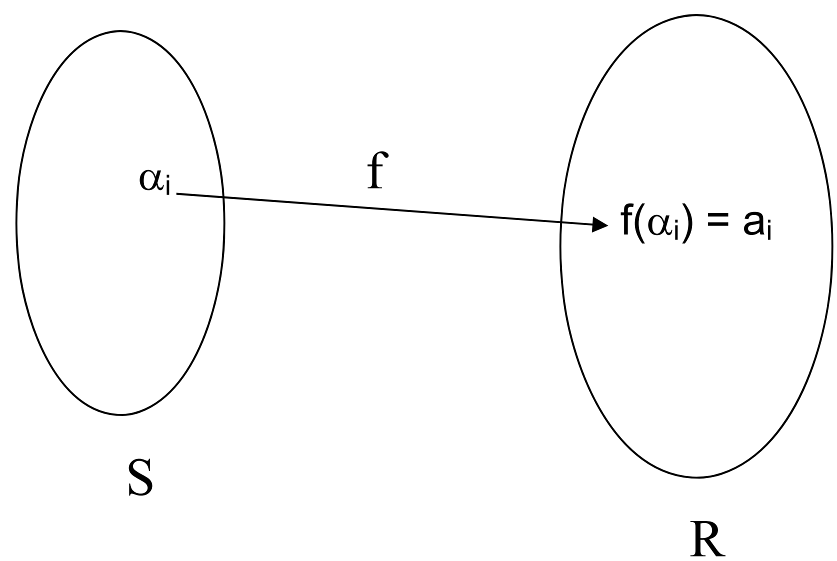
Topic 5: Sequences & Summations

Read: Chpt.2.4, Rosen

Let $S \subseteq N \cup \{0\}$ be a set of non-negative integers. Consider a real-valued function $f: S \to R$.



Observations:

- S is a well-ordered set and can be linearly ordered as $(\alpha_1, \alpha_2, \ldots, \alpha_i, \ldots)$ with $\alpha_1 < \alpha_2 < \ldots < \alpha_i < \ldots$.
- Based on this ordering in S, elements in f(S) can also be ordered as $(f(\alpha_1), f(\alpha_2), \ldots, f(\alpha_i), \ldots)$. This is an ordering induced (defined) by f.

S	$(\alpha_1, \alpha_2, \ldots, \alpha_i, \ldots)$, $\alpha_1 < \alpha_2 < \ldots < \alpha_i < \ldots$
f(S)	$(f(\alpha_1)=a_1, f(\alpha_2)=a_2, \ldots, f(\alpha_i)=a_i, \ldots)$

Dfn: The ordered set of elements $(f(\alpha_1), f(\alpha_2), \ldots, f(\alpha_i), \ldots)$, or $(a_1, a_2, \ldots, a_i, \ldots)$, forms a ***sequence defined by f***.

$$\begin{array}{lcccccc}
S: & (\alpha_1, & \alpha_2, & \ldots, & \alpha_i, & \ldots) \\
 & \downarrow & \downarrow & \ldots & \downarrow & \\
f(S) & f(\alpha_1), & f(\alpha_2), & \ldots, & f(\alpha_i), & \ldots \\
 & \downarrow & \downarrow & \ldots & \downarrow & \\
\text{Sequence} & a_1, & a_2, & \ldots, & a_i, & \ldots
\end{array}$$

Representation of a Finite Sequence:

- $(a_l, a_{l+1}, a_{l+2}, \ldots, a_u)$
- $\{a_i\}_{l \le i \le u}$.
- $\{a_i\}_{i=l}^{i=u}$.
- $\{a_i\}_{i \in I}, I = \{l, l+1, \ldots, u\}$ is the index set.

Representation of Infinite Sequence:

- $(a_l, a_{l+1}, a_{l+2}, \ldots)$
- $\{a_i\}_{l \le i \le \infty}$.
- $\{i\}_{i=l}^{i=\infty}$.
- $\{a_i\}_{i \in I}, I = \{l, l+1, \ldots\}$.

