COMP9414: Artificial Intelligence
Solutions 4: Propositional Logic

1. (i) $(\neg Ja \wedge \neg Jo) \rightarrow T$
   Where:
   Ja: *Jane is in town*
   Jo: *John is in town*
   T: *we will play tennis*

   (ii) $R \vee \neg R$
   Where:
   R: *it will rain today*

   (iii) $\neg S \rightarrow \neg P$, or $\neg(P \wedge \neg S)$ (check these are equivalent)
   Where:
   S: *you study*
   P: *you will pass this course*

2. (i) $P \rightarrow Q$
   $\neg P \vee Q$ [Remove $\rightarrow$]

   (ii) $(P \rightarrow \neg Q) \rightarrow R$
   $\neg(\neg P \vee \neg Q) \vee R$ [Remove $\rightarrow$]
   $(\neg\neg P \wedge \neg\neg Q) \vee R$ [De Morgan]
   $(P \wedge Q) \vee R$ [Double Negation]
   $(P \vee R) \wedge (Q \vee R)$ [Distribute $\vee$ over $\wedge$]

   (iii) $\neg(P \wedge \neg Q) \rightarrow (\neg R \vee \neg Q)$
   $\neg\neg(P \wedge \neg Q) \vee (\neg R \vee \neg Q)$ [Remove $\rightarrow$]
   $(P \wedge \neg Q) \vee (\neg R \vee \neg Q)$ [Double Negation]
   $(P \vee \neg R \vee \neg Q) \wedge (\neg Q \vee \neg R \vee \neg Q)$ [Distribute $\vee$ over $\wedge$]
   This can be further simplified to $(P \vee \neg R \vee \neg Q) \wedge (\neg Q \vee \neg R)$
   and even further simplified to $\neg Q \vee \neg R$, since $\neg Q \vee \neg R$ subsumes $P \vee \neg R \vee \neg Q$

3. (i)

P	Q	$P \rightarrow Q$	$\neg Q$	$\neg P$
T	T	T	F	F
T	F	F	T	F
F	T	T	F	T
F	F	T	T	T

In all rows where both $P \rightarrow Q$ and $\neg Q$ true, $\neg P$ is true. Therefore, valid inference.

(ii)

P	Q	$\neg P$	$\neg Q$	$P \rightarrow Q$	$\neg Q \rightarrow \neg P$
T	T	F	F	T	T
T	F	F	T	F	F
F	T	T	F	T	T
F	F	T	T	T	T

In all rows where both $P \rightarrow Q$ true, $\neg Q \rightarrow \neg P$ is true. Therefore, valid inference.

(iii)

P	Q	R	$P \rightarrow Q$	$Q \rightarrow R$	$P \rightarrow R$
T	T	T	T	T	T
T	T	F	T	F	F
T	F	T	F	T	T
T	F	F	F	T	F
F	T	T	T	T	T
F	T	F	T	F	T
F	F	T	T	T	T
F	F	F	T	T	T

In all rows where both $P \rightarrow Q$ and $Q \rightarrow R$ true, $P \rightarrow R$ is true. Therefore, valid inference.

4. (i) CNF$(P \to Q)$
$\Leftrightarrow \neg P \vee Q$ [Remove $\to$]

CNF$(\neg Q)$
$\Leftrightarrow \neg Q$

CNF$(\neg\neg P)$
$\Leftrightarrow P$ [Double Negation]

Proof:

1.	$\neg P \vee Q$	[Hypothesis]
2.	$\neg Q$	[Hypothesis]
3.	P	[Negation of Query]
4.	Q	1, 3 Resloution
5.	$\square$	2, 4 Resloution

(ii) CNF$(P \to Q)$
$\Leftrightarrow \neg P \vee Q$

CNF$(\neg(\neg Q \to \neg P))$
$\Leftrightarrow \neg(\neg\neg Q \vee \neg P)$ [Remove $\to$]
$\Leftrightarrow \neg(Q \vee \neg P)$ [Double Negation]
$\Leftrightarrow \neg Q \wedge \neg\neg P$ [De Morgan]
$\Leftrightarrow \neg Q \wedge P$ [Double Negation]

Proof:

1.	$\neg P \vee Q$	[Hypothesis]
2.	$\neg Q$	[Negation of Query]
3.	P	[Negation of Query]
4.	$\neg P$	1, 2 Resolution
5.	$\square$	3, 4 Resolution

(iii) $P \to Q, Q \to R \vdash P \to R$
CNF$(P \to Q)$
$\Leftrightarrow \neg P \vee Q$

CNF$(Q \to R)$
$\Leftrightarrow \neg Q \vee R$

CNF$(\neg(P \to R))$
$\Leftrightarrow \neg(\neg P \vee R)$ [Remove $\to$]
$\Leftrightarrow \neg\neg P \wedge \neg R$ [De Morgan]
$\Leftrightarrow P \wedge \neg R$ [Double Negation)

Proof:

1.	$\neg P \vee Q$	[Hypothesis]
2.	$\neg Q \vee R$	[Hypothesis]
3.	P	[Negation of Query]
4.	$\neg R$	[Negation of Query]
5.	Q	1, 3 Resolution
6.	R	2, 5 Resolution
7.	$\square$	4, 6 Resolution

5. (i)

P	Q	$\neg P$	$P \vee Q$	$(P \vee Q) \wedge \neg P$	$((P \vee Q) \wedge \neg P) \to Q$
T	T	F	T	F	T
T	F	F	T	F	T
F	T	T	T	T	T
F	F	T	F	F	T

Last column is always true no matter what truth assignment to P and Q. Therefore $((P \vee Q) \wedge \neg P) \rightarrow Q$ is a tautology.

