# Effects and Types

**Syntax.**

$$
\begin{array}{rll}
t & ::= z \mid t + t \mid \lambda x.t \mid t\ t \mid \texttt{ifz}\ t\ \texttt{then}\ t\ \texttt{else}\ t \mid \texttt{get} \mid \texttt{put}\ t \mid t\ \texttt{or}\ t & \text{Terms} \\
T & ::= \texttt{Int} \mid T \xrightarrow{E} T & \text{Types} \\
e & ::= \mathsf{g} \mid \mathsf{p} \mid \mathsf{a} & \text{Effects}
\end{array}
$$

**Evaluation.**

$$
\frac{}{z \mid s \Downarrow z \mid s}
\qquad
\frac{t_1 \mid s_1 \Downarrow z_1 \mid s_2 \quad t_2 \mid s_2 \Downarrow z_2 \mid s_3}{t_1 + t_2 \mid s_1 \Downarrow z_1 + z_2 \mid s_3}
\qquad
\frac{}{\texttt{fix}\ t \mid s \Downarrow \lambda x.t\ \texttt{fix}\ t\ x \mid s}
$$

$$
\frac{}{\lambda x.t \mid s \Downarrow \lambda x.t \mid s}
\qquad
\frac{t_1 \mid s_1 \Downarrow \lambda x.t \mid s_2 \quad t_2 \mid s_2 \Downarrow w \mid s_3 \quad t[w/x] \mid s_3 \Downarrow v \mid s_4}{t_1\ t_2 \mid s_1 \Downarrow v \mid s_4}
$$

$$
\frac{t_1 \mid s_1 \Downarrow 0 \mid s_2 \quad t_2 \mid s_2 \Downarrow v \mid s_3}{\texttt{ifz}\ t_1\ \texttt{then}\ t_2\ \texttt{else}\ t_3 \mid s_1 \Downarrow v \mid s_3}
\qquad
\frac{t_1 \mid s_1 \Downarrow z \mid s_2 \quad t_3 \mid s_2 \Downarrow v \mid s_3}{\texttt{ifz}\ t_1\ \texttt{then}\ t_2\ \texttt{else}\ t_3 \mid s_1 \Downarrow v \mid s_3}\ (z \neq 0)
$$

$$
\frac{}{\texttt{get} \mid s \Downarrow s \mid s}
\qquad
\frac{t \mid s_1 \Downarrow v \mid s_2}{\texttt{put}\ t \mid s_1 \Downarrow v \mid v}
\qquad
\frac{t_1 \mid s_1 \Downarrow v \mid s_2}{t_1\ \texttt{or}\ t_2 \mid s_1 \Downarrow v \mid s_2}
\qquad
\frac{t_2 \mid s_1 \Downarrow v \mid s_2}{t_1\ \texttt{or}\ t_2 \mid s_1 \Downarrow v \mid s_2}
$$

**Subtyping.**

$$
\frac{}{T <: T}
\qquad
\frac{T_2 <: T_1 \quad E_1 \subseteq E_2 \quad U_1 <: U_2}{T_1 \xrightarrow{E_1} U_1 <: T_2 \xrightarrow{E_2} U_2}
$$

**Typing.**

$$
\frac{}{\Gamma \vdash z : \texttt{Int}\ \&\ \emptyset}
\qquad
\frac{\Gamma \vdash t_1 : \texttt{Int}\ \&\ E_1 \quad \Gamma \vdash t_2 : \texttt{Int}\ \&\ E_2}{\Gamma \vdash t_1 + t_2 : \texttt{Int}\ \&\ E_1 \cup E_2}
$$

$$
\frac{\Gamma[x \mapsto T_1] \vdash t : T_2\ \&\ E}{\Gamma \vdash \lambda x.t : T_1 \xrightarrow{E} T_2\ \&\ \emptyset}
\qquad
\frac{\Gamma \vdash t_1 : T_1 \xrightarrow{E_3} T_2\ \&\ E_1 \quad \Gamma \vdash t_2 : T_1\ \&\ E_2}{\Gamma \vdash t_1\ t_2 : T_2\ \&\ E_1 \cup E_2 \cup E_3}
$$

