COMP9020 22T1
Week 3
Functions and Relations

- Textbook (R & W) - Ch. 1, Sec. 1.7
  Ch. 3, Sec. 3.1, 3.3
- Problem set 3 + quiz

Functions

Reminder:
We deal with functions as a set-theoretic concept, it being a special kind of correspondence (between two sets)
$f : S \longrightarrow T$ describes pairing of the sets: it means that f assigns to every element $s \in S$ a unique element $t \in T$.

S — **domain** of f, symbol: $\mathrm{Dom}(f)$
T — **codomain** of f, symbol: $\mathrm{Codom}(f)$
$\{ f(x) : x \in \mathrm{Dom}(f) \}$ — **image** of f, symbol: $\mathrm{Im}(f)$

$$\mathrm{Im}(f) \subseteq \mathrm{Codom}(f)$$

We observe that every function maps its domain **into** its codomain, but only **onto** its image.

Exercise

$\boxed{1.5.3}$ Regarding length : $\{a, b\}^* \longrightarrow \mathbb{N}$

(c) length$(\lambda) \stackrel{?}{=}$

(d) Im(length) $\stackrel{?}{=}$

$\boxed{1.5.4}$ Σ^* as above and $g(n) \stackrel{\text{def}}{=} \{\, \omega \in \Sigma^* : \text{length}(\omega) \leq n \,\}$, $n \in \mathbb{N}$
Here $g(n)$ is a function that has a complex object as its value for any given argument — it maps $\mathbb{N}$ into Pow(Σ^*)

(a) $g(0) \stackrel{?}{=}$

(b) $g(1) \stackrel{?}{=}$

(c) $g(2) \stackrel{?}{=}$

(d) Are all $g(n)$ finite?

Exercise

$\boxed{1.5.3}$ Regarding $\text{length} : \{a, b\}^* \longrightarrow \mathbb{N}$
(c) $\text{length}(\lambda) = 0$
(d) $\text{Im}(\text{length}) = \mathbb{N}$

$\boxed{1.5.4}$ Σ^* as above and $g(n) \stackrel{\text{def}}{=} \{\ \omega \in \Sigma^* : \text{length}(\omega) \leq n\ \}$, $n \in \mathbb{N}$
Here $g(n)$ is a function that has a complex object as its value for any given argument — it maps $\mathbb{N}$ into $\text{Pow}(\Sigma^*)$
(a) $g(0) = \{\lambda\}$
(b) $g(1) = \{\lambda, a, b\}$
(c) $g(2) = \{\lambda, a, b, aa, ab, ba, bb\}$
In general $g(n) = \bigcup_{i=0}^{n} \Sigma^i = \Sigma^{\leq n}$
(d) Are all $g(n)$ finite?
Yes; $|g(n)| = 2^0 + 2^1 + \ldots + 2^n = 2^{n+1} - 1$

Exercise (cont'd)

(e) Give an example of a set in $\mathrm{Pow}(\Sigma^*)$ that is not in $\mathrm{Im}(g)$

- any infinite subset of Σ^* (infinite language)
- any finite language that excludes some intermediate length words, e.g. $\{\lambda, a\}, \{a, b\}, \{\lambda, a, aa\}, \ldots$

Exercise (cont'd)

(e) Give an example of a set in $\mathrm{Pow}(\Sigma^*)$ that is not in $\mathrm{Im}(g)$

- any infinite subset of Σ^* (infinite language)
- any finite language that excludes some intermediate length words, e.g. $\{\lambda, a\}, \{a, b\}, \{\lambda, a, aa\}, \ldots$

Exercise (cont'd)

$\boxed{1.5.6}$ Regarding $\gcd : \mathbb{P} \times \mathbb{P} \longrightarrow \mathbb{P}$

(c) $\mathrm{Im}(\gcd) \stackrel{?}{=}$

$\boxed{1.5.7}$ Define $f : \mathbb{R} \longrightarrow \mathbb{R}$ by

$$f(x) = \begin{cases} x^3 & x \geq 1 \\ x & 0 \leq x < 1 \\ -x^3 & x < 0 \end{cases}$$

(c) $\mathrm{Im}(f) \stackrel{?}{=}$

Exercise (cont'd)

$\boxed{1.5.6}$ Regarding $\gcd : \mathbb{P} \times \mathbb{P} \longrightarrow \mathbb{P}$

(c) $\mathrm{Im}(\gcd) = \mathbb{P}$ since $\gcd(n, n) = n$

$\boxed{1.5.7}$ Define $f : \mathbb{R} \longrightarrow \mathbb{R}$ by

$$f(x) = \begin{cases} x^3 & x \geq 1 \\ x & 0 \leq x < 1 \\ -x^3 & x < 0 \end{cases}$$

