EECS560 Data Structures Kong

Topic 5: Priority Queues & Heaps

Read: Chpt. 6, Weiss

"Good" characteristics of BST:

- Simplicity.
- Support general search/delete as well as special searchMin(Max)/deleteMin(Max) operations.
- Can easily be sorted (inorder traversal).
- Can easily be stored (preorder traversal).
- Good average performance, $T_a(n) = O(\lg n)$.

"Bad" characteristics of BST:

- Worst-case complexity depends on height of tree. Hence, $T_w(n) = O(n)$.
- Inefficient when many items are having identical keys since height may increase.

Q: Can we design an efficient ADT:Search Tree with $T_w(n) = T_a(n) = O(\lg n)$?

A: Yes, but much more complex.

Observation: In many applications, a less powerful data structure may be sufficient. What if general delete operations are not required or if they are only performed infrequently?

设计一个快速删除最大值或者最小值的ADT

Q: Can we design an efficient ADT that will allow us to remove only those data objects with maximum (or minimum) priority whenever a delete operation is performed?

These applications lead us to the design of a class of ADTs called *priority queues*.

ADT: Priority Queue.

A collection class, whose items have all been assigned a priority, supports the following operations:

1. ***PQInsert(in newItem: PQItemType)***
2. ***PQDelete(out priorityItem: PQItemType)***
3. createPQ():
4. destroyPQ():
5. PQIsEmpty():
6. PQSize():

Min PQ:

PQDelete will delete an object with min priority.

Max PQ:

PQDelete will delete an object with max priority.

Simplest Approaches:
Sorted and unsorted array/linked list.

Better Approach:
BST.

Best Approach:
k-Heaps, $k \geq 2$.

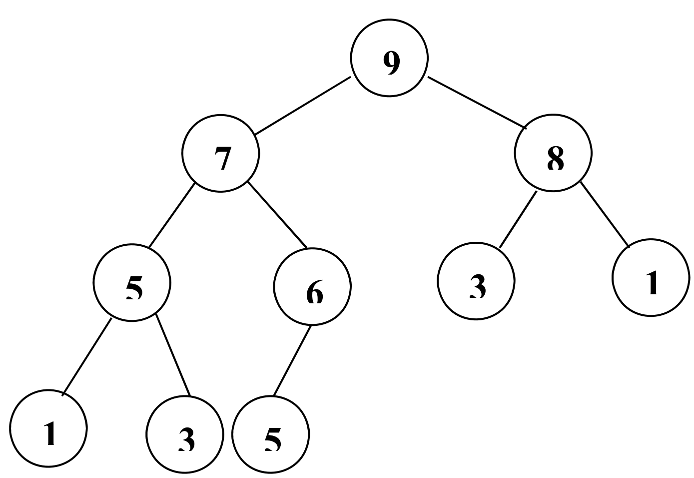
Defn: A max (min) ***k-heap*** is a k-ary tree H such that it satisfies the following properties:

(1) H is a complete k-ary tree, and

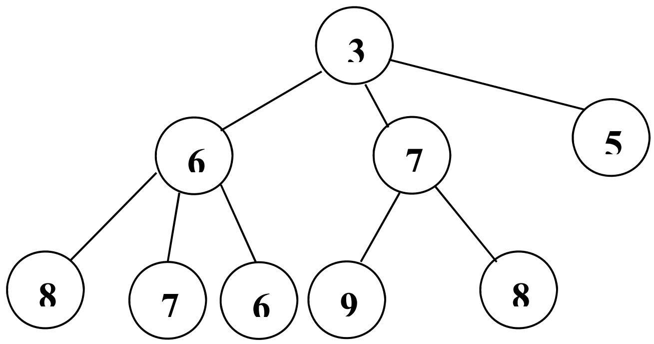
(2) ***Max (Min) Heap-Ordered Tree Property***:
Priority of any node in H $\geq$ ($\leq$) priority of all its descendants. Important!父母永远大于或小于所有的儿子

Remark: When $k = 2$, we have a 2-heap (heap).

Example: A max 2-heap H. Search: the max number O(1) = index 0 of array



Example: A min 3-heap H.



Implementations of k-Heaps, $k \geq 2$:

1. Pointer-based implementation:
   Inefficient. Why?

2. Array-based Implementation:
   For a k-heap H with n nodes, H can be implemented using an array A[0:maxSize −1] such that
   (1) Root of H at A[0],
   (2) Parent of A[i] at A[(i−1)/k] if exists,
   (3) The jth child of A[i] at A[ki+j], $1 \leq j \leq k$, if exists.

Remarks:

- When i = 0, (i−1)/k = −1, implying that A[i] is the root of H.
- For $n \geq 1$, A[i] is a leaf iff $ki \geq n-1$.

- Given A[i], the parent and the children of A[i] can be computed in O(1) time.
- H can also be implemented using a similar array structure by storing the root at A[1]. **(HW)**

PQ Operations: Two-steps process,

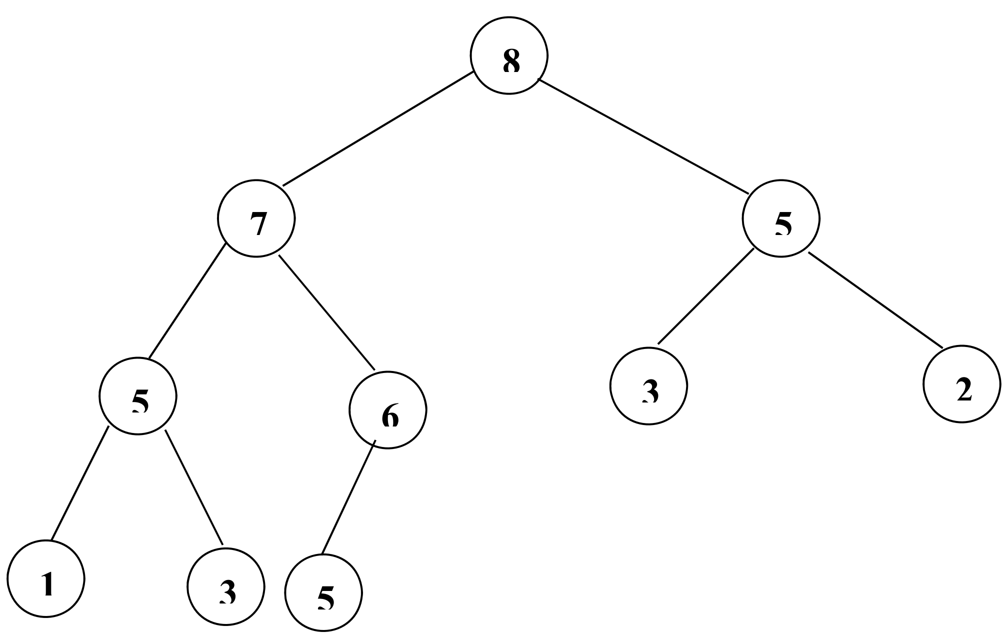
(1) After insertion/deletion, try to maintain a complete k-ary tree structure for H.

(2) Re-structure (Heapify) the complete k-ary tree from (1) so that it will satisfy the heap-ordered tree property.

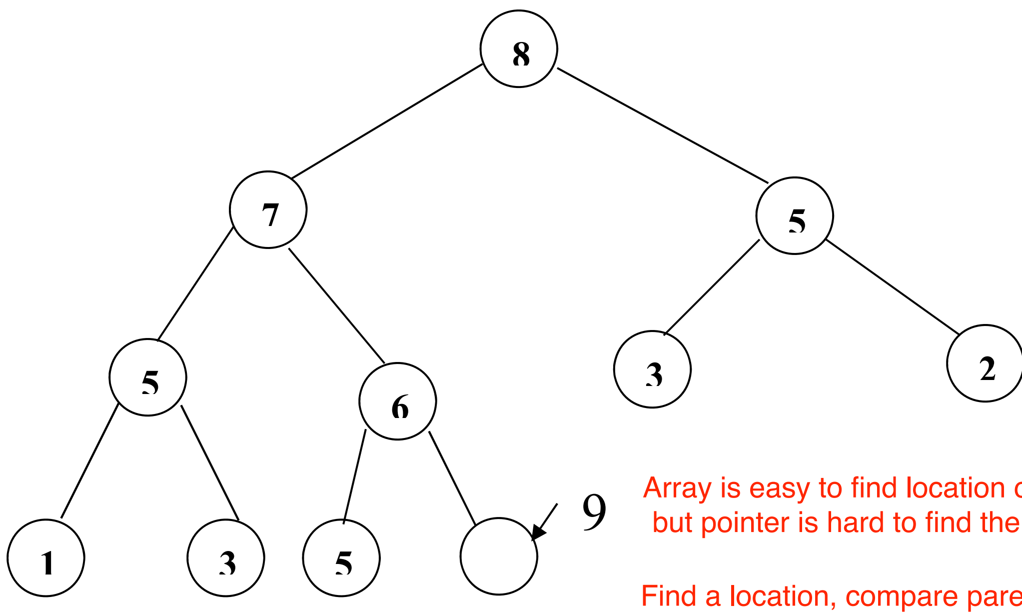
1. Insert(x,H):

Where can we find a location in H for insertion?

Consider inserting 9 into the following 2-heap H.



Q: Where will 9 go (in order to get back a complete binary tree)?



Array is easy to find location of insertion but pointer is hard to find the last node

Find a location, compare parents
parent = floor(child/2)

Heapifying/Restructuring resulting tree after insert:

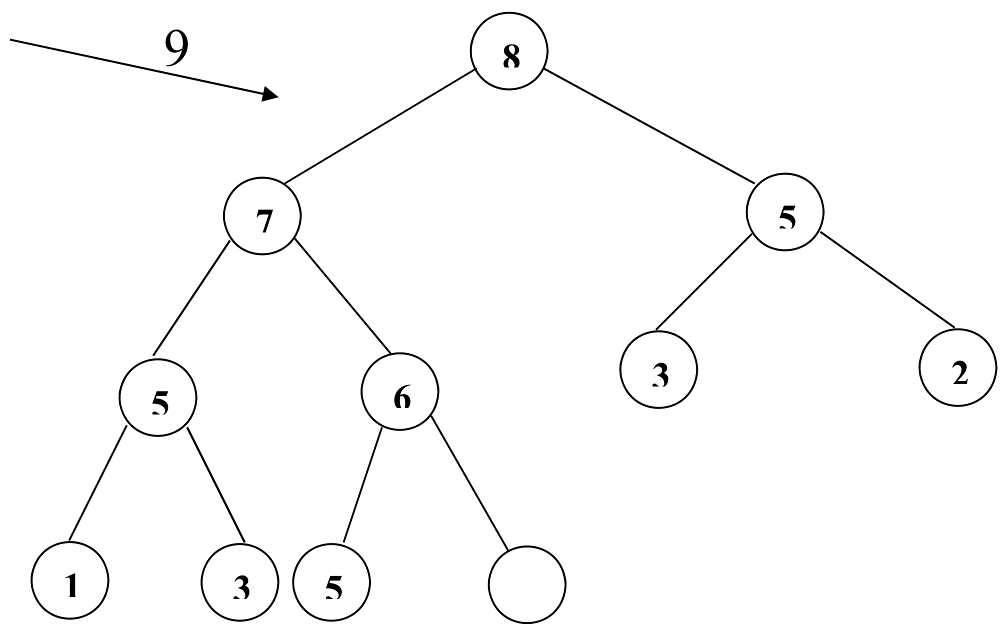
After inserting a node with x into H, the newly inserted node may or may not satisfy the heap-ordered tree property. If the heap-ordered tree property is violated, x must have priority higher than its parent. Hence, we can simply swap x with its parent and verify the heap-ordered tree property again by comparing the priority of x with its new parent.

