10703 Deep Reinforcement Learning and Control

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Slides borrowed from
Katerina Fragkiadaki

Solving known MDPs: Dynamic Programming

Markov Decision Process (MDP)

A **Markov Decision Process** is a tuple $(\mathcal{S}, \mathcal{A}, T, r, \gamma)$

- $\mathcal{S}$ is a finite set of states
- $\mathcal{A}$ is a finite set of actions
- T is a state transition probability function

$$T(s'|s,a) = \mathbb{P}[S_{t+1} = s' | S_t = s, A_t = a]$$

- r is a reward function

$$r(s,a) = \mathbb{E}[R_{t+1} | S_t = s, A_t = a]$$

- γ is a discount factor $\gamma \in [0,1]$

Solving MDPs

- **Prediction**: Given an MDP $(\mathcal{S}, \mathcal{A}, T, r, \gamma)$ and a policy

$$\pi(a|s) = \mathbb{P}[A_t = a | S_t = s]$$

  find the state and action value functions.

- **Optimal control**: given an MDP $(\mathcal{S}, \mathcal{A}, T, r, \gamma)$ find the optimal policy (aka the planning problem).

- Compare with the learning problem with missing information about rewards/dynamics.

- We still consider finite MDPs (finite $\mathcal{S}$ and $\mathcal{A}$) with known dynamics.

Outline

- Policy evaluation
- Policy iteration
- Value iteration
- Asynchronous DP

Policy Evaluation

Policy evaluation: for a given policy π, compute the state value function

$$\mathrm{v}_\pi(s) = \mathbb{E}_\pi\left[R_{t+1} + \gamma R_{t+2} + \gamma^2 R_{t+3} + \ldots | S_t = s\right]$$

where $\mathrm{v}_\pi(s)$ is implicitly given by the **Bellman equation**

$$\mathrm{v}_\pi(s) = \sum_{a \in \mathcal{A}} \pi(a|s)\left(r(s,a) + \gamma \sum_{s' \in \mathcal{S}} T(s'|s,a)\mathrm{v}_\pi(s')\right)$$

a system of $|\mathcal{S}|$ simultaneous equations.

MDPs to MRPs

MDP under a fixed policy becomes **Markov Reward Process** (MRP)

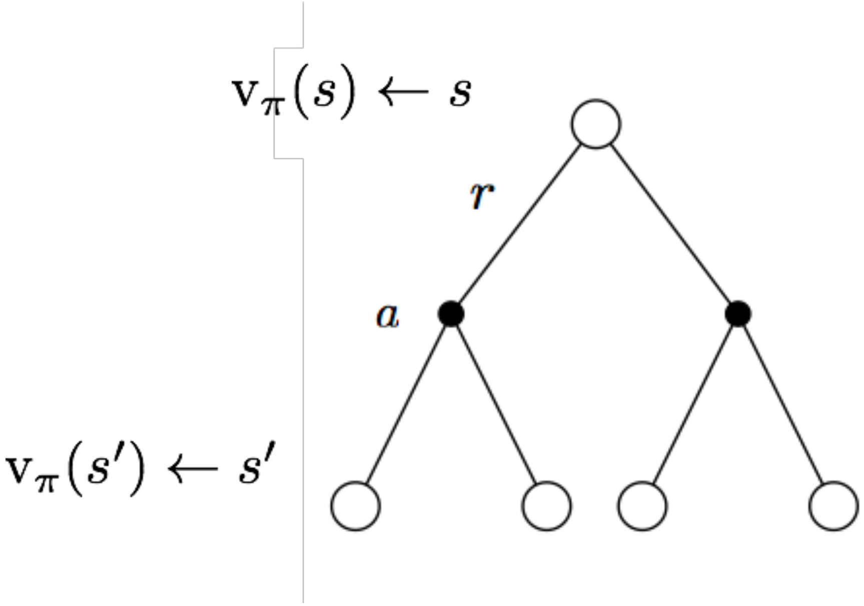
$$\begin{aligned}
\mathrm{v}_\pi(s) &= \sum_{a \in \mathcal{A}} \pi(a|s) \left(r(s,a) + \gamma \sum_{s' \in \mathcal{S}} T(s'|s,a) \mathrm{v}_\pi(s') \right) \\
&= \sum_{a \in \mathcal{A}} \pi(a|s) r(s,a) + \gamma \sum_{a \in \mathcal{A}} \pi(a|s) \sum_{s' \in \mathcal{S}} T(s'|s,a) \mathrm{v}_\pi(s') \\
&= r_s^\pi + \gamma \sum_{s' \in \mathcal{S}} T_{s's}^\pi \mathrm{v}_\pi(s')
\end{aligned}$$

