COMP9020 22T1
Week 8
Counting and Probability

- Textbook (R & W) - Ch. 5, Sec. 5.1-5.3
- Problem set week 8 + quiz

Fact

Combinatorics and probability arise in many areas of Computer Science, e.g.

- *Complexity of algorithms, data management*
- *Reliability, quality assurance*
- *Computer security*
- *Data mining, machine learning, robotics*

Counting Techniques

General idea: find methods, algorithms or precise formulae to count the number of elements in various sets or collections derived, in a structured way, from some basic sets.

Examples

Single base set $S = \{s_1, \ldots, s_n\}$, $|S| = n$; find the number of

- all subsets of S
- ordered selections of r different elements of S
- unordered selections of r different elements of S
- selections of r elements from S s.t. ...
- functions $S \longrightarrow S$ (onto, 1-1)
- partitions of S into k equivalence classes
- graphs/trees with elements of S as labelled vertices/leaves
- ...and many more

Basic Counting Rules (1)

Union rule — S and T *disjoint*

$$|S \cup T| = |S| + |T|$$

$S_1, S_2, \ldots, S_n$ pairwise disjoint ($S_i \cap S_j = \emptyset$ for $i \neq j$)

$$|S_1 \cup \ldots \cup S_n| = \sum |S_i|$$

Exercise

How many numbers in $A = [1, 2, \ldots, 999]$ are divisible by 31 or 41?

$\lfloor 999/31 \rfloor = 32$ divisible by 31.
$\lfloor 999/41 \rfloor = 24$ divisible by 41.
No number in A divisible by both.
Hence, $32 + 24 = 56$ divisible by 31 or 41.

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Basic Counting Rules (2)

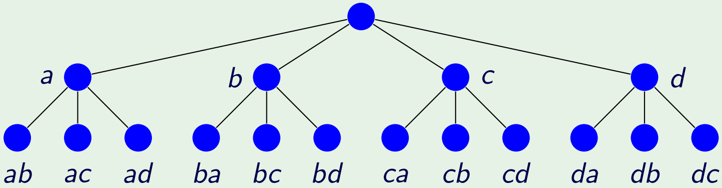
Product rule

$$|S_1 \times \ldots \times S_k| = |S_1| \cdot |S_2| \cdots |S_k| = \prod_{i=1}^{k} |S_i|$$

If all $S_i = S$ (the same set) and $|S| = m$ then $|S^k| = m^k$

Example

Let $\Sigma = \{a, b, c, d\}$.
Number of words in Σ^2 with no letter repeated: $|\Sigma| \cdot (|\Sigma| - 1) = 12$



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Exercise

Let $\Sigma = \{a, b, c, d, e, f, g\}$.
How many 5-letter words? How many with no letter repeated?

$$|\Sigma^5| = |\Sigma|^5 = 7^5 = 16,807$$

$$\prod_{i=0}^{4}(|\Sigma| - i) = 7 \cdot 6 \cdot 5 \cdot 4 \cdot 3 = 2,520$$

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Exercise

S, T finite. How many functions $S \longrightarrow T$ are there?

$$|T|^{|S|}$$

Exercise

$\boxed{5.1.19}$ Consider a *complete* graph on n vertices.

(a) No. of paths of length 3

Take any vertex to start, then every next vertex different from the preceding one. Hence $n \cdot (n-1)^3$

(b) paths of length 3 with all vertices distinct

$n(n-1)(n-2)(n-3)$

(c) paths of length 3 with all edges distinct

$n(n-1)(n-2)^2$

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Inclusion-Exclusion

This is one of the most universal counting procedures. It allows you to compute the size of

$$A_1 \cup \ldots \cup A_n$$

from the sizes of all possible intersections

$$A_{i_1} \cap A_{i_2} \cap \ldots \cap A_{i_k}, \;\; a_{i_1} < a_{i_2} < \ldots < a_{i_k}$$

Two sets $|A \cup B| = |A| + |B| - |A \cap B|$

Three sets $|A \cup B \cup C| = |A| + |B| + |C|$
$-|A \cap B| - |A \cap C| - |B \cap C|$
$+|A \cap B \cap C|$

NB

Inclusion-exclusion is often applied informally without making clear or explicit *why* certain quantities are subtracted or put back in.

