Topic 4: Relations & Functions

Read: Chpt. 9.1, 9.3, 9.5, 9.6, Rosen

Let A and B be any two sets. Sometimes we are interested in how elements in A are "related" to the elements of B with respect to a given property P.

Example: Let A = {2, 3, 4}, B = {3, 4, 5, 6, 7}. Consider a relation R defined from A to B such that an element $a \in A$ is *related to* an element $b \in B$ iff a divides b.

Hence,

$2|4$: 2 is related to 4,
$2|6$: 2 is related to 6,
$3|3$: 3 is related to 3,
$3|6$: 3 is related to 6,
$4|4$: 4 is related to 4.

Or, 2R4, 2R6, 3R3, 3R6, 4R4.

Or, we can use an ordered-pair (a,b) to indicate an element aRb. A relation from A to B can then be characterized by the following set of ordered-pairs:

R = {(2,4), (2,6), (3,3), (3,6), (4,4)}.

Remark: aRb may or may not imply bRa.

Generalization: Recall that the set of all ordered-pairs (a,b) with $a \in A$ and $b \in B$ is just the Cartesian product of $A \times B$. Hence, a relation from A to B is simply a subset of $A \times B$ and, any subset of $A \times B$ can also be interpreted as a relation from A to B.

Def: A (binary) ***relation*** R from A to B is any subset of $A \times B$.

Notation:

If a is related to b in R, then aRb and $(a,b) \in R$.

Q: Given two sets A and B. How many relations can one define from A to B?

A: Number of relations
= Number of all possible subsets of $A \times B$
$= 2^{|A|*|B|}$.

Examples:

1. Let A be the set of all students at KU and B be the set of all majors at KU.
   Define a relation R_1 from A to B such that a student $a \in A$ is related to a major $b \in B$ iff student a is majoring in b.

2. Let A be the set of all PCs and B be the set of all printers in the EECS Dept. at KU.
   Define a relation R_2 from A to B such that a computer $a \in A$ is related to a printer $b \in B$ iff a computer a is connected to a printer b.

3. Let $A = \{2, 3, 4\}$, $B = \{3, 4, 5, 6, 7\}$.
   (a) Define a relation R_3 from A to B such that an integer $a \in A$ is related to an integer $b \in B$ iff $a^2 = b$. Hence,
   $R_3 = \{(2,4)\}$.

   (b) Define a relation R_4 from A to B such that an integer $a \in A$ is related to an integer $b \in B$ iff a and b are ***relatively prime*** (the only positive integer that divides both a and b is 1 and -1).
   Hence,
   $R_4 = \{(2,3), (2,5), (2,7), (3,4), (3,5), (3,7), (4,3), (4,5), (4,7)\}$.

   (c) Define a relation R_5 from A to B such that an integer $a \in A$ is related to an integer $b \in B$ iff $a^3 = b$. Hence, $R_5 = \varnothing$.

Representing Relations:

1. Using representations for sets such as (i) English description, (ii) Listing all elements (ordered-pairs) in relation, and (iii) Set descriptor such as $R = \{(a,b) \mid aRb\}$.

2. Using graphical representations

   (a) Table (matrix) representation:

   Using a table (matrix) M with |A| rows and |B| columns such that M(a,b) = 1 iff aRb; otherwise, M(a,b) = 0.

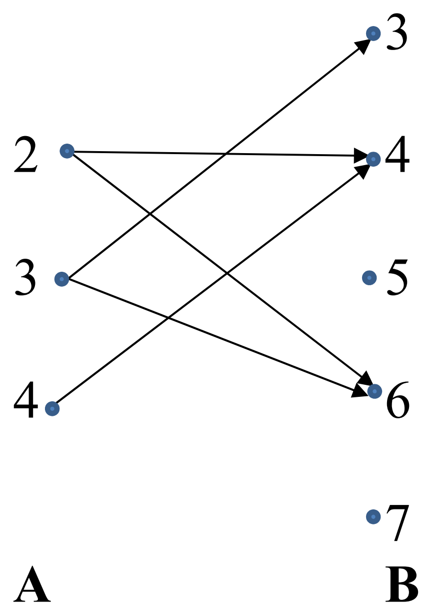
   Example: Let A = {2, 3, 4}, B = {3, 4, 5, 6, 7}, and relation R = {(2,4), (2,6), (3,3), (3,6), (4,4)}. The relation R can be represented by the following table:

R	**3**	**4**	**5**	**6**	**7**
2	0	1	0	1	0
3	1	0	0	1	0
4	0	1	0	0	0

(b) Point-Line (graph) Representation:

Each element in A and B is represented by a point and, two points a and b with $a \in A$ and $b \in B$ are joined together by an arrow from a to b iff aRb.

Example: Relation R = {(2,4), (2,6), (3,3), (3,6), (4,4)} above can be represented by the following *directed graph*:



Remark: Graphical representations give us a very useful conceptual and visual tool in illustrating a relation R but have very limited applications when |R| is large.

Recall that a relation R from A to B is a subset of $A \times B$.
Q: What if B = A?