In general, for inserting a new item x into H, we need to find a *location* to insert x along the path from x (after insertion) to the root of H by repeatedly comparing x with his parent, grandparent, ..., until either a node with priority ≥ x is found or the root of H is reached. Once the final location for x is found, it will then be inserted.

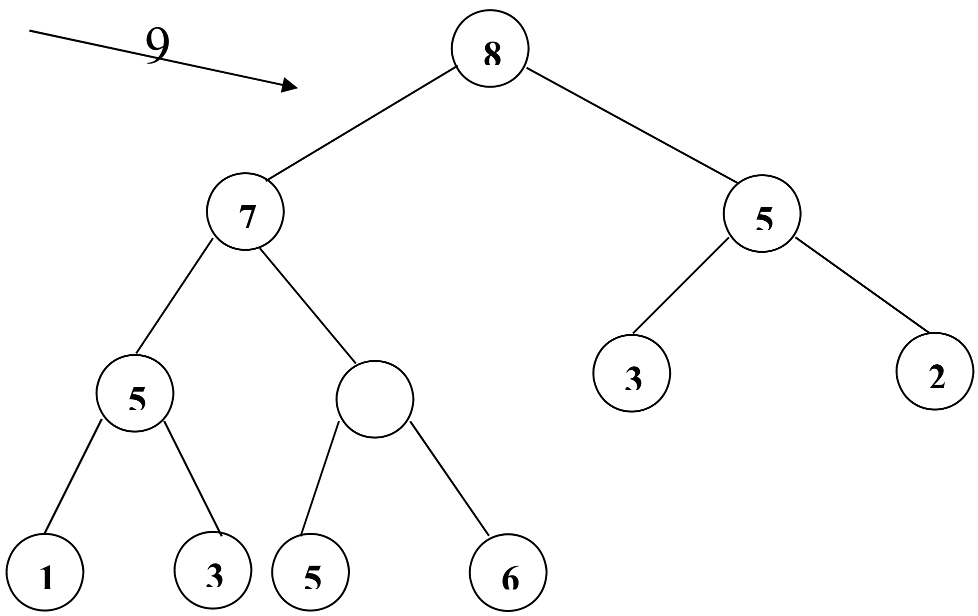
Remark: Do not insert and remove x repeatedly. Find the final location for x and then insert it. Once x is inserted, it stays.

Example: Consider inserting 9 into the heap H.

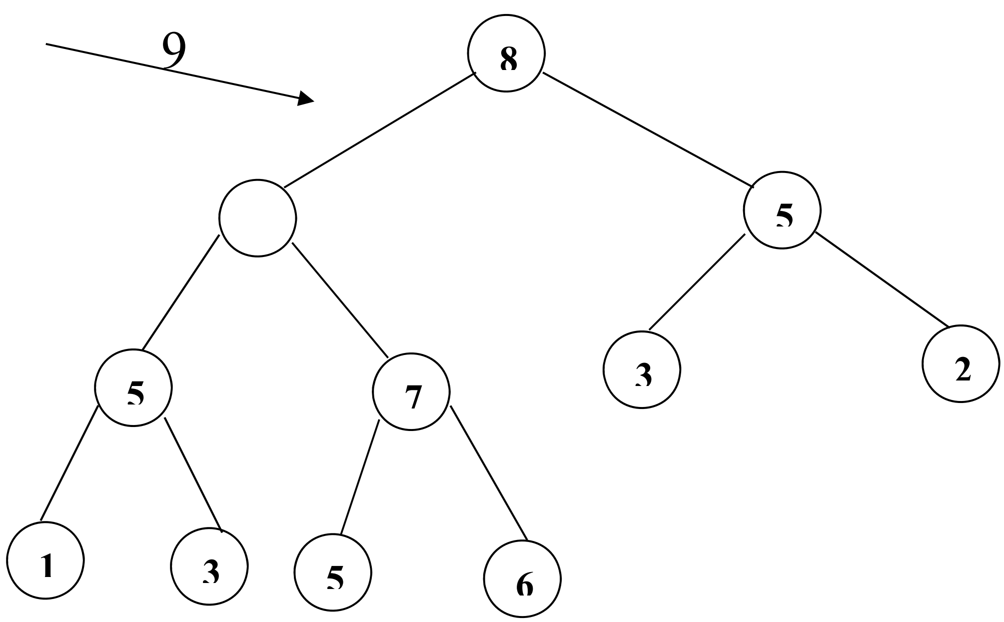
Create a new location for 9:



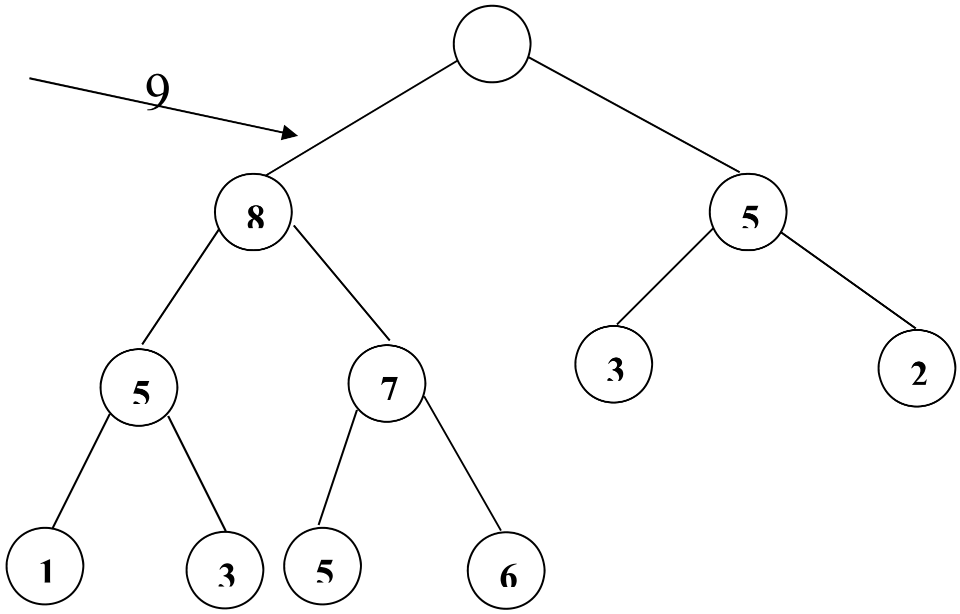
Compare 9 with its parent 6; 6 moves down:



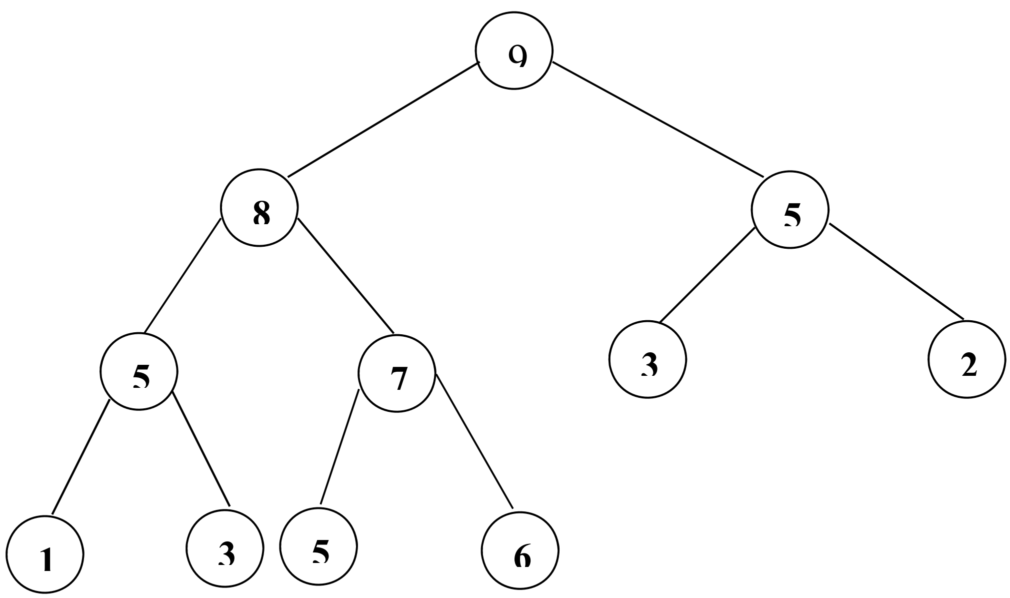
Compare 9 with its parent 7; 7 moves down:



Compare 9 with its parent 8; 8 moves down:



Insert 9 into its final location; process terminates:



Insert 通过不停的和父母比较交换位置，找到正确的位置

Complexity Analysis:

Observe that a k-heap with m nodes has height $\lfloor \log_k m \rfloor$ and requires at most $\lfloor \log_k (m+1) \rfloor$ comparisons to insert a new node into this heap.

Hence,

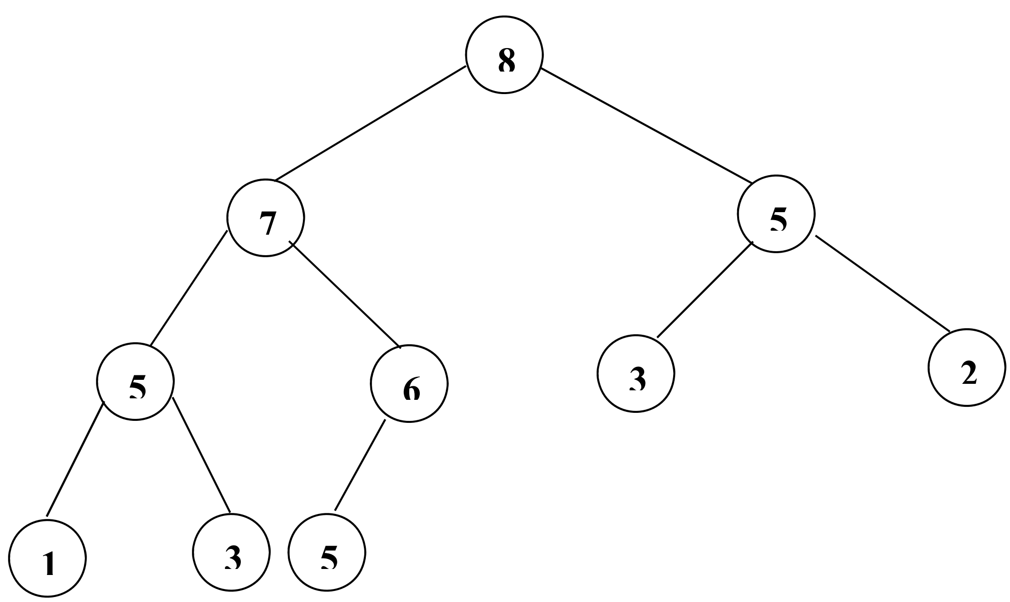
$$T_w(n) = \lfloor \log_k (n+1) \rfloor = O(\log_k n). \quad (= O(\lg n))$$

Conclusion: Larger k results in better performance in insertions.

