Social Choice

COMP4418 Knowledge Representation and Reasoning

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How to rank and make recommendations?

How to aggregate scheduling preferences?

How to agree on a budget?

Outline

Social Choice

Social Choice is the theory of aggregating preferences of agents into a socially desirable outcome.

- Mostly studied in Economics and Political Science
- Now also studied within Computer Science (computational social choice)

Applications

- **Search Engines**: to determine the most important sites based on links.
- **Recommender Systems**: to recommend a product to a user based on ratings by users.

Social Choice (cont.)

- **Multiagent Systems**: to coordinate the actions of groups of autonomous software agents.

Outline

Social Choice

- Plurality is the most common voting rule — it selects the alternative that is ranked first by most voters.
- Why do we study different voting rules when we have the *plurality* voting rule?

$$
\begin{aligned}
4 \text{ voters } &: a \succ b \succ c \succ d \\
4 \text{ voters } &: a \succ c \succ b \succ d \\
7 \text{ voters } &: b \succ d \succ c \succ a \\
7 \text{ voters } &: c \succ b \succ d \succ a
\end{aligned}
$$

Alternative a is the plurality winner. However plurality may not be the best rule. Why?

Social Choice

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\end{aligned}$$

Alternative a is the plurality winner. However plurality may not be the best rule. In the example

Social Choice (cont.)

- a majority of voters think a is the worst alternative.
- b, c, and d are better than a in pairwise majority comparisons.

This motivates a study of different voting rules.

Voting Rules

- **Plurality**: alternatives that are ranked first by most voters win.
- **Borda**: Most preferred alternative gets $m-1$ points, the second most-preferred $m-2$ points, etc. Alternatives with highest total score win.
- **Plurality with runoff**: Two alternatives that are ranked first by most voters are short-listed. Then among the shortlisted alternatives, the alternative which is preferred by a majority wins.
- **Instant Runoff**: Alternatives that are ranked first by the lowest number of voters are removed from consideration. Repeat until no more alternatives can be deleted.

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Voting Rules (cont.)

$$\begin{aligned}
33\% \text{ voters}: &\quad a \succ b \succ c \succ d \succ e \\
16\% \text{ voters}: &\quad b \succ d \succ c \succ e \succ a \\
3\% \text{ voters}: &\quad c \succ d \succ b \succ a \succ e \\
8\% \text{ voters}: &\quad c \succ e \succ b \succ d \succ a \\
18\% \text{ voters}: &\quad d \succ e \succ c \succ b \succ a \\
22\% \text{ voters}: &\quad e \succ c \succ b \succ d \succ a
\end{aligned}$$

- Plurality winner: a
- Borda winner: b
- Plurality with runoff: e (after beating a)
- Instant Runoff: d (removal: c, b, e, a)

Outline

Formal Framework

Setting:

- Set of agents/voters $N = \{1, \ldots, n\}$
- Set of alternatives $A = \{a_1, \ldots, a_m\}$
- $\succ = (\succ_1, \ldots, \succ_n)$ profile of preferences where each preference $\succ_i$ is a linear order over A.

We denote by $\mathcal{L}(A)$ the set of linear orders over A.

We denote by $\mathcal{P}_k(A)$ as the set of subsets of A of size k.

Formal Framework (cont.)

Outcome:

- single selected alternative
- collective preference (linear order)
- set of collective preferences linear orders
- a subset of selected alternatives
- probability distribution over alternatives

Formal Framework

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- Set of alternatives $A = \{a_1, \ldots, a_m\}$
- $\succ = (\succ_1, \ldots, \succ_n)$ profile of preferences where each preference $\succ_i$ is a linear order over A.

Formal Framework (cont.)

Example (Voting Setting)
$N = \{1, 2, 3\}$, $A = \{a, b, c, d\}$

$$1 : a \succ_1 b \succ_1 c \succ_1 d$$
$$2 : a \succ_2 c \succ_2 b \succ_2 d$$
$$3 : b \succ_3 d \succ_3 c \succ_3 a$$

.

