Topic 2: Proof Techniques & Math Reasoning

Read: Chpt. 1.6, Rosen

Q: Given a declarative statement. How do we establish the truthfulness of the given statement?

Consider the statement: It is cold today.

Q: How cold is cold?

Remark: In order to establish the validity of a statement, we need a *mathematical system (computational environment)* within which one can determine (*prove*) the truthfulness of the statement.

Q: What is a mathematical system?

A: ***Mathematical system*** consists of axioms, definitions, and undefined terms.

- ***Axioms*** are statements that are assumed to be true.
- ***Undefined terms*** are concepts that are not explicitly defined but are defined (characterized) implicitly by the axioms.
- ***Definitions*** are used to generate new concepts in terms of existing ones.

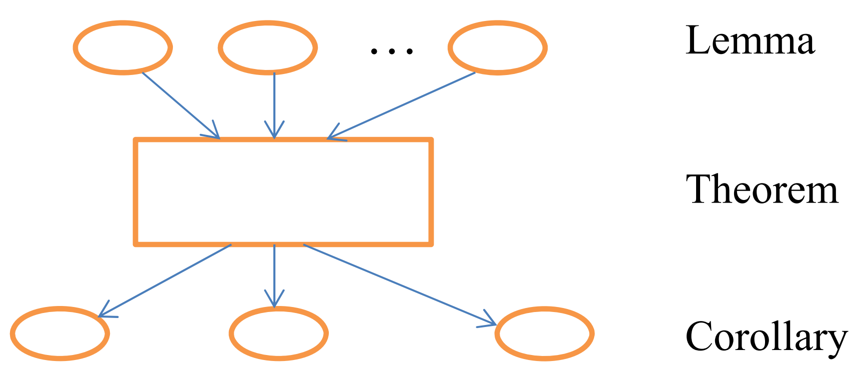
Within a mathematical system, new results (*theorems*) can then be discovered and confirmed (proven).

Q: What is a theorem?

A ***theorem*** is a statement that has been proven to be true (within a given mathematical system).

A ***lemma*** is a "simple" theorem that is used in the proof of another theorem.

A ***corollary*** is a "simple" theorem that can be established directly from a theorem that has just been proven.



Given a proposition P
Q: How do we establish (prove) the validity of P?
A: By constructing a "proof" for P.

Q: What is a proof?

A ***proof*** is a *valid argument* that can be used to establish the validity of a theorem.

An ***argument*** for P, $\langle P_1, P_2, \cdots, P_k \mid P\rangle$, is a sequence of propositions $P_1, P_2, \cdots, P_k$ that ends with the conclusion P.

A ***valid argument for P,*** is an argument $\langle P_1, P_2, \cdots, P_k \mid P\rangle$ for P such that it is impossible for $P_1, P_2, \cdots, P_k$ to be all true and the conclusion P to be false.

Hence, in a valid argument for P, $P_1 \wedge P_2 \wedge \cdots \wedge P_k \rightarrow P$ must be a tautology! (Why?)

Notations for Argument for P:

1. $\langle P_1, P_2, \cdots, P_k \mid P\rangle$.

2. $$\left.\begin{array}{l} P_1 \\ P_2 \\ . \\ . \\ . \\ P_k \end{array}\right\}\ \textbf{Hypothesis (Premises)}$$
   $$\begin{array}{l} \hline \therefore P \quad \textbf{Conclusion} \end{array}$$

Q: Given a proposition P. How do we construct a proof, which is a valid argument, for P?

A: Use axioms, existing theorems, and ***the rules of inference***.

Remark: The process of drawing a conclusion from a sequence of propositions using the rules of inference is called ***deductive reasoning***.