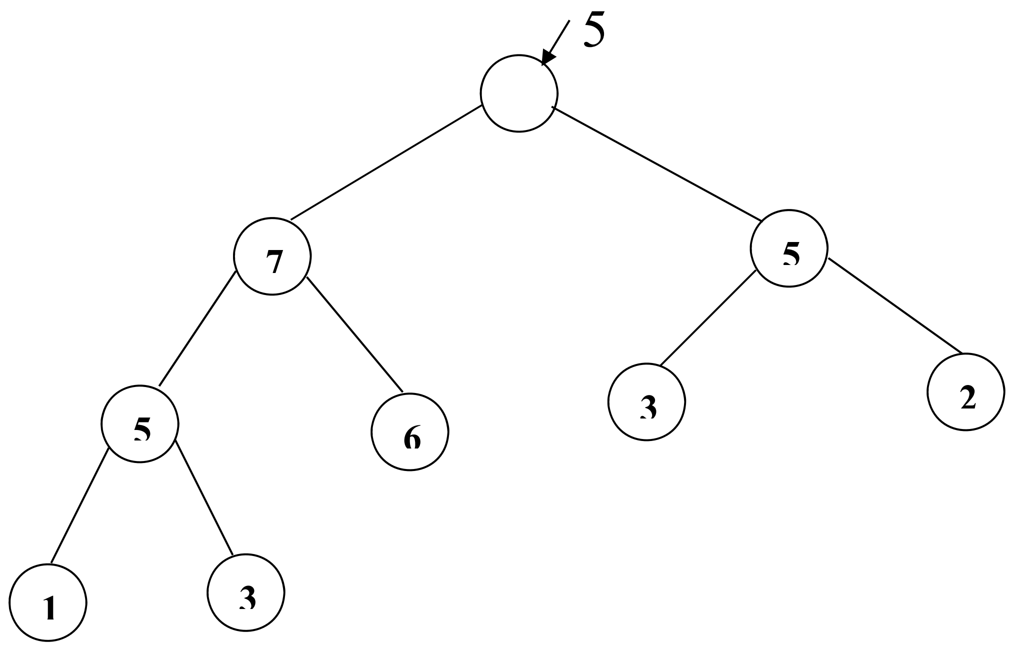
2. delete(H):

Consider deleting the highest priority item (root) from the original heap H.

Compare between a parent and children

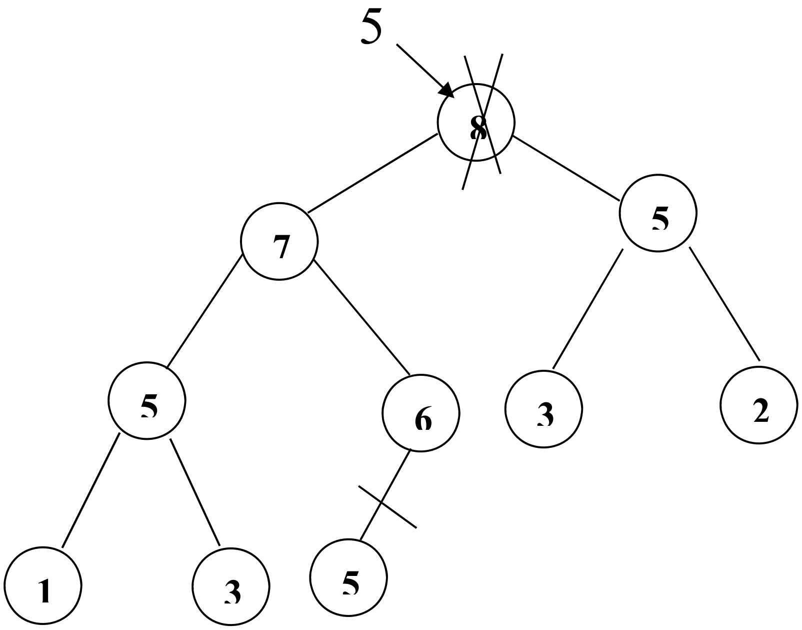


After 8 is removed, in order to get back a complete 2-ary tree, one must replace the root of H with the last item (in level order) in H.

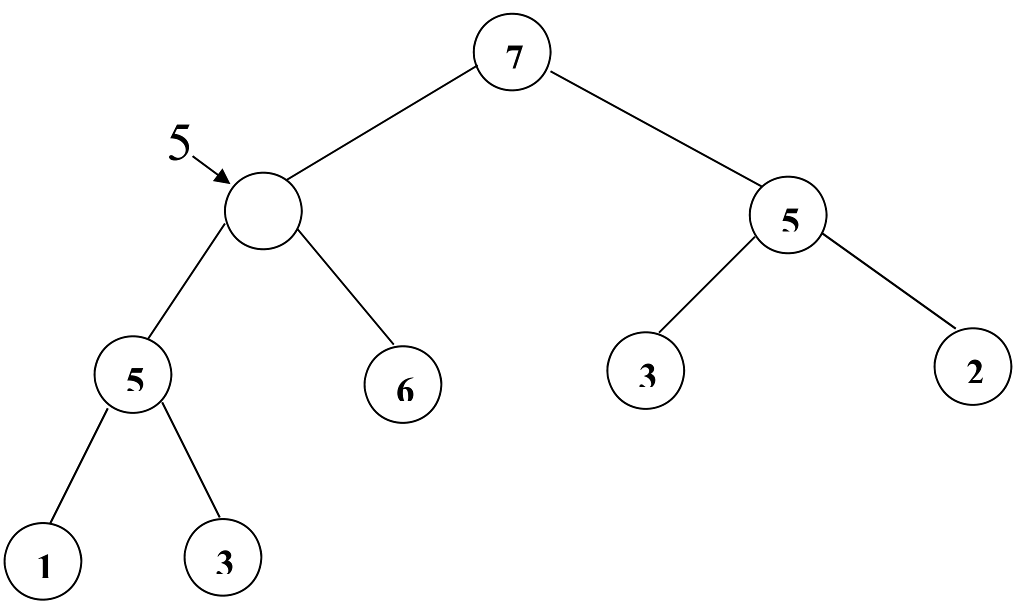


In general, we must replace the root (highest priority item) with the “last” item x and then percolate down along a path from the root to a leaf by repeatedly compare x with its child (children), swapping with the larger child if necessary, until x ≥ its children or a leaf is reached.

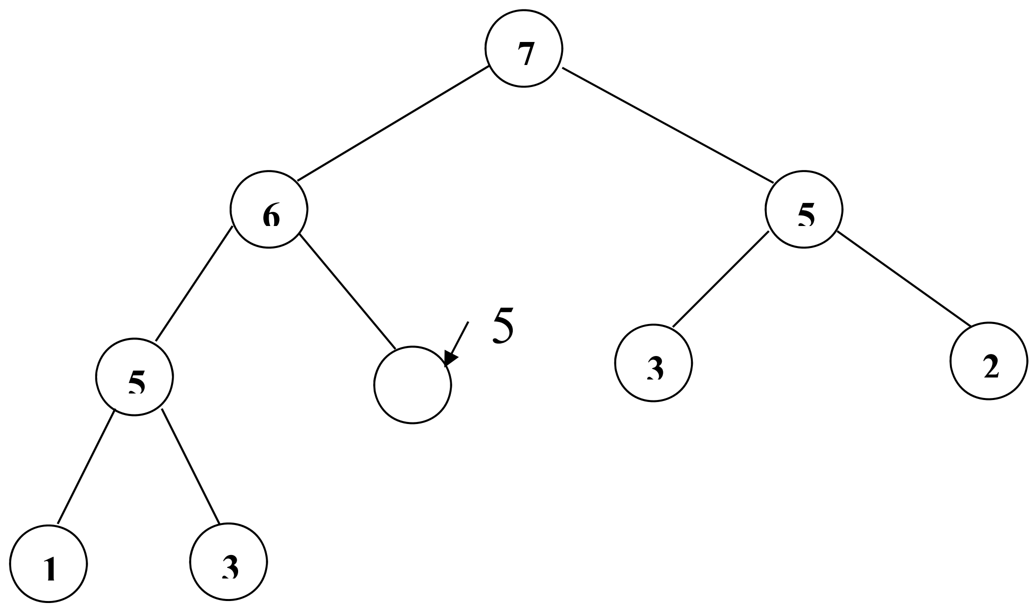
Example: Deleting the highest priority item (root).



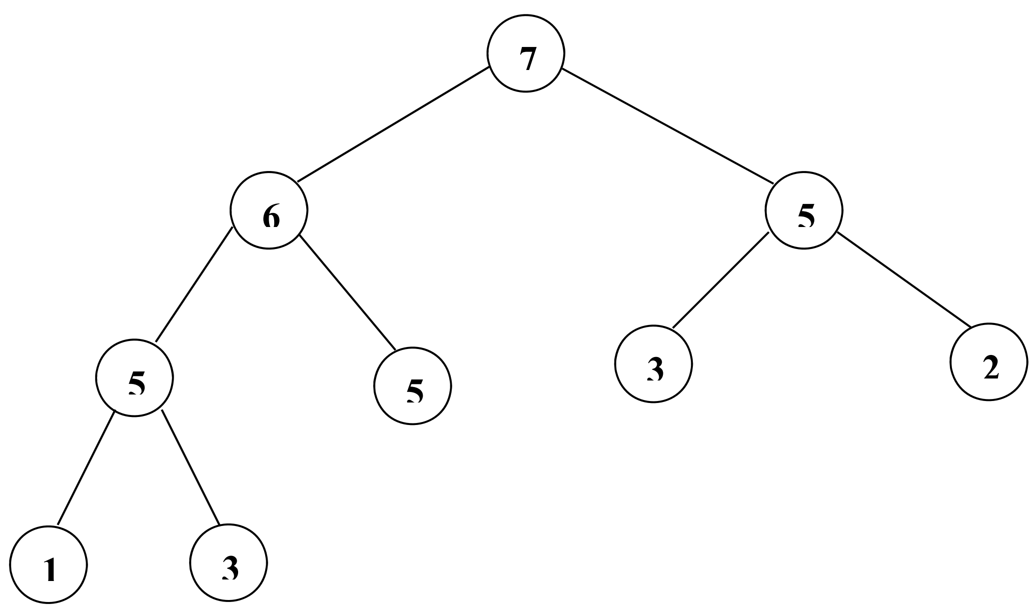
Compare 5 with its two children, 7 moves up:



Compare 5 with its two children, 6 moves up:



Insert 5 into its final location; process terminates:



delete topdown, 把最后一个元素替换到最上面，然后不停比较儿子找到适合的位置

Complexity Analysis:

Observe that to move a node down one level in a k-heap requires k comparisons. Since a heap with m nodes has height $\lfloor \log_k m \rfloor$, it requires at most $k*\lfloor \log_k m \rfloor$ comparisons to delete its root.

Hence,

$$T_w(n) = k*\lfloor \log_k n \rfloor = O(k\log_k n). \quad (= O(\lg n))$$

Conclusion: Smaller k results in better performance in deletions.

Q: Given a set of data objects S. How do we construct an initial k-heap H for S?

Consider building a k-heap, $k \geq 2$.