- **Social Welfare Function (SWF):** aggregates individual preferences into a collective preference.

$$F : \mathcal{L}(A)^n \to \mathcal{L}(A)$$

- **Social Choice Function (SCF):** aggregates individual preferences into a single selected alternative.

$$F : \mathcal{L}(A)^n \to A$$

- **Social Choice Correspondence (SCC):** aggregates individual preferences into a subset of selected alternatives.

$$F : \mathcal{L}(A)^n \to 2^A$$

Formal Framework (cont.)

- **Committee Voting Rule:** aggregates individual preferences into a subset of selected alternatives of a given size (k).

$$F : \mathcal{L}(A)^n \rightarrow \mathcal{P}_k(A)$$

- **Social Decision Scheme (SDS):** aggregates individual preferences into a probability distribution over alternatives.

$$F : \mathcal{L}(A)^n \rightarrow \Delta(A)$$

Formal Framework

Voting rule is an informal term used for social choice functions, social welfare functions etc.

- A social welfare function can be used as a social choice function: select the first alternative of the output ranking.
- A social choice correspondence can be used as a social choice function: use some tie-breaking rule over the set of alternatives selected.
- A committe voting rule can be used as a social choice function: use it for $k = 1$.

Some classes of Voting Rules

- **Based on majority pairwise comparisons**
    - Decision are made based on pairwise majority comparisons.
    - An alternative x wins a pairwise majority comparison against alternative y if $|\{i \in N \mid x \succ_i y\}| > |\{i \in N \mid y \succ_i x\}|$.
- **Based on weighted majority pairwise comparisons**
    - Decision are made based on weighted pairwise majority comparisons.
    - For any two alternatives $x, y \in A$, the weighted majority pairwise comparison for (x, y) is $|\{i \in N \mid x \succ_i y\}| - |\{i \in N \mid y \succ_i x\}|$.
- **Positional scoring rules** (next slide).

Positional Scoring Rules

A **positional scoring rule (PSR)** is given by a scoring vector $s = (s_1, \ldots, s_m)$ with $s_1 \geq s_2 \cdots \geq s_m$ and $s_1 > s_m$. When a voter puts alternative a in position j, a gets score s_j. Alternatives with the maximum total score win.

- **Borda** rule: PSR with scoring vector $(m-1, m-2, \ldots, 0)$
- **Plurality** rule: PSR with scoring vector $(1, 0, \ldots, 0)$
- **Antiplurality** rule: PSR with scoring vector $(1, 1, \ldots, 1, 0)$
- k **approval** rule: PSR with scoring vector $(\underbrace{1, \ldots, 1}_{k}, 0, \ldots, 0)$

Outline

Axioms of Voting Rules

- **Anonymity:** The voting rule treats voters equally: the outcome remains the same as long as the set of votes is the same.
- **Neutrality:** The voting rule treats alternatives equally: F is neutral if $F(\pi(\succ)) = \pi(F(\succ))$ where π is a permutation $\pi : A \longrightarrow A$.
- **Monotonicity:** A chosen alternative will still be chosen when it rises in individual preference rankings (while leaving everything else unchanged).
- **Pareto optimality:** An alternative will not be chosen if there exists another one that all voters prefer the latter to the former.

Axioms of Voting Rules (cont.)

- **Independence of Irrelevant Alternatives (IIA):** If alternative a is socially preferred to b, then this should not change when a voter changes her ranking of $c \neq a, b$.
- **Non-dictatorial:** there exists no voter such that the outcome is always identical to the preference supplied by the dictator.
- **Condorcet-extension:** if an alternative is pairwise preferred by a majority of voters over every other alternative, then that alternative is selected.
- **Strategyproof:** A voter cannot misreport his/her preference to select a more preferred alternative.

Axioms of Voting Rules

- **Anonymity:** The voting rule treats voters equally: the outcome remains the same as long as the set of votes is the same. F is anonymous if $F(\succ) = F(\pi(\succ))$ where $\pi(\succ)$ is a profile $(\succ_{\pi(1)}, \ldots, \succ_{\pi(n)})$.

Axioms of Voting Rules (cont.)

Example

$N = \{1, 2, 3\}$, $A = \{a, b, c, d\}$

$$1 : a \succ_1 b \succ_1 c \succ_1 d$$

$$2 : a \succ_2 c \succ_2 b \succ_2 d$$

$$3 : b \succ_3 d \succ_3 c \succ_3 a$$

$$3 : a \succ_1 b \succ_1 c \succ_1 d$$

$$2 : a \succ_2 c \succ_2 b \succ_2 d$$

$$1 : b \succ_3 d \succ_3 c \succ_3 a$$

An anonymous voting rule should have the same output for both preference profiles.