Some Useful Rules of Inference (Table 1 on Page 72):

1. $p \to (p \vee q)$

$$\frac{\mathbf{p}}{\therefore \mathbf{p} \vee \mathbf{q}} \qquad \textbf{Addition}$$

2. $(p \wedge q) \to p$

$$\frac{\mathbf{p} \wedge \mathbf{q}}{\therefore \mathbf{p}} \qquad \textbf{Simplification}$$

3. $((p) \wedge (q)) \to p \wedge q$

$$\frac{\begin{array}{c}\mathbf{p}\\ \mathbf{q}\end{array}}{\therefore \mathbf{p} \wedge \mathbf{q}} \qquad \textbf{Conjunction}$$

4. $(p \wedge (p \to q)) \to q$

$$\frac{\begin{array}{c}\mathbf{p}\\ \mathbf{p} \to \mathbf{q}\end{array}}{\therefore \mathbf{q}} \qquad \textbf{Modus Ponens}$$

5. $(\neg q \wedge (p \rightarrow q)) \rightarrow \neg p$

$$\begin{array}{l} \neg \mathbf{q} \\ \underline{\mathbf{p} \rightarrow \mathbf{q}} \\ \therefore \neg \mathbf{p} \end{array}$$

Modus Tollens

6. $((p \rightarrow q) \wedge (q \rightarrow r)) \rightarrow (p \rightarrow r)$

$$\begin{array}{l} \mathbf{p} \rightarrow \mathbf{q} \\ \underline{\mathbf{q} \rightarrow \mathbf{r}} \\ \therefore \mathbf{p} \rightarrow \mathbf{r} \end{array}$$

Hypothetical Syllogism

7. $((p \vee q) \wedge \neg p) \rightarrow q$

$$\begin{array}{l} \mathbf{p} \vee \mathbf{q} \\ \underline{\neg \mathbf{p}} \\ \therefore \mathbf{q} \end{array}$$

Disjunctive Syllogism

8. $((p \vee q) \wedge (\neg p \vee r))) \rightarrow (q \vee r)$

$$\begin{array}{l} \mathbf{p} \vee \mathbf{q} \\ \underline{\neg \mathbf{p} \vee \mathbf{r}} \\ \therefore \mathbf{q} \vee \mathbf{r} \end{array}$$

Resolution

Q: How do we prove the validity of these Rules of Inference?

Need to prove that $P_1 \wedge P_2 \wedge \cdots \wedge P_k \rightarrow P$ is a tautology!

Examples in Proving Rules of Inference:

1. Proving $p \rightarrow (p \vee q)$ (Addition) using Truth Table:

p	q	$p \vee q$	$p \rightarrow (p \vee q)$
T	T	T	T
T	F	T	T
F	T	T	T
F	F	F	T

$\therefore$ $p \rightarrow (p \vee q)$ is a tautology.

2. Proving $(p \wedge (p \rightarrow q)) \rightarrow q$ (Modus Ponens) using Truth Table:

p	q	$p \rightarrow q$	$p \wedge (p \rightarrow q)$	$p \wedge (p \rightarrow q)] \rightarrow q$
T	T	T	T	T
T	F	F	F	T
F	T	T	F	T
F	F	T	F	T

Hence, $(p \wedge (p \rightarrow q)) \rightarrow q$ is a tautology.

3. Proving $(p \wedge (p \rightarrow q)) \rightarrow q$ (Modus Ponens) using Direct Method:

   Proof: Consider two cases:

   (1) If $(p \wedge (p \rightarrow q))$ is true, then p is true and $(p \rightarrow q)$ is true, implying that q must also be true. Hence, $(p \wedge (p \rightarrow q)) \rightarrow q$ is true.

   (2) If $(p \wedge (p \rightarrow q))$ is false, then $(p \wedge (p \rightarrow q)) \rightarrow q$ must be true by the definition of implication.

   Hence, $(p \wedge (p \rightarrow q)) \rightarrow q$ is a tautology.

Applications of Rules of Inference:

1. Let p = Today is Saturday,
   q = All EECS majors at KU must take Dance101,
   r = We will go fishing today.

   Consider the following argument:

   (a) *Today is Saturday.*
   If today is Saturday, then we will go fishing.
   Q: If the above statements are true, what can we conclude? What is P in $<p, p \rightarrow r \mid P>$?

   Conclusion: If today is a Saturday, we will go fishing today. (Modus ponens)

   (b) What if today is a Thursday, can we conclude whether we go fishing or not? (Inconclusive.)

(c) *Today is either a Saturday or we will go fishing today.*
Today is a Thursday.
Q: If the above statements are true, what can we conclude? What is P in $<p \vee r, \neg p \mid P>$?

