Topic 1: Proposition Logic

Read: Chpt.1.1, Rosen

Def: A ***proposition*** is a ***declarative statement*** that is either true or false, but not both.

Examples:

1. Today is Tuesday.
2. Yesterday is Sunday.
3. Dance 210 is required for all CS majors at KU.
4. Have you passed EECS168 yet?
5. It is cold today.
6. All dogs go to heaven.
7. Andrew is a student in the EECS Dept. at KU.
8. $x + 10 = y$.
9. I always lie.

Q: Given a proposition, how do you determine the truthfulness of a proposition?

Forming new propositions using connectives:

Connectives are operators that can be used to transform/combine proposition(s) according to a specific set of rules.

Some useful connectives:

Negation:	$\neg$	(unary operator)
Conjunction:	$\wedge$	(binary operator)
Disjunction:	$\vee$	
Exclusive-or:	$\oplus$	
Implication:	$\rightarrow$	
Biconditional:	$\leftrightarrow$	

Notation:

p, q, …, z — propositions, T (true), F (false);
P, Q, …, Z — compound propositions formed by combining propositions together using connectives.

1. **Negation:** $\neg p$ (**Not** p)

   $\neg p$ is true when p is false; false otherwise.

Remark: The relation between a compound proposition Q(p,q,r,…) and its building propositions p, q, r, … can always be characterized using a ***truth table***.

Example: The truth table for $\neg p$:

p	**¬p**
T	**F**
F	**T**

2. **Conjunction:** $p \wedge q$ (p **and** q)
   $p \wedge q$ is true when both p and q are true; false otherwise.

p	**q**	**p ∧ q**
T	T	**T**
T	F	**F**
F	T	**F**
F	F	**F**

3. **Disjunction:** $p \vee q$ (p **or** q)
   $p \vee q$ is true when either p, or q **(or both)** is true; false otherwise.

p	**q**	**p ∨ q**
T	T	**T**
T	F	**T**
F	T	**T**
F	F	**F**

Q: What is the negation of these two propositions?

Consider the truth table for propositions $\neg(p \wedge q)$ and $\neg p \vee \neg q$.

p	**q**	**$p \wedge q$**	**$\neg(p \wedge q)$**	**$\neg p \vee \neg q$**
T	T	T	**F**	**F**
T	F	F	**T**	**T**
F	T	F	**T**	**T**
F	F	F	**T**	**T**

Observe that both $\neg(p \wedge q)$ and $\neg p \vee \neg q$ are characterized by the same truth table.

Whenever two propositions are characterized by the same truth table, they are said to be ***logically equivalent***.

HW: By using truth table, prove that $\neg(p \vee q)$ and $\neg p \wedge \neg q$ are logically equivalent.

4. **Exclusive-Or:** $p \oplus q$.

   $p \oplus q$ is true when ***exactly one*** of p or q is true; false otherwise.

p	**q**	**$p \oplus q$**
T	T	**F**
T	F	**T**
F	T	**T**
F	F	**F**

HW: Rewrite $p \oplus q$ in terms of p and q without using the $\oplus$ operator.

5. **Implication:** $p \to q$.

$p \to q$ is false when p is true and q is false; true otherwise.

p	**q**	$p \to q$
T	T	T
T	F	F
F	T	T
F	F	T

The statement p is the **hypothesis** and q is the **conclusion** of the implication $p \to q$.

Example: Consider the following implication P(p,q):

If today is Friday, then all CS graduates are millionaires.

Observe that, by definition, if today is not a Friday, then this implication is always true whether all CS graduate are millionaires or not!

However, this implication is false if today is indeed Friday but not all CS graduates are millionaires.

The following statements are the same for the implication $p \rightarrow q$:

1. p ***implies*** q
2. ***If*** p, ***then*** q
3. p ***only if*** q
4. q ***whenever*** p
5. p ***is sufficient for*** q
6. q ***is necessary for*** p

Def: The ***converse*** of $p \rightarrow q$ is $q \rightarrow p$.
The ***inverse*** of $p \rightarrow q$ is $\neg p \rightarrow \neg q$.
The ***contrapositive*** of $p \rightarrow q$ is $\neg q \rightarrow \neg p$.