Interpretation

Each A_i defined as the set of objects that satisfy some property P_i

$$A_i = \{ x \in X : P_i(x) \}$$

Union $A_1 \cup \ldots \cup A_n$ is the set of objects that satisfy **at least one** property P_i

$$A_1 \cup \ldots \cup A_n = \{ x \in X : P_1(x) \vee P_2(x) \vee \ldots \vee P_n(x) \}$$

Intersection $A_{i_1} \cap \ldots \cap A_{i_r}$ is the set of objects that satisfy **all** properties $P_{i_1}, \ldots, P_{i_r}$

$$A_{i_1} \cap \ldots \cap A_{i_r} = \{ x \in X : P_{i_1}(x) \wedge P_{i_2}(x) \wedge \ldots \wedge P_{i_r}(x) \}$$

Special case $r = 1$: $A_{i_1} = \{x \in X : P_{i_1}(x)\}$

Examples

$\boxed{5.3.1}$ 200 people. 150 swim or jog, 85 swim and 60 do both.
How many jog?

S – (set of) people who swim, J – people who jog
$|S \cup J| = |S| + |J| - |S \cap J|$; thus $150 = 85 + |J| - 60$ hence
$|J| = 125$; answer *does not* depend on the number of people overall

$\boxed{5.6.38 \text{ (supp)}}$ There are 100 problems, 75 of which are 'easy' and 40 'important'.
What's the smallest number of easy *and* important problems?

$$|E \cap I| = |E| + |I| - |E \cup I| = 75 + 40 - |E \cup I| \geq 75 + 40 - 100 = 15$$

Exercise

$\boxed{5.3.2}$ $S = [100 \ldots 999]$, thus $|S| = 900$.

(a) How many numbers have at least one digit that is a 3 or 7?

$A_3 = \{\text{at least one '3'}\}$
$A_7 = \{\text{at least one '7'}\}$

$$(A_3 \cup A_7)^c = \{\ n \in [100, 999] : n \text{ digits } \in \{0, 1, 2, 4, 5, 6, 8, 9\}\ \}$$

7 choices for the first digit and 8 choices for the later digits:

$$|(A_3 \cup A_7)^c| = |\{1, 2, 4, 5, 6, 8, 9\}| \cdot |\{0, 1, 2, 4, 5, 6, 8, 9\}|^2$$

Therefore $|A_3 \cup A_7| = 900 - 448 = 452$.

(b) How many numbers have a 3 *and* a 7?

$|A_3 \cap A_7| = |A_3| + |A_7| - |A_3 \cup A_7| =$
$(900 - 8 \cdot 9 \cdot 9) + (900 - 8 \cdot 9 \cdot 9) - 452 = 2 \cdot 252 - 452 = 52$

Exercise

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Corollaries

- If $|S \cup T| = |S| + |T|$ then S and T are disjoint
- If $|\bigcup_{i=1}^{n} S_i| = \sum_{i=1}^{n} |S_i|$ then S_i are pairwise disjoint
- If $|T \setminus S| = |T| - |S|$ then $S \subseteq T$

These properties can serve to identify cases when sets are disjoint (resp. one is contained in the other).

Proof.

$|S| + |T| = |S \cup T|$ implies $|S \cap T| = |S| + |T| - |S \cup T| = 0$

$|T \setminus S| = |T| - |S|$ implies $|S \cap T| = |S|$, which implies $S \subseteq T$ $\square$

Basic Counting Rules (3)

permutations

Ordering of all objects from a set S; equivalently: Selecting all objects while *recognising* the order of selection.

The number of permutations of n elements is

$$n! = n \cdot (n-1) \cdots 1, \quad 0! = 1! = 1$$

r-permutations

Selecting any r objects from a set S of size n without repetition while *recognising* the order of selection.