Conclusion: We will go fishing today. (Disjunctive syllogism)

Remark: If today is indeed a Saturday, we cannot conclude whether we will go fishing today or not.

(d) *We will go fishing today.*
Conclusion: If the above statement is true, then we can conclude that all EECS majors must take Dance101or we will go fishing today. (Addition)

2. Consider the following argument:

   If 2|3, then 4|9. ……. $(p \rightarrow q)$
   Therefore, 4|9. ……. (q)

   Remark: Observe that the statement "*If 2|3, then 4|9.*" is a valid statement. However, this is an invalid argument for *4|9*. Why?

3. Consider the following argument:

   If $\sqrt{2} > \frac{3}{2}$, *then* $(\sqrt{2})^2 > (\frac{3}{2})^2$.

   Since $\sqrt{2} > \frac{3}{2}$, $(\sqrt{2})^2 > (\frac{3}{2})^2 = \frac{9}{4}$.

   Hence, $2 > 2.25$.

   This is also an invalid argument for $2 > 2.25$! Why?

4. Let

   p = It is sunny this afternoon,
   q = It is colder than yesterday,
   r = We will go swimming,
   s = We will take a canoe trip,
   t = We will be home by sunset.

   Consider the following argument:

   It is not sunny this afternoon and it is colder than yesterday. $(\neg p \wedge q)$

   We will go swimming only if it is sunny this afternoon. $(r \to p)$

   If we do not go swimming, then we will take a canoe trip. $(\neg r \to s)$

   If we take a canoe trip, then we will be home by sunset. $(s \to t)$

   Q: If all given premises are valid, what is your conclusion?

Hypothesis	**Conclusion**	**Reasoning**
$\neg p \wedge q$	$\neg p$	$\neg p \wedge q \rightarrow \neg p$ (1)
$r \rightarrow p$	$\neg r$	$\neg p \wedge (r \rightarrow p) \rightarrow \neg r$ (2)
$\neg r \rightarrow s$	s	$\neg r \wedge (\neg r \rightarrow s) \rightarrow s$ (3)
$s \rightarrow t$	t	$s \wedge (s \rightarrow t) \rightarrow t$ (4)

(1) It is not sunny this afternoon and it is colder than yesterday.
Hence, it is not sunny this afternoon.

(2) We will go swimming only if it is sunny this afternoon.
Hence, we will not go swimming.

(3) If we do not go swimming, then we will take a canoe trip.
Hence, we will take a canoe trip.

(4) If we take a canoe trip, then we will be home by sunset.

Conclusion: We will be home by sunset!

More Rules of Inference for Quantified Statements (Table 2 on Page 76):

1. **Universal Instantiation:**
   $\forall x, P(x) \rightarrow P(x^*), x^* \in D_x$.

$$\frac{\forall \mathrm{x}, \mathrm{P}(\mathrm{x})}{\therefore \mathrm{P}(\mathrm{x}^*)}$$

2. **Universal Generalization:**
   $P(x^*)$, *for any arbitray* $x^* \in D_x \rightarrow \forall x, P(x)$.

$$\frac{\mathrm{P}(\mathrm{x}^*), \text{ for any } \mathrm{x}^* \in \mathrm{D}_\mathrm{x}}{\therefore \forall \mathrm{x}, \mathrm{P}(\mathrm{x})}$$

3. **Existential Instantiation:**
   $\exists x, P(x) \rightarrow P(x^*)$, *for some* $x^* \in D_x$.

$$\frac{\exists \mathrm{x}, \mathrm{P}(\mathrm{x})}{\therefore \mathrm{P}(\mathrm{x}^*), \text{ for some } \mathrm{x}^* \in \mathrm{D}_\mathrm{x}}$$

4. **Existential Generalization:**
   $P(x^*)$, *for some* $x^* \in D_x \rightarrow \exists x, P(x)$.

$$\frac{\mathrm{P}(\mathrm{x}^*), \text{ for some } \mathrm{x}^* \in \mathrm{D}_\mathrm{x}}{\therefore \exists \mathrm{x}, \mathrm{P}(\mathrm{x})}$$

HW: Study and memorize the above Rules of Inference.