Warning: In describing a sequence, the general a_i-term must be given for any $i \in I$. A sequence in the form of "1, 2, 3, ..." is not acceptable.

Examples:

1. The sequence $\{a_i\}_{1 \le i \le 210}, a_i = i$, corresponds to the sequence $1, 2, 3, \ldots, i, \ldots, 210$.

2. The sequence $\{a_i\}_{1 \le i \le \infty}, a_i = \frac{1}{i}$, corresponds to the sequence $\frac{1}{1}, \frac{1}{2}, \frac{1}{3}, \ldots, \frac{1}{i}, \ldots$

3. The sequence $a_0 = 0, a_1 = 1, a_i = a_{i-1} + a_{i-2}, \forall i > 1,$ is a recursive definition for the Fibonacci sequence 0, 1, 1, 2, 3, 5, 8, 13, ….

Σ-Notation and Summing of Sequence:

Dfn: Given a sequence $a_l, a_{l+1}, a_{l+2}, \ldots, a_u, l \le u$.

Define

$$\sum_{i=l}^{u} a_i = a_l + a_{l+1} + a_{l+2} + \ldots + a_u, \text{ where}$$

i — index of summation, l — lower index, u— upper index.

Remark: Different summation indices can be used to represent the same summation.

Example: $\sum_{i=l}^{u} a_i = \sum_{j=l}^{u} a_j = \sum_{k=l-2}^{u-2} a_{k+2} = a_l + a_{l+1} + \ldots + a_u$.

Basic Properties of Σ:

1. Given a constant C, $\sum_{i=1}^{n} C = C + C + \ldots + C = nC$.
2. Linearity Property of Σ:
   Given constants C_1, C_2, and sequences $\{a_i\}_{i\in I}$, $\{\mathrm{b}_i\}_{i\in I}$.
   $$\sum_{i=l}^{u}(C_1 a_i + C_2 b_i) = C_1 \sum_{i=l}^{u} a_i + C_2 \sum_{i=l}^{u} b_i.$$

Warnings:

(1) $\sum_{i=j}^{n} C_1 a_i + C_2 b_i = C_1 \sum_{i=j}^{n} a_i + C_2 b_i \neq C_1 \sum_{i=j}^{n} a_i + C_2 \sum_{i=j}^{n} b_i.$

(2) $\sum_{i=1}^{n} a_i * b_i \neq (\sum_{i=l}^{u} a_i) * (\sum_{i=l}^{u} b_i).$

Some Basic Techniques on Summations:
Summation Technique #1: *Rearranging* the terms in a summation so as to form and identify a simple pattern.

Consider the famous summation:

$$1 + 2 + \ldots + 100 = \sum_{i=l}^{100} i = ?$$

Q: How do we compute it?

Let's consider the more general summation $S = \sum_{i=i}^{n} i.$

Gauss: Let S = $\sum_{i=l}^{n} i$ = 1 + 2 + 3 + … + n … (1)

By rearranging the terms in S backward, we have

S = n + (n-1) + (n-2) + … + 1 … (2)

On summing (1) and (2), we have

$$2S = \begin{array}{ccccccccc} 1 & + & 2 & + & 3 & + & \ldots & + & n \\ \downarrow & & \downarrow & & \downarrow & & & & \downarrow \\ +\,n & + & n-1 & + & n-2 & + & \ldots & + & 1 \end{array}$$

$\therefore$ 2S = n(n+1)

$$S = \frac{n(n+1)}{2}.$$

Hence, $\sum_{i=1}^{n} i = \dfrac{n(n+1)}{2}.$

Examples:

1. $1 + 2 + \ldots + 100 = \sum_{i=l}^{100} i = \dfrac{(100)(101)}{2} = 5{,}050.$
2. $1 + 2 + \ldots + k^2 = \sum_{i=l}^{k^2} i = \dfrac{k^2(k^2+1)}{2}.$

Q: How about computing the sum

$$57 + 58 + \ldots + 100 = \sum_{i=57}^{100} i\,?$$

Summation Technique [#]2: *Patching* a summation.

