COMP9020 22T1
Week 4
Equivalence and Order Relations

- Textbook (R & W) - Ch. 3, Sec. 3.4-3.5
  Ch. 11, Sec. 11.1-11.2
- Problem set 4 + quiz

Equivalence Relations and Partitions

Relation $\mathcal{R}$ is called an **equivalence** relation if it satisfies (R), (S), (T).

Every equivalence $\mathcal{R}$ defines **equivalence classes** on its domain S. The equivalence class $[s]$ (w.r.t. $\mathcal{R}$) of an element $s \in S$ is

$$[s]_{\mathcal{R}} = \{ \ t \in S : t\mathcal{R}s \ \}$$

This notion is well defined only for $\mathcal{R}$ which is an equivalence relation. Collection of all equivalence classes is a *partition* of S:

$$S = \dot{\bigcup}_{s \in S} [s]_{\mathcal{R}} \qquad (\dot{\cup} \text{ denotes a } \textit{disjoint union})$$

Example

$\mathcal{R} = \{(m, n) \in \mathbb{Z} : m \mod 2 = n \mod 2\}$
$[0] = \{\ldots, -4, -2, 0, 2, 4, \ldots\}$ (same as $[-2]$, $[2]$, $\ldots$)
$[1] = \{\ldots, -3, -1, 1, 3, 5, \ldots\}$

Thus the equivalence classes are disjoint and jointly cover the entire domain. It means that every element belongs to one (and only one) equivalence class.

We call $s_1, s_2, \ldots$ *representatives* of (different) equivalence classes

For $s, t \in S$ either $[s] = [t]$, when $s\mathcal{R}t$, or $[s] \cap [t] = \emptyset$, when $s \not\mathcal{R} t$.

We commonly write $s \sim_{\mathcal{R}} t$ when s, t are in the same equivalence class.

In the opposite direction, a partition of a set defines the equivalence relation on that set. If $S = S_1 \dot\cup \ldots \dot\cup S_k$, then we specify $s \sim t$ exactly when s and t belong to the same S_i.

Example

$\mathbb{Z} = \{\ldots, -3, 0, 3, \ldots\} \dot\cup \{\ldots, -2, 1, 4, \ldots\} \dot\cup \{\ldots, -1, 2, 5, \ldots\}$

$m \sim n$ if, and only if, $m \mod 3 = n \mod 3$

$[0] = [3] = [6] = \ldots \qquad [0] \cap [1] = \emptyset = [0] \cap [2]$

If the relation $\sim$ is an equivalence on S and $[S]$ the corresponding partition, then

$$\nu : S \longrightarrow [S], \quad \nu : s \mapsto [s] = \{ \, x \in S : x \sim s \, \}$$

is called the *natural* map. It is always onto.

Exercise

When is ν also 1–1 ?

Only when $\sim$ is the identity on S.

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Exercise

When is ν also 1–1 ?

Only when $\sim$ is the identity on S.

A function $f : S \longrightarrow T$ defines an equivalence relation on S by

$$s_1 \sim s_2 \quad \text{iff} \quad f(s_1) = f(s_2)$$

These sets $f^{\leftarrow}(t)$, $t \in T$ that are nonempty form the corresponding partition

$$S = \bigcup_{t \in T} f^{\leftarrow}(t)$$

Exercise

When are all $f^{\leftarrow}(t) \neq \emptyset$?

When f is onto.

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Exercise

When are all $f^{\leftarrow}(t) \neq \emptyset$?

When f is onto.

Example: Congruence Relations

$\mathbb{Z} \longrightarrow \mathbb{Z}_p$: Partition of $\mathbb{Z}$ into classes of numbers with the same remainder $\pmod{p}$; it is particularly important for p prime

$$\mathbb{Z}_p = \{0, 1, \ldots, p-1\}$$

One can define all four arithmetic operations (with the usual properties) on $\mathbb{Z}_p$ for a prime p; division has to be restricted when p is not prime.