Example: Consider the following argument that a student in EECS210 has gotten a speeding ticket.

Linda is a student in EECS210 and she owns a red convertible.

Everyone who owns a red convertible has gotten a speeding ticket.

Hence, someone in EECS210 has gotten a speeding ticket.

Let c(x) = student x is taking EECS210,

r(x) = student x owns a red convertible,

s(x) = student x has gotten a speeding ticket.

Observe that we have the following premises c(Linda), r(Linda), and $\forall x\ (r(x) \to s(x))$, and we need to prove that $\exists x\ (c(x) \land s(x))$.

Reasoning:

c(Linda)	Premise
r(Linda)	Premise
$\forall x\ (r(x) \to s(x))$	Premise
r(Linda) $\to$ s(Linda)	Universal Instantiation
s(Linda)	Modus Ponens
c(Linda) $\land$ s(Linda)	Conjunction
$\exists x\ (c(x) \land s(x))$	Existential Generalization

Hence, the given argument is a valid argument for someone in EECS210 has gotten a speeding ticket.

Practice HW: Chpt.1.6, 3, 9, 13, 15, 19, 23.

Introduction to Proof Techniques

Read: Chpt. 1.7-1.8, Rosen

Q: Given a proposition P. How do we construct a valid argument to prove that P is a tautology?

Basic Structures of a Proposition:

I. Simple proposition: p.
II. Implication: $p \rightarrow q$.
III. Biconditional: $p \leftrightarrow q$.
IV. Quantification:
   (a) Universal quantification: $\forall x \in S, P(x)$.
   (b) Existential quantification: $\exists x \in S, P(x)$.

I. **Simple proposition, p:**

- **Direct Proof:**
  Prove directly that p is a tautology.
- **Indirect Proof (Proof by Contradiction):**
  Assume that $\neg p$ is true to obtain a contradiction.

Examples:

1. Prove that the sum of two even integers x and y must be even.
   Recall that, by definition, an integer n is even iff there exists an integer k such that $n = 2k$, and n is odd iff there exists an integer h such that $n = 2h+1$.

Direct Proof:

By definition of even integer, if x and y are two even integers, there must exist integers k_1 and k_2 such that $x = 2k_1$ and $y = 2k_2$.

$\therefore\ x + y = (2k_1) + (2k_2) = 2(k_1 + k_2) = 2K$, where $K = (k_1 + k_2)$ is also an integer.

Hence, by the definition of even integer, $x + y$ is an even integer.

Proof by Contradiction:

Assuming that x and y are two even integers yet $x + y$ is an odd integer to obtain a contradiction. Hence, by the definition of even and odd integers, there must exist integers k_1, k_2, and h such that $x = 2k_1$, $y = 2k_2$, and $x + y = 2h + 1$.

$\therefore\ x + y = (2k_1) + (2k_2) = 2h + 1$, implying that $1 = 2(k_1 + k_2 - h) = 2K$, where $K = (k_1 + k_2 - h)$ is an integer. Hence, the integer 1 must be an even integer, which is a contradiction since 1 is an odd integer. Therefore, the original assumption that $x + y$ is an odd integer must be false, implying that the sum of two even integers x and y must be even.

2. Prove that the sum of two rational numbers x and y must be rational.
Recall that a real number r is rational iff there exist two integers p and q such that $r = \frac{p}{q}, q \neq 0$.
Remark: In canonical representation of rational number, we can also assume that there is no common factors, except 1 and -1, for p and q.

Direct Proof:
By definition, if x and y are two rational numbers, there must exist integers p_1, q_1, p_2, q_2 such that

$$x = \frac{p_1}{q_1}, y = \frac{p_2}{q_2}, q_1 \neq 0, q_2 \neq 0.$$

Hence,

$$\begin{aligned} & x + y \\ &= \frac{p_1}{q_1} + \frac{p_2}{q_2} \\ &= \frac{p_1 q_2 + p_2 q_1}{q_1 q_2} \\ &= \frac{P}{Q}, \end{aligned}$$

where $P = p_1 q_2 + p_2 q_1$ and $Q = q_1 q_2 \neq 0$ are integers.
$\therefore x + y$ must be a rational number by the definition of rational.