where $\quad r_s^\pi = \sum_{a \in \mathcal{A}} \pi(a|s) r(s,a)$

$$T_{s's}^\pi = \sum_{a \in \mathcal{A}} \pi(a|s) T(s'|s,a)$$

Back Up Diagram

MDP



Back Up Diagram

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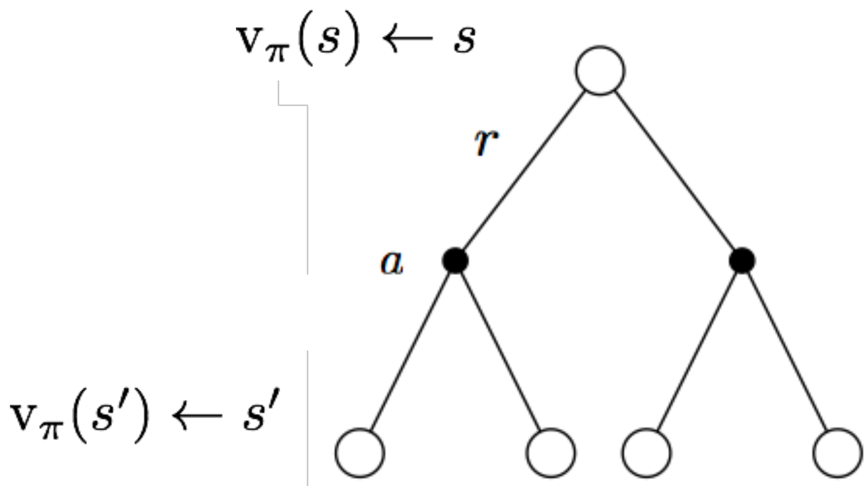
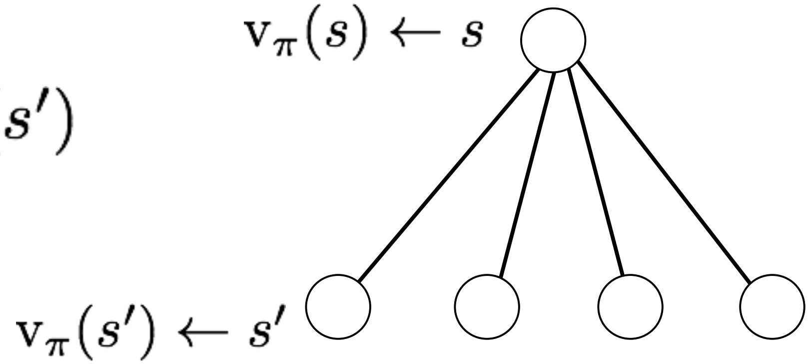


$$v_\pi(s) = r_s^\pi + \gamma \sum_{s' \in \mathcal{S}} T_{s's}^\pi v_\pi(s')$$



Matrix Form

The Bellman expectation equation can be written concisely using the induced form:

$$\mathrm{v}_\pi = r^\pi + \gamma T^\pi \mathrm{v}_\pi$$

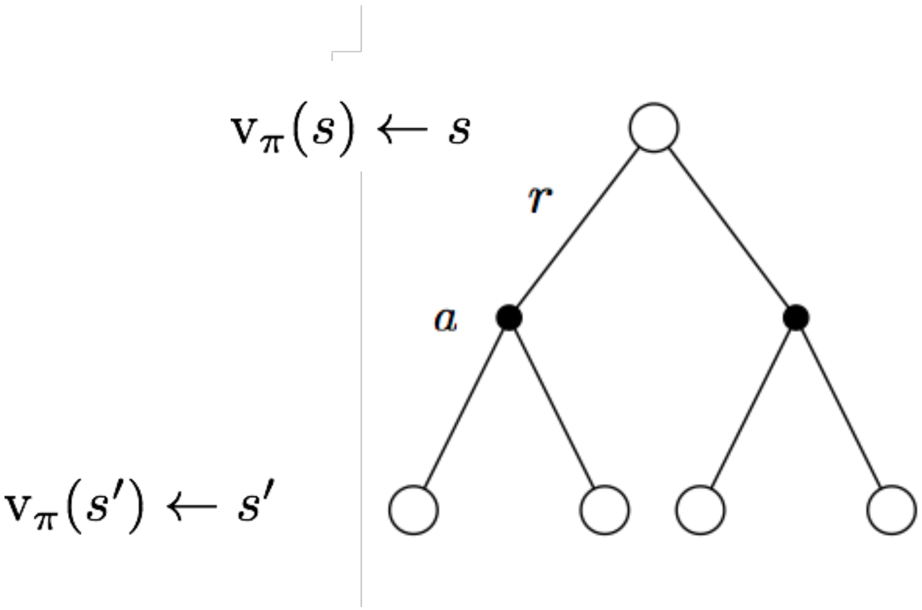
with direct solution

$$\mathrm{v}_\pi = (I - \gamma T^\pi)^{-1} r^\pi$$

of complexity $O(N^3)$

Iterative Methods: Recall the Bellman Equation

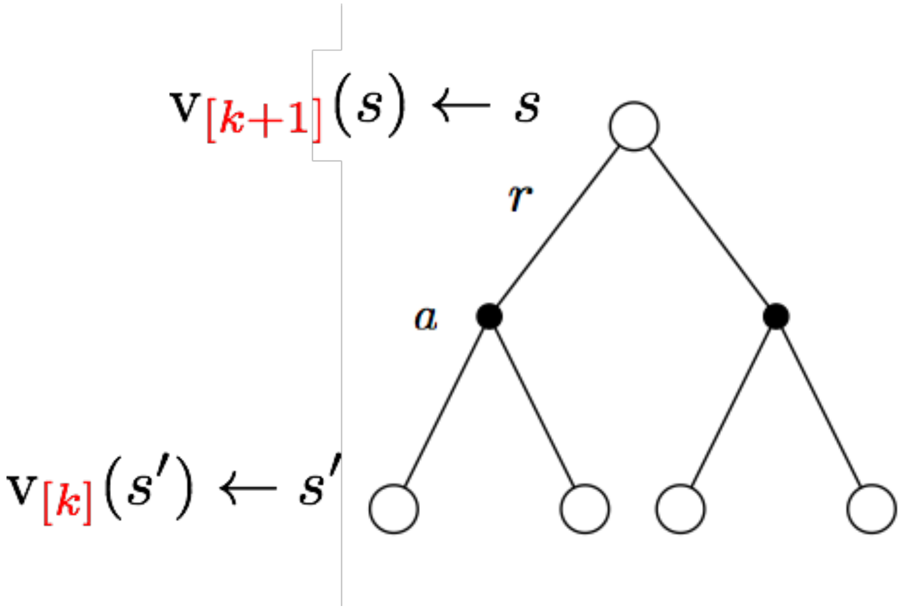
$$\mathrm{v}_\pi(s) = \sum_{a \in \mathcal{A}} \pi(a|s) \left(r(s,a) + \gamma \sum_{s' \in \mathcal{S}} T(s'|s,a) \mathrm{v}_\pi(s') \right)$$



