutation) is given by

$$\frac{n!}{m_1!m_2!\ldots m_k!}.$$

3. How many distinct character strings can be obtained by permuting the characters in the word "MISSISSIPPI"?

Let S = {1•M, 4•I, 4•S, 2•P}. Hence, |S| = 11.

#11 permutations:

$$\frac{11!}{1!4!4!2!} = 34,650.$$

**Q:** How about (unordered) combinations of multisets?

**Example:** Let S = {3•a, 2•b, 2•c}.

2-combinations are:

aa ab ac

bb bc

cc

**Theorem:** Let S be a multiset with k distinct types of objects each of which appears at least r times in S. Then the number of r-combinations of S is given by

$$\binom{r+k-1}{r}.$$

***Proof.*** Assume that there are k compartments separated by (k-1) dividers such that each compartment corresponds to exactly one type of distinct objects in S.

| | | | |
|---|---|---|---|
| Type1 | Type2 | … | Typek |

Observe that each arrangement of r objects from S corresponds to placing r (identical) markers into these k compartments and vice versa.

**Example:** Recall that for S = {3•a, 2•b, 2•c}, 2-combinations are {aa, ab, ac, bb, bc, cc}.

**Sample representations:**

aa:

| a | b | c |
|---|---|---|
| ** | | |

**||

ab:

| | | |
|---|---|---|
| * | * | |

*|*|

ac:

| | | |
|---|---|---|
| * | | * |

*||*

Let H = {r•∗, (k-1)•|} be a multiset with two types of objects (markers and dividers). There exists a one-one correspondence between any unordered arrangement of S and the permutation of the (r + k – 1) markers and dividers in H. Hence, the number of r-combinations of S is given by the number of ways one can orderly rearrange the objects in H. From previous Theorem, the (r + k – 1)-permutation of H is given by

$$\frac{(r+k-1)!}{r!(k-1)!} = \binom{r+k-1}{r},$$

which is also the number of r-combinations of S as required.

**Applications:**

4. For S = {3•a, 2•b, 2•c}, r = 2, k = 3. Hence, #2-combinations:
   $$\binom{2+3-1}{2} = \binom{4}{2} = \frac{4!}{2!2!} = 6.$$

5. There are 8 varieties of doughnuts in a doughnut shop. In how many different ways can one purchase a dozen doughnuts?

This is equivalent to computing the #12-combinations of a multiset with 8 distinct types of objects.

If there are at least 12 doughnuts of each type, r = 12, k = 8, #12-combinations:

$$\binom{12+8-1}{12} = \binom{19}{12} = \frac{19!}{12!7!} = 50,388.$$

**Q:** What if one must include each one of these 8 varieties? What are r and k?

r = 4, k = 8.

#8 combinations:

$$\binom{4+8-1}{4} = \binom{11}{4} = \frac{11!}{4!7!} = 330.$$

**Q:** What if there are only 4 special (Type1) doughnuts left?

6. How many integer solutions are there satisfying the following equations?

$$x_1 + x_2 + x_3 = 10,$$
$$x_i \geq 0, \forall i.$$

This is equivalent to selecting 10 objects from the multiset $S = \{10 \bullet x_1, 10 \bullet x_2, 10 \bullet x_3\}$.

Hence, r = 10, k = 3, and
#integer solutions = #10-combinations. Hence,

$$\binom{10+3-1}{10} = \binom{12}{10} = \frac{12!}{10!2!} = 66.$$

7. What if $x_1 \geq 1$, $x_2 \geq 2$, $x_3 \geq 3$?
Define $y_1 = x_1 - 1$, $y_2 = x_2 - 2$, $y_3 = x_3 - 3$.
The given equations are equivalent to:

$$(y_1 + 1) + (y_2 + 2) + (y_3 + 3) = 10,$$
$$y_i \geq 0, \forall i.$$

Or, $y_1 + y_2 + y_3 = 4, y_i \geq 0, \forall i.$

Hence, r = 4, k = 3.
#integer solutions:

$$\binom{4+3-1}{4} = \binom{6}{4} = \frac{6!}{4!2!} = 15.$$

8. How many different integer solutions can we have?

   $x_1 + x_2 \le n$,

   $x_1, x_2 \ge 0$.

   Consider $S = \{n* x_1, n*x_2\}$.

   # of solutions for $x_1 + x_2 = i$, $0 \le i \le n$, is:

   $$\binom{i+2-1}{i}.$$

   Total # solutions:

   $$\sum_{i=0}^{n} \binom{i+2-1}{i}$$
   $$= \sum_{i=0}^{n} \binom{i+1}{i}$$
   $$= \sum_{i=0}^{n} (i+1)$$
   $$= \frac{n(n+1)}{2} + (n+1)$$
   $$= \frac{(n+1)(n+2)}{2}.$$

**Practice HW:** Chpt.6.5, 1, 5, 9, 11, 13, 15, 19, 23, 25.

## 7. The Principle of Inclusion and Exclusion:

For finite sets A, B, $|A \cup B| = |A| + |B| - |A \cap B|$.