Their number is

$$\Pi(n, r) = n \cdot (n-1) \cdots (n-r+1) = \frac{n!}{(n-r)!}$$

r-**selections** (or: r-**combinations**)
Collecting any r distinct objects without repetition; equivalently: selecting r objects from a set S of size n and *not* recognising the order of selection.
Their number is

$$\binom{n}{r} = \frac{n!}{(n-r)!r!} = \frac{n \cdot (n-1) \cdots (n-r+1)}{1 \cdot 2 \cdots r}$$

NB

These numbers are usually called *binomial coefficients* due to

$$(a+b)^n = a^n + \binom{n}{1} a^{n-1} b + \binom{n}{2} a^{n-2} b^2 + \ldots + b^n = \sum_{i=0}^{n} \binom{n}{i} a^{n-i} b^i$$

NB

Also defined for any $\alpha \in \mathbb{R}$ as $\displaystyle\binom{\alpha}{r} = \frac{\alpha(\alpha-1)\cdots(\alpha-r+1)}{r!}$

Examples

- Number of edges in a complete graph K_n
- Number of diagonals in a convex polygon
- Number of poker hands
- Decisions in games, lotteries etc.

Example

$\boxed{5.1.2}$ Give an example of a counting problem whose answer is

(a) $\Pi(26, 10)$

(b) $\binom{26}{10}$

Draw 10 cards from a half deck (eg. black cards only)
(a) the cards are recorded in the order of appearance
(b) only the complete draw is recorded

Exercise

$\boxed{5.1.6}$ From a group of 12 men and 16 women, how many committees can be chosen consisting of

(a) 7 members? $\binom{12+16}{7}$

(b) 3 men and 4 women? $\binom{12}{3}\binom{16}{4}$

(c) 7 women or 7 men? $\binom{12}{7}+\binom{16}{7}$

$\boxed{5.1.7}$ As above, but any 4 people (male or female) out of 9 and two, Alice and Bob, unwilling to serve on the same committee.

{all committees} − {committees with both A and B}
$= \binom{9}{4} - \binom{7}{2} = 126 - 21 = 105$

equivalently, {A in, B out} + {A out, B in} + {none in}
$= \binom{7}{3} + \binom{7}{3} + \binom{7}{4} = 35 + 35 + 35 = 105$

Exercise

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equivalently, $\{$A in, B out$\}$ $+$ $\{$A out, B in$\}$ $+$ $\{$none in$\}$

$$= \binom{7}{3} + \binom{7}{3} + \binom{7}{4} = 35 + 35 + 35 = 105$$

Counting Poker Hands

Exercise

$\boxed{5.1.15}$ A poker hand consists of 5 cards drawn without replacement from a standard deck of 52 cards

$$\{A, 2\text{-}10, J, Q, K\} \times \{\text{club, spade, heart, diamond}\}$$

(a) Number of "4 of a kind" hands (e.g. 4 Jacks)

$|\text{rank of the 4-of-a-kind}| \cdot |\text{any other card}| = 13 \cdot (52 - 4)$

(b) Number of non-straight flushes, i.e. all cards of same suit but *not* consecutive (e.g. 8,9,10,J,K)

$$|\text{all flush}| - |\text{straight flush}|$$
$$= |\text{suit}| \cdot |\text{5-hand in a given suit}| - |\text{suit}| \cdot |\text{rank of a straight flush in a given suit}|$$
$$= 4 \cdot \binom{13}{5} - 4 \cdot 10$$

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Difficult Counting Problems

Example (Ramsey numbers)

An example of a *Ramsey number* is $R(3,3) = 6$, meaning that

> *"K_6 is the smallest complete graph s.t. if all edges are painted using two colours, then there must be at least one monochromatic triangle"*

This serves as the basis of a game called "Sim" (invented by G. Simmons), where two adversaries connect six dots, respectively using blue and red lines. The objective is to *avoid* closing a triangle of one's own colour. The second player has a winning strategy, but the full analysis requires a computer program.

NB

Generally, $R(m,n)$ is defined as the minimum number of people s.t. either m of them have all met each other, or n of them are complete strangers to each other (see also slide 26, week 2).