Example: Consider again the above implication P(p,q).
The converse of P(p,q) is:
If all CS graduates are millionaires, then today is a Friday.

The inverse of P(p,q) is:
If today is not a Friday, then not all CS majors are millionaires.

The contrapositive of P(p,q) is:
If not all CS majors are millionaires, then today is not a Friday.

HW: Rewrite the propositions $p \rightarrow q$, $q \rightarrow p$, and $\neg q \rightarrow \neg p$ without using the "$\rightarrow$" operator.

6. **Biconditional:** $p \leftrightarrow q$.

$p \leftrightarrow q$ is true when p and q have the same truth value; false otherwise.

p	**q**	$\mathbf{p \leftrightarrow q}$	$\mathbf{p \rightarrow q}$	$\mathbf{q \rightarrow p}$	$\mathbf{(p \rightarrow q) \wedge (q \rightarrow p)}$
T	T	**T**	T	T	**T**
T	F	**F**	F	T	**F**
F	T	**F**	T	F	**F**
F	F	**T**	T	T	**T**

Observe that $p \leftrightarrow q$ is true when both $p \rightarrow q$ and $q \rightarrow p$ are true.

HW: Rewrite the proposition $p \leftrightarrow q$ without using the "$\leftrightarrow$" operator.

The following statements are the same for $p \leftrightarrow q$:

1. p ***is equivalent to*** q
2. p implies q and q implies p
3. p ***if and only if*** q (p **iff** q)
4. p is ***both necessary and sufficient*** for q

Q: What are the advantages/disadvantages in using a truth table?

Examples:

1. Consider $(p \wedge q) \vee (\neg q \wedge r)$.

   Number of propositions = 3,
   Number of combinations = $2^3 = 8$.

p	**q**	**r**	$\mathbf{p \wedge q}$	$\mathbf{\neg q \wedge r}$	$\mathbf{(p \wedge q) \vee (\neg q \wedge r)}$
T	T	T	T	F	**T**
T	T	F	T	F	**T**
T	F	T	F	T	**T**
T	F	F	F	F	**F**
F	T	T	F	F	**F**
F	T	F	F	F	**F**
F	F	T	F	T	**T**
F	F	F	F	F	**F**

2. $(p \rightarrow q) \vee (\neg p \leftrightarrow r)$

p	**q**	**r**	$\mathbf{p \rightarrow q}$	$\mathbf{\neg p \leftrightarrow r}$	$\mathbf{(p\rightarrow q) \vee (\neg p\leftrightarrow r)}$
T	T	T	T	F	**T**
T	T	F	T	T	**T**
T	F	T	F	F	**F**
T	F	F	F	T	**T**
F	T	T	T	T	**T**
F	T	F	T	F	**T**
F	F	T	T	T	**T**
F	F	F	T	F	**T**

Remark:

If a proposition $P(q_1, q_2, \ldots, q_n)$ is formed using n (atomic) propositions, the corresponding truth table for $P(q_1, q_2, \ldots, q_n)$ will have exactly 2^n rows corresponding to the 2^n different combination of values of $q_1, q_2, \ldots, q_n$. Hence, truth table is only "practical" for propositions with few atomic propositions!

Evaluating Propositions:
For our previous operators,

Unary Operator:	$\neg$
Binary Operators:	$\wedge, \vee, \oplus$
	$\rightarrow, \leftrightarrow$

Precedence rules and operator hierarchy:

$\neg$ (highest), $\wedge$, $\vee$, $\oplus$, $\rightarrow$, $\leftrightarrow$ (lowest)

1. Binary operators are left associative.
2. Parentheses can be used to override precedence.

Example:
The proposition $\neg p \rightarrow q \vee r \leftrightarrow s \wedge \neg t \rightarrow u$ will be evaluated as $(((\neg p) \rightarrow (q \vee r)) \leftrightarrow ((s \wedge (\neg t))) \rightarrow u)$.

