COMP9020 22T1
Week 2
Logic, Proofs, Boolean Algebra

- Textbook (R & W) - Ch. 2, Sec. 2.2-2.5
  Ch. 10, Sec. 10.1-10.5
- Problem set 2 + Quiz
- *Guidelines for good mathematical writing*
- Self-guided study: Exercises Ch. 2 (R & W), ...

Reminder: Quiz Rules

Quiz on this week's problem set due Tuesday, 1 March, 5pm

Do ...

- use your own best judgement to understand & solve questions
- email me if you think Moodle is wrong (question or answer)
- discuss quizzes on the forum only **after** the deadline

Do not ...

- post specific questions about the quiz **before** the deadline
- ask me to check your answers before you submit
- agonise too much about a question that you find too difficult

NB

1. Homework and quizzes are for you to demonstrate your ability to understand and solve problems (like an exam)
2. They give you feedback on how well you have understood the contents (to prepare you for the exam)

Logical Equivalence

Two formulas ϕ, ψ are **logically equivalent**, denoted $\phi \equiv \psi$ if they have the same truth value for all values of their basic propositions.

Application: If ϕ and ψ are two formulae such that $\phi \equiv \psi$, then the digital circuits corresponding to ϕ and ψ compute the same function. Thus, proving equivalence of formulas can be used to *optimise* circuits.

Some well-known equivalences:

Theorem

Excluded Middle	$p \vee \neg p \equiv \top$	*Identity*	$p \vee \bot \equiv p$
Contradiction	$p \wedge \neg p \equiv \bot$		$p \wedge \top \equiv p$
Idempotence	$p \vee p \equiv p$		$p \vee \top \equiv \top$
	$p \wedge p \equiv p$		$p \wedge \bot \equiv \bot$
Double Negation	$\neg\neg p \equiv p$		

More well-known equivalences:

Theorem

Commutativity	$p \vee q \equiv q \vee p$
	$p \wedge q \equiv q \wedge p$
Associativity	$(p \vee q) \vee r \equiv p \vee (q \vee r)$
	$(p \wedge q) \wedge r \equiv p \wedge (q \wedge r)$
Distribution	$p \vee (q \wedge r) \equiv (p \vee q) \wedge (p \vee r)$
	$p \wedge (q \vee r) \equiv (p \wedge q) \vee (p \wedge r)$
De Morgan's laws	$\neg(p \wedge q) \equiv \neg p \vee \neg q$
	$\neg(p \vee q) \equiv \neg p \wedge \neg q$
Implication	$p \Rightarrow q \equiv \neg p \vee q$
	$p \Leftrightarrow q \equiv (p \Rightarrow q) \wedge (q \Rightarrow p)$

Example

$$
\begin{aligned}
&((r \wedge \neg p) \vee (r \wedge q)) \vee ((\neg r \wedge \neg p) \vee (\neg r \wedge q)) && \\
&\qquad \equiv \quad (r \wedge (\neg p \vee q)) \vee (\neg r \wedge (\neg p \vee q)) && \text{by Distrib.} \\
&\qquad \equiv \quad (r \vee \neg r) \wedge (\neg p \vee q) && \text{by Distrib.} \\
&\qquad \equiv \quad \top \wedge (\neg p \vee q) && \text{by Excl. Mid.} \\
&\qquad \equiv \quad \neg p \vee q && \text{by Ident.}
\end{aligned}
$$

Exercise

$\boxed{2.2.18}$ Prove or disprove:

(a) $p \Rightarrow (q \Rightarrow r) \;\equiv\; (p \Rightarrow q) \Rightarrow (p \Rightarrow r)$

(c) $(p \Rightarrow q) \Rightarrow r \;\equiv\; p \Rightarrow (q \Rightarrow r)$

Exercise

$\boxed{2.2.18}$ Prove or disprove:

(a) $(p \Rightarrow q) \Rightarrow (p \Rightarrow r)$

$$\equiv \neg(p \Rightarrow q) \vee (\neg p \vee r)$$
$$\equiv (p \wedge \neg q) \vee (\neg p \vee r)$$
$$\equiv (p \vee \neg p \vee r) \wedge (\neg q \vee \neg p \vee r)$$
$$\equiv (\top \vee r) \wedge (\neg p \vee \neg q \vee r)$$
$$\equiv \top \wedge (\neg p \vee \neg q \vee r)$$
$$\equiv p \Rightarrow (\neg q \vee r)$$
$$\equiv p \Rightarrow (q \Rightarrow r)$$

(c) $(p \Rightarrow q) \Rightarrow r \;\; \equiv \;\; p \Rightarrow (q \Rightarrow r)$

Counterexample:

p	q	r	$(p \Rightarrow q) \Rightarrow r$	$p \Rightarrow (q \Rightarrow r)$
F	T	F	F	T

Satisfiability of Formulas

A formula is **satisfiable**, if it evaluates to T for *some* assignment of truth values to its basic propositions.

Example

A	B	$\neg(A \Rightarrow B)$
F	F	F
F	T	F
T	F	T
T	T	F

Applications II: Constraint Satisfaction Problems

These are problems such as timetabling, activity planning, etc.
Many can be understood as showing that a formula is satisfiable.

Example

You are planning a party, but your friends are a bit touchy about who will be there.

1. If John comes, he will get very hostile if Sarah is there.
2. Sarah will only come if Kim will be there also.
3. Kim says she will not come unless John does.

Who can you invite without making someone unhappy?

Translation to logic: let J, S, K represent "John (Sarah, Kim) comes to the party". Then the constraints are:

1. $J \Rightarrow \neg S$
2. $S \Rightarrow K$
3. $K \Rightarrow J$

Thus, for a successful party to be possible, we want the formula $\phi = (J \Rightarrow \neg S) \wedge (S \Rightarrow K) \wedge (K \Rightarrow J)$ to be satisfiable.
Truth values for J, S, K making this true are called *satisfying assignments*, or *models*.

We figure out where the conjuncts are false, below. (so blank $=$ T)

J	K	S	$J \Rightarrow \neg S$	$S \Rightarrow K$	$K \Rightarrow J$	ϕ
F	F	F				
F	F	T		F		F
F	T	F			F	F
F	T	T			F	F
T	F	F				
T	F	T	F	F		F
T	T	F				
T	T	T	F			F

Conclusion: a party satisfying the constraints can be held. Invite nobody, or invite John only, or invite Kim and John.

Exercise

$\boxed{\text{2.7.14 (supp)}}$

Which of the following formulae are *always* true?

(a) $(p \wedge (p \Rightarrow q)) \;\Rightarrow\; q$ — always true

(b) $((p \vee q) \wedge \neg p) \;\Rightarrow\; \neg q$ — not always true

(e) $((p \Rightarrow q) \vee (q \Rightarrow r)) \;\Rightarrow\; (p \Rightarrow r)$ — not always true

(f) $(p \wedge q) \Rightarrow q$ — always true

Exercise

$\boxed{\text{2.7.14 (supp)}}$

Which of the following formulae are *always* true?

(a) $(p \wedge (p \Rightarrow q)) \Rightarrow q$ — always true

(b) $((p \vee q) \wedge \neg p) \Rightarrow \neg q$ — not always true

(e) $((p \Rightarrow q) \vee (q \Rightarrow r)) \Rightarrow (p \Rightarrow r)$ — not always true

(f) $(p \wedge q) \Rightarrow q$ — always true

Validity, Entailment, Arguments

An *argument* consists of a set of declarative sentences called *premises* and a declarative sentence called the *conclusion*.

Example

Premises:	Frank took the Ford or the Toyota.
	If Frank took the Ford he will be late.
	Frank is not late.
Conclusion:	Frank took the Toyota

An argument is *valid* if the conclusions are true *whenever* all the premises are true. Thus: if we believe the premises, we should also believe the conclusion.
(Note: we don't care what happens when one of the premises is false.)
Other ways of saying the same thing:

- The conclusion *logically follows* from the premises.
- The conclusion is a *logical consequence* of the premises.
- The premises **entail** the conclusion.

The argument above is valid. The following is invalid:

Example

Premises:	Frank took the Ford or the Toyota.
	If Frank took the Ford he will be late.
	Frank is late.
Conclusion:	Frank took the Ford.

