Topic 3: Basic Set Theory

Read: Chpt.2.1, Rosen

Def: A ***set*** is an unordered collection of objects (***elements***).

Set Representations:

(1) Use English description.
(2) List all the elements of the set inside {…}.
(3) Use propositional function and set descriptor $S = \{x \mid P(x)\}$, where S contains all elements x satisfying the given predicate P(x).

Example: Different representations of set.

S is the set of positive integers less than 5.
$S = \{1, 2, 3, 4\}$.
$S = \{x \mid x \text{ is an integer}, 0 < x < 5\}$.
$S = \{x \mid x \text{ is an integer}, 1 \leq x \leq 4\}$.

Def: If x is an element in a set S, then

(i) x is a ***member (element)*** of S,
(ii) x ***belongs to*** S, and
(iii) S ***contains*** x.

Notations:

$x \in S$ indicates that x is a member in S,
$y \notin S$ indicate that y is NOT an element in S.

Two Special Sets:

1. ***Empty/Null set*:** A set contains no element, denoted by $\varnothing$, or $\{\ \}$.
2. ***Universal set*:** A set contains all elements during computation, denoted by U.

Def: The ***cardinality (order)*** of a set S, $|S|$, is the number of elements in the set.

Def: A set is ***finite*** if it contains a finite number of elements. Otherwise, it is an infinite set.
Hence, for a finite set S, $|S| < \infty$.

Set Comparisons:

1. Equality of Sets:

Def: Two sets A and B are equal iff they contain the same elements, denoted by $A = B$.

Remark: Observe that the following statements are equivalent:

(i) $A = B$.

(ii) $(\forall x \in A, x \in B) \wedge (\forall y \in B, y \in A)$.

(iii) $(x \in A \rightarrow x \in B) \wedge (y \in B \rightarrow y \in A)$.

Example:

$A = \{1, 2, 3, 4\}$,
$B = \{1, 2, 3, 4\}$,
$C = \{4, 3, 1, 2\}$.

By the definition of equality of sets, $A = B = C$.

Remark: Observe that the order in listing the elements in a set is not important when considering the equality of sets.

Q: How about the set $D = \{1, 2, 3, 4, 1, 2, 3, 2, 2\}$?

A: Based on our definition on the equality of sets, $A = B = C = D$.

Q: How many elements are there in A and D?
$|A| = 4$,
$|D| = 9$ (or 4?).

Q: How can two equal sets having different cardinalities?

Remark: No duplicate elements should be listed in a "*simple*" set.

2. Simple vs. Multi Sets:

Simple set: Duplicate elements are excluded in the set. If D is a simple set, then A = D.

Multi set: Duplicate elements are allowed in the set. If A and D are multi sets, then $A \neq D$.

Q: How do we distinguish a simple set from a multi set?

A: We need to use a different representation for multi sets.

Let S be a multi set with n elements, among them we have m distinct types of elements $x_1, x_2, \ldots, x_m$ such that there are

k_1 copies of x_1,
k_2 copies of x_2,
…,
and k_m copies of x_m.

Representation of Multi Sets:

$S = \{ k_1 \cdot x_1, k_2 \cdot x_2, \ldots, k_m \cdot x_m \}$, where $n = k_1 + k_2 + \ldots + k_m$.

Hence, for previous example, $D = \{2 \cdot 1, 4 \cdot 2, 2 \cdot 3, 1 \cdot 4\}$.

Some Important Sets and Their Notations:

R— the set of all real numbers,
Z— the set of all integers,
N— the set of all positive integers,
(Alternate Def: The set of all non-negative integers)
Q— the set of all rational numbers
$\mathbf{R}^+$, $\mathbf{Z}^+$, $\mathbf{Q}^+$— the set of all positive elements in the set

3. Set Inclusion:

Def: Given two sets A and B, A is a ***subset of*** B, $A \subseteq B$, if an only if every element of A is also an element of B.
If A is a subset of B and $A \neq B$, then $A \subset B$.

Observation:

Since $A \subseteq B \equiv \forall x \in A, x \in B$, we have
$A = B \equiv (A \subseteq B) \wedge (B \subseteq A)$.