Proof by Contradiction:

Assume that x and y are two rational numbers yet x + y is an irrational number to obtain a contradiction. By definition, if x and y are two rational numbers, there must exist integers p_1, q_1, p_2, q_2 such that

$$x = \frac{p_1}{q_1}, y = \frac{p_2}{q_2}, q_1 \neq 0, q_2 \neq 0.$$

Hence,

$$\begin{aligned} & x + y \\ & = \frac{p_1}{q_1} + \frac{p_2}{q_2} \\ & = \frac{p_1 q_2 + p_2 q_1}{q_1 q_2} \\ & = \frac{P}{Q}, \end{aligned}$$

where $P = p_1 q_2 + p_2 q_1$ and $Q = q_1 q_2 \neq 0$ are integers. $\therefore x + y$ must be a rational number, which is a contradiction to the original assumption that x + y is an irrational number. Therefore, the sum of two rational numbers x and y must be rational.

3. Prove that $\sqrt{2}$ is irrational.

Proof by contradiction:

Assuming that $\sqrt{2}$ is rational to obtain a contradiction. Hence, by definition, there exist integers $p, q, q \neq 0$ such that $\sqrt{2} = \frac{p}{q}$ and there is no common factors, except 1 and -1, for p and q.

In computing $(\sqrt{2} = \frac{p}{q})^2$, we have $2 = \frac{p^2}{q^2}$, or $2q^2 = p^2$.

By definition of even integer, p^2 is even.

Q: Is integer p odd or even?

If p is odd, $p = 2h+1$, for some integer h. Hence, $p^2 = (2h+1)^2 = 4h^2 + 4h + 1 = 2(2h^2 + 2h) + 1$, implying that p^2 is odd, which is a contradiction to our previous computation that p^2 is even. Thus, p must be even. Let $p = 2k$, for some integer k. Since $2q^2 = p^2 = 4k^2$, or $q^2 = 2k^2$, by the same argument, q^2 and q must also be even. But if both p and q are even, they must have a common factor 2, which is a contradiction to our assumption that there is no common factors, except 1 and -1, for p and q. Hence, the original assumption that $\sqrt{2}$ is rational must be wrong and $\sqrt{2}$ must be an irrational.

II. **Implication, $p \rightarrow q$:**

Consider the truth table for $p \rightarrow q$:

p	**q**	**$p \rightarrow q$**
T	T	**T**
T	F	**F**
F	T	**T**
F	F	**T**

Observations:

1. If p is false, then $p \rightarrow q$ is always true.
   Vacuous Proof:
   Prove that p is always false.

2. If q is true, then $p \rightarrow q$ is always true.
   Trivial Proof:
   Prove that q is always true.

3. If p is true, then q must be true in order for $p \rightarrow q$ to be true.
   Direct Proof:
   Prove that if p is true, then q must always be true.

4. The only way that $p \rightarrow q$ can be false is when p is true and q is false.
   Proof by Contradiction:
   Prove that if p is true and q is false, it will always result in a contradiction.

5. Recall that $p \rightarrow q \equiv \neg q \rightarrow \neg p$.
   Proof by Contrapositive:
   Prove that if $\neg q$ is true, the $\neg p$ must also be true.

6. Observe that if $p = p_1 \vee p_2 \vee \ldots \vee p_k$, then we have
   $$
   \begin{aligned}
   & p \rightarrow q \\
   & \equiv (p_1 \vee p_2 \vee \ldots \vee p_k) \rightarrow q \\
   & \equiv (p_1 \rightarrow q) \wedge (p_2 \rightarrow q) \wedge \ldots \wedge (p_k \rightarrow q).
   \end{aligned}
   $$
   Proof by Cases:
   We can prove $p \rightarrow q$ by proving
   $p_1 \rightarrow q$,
   $p_2 \rightarrow q$,
   $\ldots$,
   $p_k \rightarrow q$ are all tautology.

Examples:

4. Define P(n): Given integer n. If $n > 1$, then $n > n^2$. Prove that P(0) is true.
   Proof: Observe that this statement is false in general. However, when $n = 0$, $0 > 1$ is always false and hence the implication must be true. Thus, P(0) is true by ***Vacuous Proof Technique***.