(c) $\mathrm{Im}(f) = \mathbb{R}_{\geq 0}$

Composition of Functions

Auxiliary notation

$$f : x \mapsto y, \quad f : A \mapsto B$$

The former means that x is mapped to y; the latter means that B is the **image** of $A \subseteq \mathrm{Dom}(f)$ under f: $B = f(A) \stackrel{\text{def}}{=} \{f(s) : \ s \in A\}$

NB

Observe the difference between $\longrightarrow$ (cf. slide 2) and $\mapsto$.
$x \mapsto y$ shorthand for $\{x\} \mapsto \{y\}$

Definition

Composition of functions is described as

$$g \circ f : x \mapsto g(f(x)), \quad \text{requiring } \mathrm{Im}(f) \subseteq \mathrm{Dom}(g)$$

If a function maps a set into itself, i.e. when $\mathrm{Dom}(f) = \mathrm{Codom}(f)$ (and thus $\mathrm{Im}(f) \subseteq \mathrm{Dom}(f)$), the function can be composed with itself — **iterated**

$$f \circ f, f \circ f \circ f, \ldots, \quad \text{also written } f^2, f^3, \ldots$$

Composition is associative

$$h \circ (g \circ f) = (h \circ g) \circ f, \quad \text{can write } h \circ g \circ f$$

Identity function on S

$$\mathrm{Id}_S(x) = x, x \in S; \mathrm{Dom}(i) = \mathrm{Codom}(i) = \mathrm{Im}(i) = S$$

For $g : S \longrightarrow T \quad g \circ \mathrm{Id}_S = g,\ \mathrm{Id}_T \circ g = g$

gcd Example

Reconsider gcd as a **higher-order function**, defined by

$$\text{gcd}(f)(m,n) = \begin{cases} m & \text{if } m = n \\ f(m-n,n) & \text{if } m > n \\ f(m,n-m) & \text{if } m < n \end{cases}$$

Its type is now $\text{gcd} : (\mathbb{P}^2 \nrightarrow \mathbb{P}) \longrightarrow (\mathbb{P}^2 \nrightarrow \mathbb{P})$
that is, it maps each partial function (from pairs of positive integers to a positive integer) to a (partial) function of the same type. The worst such function is the "nowhere defined" function

$$f_\perp(m,n) = \perp \ .$$

NB

A **partial function** $f : S \nrightarrow T$ is a function $f : S' \longrightarrow T$ for $S' \subseteq S$

gcd Example cont'd

Consider the sequence

$$f_\perp, \text{gcd}(f_\perp), \text{gcd}(\text{gcd}(f_\perp)), \ldots, \text{gcd}(\text{gcd}(\ldots(f_\perp)\ldots)), \ldots$$

and observe that the i'th element of this sequence is an approximation of the gcd function that works as long as the depth of the recursion is less than $i-1$. Since we proved that the original gcd function terminates, we can deduce that the limit of this sequence exists, and is the original gcd. It also is the **least fixpoint** of gcd i.e. the "simplest" solution f to the equation $f = \text{gcd}(f)$. This, in a nutshell, explains how the semantics of recursive procedures is defined in CS. How all this works is somewhat beyond the scope of COMP9020 but still serves the purpose of motivating why we discuss functions and their composition, iteration.

Properties of Functions

Function is called **onto** (or **surjective**) if every element of the codomain is mapped to by at least one x in the domain, i.e.

$$\mathrm{Im}(f) = T$$

Examples (of functions that are not onto)

- $f : \mathbb{N} \longrightarrow \mathbb{N}$ with $f(x) \mapsto x^2$
- $f : \{a, \ldots, z\}^* \longrightarrow \{a, \ldots, z\}^*$ with $f(\omega) \mapsto a\omega e$

1-1 Functions

Function is called **1–1** (**one-to-one**) or **injective** if different x implies different $f(x)$, i.e.

$$f(x) = f(y) \Rightarrow x = y$$

Examples (of functions that are not 1–1)

- absolute value
- floor, ceiling
- length of a word

Inverse Functions

Definition

Inverse function for a given $f : S \longrightarrow T$

$f^{-1} : T \longrightarrow S$ s.t. $f^{-1} \circ f = \mathrm{Id}_S$ (i.e. $f^{-1}(f(x)) = x$ for all $x \in S$)

exists exactly when f is both 1–1 and onto

Inverse image of $B \subseteq \mathrm{Codom}(f)$

$$f^{\leftarrow}(B) \stackrel{\text{def}}{=} \{\, s \in S : f(s) \in B \,\} \subseteq \mathrm{Dom}(f)$$

It is defined for every f

- For $t \in T$ we write $f^{\leftarrow}(t)$ for the set $f^{\leftarrow}(\{t\})$
- If f^{-1} exists then $f^{\leftarrow}(t) = \{f^{-1}(t)\}$
- $f(\emptyset) = \emptyset, f^{\leftarrow}(\emptyset) = \emptyset$

Exercise

$\boxed{1.7.5}$ f and g are 'shift' functions $\mathbb{N} \longrightarrow \mathbb{N}$ defined by $f(n) = n + 1$, and $g(n) = \max(0, n - 1)$

(c) Is f 1–1? onto?
(d) Is g 1-1? onto?
(e) Do f and g commute, i.e. $\forall n\,((f \circ g)(n) = (g \circ f)(n))$?