Dfn: A ***relation R on A*** is a relation defined from A to A, which is a subset of $A \times A$.

Examples:

1. Let A be the set of all students at KU.
   (a) Define a relation R_6 on A such that a student a is related to another student b iff both a and b have the same last name.

   (b) Define a relation R_7 on A such that a student a is related to another student b iff both a and b are taking the same class.

2. Let A = {2, 3, 4, 5, 6}.
   Define a relation R_8 on A such that $a \in A$ is related to an integer $b \in A$ iff a divides b. Hence,
   $R_8 = \{(2,2), (2,4), (2,6), (3,3), (3,6), (4,4), (5,5), (6,6)\}$.

Let R be a relation defined on a set A.

Some Important Relations defined on A:

1. R is ***reflexive*** iff $\forall a \in A, (a,a) \in R$.
2. R is ***irreflexive*** iff $\forall\ a \in A, (a,a) \notin R$.
3. R is ***symmetric*** iff $(a,b) \in R \rightarrow (b,a) \in R$.
4. R is ***asymmetric*** iff $(a,b) \in R \rightarrow (b,a) \notin R$.
5. R is ***anti-symmetric*** iff $((a,b) \in R \wedge (b,a) \in R) \rightarrow a = b$.
6. R is ***transitive*** iff $(a,b) \in R \wedge (b,c) \in R \rightarrow (a,c) \in R$.
7. R is an ***equivalence relation*** if it is reflexive, symmetric and transitive.

Example: Let A = {1, 2, 3, 4}. Define

$R_1 = \{(1,1), (1,2), (2,2), (3,3), (4,1), (4,4)\}$,

$R_2 = \{(1,2), (1,3), (2,1), (3,1), (4,4)\}$,

$R_3 = \{(1,2), (2,3), (1,3), (3,4), (1,4), (2,4)\}$,

$R_4 = \{(1,2), (2,1), (1,3)\}$,

$R_5 = \{(1,1), (2,2), (3,3), (4,4), (1,2), (2,1), (1,4), (4,1), (2,4), (4,2)\}$.

Property	**Relation**
Reflexive	R_1, R_5
Irreflexive	R_3, R_4
Symmetric	R_2, R_5
Asymmetric	R_3
Anti-Symmetric	R_1, R_3
Transitive	R_3, R_5
Equivalence	R_5

Graphical Representations for Relation R on A:

1. A relation R defined on A can also be represented by a table as before.

   Example: Table (Matrix) representation of R_5.

$\mathbf{R_5}$	**1**	**2**	**3**	**4**
1	1	1	0	1
2	1	1	0	1
3	0	0	1	0
4	1	1	0	1

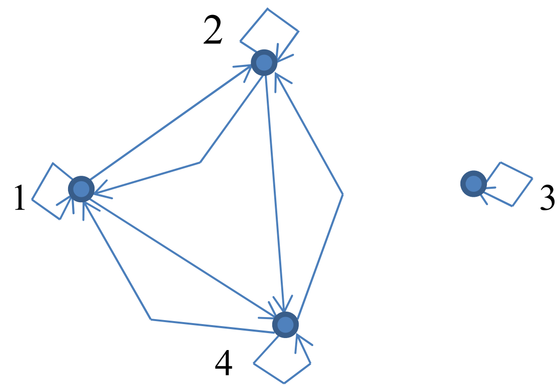
   Observe that the relation R_5 can now be represented by using a 4×4 matrix.

$$R_5 = \begin{pmatrix} 1 & 1 & 0 & 1 \\ 1 & 1 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 1 & 1 & 0 & 1 \end{pmatrix}.$$

HW: Explore the structure of the above matrix structures with respect to different types of relations.

2. A relation R defined on A can also be represented by using a directed graph with a single set of points defined by A.

   Example: Directed graph representation of R_5.



HW: Explore the structure of the above directed graph with respect to different types of relations.

More on Equivalence Relation:
Dfn: Let R be an equivalence relation defined on a set A. For any element $a \in A$, an ***equivalence class*** of a is the set $[a] = \{x \mid (a,x) \in R\}$.

Remarks:

1. Since $(a,a) \in R$, $a \in [a]$. Hence, $[a] \neq \varnothing$.
2. For any given set $A = \{a, b, \ldots, \alpha\}$, each element in A define an equivalence class $[a], [b], \ldots, [\alpha]$.
3. Some of these equivalence classes may be identical.

Dfn: Two elements x, y $\in$ A are ***equivalent*** iff they belong to the same equivalence class.
Notation: $x \sim y$.

Dfn: A ***partition*** of A is a collection of subsets A_1, A_2, …, A_k of A such that

(i) $A_i \cap A_j = \varnothing$, for $i \neq j$, $1 \leq i, j \leq k$, and
(ii) $A_1 \cup A_2 \cup \ldots \cup A_k = \bigcup_{i=1}^{k} A_i = A$.

Theorem: Given a set A and an equivalence relation R defined on A. The collection of all distinct equivalence classes of A forms a partition of A.