For finite sets A, B, C.

$$|A \cup B \cup C| = |A| + |B| + |C| - |A \cap B| - |B \cap C| - |A \cap C| + |A \cap B \cap C|.$$

In general, for finite sets $A_1, A_2, \ldots, A_n$,

$$|A_1 \cup A_2 \cup \ldots \cup A_n| = \sum_{1\le i\le n}|A_i| - \sum_{1\le i<j\le n}|A_i \cap A_j| + \sum_{1\le i<j<k\le n}|A_i \cap A_j \cap A_k| - \ldots + (-1)^{n+1}|A_1 \cap A_2 \cap \ldots \cap A_n|.$$

**Applications:**

1. Consider the following enrollment information:

    100 students taking EECS140,
    80 students taking EECS210,
    70 students taking EECS268,
    60 students taking both EECS140 and EECS210,
    50 students taking both EECS210 and EECS268,
    30 students taking both EECS140 and EECS268,
    20 students taking EECS140, EECS210, and EECS268.

#students taking any one of these courses

$= 100 + 80 + 70 - 60 - 50 - 30 + 20 = 130$.

2. How many integers between 1 and 1000, inclusively, are not divisible by 5, 6, or 8?
Let S be the set of the first 1000 positive integers.
For $1 \le i \le 3$, let $A_i$ be the set of integers in S that are divisible by 5, 6, and 8, respectively.
Observe that

$$|A_1| = \left\lfloor \frac{1000}{5} \right\rfloor = 200,$$

$$|A_2| = \left\lfloor \frac{1000}{6} \right\rfloor = 166,$$

$$|A_3| = \left\lfloor \frac{1000}{8} \right\rfloor = 125.$$

Since an integer n is divisible by integers x and y iff n is divisible by lcm(x,y), we have

$$|A_1 \cap A_2| = \left\lfloor \frac{1000}{lcm(5,6)} \right\rfloor = 33,$$

$$|A_2 \cap A_3| = \left\lfloor \frac{1000}{lcm(6,8)} \right\rfloor = 41,$$

$$|A_1 \cap A_3| = \left\lfloor \frac{1000}{lcm(5,8)} \right\rfloor = 25.$$

Also, by a similar argument,

$$|A_1 \cap A_2 \cap A_3| = \left\lfloor \frac{1000}{lcm(5,6,8)} \right\rfloor = 8.$$

Hence, # integers between 1 and 1000 not divisible by 5, 6, or 8 is given by $1000 - (200 + 166 + 125) + (33 + 41 + 25) - 8 = 600$.

**Another Form of Inclusion and Exclusion:**

Given a finite set of A. Let $A_i$ be the subset set of A such that elements in $A_i$ satisfy a given property $P_i$, $1 \le i \le n$.

**Q:** How many elements in A that will have none of the properties $P_i$, $1 \le i \le n$?

Let $N(P_i)$ be the number of elements satisfying $P_i$,

$N(P_i')$ be the number of elements not satisfying $P_i$,

$N(P_iP_j)$ be the number of elements satisfying $P_i$ & $P_j$,

$N(P_i'P_j')$ be the number of elements not satisfying $P_i$ or $P_j$,

…

$N(P_1P_2\ldots P_n)$ be the number of elements satisfying all given properties $P_1, P_2, \ldots, P_n$,

$N(P_1'P_2'\ldots P_n')$ be the number of elements satisfying none of the given properties $P_1, P_2, \ldots, P_n$.

Observe that

$$N(P_1P_2\ldots P_n) = |A_1 \cap A_2 \cap \ldots \cap A_n|, \text{ and}$$
$$N(P_1'P_2'\ldots P_n') = |A| - |A_1 \cup A_2 \cup \ldots \cup A_n|.$$

By applying the Principle of IE, we have

$N(P_1'P_2'\ldots P_n')$

$= |A| - |A_1 \cup A_2 \cup \ldots \cup A_n|$

$= |A| - \sum_{1\le i\le n}|A_i|$

$\quad + \sum_{1\le i<j\le n}|A_i \cap A_j|$

$\quad - \sum_{1\le i<j<k\le n}|A_i \cap A_j \cap A_k|$

$\quad + \ldots$

$\quad + (-1)^n|A_1 \cap A_2 \cap \ldots \cap A_n|$

$= |A| - \sum_{1\le i\le n}N(P_i) + \sum_{1\le i<j\le n}N(P_iP_j) - \sum_{1\le i<j<k\le n}N(P_iP_jP_k)$

$\quad + \ldots + (-1)^nN(P_1P_2\ldots P_n)$

**Examples:**

1. How many primes $\le 100$ are there?

Recall that if a positive integer $n \le 100$ is composite, it must be divisible by a prime $\le \lfloor\sqrt{n}\rfloor$.