Example: Given $A = \{1, 2, 3, 4\}$. Observe that

$\{1\} \subseteq A$, $\{1\} \subset A$, $1 \not\subset A$,
$\{1\} \notin A$, $1 \in A$,
$\{2, 1, 4, 3\} \subseteq A$, $\{1, 2, 3, 4\} \not\subset A$,
$A \subseteq A$ and $\varnothing \subseteq A$,
$\varnothing \subseteq \varnothing$ but $\varnothing \notin \varnothing$.

Def: A set A is a ***proper subset*** of set B iff $A \subseteq B$, $A \neq B$ and $A \neq \varnothing$.

A Simple Counting Problem:

How many distinct subsets are there in a set A with n elements?

Example: Given A = {1, 2, 3, 4}.

There are 16 subsets of A:

$\varnothing$,
{1}, {2}, {3}, {4},
{1, 2}, {1, 3}, {1, 4}, {2, 3}, {2, 4}, {3, 4},
{1, 2, 3}, {1, 2, 4}, {1, 3, 4}, {2, 3, 4},
{1, 2, 3, 4}.

Remark: In general, if A is a finite set having n elements, then there are 2^n subsets of A.

Def: The collection of all subsets of a set A is called the ***Power Set*** of A, P(A).

Example: The power set of A = {1, 2, 3, 4} is given by
P(A)
= {$\varnothing$, {1}, {2}, {3}, {4}, {1, 2}, {1, 3}, {1, 4}, {2, 3},
{2, 4}, {3, 4},{1, 2, 3}, {1, 2, 4}, {1, 3, 4},
{2, 3, 4}, {1, 2, 3, 4}}.

Two Very Important Computational Problems:

- Given a set A, how do we generate all the subsets of A?
- Given an integer k, $1 \le k \le |A|$, how do we generate all those subsets of A with order k?

4. Ordered Sets:

Recall that sets are usually unordered. If linear ordering needs to be defined on the set of objects in A, how do we represent such an ordered set A?

Def: An ***(ordered) n-tuple,*** denoted by $(x_1, x_2, \ldots, x_n)$, is an ordered set $S = \{x_1, x_2, \ldots, x_n\}$ with n elements such that x_1 is the first element in S, x_2 is the second element in S, …, and x_n is the nth element in S.

Equality of Ordered Sets:

Def: Given two ordered n-tuples $A = (a_1,a_2,\ldots,a_n)$ and $B = (b_1, b_2, \ldots, b_n)$. $A = B$ iff $a_1 = b_1$, $a_2 = b_2$, …, $a_n = b_n$.

Example: $(1, 2, 3, 4) = (1, 2, 3, 4)$,
$(1, 2, 3, 4) \neq (1, 2, 4, 3)$,
$(1, 2, 3, 4) \neq (1, 2, 3)$.

Generalization:

Given an n-tuple $S = (x_1, x_2, \ldots, x_n)$, each i^{th} element x_i may come from any set S_i, $1 \le i \le n$.

An Important Special Case:

When $n = 2$, an ordered 2-tuple (x_1, x_2) is called an ***ordered pair***.

Def: Given two (simple) sets A and B.
The ***Cartesian Product*** of A and B is the set of all ordered pairs (x,y) such that $x \in A$ and $y \in B$.

Hence,

$$A \times B = \{(x, y) \mid (x \in A) \wedge (y \in B)\}.$$

Example: Given $A = \{1, 2\}$, $B = \{a, b, c\}$.

$$A \times B = \{(1, a), (1, b), (1, c), (2, a), (2, b), (2, c)\}$$
$$B \times A = \{(a, 1), (a, 2), (b, 1), (b, 2), (c, 1), (c, 2)\}.$$

Warning: In general, $A \times B \neq B \times A$, unless $A = B$.

Extension:

Given $A_1, A_2, \ldots, A_n$. The Cartesian product of $A_1, A_2, \ldots, A_n$ is the set of ordered n–tuples
$A_1 \times A_2 \times \ldots \times A_n = \{(a_1, a_2, \ldots, a_n) \mid a_i \in A_i, i = 1, 2, \ldots, n\}$.