Example: Consider R_5 = {(1,1), (2,2), (3,3), (4,4), (1,2), (2,1), (1,4), (4,1), (2,4), (4,2)} defined on A = {1, 2, 3, 4}.
Observe that
[1] = [2] = [4] = {1, 2, 4},
[3] = {3}.
Hence, $1 \sim 2$, $1 \sim 4$, $2 \sim 4$.
Three partitions:
{[1], [3]}, {[2], [3]}, {[4], [3]}.

Closure of Relations:

Given a relation R defined on a set A and a (relation) property P.

Q: If the elements in R do not satisfy the given property P, how do we find the smallest extension of R, say R^*, such that R^* will satisfy property P?

Def: Given a relation R defined on a set A and a property P. The closure of R with respect to the property P (P-closure of R) is a relation R* defined on S such that

(1) R* satisfies the given property P,

(2) $R \subseteq R^*$, and

(3) if there exists another relation R** satisfying (1) and (2), then $|R^{**}| \geq |R^*|$.

Remarks:

1. The P-closure of R is the "smallest" relation R* defined on S that contains R and satisfying property P.
2. If R satisfies P, then $R^* = R$.
3. If $R \neq R^*$, then R* can be obtained from R by adding a minimum number of ordered pairs to R.

Example: Consider the relation R_1 = {(1,2), (1,5), (2,4), (3,1), (3,2), (4,3),(4,5), (5,2),(5,4)} defined on a set A = {1, 2, 3, 4, 5}.
The reflexive closure of R_1 is given by r(R) = {(1,2), (1,5), (2,4), (3,1), (3,2), (4,3),(4,5), (5,2),(5,4), ***(1,1), (2,2), (3,3), (4,4), (5,5)***}.

The symmetric closure of R_1 is given by s(R) = {(1,2), (1,5), (2,4), (3,1), (3,2), (4,3),(4,5), (5,2),(5,4), ***(2,1), (5,1), (4,2), (1,3), (2,3), (3,4), (2,5)***}.

The transitive closure of R_1 is given by t(R) = {(1,2), (1,5), (2,4), (3,1), (3,2), (4,3),(4,5), (5,2),(5,4), ***(1,1), (1,3), (1,4), (2,1), (2,2), (2,3), (2,5), (3,3), (3,4), (3,5), (4,1), (4,2),(4,4), (5,1), (5,3), (5,5)***}.

HW: Construct the directed graph representation for $r(R_1)$, $s(R_1)$, and $t(R_1)$.

HW: Let R be a relation defined on A. Design and analyze an efficient algorithm for computing the reflexive and symmetric closures of R.

Q: How about computing the transitive closure of a given relation R defined on A? TBA.

HW: Compute the reflexive, symmetric, and transitive closures for the following relations defined on the set $A = \{1, 2, 3, 4\}$.

$$R_1 = \{(1,2), (2,1), (1,3), (3,1), (4,4)\},$$
$$R_2 = \{(1,2), (2,3), (1,3), (3,4), (1,4), (2,4)\}.$$

Basic Operations on Relations:

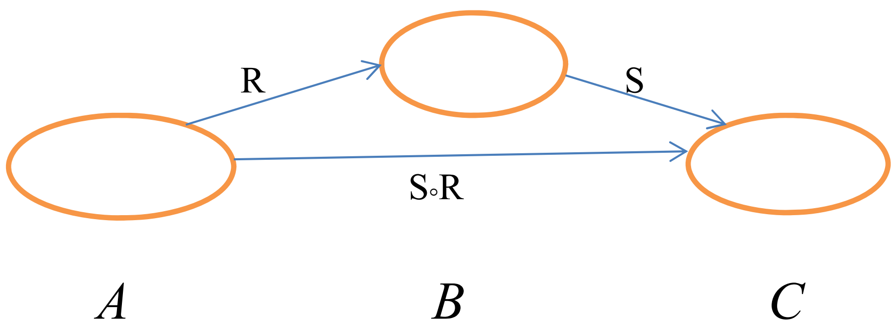
Since any relation is by itself a set, set operators can be applied to combine relations together.

Example: Given the following relations R_1 & R_2 defined on $A = \{1, 2, 3, 4\}$:

$$R_1 = \{(1,1), (1,2), (2,2), (3,3), (4,1), (4,4)\},$$
$$R_2 = \{(1,2), (1,3), (2,1), (3,1), (4,4)\}.$$

$$\begin{aligned}
R_1 \cup R_2 &= \{(1,1), (1,2), (2,2), (3,3), (4,1), (4,4), (2,1), \\
&\quad (1,3), (3,1)\}, \\
R_1 \cap R_2 &= \{(1,2), (4,4)\}, \\
R_1 - R_2 &= \{(1,1), (2,2), (3,3), (4,1)\}, \\
R_2 - R_1 &= \{(2,1), (1,3), (3,1)\}, \\
R_1 \oplus R_2 &= (R_1 - R_2) \cup (R_2 - R_1) = (R_1 \cup R_2) - (R_1 \cap R_2) \\
&= \{(1,1), (2,2), (3,3), (4,1), (2,1), (1,3), (3,1)\}.
\end{aligned}$$

Composition and Powers of Relation:
Given sets A, B, C. Let R be a relation defined from A to B and S be a relation defined from B to C.