Axioms of Voting Rules

- **Neutrality:** The voting rule treats alternatives equally: F is neutral if $F(\pi(\succ)) = \pi(F(\succ))$ where π is a permutation $\pi : A \longrightarrow A$.

Axioms of Voting Rules (cont.)

Example

$N = \{1, 2, 3\}$, $A = \{a, b, c, d\}$

$$1 : a \succ_1 b \succ_1 c \succ_1 d$$
$$2 : a \succ_2 c \succ_2 b \succ_2 d$$
$$3 : b \succ_3 d \succ_3 c \succ_3 a$$

$$3 : b \succ_3 c \succ_3 d \succ_3 a$$
$$2 : b \succ_2 d \succ_2 c \succ_2 a$$
$$1 : c \succ_1 a \succ_1 d \succ_1 b$$

If a neutral voting rule returns a for the first profile, it should return b for the second one.

Axioms of Voting Rules

- **Monotonicity:** A chosen alternative will still be chosen when it rises in individual preference rankings (while leaving everything else unchanged).

Axioms of Voting Rules (cont.)

Example

$N = \{1, 2, 3\}$, $A = \{a, b, c, d\}$

$$1 : a \succ_1 b \succ_1 c \succ_1 d$$
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If a monotonic voting rule returns a for the first profile, it should return a for the second one.

Axioms of Voting Rules

- **Pareto optimality:** An alternative will not be chosen if there exists another one that all voters prefer the latter to the former.

Example

$N = \{1, 2, 3\}$, $A = \{a, b, c, d\}$

$$1 : a \succ_1 b \succ_1 c \succ_1 d$$
$$2 : a \succ_2 c \succ_2 b \succ_2 d$$
$$3 : c \succ_3 d \succ_3 a \succ_3 b$$

A Pareto optimal voting rule will not select b because b is Pareto dominated by a.

Axioms of Voting Rules

- **Independence of Irrelevant Alternatives (IIA):** If alternative a is socially preferred to b, then this should not change when a voter changes her ranking of $c \neq a, b$.

Axioms of Voting Rules (cont.)

Example

$N = \{1, 2, 3\}$, $A = \{a, b, c, d\}$

$$1 : a \succ_1 b \succ_1 c \succ_1 d$$
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$$1 : a \succ_1 b \succ_1 d \succ_1 c$$
$$2 : a \succ_2 c \succ_2 b \succ_2 d$$
$$3 : d \succ_3 c \succ_3 a \succ_3 b$$

If a is socially preferred over b, then it should still be socially preferred over b, if ranking of d is changed.

Axioms of Voting Rules

- **Condorcet-extension:** if an alternative is pairwise preferred by a majority of voters over every other alternative, then that alternative is selected.

Example
$N = \{1, 2, 3\}$, $A = \{a, b, c, d\}$

$$1 : a \succ_1 b \succ_1 c \succ_1 d$$
$$2 : a \succ_2 c \succ_2 b \succ_2 d$$
$$3 : c \succ_3 d \succ_3 a \succ_3 b$$

A Condorcet-extension voting rule should select a.

Axioms of Voting Rules

- **Strategyproof:** A voter cannot misreport his/her preference to select a more preferred alternative.

Axioms of Voting Rules (cont.)

Example

$N = \{1, 2, 3\}$, $A = \{a, b, c, d\}$

$$1 : a \succ_1 b \succ_1 c \succ_1 d$$
$$2 : c \succ_2 a \succ_2 b \succ_2 d$$
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$$1 : a \succ_1 b \succ_1 c \succ_1 d$$
$$2 : c \succ_2 b \succ_2 d \succ_2 a$$
$$3 : c \succ_3 d \succ_3 a \succ_3 b$$

If the voting rule selects a for the first profile and c for the second profile, it is not strategyproof because 2 can manipulate.

Condorcet's Paradox

Condorcet winner: an alternative that is pairwise preferred by a majority of voters over every other alternative.

A Condorcet winner may not exist.