Two build-heap methods:

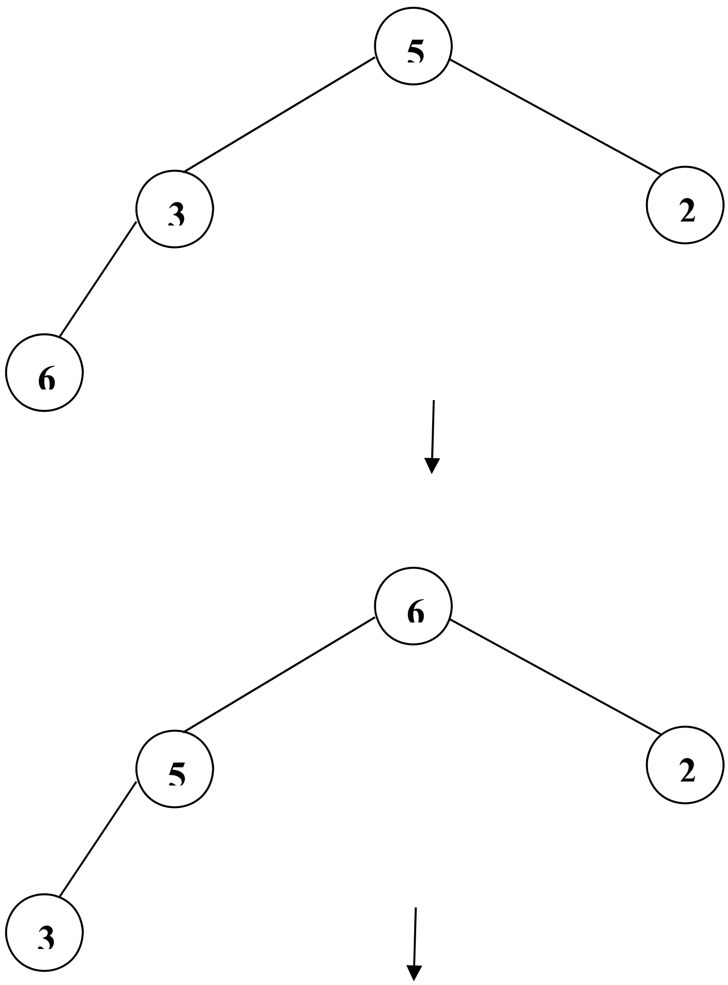
1. Top-down approach:

Insert items of S one at a time, in the order given, into an initially empty heap.

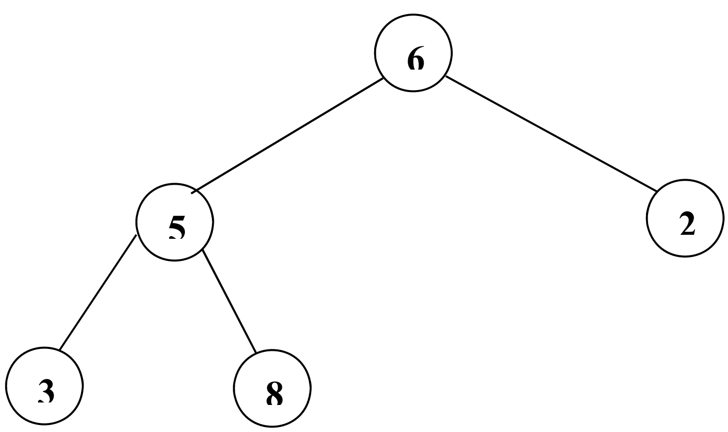
Build time O(nlgn)

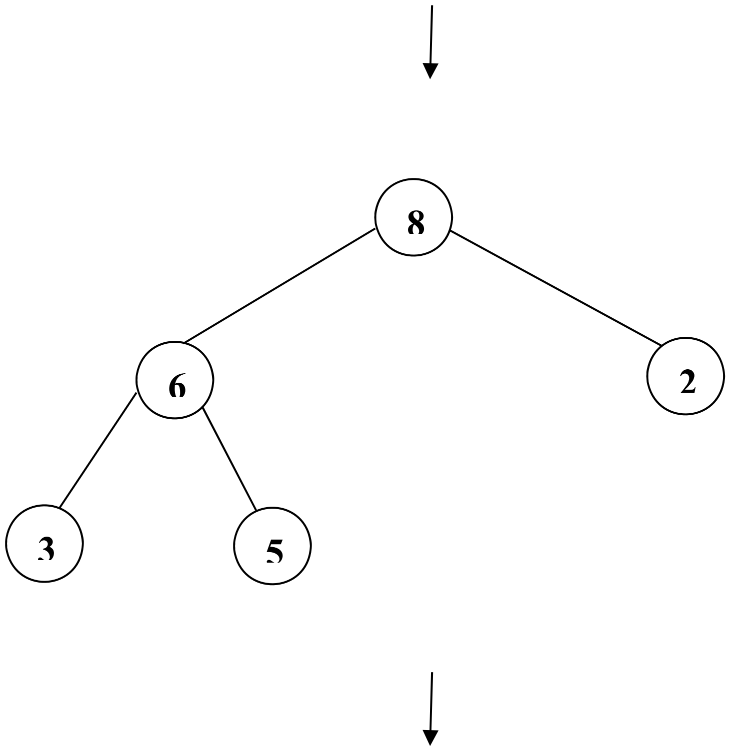
Example: Build a max 2-heap for S = {5,3,2,6,8,5,7,1,3,5}.

Insert 5, 3, 2, 6:

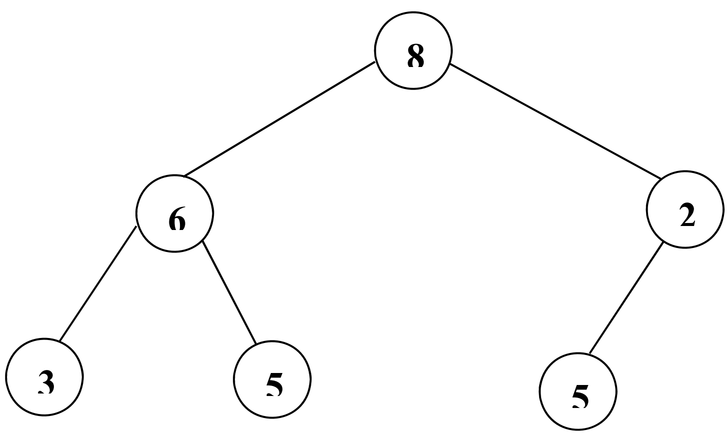


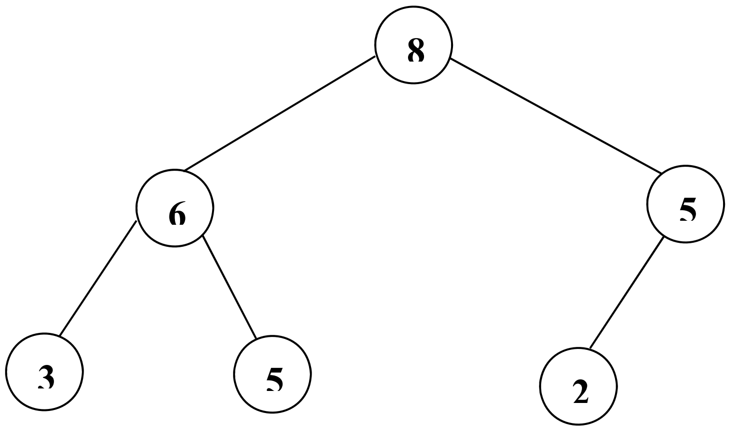
Insert 8:



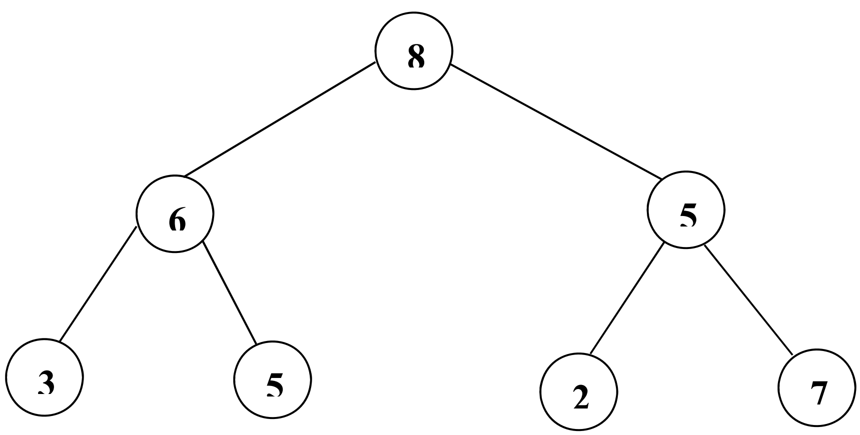


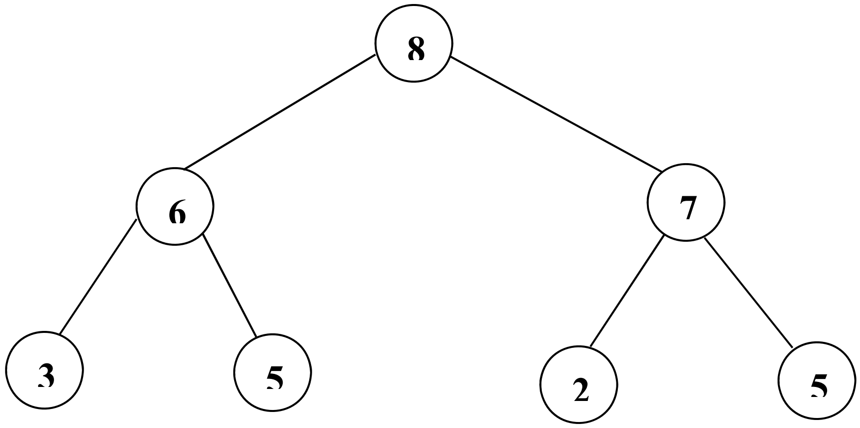
Insert 5:



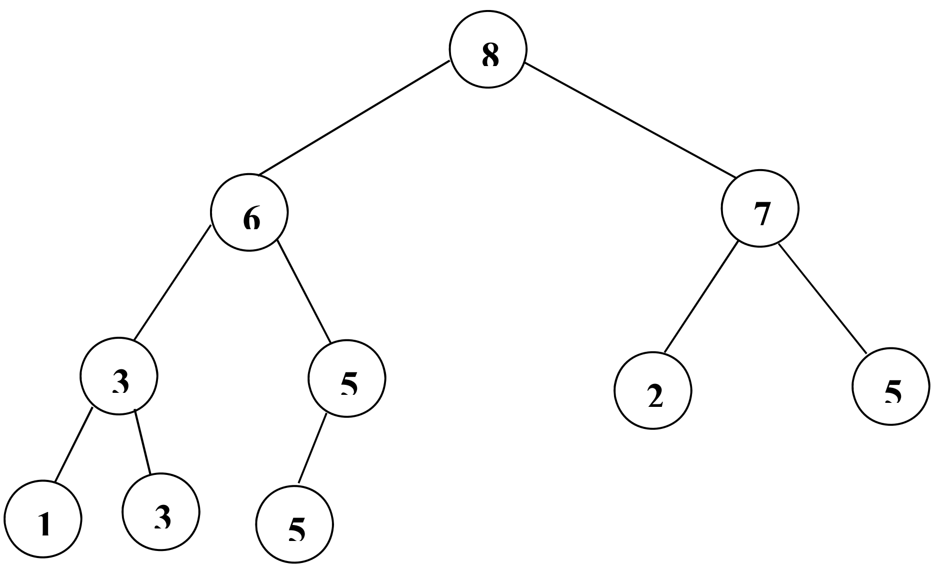


Insert 7:





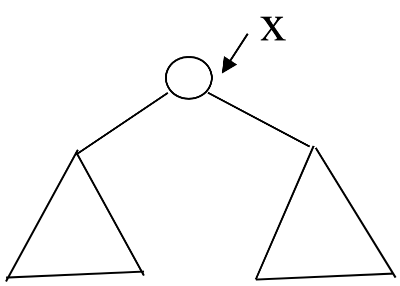
Insert 1, 3, 5:



2. Bottom-up approach:

Tn = Big O(n)

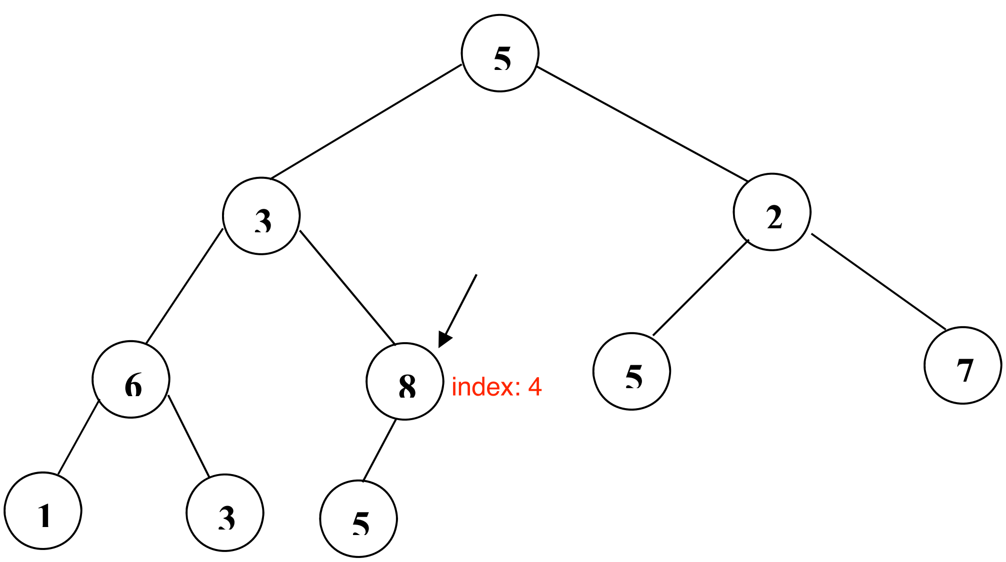
First form a complete binary tree H for S according to its given order. Observe that a leaf by itself is already a heap. If we scan the nodes of H in the reverse level order (leaf-to-root and right-to-left), two heaps can then be combined together by inserting a new element x as the new root of the resulting heap (as in delete operation). Hence, we can grow a heap for S in a bottom-up fashion by using ***heapify*** operations as in delete operation:



Example: Bottom-up approach to build a max 2-heap for S = {5,3,2,6,8,5,7,1,3,5}.

Form initial complete binary tree:

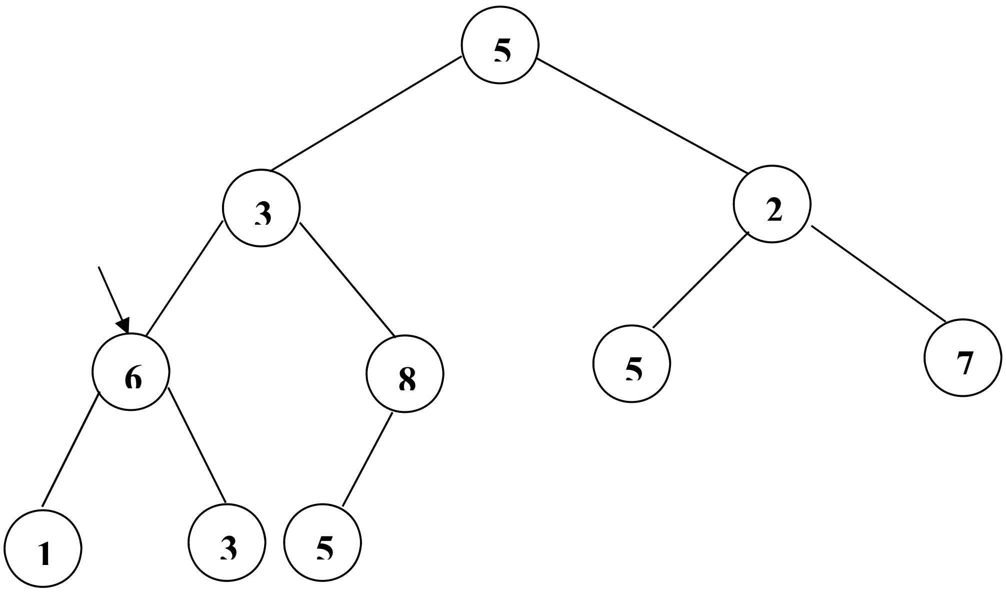
Find node is not a leaf, compare, then swap;
right side is shorter



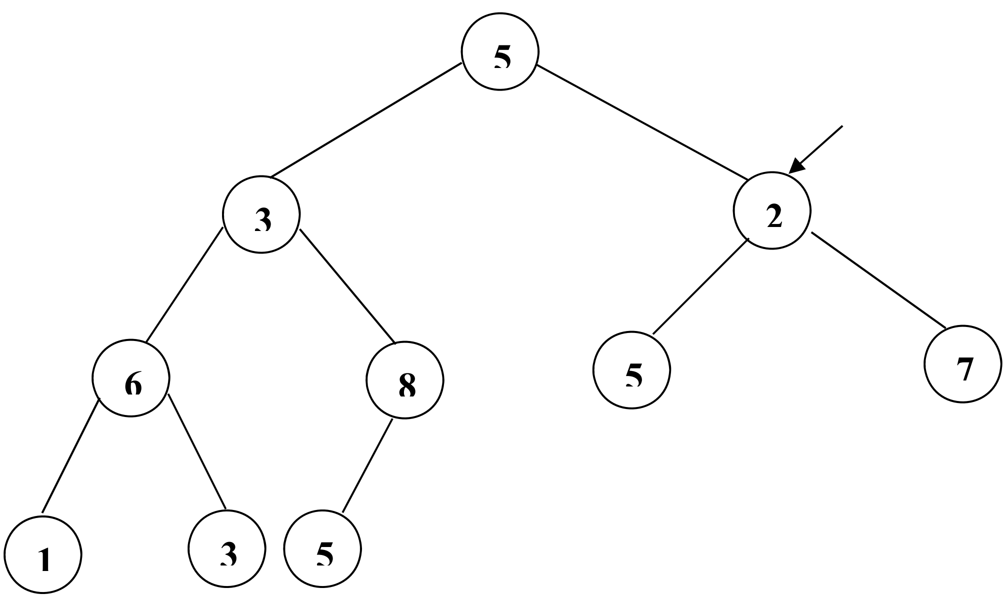
第一个下标开始有叶子的

Observe that since n = 10, the first non-leaf node needs to be checked has array index $\lfloor 10/2 \rfloor - 1 = 4$, follows by nodes with index 3, 2, 1, 0.

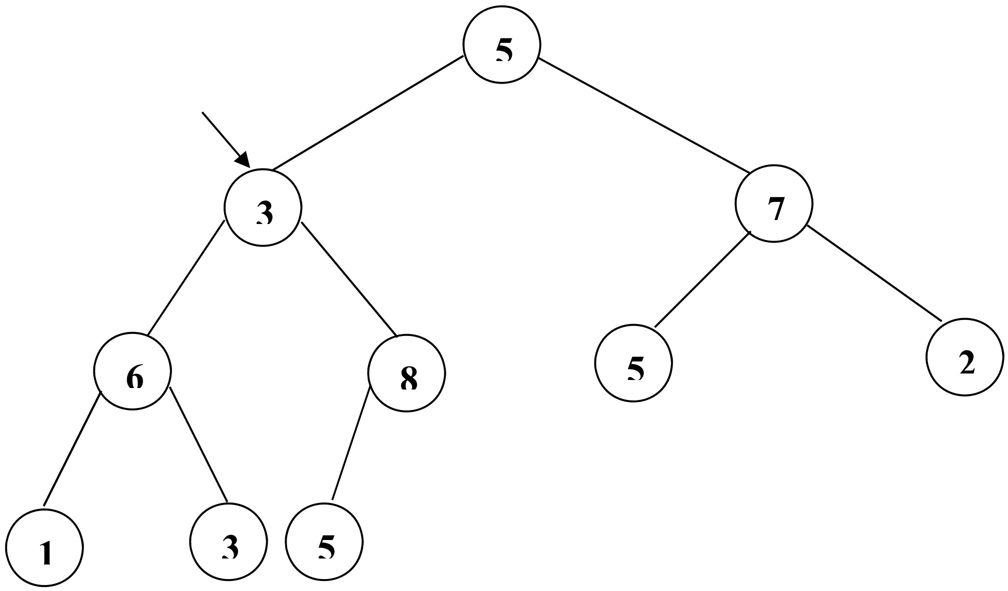
For A[4], compare 8 with 5; no swap:



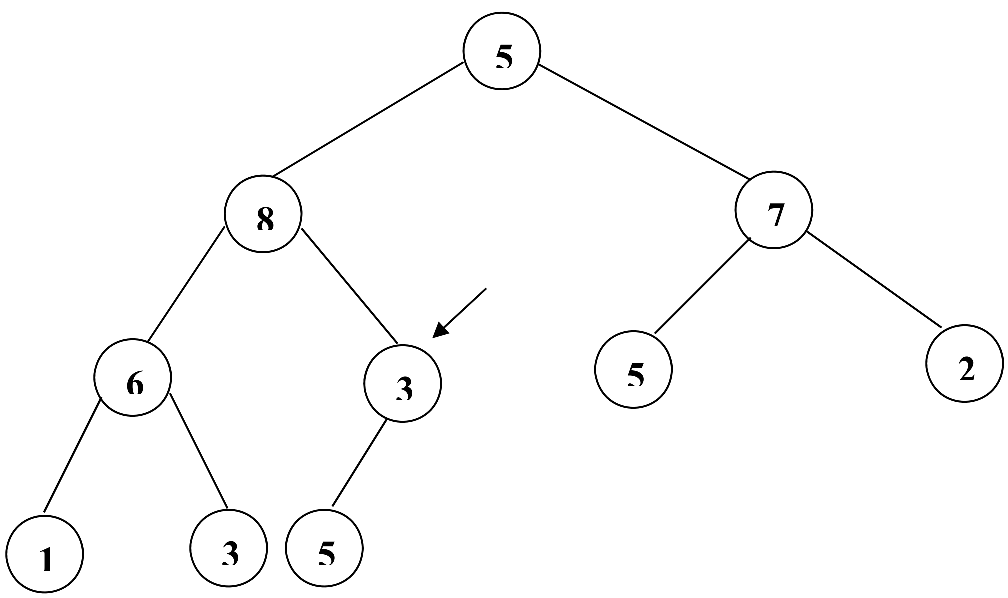
For A[3], compare 6 with 1 and 3; no swap:



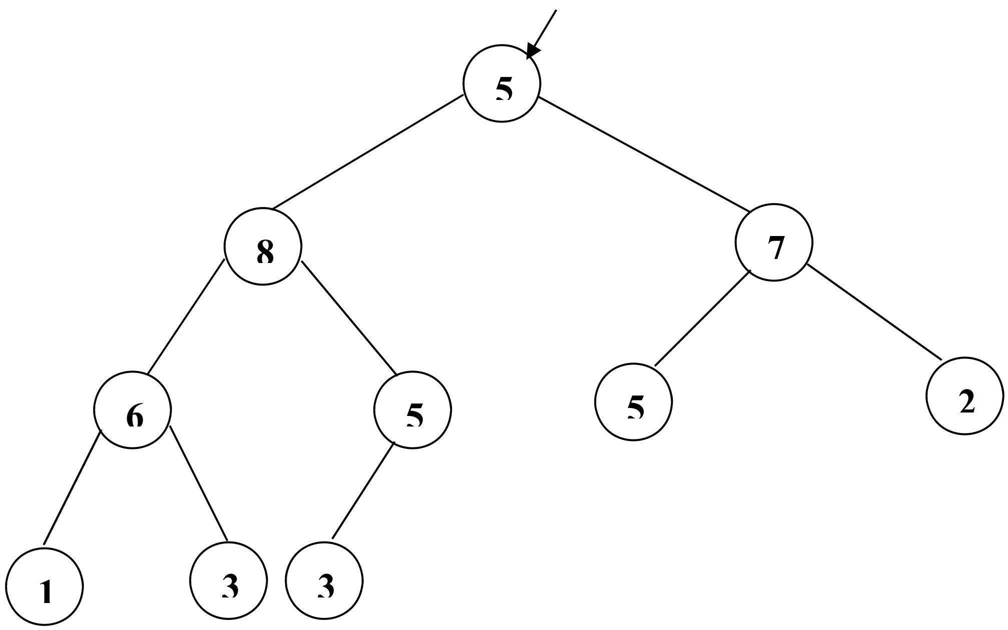
For A[2], compare 2 with 5 and 7, swap with 7:



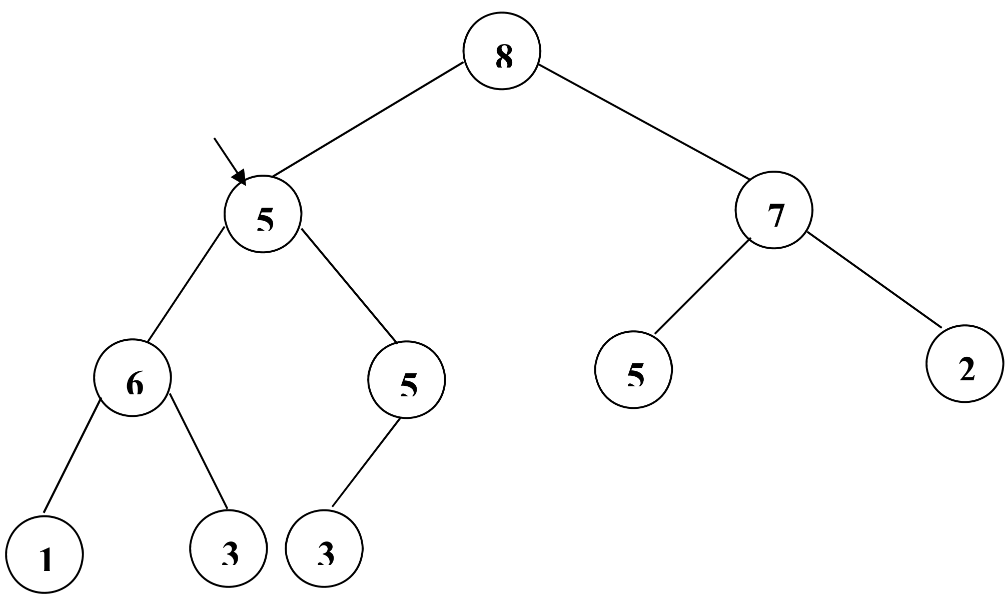
For A[1], compare 3 with 6 and 8, swap with 8:



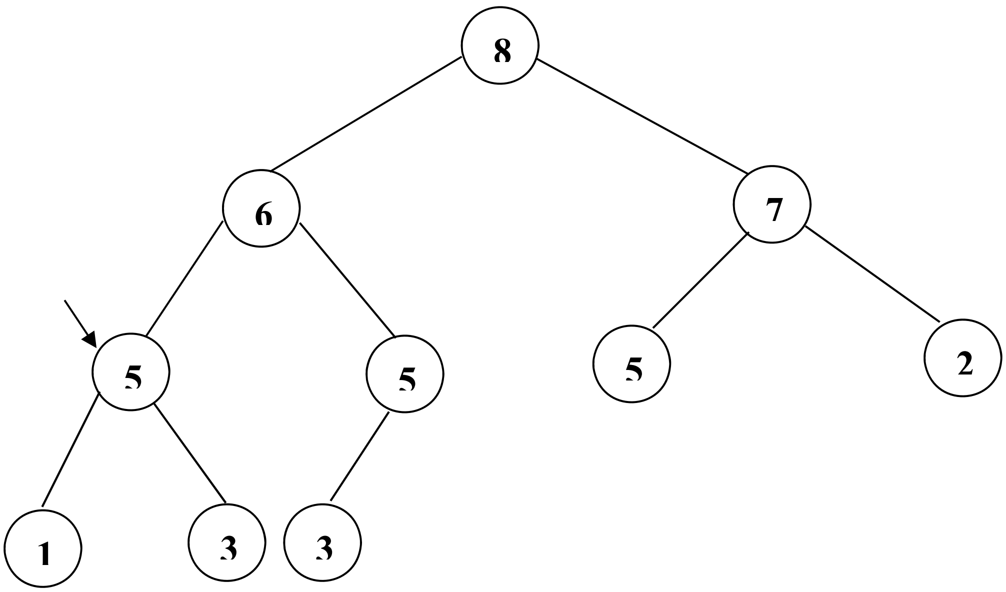
Continued: Compare 3 with 5, swap with 5:



For A[0], compare 5 with 8 and 7, swap with 8:



Continued: Compare 5 with 6 and 5, swap with 6:



Continued: Compare 5 with 1 and 3, process terminates!

HW: Redo the above examples using arrays.

Q: Which build-heap method should we use to build an initial heap?

Complexity Analysis for buildHeap Operations:

Let's compare the two buildHeap operations using a 2-heap:

1. Top-down approach using insert operations:

   Recall that it requires at most $\lfloor \lg(m+1) \rfloor$ comparisons to insert a new node into a heap with m nodes. Hence,

$T_w(n)$
$= \lfloor \lg 1 \rfloor + \lfloor \lg 2 \rfloor + \ldots + \lfloor \lg n \rfloor$
$\leq \lg 1 + \lg 2 + \ldots + \lg n$
$= O(\lg n!)$
$= O(n \lg n)$

2. Bottom-up approach using heapify operations:

   Recall that it requires at most 2 comparisons to move a node down one level in a heap. For a node x of height h(x), it will require at most 2*h(x) to heapify x. Hence,

$$T_w(n) = \sum_{x \in I(H)} 2 * h(x).$$

Observe that, in a complete binary tree, there are

$1 = 2^0$ node of height h(H),

$2 = 2^1$ nodes of height h(H)–1,

$4 = 2^2$ nodes of height h(H)–2,

…

2^i nodes of height h(H)–i,

…

$2^{h(H)-2}$ nodes of height 2,

$2^{h(H)-1}$ nodes of height 1,

$\leq 2^{h(H)}$ nodes of height 0.

By summing all the nodes according to their height,

$$
\begin{aligned}
& T_w(n) \\
&= \sum_{x \in I(H)} 2 * h(x) \\
&= 2 \sum_{i=1}^{h(H)} i * 2^{h(H)-i} \\
&= 2 * 2^{h(H)} \sum_{i=1}^{h(H)} \frac{i}{2^i} \\
&\leq 4 * 2^{h(H)} \\
&\leq 4n \\
&= \mathrm{O}(n).
\end{aligned}
$$

Conclusion: bottom up faster than top down: O(n)<O(nlgn)

Always use the O(n) bottom-up approach to build an initial heap.

Application of Priority Queue to Sorting:
PQ Sorting:

1. Build a PQ Q for S.
2. Repeatedly delete elements from Q until Q is empty.

Heap Sort:

1. Build a max-heap H for S.
2. Deletemax repeatedly until H is empty. (Swap max item with the current last item in array.)

Hence, $T(n) = O(n) + O(n\lg n) = O(n\lg n)$.

Final Remarks:

1. What is k in the k-heap?

Observe that there is a tradeoff between insert and delete operations since large k implies faster insert but slower delete.

For k-heap, $k \geq 2$:

Insert: $T_w(n) = O(\log_k n)$

Delete: $T_w(n) = O(k\log_k n)$

HW: Implement a 5-heap class.

2. Worst-Case Time Comparison of PQ Implementations:

PQ Operation	**Heap***	**BST**	**Sorted lList**	**Unsorted Array**
Bottom up ***Build/Organize***	O(n)	$O(n^2)$	$O(n^2)$	O(n)
Insert	O(lgn)	O(n)	O(n)	O(1)
Find	O(n)	O(n)	O(n)	O(n)
GeneralDelete	O(n)	O(n)	O(n)	O(n)
DeleteMax	O(lgn)	O(n)	O(n)	O(n)
DeleteMin	O(n)	O(n)	O(1)	O(n)

*Max 2-heap.

3. Other operations such as find(x), changeKey and delete(x,H) possible but inefficient. (HW)

HW: Given a min-heap H. Design and analyze a function deleteMax(H) to delete an object with max priority from H.

Extension:

How do we design an ADT that will support ***both*** deletemin(Q) and deletemax(Q) operations?

Two Heap: findMin(); O(1) findMax(); O(1)

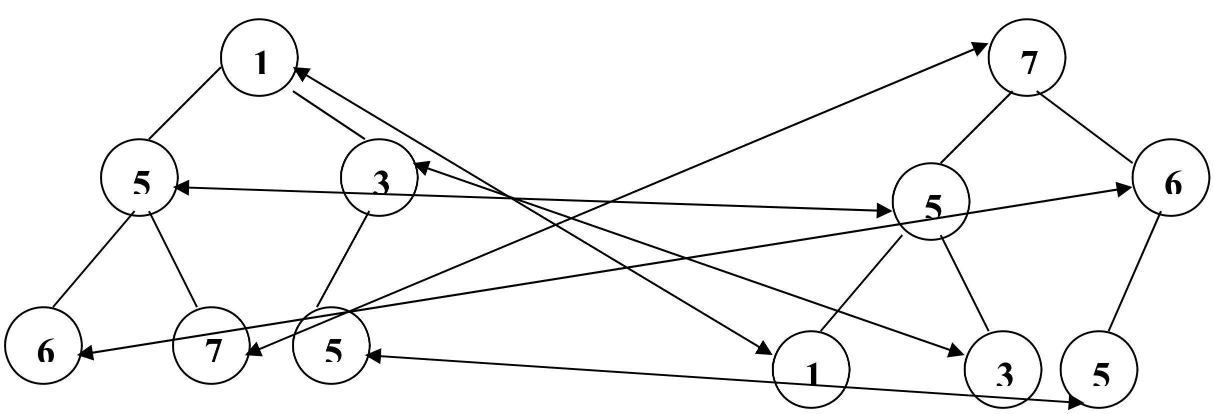
A **double-ended priority queue (DEPQ)** H is a collection of zero or more data objects together with the following operations:

1. ***findMin()***
2. ***findMax()***
3. ***insert(in newItem: DEQItemType)***
4. ***deleteMin(out priorityItem: DEQItemType)***
5. ***deleteMax(out priorityItem: DEQItemType)***
6. *createDEQ()*
7. *destroyDEQ()*
8. *DEQisEmpty()*
9. *DEQSize()*

Simplest DEPQ:

Dual heap: A pair of min heap and a max heap with corresponding pointer between each pair of corresponding elements.