Def: The ***composite of R and S*** is a relation defined from A to C such that

$$S \circ R = \{(a,c) \mid a \in A,\ c \in C, \text{ and } \exists b \in B \ni ((a,b) \in R \wedge (b,c) \in S)\}.$$

Example: Recall the relations R_1 & R_2 defined on $A = \{1, 2, 3, 4\}$ with

$R_1 = \{(1,1), (1,2), (2,2), (3,3), (4,1), (4,4)\}$,
$R_2 = \{(1,2), (1,3), (2,1), (3,1), (4,4)\}$.

$R_2 \circ R_1 = \{(1,1), (1,2), (1,3), (2,1), (3,1), (4,2), (4,3), (4,4)\}$,
$R_1 \circ R_2 = \{(1,2), (1,3), (2,1), (2,2), (3,1), (3,2), (4,1), (4,4)\}$.

In general, $R_2 \circ R_1 \neq R_1 \circ R_2$.

Let R be a relation defined on a set A.

Def: For any given positive integer n, the ***nth powers of R*** is defined recursively by

$R^1 = R$,

$R^n = R^{n-1} \circ R$, $n > 2$.

Examples:

1. Let $R = \{(1,1), (1,2), (2,2), (3,3), (4,1), (4,4)\}$.

$$
\begin{aligned}
R^1 &= \{(1,1), (1,2), (2,2), (3,3), (4,1), (4,4)\},\\
R^2 &= \{(1,1), (1,2), (2,2), (3,3), (4,1), (4,2), (4,4)\},\\
R^3 &= \{(1,1), (1,2), (2,2), (3,3), (4,1), (4,2), (4,4)\},\\
&= R^4 = R^5 = \ldots
\end{aligned}
$$

Observe that $R^2 \not\subset R$.

2. Let $S = \{(1,2), (2,3), (1,3), (3,4), (1,4), (2,4)\}$.

$$
\begin{aligned}
S^1 &= \{(1,2), (2,3), (1,3), (3,4), (1,4), (2,4)\},\\
S^2 &= \{(1,3), (1,4), (2,4)\},\\
S^3 &= \{(1,4)\},\\
S^4 &= \varnothing\\
&= S^5 = S^6 = \ldots
\end{aligned}
$$

Observe that $S^4 \subset R$.

Theorem: Any relation R defined on a set A is transitive iff $R^n \subseteq R$, $\forall\, n \in N$.

Example: The relation S above is a transitive relation.

Three Important Relations Defined on A:
Partial Ordering and Poset:

Def: Let R be a relation defined on a set S. R is a ***partial ordering***, or ***partial order***, iff R is reflexive, antisymmetric, and transitive.

The set S, together with the partial ordering R defined on it, forms a ***partially ordered set*** (***poset***), which is denoted by (S,R).

Examples:

1. Let $R_1 = \{(a,b) \mid a, b \in N, a|b\}$.
   R_1 is a partial ordering defined on N and (N, R_1) is a poset.

2. Let S be the set of all students in EECS210.
   Define a relation R_2 such that xR_2y iff x is taller than y. Since a person cannot be taller than himself/herself, R_2 is not reflexive. Hence, R_2 is not a partial ordering.

3. Let $R_3 = \{(1,1), (1,2), (2,2), (3,3), (4,1), (4,2), (4,4)\}$ be a relation defined on $S = \{1, 2, 3, 4\}$.
   Observe that R_3 forms a partial ordering on S. Hence, (S, R_3) is a poset.

4. Let R be the set of real number. Define a relation R_4 such that xR_4y iff $x \leq y$. Observe that $(R, \leq)$ forms a poset.

Def: Let (S,R) be poset. Two elements $x, y \in S$ are ***comparable*** iff either $(x,y) \in R$ or $(y,x) \in R$. Otherwise, they are ***incomparable***.

Remarks:

1. When comparing elements in a poset, we often use $\leq$ instead of R. Hence, we will use $(S, \leq)$ to denote a poset and $x \leq y$ to indicate that x is related to y.
   If x is related to y and $x \neq y$, we use $x < y$.
2. Elements in a poset may not all be comparable. In R_3 above, 1 and 4 are comparable but 3 and 4 are incomparable.
3. If all the elements in a poset are comparable, the poset $(S, \leq)$ forms a ***totally*** (***linearly***) ***ordered set*** (***chain***) and $\leq$ is a ***total (linear) ordering*** (***order***) defined on S.

Def: Let $(S, \leq)$ be a totally ordered set. An element $s \in S$ is a ***least element*** in S iff $\forall x \in S, s \leq x$.
The totally ordered set $(S, \leq)$ is a ***well-ordered set*** iff $\forall H \subseteq S, H \neq \varnothing$ implies that there exists a least element in H.

Example: Consider the set of all integers Z and the usual *less than or equal* relation ≤ defined on Z.

Observations:

(1) The relation ≤ is a partial ordering defined on Z and (Z, ≤) forms a poset.