$$
\frac{}{\Gamma \vdash \texttt{get} : \texttt{Int}\ \&\ \{\mathsf{g}\}}
\qquad
\frac{\Gamma \vdash t : \texttt{Int}\ \&\ E}{\Gamma \vdash \texttt{put}\ t : \texttt{Int}\ \&\ E \cup \{\mathsf{p}\}}
\qquad
\frac{\Gamma \vdash t_1 : T\ \&\ E_1 \quad \Gamma \vdash t_2 : T\ \&\ E_2}{\Gamma \vdash t_1\ \texttt{or}\ t_2 : T\ \&\ E_1 \cup E_2 \cup \{\mathsf{a}\}}
$$

$$
\frac{\Gamma \vdash t_1 : \texttt{Int}\ \&\ E_1 \quad \Gamma \vdash t_2 : T\ \&\ E_2 \quad \Gamma \vdash t_3 : T\ \&\ E_3}{\Gamma \vdash \texttt{ifz}\ t_1\ \texttt{then}\ t_2\ \texttt{else}\ t_3 : T\ \&\ E_1 \cup E_2 \cup E_3}
\qquad
\frac{\Gamma \vdash t : (T \xrightarrow{E_1} T) \xrightarrow{E_2} (T \xrightarrow{E_1} T)\ \&\ E_3}{\Gamma \vdash \texttt{fix}\ t : T \xrightarrow{E_1} T\ \&\ E_2 \cup E_3}
$$

$$
\frac{\Gamma \vdash t : T_1\ \&\ E_1 \quad T_1 <: T_2 \quad E_1 \subseteq E_2}{\Gamma \vdash t : T_2\ \&\ E_2}
$$

function call:left
lambda:right associate
plus: right

1. **Syntax.**
Fully parenthesize the following $\lambda$-terms.

(a) $\lambda a.\lambda b.a$

$\lambda a.(\lambda b.a)$

(b) $\lambda a.(\lambda b.b)\ a$

$\lambda a.((\lambda b.b)\ a)$

(c) $\lambda a.\lambda b.b\ a$

$\lambda a.(\lambda b.(b\ a))$

(d) $\lambda a.a + 2$

$\lambda a.(a + 2)$

(e) $(\lambda a.\lambda b.a)\ b$

$(\lambda a.(\lambda b.a))\ b$

(f) $\lambda a.\texttt{get} + (\lambda b.b)\,3$

$\lambda a.(\texttt{get} + ((\lambda b.b)\,3))$

2. **Evaluation.**
Derive the following judgments.

(a) $\texttt{put}\,(\texttt{get} + 1) \mid 3 \Downarrow 4 \mid 4$

$$\dfrac{\dfrac{\dfrac{}{\texttt{get} \mid 3 \Downarrow 3 \mid 3} \quad \dfrac{}{1 \mid 3 \Downarrow 1 \mid 3}}{\texttt{get} + 1 \mid 3 \Downarrow 4 \mid 3}}{\texttt{put}\,(\texttt{get} + 1) \mid 3 \Downarrow 4 \mid 4}$$

(b) $(\lambda x.\texttt{get} + x)\,(\texttt{put}\,4) \mid 1 \Downarrow 5 \mid 4$

$$\dfrac{\dfrac{}{\lambda x.\texttt{get} + x \mid 1 \Downarrow \lambda x.\texttt{get} + x \mid 1} \quad \dfrac{\dfrac{}{4 \mid 1 \Downarrow 4 \mid 1}}{\texttt{put}\,4 \mid 1 \Downarrow 4 \mid 4} \quad \dfrac{\dfrac{}{\texttt{get} \mid 4 \Downarrow 4 \mid 4} \quad \dfrac{}{4 \mid 4 \Downarrow 4 \mid 4}}{\texttt{get} + 4 \mid 4 \Downarrow 8 \mid 4}}{(\lambda x.\texttt{get} + x)\,(\texttt{put}\,4) \mid 1 \Downarrow 8 \mid 4}$$

(c) $\texttt{put}\,((\texttt{get}\ \texttt{or}\ 1) + 2) \mid 2 \Downarrow 4 \mid 4$

$$\dfrac{\dfrac{\dfrac{\dfrac{}{\mathtt{get} \mid 2 \Downarrow 2 \mid 2}}{\mathtt{get}\ \mathtt{or}\ 1 \mid 2 \Downarrow 2 \mid 2} \quad \dfrac{}{2 \mid 2 \Downarrow 2 \mid 2}}{((\mathtt{get}\ \mathtt{or}\ 1) + 2) \mid 2 \Downarrow 4 \mid 2}}{\mathtt{put}\ ((\mathtt{get}\ \mathtt{or}\ 1) + 2) \mid 2 \Downarrow 4 \mid 4}$$