For arguments in propositional logic, we can capture validity as follows:

Let $\phi_1, \ldots, \phi_n$ and ϕ be formulae of propositional logic. Draw a truth table with columns for each of $\phi_1, \ldots, \phi_n$ and ϕ.

The argument with premises $\phi_1, \ldots, \phi_n$ and conclusion ϕ is valid, denoted

$$\phi_1, \ldots, \phi_n \models \phi$$

if in every row of the truth table where $\phi_1, \ldots, \phi_n$ are all true, ϕ is true also.

We mark only true locations (blank = F)

Frd	$Tyta$	$Late$	$Frd \lor Tyta$	$Frd \Rightarrow Late$	$\neg Late$	$Tyta$
F	F	F		T	T	
F	F	T		T		
F	T	F	T	T	T	T
F	T	T	T	T		T
T	F	F	T		T	
T	F	T	T	T		
T	T	F	T		T	T
T	T	T	T	T		T

This shows $Frd \lor Tyta$, $Frd \Rightarrow Late$, $\neg Late \models Tyta$

The following row shows $Frd \vee Tyta$, $Frd \Rightarrow Late$, $Late \not\models Frd$

Frd	$Tyta$	$Late$	$Frd \vee Tyta$	$Frd \Rightarrow Late$	$Late$	Frd
F	T	T	T	T	T	F

Applications III: Reasoning About Requirements/Specifications

Suppose a set of English language requirements R for a software/hardware system can be formalised by a set of formulae $\{\phi_1, \ldots \phi_n\}$.
Suppose C is a statement formalised by a formula ψ. Then

1. The requirements cannot be implemented if $\phi_1 \wedge \ldots \wedge \phi_n$ is not satisfiable.
2. If $\phi_1, \ldots \phi_n \models \psi$ then every correct implementation of the requirements R will be such that C is always true in the resulting system.
3. If $\phi_1, \ldots \phi_{n-1} \models \phi_n$, then the condition ϕ_n of the specification is redundant and need not be stated in the specification.

Example

Requirements R: A burglar alarm system for a house is to operate as follows. The alarm should not sound unless the system has been armed or there is a fire. If the system has been armed and a door is disturbed, the alarm should ring. Irrespective of whether the system has been armed, the alarm should go off when there is a fire.

Conclusion C: If the alarm is ringing and there is no fire, then the system must have been armed.

Question

1. *Will every system correctly implementing requirements R satisfy C?*
2. *Is the final sentence of the requirements redundant?*

Expressing the requirements as formulas of propositional logic, with

- S = the alarm sounds = the alarm rings
- A = the system is armed
- D = a door is disturbed
- F = there is a fire

we get

Requirements:

1. $S \Rightarrow (A \vee F)$
2. $(A \wedge D) \Rightarrow S$
3. $F \Rightarrow S$

Conclusion: $(S \wedge \neg F) \Rightarrow A$

Our two questions then correspond to

1. Does $S \Rightarrow (A \vee F), (A \wedge D) \Rightarrow S, F \Rightarrow S \models (S \wedge \neg F) \Rightarrow A$?
2. Does $S \Rightarrow (A \vee F), (A \wedge D) \Rightarrow S \models F \Rightarrow S$?

Answers: problem set 2, exercise 4

Validity of Formulas

A formula ϕ is **valid**, or a **tautology**, denoted $\models \phi$, if it evaluates to T for *all* assignments of truth values to its basic propositions.

Example

A	B	$(A \Rightarrow B) \Rightarrow (\neg B \Rightarrow \neg A)$
F	F	T
F	T	T
T	F	T
T	T	T

Validity, Equivalence and Entailment

Theorem

The following are equivalent:

- $\phi_1, \ldots, \phi_n \models \psi$
- $\models (\phi_1 \wedge \ldots \wedge \phi_n) \Rightarrow \psi$
- $\models \phi_1 \Rightarrow (\phi_2 \Rightarrow \ldots (\phi_n \Rightarrow \psi) \ldots)$

Theorem

$\phi \equiv \psi$ *if and only if* $\models \phi \Leftrightarrow \psi$

Proof Rules and Methods: Proof by Cases

We want to prove that A.

