Effects and Types

Syntax.

$$\begin{array}{rll} t & ::= z \mid t + t \mid \lambda x.t \mid t\ t \mid \texttt{ifz}\ t\ \texttt{then}\ t\ \texttt{else}\ t \mid \texttt{get} \mid \texttt{put}\ t \mid t\ \texttt{or}\ t & \text{Terms} \\ T & ::= \texttt{Int} \mid T \xrightarrow{E} T & \text{Types} \\ e & ::= \mathsf{g} \mid \mathsf{p} \mid \mathsf{a} & \text{Effects} \end{array}$$

Evaluation.

$$\frac{}{z \mid s \Downarrow z \mid s} \quad \frac{t_1 \mid s_1 \Downarrow z_1 \mid s_2 \quad t_2 \mid s_2 \Downarrow z_2 \mid s_3}{t_1 + t_2 \mid s_1 \Downarrow z_1 + z_2 \mid s_3} \quad \frac{}{\texttt{fix}\ t \mid s \Downarrow \lambda x.t\ \texttt{fix}\ t\ x \mid s}$$

$$\frac{}{\lambda x.t \mid s \Downarrow \lambda x.t \mid s} \quad \frac{t_1 \mid s_1 \Downarrow \lambda x.t \mid s_2 \quad t_2 \mid s_2 \Downarrow w \mid s_3 \quad t[w/x] \mid s_3 \Downarrow v \mid s_4}{t_1\ t_2 \mid s_1 \Downarrow v \mid s_4}$$

$$\frac{t_1 \mid s_1 \Downarrow 0 \mid s_2 \quad t_2 \mid s_2 \Downarrow v \mid s_3}{\texttt{ifz}\ t_1\ \texttt{then}\ t_2\ \texttt{else}\ t_3 \mid s_1 \Downarrow v \mid s_3} \quad \frac{t_1 \mid s_1 \Downarrow z \mid s_2 \quad t_3 \mid s_2 \Downarrow v \mid s_3}{\texttt{ifz}\ t_1\ \texttt{then}\ t_2\ \texttt{else}\ t_3 \mid s_1 \Downarrow v \mid s_3}\ (z \neq 0)$$

$$\frac{}{\texttt{get} \mid s \Downarrow s \mid s} \quad \frac{t \mid s_1 \Downarrow v \mid s_2}{\texttt{put}\ t \mid s_1 \Downarrow v \mid v} \quad \frac{t_1 \mid s_1 \Downarrow v \mid s_2}{t_1\ \texttt{or}\ t_2 \mid s_1 \Downarrow v \mid s_2} \quad \frac{t_2 \mid s_1 \Downarrow v \mid s_2}{t_1\ \texttt{or}\ t_2 \mid s_1 \Downarrow v \mid s_2}$$

Subtyping.

$$\frac{}{T <: T} \qquad \frac{T_2 <: T_1 \quad E_1 \subseteq E_2 \quad U_1 <: U_2}{T_1 \xrightarrow{E_1} U_1 <: T_2 \xrightarrow{E_2} U_2}$$

Typing.

$$\frac{}{\Gamma \vdash z : \texttt{Int}\ \&\ \emptyset} \quad \frac{\Gamma \vdash t_1 : \texttt{Int}\ \&\ E_1 \quad \Gamma \vdash t_2 : \texttt{Int}\ \&\ E_2}{\Gamma \vdash t_1 + t_2 : \texttt{Int}\ \&\ E_1 \cup E_2}$$

$$\frac{\Gamma[x \mapsto T_1] \vdash t : T_2\ \&\ E}{\Gamma \vdash \lambda x.t : T_1 \xrightarrow{E} T_2\ \&\ \emptyset} \quad \frac{\Gamma \vdash t_1 : T_1 \xrightarrow{E_3} T_2\ \&\ E_1 \quad \Gamma \vdash t_2 : T_1\ \&\ E_2}{\Gamma \vdash t_1\ t_2 : T_2\ \&\ E_1 \cup E_2 \cup E_3}$$

$$\frac{}{\Gamma \vdash \texttt{get} : \texttt{Int}\ \&\ \{\mathsf{g}\}} \quad \frac{\Gamma \vdash t : \texttt{Int}\ \&\ E}{\Gamma \vdash \texttt{put}\ t : \texttt{Int}\ \&\ E \cup \{\mathsf{p}\}} \quad \frac{\Gamma \vdash t_1 : T\ \&\ E_1 \quad \Gamma \vdash t_2 : T\ \&\ E_2}{\Gamma \vdash t_1\ \texttt{or}\ t_2 : T\ \&\ E_1 \cup E_2 \cup \{\mathsf{a}\}}$$