Exercise

$\boxed{1.7.5}$ f and g are 'shift' functions $\mathbb{N} \longrightarrow \mathbb{N}$ defined by $f(n) = n + 1$, and $g(n) = \max(0, n - 1)$

(c) f is 1–1, not onto: $f(\mathbb{N}) = \mathbb{N} \setminus \{0\} = \mathbb{P}$

(d) g is onto, not 1–1: $g(0) = g(1)$

(e) f and g do not commute:
$g \circ f : n \mapsto (n + 1) - 1 = n$, thus $g \circ f = \mathrm{Id}_{\mathbb{N}}$
$f \circ g : 0 \mapsto 1$, hence $f \circ g \neq \mathrm{Id}_{\mathbb{N}}$

NB

$f \circ g$ is the identity when restricted to $\mathbb{P}$

NB

For a **finite** set S and $f : S \longrightarrow S$ the properties

1. onto, and
2. 1–1

are equivalent. (Proof suggestion?)

Exercise

$\boxed{1.7.6}$ $\Sigma = \{a, b, c\}$
(c) Is length : $\Sigma^* \longrightarrow \mathbb{N}$ onto? Yes: $\text{length}^{\leftarrow}(\{n\}) = \Sigma^n \neq \emptyset$

(d) $\text{length}^{\leftarrow}(2) = \{aa, ab, ac, bb, \ldots, cc\}$

Exercise

$\boxed{1.7.12}$ Verify that $f : \mathbb{R} \times \mathbb{R} \longrightarrow \mathbb{R} \times \mathbb{R}$ defined by $f(x, y) = (x + y, x - y)$ is invertible.

The inverse is $f^{-1}(a, b) = (\frac{a+b}{2}, \frac{a-b}{2})$; substituting shows that $f^{-1} \circ f = \text{Id}_{\mathbb{R}\times\mathbb{R}}$

Extensions:
Show that $f^k(x, y) = f^{k+2}(\frac{x}{2}, \frac{y}{2})$ holds for any k.
Show that $f(x, y) = (x, y)$ only if $x = y = 0$
(such a pair (x, y) is termed a **fixpoint** of f).

Exercise

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Show that $f(x, y) = (x, y)$ only if $x = y = 0$
(such a pair (x, y) is termed a **fixpoint** of f).

Supplementary Exercises

Exercise

$\boxed{1.8.16}$ $\Sigma = \{a, b\}$

(a) Is there an onto function of the form $\Sigma \longrightarrow \Sigma^*$?

(b) Is there an onto function of the form $\Sigma^* \longrightarrow \Sigma$?

Supplementary Exercises

Exercise

$\boxed{1.8.16}$ $\Sigma = \{a, b\}$; relate it to Σ^*:

(a) Is there an onto $\Sigma \longrightarrow \Sigma^*$? No: $|\Sigma| = 2, |\Sigma^*| = \infty$.

(b) Is there an onto $\Sigma^* \longrightarrow \Sigma$? Yes, eg $f(\omega) = a$ when $\text{length}(\omega)$ is odd, $f(\omega) = b$ when $\text{length}(\omega)$ is even.

The following is **not** completely correct $f : \omega \mapsto \langle\text{first letter of } \omega\rangle$
Reason: $f(\lambda)$ is not defined.

Matrices

An $\mathbf{m} \times \mathbf{n}$ **matrix** is a rectangular array with m horizontal rows and n vertical columns.

$$\mathbf{A} = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix}$$

NB

Matrices are important objects in Computer Science, e.g. for

- optimisation
- graphics and computer vision
- cryptography
- information retrieval and web search
- machine learning

Basic Matrix Operations

The **transpose** $\mathbf{A}^\mathsf{T}$ of an $m \times n$ matrix $\mathbf{A} = [a_{ij}]$ is the $n \times m$ matrix whose entry in the ith row and jth column is a_{ji}.