Example:



Operations:

1. inser(x): insert x into both min and max heaps, set corresponding pointers.

2. deleteMin: deletMin from min heap; follow pointer to max heap and delete corresponding min element.

3. deleteMax: deletMax from max heap; follow pointer to min heap and delete corresponding max element.

Complexity: Same as min or max heap.
$T_w(n) = O(\lg n)$.

disadvantage of dual heap

Remark: Inefficient in memory; not as fast as minMax heap.

A Better DEPQ: *Minmax Heap*

A **minmax heap** H is an extension of 2-heap by fusing a min heap and a max heap together such that

(1) H is a complete binary tree.

(2) Each node x in H is either a **min node** (x is ≤ all its descendants), or a **max node** (x is ≥ all its descendants).

(3) All nodes belong to the same level must be of the same type. **A min-level (max-level)** in H is a collection of all the min (max) nodes having the same level number.

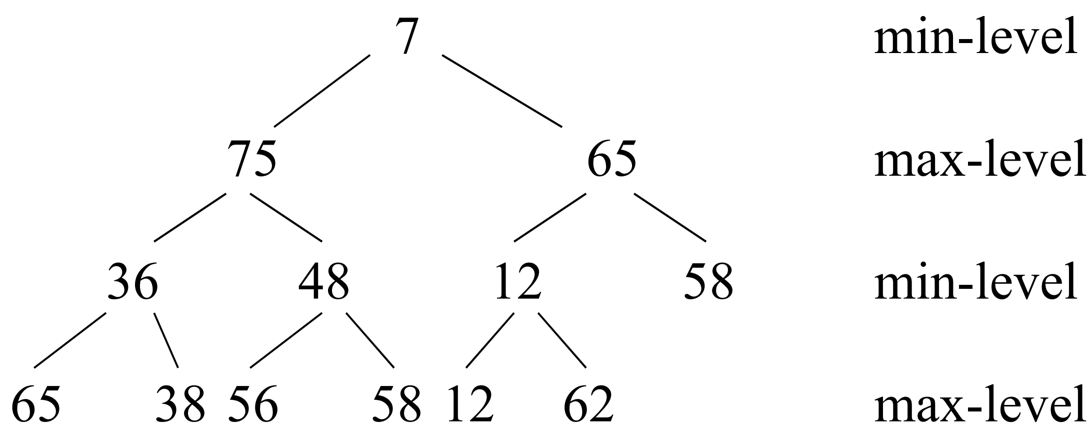
(4) Starting with the root of H at the min-level, nodes are alternating between min-level and max-level.

Observe that a minmax heap is a tree satisfying the following properties:

1. **Structural Property**: H is a complete binary tree.
2. **Relational Property**: H satisfies the minmax heap property such that, starting at the root at min-level, nodes are alternating between min-level and max-level.

Remark: A maxmin heap can be defined in a similar fashion by requiring that, starting at the root at max-level, nodes are alternating between max-level and min-level.

Example: A minmax heap H.

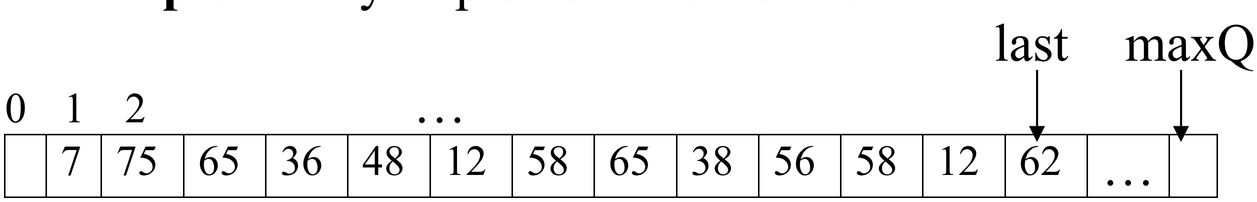


HW: Define and explore maxmin heap.

Implementation:

Sequential array implementation with root of H at A[1]. 数组从下标1来表示第一个Root

Example: Array implementation of H.



Minmax heap operations:

Two-step process as in binary heap:

1. Maintain complete binary tree structure after each insert/deleteMin/deleteMax operation.
2. Restructure resulting tree to restore minmax heap property.

1. **Insert(x,H):**

As in heap, insert x into the last+1 position and then restore the minmax heap property.

Consider the following two cases:

(1) H = ∅: Return heap with x.

(2) H ≠ ∅: Consider the parent, p(x), of x, after inserting x into H.

If x = p(x), done;

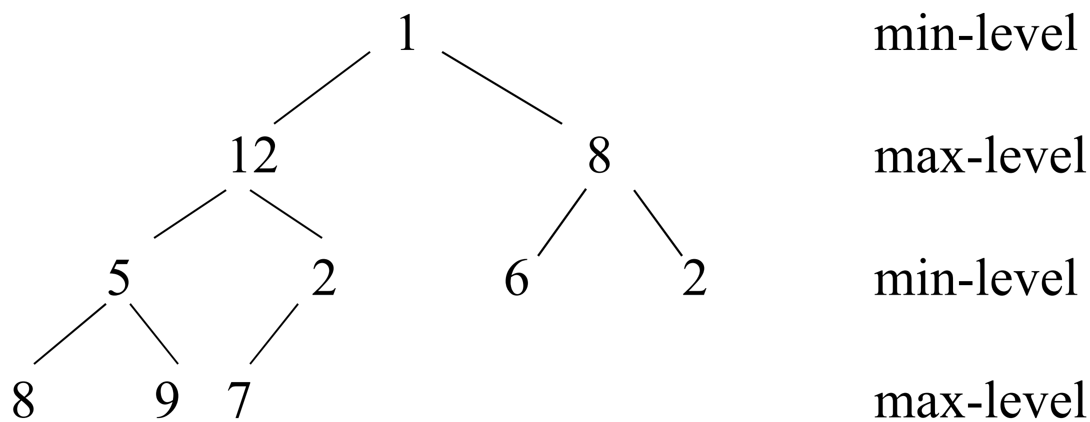
else consider x < p(x), or x > p(x).

(a) Assume x < p(x): If p(x) is a min node, then x < y for all max node y on the path from x to the root of H. Similarly, if p(x) is a max node, then x < z for all max node z on the path from x to the root of H. Hence, we need only compare x with the min nodes along the path from x to the root in order to restore the minmax heap property as in a min heap.

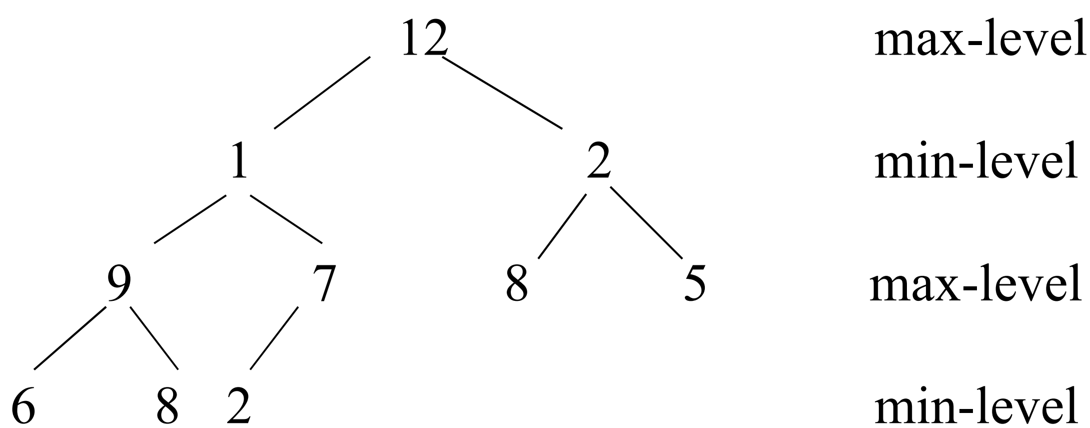
(b) Assume x > p(x): If p(x) is a min node, then x > y for all min node y on the path from x to the root of H. Similarly, if p(x) is a max node, then x > z for all min node z on the path from x to the root of H. Hence, we need only compare x with the max nodes along the path from x to the root in order to restore the minmax heap property as in a max heap.

Example: Build a minmax heap and a maxmin heap by inserting <6, 8, 5, 2, 7, 8, 2, 9, 12, 1> into an initially empty heap.

Minmax heap:



Maxmin heap:



Remark: A minmax heap can also be built using a modified bottom-up approach.

HW: Construct a minMax heap and a maxMinHeap for the given set of keys using both approaches.

Q: Given a node x at A[i]. How do you determine whether x is a min node or a max node?

Min node: $\lfloor \lg(i) \rfloor$ = even
Max node: $\lfloor \lg(i) \rfloor$ = odd

Q: How do you locate the grandparent of x if exists?

Grandparent of A[i] at A[$\lfloor \frac{i}{4} \rfloor$].

HW: Implement insert(H) for minmax heap and maxmin heap.

Consider deleteMin/deleteMax operations.

Q: Where is the min (max) element in a minmax heap?

Min element: Root of H.
Max element: A child of root if |H| > 1.

General approach: Replace the min (max) element of H by an element in H.

(1) Find the second smallest/largest element s in H and use it, or the last element in H, to patch up the hole left by the deletion of the min (max) element.
(2) Remove and then re-insert the last element x (in level order traversal) back into H to patch up the "hole" left by the deletion of s if used.

2. Deletemin(H):

MinMax Heap

Consider the following four cases:

(1) H = ∅: Return error.

(2) |H| = 1: Return ∅.

(3) |H| = 2: Replace root with its only child.

(4) |H| ≥ 3: Delete root r and then find the second smallest element s in H, which is either a child (|H| = 3), or a grandchild, of r to patch up the hole. Compare the last element x in H with s (before removing any element from H).

- (a) x ≤ s: Remove x; x becomes new root of H.
- (b) x > s: Remove s; s become the new root of H. Observe that a new "hole" is now generated in H and we will delete x (the very first time) and use it to patch up the hole.
  - (i) If s is a child of r, then x can be used to patch up the hole vacated by s and we are done. (Why?)
  - (ii) If s is a grandchild of r, s is a min node and it has a parent p(s), which is a max node, in H. Compare x with p(s).
    - (a) If x ≤ p(s), recursively use x to patch up the root of the minmax heap rooted at the original s location.