HW: Evaluate the above proposition if p, q, s are true and r, t, u are false.

Practice HW: Chpt.1.1: 5, 7, 9, 11, 17, 23, 25, 27, 29, 31.

Algebra of Propositions

Read: Chpt.1.3, Rosen

Q: Given (compound) propositions p and q.
How do we prove that p and q are logically equivalent?

Def. A proposition p is a ***tautology*** if it is always true. It is a ***contradiction*** if it is always false.

Def. Two propositions p and q are ***logically equivalent*** if they have identical truth tables.
Hence, $p \leftrightarrow q$ must be a tautology.

Notation: If p is logically equivalent to q, we denote it by using $p \Leftrightarrow q$, or $p \equiv q$.

Q: What is the difference between "$\leftrightarrow$" and "$\Leftrightarrow$"?
What is the difference between $p \leftrightarrow q$ and $p \Leftrightarrow q$?

Proving logical equivalence:

Q: How do we prove that p $\Leftrightarrow$ q?

1. Truth table (**exhaustive method**)
2. Direct method using
   (a) Case analysis
   (b) Algebraic manipulation

Examples:

1. Consider the implication p $\rightarrow$ q and its contrapositive $\neg$q $\rightarrow$ $\neg$p.

p	**q**	**$\neg$p**	**$\neg$q**	**p $\rightarrow$ q**	**$\neg$q $\rightarrow$ $\neg$p**
T	T	F	F	**T**	**T**
T	F	F	T	**F**	**F**
F	T	T	F	**T**	**T**
F	F	T	T	**T**	**T**

Observe that this truth table for p $\rightarrow$ q is identical to the truth table for $\neg$q $\rightarrow$ $\neg$p.

Hence, p $\rightarrow$ q and its contrapositive $\neg$q $\rightarrow$ $\neg$p are logically equivalent and (p $\rightarrow$ q) $\Leftrightarrow$ ($\neg$q $\rightarrow$ $\neg$p).

2. Let P(p,q,r) and Q(p,r) be two compound propositions formed by propositions p, q, r such that

$$P = (p \to q) \wedge (q \to r) \text{ and } Q = p \to r.$$

Q: Are they equivalent?

p	**q**	**r**	**p→q**	**q→r**	**(p→q) ∧ (q→r)**	**p→r**
T	T	T	T	T	T	T
T	T	F	T	F	F	F
T	**F**	**T**	**F**	**T**	**F**	**T**
T	F	F	F	T	F	F
F	T	T	T	T	T	T
F	**T**	**F**	**T**	**F**	**F**	**T**
F	F	T	T	T	T	T
F	F	F	T	T	T	T

Hence, P $\not\Leftrightarrow$ Q.

HW: What about the implications

$$((p \to q) \wedge (q \to r)) \to (p \to r) \text{ and}$$
$$(p \to r) \to ((p \to q) \wedge (q \to r))?$$

Q: How do we prove the logical equivalence of two propositions without using truth tables?

Direct methods

3. Prove that $p \rightarrow q \Leftrightarrow \neg q \rightarrow \neg p$.

Proof. Consider the proposition $p \rightarrow q$.

If $p \rightarrow q = F$, then $p = T$ and $q = F$. Hence, $\neg q \rightarrow \neg p = F$ ($T \rightarrow F = F$).

If $p \rightarrow q = T$, we have the following two cases.

(1) If $p = T$, then q must also be T, implying that $\neg q \rightarrow \neg p = T$ ($F \rightarrow F = T$).

(2) If $p = F$, then $\neg p = T$, implying that $\neg q \rightarrow \neg p$ is always T ($\neg q \rightarrow T = T$).

Hence, $p \rightarrow q \leftrightarrow \neg q \rightarrow \neg p$ is a tautology and $p \rightarrow q \Leftrightarrow \neg q \rightarrow \neg p$.

Remark: We can also study and identify some useful logical equivalence relations and then use them to discover the logical equivalence relations of other propositions.