Example

$$\mathbf{A} = \begin{bmatrix} 2 & -1 & 0 & 4 \\ 3 & 2 & -1 & 2 \\ 4 & 0 & 1 & 3 \end{bmatrix} \qquad \mathbf{A}^\mathsf{T} = \begin{bmatrix} 2 & 3 & 4 \\ -1 & 2 & 0 \\ 0 & -1 & 1 \\ 4 & 2 & 3 \end{bmatrix}$$

NB

A matrix $\mathbf{M}$ is called symmetric if $\mathbf{M}^\mathsf{T} = \mathbf{M}$

The **sum** of two $m \times n$ matrices $\mathbf{A} = [a_{ij}]$ and $\mathbf{B} = [b_{ij}]$ is the $m \times n$ matrix whose entry in the ith row and jth column is $a_{ij} + b_{ij}$.

Example

$$\mathbf{A} = \begin{bmatrix} 2 & -1 & 0 & 4 \\ 3 & 2 & -1 & 2 \\ 4 & 0 & 1 & 3 \end{bmatrix} \qquad \mathbf{B} = \begin{bmatrix} 1 & 0 & 5 & 3 \\ 2 & 3 & -2 & 1 \\ 4 & -2 & 0 & 2 \end{bmatrix}$$

$$\mathbf{A} + \mathbf{B} = \begin{bmatrix} 3 & -1 & 5 & 7 \\ 5 & 5 & -3 & 3 \\ 8 & -2 & 1 & 5 \end{bmatrix}$$

Fact

$\mathbf{A} + \mathbf{B} = \mathbf{B} + \mathbf{A}$ *and* $(\mathbf{A} + \mathbf{B}) + \mathbf{C} = \mathbf{A} + (\mathbf{B} + \mathbf{C})$

Exercise

$\boxed{3.3.6}$ Define 3×3 matrices by $\mathbf{A}[i,j] = i \cdot j$ and $\mathbf{B}[i,j] = i + j^2$.

(a) Find $\mathbf{A} + \mathbf{B}$

$$\begin{bmatrix} 1 & 2 & 3 \\ 2 & 4 & 6 \\ 3 & 6 & 9 \end{bmatrix} + \begin{bmatrix} 2 & 5 & 10 \\ 3 & 6 & 11 \\ 4 & 7 & 12 \end{bmatrix} = \begin{bmatrix} 3 & 7 & 13 \\ 5 & 10 & 17 \\ 7 & 13 & 21 \end{bmatrix}$$

(b) Calculate $\sum_{i=1}^{3} \mathbf{A}[i,i] = 14$

(c) Does $\mathbf{A}$ equal $\mathbf{A}^\mathbf{T}$? yes

(d) Does $\mathbf{B}$ equal $\mathbf{B}^\mathbf{T}$? no

Exercise

$\boxed{3.3.6}$ Define 3×3 matrices by $\mathbf{A}[i,j] = i \cdot j$ and $\mathbf{B}[i,j] = i + j^2$.

(a) Find $\mathbf{A} + \mathbf{B}$

$$\begin{bmatrix} 1 & 2 & 3 \\ 2 & 4 & 6 \\ 3 & 6 & 9 \end{bmatrix} + \begin{bmatrix} 2 & 5 & 10 \\ 3 & 6 & 11 \\ 4 & 7 & 12 \end{bmatrix} = \begin{bmatrix} 3 & 7 & 13 \\ 5 & 10 & 17 \\ 7 & 13 & 21 \end{bmatrix}$$

(b) Calculate $\sum_{i=1}^{3} \mathbf{A}[i,i] = 14$

(c) Does $\mathbf{A}$ equal $\mathbf{A}^{\mathbf{T}}$? yes

(d) Does $\mathbf{B}$ equal $\mathbf{B}^{\mathbf{T}}$? no

Given $m \times n$ matrix $\mathbf{A} = [a_{ij}]$ and $c \in \mathbb{R}$, the **scalar product** $c\mathbf{A}$ is the $m \times n$ matrix whose entry in the ith row and jth column is $c \cdot a_{ij}$.

Example

$$\mathbf{A} = \begin{bmatrix} 2 & -1 & 0 & 4 \\ 3 & 2 & -1 & 2 \\ 4 & 0 & 1 & 3 \end{bmatrix} \qquad 2\mathbf{A} = \begin{bmatrix} 4 & -2 & 0 & 8 \\ 6 & 4 & -2 & 4 \\ 8 & 0 & 2 & 6 \end{bmatrix}$$

The **product** of an $m \times n$ matrix $\mathbf{A} = [a_{ij}]$ and an $n \times p$ matrix $\mathbf{B} = [b_{jk}]$ is the $m \times p$ matrix $\mathbf{C} = [c_{ik}]$ defined by

$$c_{ik} = \sum_{j=1}^{n} a_{ij} b_{jk} \qquad \text{for } 1 \leq i \leq m \text{ and } 1 \leq k \leq p$$

Example

$$\begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} \cdot \begin{bmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{bmatrix} = \begin{bmatrix} a_{11}b_{11} + a_{12}b_{21} & a_{11}b_{12} + a_{12}b_{22} \\ a_{21}b_{11} + a_{22}b_{21} & a_{21}b_{12} + a_{22}b_{22} \end{bmatrix}$$

NB

The rows of **A** must have the same number of entries as the columns of **B**.