(ii) $S = ((P \rightarrow Q) \wedge \neg(P \rightarrow R)) \rightarrow (P \rightarrow Q)$

P	Q	R	$P \rightarrow Q$	$P \rightarrow R$	$\neg(P \rightarrow R)$	$(P \rightarrow Q) \wedge \neg(P \rightarrow R)$	S
T	T	T	T	T	F	F	T
T	T	F	T	F	T	T	T
T	F	T	F	T	F	F	T
T	F	F	F	F	T	F	T
F	T	T	T	T	F	F	T
F	T	F	T	T	F	F	T
F	F	T	T	T	F	F	T
F	F	F	T	T	F	F	T

Last column is always true no matter what truth assignment to P, Q and R. Therefore $((P \rightarrow Q) \wedge \neg(P \rightarrow R)) \rightarrow (P \rightarrow Q)$ is a tautology.

(iii)

P	$\neg P$	$\neg P \wedge P$	$\neg(\neg P \wedge P)$	$\neg(\neg P \wedge P) \wedge P$
T	F	F	T	T
F	T	F	T	F

Last column is not always true. Therefore $\neg(\neg P \wedge P) \wedge P$ is not a tautology.

(iv) $(P \vee Q) \rightarrow \neg(\neg P \wedge \neg Q)$

P	Q	$\neg P$	$\neg Q$	$P \vee Q$	$\neg P \wedge \neg Q$	$\neg(\neg P \wedge \neg Q)$	$(P \vee Q) \rightarrow \neg(\neg P \wedge \neg Q)$
T	T	F	F	T	F	T	T
T	F	F	T	T	F	T	T
F	T	T	F	T	F	T	T
F	F	T	T	F	T	F	T

Last column is always true no matter what truth assignment to P and Q. Therefore $(P \vee Q) \rightarrow \neg(\neg P \wedge \neg Q)$ is a tautology.

6. (i) CNF($\neg(((P \vee Q) \wedge \neg P) \rightarrow Q)$)
$\Leftrightarrow \neg(\neg((P \vee Q) \wedge \neg P) \vee Q)$ [Remove $\rightarrow$]
$\Leftrightarrow \neg\neg((P \vee Q) \wedge \neg P) \wedge \neg Q$ [De Morgan]
$\Leftrightarrow (P \vee Q) \wedge \neg P \wedge \neg Q$ [Double Negation]

Proof:

1. $P \vee Q$ [Negated Query]
2. $\neg P$ [Negated Query]
3. $\neg Q$ [Negated Query]
4. Q 1, 2 Resolution
5. $\square$ 3, 4 Resolution

Therefore $\neg(((P \vee Q) \wedge \neg P) \rightarrow Q)$ is a tautology.

(ii) CNF($\neg(((P \rightarrow Q) \wedge \neg(P \rightarrow R)) \rightarrow (P \rightarrow Q))$)
$\Leftrightarrow \neg(\neg((\neg P \vee Q) \wedge \neg(\neg P \vee R)) \vee (\neg P \vee Q))$ [Remove $\rightarrow$]
$\Leftrightarrow \neg\neg((\neg P \vee Q) \wedge \neg(\neg P \vee R)) \wedge \neg(\neg P \vee Q)$ [De Morgan]
$\Leftrightarrow (\neg P \vee Q) \wedge (\neg\neg P \wedge \neg R) \wedge (\neg\neg P \wedge \neg Q)$ [Double Negation and De Morgan]
$\Leftrightarrow (\neg P \vee Q) \wedge (P \wedge \neg R) \wedge (P \wedge \neg Q)$ [Double Negation]

Proof:

1. $\neg P \vee Q$ [Negated Query]
2. P [Negated Query]
3. $\neg R$ [Negated Query]
4. $\neg Q$ [Negated Query]
5. Q 1, 2 Resolution
6. $\square$ 4, 5 Resolution

Therefore $((P \rightarrow Q) \wedge \neg(P \rightarrow R)) \rightarrow (P \rightarrow Q)$ is a tautology.

(iii) CNF($\neg(\neg(\neg P \wedge P) \wedge P)$)
$\Leftrightarrow \neg\neg(\neg P \wedge P) \vee \neg P$ [De Morgan]
$\Leftrightarrow (\neg P \wedge P) \vee \neg P$ [Double Negation]
$\Leftrightarrow (\neg P \vee \neg P) \wedge (P \vee \neg P)$ [Distribute $\wedge$ over $\vee$]
$\Leftrightarrow \neg P$ [Remove repetition and tautologies]

Proof:

1.	$\neg P$	(Negated Query)

Cannot obtain empty clause using resolution so $\neg(\neg P \wedge P) \wedge P$ is not a tautology.

(iv) CNF($\neg((P \vee Q) \rightarrow \neg(\neg P \wedge \neg Q))$)
$\Leftrightarrow \neg(\neg(P \vee Q) \vee \neg(\neg P \wedge \neg Q))$ [Remove $\rightarrow$]
$\Leftrightarrow \neg\neg(P \vee Q) \wedge \neg\neg(\neg P \wedge \neg Q)$ [De Morgan]
$\Leftrightarrow (P \vee Q) \wedge \neg P \wedge \neg Q$ [Double Negation]

Proof:

1.	$P \vee Q$	[Negated Query]
2.	$\neg Q$	[Negated Query]
3.	$\neg P$	[Negated Query]
4.	P	1, 2 Resolution
5.	$\square$	3, 4, Resolution

Therefore $(P \vee Q) \rightarrow \neg(\neg P \wedge \neg Q)$ is a tautology.