5. Define P(n): Given integer n, real numbers x and y. If $x \geq y \geq 1$, then $x^n = y^n$. Prove that P(0) is true.
   Proof: Observe again that this statement is false in general. However, when $n = 0$, $x^0 = y^0 = 1$. Since the conclusion $x^n = y^n$ is always true, P(0) is true by ***Trivial Proof Technique.***

6. Prove that if n is even, $123n + 210$ must also be even.
   Direct Proof:
   Assume that n is even, there exists an integer k such that $n = 2k$.
   $\therefore 123n + 210$
   $= 123(2k) + 210$
   $= 2(123k + 105)$
   $= 2K$, where $K = 123k+105$ is an integer.
   Hence, by definition, $123n + 210$ must also be even.

Proof by Contradiction:

Assume that n is even yet 123n + 210 is odd to obtain a contradiction.

If n is even, there exists an integer k such that n = 2k.

Also, if 123n + 210 is odd, there exists an integer h such that 123n + 210 = 2h + 1.

$\therefore$ 123n + 210

= 123(2k) + 210

= 2h + 1.

Since 123(2k) +210 = 2h + 1, we have

2(123k + 105 – h) = 2K = 1, where

K = 123k + 105 – h is an integer.

But if 1 = 2K, the integer 1 must be an even integer, which is a contradiction since 1 is an odd integer. Hence, the original assumption that 123n + 210 is odd must be wrong, implying that if n is even, 123n + 210 must also be even.

Proof by Contrapositive:
Assume that $123n + 210$ is odd. We must now prove that n must also be odd.
Assume that $123n + 210$ is odd yet n is even to obtain a contradiction.
By definition, if $123n + 210$ is odd, there exists an integer h such that $123n + 210 = 2h + 1$.
Also, if n is even, there exists an integer k such that $n = 2k$.

$$\begin{aligned} \therefore\ & 123n + 210 \\ &= 123(2k) + 210 \\ &= 2h + 1. \end{aligned}$$

Hence, $123(2k) + 210 = 2h + 1$, implying that $2(123k + 105 - h) = 2K = 1$, where $K = 123k + 105 - h$ is an integer.
But if $1 = 2K$, the integer 1 must be an even integer, which is a contradiction since 1 is an odd integer. Hence, the original assumption that n is even must be wrong and n must be odd, implying that if $123n + 210$ is odd then n must also be odd.

7. Let n be an integer. Prove that if n is even, then n^2-8n+7 is odd.

   Direct Proof:

   Assume that n is even, n = 2k, for some integer k. Hence, $n^2- 8n+7 = (2k)^2- 8(2k)+7 = 4k^2- 16k+7 = 2(2k^2- 8k+3)+1 = 2K+1$, where $K = 2k^2- 8k+3$ is an integer, implying that $n^2- 8n+7$ must be odd.

   Proof by Contradiction:

   Assume that n is even and $n^2- 8n+7$ is also even to obtain a contradiction. Since n is even, n = 2k, for some integer k. Hence, $n^2- 8n+7 = (2k)^2- 8(2k)+7 = 4k^2- 16k+7 = 2(2k^2- 8k+3)+1 = 2K+1$, $K = 2k^2- 8k+3$ is an integer, implying that $n^2- 8n+7$ must be odd. A contradiction is now reached since, by assumption, $n^2- 8n+7$ is even.

   Proof by Contrapositive:

   Assume that $n^2- 8n+7$ is even, we need to prove that n must be odd. By definition, there exists an integer k such that $n^2- 8n+7 = 2k$.
   In order to prove that n is odd, we need to isolate n from $n^2- 8n+7 = 2k$ which is not easy.

