Question 1

(a) $f(x,y) = a_1x^2y^2 + a_4xy + a_5x + a_7$

first order derivatives:

$f_x'(x,y) = 2a_1xy^2 + a_4y + a_5$

$f_y'(x,y) = 2a_1x^2y + a_4x$

second order derivatives:

$\frac{\partial^2 f}{\partial x^2} = 2a_1y^2$

$\frac{\partial^2 f}{\partial y^2} = 2a_1x^2$

$\frac{\partial^2 f}{\partial x \partial y} = 4a_1xy + a_4$

$\frac{\partial^2 f}{\partial y \partial x} = 4a_1xy + a_4$

(b) $f(x,y) = a_1x^2y^2 + a_2x^2y + a_3xy^2 + a_4xy + a_5x + a_6y + a_7$

first order derivatives:

$f_x'(x,y) = 2a_1xy^2 + 2a_2xy + a_3y^2 + a_4y + a_5$

$f_y'(x,y) = 2a_1x^2y + a_2x^2 + 2a_3xy + a_4x + a_6$

second order derivatives:

$\frac{\partial^2 f}{\partial x^2} = 2a_1y^2 + 2a_2y$

$\frac{\partial^2 f}{\partial y^2} = 2a_1x^2 + 2a_3x$

$\frac{\partial^2 f}{\partial x \partial y} = 4a_1xy + 2a_2x + 2a_3y + a_4$

$\frac{\partial^2 f}{\partial y \partial x} = 4a_1xy + 2a_2x + 2a_3y + a_4$

(c) $\sigma(x) = \frac{1}{1+e^{-x}}$

$$\sigma'(x) = \frac{\partial \sigma}{\partial x} = \frac{e^{-x}}{(1+e^{-x})^2} = \frac{1}{1+e^{-x}} \cdot \frac{e^{-x}}{1+e^{-x}} = \frac{1}{1+e^{-x}} \cdot \left(\frac{1+e^{-x}-1}{1+e^{-x}}\right) = \frac{1}{1+e^{-x}} \cdot \left(1 - \frac{1}{1+e^{-x}}\right)$$

$$= \sigma(x)(1-\sigma(x))$$

(d) $y_1 = 4x^2 - 3x + 3$

$y_1' = 8x - 3 \quad y_1'' = 8 > 0$

$y_1' = 0 \Rightarrow x = \frac{8}{3} \quad y_1 = 4\times(\frac{8}{3})^2 - 3\times\frac{8}{3} + 3 = \frac{211}{9}$

so mimimum point $(\frac{8}{3}, \frac{211}{9})$

$y_2 = 3x^4 - 2x^3$

$y_2' = 12x^3 - 6x^2 \quad y_2' = 0 \Rightarrow x = 0$ or $x = \frac{1}{2}$

$y_2'' = 36x^2 - 12x \quad \begin{cases} x=0, & y_2''=0 \\ x=\frac{1}{2}, & y_2''=3>0 \end{cases}$

so minimum point $(\frac{1}{2}, -\frac{1}{16})$

stationary point $(0,0)$

$y_3 = 4x + \sqrt{1-x}$

$y_3' = 4 - \frac{1}{\sqrt{1-x}} \quad y_3' = 0 \Rightarrow x = \frac{15}{16}$

$y_3'' > 0$ and when $x = \frac{15}{16}$, $y_3 = 4$

so minimum point $(\frac{15}{16}, 4)$

$y_4 = x + x^{-1}$

$y_4' = 1 - x^{-2} \quad y_4' = 0 \Rightarrow x = 1$ or $x = -1$

$y_4'' = 2x^{-3} \quad \begin{cases} x=1, & y_4''=2>0 \\ x=-1, & y_4''=-2<0 \end{cases}$

so minimum point $(1,2)$

maximum point $(-1,-2)$

Question 2

(a) $P(A) = 20\% + 10\% = 30\%$

$P(B) = (1 - 20\% - 10\% - 40\%) + 10\% = 40\%$

$P(A \cup B) = 10\%$

$P(\bar{A}\bar{B}) = 40\%$

(b) (i) $r = 1 - \frac{1}{6} - \frac{1}{6} - \frac{1}{12} - \frac{1}{12} - \frac{1}{6} = \frac{1}{3}$

(ii) $P(X=2, Y=3) = \frac{1}{6}$

(iii) $P(X=3) = 0 + r + 0 = \frac{1}{3}$ $\quad P(X=3 \mid Y=2) = \frac{r}{\frac{1}{12}+r} = \frac{4}{5}$

(iv) $E[X] = (\frac{1}{6} + \frac{1}{12} + \frac{1}{12}) \times 1 + (\frac{1}{6} + \frac{1}{6}) \times 2 + \frac{1}{3} \times 3 = 2$

$E[Y] = (\frac{1}{6} + \frac{1}{6}) \times 1 + (\frac{1}{12} + \frac{1}{3}) \times 2 + (\frac{1}{12} + \frac{1}{6}) \times 3 = \frac{23}{12}$

$E[XY] = 1 \times \frac{1}{6} + 2 \times (\frac{1}{6} + \frac{1}{12}) + 3 \times \frac{1}{12} + 6 \times (\frac{1}{6} + \frac{1}{3}) = \frac{47}{12}$

(v) $E[X^2] = (\frac{1}{6} + \frac{1}{12} + \frac{1}{12}) \times 1 + (\frac{1}{6} + \frac{1}{6}) \times 4 + \frac{1}{3} \times 9 = \frac{14}{3}$

$E[Y^2] = (\frac{1}{6} + \frac{1}{6}) \times 1 + (\frac{1}{12} + \frac{1}{3}) \times 4 + (\frac{1}{12} + \frac{1}{6}) \times 9 = \frac{17}{4}$