Standard notation: $\mathbf{m} \equiv \mathbf{n} \pmod{\mathbf{p}}$
$\stackrel{\text{def}}{=}$ remainder of dividing m by p = remainder of dividing n by p

NB

$(\mathbb{Z}_p, +, \cdot, 0, 1)$ are fundamental algebraic structures known as **rings**. These structures are very important in coding theory and cryptography.

Modular Arithmetic

Example

$\mathbb{Z}_5 = \{0, 1, 2, 3, 4\}$

$+_5$	0	1	2	3	4
0	0	1	2	3	4
1	1	2	3	4	0
2	2	3	4	0	1
3	3	4	0	1	2
4	4	0	1	2	3

n	$-n$
0	0
1	4
2	3
3	2
4	1

$*_5$	0	1	2	3	4
0	0	0	0	0	0
1	0	1	2	3	4
2	0	2	4	1	3
3	0	3	1	4	2
4	0	4	3	2	1

n	n^{-1}
0	–
1	1
2	3
3	2
4	4

Exercise

$\boxed{3.5.6}$ Calculate the following in $\mathbb{Z}_7$.

(b) $5 + 6 =$ 4

(c) $4 * 4 =$ 2

(d) for any $k \in \mathbb{Z}_7$, $0 + k =$ k

(e) for any $k \in \mathbb{Z}_7$, $1 * k =$ k

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(e) for any $k \in \mathbb{Z}_7$, $1 * k = k$

Exercise

Solve the following for x in $\mathbb{Z}_5$.

(a) $2 + x = 1$ $\quad \Rightarrow x = 4$

(b) $2 * x = 1$ $\quad \Rightarrow x = 2^{-1} = 3$

(c) $2 * x = 3$ $\quad \Rightarrow x = 3 * 2^{-1} = 3 * 3 = 4$

Exercise

Solve the following for x in $\mathbb{Z}_6$.

(d) $5 + x = 1$

(e) $5 * x = 1$

(e) $2 * x = 1$

Exercise

Solve the following for x in $\mathbb{Z}_5$.

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(c) $2 * x = 3 \quad \Rightarrow x = 3 * 2^{-1} = 3 * 3 = 4$

Exercise

Solve the following for x in $\mathbb{Z}_6$.

(d) $5 + x = 1 \quad \Rightarrow x = 2$

(e) $5 * x = 1 \quad \Rightarrow x = 5 \quad (\text{since } 25 \mod 6 = 1)$

(e) $2 * x = 1 \quad$ undefined $\quad (\text{since } 2 \cdot k \mod 6 \neq 1 \text{ for all } k \in \mathbb{Z}_6)$

Exercise

Solve the following for x in $\mathbb{Z}_5$.

(a) $2 + x = 1 \quad \Rightarrow x = 4$

(b) $2 * x = 1 \quad \Rightarrow x = 2^{-1} = 3$

(c) $2 * x = 3 \quad \Rightarrow x = 3 * 2^{-1} = 3 * 3 = 4$

Exercise

Solve the following for x in $\mathbb{Z}_6$.

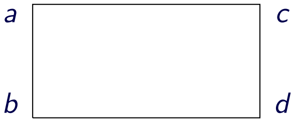
(d) $5 + x = 1 \quad \Rightarrow x = 2$

(e) $5 * x = 1 \quad \Rightarrow x = 5 \quad (\text{since } 25 \mod 6 = 1)$

(e) $2 * x = 1$ undefined (since $2 \cdot k \mod 6 \neq 1$ for all $k \in \mathbb{Z}_6$)

Example: Equivalence Classes in Geometry

Starting from the rectangle

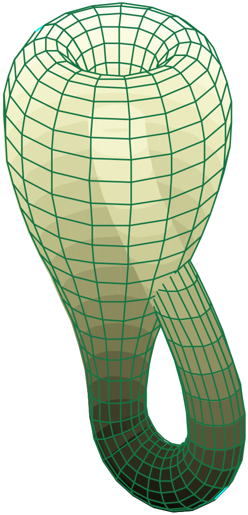


and identifying the points in the interval $[a, b]$ with the points in $[c, d]$ (in this direction) results in a cylinder, while identifying also $[a, c]$ and $[b, d]$ gives a torus.