Using Programs to Count

Two dice, a red die and a black die, are rolled.
(Note: one *die*, two or more *dice*)

Write a program to list all the pairs $\{(R, B) : R > B\}$

Similarly, for three dice, list all triples $R > B > G$

Generally, for n dice, all of which are m-sided ($n \leq m$), list all *decreasing* n-tuples

NB

In order to just find the number of such n-tuples, it is not necessary to list them all. One can write a recurrence relation for these numbers and compute (or try to solve) it.

Approximate Counting

NB

A *Count* may be a precise value or an **estimate**.

The latter should be asymptotically correct or at least give a good asymptotic bound, whether upper or lower. If S is the base set, $|S| = n$ its size, and we denote by $c(S)$ some collection of objects from S we are interested in, then the estimate $\mathrm{est}(c(S))$ is asymptotically correct if

$$\lim_{n \to \infty} \frac{\mathrm{est}(|c(S)|)}{|c(S)|} = 1$$

Elementary Probability

Sample space:

$$\Omega = \{\omega_1, \ldots, \omega_n\}$$

Each point represents an outcome, each outcome ω_i equally likely:

$$P(\omega_1) = P(\omega_2) = \ldots = P(\omega_n) = \frac{1}{n}$$

This a called a **uniform probability distribution** over Ω

Examples

Tossing a coin: $\Omega = \{H, T\}$

$$P(H) = P(T) = 0.5$$

Rolling a die: $\Omega = \{1, 2, 3, 4, 5, 6\}$

$$P(1) = P(2) = P(3) = P(4) = P(5) = P(6) = \frac{1}{6}$$

Non-uniform Probability

Slight modification is needed to define an arbitrary (in general non-uniform) probability distribution:

$$\Omega = \{\omega_1, \ldots, \omega_n\}$$

Let

$$P(\omega_1) = p_1, P(\omega_2) = p_2, \ldots, P(\omega_n) = p_n$$

Then

$$\sum_{i=1}^{n} p_i = 1$$

Events

Definition

Event — a collection of outcomes = subset of Ω

Probability of an event:

$$P(E) = \sum_{\omega \in E} P(\omega)$$

Fact

$$P(\emptyset) = 0, \quad P(\Omega) = 1, \quad P(E^c) = 1 - P(E)$$

Exercise

$\boxed{5.2.7}$ Suppose an experiment leads to events A, B with probabilities $P(A) = 0.5, P(B) = 0.8, P(A \cap B) = 0.4$.
Find

- $P(B^c) = 1 - P(B) = 0.2$
- $P(A \cup B) = P(A) + P(B) - P(A \cap B) = 0.9$
- $P(A^c \cup B^c) = 1 - P((A^c \cup B^c)^c) = 1 - P(A \cap B) = 0.6$

Exercise

$\boxed{5.2.8}$ Given $P(A) = 0.6, \ P(B) = 0.7$, show $P(A \cap B) \geq 0.3$

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$$\begin{aligned} P(A \cap B) &= P(A) + P(B) - P(A \cup B) \\ &= 0.6 + 0.7 - P(A \cup B) \\ &\geq 0.6 + 0.7 - 1 = 0.3 \end{aligned}$$

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Computing Probabilities by Counting

Computing probabilities with respect to a *uniform* distribution comes down to counting the size of the event.
If $E = \{e_1, \ldots, e_k\}$ then

$$P(E) = \sum_{i=1}^{k} P(e_i) = \sum_{i=1}^{k} \frac{1}{|\Omega|} = \frac{|E|}{|\Omega|}$$

Most of the counting rules carry over to probabilities wrt. a uniform distribution.

NB

The expression "selected at random", when not further qualified, means:
"subject to / according to / … a *uniform* distribution."

Example

$\boxed{\text{5.6.38 (supp)}}$ Of 100 problems, 75 are 'easy' and 40 'important'.
(b) n problems chosen randomly. What is the probability that all n are important?

$$p = \frac{\binom{40}{n}}{\binom{100}{n}} = \frac{40 \cdot 39 \cdots (41-n)}{100 \cdot 99 \cdots (101-n)}$$

Exercise

$\boxed{5.2.3}$ A 4-letter word is selected at random from Σ^4, where $\Sigma = \{a, b, c, d, e\}$. What is the probability that

(a) the letters in the word are all distinct?