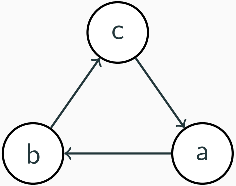
Condorcet's Paradox (cont.)

Example (Condorcet's Paradox)

$$1 : a \succ_1 b \succ_1 c$$

$$2 : b \succ_2 c \succ_2 a$$

$$3 : c \succ_3 a \succ_3 b$$



Axiomatic method

Formal approach

- **Characterisation Theorems:** show that a particular (class of) rules is the only one satisfying a given set of axioms.
- **Impossibility Theorems:** show which sets of axioms are incompatible.

Outline

Majority Graph

Given $(N, A, \succ)$, the **corresponding majority graph** is a directed graph (V, E) where $V = A$ and in which $(x, y) \in E$ if and only if x is preferred over y by a majority of voters. If $(x, y) \in E$, we say that x *dominates* y. We will denote $D(x) = \{y \mid (x, y) \in E\}$.

Assuming there is an odd number of voters, the pairwise majority graph is a tournament (complete and asymmetric graph).

Majority Graph (cont.)

Example (Tournament)
$N = \{1, 2, 3\}$, $A = \{a, b, c, d\}$.

$$1 : a \succ_1 b \succ_1 c \succ_1 d$$
$$2 : a \succ_2 c \succ_2 b \succ_2 d$$
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What is the induced majority tournament?

Majority Graph

Example (Tournament)
$N = \{1, 2, 3\}$, $A = \{a, b, c, d\}$.

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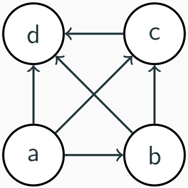

Copeland Rule

The Copeland rule selects alternatives based on the number of other alternatives they dominate. Define the **Copeland score** of an alternative x in tournament $T = (V, E)$ as the outdegree of the alternative.

The set of **Copeland winners** $CO(T)$ then consists of all alternatives that have maximal Copeland score.

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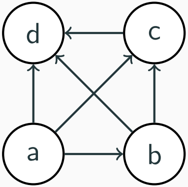
Copeland Rule (cont.)

Example (Tournament)

$CO(T) = ?$



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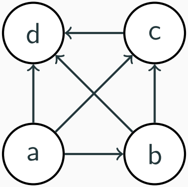
Copeland Rule (cont.)

Example (Tournament)



$CO(T) = \{a\}.$

Top Cycle

An alternative is in the top cycle iff it can reach every other alternative by a path in the tournament.

The top cycle rule is a Condorcet-extension.

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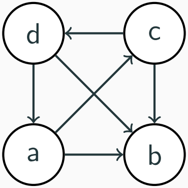
Example (Tournament)
Top cycle=?

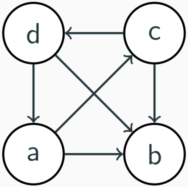


Top Cycle

An alternative is in the top cycle iff it can reach every other alternative by a path in the tournament.

The top cycle rule is a Condorcet-extension.

Example (Tournament)
Top cycle: $\{a, c, d\}$



Uncovered Set

The **Uncovered Set** of a tournament $T = (V, E)$, denoted by $UC(T)$, is the set of alternative that can reach every other alternative in at most two steps.

The alternatives in the uncovered set are also referred to as kings.

The uncovered set rule is a Condorcet-extension.

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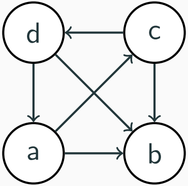
The uncovered set rule is a Condorcet-extension.

Uncovered Set (cont.)

Example (Tournament)
Uncovered Set: $\{a, c, d\}$



Banks

Under the Banks rule, an alternative x is a **Banks winner** if it is a top element in a maximal acyclic vertex-induced subgraph of the tournament. The set of Banks winners of a tournament T is denoted by $BA(T)$.

The Bank rule is a Condorcet-extension.

Computing some Banks winner is easy: grow the set of alternative as long as the graph is acyclic. The top element of the set is a Banks winner.

Banks (cont.)

Theorem (Woeginger, 2003)
Checking whether a certain alternative is a Banks winner is NP-complete.

Relations between Tournament Solutions

Theorem
Any Copeland winner is a member of the uncovered set.

$$CO(T) \subseteq UC(T)$$

Relations between Tournament Solutions (cont.)