Iterative Methods: Backup Operation

Given an expected value function at iteration *k*, we back up the expected value function at iteration *k+1:*

$$\mathrm{v}_{[k+1]}(s) = \sum_{a \in \mathcal{A}} \pi(a|s) \left(r(s,a) + \gamma \sum_{s' \in \mathcal{S}} T(s'|s,a) \mathrm{v}_{[k]}(s') \right)$$



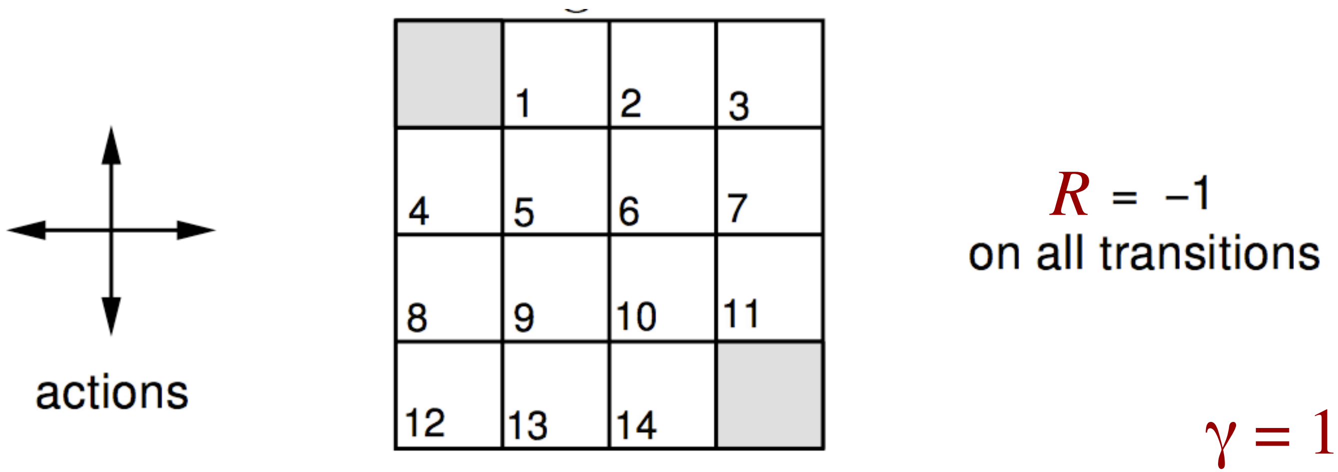
Iterative Methods: Sweep

A **sweep** consists of applying the backup operation $\mathbf{v} \rightarrow \mathbf{v}'$ for all the states in $\mathcal{S}$

Applying the back up operator iteratively

$$\mathbf{v}_{[0]} \rightarrow \mathbf{v}_{[1]} \rightarrow \mathbf{v}_{[2]} \rightarrow \ldots \mathbf{v}_{\pi}$$

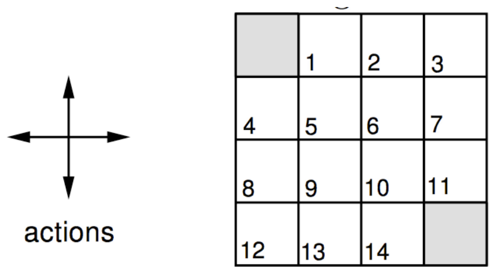
A Small-Grid World



- An undiscounted episodic task
- Nonterminal states: 1, 2, … , 14
- Terminal state: one, shown in shaded square
- Actions that would take the agent off the grid leave the state unchanged
- Reward is -1 until the terminal state is reached

Iterative Policy Evaluation

Policy π, an equiprobable random action



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$\mathbf{V}[k]$ for the random policy