To prove it, we find a set of cases $B_1, B_2, \ldots, B_n$ such that

1. $B_1 \vee \ldots \vee B_n$, and
2. $B_i \Rightarrow A$ for each $i = 1..n$.

(Hard Part: working out what the B_i should be.)

(Often $n = 2$ and $B_2 = \neg B_1$, then $B_1 \vee B_2 = B_1 \vee \neg B_1$ holds trivially.)

Example

Every group of 6 people includes 3 who have met or 3 strangers.

Proof: Let x denote one of the 6 people.

Case 1: At least 3 of the other 5 have met x.

    Case 1.1: No pair among the 3 have met each other.

    Case 1.2: Some pair among the 3 have met each other.

Case 2: At least 3 of the other 5 have not met x.

    Case 1.1: Every pair among the 3 have met each other.

    Case 1.2: Some pair among the 3 have not met each other.

Quantifiers

We've made quite a few statements of the kind

"If there exists a satisfying assignment ..." or

"Every natural number greater than 2 ..."

without formally capturing these quantitative aspects.

Notation: $\forall$ means "for all" and $\exists$ means "there exist(s)"

Example

Goldbach's conjecture

$$\forall n \in 2\mathbb{N}\,(n > 2 \Rightarrow \exists p, q \in \mathbb{N}\,(p, q \in \textsc{Primes} \land n = p + q))$$

Exercise

Which of the following is a tautology?

- $\forall x\,(\exists y\,(P(x,y))) \Rightarrow \exists y\,(\forall x\,(P(x,y)))$ not always true
- $\exists y\,(\forall x\,(P(x,y))) \Rightarrow \forall x\,(\exists y\,(P(x,y)))$ always true

Example

$\forall x \in \mathbb{Z}\,(\exists y \in \mathbb{Z}\,(y \leq x))$ but $\neg\exists y \in \mathbb{Z}\,(\forall x \in \mathbb{Z}\,(y \leq x))$

$\exists y \in \mathbb{N}\,(\forall x \in \mathbb{N}\,(y \leq x))$, hence $\forall x \in \mathbb{N}\,(\exists y \in \mathbb{N}\,(y \leq x))$

Exercise

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Example

$\forall x \in \mathbb{Z}\,(\exists y \in \mathbb{Z}\,(y \leq x))$ but $\neg\exists y \in \mathbb{Z}\,(\forall x \in \mathbb{Z}\,(y \leq x))$

$\exists y \in \mathbb{N}\,(\forall x \in \mathbb{N}\,(y \leq x))$, hence $\forall x \in \mathbb{N}\,(\exists y \in \mathbb{N}\,(y \leq x))$

Proof Rules and Methods: Proof of the Contrapositive

We want to prove $A \Rightarrow B$.
To prove it, we show $\neg B \Rightarrow \neg A$ and invoke the equivalence $(A \Rightarrow B) \equiv (\neg B \Rightarrow \neg A)$.

Example

The square root of an irrational number is irrational:
$\forall x \in \mathbb{R} \left(x \notin \mathbb{Q} \;\Rightarrow\; \sqrt{x} \notin \mathbb{Q} \right)$

Proof Rules and Methods: Proof by Contradiction

We want to prove A.
To prove it, we assume $\neg A$, and derive both B and $\neg B$ for some proposition B.
(Hard part: working out what B should be.)

Examples

- $\sqrt{2}$ is irrational
- There exist an infinite number of primes

Substitution

Substitution is the process of replacing every occurrence of some symbol by an expression.

Examples

The result of substituting 3 for x in

$$x^2 + 7y = 2xz$$

is

$$3^2 + 7y = 2 \cdot 3 \cdot z$$

The result of substituting $2k + 3$ for x in

$$x^2 + 7y = 2xz$$

is

$$(2k + 3)^2 + 7y = 2 \cdot (2k + 3) \cdot z$$

We can substitute logical expressions for logical variables:

Example

The result of substituting $P \wedge Q$ for A in

$$(A \wedge B) \Rightarrow A$$

is

$$((P \wedge Q) \wedge B) \Rightarrow (P \wedge Q)$$

Substitution Rules

(a) If we substitute an expression for *all* occurrences of a logical variable in a tautology then the result is still a tautology.