(d) $(\lambda a.a + \mathtt{get})\ ((\mathtt{put}\ 3)\ \mathtt{or}\ 3) \mid 1 \Downarrow 6 \mid 3$

$$\dfrac{\dfrac{}{\lambda a.a + \mathtt{get} \mid 1 \Downarrow \lambda a.a + \mathtt{get} \mid 1} \quad \dfrac{\dfrac{\dfrac{}{3 \mid 1 \Downarrow 3 \mid 1}}{\mathtt{put}\ 3 \mid 1 \Downarrow 3 \mid 3}}{(\mathtt{put}\ 3)\ \mathtt{or}\ 3 \mid 1 \Downarrow 3 \mid 3} \quad \dfrac{\dfrac{}{3 \mid 3 \Downarrow 3 \mid 3} \quad \dfrac{}{\mathtt{get} \mid 3 \Downarrow 3 \mid 3}}{3 + \mathtt{get} \mid 3 \Downarrow 6 \mid 3}}{(\lambda a.a + \mathtt{get})\ ((\mathtt{put}\ 3)\ \mathtt{or}\ 3) \mid 1 \Downarrow 6 \mid 3}$$

(e) $(\lambda a.a + \mathtt{get})\ ((\mathtt{put}\ 3)\ \mathtt{or}\ 3) \mid 1 \Downarrow 4 \mid 1$

$$\dfrac{\dfrac{}{\lambda a.a + \mathtt{get} \mid 1 \Downarrow \lambda a.a + \mathtt{get} \mid 1} \quad \dfrac{\dfrac{}{3 \mid 1 \Downarrow 3 \mid 1}}{(\mathtt{put}\ 3)\ \mathtt{or}\ 3 \mid 1 \Downarrow 3 \mid 1} \quad \dfrac{\dfrac{}{3 \mid 1 \Downarrow 3 \mid 1} \quad \dfrac{}{\mathtt{get} \mid 1 \Downarrow 1 \mid 1}}{3 + \mathtt{get} \mid 1 \Downarrow 4 \mid 1}}{(\lambda a.a + \mathtt{get})\ ((\mathtt{put}\ 3)\ \mathtt{or}\ 3) \mid 1 \Downarrow 4 \mid 1}$$

3. **Subtyping.**

Derive the following judgments

(a) $\mathtt{Int} \xrightarrow{\emptyset} \mathtt{Int} <: \mathtt{Int} \xrightarrow{\mathrm{g}} \mathtt{Int}$

$$\dfrac{\dfrac{\begin{matrix}\text{T2 <: T1}\\ \text{Int <: Int}\end{matrix}}{\mathtt{Int} :> \mathtt{Int}} \quad \dfrac{}{\emptyset \subseteq \{\mathrm{g}\}} \quad \dfrac{\begin{matrix}\text{U1 <: U2}\\ \text{Int <: Int}\end{matrix}}{\mathtt{Int} <: \mathtt{Int}}}{\mathtt{Int} \xrightarrow{\emptyset} \mathtt{Int} <: \mathtt{Int} \xrightarrow{\mathrm{g}} \mathtt{Int}}$$

(b) $\mathtt{Int} \xrightarrow{\emptyset} (\mathtt{Int} \xrightarrow{\emptyset} \mathtt{Int}) <: \mathtt{Int} \xrightarrow{\emptyset} (\mathtt{Int} \xrightarrow{\mathrm{g}} \mathtt{Int})$

$$\dfrac{\dfrac{}{\mathtt{Int} :> \mathtt{Int}} \quad \dfrac{}{\emptyset \subseteq \emptyset} \quad \dfrac{\dfrac{}{\mathtt{Int} :> \mathtt{Int}} \quad \dfrac{}{\emptyset \subseteq \{\mathrm{g}\}} \quad \dfrac{}{\mathtt{Int} <: \mathtt{Int}}}{\mathtt{Int} \xrightarrow{\emptyset} \mathtt{Int} <: \mathtt{Int} \xrightarrow{\mathrm{g}} \mathtt{Int}}}{\mathtt{Int} \xrightarrow{\emptyset} (\mathtt{Int} \xrightarrow{\emptyset} \mathtt{Int}) <: \mathtt{Int} \xrightarrow{\emptyset} (\mathtt{Int} \xrightarrow{\mathrm{g}} \mathtt{Int})}$$