(2) The relation ≤ also defines a total ordering on Z and (Z, ≤) also forms a totally ordered set.

(3) The relation (Z, ≤) is not a well-ordered set. Why?

(4) If we restricted our set to Z^+, $(Z^+, \leq)$ forms a well-ordered set.

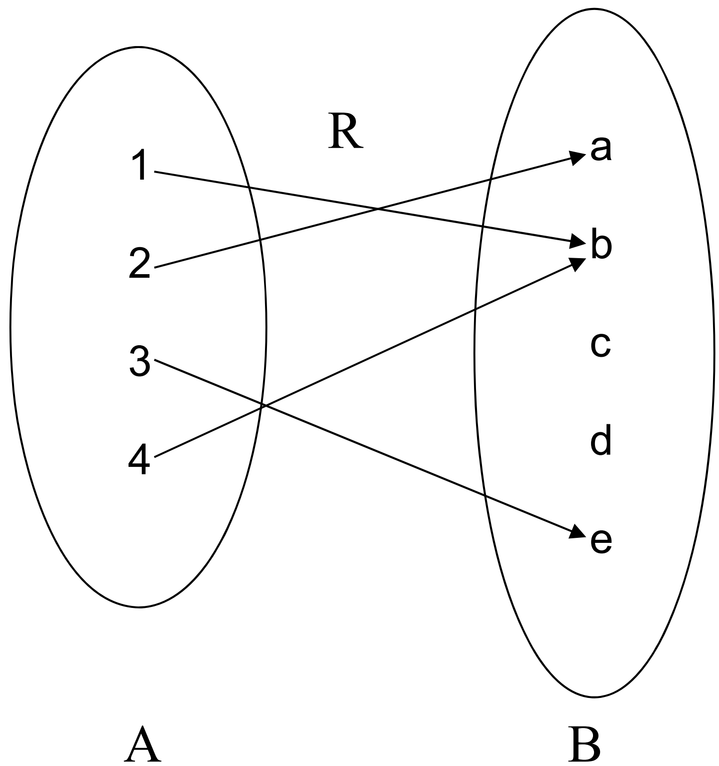
Practice HW:

Chpt. 9.1: 1, 3, 7, 15, 17, 27, 31, 33, 35
Chpt. 9.3: 1, 3, 7, 13, 15, 19, 23, 25, 27
Chpt. 9.5: 1, 7, 15, 21, 23, 41, 45
Chpt. 9.6: 1, 3, 5, 7, 9,15

Functions

Read: Chpt.2.3, Rosen

Given two sets A = {1, 2, 3, 4} and B = {1, b, c, d, e}. Consider the following relation R defined from A to B:



Consider the relation R = {(1,b), (2,a), (3,e), (4,b)}. Observe that for each element x $\in$ A, there exists a unique element y$\in$ B such that xRy. This type of special relation defines a function f from A to B.

Def. Let A and B be sets. A ***function*** (***mapping***) f from A to B is an assignment of elements from A to B such that for every element $a \in A$, there corresponds to exactly one element $b \in B$.

Notation: $f: A \to B$, and $b = f(a)$.

A — ***domain*** of f,

B — ***codomain*** of f,

b — ***image*** of a,

a — ***pre-image*** of b.

Remarks:

- $f: A \to B$ is a function from A to B iff $\forall a \in A$, $\exists\, !b \in B \ni b = f(a)$.

  Notation:

  !: unique
- Function is a special kind of relation such that very element in A must be related to exactly one element in B.
- Different elements in A can have the same image in B.
- Not every element in B is an image of an element in A.

Let f: $A \to B$ be a function.

Q: How do we describe a function?

Approach 1: Use English description to specify the image for all elements $a \in A$.

Approach 2: Use mathematical equation to specify the image $f(a)$ for all element $a \in A$.

Approach 3: Use set notation to construct the set of all ordered pairs $\{(a,b) \mid a \in A, b = f(a)\}$.

Examples:

1. (a) Let f: $\mathbf{R} \to \mathbf{R}$ be a function defined on $\mathbf{R}$ such that for every real number $x \in \mathbf{R}$, f assigns the square of x.

   $f(2) = 4$,
   $f(0.5) = 0.25$,
   $f(-1) = 1$.

   (b) Let f: $\mathbf{R} \to \mathbf{R}$ such that $f(x) = x^2, \forall x \in \mathbf{R}$.

   (c) Let f: $\mathbf{R} \to \mathbf{R}$ such that $f = \{(x,x^2) \mid x \in \mathbf{R}\}$.

2. (a) Let f: R $\to$ Z be a function that maps R to Z such that for every given real number x, f assigns the smallest integer that is larger or equal to x.

$f(3.5) = 4$,
$f(5) = 5$,
$f(-2) = -2$,
$f(-2.5) = -2$.

This is called the ***ceiling function***, $\lceil x \rceil$.

(b) Let f: R $\to$ Z such that $f(x) = \lceil x \rceil$, $\forall x \in R$.

(c) Let f: R $\to$ Z such that $f = \{(x, \lceil x \rceil) \mid x \in R\}$.