Example: Given $A = \{1, 2\}$, $B = \{a, b, c\}$, $C = \{x, y\}$.

$$\begin{aligned} A \times B \times C = \{&(1, a, x), (1, a, y), \\ &(1, b, x), (1, b, y), \\ &(1, c, x), (1, c, y), \\ &\quad (2, a, x), (2, a, y), \\ &\quad (2, b, x), (2, b, y), \\ &\quad (2, c, x), (2, x, y)\}. \end{aligned}$$

Q: How many elements are there in $A_1 \times A_2 \times \ldots \times A_n$?

A: $|A_1| * |A_2| * \ldots * |A_n|$.

Practice HW: Chpt.2.1: 7, 9, 11, 17, 19, 21, 23, 25, 27, 31.

Operations on Sets:

Read: Chpt.2.2, Rosen

Simple Set Operations:
Given (simple) sets A, B, C, ….

Def: The ***union*** of A and B:

$$A \cup B = \{x \mid (x \in A) \vee (x \in B)\}.$$

Def: The ***intersection*** of A and B:

$$A \cap B = \{x \mid (x \in A) \wedge (x \in B)\}.$$

Def: Two sets A and B are ***disjoint*** iff $A \cap B = \varnothing$.

Q: Given two finite sets A and B. How are $|A|$, $|B|$, $|A \cup B|$, $|A \cap B|$ related?

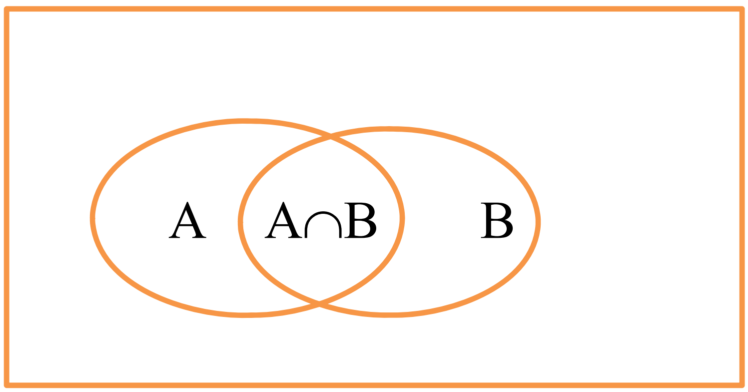
$$|A| + |B| = |A \cup B| + |A \cap B|, \text{ or}$$
$$|A \cup B| = |A| + |B| - |A \cap B|.$$

This is the simplest form of the ***Principle of Inclusion–Exclusion***.

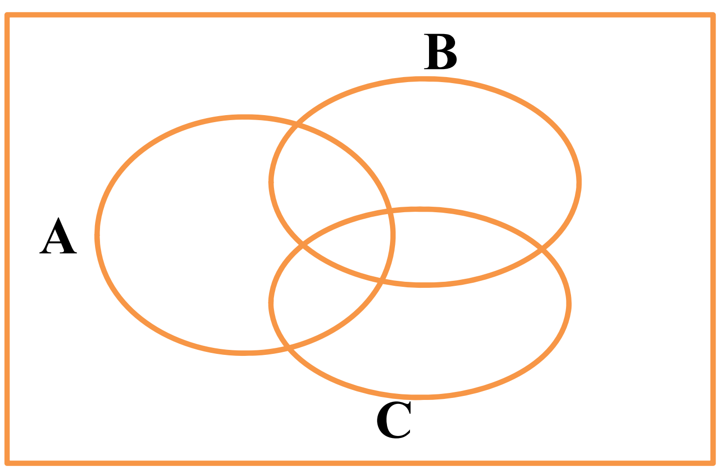
Graphical Representation of Sets:

Venn diagram: When representing more than one set, all the sets must "intersect" each other in the Venn diagram.

Examples: Venn diagrams representing 2 and 3 sets.



U: universal set containing all objects



HW: Review Venn diagrams in Rosen.

Warning: Venn diagram is merely a graphical tool used in illustration only. You cannot prove any set identity using Venn diagram!!!