Points in the interior of the rectangle are not 'glued' together; thus the equivalence classes (for the cylinder) have either one or two elements, while for the torus there is also one class with four elements (which are the four elements?)

Identifying $[a, b]$ with $[d, c]$ (in that direction) gives a *one-sided* Moebius strip; furthermore, putting it together with identifying $[a, c]$ and $[b, d]$ gives a one-sided closed surface ("Klein bottle").

A Klein bottle cannot be embedded in 3 dimensions without self-intersection.



Supplementary Exercise

Exercise

$\boxed{3.6.6}$ Show that $m \sim n$ iff $m^2 \equiv n^2 \pmod 5$ is an equivalence on $S = \{1, \ldots, 7\}$. Find all the equivalence classes.

(a) It just means that $m \equiv n \pmod 5$ or $m \equiv -n \pmod 5$, e.g. $1 \equiv -4 \pmod 5$. This satisfies (R), (S), (T).

(b) We have
$[1] = \{1, 4, 6\}$
$[2] = \{2, 3, 7\}$
$[5] = \{5\}$

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It is often necessary to define a function on $[S]$ by describing it on the individual representatives $t \in [s]$ for each equivalence class $[s]$. If $\phi : [S] \longrightarrow X$ is to be defined in this way, one must

- define $\phi(t)$ for all $t \in S$, making sure that $\phi(t) \in X$
- make sure that $\phi(t_1) = \phi(t_2)$ whenever $t_1 \sim t_2$, ie. when $[t_1] = [t_2]$
- define $\phi([s]) \stackrel{\text{def}}{=} \phi(s)$.

The second condition is critical for ϕ to be well-defined.

Example

$[S] = \{0, 4, 8, \ldots\} \mathbin{\dot\cup} \{1, 5, 9, \ldots\} \mathbin{\dot\cup} \{2, 6, 10, \ldots\} \mathbin{\dot\cup} \{3, 7, 11, \ldots\}$

$\phi : [S] \longrightarrow \mathbb{Z}_2$ defined by $\phi(n) = n \bmod 2$

$\phi(0) = 0 = \phi(4) = \phi(8) = \ldots$

Example

Example of a not well-defined 'function' on equivalence classes:

$\phi : \{0, 3, 6, \ldots\} \,\dot\cup\, \{1, 4, 7, \ldots\} \,\dot\cup\, \{2, 5, 8, \ldots\} \longrightarrow \mathbb{Z}_5$
$\phi(n) \stackrel{?}{=} n \mod 5$

Problem: $[0] = [3] = [6] = \ldots$ in $\mathbb{Z}_3$; however,
$0 \mod 5 = 0$, $3 \mod 5 = 3$, $6 \mod 5 = 1 \ \ldots$

Supplementary Exercise

Exercise

$\boxed{3.6.10}$

$\mathcal{R}$ is a binary relation on $\mathbb{N} \times \mathbb{N}$, i.e. it is a subset of $\mathbb{N}^4$

$(m, n)\,\mathcal{R}\,(p, q)$ if $m \equiv p \pmod 3$ or $n \equiv q \pmod 5$.

(a) $\mathcal{R} \in (\mathrm{R})$?

Yes: $(m, n) \sim (m, n)$ iff $m \equiv m \pmod 3$ or $n \equiv n \pmod 5$ iff true or true.

(b) $\mathcal{R} \in (\mathrm{S})$?

Yes: by symmetry of $. \equiv . \pmod n$.

(c) $\mathcal{R} \in (\mathrm{T})$?

No — for arbitrary two pairs (m_1, n_1) and (m_2, n_2) one can create a chain $(m_1, n_1)\mathcal{R}(m_2, n_1)$ and $(m_2, n_1)\mathcal{R}(m_2, n_2)$, but $(m_1, n_1) \not\mathcal{R} (m_2, n_2)$.