Proof.

- Supposed that an alternative x is a Copeland winner but not a member of the uncovered set.
- This means that $x \cup D(x) \cup D(D(x)) \neq A$.
- Hence there exists some $y \in A \setminus (\{x\} \cup D(x) \cup D(D(x)))$.
- Thus, $D(y) = \{x\} \cup D(x)$ so that Copeland score of y is more than that of x.

$\square$

Relations between Tournament Solutions

Theorem
Any member of the uncovered set is a member of the top cycle.

$$UC(T) \subseteq TC(T).$$

Proof.
If an alternative is a member of the uncovered set, then it can reach each other alternative in at most two steps so it reaches all other alternatives. $\square$

Relations between Tournament Solutions

Theorem
Any member of the Banks set is a member of the uncovered set.

$$BA(T) \subseteq UC(T)$$

Relations between Tournament Solutions (cont.)

Proof.

- Consider a Banks winner b which is the the top element of a maximal acyclic subgraph of the tournament with vertex set V'. Note that b dominates each vertex in $V' \setminus \{b\}$.
- Consider any vertex $v \in V \setminus V'$.
- Then the graph induced by $V' \cup \{v\}$ is not acyclic which means that there exist $x, y \in V'$ such that $(x, y) \in E$, $(y, v) \in E$ and $(v, x) \in E$.
  - If $b = y$, then b dominates y.
  - If $b \neq y$, b dominates y which dominates v.

  In either case, b reaches v in at most two steps.

□

Outline

Positional Scoring Rules are not Condorcet-extensions

Theorem (Condorcet, 1785)
Borda's rule is not a Condorcet-extension when there are 3 or more alternatives.

Theorem (Fishburn, 1973)
No positional scoring rule is a Condorcet-extension when there are 3 or more alternatives.

Consider the following profile:

$$\begin{aligned} 6 \text{ voters } &: a \succ b \succ c \\ 3 \text{ voters } &: c \succ a \succ b \\ 4 \text{ voters } &: b \succ a \succ c \\ 4 \text{ voters } &: b \succ c \succ a \end{aligned}$$

Positional Scoring Rules are not Condorcet-extensions (cont.)

- Score of a: $6s_1 + 7s_2 + 4s_3$
- Score of b: $8s_1 + 6s_2 + 3s_3$
- Score of c: $3s_1 + 4s_2 + 10s_3$

Alternative b is the winner under every PSR. However a is the Condorcet winner.

Theorem (Arrow's Theorem)
Any social welfare function (SWF) for three or more alternatives cannot satisfy all the three axioms:

1. *Pareto optimality*
2. *Independence of Irrelevant Alternatives (IIA)*
3. *Non-dictatorship*

Arrow's Theorem (cont.)

Gibbard–Satterthwaite Theorem

Theorem (Gibbard–Satterthwaite Theorem)
Any social choice function for three or more alternatives cannot satisfy all the three axioms:

1. *Onto*
2. *Strategyproofness*
3. *Non-dictatorship*

Outline

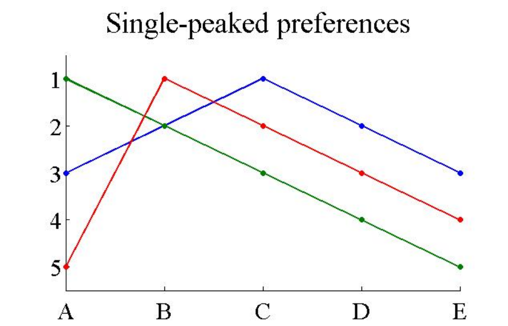
Domain Restriction

A preference profile has **single-peaked preferences** if there exists a left to right ordering $>$ on the alternatives such that any voter prefers a to b if a is between b and her top alternative. We say that the preferences are single-peaked with respect to the ordering $>$.

Examples

- Airconditioner temperature.
- Political spectrum.