$$\frac{\Gamma \vdash t_1 : \texttt{Int}\ \&\ E_1 \quad \Gamma \vdash t_2 : T\ \&\ E_2 \quad \Gamma \vdash t_3 : T\ \&\ E_3}{\Gamma \vdash \texttt{ifz}\ t_1\ \texttt{then}\ t_2\ \texttt{else}\ t_3 : T\ \&\ E_1 \cup E_2 \cup E_3} \quad \frac{\Gamma \vdash t : (T \xrightarrow{E_1} T) \xrightarrow{E_2} (T \xrightarrow{E_1} T)\ \&\ E_3}{\Gamma \vdash \texttt{fix}\ t : T \xrightarrow{E_1} T\ \&\ E_2 \cup E_3}$$

$$\frac{\Gamma \vdash t : T_1\ \&\ E_1 \quad T_1 <: T_2 \quad E_1 \subseteq E_2}{\Gamma \vdash t : T_2\ \&\ E_2}$$

1. **Syntax.**
   Fully parenthesize the following λ-terms.
   (a) $\lambda a.\lambda b.a$
   (b) $\lambda a.(\lambda b.b)\, a$
   (c) $\lambda a.\lambda b.b\; a$
   (d) $\lambda a.a + 2$
   (e) $(\lambda a.\lambda b.a)\, b$
   (f) $\lambda a.\texttt{get} + (\lambda b.b)\, 3$

2. **Evaluation.**
   Derive the following judgments.
   (a) $\texttt{put}\,(\texttt{get} + 1) \mid 3 \Downarrow 4 \mid 4$
   (b) $(\lambda x.\texttt{get} + x)\,(\texttt{put}\,4) \mid 1 \Downarrow 5 \mid 4$
   (c) $\texttt{put}\,((\texttt{get}\;\texttt{or}\;1) + 2) \mid 2 \Downarrow 4 \mid 4$
   (d) $(\lambda a.a + \texttt{get})\,((\texttt{put}\,3)\;\texttt{or}\;3) \mid 1 \Downarrow 6 \mid 3$
   (e) $(\lambda a.a + \texttt{get})\,((\texttt{put}\,3)\;\texttt{or}\;3) \mid 1 \Downarrow 4 \mid 1$

3. **Subtyping.**
   Derive the following judgments
   (a) $\texttt{Int} \xrightarrow{\emptyset} \texttt{Int} <: \texttt{Int} \xrightarrow{\mathsf{g}} \texttt{Int}$
   (b) $\texttt{Int} \xrightarrow{\emptyset} (\texttt{Int} \xrightarrow{\emptyset} \texttt{Int}) <: \texttt{Int} \xrightarrow{\emptyset} (\texttt{Int} \xrightarrow{\mathsf{g}} \texttt{Int})$
   (c) $(\texttt{Int} \xrightarrow{\mathsf{g}} \texttt{Int}) \xrightarrow{\emptyset} \texttt{Int} <: (\texttt{Int} \xrightarrow{\emptyset} \texttt{Int}) \xrightarrow{\emptyset} \texttt{Int}$
   (d) $(\texttt{Int} \xrightarrow{\mathsf{gp}} \texttt{Int}) \xrightarrow{\emptyset} (\texttt{Int} \xrightarrow{\emptyset} \texttt{Int}) <: (\texttt{Int} \xrightarrow{\mathsf{g}} \texttt{Int}) \xrightarrow{\emptyset} (\texttt{Int} \xrightarrow{\mathsf{a}} \texttt{Int})$

4. **Typing.**
   Derive the following judgments
   (a) $\{x \mapsto \texttt{Int}\} \vdash \texttt{put}\,x\;\texttt{or}\;3 : \texttt{Int}\ \&\ \{\mathsf{g}, \mathsf{a}\}$
   (b) $\emptyset \vdash \lambda a.\texttt{put}\,a : \texttt{Int} \xrightarrow{\mathsf{pa}} \texttt{Int}\ \&\ \emptyset$
   (c) $\{x \mapsto \texttt{Int}\} \vdash \texttt{ifz}\;x\;\texttt{then}\;\texttt{get}\;\texttt{else}\;\texttt{put}\;x : \texttt{Int}\ \&\ \{\mathsf{g}, \mathsf{p}\}$
   (d) $\{x \mapsto \texttt{Int}\} \vdash \texttt{ifz}\;x\;\texttt{then}\;\lambda y.\texttt{get}\;\texttt{else}\;\lambda y.\texttt{put}\;x : \texttt{Int} \xrightarrow{\mathsf{gp}} \texttt{Int}\ \&\ \emptyset$
   (e) $\emptyset \vdash (\lambda a.\texttt{put}\,a)\,3 : \texttt{Int}\ \&\ \{\mathsf{p}\}$
   (f) $\emptyset \vdash (\lambda a.(\lambda b.\lambda c.\texttt{put}\,(b + c))\,\texttt{get})\,3 : \texttt{Int} \xrightarrow{\mathsf{p}} \texttt{Int}\ \&\ \{\mathsf{g}\}$