(b) there are no vowels ("a", "e") in the word?

(c) the word begins with a vowel?

(a) $|E| = \Pi(5,4), \quad P(E) = \frac{5 \cdot 4 \cdot 3 \cdot 2}{5^4} = \frac{120}{625} \approx 19\%$

(b) $|E| = 3^4, \quad P(E) = \frac{81}{625} \approx 13\%$

(c) $|E| = 2 \cdot 5^3, \quad P(E) = \frac{2}{5}$

Example

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Exercise

$\boxed{5.2.5}$ An urn contains 3 red and 4 black balls. 3 balls are removed without replacement. What are the probabilities that
(a) all 3 are red
(b) all 3 are black
(c) one is red, two are black

All probabilities are computed using the same sample space: all possible ways to draw three balls without replacement.

The size of the sample space is $\dfrac{7 \cdot 6 \cdot 5}{3!} = 35$

(a) $E =$ All balls are red: 1 combination
(b) $E =$ All balls are black: $\binom{4}{3} = 4$ combinations
(c) $E =$ One red and two black: $\binom{3}{1} \cdot \binom{4}{2} = 18$ combinations

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Exercise

$\boxed{5.2.11}$ Two dice, a red die and a black die, are rolled.
What is the probability that

(a) the sum of the values is even?

$P\big(R + B \in \{2, 4, \ldots, 12\}\big) = \frac{18}{36} = \frac{1}{2}$

(b) the number on the red die is bigger than on the black die?

$P(R > B) = P(R < B)$; also $P(R = B) = \frac{1}{6}$
Therefore $P(R < B) = \frac{1}{2}(1 - P(R = B)) = \frac{5}{12}$

(c) the number on the black die is twice the one on the red die?

$P(R = 2 \cdot B) = P(\{(2,1), (4,2), (6,3)\}) = \frac{3}{36} = \frac{1}{12}$

$\boxed{5.2.12}$ (a) the maximum of the numbers is 4?
(b) their minimum is 4?

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$\boxed{5.2.11}$ Two dice, a red die and a black die, are rolled.
What is the probability that

(a) the sum of the values is even?

$P\big(R + B \in \{2, 4, \dots, 12\}\big) = \frac{18}{36} = \frac{1}{2}$

(b) the number on the red die is bigger than on the black die?

$P(R > B) = P(R < B)$; also $P(R = B) = \frac{1}{6}$

Therefore $P(R < B) = \frac{1}{2}(1 - P(R = B)) = \frac{5}{12}$

(c) the number on the black die is twice the one on the red die?

$P(R = 2 \cdot B) = P(\{(2, 1), (4, 2), (6, 3)\}) = \frac{3}{36} = \frac{1}{12}$

$\boxed{5.2.12}$ (a) the maximum of the numbers is 4? $P(E_1) = \frac{7}{36}$

(b) their minimum is 4? $P(E_2) = \frac{5}{36}$

Check:

$P(E_1 \cup E_2) = \frac{7}{36} + \frac{5}{36} - P(E_1 \cap E_2) = \frac{7+5-1}{36} = \frac{11}{36}$

$P(\text{at least one ‘4’}) = 1 - P(\text{no ‘4’}) = 1 - \frac{5}{6} \cdot \frac{5}{6} = \frac{11}{36}$

Asymptotic Estimate of Relative Probabilities

Example

Event $A \stackrel{\text{def}}{=}$ one die rolled n times and you obtain two 6's
Event $B \stackrel{\text{def}}{=}$ n dice rolled simultaneously and you obtain one 6

$$P(A) = \frac{\binom{n}{2} \cdot 5^{n-2}}{6^n} \qquad P(B) = \frac{\binom{n}{1} \cdot 5^{n-1}}{6^n}$$

Therefore $\frac{P(A)}{P(B)} = \frac{\binom{n}{2}}{\binom{n}{1}} \cdot \frac{1}{5} = \frac{n(n-1)}{2} \cdot \frac{1}{5n} = \frac{n-1}{10} \in \Theta(n)$

n	1	2	3	4	...	11	...	20	...
$P(A)$	0	$\frac{1}{36}$	$\frac{5}{72}$	$\frac{25}{216}$	...	0.296	...	0.198	...
$P(B)$	$\frac{1}{6}$	$\frac{10}{36}$	$\frac{25}{72}$	$\frac{125}{324}$	...	0.296	...	0.104	...