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(b) $\mathcal{R} \in$ (S)?

Yes: by symmetry of $. \equiv . \pmod n$.

(c) $\mathcal{R} \in$ (T)?

No — for arbitrary two pairs (m_1, n_1) and (m_2, n_2) one can create a chain $(m_1, n_1)\mathcal{R}(m_2, n_1)$ and $(m_2, n_1)\mathcal{R}(m_2, n_2)$, but $(m_1, n_1) \not\mathcal{R}(m_2, n_2)$.

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Order Relations

Total order $\leq$ on S
(R) $x \leq x$ for all $x \in S$
(AS) $x \leq y, y \leq x \Rightarrow x = y$
(T) $x \leq y, y \leq z \Rightarrow x \leq z$
(L) *Linearity* — any two elements are comparable:
for all x, y either $x \leq y$ or $y \leq x$ (and both if $x = y$)

On a finite set all total orders are "isomorphic"

$$x_1 \leq x_2 \leq \cdots \leq x_n$$

On an infinite set there is quite a variety of possibilities.

Examples

- discrete with a least element, e.g. $\mathbb{N} = \{0, 1, 2, \dots\}$
- discrete without a least element, e.g. $\mathbb{Z} = \{\dots, 0, 1, 2, \dots\}$
- various dense/locally dense orders
  - rational numbers $\mathbb{Q} : \forall p, q \in \mathbb{Q}\,(p < q \Rightarrow \exists r \in \mathbb{Q}\,(p < r < q))$
  - $S = [a, b]$ — both least and greatest elements
  - $S = (a, b]$ — no least element
  - $S = [a, b)$ — no greatest element
  - other $[0, 1] \cup [2, 3] \cup [4, 5] \cup \dots$

Partial Order

Definition

A **partial order** $\preceq$ on S satisfies (R), (AS), (T); need not be (L)

$(S, \preceq)$ is called a **poset** — partially ordered set

To each (partial) order one can associate a unique **quasi-order**

$$x \prec y \text{ iff } x \preceq y \text{ and } x \neq y$$

It satisfies (AS) and (T); it satisfies (L) if it corresponds to a total order (we could call it a total quasi-order); it does not satisfy (R) for any pair x, y.

Example

Exercise

$\boxed{11.1.8}$ For $\omega_1, \omega_2 \in \Sigma^*$ define $\omega_1 \preceq \omega_2$ when $\omega_2 = \nu\omega_1\nu'$ for some ν, ν'.
Is $(\Sigma^*, \preceq)$ a partial order?

Yes.
Relation $\preceq$ means being a substring; it is a partial order:

(R) $\omega = \lambda\omega\lambda$, hence $\omega \preceq \omega$
(AS) if $\omega_1 = \nu\omega_2\nu'$ and $\omega_2 = \chi\omega_1\chi'$ for some ν, ν', χ, χ' then $\nu = \nu' = \chi = \chi' = \lambda$, hence $\omega_1 = \omega_2$
(T) if $\omega_1 = \nu\omega_2\nu'$ and $\omega_2 = \chi\omega_3\chi'$ then $\omega_1 = \nu\chi\omega_3\chi'\nu'$

Example

Exercise

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(T) if $\omega_1 = \nu\omega_2\nu'$ and $\omega_2 = \chi\omega_3\chi'$ then $\omega_1 = \nu\chi\omega_3\chi'\nu'$

Exercise

$\boxed{11.6.16}$ Properties of four relations defined on $\mathbb{P} = \{1, 2, \ldots\}$?

- $\mathcal{R}_1$ if $m|n$
- $\mathcal{R}_2$ if $|m-n| \leq 2$
- $\mathcal{R}_3$ if $2|m+n$
- $\mathcal{R}_4$ if $3|m+n$

	$\mathcal{R}_1$	$\mathcal{R}_2$	$\mathcal{R}_3$	$\mathcal{R}_4$
(R)				
(S)				
(AS)				
(T)				
Equivalence	?	?	?	?
Partial order	?	?	?	?