Some Useful Laws of Logical Equivalence:

1. $p \vee F \Leftrightarrow p$ **Identity Laws**
   $p \wedge T \Leftrightarrow p$

2. $p \vee T \Leftrightarrow T$ **Domination Laws**
   $p \wedge F \Leftrightarrow F$

3. $p \vee p \Leftrightarrow p$ **Idempotent Laws**
   $p \wedge p \Leftrightarrow p$

4. $\neg(\neg p) \Leftrightarrow p$ **Involution Law (Double Negation Law)**

5. $p \vee \neg p \Leftrightarrow T$ **Complement Laws (Negation Laws)**
   $p \wedge \neg p \Leftrightarrow F$

6. $(p \vee q) \vee r \Leftrightarrow p \vee (q \vee r)$ **Associative Laws**
   $(p \wedge q) \wedge r \Leftrightarrow p \wedge (q \wedge r)$

7. $p \vee q \Leftrightarrow q \vee p$ **Commutative Laws**
   $p \wedge q \Leftrightarrow q \wedge p$

8. $p \vee (q \wedge r) \Leftrightarrow (p \vee q) \wedge (p \vee r)$ **Distributive Laws**
   $p \wedge (q \vee r) \Leftrightarrow (p \wedge q) \vee (p \wedge r)$

9. $\neg(p \land q) \Leftrightarrow \neg p \lor \neg q$ **De Morgan's Laws**

$\neg(p \lor q) \Leftrightarrow \neg p \land \neg q$

10. $p \lor (p \land q) \Leftrightarrow p$ **Absorption Laws**

$p \land (p \lor q) \Leftrightarrow p$

Generalization and Extension:

Generalized Associative Laws:

$((p \lor q) \lor r) \lor \ldots \Leftrightarrow (p \lor (q \lor r)) \lor \ldots \Leftrightarrow p \lor q \lor r \lor \ldots$

$((p \land q) \land r) \land \ldots \Leftrightarrow (p \land (q \land r)) \land \ldots \Leftrightarrow p \land q \land r \land \ldots$

Generalized Commutative Laws:

$p \lor q \lor r \lor \ldots \Leftrightarrow q \lor r \lor p \lor \ldots \Leftrightarrow r \lor p \lor q \lor \ldots \Leftrightarrow \ldots$

$p \land q \land r \land \ldots \Leftrightarrow q \land r \land p \land \ldots \Leftrightarrow r \land p \land q \land \ldots \Leftrightarrow \ldots$

Generalized De Morgan's Laws:

$\neg(p \land q \land \ldots \land z \land \ldots) \Leftrightarrow \neg p \lor \neg q \lor \ldots \lor \neg z \lor \ldots$

$\neg(p \lor q \lor \ldots \lor z \lor \ldots) \Leftrightarrow \neg p \land \neg q \land \ldots \land \neg z \land \ldots$

HW: Study and memorize the laws of logical equivalences given in Tables 6-8 on Page 27-28.

Examples in Proving the Laws of Equivalence:

1. $\neg(\neg p) \Leftrightarrow p$ (Involution Law):

p	**¬p**	**¬(¬p)**
T	F	T
F	T	F

2. $p \vee \neg p \Leftrightarrow T$ (Complement Law):

p	**¬p**	**p ∨ ¬p**
T	F	T
F	T	T

Hence, $p \vee \neg p$ is a tautology!

3. $p \wedge (q \vee r) \Leftrightarrow (p \wedge q) \vee (p \wedge r)$ (Distribution Law)

p	**q**	**r**	**q ∨ r**	**p ∧ (q∨r)**	**p ∧ q**	**p ∧ r**	**(p∧q) ∨ (p∧r)**
T	T	T	T	T	T	T	T
T	T	F	T	T	T	F	T
T	F	T	T	T	F	T	T
T	F	F	F	F	F	F	F
F	T	T	T	F	F	F	F
F	T	F	T	F	F	F	F
F	F	T	T	F	F	F	F
F	F	F	F	F	F	F	F

Hence, $p \wedge (q \vee r) \Leftrightarrow (p \wedge q) \vee (p \wedge r)$.