Let S = $\sum_{i=1}^{n} a_i$. Observe that for any constant K,
S = (S + K) – K. Hence, we can always add a number of terms with sum K to S and subtract the same terms from S to maintain equality so as to convert (S + K) into a "known" summation.

Observe that

$$57 + 58 + \ldots + 100$$
$$= [(57 + 58 + \ldots + 100) + (1 + 2 + \ldots + 56)] - (1 + 2 + \ldots + 56)$$
$$= \sum_{i=1}^{100} i - \sum_{i=1}^{56} i$$
$$= \frac{(100)(101)}{2} - \frac{(56)(57)}{2}$$
$$= 3{,}454.$$

In general, $\sum_{i=j}^{n} i$

$$= \left(\sum_{i=1}^{n} i + \sum_{i=1}^{j-1} i\right) - \sum_{i=1}^{j-1} i$$
$$= \sum_{i=1}^{n} i - \sum_{i=1}^{j-1} i$$
$$= \frac{n(n+1)}{2} - \frac{(j-1)(j)}{2}$$
$$= \frac{n^2 - j^2 + n + j}{2}$$
$$= \frac{(n+j)(n-j+1)}{2}.$$

Example:

$$210 + 211 + \ldots + 1128$$

$$= \sum_{i=210}^{1128} i$$

$$= \sum_{i=1}^{1128} i - \sum_{i=1}^{209} i$$

$$= \frac{(1128)(1129)}{2} - \frac{(209)(210)}{2}$$

$$= 614,811.$$

Summation Technique #3: ***Shifting*** the index of a summation. Observe that

$$S = a_l + a_{l+1} + \ldots + a_u = \sum_{i=l}^{u} a_i = \sum_{j=l+t}^{u+t} a_{j-t} = \sum_{k=l-t}^{u-t} a_{k+t}.$$

By shifting the index of a given summation S, one may be able to convert S into a known summation.

Example:

$$57 + 58 + \ldots + 100$$

$$= \sum_{i=57}^{100} i$$

$$= \sum_{i=1}^{44} (i + 56)$$

$$= \sum_{i=1}^{44} i + \sum_{i=1}^{44} 56$$

$$= \frac{(44)(45)}{2} + (44)(56)$$

$$= 3,454.$$

HW: Compute $\sum_{i=210}^{1128} i$ by shifting the summation index.

Q: What if we only want to sum all the odd numbers between 57 and 100?

Observe that

$$\begin{aligned}
&57 + 59 + 61 + \ldots + 99 \\
&= (55+2) + (55+4) + (55+6) \ldots + (55+44) \\
&= \sum_{i=1}^{22} (55 + 2i) \\
&= \sum_{i=1}^{22} 55 + 2\sum_{i=1}^{22} i \\
&= (22)(55) + (22)(23) \\
&= 1{,}716.
\end{aligned}$$

Another Example:

$$\begin{aligned}
&36 + 38 + 40 + \ldots + 100 \\
&= 2(18 + 19 + 20 + \ldots + 50) \\
&= 2\sum_{i=18}^{50} i \\
&= 2(\sum_{i=1}^{50} i - \sum_{i=1}^{17} i) \\
&= 50 * 51 - 17 * 18 \\
&= 2{,}244.
\end{aligned}$$

Q: How about the sum 1 – 2 + 3 – 4 + … + 99 – 100?

Summation Technique #4: Grouping the terms in a summation so as to identify and form an identifiable pattern.

Observe that

$1 - 2 + 3 - 4 + \ldots + 99 - 100$

$= (1 - 2) + (3 - 4) + \ldots + (99 - 100)$

$= (-1) * (50)$

$= -50.$

Another application of grouping:

$$1-2+3-4+\ldots+99-100$$
$$=(1+3+5+\ldots+99)-(2+4+6+\ldots+100)$$
$$=\sum_{i=1}^{50}(2i-1)-\sum_{i=1}^{50}2i$$
$$=\sum_{i=1}^{50}2i-\sum_{i=1}^{50}1-\sum_{i=1}^{50}2i$$
$$=-50.$$

Q: What if we need to compute the sum

$$2 + 2^2 + 2^3 + 2^{100} \ ?$$

Summation Technique #5: Scaling a summation by multiplying a given summation S by a constant factor K and then add it back to the original summation S so as to obtain a general pattern that can be used to simplify the sum.