Inclusion-Exclusion and Probability

Inclusion-Exclusion is a very common method for deriving probabilities from other probabilities.

Two sets

$$P(A \cup B) = P(A) + P(B) - P(A \cap B)$$

Three sets

$$\begin{aligned} P(A \cup B \cup C) = P(A) + P(B) + P(C) \\ - P(A \cap B) - P(A \cap C) - P(B \cap C) \\ + P(A \cap B \cap C) \end{aligned}$$

Example

A four-digit number n is selected at random (i.e. randomly from $[1000 \ldots 9999]$). Find the probability p that n has each of 0, 1, 2 among its digits.

Let $q = 1 - p$ be the complementary probability and define

$$A_i = \{n : \text{no digit } i\}, A_{ij} = \{n : \text{no digits } i, j\}, A_{ijk} = \{n : \text{no } i, j, k\}$$

Then define

$T = A_0 \cup A_1 \cup A_2 = \{n : \text{ missing at least one of } 0, 1, 2\}$

$S = (A_0 \cup A_1 \cup A_2)^c = \{n : \text{ containing each of } 0, 1, 2\}$

Example (cont'd)

Once we find the cardinality of T, the solution is

$$q = \frac{|T|}{9000}, \; p = 1 - q$$

To find $|A_i|, |A_{ij}|, |A_{ijk}|$ we reflect on how many choices are available for the first digit, for the second etc. A special case is the leading digit, which must be $1, \ldots, 9$

Example (cont'd)

$$
\begin{aligned}
|A_0| &= 9^4, \quad |A_1| = |A_2| = 8 \cdot 9^3 \\
|A_{01}| &= |A_{02}| = 8^4, \quad |A_{12}| = 7 \cdot 8^3 \\
|A_{012}| &= 7^4
\end{aligned}
$$

$$
\begin{aligned}
|T| &= |A_0 \cup A_1 \cup A_2| \\
&= |A_0| + |A_1| + |A_2| - |A_0 \cap A_1| - |A_0 \cap A_2| - |A_1 \cap A_2| \\
&\qquad + |A_0 \cap A_1 \cap A_2| \\
&= 9^4 + 2 \cdot 8 \cdot 9^3 - 2 \cdot 8^4 - 7 \cdot 8^3 + 7^4 \\
&= 25 \cdot 9^3 - 23 \cdot 8^3 + 7^4 = 8850
\end{aligned}
$$

$$
q = \frac{8850}{9000}, \quad p = 1 - q \quad \approx 0.01667
$$

Previous example generalised: Probability of an r-digit number having all of 0,1,2,3 among its digits.
We use the previous notation: A_i — set of numbers n *missing* digit i, and similarly for all $A_{ij\ldots}$
We aim to find the size of $T = A_0 \cup A_1 \cup A_2 \cup A_3$, and then to compute $|S| = 9 \cdot 10^{r-1} - |T|$.

$$\begin{aligned} |A_0 \cup A_1 \cup A_2 \cup A_3| &= \text{sum of } |A_i| \\ &- \text{sum of } |A_i \cap A_j| \\ &+ \text{sum of } |A_i \cap A_j \cap A_k| \\ &- \text{sum of } |A_i \cap A_j \cap A_k \cap A_l| \end{aligned}$$

Probability of Sequential Outcomes

Example

Team A has probability $p = 0.5$ of winning a game against B.
What is the probability P_p of A winning a best-of-seven match if
(a) A already won the first game?
(b) A already won the first two games?
(c) A already won two out of the first three games?