Applications:

1. $p \vee \neg(p \wedge q) \Leftrightarrow ?$

$p \vee \neg(p \wedge q)$	
$\Leftrightarrow p \vee (\neg p \vee \neg q)$	De Morgan's Law
$\Leftrightarrow (p \vee \neg p) \vee \neg q$	Associative Law
$\Leftrightarrow T \vee \neg q$	Complement Law
$\Leftrightarrow T$	Domination Law

Hence, $p \vee \neg (p \wedge q)$ is a tautology.

2. $\neg(p \vee q) \vee (\neg p \wedge q) \Leftrightarrow ?$

$\neg(p \vee q) \vee (\neg p \wedge q)$	
$\Leftrightarrow (\neg p \wedge \neg q) \vee (\neg p \wedge q)$	De Morgan's Law
$\Leftrightarrow \neg p \wedge (\neg q \vee q)$	Distributive Law
$\Leftrightarrow \neg p \wedge T$	Complement Law
$\Leftrightarrow \neg p$	Identity Laws

3. $\neg q \wedge (p \wedge (\neg p \vee q)) \Leftrightarrow ?$

$\neg q \wedge (p \wedge (\neg p \vee q))$	
$\Leftrightarrow \neg q \wedge ((p \wedge \neg p) \vee (p \wedge q))$	Distributive Law
$\Leftrightarrow \neg q \wedge (F \vee (p \wedge q))$	Complement Law
$\Leftrightarrow \neg q \wedge (p \wedge q)$	Identity Law
$\Leftrightarrow \neg q \wedge (q \wedge p)$	Commutative Law
$\Leftrightarrow (\neg q \wedge q) \wedge p$	Associative Law
$\Leftrightarrow F \wedge p$	Complement Law
$\Leftrightarrow F$	Domination Law

4. $(p \wedge \neg q) \vee (\neg p \wedge q) \Leftrightarrow \neg(p \wedge q) \wedge (p \vee q)$

$(p \wedge \neg q) \vee (\neg p \wedge q)$

$\Leftrightarrow ((p \wedge \neg q) \vee \neg p) \wedge ((p \wedge \neg q) \vee q)$ Distributive Law

$\Leftrightarrow (\neg p \vee (p \wedge \neg q)) \wedge (q \vee (p \wedge \neg q))$ Commutative Law

$\Leftrightarrow ((\neg p \vee p) \wedge (\neg p \vee \neg q)) \wedge ((q \vee p) \wedge (q \vee \neg q))$ Distributive Law

$\Leftrightarrow (T \wedge (\neg p \vee \neg q)) \wedge ((q \vee p) \wedge T)$ Complement Law

$\Leftrightarrow (\neg p \vee \neg q) \wedge (q \vee p)$ Identity Law

$\Leftrightarrow \neg(p \wedge q) \wedge (q \vee p)$ De Morgan Law

$\Leftrightarrow \neg(p \wedge q) \wedge (p \vee q)$ Commutative Law

Practice HW: Chpt.1.3: 9, 11, 13, 15, 17, 23, 31, 35, 41, 43, 45, 61.

Give a proposition P formed by a small number of (simple) propositions, we can usually construct a truth table for P easily.

Q: Given a truth table, how do we construct a proposition P such that P has the truth values as characterized by the truth table?

Example: $P(p,q) = p \oplus q$.

p	**q**	$\mathbf{p \oplus q}$
T	T	F
T	F	T
F	T	T
F	F	F

Q: For what values of p and q will P be true?

A: Observe from the truth table for P that P is true when ($p = T$ and $q = F$) or ($p = F$ and $q = T$).
Hence, $P(p,q) \equiv (p \wedge \neg q) \vee (\neg p \wedge q)$.

This expression is called the **disjunctive normal form (DNF)** of P, which is a disjunction of conjunctions of propositions such that each proposition can appear at most once in each conjunction of propositions.

Similarly, we can consider the values of p and q for which P is false.

Observe that P is false when (p = T and q = T) or (p = F and q = F).