Let S = $\sum_{i=1}^{n} a_i$.

Compute S + kS, k is a constant (positive or negative).

Let's consider summing a geometric series with common ration r.

$$\sum_{i=o}^{n} r^i = r^0 + r^1 + r^2 + \ldots + r^n.$$

$$\begin{aligned} \text{Let } S &= r^0 + r^1 + r^2 + \ldots + r^{n-1} + r^n \\ rS &= \phantom{r^0 + {}} r^1 + r^2 + \ldots + r^{n-1} + r^n + r^{n+1} \end{aligned}$$

$$rS - S = r^{n+1} - r^0$$

$$S = \frac{r^{n+1} - r^0}{r - 1}, r \neq 1.$$

Theorem:

$$\sum_{i=0}^{n} r^i = \frac{r^{n+1} - r^0}{r-1} = \frac{r^0 - r^{n+1}}{1-r}, r \neq 1.$$

$$\text{As n} \to \infty,\ \sum_{i=0}^{n} r^i \to \infty,\ \text{if } |r| < 1,\ \text{and}$$

$$\sum_{i=0}^{n} r^i \to \frac{1}{1-r},\ \text{if } |r| < 1.$$

Example:

$$\sum_{i=3}^{100} \frac{5}{2^i}$$

$$= \sum_{i=3}^{100} \frac{5}{2^i} + (5 + \frac{5}{2} + \frac{5}{4}) - (5 + \frac{5}{2} + \frac{5}{4})$$

$$= \sum_{i=0}^{100} \frac{5}{2^i} - (5 + \frac{5}{2} + \frac{5}{4})$$

$$= 5\sum_{i=0}^{100} (\frac{1}{2})^i - \frac{35}{4}$$

$$= 5[\frac{1 - (\frac{1}{2})^{101}}{1 - \frac{1}{2}}] - \frac{35}{4}$$

$$= 10[1 - (\frac{1}{2})^{101}] - \frac{35}{4}$$

More Examples:

Compute $\sum_{i=0}^{n} i*2^i = 1*2^1 + 2*2^2 + 3*2^3 + \ldots + n*2^n$.

Let

$$S = 1*2^1 + 2*2^2 + 3*2^3 + \ldots + (n-1)*2^{n-1} + n*2^n.$$

$$2S = 1*2^2 + 2*2^3 + 3*2^4 + \ldots + (n-1)*2^n + n*2^{n+1}.$$

On subtracting, we obtain

$$\begin{aligned}
-S &= 2^1 + 2^2 + 2^3 + \ldots + 2^n - n*2^{n+1} \\
&= \sum_{i=1}^{n} 2^i - n*2^{n+1} \\
&= \sum_{i=0}^{n} 2^i - n*2^{n+1} - 1 \\
&= \left(\frac{2^{n+1}-1}{2-1}\right) - n*2^{n+1} - 1 \\
&= 2^{n+1} - n*2^{n+1} - 2
\end{aligned}$$

$$\begin{aligned}
\therefore S &= n2^{n+1} - 2^{n+1} + 2 \\
&= 2^{n+1}(n-1) + 2
\end{aligned}$$

Q: How do we compute

$$\sum_{i=0}^{n} i^2 * 2^i = 1^2 * 2^1 + 2^2 * 2^2 + 3^2 * 2^3 + \ldots + n^2 * 2^n?$$

Let's use a similar approach.