Exercise

$\boxed{11.6.16}$ Properties of four relations defined on $\mathbb{P} = \{1, 2, \ldots\}$

- $\mathcal{R}_1$ if $m|n$
- $\mathcal{R}_2$ if $|m - n| \leq 2$
- $\mathcal{R}_3$ if $2|m + n$
- $\mathcal{R}_4$ if $3|m + n$

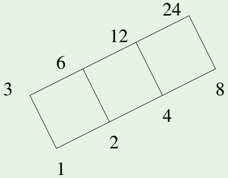
	$\mathcal{R}_1$	$\mathcal{R}_2$	$\mathcal{R}_3$	$\mathcal{R}_4$
(R)	Yes	Yes	Yes	
(S)		Yes	Yes	Yes
(AS)	Yes			
(T)	Yes		Yes	
Equivalence			Yes	
Partial order	Yes			

Hasse Diagram

Every finite poset can be represented as a **Hasse diagram**, where a line is drawn *upward* from x to y if $x \prec y$ and there is no z such that $x \prec z \prec y$

Example

$\boxed{11.1.1(a)}$ Hasse diagram for positive divisors of 24



$p \preceq q$ if, and only if, $p \mid q$

(Named after mathematician Helmut Hasse (Germany), 1898–1979)

Ordering Concepts

Definition

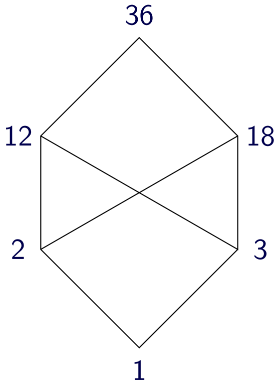
- *Minimal* and *maximal* elements (they always exist in every finite poset)
- *Minimum* and *maximum* — unique minimal and maximal element
- *lub* (least upper bound) and *glb* (greatest lower bound) of a subset $A \subseteq S$ of elements
  lub(A) — smallest element $x \in S$ s.t. $x \succeq a$ for all $a \in A$
  glb(A) — greatest element $x \in S$ s.t. $x \preceq a$ for all $a \in A$
- *Lattice* — a poset where lub and glb exist for every pair of elements
  (by induction, they then exist for every *finite* subset of elements)

Examples

- $\mathrm{Pow}(\{a, b, c\})$ with the order $\subseteq$
  $\emptyset$ is minimum; $\{a, b, c\}$ is maximum
- $\boxed{11.1.4}$
  $\mathrm{Pow}(\{a, b, c\}) \setminus \{\{a, b, c\}\}$ (proper subsets of $\{a, b, c\}$)
  Each two-element subset $\{a, b\}, \{a, c\}, \{b, c\}$ is maximal.
  - But there is no maximum
- $\{1, 2, 3, 4, 6, 8, 12, 24\}$ partially ordered by divisibility is a lattice
  - e.g. $\mathrm{lub}(\{4, 6\}) = 12$; $\mathrm{glb}(\{4, 6\}) = 2$
- $\{1, 2, 3\}$ partially ordered by divisibility is not a lattice
  - $\{2, 3\}$ has no lub
- $\{2, 3, 6\}$ partially ordered by divisibility is not a lattice
  - $\{2, 3\}$ has no glb

Examples

- $\{1, 2, 3, 12, 18, 36\}$ partially ordered by divisibility is not a lattice
  - $\{2, 3\}$ has no lub ($12, 18$ are minimal upper bounds)



NB

An infinite lattice need not have a lub (or no glb) for an arbitrary (infinite!) subset of its elements, in particular no such bound may exist for **all** its elements.