$k = 0$

0.0	0.0	0.0	0.0
0.0	0.0	0.0	0.0
0.0	0.0	0.0	0.0
0.0	0.0	0.0	0.0

$k = 1$

0.0	-1.0	-1.0	-1.0
-1.0	-1.0	-1.0	-1.0
-1.0	-1.0	-1.0	-1.0
-1.0	-1.0	-1.0	0.0

$k = 2$

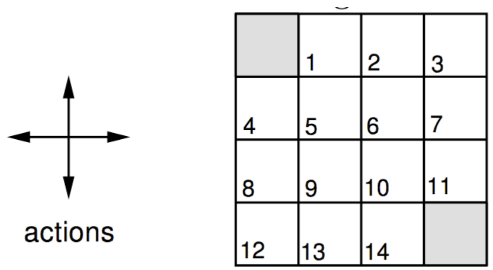
$k = 3$

$k = 10$

$k = \infty$

Iterative Policy Evaluation

Policy π, an equiprobable random action



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$k = 0$

0.0	0.0	0.0	0.0
0.0	0.0	0.0	0.0
0.0	0.0	0.0	0.0
0.0	0.0	0.0	0.0

$k = 1$

0.0	-1.0	-1.0	-1.0
-1.0	-1.0	-1.0	-1.0
-1.0	-1.0	-1.0	-1.0
-1.0	-1.0	-1.0	0.0

$k = 2$

0.0	-1.7	-2.0	-2.0
-1.7	-2.0	-2.0	-2.0
-2.0	-2.0	-2.0	-1.7
-2.0	-2.0	-1.7	0.0

$k = 3$

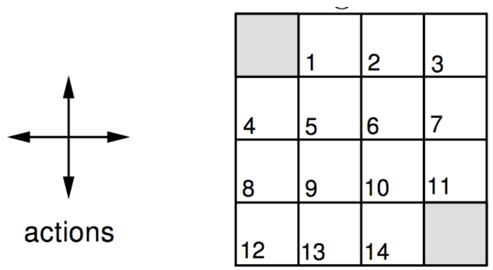
0.0	-2.4	-2.9	-3.0
-2.4	-2.9	-3.0	-2.9
-2.9	-3.0	-2.9	-2.4
-3.0	-2.9	-2.4	0.0

$k = 10$

$k = \infty$

Iterative Policy Evaluation

Policy π, an equiprobable random action



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$\mathbf{V}[k]$ for the random policy

$k = 0$

0.0	0.0	0.0	0.0
0.0	0.0	0.0	0.0
0.0	0.0	0.0	0.0
0.0	0.0	0.0	0.0

$k = 1$

0.0	-1.0	-1.0	-1.0
-1.0	-1.0	-1.0	-1.0
-1.0	-1.0	-1.0	-1.0
-1.0	-1.0	-1.0	0.0

$k = 2$

0.0	-1.7	-2.0	-2.0
-1.7	-2.0	-2.0	-2.0
-2.0	-2.0	-2.0	-1.7
-2.0	-2.0	-1.7	0.0

$k = 3$

0.0	-2.4	-2.9	-3.0
-2.4	-2.9	-3.0	-2.9
-2.9	-3.0	-2.9	-2.4
-3.0	-2.9	-2.4	0.0

$k = 10$

0.0	-6.1	-8.4	-9.0
-6.1	-7.7	-8.4	-8.4
-8.4	-8.4	-7.7	-6.1
-9.0	-8.4	-6.1	0.0

$k = \infty$

0.0	-14.	-20.	-22.
-14.	-18.	-20.	-20.
-20.	-20.	-18.	-14.
-22.	-20.	-14.	0.0

Contraction Mapping Theorem

An operator F on a normed vector space $\mathcal{X}$ is a γ**-contraction**, for $0 < \gamma < 1$, provided for all $x, y \in \mathcal{X}$

$$||T(x) - T(y)|| \leq \gamma||x - y||$$

Contraction Mapping Theorem

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$$||T(x) - T(y)|| \leq \gamma||x - y||$$

Theorem (Contraction mapping)
For a γ-contraction $\boldsymbol{F}$ in a complete normed vector space $\boldsymbol{\mathcal{X}}$

- $\boldsymbol{F}$ converges to a unique fixed point in $\boldsymbol{\mathcal{X}}$
- at a linear convergence rate γ

Remark. In general we only need metric (vs normed) space

Value Function Sapce

- Consider the vector space V over value functions
- There are $|\mathcal{S}|$ dimensions
- Each point in this space fully specifies a value function $\mathbf{v}(s)$
- Bellman backup brings value functions closer in this space?
- And therefore the backup must converge to a unique solution

Value Function ∞-Norm

- We will measure distance between state-value functions $\mathbf{u}$ and $\mathbf{v}$ by the ∞-norm
- i.e. the largest difference between state values:

$$||\mathbf{u} - \mathbf{v}||_\infty = \max_{s \in \mathcal{S}} |\mathbf{u}(s) - \mathbf{v}(s)|$$

Bellman Expectation Backup is a Contraction

- Define the Bellman expectation backup operator

$$F^{\pi}(\mathrm{v}) = r^{\pi} + \gamma T^{\pi}\mathrm{v}$$

- This operator is a γ-contraction, i.e. it makes value functions closer by at least γ,

$$\begin{aligned}
||F^{\pi}(\mathrm{u}) - F^{\pi}(\mathrm{v})||_{\infty} &= ||(r^{\pi} + \gamma T^{\pi}\mathrm{u})||_{\infty} - ||(r^{\pi} + \gamma T^{\pi}\mathrm{v})||_{\infty} \\
&= ||\gamma T^{\pi}(\mathrm{u} - \mathrm{v})||_{\infty} \\
&\leq ||\gamma T^{\pi}||\mathrm{u} - \mathrm{v}||_{\infty}||_{\infty} \\
&\leq \gamma||\mathrm{u} - \mathrm{v}||_{\infty}
\end{aligned}$$

Convergence of Iter. Policy Evaluation and Policy Iteration

- The Bellman expectation operator F^{π} has a unique fixed point
- $\mathbf{v}_{\pi}$ is a fixed point of F^{π} (by Bellman expectation equation)
- By contraction mapping theorem
- Iterative policy evaluation converges on $\mathbf{v}_{\pi}$

Policy Improvement

- Suppose we have computed v_π for a deterministic policy π
- For a given state s, would it be better to do an action $a \neq \pi(s)$?
- It is better to switch to action a for state s if and only if

$$q_\pi(s, a) > \text{v}_\pi(s)$$

- And we can compute $q_\pi(s, a)$ from v_π by:

$$\begin{aligned} q_\pi(s, a) &= \mathbb{E}[R_{t+1} + \gamma \text{v}_\pi(S_{t+1})|S_t = s, A_t = a] \\ &= r(s, a) + \gamma \sum_{s' \in \mathcal{S}} T(s'|s, a)\text{v}_\pi(s') \end{aligned}$$

```
q_\pi(s, a) & = \mathbb{E}[R_{t+1} + \gamma \text{v}_\pi(S_{t+1})|S_t=s,A_t=a] \\
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```
q_\pi(s, a) & = \mathbb{E}[R_{t+1} + \gamma \text{v}_\pi(S_{t+1})|S_t=s,A_t=a] \\
```

Policy Improvement Cont.