(a) Sample space S — 6-sequences, formed from wins (W) and losses (L)

$$|S| = 2^6 = 64$$

Favourable sequences F — those with three to six W

$$|F| = \binom{6}{3} + \binom{6}{4} + \binom{6}{5} + \binom{6}{6} = 20 + 15 + 6 + 1 = 42$$

Therefore $P_{0.5} = \frac{42}{64} \approx 66\%$

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Exercise

(b) Sample space S? — 5-sequences of W and L

$$|S| = 2^5 = 32$$

Favourable sequences F? — those with two to five W

$$|F| = \binom{5}{2} + \binom{5}{3} + \binom{5}{4} + \binom{5}{5} = 10 + 10 + 5 + 1 = 26$$

Therefore $P_{0.5} = \frac{26}{32} \approx 81\%$

(c)

$$|S| = 2^4 = 16$$

$$|F| = \binom{4}{2} + \binom{4}{3} + \binom{4}{4} = 6 + 4 + 1 = 11$$

Therefore $P_{0.5} = \frac{11}{16} \approx 69\%$

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Example (cont'd)

Redo for arbitrary p

(a)

$$P_p = \binom{6}{3} p^3 (1-p)^3 + \binom{6}{4} p^4 (1-p)^2 + \binom{6}{5} p^5 (1-p) + \binom{6}{6} p^6$$

(b)

$$P_p = \binom{5}{2} p^2 (1-p)^3 + \binom{5}{3} p^3 (1-p)^2 + \binom{5}{4} p^4 (1-p) + \binom{5}{5} p^5$$

(c)

$$P_p = \binom{4}{2} p^2 (1-p)^2 + \binom{4}{3} p^3 (1-p) + \binom{4}{4} p^4$$

Use of Recursion in Probability Computations

Question

Given n tosses of a coin, what is the probability of two HEADS *in a row? Compute for $n = 5, 10, 20, \ldots$*

Approaches:

I. Write down all possibilities — 32 for $n = 5$, 1024 for $n = 10$, ...

II. Write a program; running time $\mathcal{O}(2^n)$ — why?

III. Inter-relate the numbers of relevant possibilities

$N_n \stackrel{\text{def}}{=}$ No. of sequences of n tosses *without* ...HH... pattern

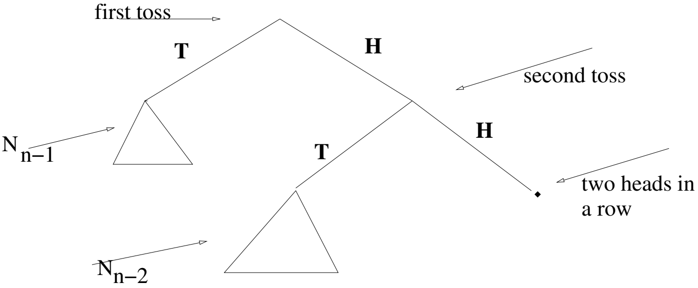
Initial values:

$N_0 = 1$ (empty sequence, has no "HH") $N_1 = 2$

$N_2 = 3$ (all except "HH") $N_3 = 5$ (check!) $N_4 = 8$ (check!)

Answer

We can summarise all possible outcomes in a **recursive tree**



$N_n = N_{n-1} + N_{n-2}$ — Fibonacci recurrence: $N_n = \text{FIB}(n+2)$

$N_n \approx \frac{1}{\sqrt{5}}\left(\frac{\sqrt{5}+1}{2}\right)^{n+2} \approx 0.72 \cdot (1.6)^{n+1}$

$p_n = \frac{2^n - \text{FIB}(n+2)}{2^n} \approx 1 - 0.72 \cdot (0.8)^{n+1}$

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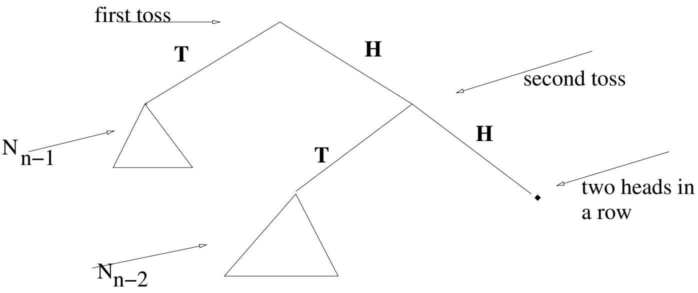

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Example

Question

Given n tosses, what is the probability q_n of at least one HHH?