Parametric Polymorphism

Syntax.

$$\begin{array}{rll} \mathcal{T} \ni t ::= & z \mid t + t \mid x \mid \lambda x.t \mid t\ t \mid (t, t) \mid \texttt{fst}\ t \mid \texttt{snd}\ t \mid \texttt{let}\ x = t\ \texttt{in}\ t & \text{Terms} \\ \mathcal{Y} \ni T ::= & \alpha \mid \texttt{Int} \mid T \to T \mid (T, T) & \text{Types} \\ \mathcal{S} \ni S ::= & T \mid \forall \alpha.S & \text{Type schemes} \end{array}$$

Typing.

$$\frac{}{\Gamma \vdash z : \texttt{Int}} \qquad \frac{\Gamma \vdash t_1 : \texttt{Int} \quad \Gamma \vdash t_2 : \texttt{Int}}{\Gamma \vdash t_1 + t_2 : \texttt{Int}}$$

$$\frac{}{\Gamma \vdash x : \Gamma(x)} \qquad \frac{\Gamma[x \mapsto T_1] \vdash t : T_2}{\Gamma \vdash \lambda x.t : T_1 \to T_2} \qquad \frac{\Gamma \vdash t_1 : T_1 \to T_2 \quad \Gamma \vdash t_2 : T_1}{\Gamma \vdash t_1\ t_2 : T_2}$$

$$\frac{\Gamma \vdash t_1 : T_1 \quad \Gamma \vdash t_2 : T_2}{\Gamma \vdash (t_1, t_2) : (T_1, T_2)} \qquad \frac{\Gamma \vdash t : (T_1, T_2)}{\Gamma \vdash \texttt{fst}\ t : T_1} \qquad \frac{\Gamma \vdash t : (T_1, T_2)}{\Gamma \vdash \texttt{snd}\ t : T_2}$$

$$\frac{\Gamma \vdash t_1 : S \quad \Gamma[x \mapsto S] \vdash t_2 : T}{\Gamma \vdash \texttt{let}\ x = t_1\ \texttt{in}\ t_2 : T} \qquad \frac{\Gamma \vdash t : S}{\Gamma \vdash t : \forall \alpha.S}\ (\alpha \notin fv(\Gamma)) \qquad \frac{\Gamma \vdash t : \forall \alpha.S}{\Gamma \vdash t : S[T/\alpha]}$$

Instances.

$$\lfloor T \rfloor = \{T\} \qquad \lfloor \forall \alpha.S \rfloor = \bigcup_{T \in \mathcal{Y}} \lfloor S[T/\alpha] \rfloor$$

5. **Type schemes.**

   Justify the following (in)equalities.

   (a) $\lfloor \forall\alpha.\alpha \to \alpha \rfloor = \lfloor \forall\beta.\beta \to \beta \rfloor$.

   (b) $\lfloor \forall\alpha.\alpha \rfloor \supsetneq \lfloor \forall\alpha.\alpha \to \alpha \rfloor$.

   (c) $\lfloor \forall\alpha.\forall\beta.(\alpha, \beta) \rfloor = \lfloor \forall\beta.\forall\alpha.(\alpha, \beta) \rfloor$

6. **Typing with polymorphism.**

   Derive the following judgments.

   (a) $\emptyset \vdash \lambda a.\mathtt{fst}\; a : \forall\alpha.\forall\beta.(\alpha, \beta) \to \alpha$

   (b) $\emptyset \vdash \lambda a.\lambda b.a : \forall\alpha.\forall\beta.\alpha \to \beta \to \alpha$

   (c) $\emptyset \vdash \lambda a.\lambda b.a : \forall\alpha.\alpha \to \alpha \to \alpha$

   (d) $\emptyset \vdash \lambda a.\lambda b.\mathtt{fst}\; a + b : \forall\alpha.(\mathtt{Int}, \alpha) \to \mathtt{Int} \to \mathtt{Int}$