(c) $(\mathtt{Int} \xrightarrow{\mathrm{g}} \mathtt{Int}) \xrightarrow{\emptyset} \mathtt{Int} <: (\mathtt{Int} \xrightarrow{\emptyset} \mathtt{Int}) \xrightarrow{\emptyset} \mathtt{Int}$

$$\dfrac{\dfrac{\dfrac{}{\mathtt{Int} <: \mathtt{Int}} \quad \dfrac{\text{E1 ....E2}}{\{\mathrm{g}\} \supseteq \emptyset} \quad \dfrac{}{\mathtt{Int} :> \mathtt{Int}}}{\mathtt{Int} \xrightarrow{\mathrm{g}} \mathtt{Int} :> \mathtt{Int} \xrightarrow{\emptyset} \mathtt{Int}} \quad \dfrac{}{\emptyset \subseteq \emptyset} \quad \dfrac{}{\mathtt{Int} <: \mathtt{Int}}}{(\mathtt{Int} \xrightarrow{\mathrm{g}} \mathtt{Int}) \xrightarrow{\emptyset} \mathtt{Int} <: (\mathtt{Int} \xrightarrow{\emptyset} \mathtt{Int}) \xrightarrow{\emptyset} \mathtt{Int}}$$

(d) $(\mathtt{Int} \xrightarrow{\mathsf{gp}} \mathtt{Int}) \xrightarrow{\emptyset} (\mathtt{Int} \xrightarrow{\emptyset} \mathtt{Int}) <: (\mathtt{Int} \xrightarrow{\mathsf{g}} \mathtt{Int}) \xrightarrow{\emptyset} (\mathtt{Int} \xrightarrow{\mathsf{a}} \mathtt{Int})$

$$\dfrac{\dfrac{\dfrac{}{\mathtt{Int} <: \mathtt{Int}} \quad \dfrac{}{\{\mathsf{g},\mathsf{p}\} \supseteq \{\mathsf{g}\}} \quad \dfrac{}{\mathtt{Int} :> \mathtt{Int}}}{\mathtt{Int} \xrightarrow{\mathsf{gp}} \mathtt{Int} :> \mathtt{Int} \xrightarrow{\mathsf{g}} \mathtt{Int}} \quad \dfrac{}{\emptyset \subseteq \emptyset} \quad \dfrac{\dfrac{}{\mathtt{Int} :> \mathtt{Int}} \quad \dfrac{}{\emptyset \subseteq \{\mathsf{a}\}} \quad \dfrac{}{\mathtt{Int} <: \mathtt{Int}}}{\mathtt{Int} \xrightarrow{\emptyset} \mathtt{Int} <: \mathtt{Int} \xrightarrow{\mathsf{a}} \mathtt{Int}}}{(\mathtt{Int} \xrightarrow{\mathsf{gp}} \mathtt{Int}) \xrightarrow{\emptyset} (\mathtt{Int} \xrightarrow{\emptyset} \mathtt{Int}) <: (\mathtt{Int} \xrightarrow{\mathsf{g}} \mathtt{Int}) \xrightarrow{\emptyset} (\mathtt{Int} \xrightarrow{\mathsf{a}} \mathtt{Int})}$$

4. **Typing.**
Derive the following judgments

(a) $\{x \mapsto \mathtt{Int}\} \vdash \mathtt{put}\ x\ \mathtt{or}\ 3 : \mathtt{Int}\ \&\ \{\mathsf{g},\mathsf{a}\}$

$$\dfrac{\dfrac{\dfrac{}{\{x \mapsto \mathtt{Int}\} \vdash x : \mathtt{Int}\ \&\ \emptyset} \quad \dfrac{}{\{x \mapsto \mathtt{Int}\} \vdash 3 : \mathtt{Int}\ \&\ \emptyset}}{\{x \mapsto \mathtt{Int}\} \vdash x\ \mathtt{or}\ 3 : \mathtt{Int}\ \&\ \{\mathsf{a}\}}}{\{x \mapsto \mathtt{Int}\} \vdash \mathtt{put}\ x\ \mathtt{or}\ 3 : \mathtt{Int}\ \&\ \{\mathsf{g},\mathsf{a}\}}$$