3. (a) Let f: R $\to$ Z be a function that maps R to Z such that for every given real number x, f assigns the greatest integer that is smaller or equal to x.

$f(3.5) = 3$,
$f(5) = 5$,
$f(-2) = -2$,
$f(-2.5) = -3$.

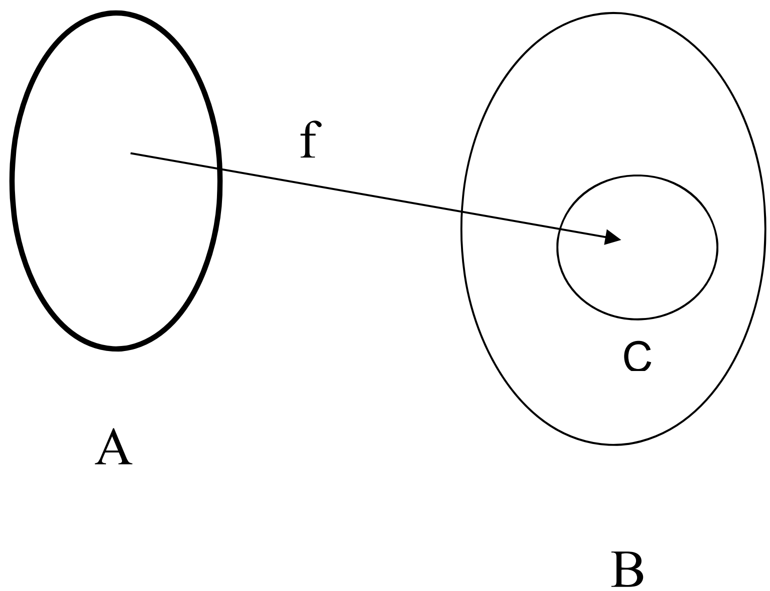
This is called the ***floor function***, $\lfloor x \rfloor$.

(b) Let f: R $\to$ Z such that $f(x) = \lfloor x \rfloor$, $\forall x \in R$.

(c) Let f: R $\to$ Z such that $f = \{(x, \lfloor x \rfloor) \mid x \in R\}$.

HW: Review ceiling and floor functions.

Recall that not all elements in B are images of some elements in A. Let's restrict B to the set of elements $C \subseteq B$ that are images of A under f.



Def. Let $C \subseteq B$ be the set containing all the images of the elements in A. The set C is the ***range*** of f.

Hence, $b \in C \leftrightarrow \exists\, a \in A \ni f(a) = b$.

Image of a Set:

Def. Let $S \subseteq A$. The set $f(S) = \{f(s) \mid s \in S\}$ is the ***image*** of S.

Some Important Functions:

Let f: $A \to B$ be a given function.

Def. A function f is an ***injection***, or ***one-to-one function***, iff distinct elements in A must have distinct images in B.

Hence, $\forall\ x, y \in A,\ x \neq y \Leftrightarrow f(x) \neq f(y)$.

Equivalently, $\forall\ x, y \in A,\ f(x) = f(y) \Leftrightarrow x = y$.

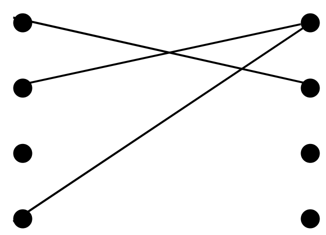
Def. A function f is a ***surjection***, or ***onto function***, iff every element in B is the image of some element(s) in A.

Hence, $\forall\ b \in B,\ \exists\ a \in A \ni b = f(a)$.

Def. A function f is a ***bijection***, or ***one-to-one correspondence***, iff it is both an injection and a surjection.

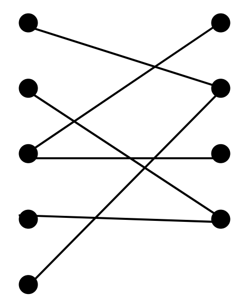
Example:

1.



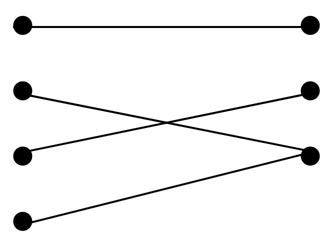
Not a function

2.



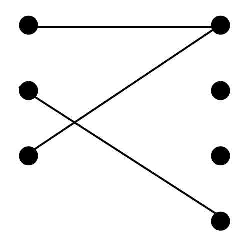
Not a function

3.



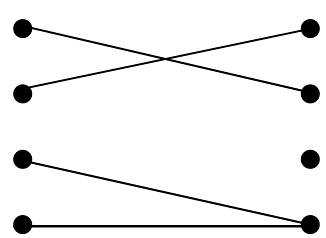
Onto but not 1-1

4.



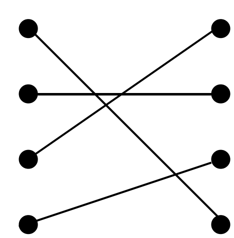
1-1 but not onto

5.



Not onto, not 1-1

6.