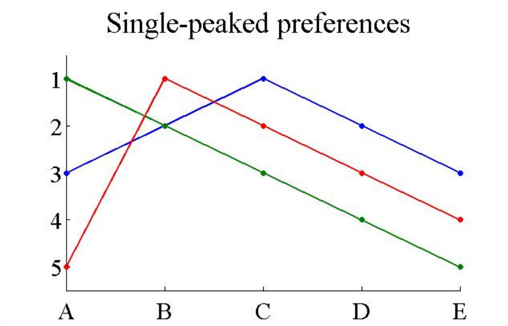
Domain Restriction (cont.)



Theorem (Bartholdi, III and Trick [1986])
It can be checked in polynomial time whether a given preference profile is single-peaked or not.

Domain Restriction

Given a left-to-right ordering $>$, the **median voter rule** asks each voter for their top alternative and elects the alternative proposed by the voter corresponding to the median with respect to $>$.



Domain Restriction

Given a left-to-right ordering $>$ such that the preferences are single-peaked with the respect to the ordering, the **median voter rule** asks each voter for their top alternative and elects the alternative proposed by the voter corresponding to the median top alternative with respect to $>$.

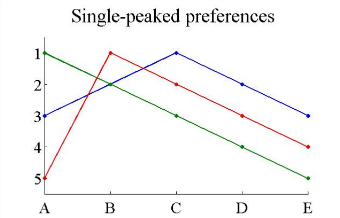
Theorem (Black's Theorem, 1948)
If an odd number of voters submit single-peaked preferences, then there exists a Condorcet winner and it will get elected by the median voter rule.

Domain Restriction (cont.)

$$1 : A \succ B \succ C \succ D \succ E$$

$$2 : B \succ C \succ D \succ E \succ A$$

$$3 : C \succ B \succ D \succ A \succ E$$



Outline

Social Decision Schemes

Social Decision Scheme: aggregates individual preferences into a probability distribution over alternatives.

$$F : \mathcal{L}(A)^n \to \Delta(A)$$

Social Decision Schemes

Random Dictatorship: probability of an alternative is proportional to its plurality score.

Example (Random Dictatorship)

$$
\begin{aligned}
1 &: \quad a \succ b \succ c \\
2 &: \quad b \succ a \succ c \\
3 &: \quad b \succ c \succ a
\end{aligned}
$$

Social Decision Schemes

Random Dictatorship: probability of an alternative is proportional to its plurality score.

Example (Random Dictatorship)

$$
\begin{aligned}
1: &\quad a \succ b \succ c \\
2: &\quad b \succ a \succ c \\
3: &\quad b \succ c \succ a
\end{aligned}
$$

$$p(a) = 1/3, p(b) = 2/3, p(c) = 0.$$

Social Decision Schemes

Random Dictatorship: probability of an alternative is proportional to its plurality score.

Theorem
Random dictatorship is the only social decision scheme that is anonymous, strategyproof, and has Pareto optimal alternatives in the support.

Social Decision Schemes

Borda Proportional: probability of an alternative is proportional to its Borda score.

Example (Borda Proportional)

$$
\begin{aligned}
1:&\quad a \succ b \succ c\\
2:&\quad b \succ a \succ c\\
3:&\quad b \succ c \succ a
\end{aligned}
$$

Borda score of a is 3; Borda score of b is 5; Borda score of c is 1;

$$p(a) = 3/9, p(b) = 5/9, p(c) = 1/9.$$

Borda Proportional: probability of an alternative is proportional to its Borda score.

Theorem
Borda Proportional is strategyproof but may give Pareto dominated alternatives non-zero probability.

Outline

Kemeny

- A Kemeny ranking is

$$\arg\min_{>\in\mathcal{L}} \sum_{x,y\in A, x>y} |\{i \in N \mid y \succ_i x\}|$$

- A Kemeny winner is the maximal alternative of a Kemeny ranking.

The Kemeny rule is a Condorcet-extension.

Theorem (Bartholdi et al., 1989)
Finding a Kemeny ranking and a Kemeny winner is NP-hard.