Examples

- $(\mathbb{Z}, \leq)$ — neither lub nor glb;
- $(\mathbb{F}(\mathbb{N}), \subseteq)$ — all finite sets of natural numbers
  has no *arbitrary* lub property
  glb exists for infinitely many elements: it is the intersection, hence always finite;
- $(\mathbb{I}(\mathbb{N}), \subseteq)$ — all infinite sets of natural numbers
  may not have an arbitrary glb
  lub exists for infinitely many elements: it is the union, which is always infinite.

Exercise

$\boxed{11.1.5}$ Consider poset $(\mathbb{R}, \leq)$

(a) Is this a lattice?

(b) Give an example of a non-empty subset of $\mathbb{R}$ that has no upper bound.

(c) Find $\text{lub}(\{\ x \in \mathbb{R} : x < 73\ \})$

(d) Find $\text{lub}(\{\ x \in \mathbb{R} : x \leq 73\ \})$

(e) Find $\text{lub}(\{\ x : x^2 < 73\ \})$

(f) Find $\text{glb}(\{\ x : x^2 < 73\ \})$

Exercise

(a) It is a lattice.
(b) subset with no upper bound: $\mathbb{R}_{>0} = \{ r \in \mathbb{R} : r > 0 \}$
(c) and (d) $\text{lub}(\{ x : x < 73 \}) = \text{lub}(\{ x : x \leq 73 \}) = 73$
(e) $\text{lub}(\{ x : x^2 < 73 \}) = \sqrt{73}$
(f) $\text{glb}(\{ x : x^2 < 73 \}) = -\sqrt{73}$

Exercise

$\boxed{11.1.13}$ $\mathbb{F}(\mathbb{N})$ — collection of all *finite* subsets of $\mathbb{N}$; $\subseteq$-order

(a) Does it have a maximal element?

(b) Does it have a minimal element?

(c) Given $A, B \in \mathbb{F}(\mathbb{N})$, does $\{A, B\}$ have a lub in $\mathbb{F}(\mathbb{N})$?

(d) Given $A, B \in \mathbb{F}(\mathbb{N})$, does $\{A, B\}$ have a glb in $\mathbb{F}(\mathbb{N})$?

(e) Is $(\mathbb{F}(\mathbb{N}), \subseteq)$ a lattice?

Exercise

$\boxed{11.1.13}$ $\mathbb{F}(\mathbb{N})$ — collection of all *finite* subsets of $\mathbb{N}$; $\subseteq$-order

(a) No maximal elements

(b) $\emptyset$ is the minimum

(c) $\mathrm{lub}(A, B) = A \cup B$

(d) $\mathrm{glb}(A, B) = A \cap B$

(e) $(\mathbb{F}(\mathbb{N}), \subseteq)$ is a lattice — is has *finite* union and intersection properties.

Exercise

$\boxed{11.1.14}$ $\mathbb{I}(\mathbb{N}) = \text{Pow}(\mathbb{N}) \setminus \mathbb{F}(\mathbb{N})$ — all *infinite* subsets of $\mathbb{N}$

(a) Does it have a maximal element?

(b) Does it have a minimal element?

(c) Given $A, B \in \mathbb{I}(\mathbb{N})$, does $\{A, B\}$ have a lub in $\mathbb{I}(\mathbb{N})$?

(d) Given $A, B \in \mathbb{I}(\mathbb{N})$, does $\{A, B\}$ have a glb in $\mathbb{I}(\mathbb{N})$?

(e) Is $(\mathbb{I}(\mathbb{N}), \subseteq)$ a lattice?

Exercise

$\boxed{11.1.14}$ $\mathbb{I}(\mathbb{N}) = \mathrm{Pow}(\mathbb{N}) \setminus \mathbb{F}(\mathbb{N})$ — all *infinite* subsets of $\mathbb{N}$
(a) $\mathbb{N}$ is the maximum
(b) No minimal elements ($\emptyset$ is not in $\mathbb{I}(\mathbb{N})$)
(c) $\mathrm{lub}(A, B) = A \cup B$
(d) $\mathrm{glb}(A, B) = A \cap B$ *if it exists*; it does not exist when $A \cap B$ is finite, eg. when empty.
(e) $(\mathbb{I}(\mathbb{N}), \subseteq)$ is not a lattice — it has finite union but not finite intersection property; eg. sets $2\mathbb{N}$ and $2\mathbb{N} + 1$ have the empty intersection.