The product of a $1 \times n$ matrix and an $n \times 1$ matrix is usually called the **inner product** of two **n-dimensional vectors.**

Exercise

Consider

$$\mathbf{A} = \begin{bmatrix} 1 & 2 \\ 2 & 4 \end{bmatrix} \qquad \mathbf{B} = \begin{bmatrix} 2 & -1 \\ -6 & 3 \end{bmatrix}$$

Calculate $\mathbf{AB}$, $\mathbf{BA}$

$$\mathbf{AB} = \begin{bmatrix} -10 & 5 \\ -20 & 10 \end{bmatrix} \qquad \mathbf{BA} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$$

NB

In general, $\mathbf{A} \cdot \mathbf{B} \neq \mathbf{B} \cdot \mathbf{A}$

Exercise

Consider

$$\mathbf{A} = \begin{bmatrix} 1 & 2 \\ 2 & 4 \end{bmatrix} \qquad \mathbf{B} = \begin{bmatrix} 2 & -1 \\ -6 & 3 \end{bmatrix}$$

Calculate **AB**, **BA**

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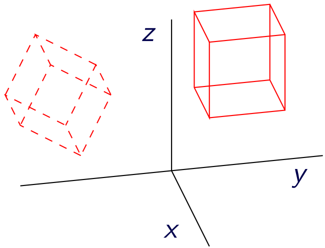
NB

In general, $\mathbf{A} \cdot \mathbf{B} \neq \mathbf{B} \cdot \mathbf{A}$

Example: Computer Graphics

Rotating an object w.r.t. the x axis by degree α:

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos\alpha & -\sin\alpha \\ 0 & \sin\alpha & \cos\alpha \end{bmatrix} \cdot \begin{bmatrix} 5 & 5 & 7 & 7 & 5 & 7 & 5 & 7 \\ 1 & 1 & 1 & 1 & 3 & 3 & 3 & 3 \\ 9 & 7 & 7 & 9 & 7 & 7 & 9 & 9 \end{bmatrix}$$



Relations and their Representation

Relations are an abstraction used to capture the idea that the objects from certain domains (often the same domain for several objects) are *related*. These objects may

- influence one another (each other for binary relations; self(?) for unary)
- share some common properties
- correspond to each other precisely when some constraints are satisfied

In general, relations formalise the concept of interaction among objects from various domains; however, there must be a specified domain for each type of objects.

An **n-ary relation** is a subset of the cartesian product of n sets.

$$\mathcal{R} \subseteq S_1 \times S_2 \times \ldots \times S_n$$

$$x \in \mathcal{R} \Rightarrow x = (x_1, x_2, \ldots x_n) \text{ where each } x_i \in S_i$$

If $n = 2$ we have a **binary** relation $\mathcal{R} \subseteq S \times T$.
(mostly we consider binary relations)
equivalent notations: $(x_1, x_2, \ldots x_n) \in \mathcal{R} \iff \mathcal{R}(x_1, x_2, \ldots x_n)$
for binary relations: $(x, y) \in \mathcal{R} \iff \mathcal{R}(x, y) \iff x\mathcal{R}y$.

Database Examples

Example (course enrolments)

S = set of CSE students
(S can be a subset of the set of all students)
C = set of CSE courses
(likewise)
E = enrolments = $\{ (s, c) : s \text{ takes } c \}$

$$E \subseteq S \times C$$

In practice, almost always there are various 'onto' (nonemptiness) and 1–1 (uniqueness) constraints on database relations.

Example (class schedule)

C = CSE courses
T = starting time (hour & day)
R = lecture rooms
S = schedule =

$$\{\,(c,t,r) : c \text{ is at } t \text{ in } r\,\} \subseteq C \times T \times R$$

Example (sport stats)

$$\mathcal{R} \subseteq \text{competitions} \times \text{results} \times \text{years} \times \text{athletes}$$

Applications

Relations are ubiquitous in Computer Science

- Databases are collections of relations
- Common data structures (e.g. graphs) are relations
- Any ordering is a relation
- Functions/procedures/programs compute relations between their input and output

Relations are therefore used in most problem specifications and to describe formal properties of programs.
For this reason, studying relations and their properties helps with formalisation, implementation and verification of programs.

n-ary Relations

Relations can be defined linking $k \geq 1$ domains $D_1, \ldots, D_k$ simultaneously.

In database situations one also allows for *unary* $(n = 1)$ relations.

Most common are **binary** relations

$$\mathcal{R} \subseteq S \times T; \quad \mathcal{R} = \{(s,t) : \text{ “some property that links } s,t\text{”}\}$$

For related s, t we can write $(s,t) \in \mathcal{R}$ or $s\mathcal{R}t$; for unrelated items either $(s,t) \notin \mathcal{R}$ or $s \not\mathcal{R} t$.