Borda versus Condorcet

- Jean-Charles, chevalier de Borda (1733 – February 1799): French mathematician, physicist, political scientist, and sailor.
- Marquis de Condorcet (1743 – 1794): French philosopher, mathematician, and early political scientist

Combining Borda and Condorcet:

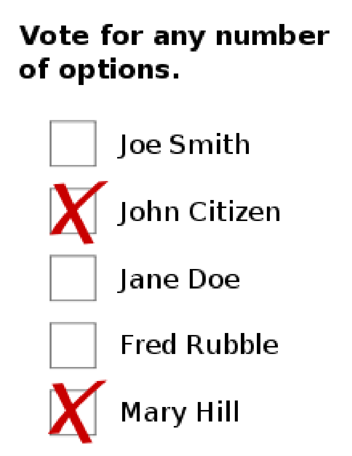
- **Black's rule:** Return the Condorcet winner if one exists and the Borda winner otherwise.
- **Nanson's rule:** Runoff rule in which alternatives with alternatives with less than the average Borda score are deleted until no more deletions are possible.

Summary

- There are many interesting voting rules with relative merits.
- An axiomatic study of voting rules helps understand the relative merits.
- Voting rules have different computational complexity as well.
- Social choice has many famous impossibility results.
- Considering restricted domains and randomisation are two possible ways to avoid impossibility results.

Outline

Approval Voting



Approval Voting

- $N = \{1, \ldots, n\}$
- $A = \{a_1, \ldots, a_m\}$
- $(A_1, \ldots, A_n)$ where $A_i \subset A$ is the approval set of voter i.

Approval-based Committee Voting

- $N = \{1, \ldots, n\}$
- $A = \{c_1, \ldots, c_m\}$
- $(A_1, \ldots, A_n)$ where $A_i \subset A$ is the approval set of voter i.

Goal: Select specified number of k alternatives. The outcome is called a *committee* or *winning set*.

Applications

- Selecting a set of movies
- Ordering some food dishes
- Generalisation of parliamentary apportionment.

Approval-based Committee Voting

- $N = \{1, \ldots, n\}$
- $A = \{c_1, \ldots, c_m\}$
- $(A_1, \ldots, A_n)$ where $A_i \subset A$ is the approval set of voter i.

Goal: Select specified number of k alternatives.

- Why not simply use AV (return the k alternatives with the most approvals)?
- AV used used by American Mathematical Society (AMS), the Institute of Electrical and Electronics Engineers (IEEE), and the International Joint Conference on Artificial Intelligence (IJCAI).

Approval-based Committee Voting (cont.)

Example

- $N = \{1, \ldots, 100\}$, $A = \{a, b, c, d\}$, $k = 2$
    - 49 voters approve of c and d.
    - 51 voters approve of a and b.

The solution of $\{a, b\}$ is unfair to the minority.

Other Rules for Approval Voting Committee Voting

- S. J. Brams and D. M. Kilgour. Satisfaction approval voting. Voting Power and Procedures, Studies in Choice and Welfare, Springer, 2014.
- S. J. Brams, D. M. Kilgour, and R. M. Sanver. A minimax procedure for electing committees. Public Choice, 132(3-4), 2007.
- D. M. Kilgour. Approval balloting for multi-winner elections. In Handbook on Approval Voting, chapter 6. Springer, 2010.
- D. M. Kilgour and E. Marshall. Approval balloting for fixed-size committees. Electoral Systems, Studies in Choice and Welfare, Springer.

Approval-based voting rules

- **SAV (Satisfaction Approval Voting)** [Brams & Kilgour 2014]
  - Voter i gives committee W score $|A_i \cap W|/|A_i|$
  - Return committee $\arg\max_{W \in \mathcal{P}_k(A)} \sum_{i \in N}(|A_i \cap W|/|A_i|)$

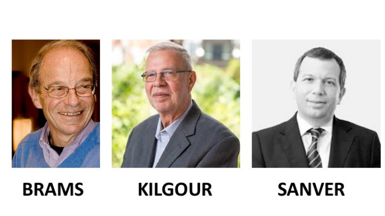
Approval-based voting rules (cont.)

Example

- $N = \{1, \ldots, 100\}$, $A = \{a, b, c, d\}$, $k = 2$
  - 49 voters approve of c and d
  - 51 voters approve of a and b

- $\{a, c\}$ gets score $\frac{1}{2}100 = 50$.
- $\{a, b\}$ gets score 51

Approval-based voting rules



- **MinimaxAV (Minimax Approval Voting)** [Brams, Kilgour & Sanver 2007]:
  - Distance of A_i from W is $d(A_i, W) = |A_i \setminus W| + |W \setminus A_i|$.
  - Select a k-sized committee: $\arg\min_{W \in \mathcal{P}_k(A)}(\max_{i \in N} d(A_i, W))$

Approval-based voting rules (cont.)