Def. The ***difference*** of A and B:

$$A - B = \{x \mid (x \in A) \wedge (x \notin B)\}.$$

Observe that, in general, $A - B \neq B - A$.

Def. If the universal set U is specified, we can define the ***complement*** of A to be

$$\overline{A} = U - A = \{x \mid (x \in U) \wedge (x \notin A)\}.$$

Def. The ***symmetric difference*** of A and B is a set of elements x with $x \in A$ <u>or</u> $x \in B$, but NOT both.

$$A \oplus B = \{x \mid (x \in A \wedge x \notin B) \vee (x \in B \wedge x \notin A)\},$$
$$= (A - B) \cup (B - A)$$

Example: Given U = {b, 2, 1, c, a, 6, 7, 8}, A = {a, 2, 8}, B = {1, 2, 8, b}, D = {a, b, c}.

$A - B = \{a\}$,
$B - A = \{1, b\}$,
$A \oplus B = \{a, 1, b\}$,
$\overline{D} = \{1, 2, 6, 7, 8\}$,
$A \cup B = \{1, 2, 8, a, b\}$, $|A \cup B| = 5$,
$A \cap B = \{2, 8\}$, $|A \cap B| = 2$,
$|A \cup B| = |A| + |B| - |A \cap B| = 3 + 4 - 2 = 5.$

Q: How do we prove the equality of two set expressions?

Proving Set Identities:

Some Possible Approaches:

1. Use set definitions and direct proof technique.
2. Use properties of sets and Laws of Logical Equivalence for propositions.
3. Use membership (truth) tables.
4. Use set identities.

Examples:

1. Prove that $A \cap (A \cup B) = A$ using set definitions and direct proof technique.

   Proof: We need to prove that (i) $A \cap (A \cup B) \subseteq A$, and (ii) $A \subseteq A \cap (A \cup B)$.

   (i) Let $x \in A \cap (A \cup B)$. By definition of sets intersection, $x \in A$ and $x \in A \cup B$. Hence, $x \in A$ implying that $A \cap (A \cup B) \subseteq A$.

   (ii) Let $x \in A$. Hence, $x \in A$ and $x \in (A \cup B)$. By definition of sets intersection, $x \in A \cap (A \cup B)$, implying that $A \subseteq A \cap (A \cup B)$.

   Since $A \cap (A \cup B) \subseteq A$ and $A \subseteq A \cap (A \cup B)$, $A \cap (A \cup B) = A$.

2. Prove that $A \cap (A \cup B) = A$ using Laws of Equivalence for propositions.

Proof:

$A \cap (A \cup B)$

$= \{x \mid x \in A \cap (A \cup B)\}$	Def of $A \cap (A \cup B)$
$= \{x \mid (x \in A) \wedge (x \in (A \cup B))\}$	Def of $\cap$
$= \{x \mid (x \in A) \wedge ((x \in A) \vee (x \in B))\}$	Def of $\cup$
$= \{x \mid x \in A\}$	Absorption Law
$= A$	

3. $\overline{(A \cup B)} = \overline{A} \cap \overline{B}$.

Proof:

$\overline{(A \cup B)}$

$= \{x \mid \neg(x \in A \vee x \in B)\}$	Def of $\overline{(A \cup B)}$
$= \{x \mid \neg(x \in A) \wedge \neg(x \in B)\}$	De Morgan(log. eq.)
$= \{x \mid x \notin A \wedge x \notin B\}$	Def of negation
$= \{x \mid x \in \overline{A} \wedge x \in \overline{B}\}$	Def of set complement
$= \overline{A} \cap \overline{B}$.	Def of set intersection

4. $A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$.

$A \cap (B \cup C)$

$= \{x \mid (x \in A) \wedge x \in (B \cup C)\}$ Def of $A \cap (B \cup C)$

$= \{x \mid (x \in A) \wedge ((x \in B) \vee (x \in C))$

Def of $B \cup C$

$= \{x \mid ((x \in A) \wedge (x \in B)) \vee ((x \in A) \wedge (x \in C))\}$

Distrib. Law (log. eq.)