$$S = \sum_{i=0}^{n} i^2 * 2^i$$

$$= 1^2 * 2^1 + 2^2 * 2^2 + 3^2 * 2^3 + \ldots + (n-1)^2 * 2^{n-1} + n^2 * 2^n$$

$$2S = 1^2 * 2^2 + 2^2 * 2^3 + 3^2 * 2^4 + \ldots + (n-1)^2 * 2^n + n^2 * 2^{n+1}$$

On subtracting, we obtain

$$-S = (1^2 - 0^2)2^1 + (2^2 - 1^2)2^2 + (3^2 - 2^2)2^3 + \ldots + (n^2-(n-1)^2)2^n - n^2 * 2^{n+1}$$

$$= 1*2^1 + 3*2^2 + 5*2^3 + \ldots + (2n-1)2^n - n^2 * 2^{n+1}$$

$$= \sum_{i=1}^{n} (2i-1)2^i - n^2 * 2^{n+1}$$

$$= 2\sum_{i=0}^{n} i2^i - \sum_{i=o}^{n} 2^i + 1 - n^2 * 2^{n+1}$$

$$= 2[2^{n+1}(n-1) + 2] - (2^{n+1} - 1) - n^2 * 2^{n+1} + 1$$

$$\therefore S = n^2 * 2^{n+1} + (2^{n+1} - 1) - 2[2^{n+1}(n-1) + 2] - 1$$

HW: Compute $\sum_{i=0}^{n} i^3 2^i$, $\sum_{i=0}^{n} \frac{2^i}{i^2}$.

Some Simple Summations:

1. $\sum_{i=j}^{n} C = C\sum_{i=j}^{n} 1 = C(n-j+1),\ C - \text{constant}.$

2. $\sum_{i=j}^{n} (C_1 a_i + C_2 b_i) = C_1\sum_{i=j}^{n} a_i + C_2\sum_{i=j}^{n} b_i,\ C_1, C_2 - \text{constants}.$

3. $\sum_{i=1}^{n} i = \frac{n(n+1)}{2}.$

4. $\sum_{i=j}^{n} i = \frac{n^2 - j^2 + n + j}{2} = \frac{(n-j+1)(n+j)}{2}.$

5. $\sum_{i=1}^{n} i^2 = \frac{n(n+1)(2n+1)}{6}.$

6. $\sum_{i=1}^{n} i^3 = \frac{n^2(n+1)^2}{4}.$

7. $\sum_{i=1}^{n} i^4 = \frac{n(n+1)(2n+1)(3n^2+3n-1)}{30}.$

8. $\sum_{i=0}^{n} r^i = \frac{r^{n+1}-1}{r-1} = \frac{1-r^{n+1}}{1-r}.$

9. $\sum_{i=0}^{n} ir^i = \frac{r}{(1-r)^2}[1-(n+1)r^n + nr^{n+1}].$

10. $\sum_{i=0}^{n} i^2 r^i = \frac{r}{(1-r)^3}[(1+r)-(n+1)^2 r^n + (2n^2+2n-1)r^{n+1} - n^2 r^{n+2}].$

A Special Summation:

Given a sequence $\{a_i\}_{i \in I}$. The sum $\sum_{i=1}^{n} (a_i - a_{i-1})$ is called a ***telescoping sum***.

Observe that

$$\sum_{i=1}^{n} (a_i - a_{i-1})$$

$$\begin{aligned} = a_1 &- a_0 + \\ a_2 &- a_1 + \\ a_3 &- a_2 + \\ &\dots \\ a_{n-1} &- a_{n-2} + \\ a_n &- a_{n-1} \end{aligned}$$

$$= a_n - a_o$$

HW: Verify that $\sum_{i=1}^{n} (a_{i-1} - a_i) = a_o - a_n$.

Summation Technique #6: Telescoping summation.

Convert a given summation S into a telescoping summation and then sum it using the above approach.

Applications of Telescopic Summations:

Let's look at the sum $\sum_{i=1}^{n} i = \frac{n(n+1)}{2}$ again.

Define $a_i = i^2$.