Well-Ordered Sets

Well-ordered set: every subset has a least element.

NB

The greatest element is not required.

Examples

- $\mathbb{N} = \{0, 1, \ldots\}$
- $\mathbb{N}_1 \dot\cup \mathbb{N}_2 \dot\cup \mathbb{N}_3 \dot\cup \ldots$, where each $\mathbb{N}_i \simeq \mathbb{N}$ and $\mathbb{N}_1 < \mathbb{N}_2 < \mathbb{N}_3 \cdots$

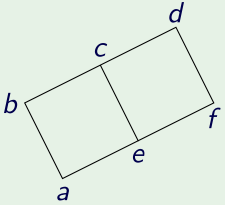
NB

Well-order sets are an important mathematical tool to prove termination of programs.

Ordering of a Poset — Topological Sort

For a poset $(S, \preceq)$ any linear order $\leq$ that is consistent with $\preceq$ is called **topological sort**. Consistency means that $a \preceq b \Rightarrow a \leq b$.

Example



The following all are topological sorts:

$a \leq b \leq e \leq c \leq f \leq d$

$a \leq e \leq b \leq f \leq c \leq d$

. . . . . .

$a \leq e \leq f \leq b \leq c \leq d$

Combining Orders

Product order — can combine any partial orders. In general, it is only a *partial order*, even if combining total orders.
For $s, s' \in S$ and $t, t' \in T$ define

$$(s,t) \preceq (s',t') \quad \text{if } s \preceq s' \text{ and } t \preceq t'$$

Exercise

$\boxed{11.2.1}$ Let $A = \{1, 2, 3, 4\}$ with the usual order, and $S = A \times A$ with the product order.

(a) A chain with seven elements?

(b) A chain with eight elements?

Exercise

$\boxed{11.2.1}$ Let $A = \{1, 2, 3, 4\}$ with the usual order, and $S = A \times A$ with the product order.

(a) A chain with seven elements?
$(1, 1)\,(1, 2)\,(2, 2)\,(2, 3)\,(2, 4)\,(3, 4)\,(4, 4)$ (other solutions exist)

(b) A chain with eight elements?
The above is a maximal chain.
No chains of eight elements.

Example

$\boxed{11.2.4}$ Take $(S, \preceq_1)$, $(T, \preceq_2)$ to be any total orders of more than one element. Then $S \times T$ with the product order is not a total order: for any $s_1 \prec s_2, t_1 \prec t_2$ the pair (s_1, t_2) and (s_2, t_1) are *not* comparable.

Exercise

$\boxed{11.2.10}$ Take $(S, \preceq_1)$, $(T, \preceq_2)$ to be any posets (even chains) and define the combined order $\sqsubseteq$ on $S \times T$

$$(s, t) \sqsubseteq (s', t') \text{ if } \; s \preceq s' \text{ or } \; t \preceq t'$$

Is $\sqsubseteq$ a partial order?

Not a partial order, as it may not satisfy (AS) nor (T).

Exercise

$\boxed{11.2.10}$ Take $(S, \preceq_1)$, $(T, \preceq_2)$ to be any posets (even chains) and define the combined order $\sqsubseteq$ on $S \times T$

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Is $\sqsubseteq$ a partial order?
Not a partial order, as it may not satisfy (AS) nor (T).

Ordering of Functions

S — arbitrary set (no order required)
$(T, \preceq_T)$ — partially ordered set
$M = \{f : S \longrightarrow T\}$ — set of all functions from S to T
It has a natural partial order:

$$f \preceq g \quad \text{iff} \quad \forall s \in S\,(f(s) \preceq_T g(s))$$

NB

For finite $S = \{s_1, \ldots, s_n\}$ this is essentially a generalised product order: $(f(s_1), \ldots, f(s_n)) \preceq (g(s_1), \ldots, g(s_n))$.