(b) $\emptyset \vdash \lambda a.\mathtt{put}\ a : \mathtt{Int} \xrightarrow{\mathsf{pa}} \mathtt{Int}\ \&\ \emptyset$

$$\dfrac{\dfrac{\dfrac{\dfrac{}{\{a \mapsto \mathtt{Int}\} \vdash a : \mathtt{Int}\ \&\ \emptyset}}{\{a \mapsto \mathtt{Int}\} \vdash \mathtt{put}\ a : \mathtt{Int}\ \&\ \{\mathsf{p}\}}}{\{a \mapsto \mathtt{Int}\} \vdash \mathtt{put}\ a : \mathtt{Int}\ \&\ \{\mathsf{p},\mathsf{a}\}}\ (\{\mathsf{p}\} \subseteq \{\mathsf{p},\mathsf{a}\})}{\emptyset \vdash \lambda a.\mathtt{put}\ a : \mathtt{Int} \xrightarrow{\mathsf{pa}} \mathtt{Int}\ \&\ \emptyset}$$

(c) $\{x \mapsto \mathtt{Int}\} \vdash \mathtt{ifz}\ x\ \mathtt{then}\ \mathtt{get}\ \mathtt{else}\ \mathtt{put}\ x : \mathtt{Int}\ \&\ \{\mathsf{g},\mathsf{p}\}$

$$\dfrac{\dfrac{}{\{x \mapsto \mathtt{Int}\} \vdash x : \mathtt{Int}\ \&\ \emptyset} \quad \dfrac{}{\{x \mapsto \mathtt{Int}\} \vdash \mathtt{get} : \mathtt{Int}\ \&\ \{\mathsf{g}\}} \quad \dfrac{\dfrac{}{\{x \mapsto \mathtt{Int}\} \vdash x : \mathtt{Int}\ \&\ \emptyset}}{\{x \mapsto \mathtt{Int}\} \vdash \mathtt{put}\ x : \mathtt{Int}\ \&\ \{\mathsf{p}\}}}{\{x \mapsto \mathtt{Int}\} \vdash \mathtt{ifz}\ x\ \mathtt{then}\ \mathtt{get}\ \mathtt{else}\ \mathtt{put}\ x : \mathtt{Int}\ \&\ \{\mathsf{g},\mathsf{p}\}}$$

(d) $\{x \mapsto \mathtt{Int}\} \vdash \mathtt{ifz}\ x\ \mathtt{then}\ \lambda y.\mathtt{get}\ \mathtt{else}\ \lambda y.\mathtt{put}\ x : \mathtt{Int} \xrightarrow{\mathsf{gp}} \mathtt{Int}\ \&\ \emptyset$

$$\dfrac{\dfrac{}{\{x \mapsto \mathtt{Int}\} \vdash x : \mathtt{Int}\ \&\ \emptyset} \quad \dfrac{\dfrac{\dfrac{}{\{x \mapsto \mathtt{Int}, y \mapsto \mathtt{Int}\} \vdash \mathtt{get} : \mathtt{Int}\ \&\ \{\mathsf{g}\}}}{\{x \mapsto \mathtt{Int}, y \mapsto \mathtt{Int}\} \vdash \mathtt{get} : \mathtt{Int}\ \&\ \{\mathsf{g},\mathsf{p}\}}}{\{x \mapsto \mathtt{Int}\} \vdash \lambda y.\mathtt{get} : \mathtt{Int} \xrightarrow{\mathsf{gp}} \mathtt{Int}\ \&\ \emptyset} \quad \dfrac{\dfrac{\dfrac{\dfrac{}{\{x \mapsto \mathtt{Int}, y \mapsto \mathtt{Int}\} \vdash x : \mathtt{Int}\ \&\ \emptyset}}{\{x \mapsto \mathtt{Int}, y \mapsto \mathtt{Int}\} \vdash \mathtt{put}\ x : \mathtt{Int}\ \&\ \{\mathsf{p}\}}}{\{x \mapsto \mathtt{Int}, y \mapsto \mathtt{Int}\} \vdash \mathtt{put}\ x : \mathtt{Int}\ \&\ \{\mathsf{g},\mathsf{p}\}}}{\{x \mapsto \mathtt{Int}\} \vdash \lambda y.\mathtt{put}\ x : \mathtt{Int} \xrightarrow{\mathsf{gp}} \mathtt{Int}\ \&\ \emptyset}}{\{x \mapsto \mathtt{Int}\} \vdash \mathtt{ifz}\ x\ \mathtt{then}\ \lambda y.\mathtt{get}\ \mathtt{else}\ \lambda y.\mathtt{put}\ x : \mathtt{Int} \xrightarrow{\mathsf{gp}} \mathtt{Int}\ \&\ \emptyset}$$