Observe that

$$
\begin{aligned}
& a_i - a_{i-1} \\
&= i^2 - (i-1)^2 \\
&= i^2 - i^2 + 2i - 1 \\
&= 2i - 1.
\end{aligned}
$$

Summing both sides, we have

$$\sum_{i=1}^{n} [i^2 - (i-1)^2] = \sum_{i=1}^{n} (2i-1)$$

$$n^2 - 0^2 = 2\sum_{i=1}^{n} i - \sum_{i=1}^{n} 1$$

$$\therefore 2\sum_{i=1}^{n} i = n^2 + n$$

$$\therefore \sum_{i=1}^{n} i = \frac{n(n+1)}{2}.$$

Q: How about $\sum_{i=1}^{n} i^2$?

Define $a_i = i^3$.

Observe that

$$
\begin{aligned}
& a_i - a_{i-1} \\
&= i^3 - (i^3 - 3i^2 + 3i - 1) \\
&= 3i^2 - 3i + 1.
\end{aligned}
$$

On summing both sides, we have

$$
\begin{aligned}
\sum_{i=1}^{n} [i^3 - (i-1)^3] &= \sum_{i=1}^{n} (3i^2 - 3i + 1) \\
&= 3\sum_{i=1}^{n} i^2 - 3\sum_{i=1}^{n} i + \sum_{i=1}^{n} 1 \\
n^3 - 0^3 &= 3\sum_{i=1}^{n} i^2 - \frac{3n(n+1)}{2} + n
\end{aligned}
$$

$$
\begin{aligned}
\therefore \sum_{i=1}^{n} i^2 &= \frac{1}{3}\left[n^3 + \frac{3n(n+1)}{2} - n\right] \\
&= \frac{n}{6}(2n^2 + 3n + 3 - 2) \\
&= \frac{n}{6}(2n^2 + 3n + 1) \\
&= \frac{n(n+1)(2n+1)}{6}
\end{aligned}
$$

Example: $\frac{1}{1*2} + \frac{1}{2*3} + \frac{1}{3*4} + \ldots + \frac{1}{(n-1)*n} = \sum_{i=1}^{n-1} \frac{1}{i*(i+1)}$.

Observe that $\frac{1}{i*(i+1)} = \frac{1}{i} - \frac{1}{(i+1)}$.

By letting $a_i = \frac{1}{i}$, we have a telescoping sum $\sum_{i=1}^{n-1} (a_i - a_{i+1})$ which sums to $a_1 - a_n$.

$$\therefore \sum_{i=1}^{n-1} \frac{1}{i*(i+1)} = 1 - \frac{1}{n}$$

Or, by expanding,

$$\sum_{i=1}^{n-1} \frac{1}{i*(i+1)}$$

$$= \left(\frac{1}{1} - \frac{1}{2}\right) + \left(\frac{1}{2} - \frac{1}{3}\right) + \left(\frac{1}{3} - \frac{1}{4}\right) + \ldots + \left(\frac{1}{n-1} - \frac{1}{n}\right)$$

$$= 1 - \frac{1}{n}.$$

HW: Compute $\sum_{i=1}^{n} i^3$ and $\sum_{i=2}^{n} \frac{1}{(i-1)(i+1)}$ using telescopic summations.

Multiple Summations:

$$\sum_i \sum_j a_{i,j} = \sum_i (\sum_j a_{i,j})$$

$$\ldots \sum_i \sum_j \sum_k a_{i,j,k} = \ldots(\sum_i (\sum_j (\sum_k a_{i,j,k})))\ldots$$

Examples:

1. $$\sum_{i=1}^{2} \sum_{j=1}^{2} ij = \sum_{i=1}^{2} (\sum_{j=1}^{2} ij)$$
$$= \sum_{i=1}^{2} (i*1+i*2)$$
$$= (1*1 + 1*2) + (2*1 + 2*2)$$
$$= 9$$