- Do this for all states to get a new policy $\pi' \geq \pi$ that is greedy with respect to v_π:

$$\begin{aligned}
\pi'(s) &= \arg\max_a q_\pi(s, a) \\
&= \arg\max_a \mathbb{E}[R_{t+1} + \gamma \mathrm{v}_\pi(s') | S_t = s, A_t = a] \\
&= \arg\max r(s, a) + \gamma \sum_{s' \in \mathcal{S}} T(s'|s, a) \mathrm{v}_\pi(s')
\end{aligned}$$

- What if the policy is unchanged by this?
  - Then the policy must be optimal.

```
\pi'(s) & =  \arg\max_{a} q_\pi(s, a) \\
```

Policy Iteration

$$\pi_0 \xrightarrow{\mathrm{E}} v_{\pi_0} \xrightarrow{\mathrm{I}} \pi_1 \xrightarrow{\mathrm{E}} v_{\pi_1} \xrightarrow{\mathrm{I}} \pi_2 \xrightarrow{\mathrm{E}} \cdots \xrightarrow{\mathrm{I}} \pi_* \xrightarrow{\mathrm{E}} v_*$$

policy evaluation

policy improvement
"greedification"

Policy Iteration

1. Initialization

   $V(s) \in \mathbb{R}$ and $\pi(s) \in \mathcal{A}(s)$ arbitrarily for all $s \in \mathcal{S}$

2. Policy Evaluation

   Repeat

   $\Delta \leftarrow 0$

   For each $s \in \mathcal{S}$:

   $v \leftarrow V(s)$

   $V(s) \leftarrow \boxed{\Sigma_{a \in \mathcal{A}} \pi(a|s) \left(r(s,a) + \gamma \Sigma_{s' \in \mathcal{S}} T(s'|s,a) V(s') \right)}$

   $\Delta \leftarrow \max(\Delta, |v - V(s)|)$

   until $\Delta < \theta$ (a small positive number)

3. Policy Improvement

   policy-stable $\leftarrow$ *true*

   For each $s \in \mathcal{S}$:

   $a \leftarrow \pi(s)$

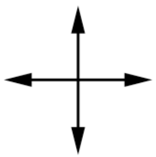
   $\pi(s) \leftarrow \boxed{\arg\max r(s,a) + \gamma \Sigma_{s' \in \mathcal{S}} T(s'|s,a) \mathrm{v}_\pi(s')}$

   If $a \neq \pi(s)$, then *policy-stable* $\leftarrow$ *false*

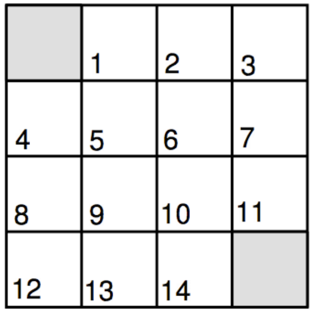
   If *policy-stable*, then stop and return V and π; else go to 2

Iterative Policy Eval for the Small Gridworld

Policy π, an equiprobable random action



actions



$R = -1$
on all transitions

$\gamma = 1$

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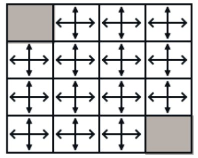
V_k for the Random Policy

Greedy Policy w.r.t. V_k

$k = 0$

0.0	0.0	0.0	0.0
0.0	0.0	0.0	0.0
0.0	0.0	0.0	0.0
0.0	0.0	0.0	0.0



random policy

$k = 1$

0.0	-1.0	-1.0	-1.0
-1.0	-1.0	-1.0	-1.0
-1.0	-1.0	-1.0	-1.0
-1.0	-1.0	-1.0	0.0

$k = 2$

0.0	-1.7	-2.0	-2.0
-1.7	-2.0	-2.0	-2.0
-2.0	-2.0	-2.0	-1.7
-2.0	-2.0	-1.7	0.0

$k = 3$

0.0	-2.4	-2.9	-3.0
-2.4	-2.9	-3.0	-2.9
-2.9	-3.0	-2.9	-2.4
-3.0	-2.9	-2.4	0.0

$k = 10$

0.0	-6.1	-8.4	-9.0
-6.1	-7.7	-8.4	-8.4
-8.4	-8.4	-7.7	-6.1
-9.0	-8.4	-6.1	0.0

$k = \infty$

0.0	-14.	-20.	-22.
-14.	-18.	-20.	-20.
-20.	-20.	-18.	-14.
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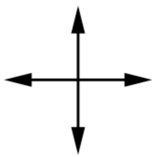
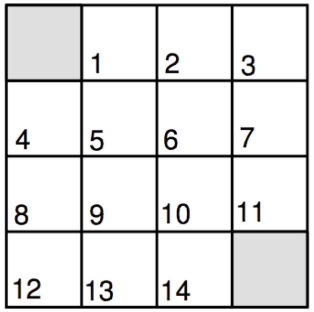
Iterative Policy Eval for the Small Gridworld