(e) $\emptyset \vdash (\lambda a.\mathtt{put}\ a)\,3 : \mathtt{Int}\ \&\ \{\mathsf{p}\}$

$$\dfrac{\dfrac{\dfrac{\dfrac{}{\{a \mapsto \texttt{Int}\} \vdash a : \texttt{Int} \ \& \ \emptyset}}{\{a \mapsto \texttt{Int}\} \vdash \texttt{put}\ a : \texttt{Int} \ \& \ \{\mathsf{p}\}}}{\emptyset \vdash \lambda a.\texttt{put}\ a : \texttt{Int} \xrightarrow{\mathsf{p}} \texttt{Int} \ \& \ \emptyset} \quad \dfrac{}{\emptyset \vdash 3 : \texttt{Int} \ \& \ \emptyset}}{\emptyset \vdash (\lambda a.\texttt{put}\ a)\,3 : \texttt{Int} \ \& \ \{\mathsf{p}\}}$$

(f) $\emptyset \vdash (\lambda a.(\lambda b.\lambda c.\texttt{put}\,(b + c))\,\texttt{get})\,3 : \texttt{Int} \xrightarrow{\mathsf{p}} \texttt{Int} \ \& \ \{\mathsf{g}\}$

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{}{\{a \mapsto \texttt{Int}, b \mapsto \texttt{Int}, c \mapsto \texttt{Int}\} \vdash b : \texttt{Int} \ \& \ \emptyset} \quad \dfrac{}{\{a \mapsto \texttt{Int}, b \mapsto \texttt{Int}, c \mapsto \texttt{Int}\} \vdash c : \texttt{Int} \ \& \ \emptyset}}{\{a \mapsto \texttt{Int}, b \mapsto \texttt{Int}, c \mapsto \texttt{Int}\} \vdash \texttt{put}\,(b + c) : \texttt{Int} \ \& \ \{\mathsf{p}\}}}{\{a \mapsto \texttt{Int}, b \mapsto \texttt{Int}\} \vdash \lambda c.\texttt{put}\,(b + c) : \texttt{Int} \xrightarrow{\mathsf{p}} \texttt{Int} \ \& \ \emptyset} \quad \dfrac{}{\{a \mapsto \texttt{Int}\} \vdash \texttt{get} : \texttt{Int} \ \& \ \{\mathsf{g}\}}}{\{a \mapsto \texttt{Int}\} \vdash (\lambda b.\lambda c.\texttt{put}\,(b + c))\,\texttt{get}) : \texttt{Int} \xrightarrow{\emptyset} \texttt{Int} \xrightarrow{\mathsf{p}} \texttt{Int} \ \& \ \{\mathsf{g}\}}}{\emptyset \vdash (\lambda a.(\lambda b.\lambda c.\texttt{put}\,(b + c))\,\texttt{get}) : \texttt{Int} \xrightarrow{\mathsf{g}} (\texttt{Int} \xrightarrow{\mathsf{p}} \texttt{Int}) \ \& \ \emptyset} \quad \dfrac{}{\emptyset \vdash 3 : \texttt{Int} \ \& \ \emptyset}}{\emptyset \vdash (\lambda a.(\lambda b.\lambda c.\texttt{put}\,(b + c))\,\texttt{get})\,3 : \texttt{Int} \xrightarrow{\mathsf{p}} \texttt{Int} \ \& \ \{\mathsf{g}\}}$$

## Parametric Polymorphism

**Syntax.**

$$\begin{array}{ll} \mathcal{T} \ni t ::= z \mid t + t \mid x \mid \lambda x.t \mid t\ t \mid (t, t) \mid \texttt{fst}\ t \mid \texttt{snd}\ t \mid \texttt{let}\ x = t\ \texttt{in}\ t & \text{Terms} \\ \mathcal{Y} \ni T ::= \alpha \mid \texttt{Int} \mid T \to T \mid (T, T) & \text{Types} \\ \mathcal{S} \ni S ::= T \mid \forall \alpha.S & \text{Type schemes} \end{array}$$