2. $$\sum_{j=1}^{100} \sum_{i=1}^{5} (i+j) = \sum_{j=1}^{100} (\sum_{i=1}^{5} (i+j))$$
$$= \sum_{j=1}^{100} (\sum_{i=1}^{5} i + \sum_{i=1}^{5} j)$$
$$= \sum_{j=1}^{100} [(5*6)/2 + 5*\mathrm{j}]$$
$$= 15\sum_{j=1}^{100} 1 + 5\sum_{j=1}^{100} \mathrm{j}$$
$$= 15*100 + 5*[(100*101)/2]$$
$$= 26{,}750$$

Product (Π) Notation:

Dfn: Given a sequence $a_l, a_{l+1}, \ldots, a_u$, $l \le u$.

$$a_l * a_{l+1} * \ldots * a_u$$

$$= \prod_{l \le i \le u} a_i.$$

Examples:

1. $\prod_{0 \le i \le 5} \frac{1}{(i+1)} = \frac{1}{1} * \frac{1}{2} * \frac{1}{3} * \frac{1}{4} * \frac{1}{5} * \frac{1}{6} = \frac{1}{720}.$

2. $n! = 0$, if $n = 0$,
   $= 1*2*\ldots*n = \prod_{1 \le i \le n} i.$

Practice HW: Chpt.2.4, 3, 5, 9, 29, 33, 37, 43.

(Optional) Countable and Uncountable Sets

Read: Chpt. 2.5, Rosen

Recall that wo finite sets A and B are said to have the same cardinality iff they have the same number of elements.

Q: Can we extend this concept of "same cardinality" to infinite sets?

Observe that if two finite sets A and B are having the same cardinality, then we can always define a bijection from A to B.

$$\begin{array}{cccc} a_1 & a_2 & a_3 \ldots & a_{|A|} \\ \updownarrow & \updownarrow & \updownarrow & \updownarrow \\ b_1 & b_2 & b_3 \ldots & b_{|A|} \end{array}$$

Hence, one may compare the cardinalities of two sets based on the existence of a bijection between them.

Dfn: A set S is ***countable*** if either S is a finite set or if there exists a bijection from **N** to S. Otherwise, S is ***uncountable***.

Theorem: If S is countable, then any subset of S is also countable.

Theorem: If $S_1, S_2, \ldots, S_k$ are countable, where k is a fixed positive integer, then $S_1 \cup S_2 \cup \ldots \cup S_k$ is also countable.

Examples:

1. S = {a, 8, b, c, 10} is countable.

2. **N** is countable since we can define a bijection $f: \mathbf{N} \to \mathbf{N}$ such that $f(i) = i, \forall\, i \in \mathbf{N}$.

3. The set of all positive odd integers $\mathbf{N_o}$ is countable since we can define a bijection $f: \mathbf{N} \to \mathbf{N_o}$ such that $f(i) = 2i - 1, \forall\, i \in \mathbf{N}$.

$$\begin{array}{ccc} 1 & \longrightarrow & 1 \\ 2 & \longrightarrow & 3 \\ 3 & \longrightarrow & 5 \\ 4 & \longrightarrow & 7 \\ & \ldots & \end{array}$$

4. The set of all positive even integers $\mathbf{N_e}$ is countable since we can define a bijection f: $\mathbf{N} \rightarrow \mathbf{N_e}$ such that $f(i) = 2i - 2, \forall i \in \mathbf{N}$.

$$\begin{array}{ccc} 1 & \longrightarrow & 0 \\ 2 & \longrightarrow & 2 \\ 3 & \longrightarrow & 4 \\ 4 & \longrightarrow & 6 \\ \ldots & & \end{array}$$

5. The set of all odd integers $\mathbf{Z_o}$ is countable since we can define a bijection f: $\mathbf{N} \rightarrow Z_o$ such that