1-1 correspondence

Operations on Functions:
Sum, Product, and Composition of Functions:
Let f_1 and f_2 be two real-valued functions defined from $S \to R$.

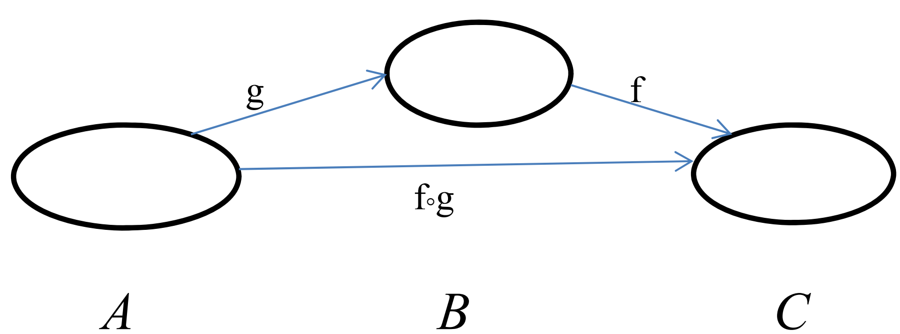
Def: The ***sum of f_1 and f_2*** is a function $(f_1+f_2):S \to R$ defined by $(f_1+f_2)(x) = f_1(x)+f_2(x)$.

Def: The ***product of f_1 and f_2*** is a function $(f_1*f_2):S \to R$ defined by $(f_1*f_2)(x) = f_1(x)*f_2(x)$.

Let $g: A \to B$ and $f: B \to C$ be two given functions.

Def. The ***composition of f and g*** is a function $(f\circ g): A \to C$ defined by $(f\circ g)(x) = f(g(x))$.

The composition of f and g, $(f\circ g)$: $A \to C$:



Example: Let $f_1: R \to R$ and $f_2: R \to R$ be two real-valued functions defined by $f_1(x) = x+1$ and $f_2(x) = 2x-5$.

$$
\begin{aligned}
(f_1+f_2)(x) &= f_1(x)+f_2(x) \\
&= (x+1)+(2x-5) \\
&= 3x-4
\end{aligned}
$$

$$
\begin{aligned}
(f_1 * f_2)(x) &= f_1(x) * f_2(x) \\
&= (x+1) * (2x-5) \\
&= 2x^2-3x-5
\end{aligned}
$$

$$
\begin{aligned}
(f_1 \circ f_2)(x) &= f_1(f_2(x)) \\
&= f_1(2x-5) \\
&= (2x-5)+1 \\
&= 2x-4
\end{aligned}
$$

$$
\begin{aligned}
(f_2 \circ f_1)(x) &= f_2(f_1(x)) \\
&= f_2(x+1) \\
&= 2(x+1)-5 \\
&= 2x-3
\end{aligned}
$$

Observe that $(f_1 \circ f_2) \neq (f_2 \circ f_1)$.

Given a bijection $f: A \to B$.

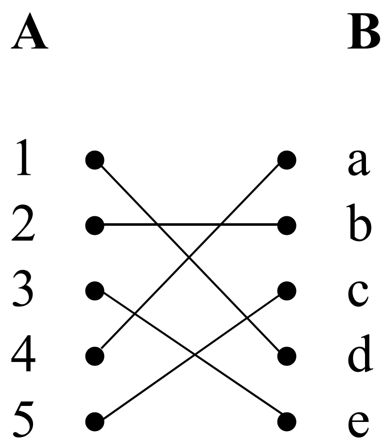
Observations:

1. If both A and B are finite sets, then $|A| = |B|$.
2. Every element $a \in A$ has a unique image $b \in B$.
3. Every element $b \in B$ has a unique pre-image $a \in A$ such that $b = f(a)$.
4. If $f = \{(a,f(a)) \mid a \in A\}$, then $f^{-1} = \{(f(a),a) \mid a \in A\}$ defines a function from B to A; $f^{-1}: B \to A$ is the ***inverse function*** of f.
5. If f^{-1} is an inverse function of f, then $(a,b) \in f$ iff $(b,a) \in f^{-1}$. Also, $f(a) = b$ iff $f^{-1}(b) = a$.

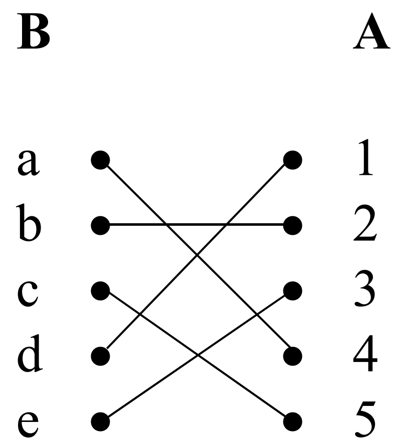
Def: Given a function $f: A \to B$. The function $f^{-1}: B \to A$ is the ***inverse function*** of f iff $\forall\, b \in B$, $(f \circ f^{-1})(b) = b$ and $\forall\, a \in A$, $(f^{-1} \circ f)(a) = a$. If the inverse function of f exists, f is an ***invertible function***.

HW: Review logarithmic and exponential functions.