Hence,

$$\begin{aligned} P(p,q) &\equiv \neg((p \land q) \lor (\neg p \land \neg q)) & \\ &\equiv \neg(p \land q) \land \neg(\neg p \land \neg q) & \text{De Morgan Law} \\ &\equiv (\neg p \lor \neg q) \land (\neg(\neg p) \lor \neg(\neg q)) & \text{De Morgan Law} \\ &\equiv (\neg p \lor \neg q) \land (p \lor q) & \text{Involution Law} \end{aligned}$$

The expression $(\neg p \lor \neg q) \land (p \lor q)$ is called the **conjunctive normal form (CNF)** of P, which is a conjunction of disjunctions of propositions such that each proposition can appear at most once in each disjunction of propositions.

Hence, $p \oplus q \equiv (p \land \neg q) \lor (\neg p \land q)$

$$\equiv (\neg p \lor \neg q) \land (p \lor q).$$

Observation: Any proposition can be formed by using negation, disjunction, and conjunction operators only!

Example:
Construct a propositional function P(p,q,r) such that P is true whenever exactly two of p, q, and r are true.

From the given functional description of P(p,q,r), one can construct the following truth table for P(p,q,r).

p	**q**	**r**	**P**
T	T	T	F
T	T	F	T
T	F	T	T
T	F	F	F
F	T	T	T
F	T	F	F
F	F	T	F
F	F	F	F

DNF of P(p,q,r):

$$P(p,q,r) = (p \wedge q \wedge \neg r) \vee (p \wedge \neg q \wedge r) \vee (\neg p \wedge q \wedge r).$$

CNF of P(p,q,r):

$$\begin{aligned} P(p,q,r) = {} & (\neg p \vee \neg q \vee \neg r) \wedge (\neg p \vee q \vee r) \wedge \\ & (p \vee \neg q \vee r) \wedge (p \vee q \vee \neg r) \wedge (p \vee q \vee r). \end{aligned}$$

Application to Digital Circuit Synthesis:

Proposition ↔ Boolean function

True-value ↔ 1

False-value ↔ 0

Basic Design

↓

Functional Description

↓

Truth table

↓

Logical function

↓

Simplification

↓

Circuit

Conclusion:

Any digital circuit can be constructed by using NOT-gate, OR-gate, and AND-gate only! They form a **Complete Family of Logic Gates**!

HW: Read Chpt.1.2, Rosen, and review logic circuits.

More DNF and CNF:

1. Given the truth table for P(p,q).

p	q	?
T	T	T
T	F	T
F	T	F
F	F	T

DNF of P(p,q): $(p\wedge q) \vee (p\wedge\neg q) \vee (\neg p\wedge\neg q)$
CNF of P(p,q): $(p\vee\neg q)$

2. Given the truth table for P(p,q).

p	q	?
T	T	T
T	F	T
F	T	T
F	F	T

DNF of P(p,q): $(p\wedge q) \vee (p\wedge\neg q) \vee (\neg p\wedge q) \vee (\neg p\wedge\neg q)$
CNF of P(p,q): Doesn't exist (or just T).

3. Eliminating the disjunction operator in P(p,q).
P(p,q)
$= (p\wedge q) \vee (p\wedge\neg q) \vee (\neg p\wedge\neg q)$
$\Leftrightarrow \neg\neg((p\wedge q) \vee (p\wedge\neg q) \vee (\neg p\wedge\neg q))$
$\Leftrightarrow \neg(\ \neg(p\wedge q) \wedge \neg(p\wedge\neg q) \wedge \neg(\neg p\wedge\neg q))$

More Application:
Recall that, in any programming language, the order of execution can be altered by using conditional branching.

Example:

```
if ( boolean-expression )
    yes_statements
else
    no_statements
```

Must know how to evaluate the Boolean expression, which is just a proposition!

```
int x = 1;
int y = 6;
int z = 8;
if ((!(x > y) || (z - x) >= y) && (x < z))
   x = x + 2*y;
else
   y = y + z;
```

Propositional Functions and Quantifiers

Read: Chpt.1.4, Rosen

Consider the following statements:

1. $x > 4$.
2. y is a prime number.
3. $a + 10 = b$
4. Andrew is a student in the EECS Dept. at KU.
5. $c^2 + d^2 = 1$

These are **not** propositions. *Why not?*

Example:

In (1), if $x = 1$, then $x > 4$ is false.
But if $x = 10$, then $x > 4$ is true.