Since $\lfloor\sqrt{100}\rfloor = 9$, any composite number $\le 100$ must be divisible by a prime from $\{2, 3, 5, 7\}$.

Let $A = \{2, 3, \ldots, 100\}$.

Define: $P_1$ be the property that n is divisible by 2,
$P_2$ be the property that n is divisible by 3,
$P_3$ be the property that n is divisible by 5,
$P_4$ be the property that n is divisible by 7.

Hence, #primes in $A = 4 + N(P_1'P_2'P_3'P_4')$.

Why?

$N(P_1'P_2'P_3'P_4')$

$= |A| - \sum_{1\le i\le 4} N(P_i) + \sum_{1\le i<j\le 4} N(P_iP_j) - \sum_{1\le i<j<k\le 4} N(P_iP_jP_k)$

$\quad + N(P_1P_2\ldots P_n)$

$= 99 - \lfloor \frac{100}{2} \rfloor - \lfloor \frac{100}{3} \rfloor - \lfloor \frac{100}{5} \rfloor - \lfloor \frac{100}{7} \rfloor$

$\quad + \lfloor \frac{100}{2*3} \rfloor + \lfloor \frac{100}{2*5} \rfloor + \lfloor \frac{100}{2*7} \rfloor + \lfloor \frac{100}{3*5} \rfloor + \lfloor \frac{100}{3*7} \rfloor + \lfloor \frac{100}{5*7} \rfloor$

$\quad - \lfloor \frac{100}{2*3*5} \rfloor - \lfloor \frac{100}{2*3*7} \rfloor - \lfloor \frac{100}{2*5*7} \rfloor - \lfloor \frac{100}{3*5*7} \rfloor$

$\quad + \lfloor \frac{100}{2*3*5*7} \rfloor$

$= 99 - 50 - 33 - 20 - 14$

$\quad + 16 + 10 + 7 + 6 + 4 + 2$

$\quad - 3 - 2 - 1 - 0$

$\quad + 0$

$= 21$

Hence, there are $4 + 21 = 25$ primes $\le 100$.

2. Given two sets A and B with $|A| = m$ and $|B| = n$ we have
   (a) # of relations defined from A to B = $2^{m\times n}$.
   (b) # of functions defined from A to B = $n^m$.
   (c) If $m > n$, there is no injection from A to B.
   (d) If $m < n$, there is no surjection from A to B.
   (e) If $m \ne n$, there is no bijection from A to B.
   (f) If $m = n$, there are $n!$ bijections from A to B.
   (g) If $m \le n$, there are $(n)_m$ injections from A to B.

**Q:** How many surjections are there from A to B?

(h) If m ≥ n, the number of surjections from A to B is:

$$n^m - C(n,1)(n-1)^m + C(n,2)(n-2)^m - \ldots + (-1)^{n-1}C(n,n-1)1^m.$$

***Proof.*** Let B = $\{b_1, b_2, \ldots, b_n\}$. If a function f from A to B is not a surjection, one or more $b_i$'s must not have a pre-image in A.

Define:

$P_1$ be the property that $b_1$ has no pre-image in A,
$P_2$ be the property that $b_2$ has no pre-image in A,
…
$P_n$ be the property that $b_n$ has no pre-image in A.

Hence, # of surjections

$= N(P_1'P_2'\ldots P_n')$
$= n^m - \sum_{1\le i\le n} N(P_i) + \sum_{1\le i<j\le n} N(P_iP_j) - \sum_{1\le i<j<k\le n} N(P_iP_jP_k) + \ldots + (-1)^n N(P_1P_2\ldots P_n)$.

Observe that

$N(P_i) = (n-1)^m$, (# functions; $b_i$ has a pre-image)
$N(P_iP_j) = (n-2)^m$,
$N(P_iP_jP_k) = (n-3)^m$,
…
$N(P_1P_2\ldots P_n) = (n-n)^m = 0$.

Hence,

\# of surjections
$= n^m - \sum_{1\le i\le n} N(P_i) + \sum_{1\le i<j\le n} N(P_iP_j) - \sum_{1\le i<j<k\le n} N(P_iP_jP_k) + \ldots + (-1)^n N(P_1P_2\ldots P_n)$
$= n^m - C(n,1)(n-1)^m + C(n,2)(n-2)^m - \ldots + (-1)^{n-1}C(n,n-1)1^m.$

3. In how many different ways can we assign five different jobs to four different employees such that each employee must perform at least one job?

Define A = set of jobs, $|A| = 5$,
B = set of employees, $|B| = 4$.

Job assignment $\Leftrightarrow$ surjection from A to B.

Hence,
# different assignments
$= 4^5 - C(4,1)*(4-1)^5 + C(4,2)*(4-2)^5 - C(4,3)*(4-3)^5$
$= 1024 - 972 + 192 - 4$
$= 240$

**Practice HW**: Chpt.8.5, 1, 3, 5, 7, 9, 11, 15, 19.
Chpt.8.6, 1, 5, 9, 11.