# HW4: Mathematical Induction

**Q1: (Tower of Hanoi) You have three pegs and a collection of disks of different sizes. Initially all of the disks are stacked on top on each other according to size on the first peg - the largest disk being on the bottom, and the smallest on the top. A move in this game consists of moving a disk from one peg to another, subject to the condition that a larger disk may never rest on a smaller one. The objective of the game is to find a number of permissible moves that will transfer all of the disks from the first peg to the third peg, making sure that the disks are assembled on the third peg according to size. The second peg is used as an intermediate peg.**

**Prove it takes $\mathbf{2^n - 1}$ moves to move $\boldsymbol{n}$ disks from the first peg to the third peg.**

**A1:** We will prove the statement using math induction.

(*Basis step*): When $n = 1$, we can directly move the disk to the third peg, which takes 1 move. Since $2^1 - 1 = 2 - 1 = 1$, the statement is true when $n = 1$.

(*Induction step*): We assume that, for $k \geq 1$, we can move $k$ disks from one peg to another using $2^k - 1$ moves, provided that <u>there exists one empty peg, or an empty peg whose top disk is larger than any of the $k$ disks to be moved</u>, serving as the intermediate peg. Now we consider the move of the top $k + 1$ disks.
We can do the following:

- First, move the top $k$ disks from the first peg to the second peg, using the third peg as the intermediate. Clearly, the third peg is empty, therefore we can do this with $2^k - 1$ moves, *by induction*.
- Second, move disk $k + 1$ from the first peg to the third peg. This is trivial and requires 1 move.
- Third, move the $k$ disks on the second peg to the third peg, using the first peg as the intermediate. Note that the top disk on the first peg is $k + 2$, which is larger than any of the $k$ disks. Therefore, the first peg serves well as an intermediate. This phase also requires $2^k - 1$ moves, *by induction*.

Taken together, we can spend $(2^k - 1) + 1 + (2^k - 1) = 2^{k+1} - 1$ moves to move the top $k + 1$ disks from the first peg to the third peg.

(*Conclusion step*): Therefore, it takes $2^n - 1$ moves to move $n$ disks from the first peg to the third peg.