If $\models \phi(P)$ then $\models \phi(\alpha)$.

Examples

$\models P \Rightarrow (P \vee Q)$, so

$$\models (A \vee B) \Rightarrow ((A \vee B) \vee Q)$$

$\boxed{2.5.7}$

$\models \neg Q \Rightarrow (Q \Rightarrow P)$, so

$$\models \neg(P \Rightarrow Q) \Rightarrow ((P \Rightarrow Q) \Rightarrow P)$$

(b) If a logical formula ϕ contains a formula α, and we replace (an occurrence of) α by a logically equivalent formula β, then the result is logically equivalent to ϕ.

If $\alpha \equiv \beta$ then $\phi(\alpha) \equiv \phi(\beta)$.

Example

$P \Rightarrow Q \equiv \neg P \vee Q$, so

$$Q \Rightarrow (P \Rightarrow Q) \equiv Q \Rightarrow (\neg P \vee Q)$$

Boolean Functions

Formulae can be viewed as **Boolean functions** mapping valuations of their propositional letters to truth values.

A Boolean function of one variable is also called **unary**.
A function of two variables is called **binary**.
A function of n input variables is called **n-ary**.

(Named after mathematician George Boole (England), 1815–1864)

Question

How many unary Boolean functions are there?
How many binary functions? n-ary?

Question

What connectives do we need to express all of them?

Boolean Arithmetic

Consider truth values with operations $\wedge, \vee, \neg$ as an **algebraic structure**:

- $\mathbb{B} = \{0, 1\}$ with 'Boolean' arithmetic

$$a \cdot b, \ a + b, \ a' = 1 - a$$

NB

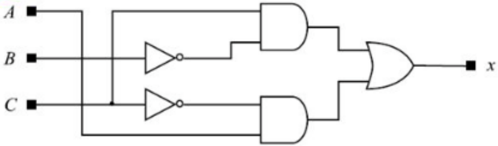
We often write pq for $p \cdot q$.
In electrical and computer engineering, the notation $\overline{p}$ is more common than p', which is often used in mathematics.
Observe that using $\overline{(\cdot)}$ obviates the need for some parentheses.

Applications IV: Digital Circuits

A formula can be viewed as defining a digital circuit, which computes a Boolean function of the input propositions. The function is given by the truth table of the formula.

A	B	C	x
0	0	0	0
0	0	1	1
0	1	0	0
0	1	1	0
1	0	0	1
1	0	1	1
1	1	0	1
1	1	1	0

$$x = \overline{A}\,\overline{B}C + A\overline{B}\,\overline{C} + A\overline{B}C + AB\overline{C} = \overline{B}C + A\overline{C}$$



Definition: Boolean Algebra

Every structure consisting of a set T with operations *join*: $a, b \mapsto a + b$, *meet*: $a, b \mapsto a \cdot b$ and *complementation*: $a \mapsto a'$, and distinct elements 0 and 1, is called a **Boolean algebra** if it satisfies the following laws, for all $x, y, z \in T$:

commutative:
- $x + y = y + x$
- $x \cdot y = y \cdot x$

associative:
- $(x + y) + z = x + (y + z)$
- $(x \cdot y) \cdot z = x \cdot (y \cdot z)$

distributive:
- $x + (y \cdot z) = (x + y) \cdot (x + z)$
- $x \cdot (y + z) = (x \cdot y) + (x \cdot z)$

identity: $x + 0 = x, \quad x \cdot 1 = x$

complementation: $x + x' = 1, \quad x \cdot x' = 0$

Exercise

$\boxed{\text{Example 10.1.2}}$ Define a Boolean algebra for 2-bit vectors $\mathbb{B}^2$

$0 \stackrel{\text{def}}{=} (0,0)$

$1 \stackrel{\text{def}}{=} (1,1)$

join: $(a_1, a_2) + (b_1, b_2) \mapsto (a_1 + b_1, a_2 + b_2)$

meet: $(a_1, a_2) \cdot (b_1, b_2) \mapsto (a_1 \cdot b_1, a_2 \cdot b_2)$

complementation: $(a_1, a_2)' \mapsto (a_1{}', a_2{}')$

Check that all Boolean algebra laws hold for $x, y \in \mathbb{B} \times \mathbb{B}$