$\mathcal{R}$ can be defined by

- explicit enumeration of interrelated k-tuples (ordered pairs in case of binary relations);
- properties that identify relevant tuples within the entire $D_1 \times D_2 \times \ldots \times D_k$;
- construction from other relations.

Functions as Relations

Any function $f : S \longrightarrow T$ can be viewed as a binary relation

$$\{ (s, f(s)) : s \in S \} \subseteq S \times T$$

If a subset of $S \times T$ corresponds to a function, it must satisfy certain conditions w.r.t. S and T (which?)

Binary Relations

A binary relation, say $\mathcal{R} \subseteq S \times T$, can be presented as a matrix with rows enumerated by (the elements of) S and the columns by T; eg. for $S = \{s_1, s_2, s_3\}$ and $T = \{t_1, t_2, t_3, t_4\}$ we may have

$$\begin{bmatrix} \bullet & \circ & \bullet & \bullet \\ \circ & \bullet & \bullet & \bullet \\ \bullet & \bullet & \circ & \circ \end{bmatrix}$$

Example

Exercise

$\boxed{3.1.2(e)}$ Write the following relation on $A = \{0, 1, 2\}$ as a matrix.

$(m, n) \in \mathcal{R}$ if $m \cdot n = m$

$$\begin{array}{c c} & \begin{array}{ccc} 0 & 1 & 2 \end{array} \\ \begin{array}{c} 0 \\ 1 \\ 2 \end{array} & \begin{bmatrix} \bullet & \bullet & \bullet \\ \circ & \bullet & \circ \\ \circ & \bullet & \circ \end{bmatrix} \end{array}$$

Example

Exercise

$\boxed{\text{3.1.2(e)}}$ Write the following relation on $A = \{0, 1, 2\}$ as a matrix.

$(m, n) \in \mathcal{R}$ if $m \cdot n = m$

$$\begin{array}{cc} & \begin{array}{ccc} 0 & 1 & 2 \end{array} \\ \begin{array}{c} 0 \\ 1 \\ 2 \end{array} & \begin{bmatrix} \bullet & \bullet & \bullet \\ \circ & \bullet & \circ \\ \circ & \bullet & \circ \end{bmatrix} \end{array}$$

Relations on a Single Domain

Particularly important are binary relationships between the elements of the same set. We say that '$\mathcal{R}$ is a binary relation on S' if

$$\mathcal{R} \subseteq S \times S$$

Special (Trivial) Relations

(all w.r.t. set S)

Identity (diagonal, equality) $E = \{ (x, x) : x \in S \}$

Empty $\emptyset$

Universal $U = S \times S$

Important Properties of Binary Relations $\mathcal{R} \subseteq S \times S$

(R)	reflexive	$(x, x) \in \mathcal{R}$	$\forall\, x \in S$
(AR)	antireflexive	$(x, x) \notin \mathcal{R}$	$\forall\, x \in S$
(S)	symmetric	$(x, y) \in \mathcal{R} \Rightarrow (y, x) \in \mathcal{R}$	$\forall\, x, y \in S$
(AS)	antisymmetric	$(x, y), (y, x) \in \mathcal{R} \Rightarrow x = y$	$\forall\, x, y \in S$
(T)	transitive	$(x, y), (y, z) \in \mathcal{R} \Rightarrow (x, z) \in \mathcal{R}$	$\forall\, x, y, z \in S$

NB

An object, notion etc. is considered to satisfy a property if none of its instances violates any defining statement of that property.

Examples

(R) reflexive $(x, x) \in \mathcal{R}$ for all $x \in S$ $\begin{bmatrix} \bullet & \bullet & \circ \\ \circ & \bullet & \circ \\ \bullet & \circ & \bullet \end{bmatrix}$

(AR) antireflexive $(x, x) \notin \mathcal{R}$ $\begin{bmatrix} \circ & \bullet & \bullet \\ \circ & \circ & \circ \\ \bullet & \circ & \circ \end{bmatrix}$

(S) symmetric $(x, y) \in \mathcal{R} \Rightarrow (y, x) \in \mathcal{R}$ $\begin{bmatrix} \bullet & \circ & \bullet \\ \circ & \circ & \bullet \\ \bullet & \bullet & \circ \end{bmatrix}$

(AS) antisymmetric $(x, y), (y, x) \in \mathcal{R} \Rightarrow x = y$
$\begin{bmatrix} \bullet & \bullet & \circ \\ \circ & \circ & \bullet \\ \bullet & \circ & \circ \end{bmatrix}$

(T) transitive $(x, y), (y, z) \in \mathcal{R} \Rightarrow (x, z) \in \mathcal{R}$
$\begin{bmatrix} \circ & \circ & \bullet \\ \bullet & \bullet & \bullet \\ \circ & \circ & \circ \end{bmatrix}$

Example

Exercise

$\boxed{3.1.1}$ The following relations are on $S = \{1, 2, 3\}$.
Which of the properties (R), (AR), (S), (AS), (T) does each satisfy?