The truth value of the above statements depends on the value of the variables x, y, a, b, John, c, and d.

They are propositional functions of the variables x, y, a, b, Andrew, c, d.

Defn: A statement $P(x_1,x_2,\ldots,x_n)$ is **a propositional function with n variables** $x_1,x_2,\ldots,x_n$ and **predicate** P if $P(x_1,x_2,\ldots,x_n)$ becomes a proposition whenever fixed values are assigned to $x_1, x_2, \ldots, x_n$.

The collection of values that the variable(s) can have is the **domain of disclosure** (universe of disclosure) of the variable(s).

Propositional function P(**x**) consist of

- Variable(s), **x**
- Domain of Disclosure, D
- Predicate (property that the variable(s) must meet), P

Examples:

Function	**Predicate**	**Variable(s)**	**DOD**
1. P(x)	$x > 4$	x	all real #
2. P(y)	y is prime	y	all integer > 1
3. P(a,b)	$a + 10 = b$	a, b	all integers
4. P(Andrew)	Andrew is a …	Andrew	all students
5. P(c,d)	$c^2 + d^2 = 1$	c,d	all real #

Evaluating Propositional Functions:

1. Let P(x) be the statement $x > 4$.
   P(1) = F and P(10) = T.

2. Let P(a,b) be the statement $a + 10 = b$.
   P(5,15) = T and P(2,8) = F.

Given a propositional function P(x) with domain of disclosure D.

Q: What will happen if we compute P(x) for each and every possible value of x in D?

Consider the following two cases:

1. For all x in D, P(x) is a tautology.
2. There exists x in D such that P(x) is true.

Remark: Other cases can be generated from the above two cases using logical operators.

By restricting the values of x in its domain of disclosure D, new proposition can be constructed from P(x).

Defn: Universal Quantification
For all values of x in D, P(x) is true.

Notation: $\forall x, P(x)$.

$\forall$: universal quantifier.

$\forall x$: for all x;
for every x.

Remark: If we can list all values of x in D as x_1, x_2, …, x_n, then we have

$$\forall x, P(x) \Leftrightarrow P(x_1) \wedge P(x_2) \wedge \ldots \wedge P(x_n).$$

Examples:

1. Let $P(x)$ be the propositional function $x > 4$ with D = all real numbers.
   Then, the proposition $\forall x, P(x)$ is the same as the statement: For every real number x, $x > 4$. Hence, it is false.
   However, if we take D = all real numbers greater than 10, then the proposition $\forall x, P(x)$ is true.

2. Let $P(x)$ be the propositional function $x^2 + 1 \geq 1$ with D = all real numbers.
   Then, the proposition $\forall x, P(x)$ is true.

Defn: Existential Quantification
There exists **x** in D such that P(**x**) is true.

Notation: $\exists x, P(x)$.

- $\exists$: existential quantifier.
- $\exists x$: there exist an x;
  there exist at least one x;
  for some x.

Remark: If we can list all values of x in D as x_1, x_2, …, x_n, then we have

$$\exists x, P(x) \Leftrightarrow P(x_1) \vee P(x_2) \vee \ldots \vee P(x_n).$$

Examples:

1. Let P(x) be the propositional function $x > 4$ with D = all real numbers.
   The proposition $\exists x, P(x)$ is true since P(5) is true.

2. Let P(x) be the propositional function $x^2 + 1 \geq 1$ with D = all real numbers.
   The proposition $\exists x, P(x)$ is true since P(1) is true.

3. Let P(x) be the propositional function $x^2 + 1 < 1$ with D = all real numbers.
   The proposition $\exists x, P(x)$ is false.

Remark: We can also apply logical operators to propositions with quantifiers.