(vi) $Cov[X, Y] = E(XY) - E(X)E(Y) = \frac{1}{12}$

(vii) $Var(X) = E[(X - E[X])^2] = \frac{2}{3}$

$Var(Y) = E[(Y - E[Y])^2] = \frac{23}{24}$

(viii) $corr(X, Y) = \frac{cov(X, Y)}{\sqrt{Var(X) \cdot Var(Y)}} = \frac{\sqrt{23}}{46}$

(IX) $E(X+Y) = E(X) + E(Y) = \frac{47}{12}$

$E[X+Y^2] = E(X) + E[Y^2] = \frac{25}{4}$

$Var(X+Y) = Var(X) + Var(Y) = \frac{13}{8}$

$Var(X+Y^2) = Var(X) + Var^2(Y) = \frac{913}{576}$

Question 3

(a) $\dim(A)=2 \quad \dim(b)=1 \quad \dim(A^T)=2$

(b)(i) $A\cancel{B}=\begin{bmatrix}1&3&4\\2&2&1\\6&4&3\end{bmatrix}$ $B=\begin{bmatrix}2&4\\1&1\end{bmatrix}$ A and B can not multiply

$3\times3 \qquad 2\times2$

(ii) $AC=\begin{bmatrix}\cancel{1}&\cancel{3}&\cancel{4}\\2&2&1\\6&4&3\end{bmatrix}\cdot\begin{bmatrix}7&3&3\\2&1&1\\2&2&2\end{bmatrix}=\begin{bmatrix}21&14&14\\20&12&12\\50&28&28\end{bmatrix}$

$CA=\begin{bmatrix}7&3&3\\2&1&1\\2&2&2\end{bmatrix}\cdot\begin{bmatrix}1&3&4\\2&2&1\\6&4&3\end{bmatrix}=\begin{bmatrix}31&39&40\\10&12&12\\18&18&16\end{bmatrix}$

(iii) $AD=\begin{bmatrix}1&3&4\\2&2&1\\6&4&3\end{bmatrix}\cdot\begin{bmatrix}4&2\\4&6\\1&3\end{bmatrix}=\begin{bmatrix}20&32\\17&19\\43&45\end{bmatrix}$ DA error

(iv) DC error $CD=\begin{bmatrix}7&3&3\\2&1&1\\2&2&2\end{bmatrix}\begin{bmatrix}4&2\\4&6\\1&3\end{bmatrix}=\begin{bmatrix}43&41\\13&13\\18&22\end{bmatrix}$

$D^TC=\begin{bmatrix}4&4&1\\2&6&3\end{bmatrix}\begin{bmatrix}7&3&3\\2&1&1\\2&2&2\end{bmatrix}=\begin{bmatrix}38&18&18\\32&18&18\end{bmatrix}$

(v) $Bu=\begin{bmatrix}2&4\\1&1\end{bmatrix}\begin{bmatrix}1\\3\end{bmatrix}=\begin{bmatrix}14\\4\end{bmatrix}$

uB error

(vi) Au error

(vii) $Av=\begin{bmatrix}1&3&4\\2&2&1\\6&4&3\end{bmatrix}\begin{bmatrix}2\\4\\1\end{bmatrix}=\begin{bmatrix}18\\13\\31\end{bmatrix}$

vA error

(viii) $Av+Bu$

$=\begin{bmatrix}18\\13\\31\end{bmatrix}$ + error = error

(c)(i) $u=\begin{bmatrix}1\\3\end{bmatrix}$ $\|u\|_1=1+3=4$ $\|u\|_\infty=\max(1,3)=3$

(ii) $v=\begin{bmatrix}2\\4\\1\end{bmatrix}$ $\|v\|=2+4+1=7$ $\|v\|_\infty=\max(2,4,1)=4$

(iii) $v+w=\begin{bmatrix}3\\2\\3\end{bmatrix}$ $\|v+w\|_1=3+2+3=8$ $\|v+w\|_\infty=\max(3,2,3)=3$

(d) $u=\begin{bmatrix}1\\2\end{bmatrix}$ $v=\begin{bmatrix}1\\1\end{bmatrix}$ $w=\begin{bmatrix}-1\\\frac{1}{2}\end{bmatrix}$

$\langle u,v\rangle=u^Tv=3$ $\quad\langle u,w\rangle=u^Tw=0$

$\langle v,u\rangle=v^Tu=3$ $\quad\langle v,w\rangle=v^Tw=-\frac{1}{2}$

$\langle w,u\rangle=w^Tu=0$ $\quad\langle w,v\rangle=w^Tv=-\frac{1}{2}$

(e) The dot product is always one dimensional and always expresses a certain aspect of all its values or meanings.

(f) $A=\begin{bmatrix}1&3\\4&1\end{bmatrix}$

$|A|=1\times1-3\times4=-11\neq0$

$\therefore A^{-1}=\frac{1}{|A|}A^*=\frac{1}{-11}\begin{bmatrix}1&-3\\-4&1\end{bmatrix}=\begin{bmatrix}-\frac{1}{11}&\frac{3}{11}\\\frac{4}{11}&-\frac{1}{11}\end{bmatrix}$

(g) $A=\begin{bmatrix}3&3\\4&4\end{bmatrix}$ $|A|=3\times4-3\times4=0$

so A can't be inversed

(h) $(X^TX)^T=(X^T)^T\cdot X^T$

$=X\cdot X^T$

when X is a matrix with the form of $m\times m$

$X\cdot X^T=X^T\cdot X$

so $(X^TX)^T=X\cdot X^T=X^TX$

$\therefore X^TX$ is always symmetric.