(a) $(m, n) \in \mathcal{R}$ if $m + n = 3$

(AR) and (S)

(e) $(m, n) \in \mathcal{R}$ if $\max\{m, n\} = 3$

(S)

$\boxed{3.1.2(b)}$ $(m, n) \in \mathcal{R}$ if $m < n$

(AR), (AS), (T)

Example

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Interaction of Properties

A relation *can* be both symmetric and antisymmetric. Namely, when $\mathcal{R}$ consists only of some pairs $(x, x), x \in S$.
A relation *cannot* be simultaneously reflexive and antireflexive (unless $S = \emptyset$).

NB

$$\left.\begin{array}{r}\text{nonreflexive}\\ \text{nonsymmetric}\end{array}\right\} \text{ is not the same as } \left\{\begin{array}{l}\text{antireflexive/irreflexive}\\ \text{antisymmetric}\end{array}\right.$$

A relation *can* be nonreflexive without being antireflexive.
A relation *can* be nonsymmetric without being antisymmetric.

Most important kinds of relations on S

- total order $\begin{bmatrix} \bullet & \bullet & \bullet \\ \circ & \bullet & \bullet \\ \circ & \circ & \bullet \end{bmatrix}$ e.g. $\leq$ on numbers
- partial order $\begin{bmatrix} \bullet & \bullet & \bullet \\ \circ & \bullet & \circ \\ \circ & \circ & \bullet \end{bmatrix}$, $\begin{bmatrix} \bullet & \bullet & \circ \\ \circ & \bullet & \circ \\ \circ & \circ & \bullet \end{bmatrix}$ e.g. $\subseteq$ on sets
- equivalence $\begin{bmatrix} \bullet & \bullet & \circ \\ \bullet & \bullet & \circ \\ \circ & \circ & \bullet \end{bmatrix}$
- identity $\begin{bmatrix} \bullet & \circ & \circ \\ \circ & \bullet & \circ \\ \circ & \circ & \bullet \end{bmatrix}$

NB

Some of those are special cases of the others, eg. ‘total order’ of a ‘partial order’, ‘identity’ of an ‘equivalence’.

Examples

Exercise

$\boxed{\text{3.1.10(a)}}$ Give examples of relations with specified properties.
(AS), (T), $\neg$(R).

Examples over $\mathbb{N}$, Pow($\mathbb{N}$):

- strict order of numbers $x < y$
- simple (weak) order, but with some pairs (x, x) removed from $\mathcal{R}$
- being a prime divisor
  $(p, n) \in \mathcal{R}$ iff p is prime and $p|n$
  - not reflexive: $(1, 1) \notin \mathcal{R}, (4, 4) \notin \mathcal{R}, (6, 6) \notin \mathcal{R}$
  - transitivity is meaningful only for the pairs $(p, p), (p, n)$, $p|n$ for p prime

Examples

Exercise

$\boxed{\text{3.1.10(a)}}$ Give examples of relations with specified properties. (AS), (T), ¬(R).

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More Examples

Exercise

$\boxed{3.1.10(b)}$ Give examples of relations with specified properties. (S), $\neg$(R), $\neg$(T).

Easiest examples: inequality

- $\mathcal{R} = \{(x, y) | x \neq y, \ x, y \in \mathbb{N}\}$
- $\mathcal{R} = \{(A, B) | A \neq B, \ A, B \subseteq S\}$

More Examples

Exercise

$\boxed{\text{3.1.10(b)}}$ Give examples of relations with specified properties. (S), $\neg$(R), $\neg$(T).

Easiest examples: inequality

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Relation $\mathcal{R}$ as Correspondence From S to T

Definition

Converse relation $\mathcal{R}^{\leftarrow}$:

$$\mathcal{R}^{\leftarrow} = \{(t, s) \in T \times S : (s, t) \in \mathcal{R}\}$$

Note that $\mathcal{R}^{\leftarrow} \subseteq T \times S$.
Observe that $(\mathcal{R}^{\leftarrow})^{\leftarrow} = \mathcal{R}$.

For subsets:

Definition

$\mathcal{R}(A) \stackrel{\text{def}}{=} \{t \in T : (s, t) \in \mathcal{R} \text{ for some } s \in A \subseteq S\}$
$\mathcal{R}^{\leftarrow}(B) \stackrel{\text{def}}{=} \{s \in S : (s, t) \in \mathcal{R} \text{ for some } t \in B \subseteq T\}$

NB

Viewed this way $\mathcal{R}$ becomes a function from $\mathrm{Pow}(S)$ to $\mathrm{Pow}(T)$. However, *not* every $g : \mathrm{Pow}(S) \longrightarrow \mathrm{Pow}(T)$ can be matched to a relation.