Example:



f



$\mathbf{f^{-1}}$

$f \;=\; \{(1,d), (2,b), (3,e), (4,a), (5,c)\}$

$f^{-1} = \{(a,4), (b,2), (c,5), (d,1), (e,3)\}$

Q: Given a function f: A $\rightarrow$ B. How do we prove that f is an injection, surjection, or bijection?

Examples:

1. Consider the function $f: R \to R$ with $f(x) = (3x-4)/5$.

   (a) Proving that f is an injection:

   Need to show that $\forall x, y \in R$, $f(x) = f(y)$ implies $x = y$.

$$\begin{aligned} f(x) = f(y) &\to (3x-4)/5 = (3y-4)/5 \\ &\to 3x-4 = 3y-4 \\ &\to 3x = 3y \\ &\to x = y \end{aligned}$$

   Hence, f is an injection.

   (b) Proving that f is a surjection:

   Need to show that $\forall\ y \in R, \exists\ x \in R, \ni y = f(x)$.

$$\begin{aligned} \text{Observe that } f(x) = y &\to (3x-4)/5 = y \\ &\to 3x-4 = 5y \\ &\to x = (5y+4)/3 \end{aligned}$$

   Verification:

$$\begin{aligned} f(x) &= f((5y+4)/3) \\ &= \{3[(5y+4)/3]-4\}/5 \\ &= y \end{aligned}$$

   Since $\forall\ y \in R$, we can choose a real number $x = (5y+4)/3$ such that $y = f(x)$, f is indeed a surjection.

(c) Computing the inverse of f if f is invertible:
Define $f^{-1}(y) = (5y+4)/3$. Claim that f^{-1} is the inverse function of f.

Proof:

$$(f \circ f^{-1})(y) = f(f^{-1}(y) = f((5y+4)/3) = \{3[(5y+4)/3]-4\}/5 = y$$

$$(f^{-1} \circ f)(x) = f^{-1}(f(x) = f^{-1}((3x-4)/5) = \{5[(3x-4)/5]+4\}/3 = x$$

Since $(f \circ f^{-1})(y) = y$ and $(f^{-1} \circ f)(x) = x$, by definition of inverse function, f^{-1} is the inverse function of f.

2. Consider the assignment g: R → R with $g(x)=\dfrac{3x-4}{x+5}$.

(a) Observe that since x ≠ -5, g is NOT a function.

(b) Let's restricted the domain of g to R−{-5}. Hence, g: R−{-5} → R with $g(x)=\dfrac{3x-4}{x+5}$.
Observe now that g has become a function. (Why?)

(c) Proving that g is an injection:
Need to show that ∀x, y ∈ R−{-5}, g(x) = g(y) implies x = y.

$$g(x)=g(y)$$
$$\rightarrow \frac{3x-4}{x+5}=\frac{3y-4}{y+5}$$
$$\rightarrow (3x-4)(y+5)=(3y-4)(x+5)$$
$$\rightarrow 3xy+15x-4y-20=3xy+15y-4x-20$$
$$\rightarrow 15x+4x=15y+4y$$
$$\rightarrow 19x=19y$$
$$\rightarrow x=y.$$

Hence, g is an injection.

(d) Proving that g is a surjection:

Need to show that $\forall$ y $\in$ R, $\exists$ x $\in$ R$-\{$-5$\}$ ∍ g(x) = y.

$$g(x) = y$$

$$\to \frac{3x-4}{x+5} = y$$

$$\to 3x-4 = y(x+5)$$

$$\to 3x-4 = xy+5y$$

$$\to 3x-xy = 5y+4$$

$$\to x = \frac{5y+4}{3-y}.$$

Observe that y $\neq$3 $\in$ R. Hence, g is NOT a surjection.

(e) By restricting the range of g to R$-\{3\}$, the function g: R$-\{$-5$\}$ $\to$ R$-\{3\}$ with $g(x) = \frac{3x-4}{x+5}$ forms a bijection.

(f) Computing the inverse of g:

Take $g^{-1}(y) = \frac{5y+4}{3-y}$.

Verification of inverse:

$$
\begin{aligned}
&(g \circ g^{-1})(y) \\
&= g((g^{-1}(y)) \\
&= g(\frac{5y+4}{3-y}) \\
&= \frac{3(\frac{5y+4}{3-y}) - 4}{(\frac{5y+4}{3-y}) + 5} \\
&= \frac{15y + 12 - 12 + 4y}{5y + 4 + 15 - 5y} \\
&= y.
\end{aligned}
$$

$$
\begin{aligned}
&(g^{-1} \circ g)(x) \\
&= g^{-1}((g(x)) \\
&= g(\frac{3x-4}{x+5}) \\
&= \frac{5(\frac{3x-4}{x+5}) + 4}{3 - (\frac{3x-4}{x+5})} \\
&= \frac{15x - 20 + 4x + 20}{3x + 15 - 3x + 4} \\
&= x.
\end{aligned}
$$

Practice HW: Chpt.2.3: 1, 7, 9, 11, 13, 15, 21, 23, 25.

9/17/17