Policy π, an equiprobable random action





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V_k for the Random Policy

$k = 0$

0.0	0.0	0.0	0.0
0.0	0.0	0.0	0.0
0.0	0.0	0.0	0.0
0.0	0.0	0.0	0.0

$k = 1$

0.0	-1.0	-1.0	-1.0
-1.0	-1.0	-1.0	-1.0
-1.0	-1.0	-1.0	-1.0
-1.0	-1.0	-1.0	0.0

$k = 2$

0.0	-1.7	-2.0	-2.0
-1.7	-2.0	-2.0	-2.0
-2.0	-2.0	-2.0	-1.7
-2.0	-2.0	-1.7	0.0

$k = 3$

0.0	-2.4	-2.9	-3.0
-2.4	-2.9	-3.0	-2.9
-2.9	-3.0	-2.9	-2.4
-3.0	-2.9	-2.4	0.0

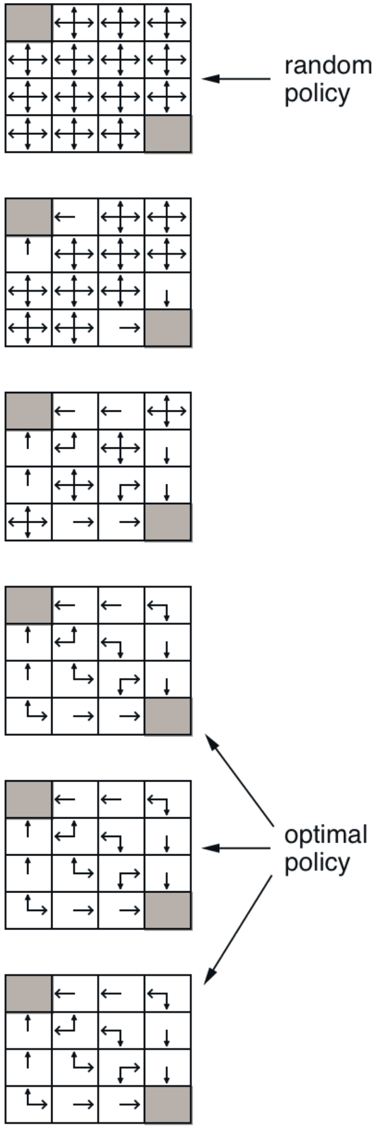
$k = 10$

0.0	-6.1	-8.4	-9.0
-6.1	-7.7	-8.4	-8.4
-8.4	-8.4	-7.7	-6.1
-9.0	-8.4	-6.1	0.0

$k = \infty$

0.0	-14.	-20.	-22.
-14.	-18.	-20.	-20.
-20.	-20.	-18.	-14.
-22.	-20.	-14.	0.0

Greedy Policy w.r.t. V_k

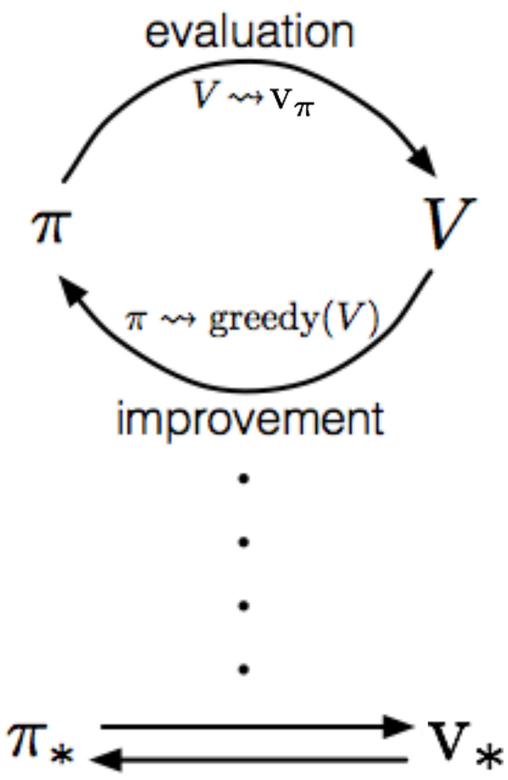


Generalized Policy Iteration

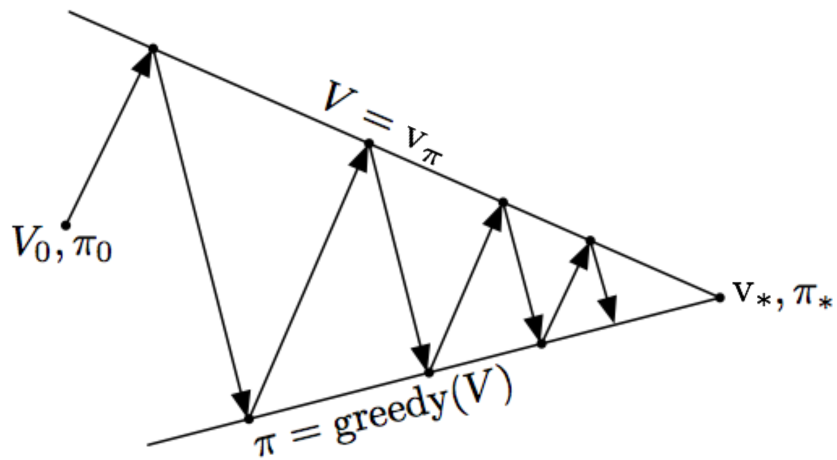
- Does policy evaluation need to converge to v_π ?
- Or should we introduce a stopping condition
  - e.g. ϵ-convergence of value function
- Or simply stop after k iterations of iterative policy evaluation?
- For example, in the small grid world $k = 3$ was sufficient to achieve optimal policy
- Why not update policy every iteration? i.e. stop after $k = 1$
  - This is equivalent to value iteration (next section)

Generalized Policy Iteration

Generalized Policy Iteration (GPI): any interleaving of policy evaluation and policy improvement, independent of their granularity.