Exercise

$\boxed{\text{Example 10.1.2}}$ Define a Boolean algebra for 2-bit vectors $\mathbb{B}^2$

$0 \stackrel{\text{def}}{=} (0, 0)$
$1 \stackrel{\text{def}}{=} (1, 1)$
join: $(a_1, a_2) + (b_1, b_2) \mapsto (a_1 + b_1, a_2 + b_2)$
meet: $(a_1, a_2) \cdot (b_1, b_2) \mapsto (a_1 \cdot b_1, a_2 \cdot b_2)$
complementation: $(a_1, a_2)' \mapsto (a_1{}', a_2{}')$

Check that all Boolean algebra laws hold for $x, y \in \mathbb{B} \times \mathbb{B}$

Boolean Expressions

Boolean algebra (BA) notation for propositional formulae:

	PL	**BA**
propositional atoms	$p, q, \ldots$	$p, q, \ldots$
conjunction	$p \wedge q$	$p \cdot q$ or pq
disjunction	$p \vee q$	$p + q$
negation	$\neg p$	p'

Example

$$(p \vee q) \wedge (\neg(p \vee \neg q) \vee \neg(\neg(r \wedge (p \vee \neg q))))$$

$$\begin{aligned} & (p+q) \cdot ((p+q')' + (r \cdot (p+q'))'') \\ = & (p+q)((p+q')' + (r(p+q'))'') \end{aligned}$$

Terminology and Rules

- A **literal** is an expression p or p', where p is a propositional atom.
- An expression is in CNF (conjunctive normal form) if it has the form
$$\prod_i C_i$$
where each **clause** C_i is a disjunction of literals e.g. $p + q + r'$.
- An expression is in DNF (disjunctive normal form) if it has the form
$$\sum_i C_i$$
where each clause C_i is a conjunction of literals e.g. pqr'.

NB

A clause (i.e. a conjunction or disjunction) can be a single literal.

- CNF and DNF are named after their top level operators; no deeper nesting of $\cdot$ or $+$ is permitted.
- We can assume in every clause (disjunct for the CNF, conjunct for the DNF) any given variable (literal) appears only once; preferably, no literal and its negation together.
  - $x + x = x, \quad xx = x$
  - $xx' = 0, \quad x + x' = 1$
  - $x \cdot 0 = 0,\ x \cdot 1 = x,\ x + 0 = x,\ x + 1 = 1$
- A preferred form for an expression is DNF, with as few terms as possible. In deriving such minimal simplifications the two basic rules are **absorption** and **combining the opposites**.

Fact

1. $x + xy = x$ *(absorption)*
2. $xy + xy' = x$ *(combining the opposites)*

Theorem

For every Boolean expression ϕ, there exists an equivalent expression in conjunctive normal form and an equivalent expression in disjunctive normal form.

Proof.

We show how to apply the equivalences already introduced to convert any given formula to an equivalent one in CNF, DNF is similar. $\square$

Step 1: Push Negations Down

Using **De Morgan's** laws and the **double negation** rule

$$(x + y)' = x' \cdot y'$$
$$(x \cdot y)' = x' + y'$$
$$(x')' = x$$

we push negations down towards the atoms until we obtain a formula that is formed from literals using only $\cdot$ and $+$.

Step 2: Use Distribution to Convert to CNF

Using the distribution rules

$$x + (y_1 \cdot \ldots \cdot y_n) = (x + y_1) \cdot \ldots \cdot (x + y_n)$$
$$(y_1 \cdot \ldots \cdot y_n) + x = (y_1 + x) \cdot \ldots \cdot (y_n + x)$$

we obtain a CNF formula.