**Typing.**

$$\frac{}{\Gamma \vdash z : \texttt{Int}} \qquad \frac{\Gamma \vdash t_1 : \texttt{Int} \quad \Gamma \vdash t_2 : \texttt{Int}}{\Gamma \vdash t_1 + t_2 : \texttt{Int}}$$

$$\frac{}{\Gamma \vdash x : \Gamma(x)} \qquad \frac{\Gamma[x \mapsto T_1] \vdash t : T_2}{\Gamma \vdash \lambda x.t : T_1 \to T_2} \qquad \frac{\Gamma \vdash t_1 : T_1 \to T_2 \quad \Gamma \vdash t_2 : T_1}{\Gamma \vdash t_1\ t_2 : T_2}$$

$$\frac{\Gamma \vdash t_1 : T_1 \quad \Gamma \vdash t_2 : T_2}{\Gamma \vdash (t_1, t_2) : (T_1, T_2)} \qquad \frac{\Gamma \vdash t : (T_1, T_2)}{\Gamma \vdash \texttt{fst}\ t : T_1} \qquad \frac{\Gamma \vdash t : (T_1, T_2)}{\Gamma \vdash \texttt{snd}\ t : T_2}$$

$$\frac{\Gamma \vdash t_1 : S \quad \Gamma[x \mapsto S] \vdash t_2 : T}{\Gamma \vdash \texttt{let}\ x = t_1\ \texttt{in}\ t_2 : T} \qquad \frac{\Gamma \vdash t : S}{\Gamma \vdash t : \forall \alpha.S}\ (\alpha \notin \mathit{fv}(\Gamma)) \qquad \frac{\Gamma \vdash t : \forall \alpha.S}{\Gamma \vdash t : S[T/\alpha]}$$

**Instances.**

$$\lfloor T \rfloor = \{T\} \qquad \lfloor \forall \alpha.S \rfloor = \bigcup_{T \in \mathcal{Y}} \lfloor S[T/\alpha] \rfloor$$

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5. **Type schemes.**

Justify the following (in)equalities.

(a) $\lfloor \forall \alpha . \alpha \to \alpha \rfloor = \lfloor \forall \beta . \beta \to \beta \rfloor$.

$$\begin{aligned}
\lfloor \forall \alpha . \alpha \to \alpha \rfloor &= \bigcup_{T \in \mathcal{Y}} \lfloor T \to T \rfloor \\
&= \bigcup_{T \in \mathcal{Y}} \{T \to T\} \\
&= \{T \to T \mid T \in \mathcal{Y}\} \\
&= \bigcup_{U \in \mathcal{Y}} \{U \to U\} \\
&= \bigcup_{U \in \mathcal{Y}} \lfloor U \to U \rfloor = \lfloor \forall \beta . \beta \to \beta \rfloor
\end{aligned}$$

(b) $\lfloor \forall \alpha . \alpha \rfloor \supsetneq \lfloor \forall \alpha . \alpha \to \alpha \rfloor$.

As above, we have $\lfloor \forall \alpha . \alpha \to \alpha \rfloor = \{T \to T \mid T \in \mathcal{Y}\}$. We can derive

$$\begin{aligned}
\lfloor \forall \alpha . \alpha \rfloor &= \bigcup_{T \in \mathcal{Y}} \lfloor T \rfloor \\
&= \bigcup_{T \in \mathcal{Y}} \{T\} \\
&= \mathcal{Y}
\end{aligned}$$

To show the containment, we can observe that since if $T \in \mathcal{Y}$ then $T \to T \in \mathcal{Y}$, for each $U \in \lfloor \forall \alpha . \alpha \to \alpha \rfloor$ we must also have $U \in \mathcal{Y}$. To show the inequality, we need to find a type $U \in \mathcal{Y}$ such that $U \notin \lfloor \forall \alpha . \alpha \to \alpha \rfloor$; `Int` will do.