$q_0 = q_1 = q_2 = 0; q_3 = \frac{1}{8}$

Then recursive computation:

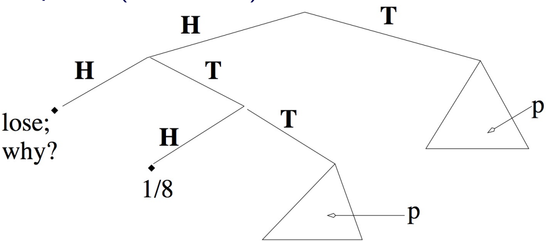
$$
\begin{aligned}
q_n = & \frac{1}{2} q_{n-1} && \text{(initial: T)} \\
& + \frac{1}{4} q_{n-2} && \text{(initial: HT)} \\
& + \frac{1}{8} q_{n-3} && \text{(initial: HHT)} \\
& + \frac{1}{8} && \text{(start with: HHH)}
\end{aligned}
$$

Example

Question

A coin is tossed 'indefinitely'. Which pattern is more likely (and by how much) to appear first, HTH or HHT?

let $p = P(HTH \text{ first})$



$p = \frac{1}{8} + \frac{1}{8}p + \frac{1}{2}p \;\Rightarrow\; \frac{3}{8}p = \frac{1}{8} \;\Rightarrow\; p = \frac{1}{3}$

NB

Probability that either pattern would appear at a given, *prespecified* point in the sequence of tosses is, obviously, the same.

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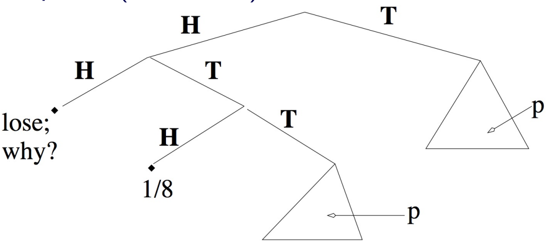

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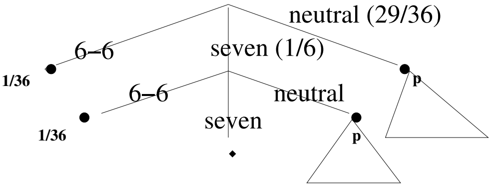
Two dice are rolled repeatedly. What is the probability that '6–6' will occur before two consecutive (back-to-back) 'totals seven'?

NB

The probability of either occurring at a given roll is the same: $\frac{1}{36}$.

Let $p = P(6\text{–}6 \text{ first})$



$$p = \frac{1}{36} + \frac{1}{6}\cdot\frac{1}{36} + \frac{1}{6}\cdot\frac{29}{36}p + \frac{29}{36}p \Rightarrow 216p = 7 + 203p \Rightarrow p = \frac{7}{13}$$

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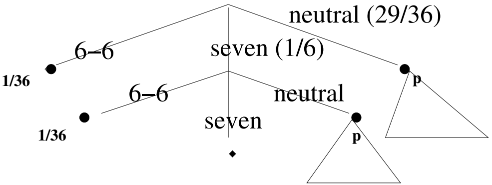

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NB

The majority of problems in probability and statistics *do not have* such elegant solutions. Hence the use of computers for either precise calculations or approximate simulations is mandatory. However, it is the use of recursion that simplifies such computing or, quite often, makes it possible in the first place.

Summary

- Counting
  - union rule, product rule, $n!$, $\Pi(n, r)$, $\binom{n}{r}$
  - inclusion-exclusion principle
- Probability
  - events
  - inclusion-exclusion
  - recursion

Coming up ...

- Ch. 9, Sec. 9.1-9.4 (Conditional Probability, Expectation)