8. Let x and y be two positive real numbers.
Assume that if a real number $x > 0$, then $\sqrt{x} > 0$.
Prove that if $x \le y$, then $\sqrt{x} \le \sqrt{y}$.
Direct Proof:
Assume that $x \le y$, then $x - y \le 0$. Or,
$(\sqrt{x})^2 - (\sqrt{y})^2 \le 0$,
$(\sqrt{x} - \sqrt{y})(\sqrt{x} + \sqrt{y}) \le 0$.
Since x and y are positive real numbers, $\sqrt{x} + \sqrt{y} > 0$.
By dividing both sides by $\sqrt{x} + \sqrt{y}$, we have
$\sqrt{x} - \sqrt{y} \le 0$, or $\sqrt{x} \le \sqrt{y}$.

Proof by Contradiction:
Assume that $x \le y$ yet $\sqrt{x} > \sqrt{y}$ to obtain a contradiction.
Since $x \le y$, $x - y \le 0$. Or,
$(\sqrt{x})^2 - (\sqrt{y})^2 \le 0$,
$(\sqrt{x} - \sqrt{y})(\sqrt{x} + \sqrt{y}) \le 0$.
Since x and y are positive real numbers, $\sqrt{x} + \sqrt{y} > 0$.
By dividing both sides by $\sqrt{x} + \sqrt{y}$, we have
$\sqrt{x} - \sqrt{y} \le 0$, implying that $\sqrt{x} \le \sqrt{y}$, which is a contradiction to the assumption that $\sqrt{x} > \sqrt{y}$.

Proof by Contrapositive:

Assume that $\sqrt{x} > \sqrt{y}$. we need to prove that $x > y$. Since $\sqrt{x} > \sqrt{y}$, $\sqrt{x} - \sqrt{y} > 0$. By assumption, x and y are positive real numbers, $\sqrt{x} + \sqrt{y} > 0$.

Hence,

$(\sqrt{x} - \sqrt{y})(\sqrt{x} + \sqrt{y}) > 0$,

$(\sqrt{x})^2 - (\sqrt{y})^2 > 0$,

$x - y > 0$, or

$x > y$.

Hence, if $x \leq y$, then $\sqrt{x} \leq \sqrt{y}$.

9. Prove that there exists no positive integer n such that $n^3 + 5n - 1 = 210$.

   Proof by Cases:

   Observe that $n^3 + 5n - 1$ is a monotonic increasing function. Since $6^3 = 216$, $n^3 + 5n - 1 > 210$, for all integers $n > 5$, we only need to verify the given equality for integers $n \leq 5$.

   Case 1: When $n = 1, n^3 + 5n - 1 = 5 \neq 210$.

   Case 2: When $n = 2, n^3 + 5n - 1 = 17 \neq 210$.

   Case 3: When $n = 3, n^3 + 5n - 1 = 41 \neq 210$.

   Case 4: When $n = 4, n^3 + 5n - 1 = 83 \neq 210$.

   Case 5: When $n = 5, n^3 + 5n - 1 = 149 \neq 210$.

   Hence, there exists no positive integer n such that $n^3 + 5n - 1 = 210$.

III. Biconditional: $P = p \leftrightarrow q$

Recall that $(p \leftrightarrow q) \equiv ((p \rightarrow q) \wedge (q \rightarrow p))$.

To prove that $p \leftrightarrow q$, we need to prove $p \rightarrow q$ (p is sufficient for q; ***sufficiency***) and $q \rightarrow p$ (q is necessary for p; ***necessity***). All techniques developed so far can be used to prove the implications.

Proving Equivalent Propositions:

Given a set of propositions $\{p_1, p_2, \ldots, p_k\}$. When proving that they are all logically equivalence (having the same truth value), we need to prove that

$p_1 \leftrightarrow p_2$,
$p_2 \leftrightarrow p_3$,
$p_3 \leftrightarrow p_4$,
…
$p_{k-1} \leftrightarrow p_k$.

A Better Approach:

$p_1 \rightarrow p_2$,
$p_2 \rightarrow p_3$,
$p_3 \rightarrow p_4$,
…
$p_{k-1} \rightarrow p_k$,
$p_k \rightarrow p_1$.

Remark: We can use previous proof techniques for implications to prove Equivalent Propositions.

IV. **Quantifications:**

(a) **Universal quantification:** $P = \forall x \in S, P(x)$

- **Direct Proof:**
  Prove directly that $\forall x \in S, P(x)$ is a tautology.