$= \{x \mid (x \in A \cap B) \vee (x \in A \cap C)\}$

Def of set intersection

$= (A \cap B) \cup (A \cap C)$ Def of set union

Observe that above proofs are based on the information of whether an arbitrarily given element belongs to a set. Hence, we can prove these identities using a membership table, which is similar to a truth table but with the following modifications.

Truth Table	**Membership Table**
Proposition	Set
T	1
F	0

5. $\overline{(A \cup B)} = \overline{A} \cap \overline{B}$.

A	**B**	$\overline{A}$	$\overline{B}$	**A ∪ B**	$\overline{(A \cup B)}$	$\overline{A} \cap \overline{B}$
1	1	0	0	1	**0**	**0**
1	0	0	1	1	**0**	**0**
0	1	1	0	1	**0**	**0**
0	0	1	1	0	**1**	**1**

6. A ∩ (B ∪ C) = (A ∩ B) ∪ (A ∩ C)

A	**B**	**C**	**B∪C**	**A∩B**	**A∩C**	**A∩(B∪C)**	**(A∩B)∪(A∩C)**
1	1	1	1	1	1	**1**	**1**
1	1	0	1	1	0	**1**	**1**
1	0	1	1	0	1	**1**	**1**
1	0	0	0	0	0	**0**	**0**
0	1	1	1	0	0	**0**	**0**
0	1	0	1	0	0	**0**	**0**
0	0	1	1	0	0	**0**	**0**
0	0	0	0	0	0	**0**	**0**

Some Useful Set Identities:

1. $A \cup \varnothing = A$ $A \cap U = A$	**Identity Laws**
2. $A \cup U = U$ $A \cap \varnothing = \varnothing$	**Domination Laws**
3. $A \cup A = A$ $A \cap A = A$	**Idempotent Laws**
4. $(\overline{\overline{A}}) = A.$	**Involution Law**
5. $A \cup \overline{A} = U$ $A \cap \overline{A} = \varnothing$	**Complement Laws**
6. $A \cup (B \cup C) = (A \cup B) \cup C$ $A \cap (B \cap C) = (A \cap B) \cap C$	**Associative Laws**
7. $A \cup B = B \cup A$ $A \cap B = B \cap A$	**Commutative Laws**
8. $A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$ $A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$	**Distributive Laws**
9. $A \cup (A \cap B) = A$ $A \cap (A \cup B) = A$	**Absorption Laws**

10. $\overline{(A \cup B)} = \overline{A} \cap \overline{B}$ **De Morgan's Law**

$\overline{(A \cap B)} = \overline{A} \cup \overline{B}$

Remark: The Associative, Commutative, and De Morgan's Laws can be generalized and extended to multiple sets.

Generalized De Morgan's Law

$$\overline{(A \cap B \cap C \cap \ldots)} = \overline{A} \cup \overline{B} \cup \overline{C} \cup \ldots$$

$$\overline{(A \cup B \cup C \cup \ldots)} = \overline{A} \cap \overline{B} \cap \overline{C} \cap \ldots$$

Remark:

The set identities above can be obtained from the corresponding laws of logical equivalence by the following transformation.

Logical Equivalence	**Set Identity**
Proposition	Set
$\wedge$	$\cap$
$\vee$	$\cup$
T	U
F	$\varnothing$
Negation	Complement

More Examples in Using Set Identities:

1. $A \cup \overline{(A \cap B)} = U$

$A \cup \overline{(A \cap B)}$

$= A \cup (\overline{A} \cup \overline{B})$ De Morgan's Law

$= (A \cup \overline{A}) \cup \overline{B}$ Associative Law

$= U \cup \overline{B}$ Complement Law

$= U$ Domination Law

2. $(\overline{A} \cup B) \cup \overline{(A \cap B)} = U$

$(\overline{A} \cup B) \cup \overline{(A \cap B)}$

$= (\overline{A} \cup B) \cup (\overline{A} \cup \overline{B})$ De Morgan's Law

$= (\overline{A} \cup \overline{A}) \cup (B \cup \overline{B})$ Gen. Associative Law

$= \overline{A} \cup U$ Complement Law

$= U$ Domination Law

Practice HW: Chpt.2.2: 7, 9, 13, 17, 19, 25, 27, 29, 31, 37, 39.