A geometric metaphor for convergence of GPI:



Principle of Optimality

- Any optimal policy can be subdivided into two components:
  - An optimal first action $\mathcal{A}_*$
  - Followed by an optimal policy from successor state $\mathcal{S}'$

- Theorem (Principle of Optimality)
  - A policy $\pi(a|s)$ achieves the optimal value from state s, $\mathrm{v}_\pi(s) = \mathrm{v}_*(s)$, if and only if
  - For any state s' reachable from s, π achieves the optimal value from state s', $\mathrm{v}_\pi(s') = \mathrm{v}_*(s')$

Value Iteration

- If we know the solution to subproblems $\mathrm{v}_*(s')$
- Then solution $\mathrm{v}_*(s')$ can be found by one-step lookahead

$$\mathrm{v}_*(s) \leftarrow \max_{a \in \mathcal{A}} r(s,a) + \gamma \sum_{s' \in \mathcal{S}} T(s'|s,a)\mathrm{v}_*(s')$$

- The idea of value iteration is to apply these updates iteratively
- Intuition: start with final rewards and work backwards
- Still works with loopy, stochastic MDPs

\text{v}_*(s) \leftarrow \max_{a \in \mathcal{A}}{ r(s,a) + \gamma \sum_{s'\in \mathcal{S}} T(s'|s,a) {\text{v}_*(s')} }

Example: Shortest Path

g			

Problem

0	0	0	0
0	0	0	0
0	0	0	0
0	0	0	0

V_1

0	-1	-1	-1
-1	-1	-1	-1
-1	-1	-1	-1
-1	-1	-1	-1

V_2

0	-1	-2	-2
-1	-2	-2	-2
-2	-2	-2	-2
-2	-2	-2	-2

V_3

0	-1	-2	-3
-1	-2	-3	-3
-2	-3	-3	-3
-3	-3	-3	-3

V_4

0	-1	-2	-3
-1	-2	-3	-4
-2	-3	-4	-4
-3	-4	-4	-4

V_5

0	-1	-2	-3
-1	-2	-3	-4
-2	-3	-4	-5
-3	-4	-5	-5

V_6

0	-1	-2	-3
-1	-2	-3	-4
-2	-3	-4	-5
-3	-4	-5	-6

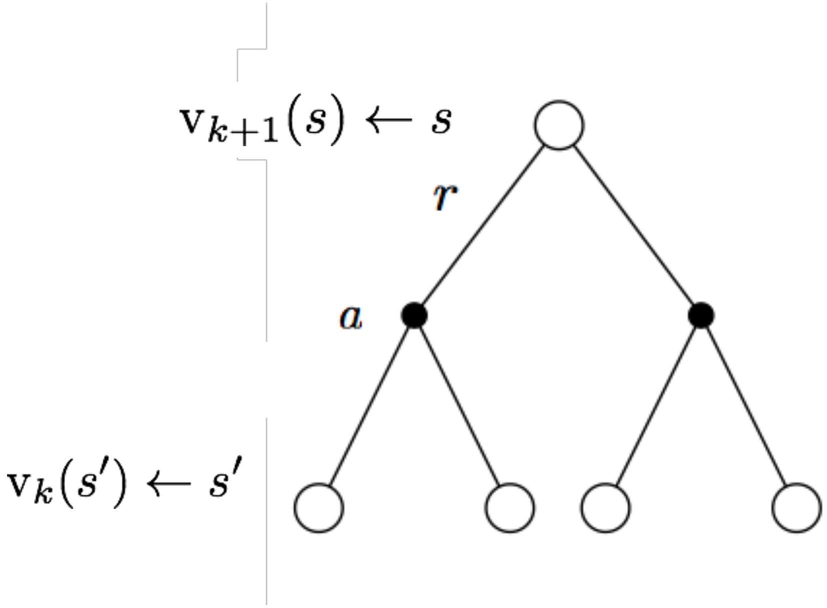
V_7

Value Iteration

- Problem: find optimal policy π
- Solution: iterative application of Bellman optimality backup
- $\text{v}_1 \rightarrow \text{v}_2 \rightarrow ... \rightarrow \text{v}_*$
- Using synchronous backups
  - At each iteration $k + 1$
  - For all states $s \in \mathcal{S}$
  - Update $\text{v}_{k+1}(s)$ from $\text{v}_k(s')$

\text{v}_1 \rightarrow \text{v}_2 \rightarrow ... \rightarrow \text{v}_*

Value Iteration (2)



$$\text{v}_{k+1}(s) = \max_{a \in \mathcal{A}} \left(r(s,a) + \gamma \sum_{s' \in \mathcal{S}} T(s'|s,a)\text{v}_k(s') \right)$$

$$\text{v}_{k+1} = \max_{a \in \mathcal{A}} r(a) + \gamma T(a)\text{v}_k$$

```
\text{v}_{k+1}(s) = \max_{a \in \mathcal{A}}{ \left( r(s,a) + \gamma \sum_{s'\in \mathcal{S}} T(s'|s,a) {\text{v}_k(s')} \right) }
```

Bellman Optimality Backup is a Contraction

- Define the Bellman optimality backup operator F^*,

$$F^*(\mathrm{v}) = \max_{a \in \mathcal{A}} r(a) + \gamma T(a)\mathrm{v}$$

- This operator is a γ-contraction, i.e. it makes value functions closer by at least γ (similar to previous proof)

$$||F^*(\mathrm{u}) - F^*(\mathrm{v})||_\infty \leq \gamma ||\mathrm{u} - \mathrm{v}||_\infty$$