(Why? Using a small domain like $S = \{a, b\}$, provide an example of a function $g : \mathrm{Pow}(S) \longrightarrow \mathrm{Pow}(S)$ which does not correspond to any relation on S! Can you even do it with $S' = \{a\}$?)

NB

The order of axes — S and T — is important. For $\mathcal{R} \subseteq S \times S$, its converse $\mathcal{R}^{\leftarrow}$ is usually quite different from $\mathcal{R}$.

Example: divisibility relation on $\mathbb{P}$

$$\begin{aligned} D &\stackrel{\text{def}}{=} \{\ (p, q) : p \mid q\ \} = \{(1,1), (1,2), \ldots, (2,2), (2,4), \ldots\} \\ D^{\leftarrow} &= \{\ (p, q) : p \in q\mathbb{P}\ \} \\ &= \{(1,1), (2,1), (2,2), (3,1), (3,3), (4,1), (4,2), \ldots\} \end{aligned}$$

For every $n \in \mathbb{P}$, $D(\{n\})$ is infinite, $D^{\leftarrow}(\{n\})$ is finite.

Exercise

Consider function $f : S \longrightarrow T$ as a relation $\{(s, f(s)) : s \in S\}$.
Converse relation is $f^{\leftarrow} = \{(f(s), s) : s \in S\} \subseteq T \times S$.
When is it also a function?

Exercise

$\boxed{3.1.9}$ Find the properties of the *empty relation* $\emptyset \subset S \times S$ and the *universal relation* $U = S \times S$. Assume that S is a nonempty domain.

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When is it also a function?

When f is 1-1 and onto.

Exercise

$\boxed{3.1.9}$ Find the properties of the *empty relation* $\emptyset \subset S \times S$ and the *universal relation* $U = S \times S$. Assume that S is a nonempty domain.
(a) $\emptyset$ is (AR), (S), (AS), (T); if $S = \emptyset$ itself then $\emptyset$ is also (R).
(b) U is (R), (S), (T); if $|S| \leq 1$ then also (AS)

Combining Relations

$\boxed{3.1.14}$ Which properties (reflexivity, symmetry, transitivity) carry from individual relations to their union?

(a) $\mathcal{R}_1, \mathcal{R}_2 \in (\mathrm{R}) \Rightarrow \mathcal{R}_1 \cup \mathcal{R}_2 \in (\mathrm{R})$

(b) $\mathcal{R}_1, \mathcal{R}_2 \in (\mathrm{S}) \Rightarrow \mathcal{R}_1 \cup \mathcal{R}_2 \in (\mathrm{S})$

(c) $\mathcal{R}_1, \mathcal{R}_2 \in (\mathrm{T}) \not\Rightarrow \mathcal{R}_1 \cup \mathcal{R}_2 \in (\mathrm{T})$
Eg. $S = \{a, b, c\}, a\mathcal{R}_1 b, b\mathcal{R}_2 c$
and no other relationships

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Composition of Relations

Definition

Composition of binary relations $\mathcal{R}_1 \subseteq S \times T$ and $\mathcal{R}_2 \subseteq T \times U$

$$\mathcal{R}_1;\mathcal{R}_2 = \{(s,u) \colon (s,t) \in \mathcal{R}_1 \wedge (t,u) \in \mathcal{R}_2 \text{ for some } t \in T\} \subseteq S \times U$$

Example (course enrolments)

S = set of CSE students
C = set of CSE courses
T = starting time

E = enrolments = $\{ (s,c) \in S \times C : s \text{ takes } c \}$
L = lectures = $\{ (c,t) \in C \times T : c \text{ is at } t \}$

$$E;L = \{ (s,t) \in S \times T : s \text{ has a lecture at time } t \}$$

Summary

- Functions
  (co-)domain, image, composition $f \circ g$, f^{-1}, $f^{\leftarrow}$, identity Id_S
  notation for subsets: $f(A)$, $f^{-1}(B)$
- Properties of functions: onto, 1-1
- Matrix operations: transposition, sum, scalar product, product
- Relations
  converse $\mathcal{R}^{\leftarrow}$, composition $\mathcal{R}_1;\mathcal{R}_2$
  notation for subsets $\mathcal{R}(A)$, $\mathcal{R}^{\leftarrow}(B)$
- Properties of binary relations: (R), (AR); (S), (AS); (T)

Coming up ...

- Ch. 3, Sec. 3.4-3.5 (Equivalence relations)
- Ch. 11, Sec. 11.1-11.2 (Orderings)