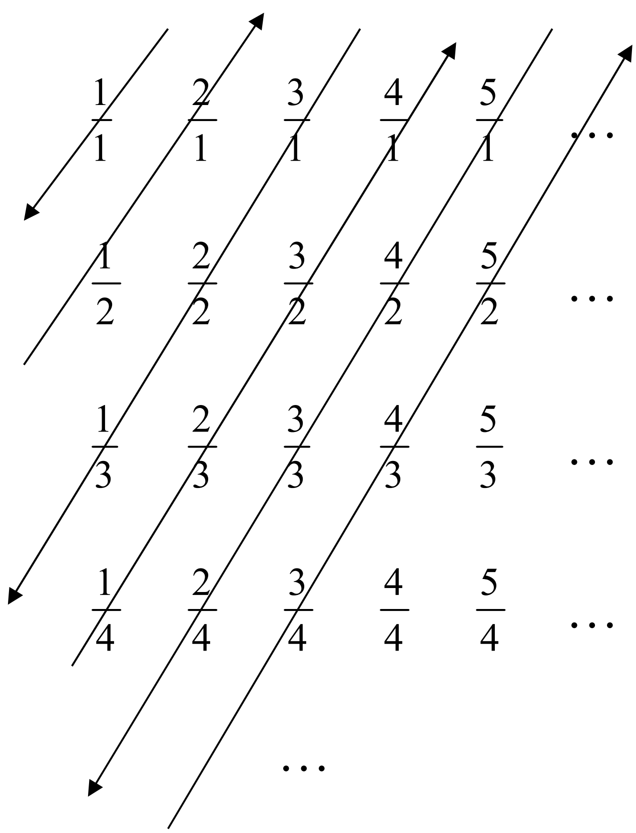
$$\begin{array}{ccr} 1 & \longrightarrow & 1 \\ 2 & \longrightarrow & -1 \\ 3 & \longrightarrow & 3 \\ 4 & \longrightarrow & -3 \\ 5 & \longrightarrow & 5 \\ 6 & \longrightarrow & -5 \\ & \ldots & \end{array}$$

Q: Can you define the bijection?

6. The set of all even integers Z_e is countable.
Q: Can you prove it?

7. The set of all positive rational numbers Q^+, which is the set of all real numbers that can be written as $\frac{p}{q}$, where p and q are positive integers, is countable.
We will define a bijection f: $\mathbf{N} \to Q^+$ and use f to induce a sequence $(a_1, a_2, a_3, \ldots)$ on Q^+.

Construction:



Q^+ will be ordered as $(\frac{1}{1}, \frac{1}{2}, \frac{2}{1}, \frac{3}{1}, \frac{1}{3}, \frac{1}{4}, \frac{2}{3}, \frac{3}{2}, \frac{4}{1}, \ldots)$, skipping any rational number that has previously been included in this construction.

8. The set of all positive real numbers R^+ is uncountable.
Proof by contradiction:
Assume that R^+ is countable to obtain a contradiction.
If R^+ is countable, then the set of all positive real numbers, $R_{(0,1)}$, between 0 and 1 must also be countable.
Hence, there must exist a bijection f from **N** to $R_{(0,1)}$ and f defines a sequence $(a_1, a_2, a_3, \ldots)$ in $R_{(0,1)}$.
Consider the decimal number representation of the numbers in this sequence:

$$\begin{aligned} a_1 &= 0.d_{11}d_{12}d_{13}d_{14}\ldots \\ a_2 &= 0.d_{21}d_{22}d_{23}d_{24}\ldots \\ a_3 &= 0.d_{31}d_{32}d_{33}d_{34}\ldots \\ a_4 &= 0.d_{41}d_{42}d_{43}d_{44}\ldots \\ &\ldots \end{aligned}$$

Observe that $d_{ij} \in \{0, 1, 2, \ldots, 9\}$, $\forall$ i, j $\in$ **N**.
We will now construct a real number $x = 0.d_1d_2d_3d_4\ldots \in R_{(0,1)}$ such that x can not be an element in this sequence to obtain a contradiction.
Define $d_i = 1$, if $d_{ii} \neq 1$,
$d_i = 2$, if $d_{ii} = 1$.
Observe that since $d_i \neq d_{ii}$, $x \neq a_i$, $\forall$ i $\in$ **N**.
Hence, x is not an element in this sequence and a contradiction is reached.

Practice HW: Chpt.2.5, 3, 5, 7, 11, 13, 15, 21.