In most applications S has a linear ordering; however, it does not affect the order of the functions defined on S (only the order on T matters).

Practical Orderings

They are, effectively, *total* orders on the *product* of ordered sets.

- **Lexicographic order** — defined on all of Σ^*. It extends a total order already assumed to exist on Σ.
- **Lenlex** — the order on (potentially) the entire Σ^*, where the elements are ordered first by length.
  $\Sigma^{(1)} \prec \Sigma^{(2)} \prec \Sigma^{(3)} \prec \cdots$, then lexicographically within each $\Sigma^{(k)}$. In practice it is applied only to the finite subsets of Σ^*.
- **Filing order** — lexicographic order confined to the strings of the same length.
  It defines total orders on Σ^i, separately for each i.

Examples

Exercise

$\boxed{11.2.5}$ Let $\mathbb{B} = \{0, 1\}$ with the usual order $0 < 1$. List the elements $101, 010, 11, 000, 10, 0010, 1000$ of $\mathbb{B}^*$ in the

(a) Lexicographic order

$000, 0010, 010, 10, 1000, 101, 11$

(b) Lenlex order

$10, 11, 000, 010, 101, 0010, 1000$

$\boxed{11.2.8}$ When are the lexicographic order and *lenlex* on Σ^* the same?

Only when $|\Sigma| = 1$.

Examples

Exercise

$\boxed{11.2.5}$ Let $\mathbb{B} = \{0, 1\}$ with the usual order $0 < 1$. List the elements $101, 010, 11, 000, 10, 0010, 1000$ of $\mathbb{B}^*$ in the

(a) Lexicographic order

$000, 0010, 010, 10, 1000, 101, 11$

(b) Lenlex order

$10, 11, 000, 010, 101, 0010, 1000$

$\boxed{11.2.8}$ When are the lexicographic order and *lenlex* on Σ^* the same?

Only when $|\Sigma| = 1$.

Supplementary Exercise

$\boxed{11.6.12}$ Draw a Hasse diagram for a poset with exactly 5 members, 2 of which are maximal and 1 of which is the poset's minimum.

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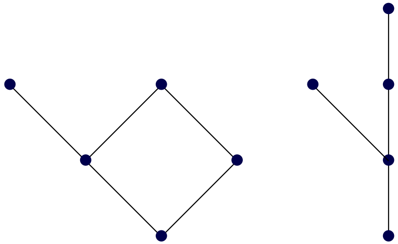

and more . . .

Supplementary Exercise

Exercise

$\boxed{11.6.6}$ True or false?

(a) If a set Σ is totally ordered, then the corresponding lexicographic partial order on Σ^* also must be totally ordered.

(b) If a set Σ is totally ordered, then the corresponding lenlex order on Σ^* also must be totally ordered.

(c) Every finite partially ordered set has a Hasse diagram.

(d) Every finite partially ordered set has a topological sorting.

(e) Every finite partially ordered set has a minimum.

(f) Every finite totally ordered set has a maximum.

(g) An infinite partially ordered set cannot have a maximum.

Supplementary Exercise

Exercise

$\boxed{11.6.6}$

(a) and (b) – True; this is the idea behind various lex-sorts
(c) Yes.
(d) Yes.
(e) False – consider a two-element set with the identity as p.o.
(f) True – due to the finiteness
(g) False, eg. $\mathbb{Z}_{<0}$

Summary

- Equivalence relations $\sim$, equivalence classes $[S]$
- Special equivalence relations on $\mathbb{Z}$: notation $m \equiv n \pmod{p}$; $\mathbb{Z}_p$
- Orderings: total, partial; lub, glb, lattice, topological sort
- Hasse diagrams
- Specific orderings: product, lexicographic, lenlex

Coming up ...

- Ch. 3, Sec. 3.2; Ch. 6, Sec. 6.1-6.5 (Graphs)