(c) $\lfloor \forall \alpha . \forall \beta . (\alpha, \beta) \rfloor = \lfloor \forall \beta . \forall \alpha . (\alpha, \beta) \rfloor$

$$\begin{aligned}
\lfloor \forall \alpha . \forall \beta . (\alpha, \beta) \rfloor &= \bigcup_{T \in \mathcal{Y}} \lfloor \forall \beta . (T, \beta) \rfloor \\
&= \bigcup_{T \in \mathcal{Y}} \bigcup_{U \in \mathcal{Y}} \lfloor (T, U) \rfloor \\
&= \bigcup_{T \in \mathcal{Y}} \bigcup_{U \in \mathcal{Y}} \{(T, U)\} \\
&= \bigcup_{U \in \mathcal{Y}} \bigcup_{T \in \mathcal{Y}} \{(T, U)\} \\
&= \bigcup_{U \in \mathcal{Y}} \bigcup_{T \in \mathcal{Y}} \lfloor (T, U) \rfloor \\
&= \bigcup_{U \in \mathcal{Y}} \lfloor \forall \alpha . (\alpha, U) \rfloor = \lfloor \forall \beta . \forall \alpha . (\alpha, \beta) \rfloor
\end{aligned}$$

6. **Typing with polymorphism.**

Derive the following judgments.

(a) $\emptyset \vdash \lambda a.\texttt{fst}\ a : \forall\alpha.\forall\beta.(\alpha,\beta) \to \alpha$

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{}{\{a \mapsto (\alpha,\beta)\} \vdash a : (\alpha,\beta)}}{\{a \mapsto (\alpha,\beta)\} \vdash \texttt{fst}\ a : \alpha}}{\emptyset \vdash \lambda a.\texttt{fst}\ a : (\alpha,\beta) \to \alpha}}{\emptyset \vdash \lambda a.\texttt{fst}\ a : \forall\beta.(\alpha,\beta) \to \alpha}}{\emptyset \vdash \lambda a.\texttt{fst}\ a : \forall\alpha.\forall\beta.(\alpha,\beta) \to \alpha}$$

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(b) $\emptyset \vdash \lambda a.\lambda b.a : \forall\alpha.\forall\beta.\alpha \to \beta \to \alpha$

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{}{\{a \mapsto \alpha, b \mapsto \beta\} \vdash a : \alpha}}{\{a \mapsto \alpha\} \vdash \lambda b.a : \beta \to \alpha}}{\emptyset \vdash \lambda a.\lambda b.a : \alpha \to \beta \to \alpha}}{\emptyset \vdash \lambda a.\lambda b.a : \forall\beta.\alpha \to \beta \to \alpha}}{\emptyset \vdash \lambda a.\lambda b.a : \forall\alpha.\forall\beta.\alpha \to \beta \to \alpha}$$

(c) $\emptyset \vdash \lambda a.\lambda b.a : \forall\alpha.\alpha \to \alpha \to \alpha$

$$\dfrac{\dfrac{\dfrac{\dfrac{}{\{a \mapsto \alpha, b \mapsto \alpha\} \vdash a : \alpha}}{\{a \mapsto \alpha\} \vdash \lambda b.a : \alpha \to \alpha}}{\emptyset \vdash \lambda a.\lambda b.a : \alpha \to \alpha \to \alpha}}{\emptyset \vdash \lambda a.\lambda b.a : \forall\alpha.\alpha \to \alpha \to \alpha}$$

(d) $\emptyset \vdash \lambda a.\lambda b.\texttt{fst}\ a + b : \forall\alpha.(\texttt{Int},\alpha) \to \texttt{Int} \to \texttt{Int}$

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{}{\{a \mapsto (\texttt{Int},\alpha), b \mapsto \texttt{Int}\} \vdash a : (\texttt{Int},\alpha)}}{\{a \mapsto (\texttt{Int},\alpha), b \mapsto \texttt{Int}\} \vdash \texttt{fst}\ a : \texttt{Int}} \quad \dfrac{}{\{a \mapsto (\texttt{Int},\alpha), b \mapsto \texttt{Int}\} \vdash b : \texttt{Int}}}{\{a \mapsto (\texttt{Int},\alpha), b \mapsto \texttt{Int}\} \vdash \texttt{fst}\ a + b : \texttt{Int}}}{\{a \mapsto (\texttt{Int},\alpha)\} \vdash \lambda b.\texttt{fst}\ a + b : \texttt{Int} \to \texttt{Int}}}{\emptyset \vdash \lambda a.\lambda b.\texttt{fst}\ a + b : (\texttt{Int},\alpha) \to \texttt{Int} \to \texttt{Int}}}{\emptyset \vdash \lambda a.\lambda b.\texttt{fst}\ a + b : \forall\alpha.(\texttt{Int},\alpha) \to \texttt{Int} \to \texttt{Int}}$$