(b) If $x > p(s)$, replace $p(s)$ with x in H and then recursively use $p(x)$ to patch up the root of the minmax heap rooted at the original s location.

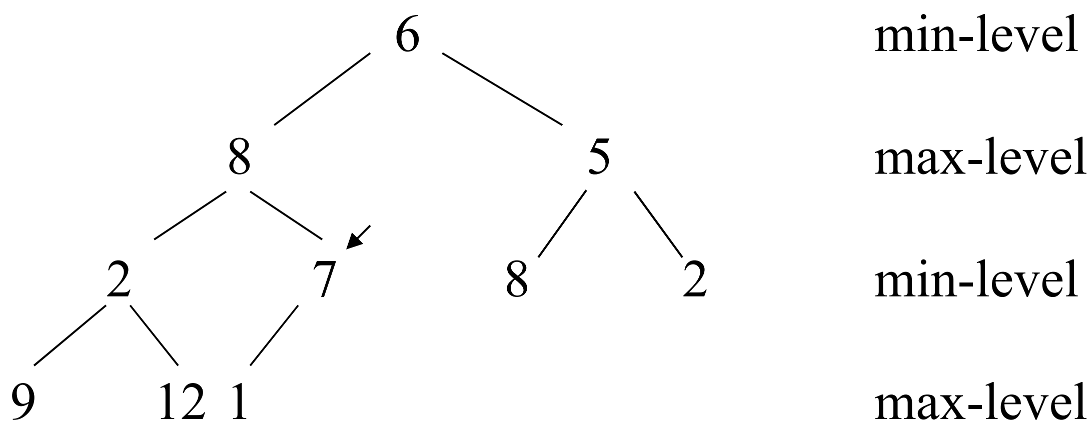
Q: How about deletemax(H)?
Similar method except using maxmin heap concept instead.

HW: Implement the deletemin(H) and deletemax(H) operations.

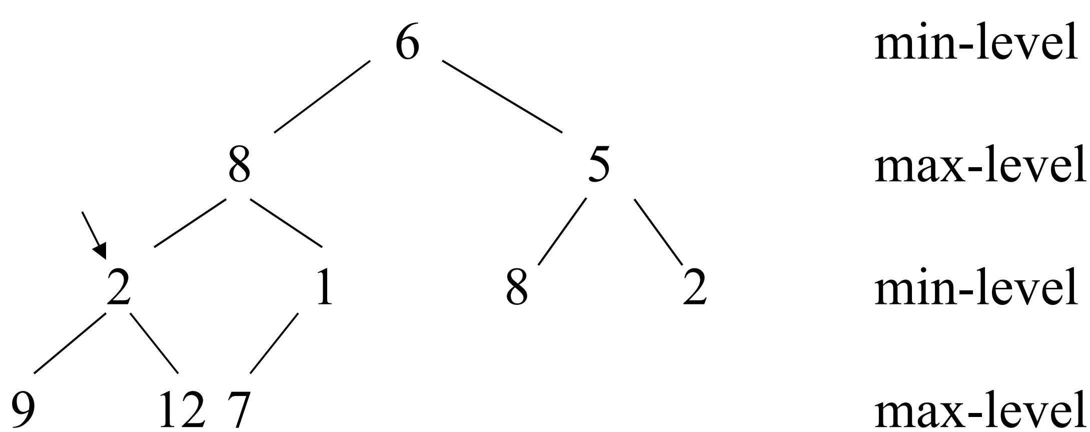
Complexity Analysis:
For insert, deleteMin, deleteMax operations, $T_w(n) = O(h(H)) = O(\lg n)$.

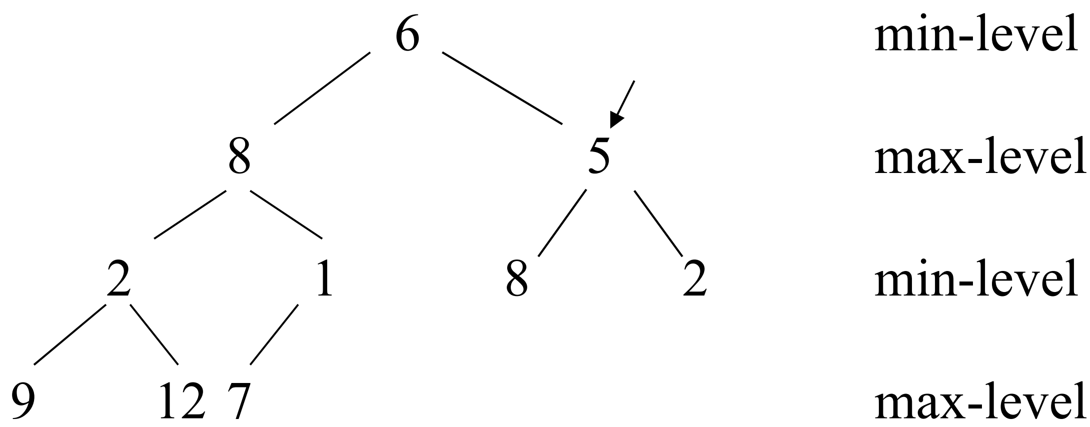
3. Build-Heap(H):
Two approaches similar to building a binary heap:
1. Top-down $O(n \lg n)$ approach using insert operations.
2. Bottom-up $O(n)$ approach using a modified heapify operations.

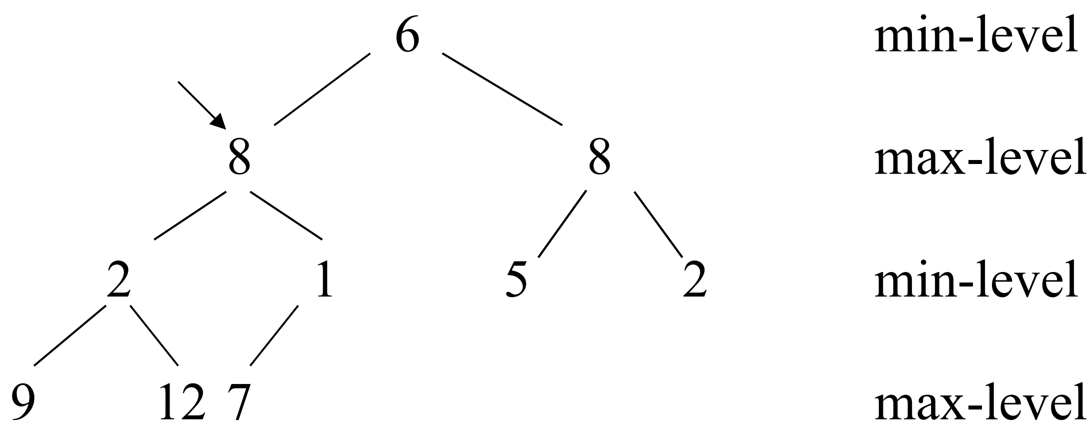
Example: Build a minmax heap for the set S = {6, 8, 5, 2, 7, 8, 2, 9, 12, 1} using a modified bottom-up O(n) buildMinMaxHeap operation.



Considering delete and then re-insert the node as in deleteMin or deleteMax operations:




min-level
max-level
min-level
max-level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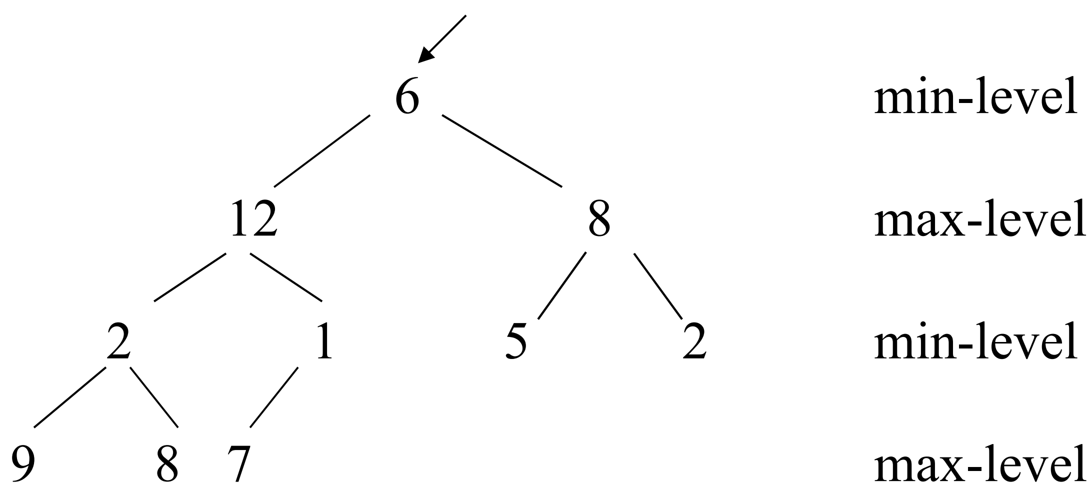
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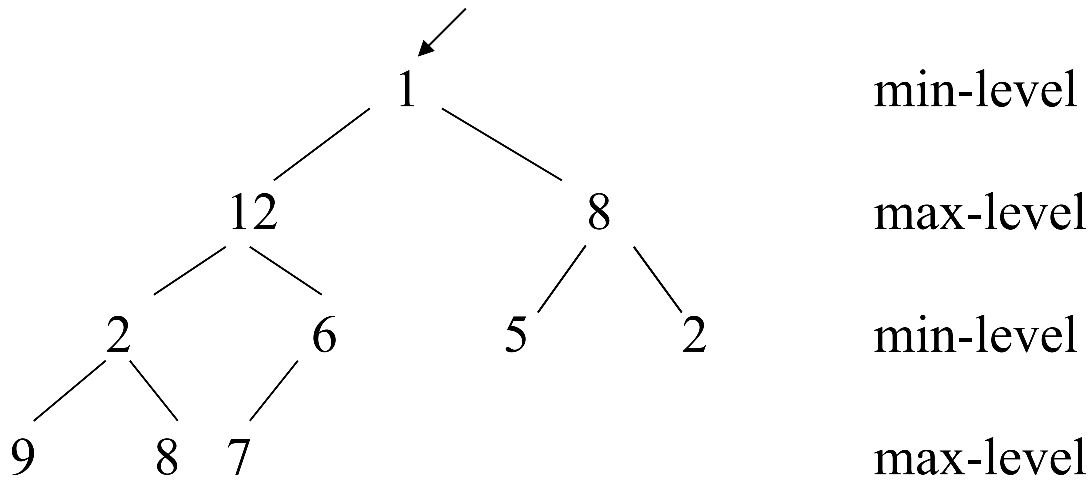

HW:

1. Build a minmax heap by inserting <3, 8, 10, 2, 7, 24, 5, 12, 26, 1, 28> into an initially empty heap.
2. Build a minmax heap for {13, 8, 10, 22, 7, 24, 5, 12, 26, 1, 28, 6} using the bottom-up O(n) approach.
3. Repeat (1) & (2) by building a maxmin heap.
4. Given a minmax heap [1, 28, 24, 3, 2, 10, 5, 8, 12, 7, 26]. Perform deletemin until the heap is empty.
5. Given a minmax heap [1, 28, 24, 12, 6, 10, 15, 18, 22, 7, 8]. Perform deletemax until the heap is empty.