Example

- $N = \{1, \ldots, 100\}$, $A = \{a, b, c, d\}$, $k = 2$
  - 49 voters approve of c and d
  - 51 voters approve of a and b

The outcome is any committee that represents each voter once.

Approval-based voting rules

THIELE

- **PAV (Proportional Approval Voting)** [Thiele 1895, Simmons 2001]
  - Let H be a function defined on integers such that $H(p) = 0$ for $p = 0$ and $H(p) = \sum_{j=1}^{p} \frac{1}{j}$ otherwise.
  - $u_i(W) = H(|W \cap A_i|)$
  - Select a size k committee W that is $\arg\max_{W \in \mathcal{P}_k(A)} \sum_{i \in N} u_i(W)$.

Approval-based voting rules (cont.)

Example

- $N = \{1, \ldots, 9\}$, $A = \{a, b, c\}$, $k = 2$
  - $A_1, \ldots, A_5 = \{a, b\}$
  - $A_6, \ldots, A_9 = \{c\}$.
- The outcome of AV is $\{a, b\}$
- The outcome of PAV is $\{a, c\}$ or $\{b, c\}$.

Approval-based voting rules

- **SeqPAV (Sequential Proportional Approval Voting)** [Thiele 1895]
  - Sequential version of PAV in which keep selecting alternatives until k are selected.
  - $w(i) = 1$ for each $i \in N$ initially and $W = \emptyset$
  - add an alternative to W with maximum approval weight $w(c) = \sum_{c \in A_i} w(i)$
  - Update weight of each voter to $w(i) = 1/(1 + (|A_i \cap W|))$.

Approval-based voting rules (cont.)

Example

- $N = \{1, \ldots, 9\}$, $A = \{a, b, c\}$, $k = 2$
  - $A_1, \ldots, A_5 = \{a, b\}$
  - $A_6, \ldots, A_9 = \{c\}$.
- The outcome of AV is $\{a, b\}$
- The outcome of SeqPAV is $\{a, c\}$ or $\{b, c\}$.

Approval-based voting rules

- **AV**: select k alternatives with the highest number of approvals.
- **SAV (Satisfaction Approval Voting)** [Brams & Kilgour 2014]
  - Voter i gives committee W score $|A_i \cap W|/|A_i|$
  - Committee with the highest total score is selected.
- **MinimaxAV (Minimax Approval Voting)** [Brams, Kilgour & Sanver 2007]
  - Distance of A_i from W is $d(A_i, W) = |A_i \setminus W| + |W \setminus A_i|$.
  - Select a k-sized committee W that minimizes $\max_i d(A_i, W)$.
  - NP-hard
- **PAV (Proportional Approval Voting)** [Thiele 1895, Simmons 2001]

Approval-based voting rules (cont.)

- $u_i(W) = H(|W \cap A_i|)$ where $H(p) = 0$ for $p = 0$ and $H(p) = \sum_{j=1}^{p} \frac{1}{j}$ otherwise.
- Select a size k committee W that maximizes $\sum_{i \in N} u_i(W)$.
- NP-hard

- **SeqPAV (Sequential Proportional Approval Voting)** [Thiele 1895]
    - Sequential version of PAV
    - $w(i) = 1$ initially and $W = \emptyset$
    - add an alternative to W with maximum approval weight $w(c) = \sum_{i \text{ approves } c} w(i)$
    - Update weight of each voter to $w(i) = 1/(1 + (1 + |A_i \cap W|))$.

Outline

Reading

Social choice chapters of the following books:

- Y. Shoham and K. Leyton-Brown. Multiagent Systems: Algorithmic, Game-Theoretic, and Logical Foundations. 2009. http://www.masfoundations.org
- N. Nisan, T. Roughgarden, E. Tardos, and V.V. Vazirani. Algorithmic Game Theory. Cambridge University Press, 2007. www.cambridge.org/journals/nisan/downloads/Nisan_Non-printable.pdf

References

J. Bartholdi, III and M. Trick. Stable matching with preferences derived from a psychological model. *Operations Research Letters*, 5(4):165–169, 1986.