Consider the negations of these two quantifiers:

$\forall x, P(x)$: For every x, P(x) is true.
$\neg(\forall x, P(x))$: There exists at least one x such that P(x) is false, which is $\exists x, \neg P(x)$.

$\exists x, P(x)$: There is an x such that P(x) is true.
$\neg(\exists x, P(x))$: There is no x such that P(x) is true.
Or, for every x, P(x) is false.
Hence, $\forall x, \neg P(x)$.

Thus,

Proposition	Negation
$\forall x, P(x)$ $\exists x, P(x)$	$\exists x, \neg P(x)$ $\forall x, \neg P(x)$

Remark: Recall that if we can list all values of x in D as $x_1, x_2, \ldots, x_n$, then we have

$$\forall x, P(x) \Leftrightarrow P(x_1) \wedge P(x_2) \wedge \ldots \wedge P(x_n), \text{ and}$$
$$\exists x, P(x) \Leftrightarrow P(x_1) \vee P(x_2) \vee \ldots \vee P(x_n).$$

Using Generalized De Morgan's Laws, we have

$$\begin{aligned} \neg(\forall x, P(x)) &\Leftrightarrow \exists x, \neg P(x) \\ &\Leftrightarrow \neg(P(x_1) \wedge P(x_2) \wedge \ldots \wedge P(x_n)) \\ &\Leftrightarrow \neg P(x_1) \vee \ldots \vee \neg P(x_n) \end{aligned}$$

$$\begin{aligned} \neg(\exists x, P(x)) &\Leftrightarrow \forall x, \neg P(x) \\ &\Leftrightarrow \neg(P(x_1) \vee P(x_2) \vee \ldots \vee P(x_n)) \\ &\Leftrightarrow \neg P(x_1) \wedge \ldots \wedge \neg P(x_n). \end{aligned}$$

Practice HW: Chpt.1.4: 7, 9, 11, 13, 17, 19, 21, 23, 25, 31, 33.

Propositions with Multiple and Nested Quantifiers

Read: Chpt.1.5, Rosen

Let P(x,y) be a propositional function with variables x, and y, and domain of disclosures D_x, and D_y, respectively.
We have

(1) $\forall x, y, P(x,y) \Leftrightarrow \forall x, \forall y, P(x,y)$
$\qquad\qquad\qquad\qquad \Leftrightarrow \forall y, \forall x, P(x,y)$,

which is true when P(x,y) is true for every pair of x, y.

(2) $\exists x, y, P(x,y) \Leftrightarrow \exists x, \exists y, P(x,y)$
$\qquad\qquad\qquad\qquad \Leftrightarrow \exists y, \exists x, P(x, y)$,

which is true when P(x,y) is true for a pair of x, y.

Remark: The above results hold for propositional functions with n variables, $n \geq 2$.

However,

$\boldsymbol{\forall x, \exists y, P(x,y) \not\Leftrightarrow \exists y, \forall x, P(x,y)}$.

Let's look at these two propositions more closely.

The proposition $\forall x, \exists y, P(x,y)$ is true whenever for every x in D_x, there exists a y in D_y such that $P(x,y)$ is true. Hence, ***different x may have different y***.

However, the proposition $\exists y, \forall x, P(x, y)$ is true whenever there exists a y in D_y such that for every x in D_x, $P(x,y)$ is true.
Hence, *the same y corresponds to all the x* in D_x.

Example:
Let $P(x,y)$ be the statement $x + y = 0$, D = all real numbers.

$\forall x, \exists y, P(x, y)$
$\Leftrightarrow$ For every real number x, there exists a real number y such that $x + y = 0$.
This is true since we can choose $y = -x$.

$\exists y, \forall x, P(x, y)$
$\Leftrightarrow$ There exist a real number y such that for every real number x, $x + y = 0$.
This is false since the only x that can be found to force $x + y = 0$ is $x = -y$.
Hence, $\forall x, \exists y\ P(x, y) \not\Leftrightarrow \exists y, \forall x\ P(x, y)$.

Practice HW: Chpt.1.5: 1, 3, 5, 9, 11, 25, 27.