- **Indirect Proof (Proof by Contradiction):**
  Assume that $\exists x \in S$ such that $P(x)$ is false to obtain a contradiction.

- **Proof by Cases:**
  If $S = \{x_1, x_1, \ldots, x_k\}$, then
  $\forall x \in S, P(x) \equiv P(x_1) \wedge P(x_2) \wedge \ldots \wedge P(x_k)$.
  Prove that $P(x_1)$ is true, $P(x_2)$ is true, …, and $P(x_1)$ is true.

- **Disproving by Counter-Example:**
  To disprove P, all you need to do is to show the existence of an element $x \in S$ such that $P(x)$ is false.

- **Mathematical Induction:**
  TBA.

(b) **Existential quantification:** $P = \exists x \in S, P(x)$

- **Direct Proof:**
  Prove directly that $\exists x \in S, P(x)$.

  (i) Constructive Proof:
  Show the actual construction of an element x such that $P(x)$ is true.

  (ii) Non-Constructive Proof:
  Need only show that an element x exists such that $P(x)$ is true. There is no need to show the construction of x.

- **Indirect Proof (Proof by Contradiction):**
  Assume that $\forall x \in S, P(x)$ is false to obtain a contradiction.

10. Prove that for all odd integer n, 3n+210 must also be odd.

Direct Proof:

Let n be any given odd integer. By definition, there exists an integer k such that $n = 2k+1$.

$\therefore 3n + 210$

$= 3(2k+1) + 210$

$= 6k + 213$

$= 2(3k + 106) + 1$

$= 2K + 1$, where $K = 3k + 106$ is an integer.

Hence, $3n + 210$ must also be odd.

Indirect Proof:

Assume that there exists an odd integer n such that 3n+210 is even to obtain a contradiction.

If n is odd, by definition, there exists an integer k such that $n = 2k+1$.

$\therefore 3n + 210$

$= 3(2k+1) + 210$

$= 6k + 213$

$= 2(3k + 106) + 1$

$= 2K + 1$, where $K = 3k + 106$ is an integer.

Hence, $3n + 210$ must also be odd, which is a contradiction to the original assumption that 3n+210 is even.

11. Prove that for all positive integer $n < 29$, $2n^2 + 29$ is prime.
Recall that an integer $n > 1$ is a prime iff its only positive divisors are 1 and itself.
Proof: Consider the 28 cases when $n = 1, 2, \ldots, 28$.
Case 1: When $n = 1$, $2n^2 + 29 = 31$ is prime.
Case 2: When $n = 2$, $2n^2 + 29 = 37$ is prime.
Case 3: When $n = 3$, $2n^2 + 29 = 47$ is prime.
…
Case 28: When $n = 28$, $2n^2 + 29 = 1597$ is prime.
Hence, for all positive integer $n < 29$, $2n^2 + 29$ is prime.

12. Prove that for all positive integer n, $2n^2 + 29$ is prime.
Disproving by Counter-Example:
Observe that although this statement is true for all integers $n \leq 28$. However, when $n = 29$, $29 \mid 2(29)^2 + 29$. Hence, the given universal quantification is false in general.

13. Prove that there exists a positive integer n such that $2n^2 + 29$ is prime.
Constructive Proof:
Take $n = 1$, $2n^2 + 29 = 31$ is a prime. Hence, the given existential quantification is true.

14. Prove that there are infinitely many primes.

Non-Constructive Proof:

Assume that there are only finitely many primes $p_1, p_2, \ldots, p_k$, $p_i > 2$, $\forall i$. Consider the integer defined by $N = p_1 p_2 \ldots p_k + 1$, which is either a prime or must be a composite integer. If N is a prime, N is a new prime bigger than all of the existing primes; a contradiction to the assumption that $p_1, p_2, \ldots, p_k$ are the only prime numbers. Else, if N is composite, N must be divisible by one of these primes, say p. Since $p \mid N$ and $p \mid (p_1 p_2 \ldots p_k)$, $p \mid (N - p_1 p_2 \ldots p_k)$.

But $N - p_1 p_2 \ldots p_k = 1$, implying that $p \mid 1$, which is again a contradiction. Hence, there must be infinitely many primes.