Convergence of Value Iteration

- The Bellman optimality operator F^* has a unique fixed point
- $\mathbf{v}_*$ is a fixed point of F^* (by Bellman optimality equation)
- By contraction mapping theorem
- Value iteration converges on $\mathbf{v}_*$

Synchronous Dynamic Programming Algorithms

Problem	Bellman Equation	Algorithm
Prediction	Bellman Expectation Equation	Iterative Policy Evaluation
Control	Bellman Expectation Equation + Greedy Policy Improvement	Policy Iteration
Control	Bellman Optimality Equation	Value Iteration

- Algorithms are based on state-value function $v_\pi(s)$ or $v_*(s)$
- Complexity $O(mn^2)$ per iteration, for m actions and n states
- Could also apply to action-value function $q_\pi(s,a)$ or $q_*(s,a)$

Efficiency of DP

- To find an optimal policy is polynomial in the number of states
- BUT, the number of states is often astronomical, e.g., often growing exponentially with the number of state variables (what Bellman called “the curse of dimensionality”).
- In practice, classical DP can be applied to problems with a few millions of states.

Asynchronous DP

- All the DP methods described so far require exhaustive sweeps of the entire state set.
- Asynchronous DP does not use sweeps. Instead it works like this:
  - Repeat until convergence criterion is met:
    - Pick a state at random and apply the appropriate backup
- Still need lots of computation, but does not get locked into hopelessly long sweeps
- Guaranteed to converge if all states continue to be selected
- Can you select states to backup intelligently? YES: an agent's experience can act as a guide.

Asynchronous Dynamic Programming

- Three simple ideas for asynchronous dynamic programming:
  - In-place dynamic programming
  - Prioritized sweeping
  - Real-time dynamic programming

In-Place Dynamic Programming

- Synchronous value iteration stores two copies of value function
  - for all s in $\mathcal{S}$

$$\text{v}_{new}(s) \leftarrow \max_{a \in \mathcal{A}} \left(r(s,a) + \gamma \sum_{s' \in \mathcal{S}} T(s'|s,a) \text{v}_{old}(s') \right)$$

$$\text{v}_{old} \leftarrow \text{v}_{new}$$

- In-place value iteration only stores one copy of value function
  - for all s in $\mathcal{S}$

$$\text{v}(s) \leftarrow \max_{a \in \mathcal{A}} \left(r(s,a) + \gamma \sum_{s' \in \mathcal{S}} T(s'|s,a) \text{v}(s') \right)$$

```
\text{v}_{new}(s) \leftarrow \max_{a \in \mathcal{A}} {\left( r(s,a) + \gamma \sum_{s'\in \mathcal{S}} T(s'|s,a) {\text{v}_{old}(s')} \right)}
```

Prioritized Sweeping

- Use magnitude of Bellman error to guide state selection, e.g.

$$\left\lvert \max_{a \in \mathcal{A}} \left(r(s,a) + \gamma \sum_{s' \in \mathcal{S}} T(s'|s,a)\mathrm{v}(s') \right) - \mathrm{v}(s) \right\rvert$$

- Backup the state with the largest remaining Bellman error
- Requires knowledge of reverse dynamics (predecessor states)
- Can be implemented efficiently by maintaining a priority queue

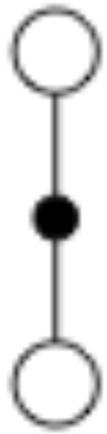
```
\left\lvert \max_{a \in \mathcal{A}} {\left( r(s,a) + \gamma \sum_{s'\in \mathcal{S}} T(s'|s,a) textcolo\r{red}{\text{v}(s')} \righ
```

Real-time Dynamic Programming

- Idea: update only states that are relevant to agent
- Use agent's experience to guide the selection of states
- After each time-step $\mathcal{S}_t, \mathcal{A}_t, r_{t+1}$
- Backup the state $\mathcal{S}_t$

$$\mathrm{v}(\mathcal{S}_t) \leftarrow \max_{a \in \mathcal{A}} \left(r(\mathcal{S}_t, a) + \gamma \sum_{s' \in \mathcal{S}} T(s' | \mathcal{S}_t, a) \mathrm{v}(s') \right)$$

Sample Backups



- In subsequent lectures we will consider sample backups
- Using sample rewards and sample transitions $\langle\mathcal{S}, \mathcal{A}, r, \mathcal{S}'\rangle$
- Instead of reward function and transition dynamics
- Advantages:
  - Model-free: no advance knowledge of MDP required
  - Breaks the curse of dimensionality through sampling
  - Cost of backup is constant, independent of $n = |\mathcal{S}|$

Approximate Dynamic Programming

- Approximate the value function
- Using a function approximate $\hat{\mathrm{v}}(s, \mathrm{w})$
- Apply dynamic programming to $\hat{\mathrm{v}}(\cdot\,, \mathrm{w})$
- e.g. Fitted Value Iteration repeats at each iteration ***k***,
  - Sample states $\tilde{\mathcal{S}} \subseteq \mathcal{S}$
  - For each state $s \in \tilde{\mathcal{S}}$, estimate target value using Bellman optimality equation,

$$\tilde{\mathrm{v}}_k(s) = \max_{a \in \mathcal{A}} \left(r(s, a) + \gamma \sum_{s' \in \mathcal{S}} T(s'|s, a) \hat{\mathrm{v}}(s', \mathrm{w}_k) \right)$$

  - Train next value function $\hat{\mathrm{v}}(\cdot\,, \mathrm{w}_{k+1})$ using targets $\{\langle s, \tilde{\mathrm{v}}_k(s) \rangle\}$