CNF/DNF in Propositional Logic

Using the equivalence

$$A \Rightarrow B \quad \equiv \quad \neg A \vee B$$

we first eliminate all occurrences of $\Rightarrow$

Example

$$\neg(\neg p \wedge ((r \wedge s) \Rightarrow q)) \equiv \neg(\neg p \wedge (\neg(r \wedge s) \vee q))$$

Step 1:

Example

$$
\begin{aligned}
(p'((rs)' + q))' &= (p')' + ((rs)' + q)' \\
&= p + (rs)'' \cdot q' \\
&= p + rsq'
\end{aligned}
$$

Step 2:

Example

$$
\begin{aligned}
p + rsq' &= (p + r)(p + sq') \\
&= (p + r)(p + s)(p + q') \qquad \text{CNF}
\end{aligned}
$$

Canonical Form DNF

Given a Boolean expression E, we can construct an equivalent DNF E^{dnf} from the lines of the truth table where E is true:
Given an assignment π of $0, 1$ to variables $x_1 \dots x_i$, define the literal

$$\ell_i = \begin{cases} x_i & \text{if } \pi(x_i) = 1 \\ x_i{}' & \text{if } \pi(x_i) = 0 \end{cases}$$

and a product $t_\pi = \ell_1 \cdot \ell_2 \cdot \ldots \cdot \ell_n$.

Example

If $\pi(x_1) = 1$ and $\pi(x_2) = 0$ then $t_\pi = x_1 \cdot x_2{}'$

The **canonical DNF** of E is

$$E^{dnf} = \sum_{E(\pi)=1} t_\pi$$

Example

If E is defined by

x	y	E
0	0	1
0	1	0
1	0	1
1	1	1

then $E^{dnf} = x'y' + xy' + xy$

Note that this can be simplified to

$$x + y'$$

Exercise

$\boxed{10.2.3}$ Find the canonical DNF form of each of the following expressions in variables x, y, z

- xy
- z'
- $xy + z'$
- 1

Exercise

$\boxed{10.2.3}$ Find the canonical DNF form of the following expressions in the three variables x, y, z.

$$\begin{aligned} xy &= xy \cdot 1 = xy \cdot (z + z') = xyz + xyz' \\ z' &= xyz' + xy'z' + x'yz' + x'y'z' \\ xy + z' &= \text{combine all of the 5 different product terms above} \\ 1 &= \text{sum of all 8 possible product terms:} \\ &\qquad xyz + x'yz + \ldots + x'y'z' \end{aligned}$$

NB

Obviously, preferred in practice are the expressions with as few terms as possible.
However, the existence of a uniform representation as the sum of (quite a few) product terms is important for proving the properties of Boolean expressions.

Karnaugh Maps

For up to four variables (propositional symbols) a diagrammatic method of simplification called **Karnaugh maps** works quite well. For every propositional function of $k = 2, 3, 4$ variables we construct a rectangular array of 2^k cells. We mark the squares corresponding to the value 1 with eg "$+$" and try to cover these squares with as few rectangles with sides 1 or 2 or 4 as possible.

Example

$\boxed{10.4.2}$ Use a K-map to find an optimised form.

	yz	yz'	$y'z'$	$y'z$
x	$+$	$+$		$+$
x'	$+$		$+$	$+$

For optimisation, the idea is to cover the '$+$ squares' with the minimum number of rectangles. One *cannot* cover any empty cells (they indicate where $f(w, x, y, z)$ is 0).

- The rectangles can go 'around the corner'/the actual map should be seen as a *torus*.
- Rectangles must have sides of 1, 2 or 4 squares (three adjacent cells are useless).
- Each rectangle must be as large as possible.

Exercise

	yz	yz'	$y'z'$	$y'z$
x	$+$	$+$		$+$
x'	$+$		$+$	$+$

$f = xy + x'y' + z$

Canonical form would consist of writing all cells separately:
$xyz + xyz' + xy'z + x'yz + x'y'z' + x'y'z$

For optimisation, the idea is to cover the '$+$ squares' with the minimum number of rectangles. One *cannot* cover any empty cells (they indicate where $f(w, x, y, z)$ is 0).

- The rectangles can go 'around the corner'/the actual map should be seen as a *torus*.
- Rectangles must have sides of 1, 2 or 4 squares (three adjacent cells are useless).
- Each rectangle must be as large as possible.

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Supplementary Exercise

Exercise

$\boxed{10.6.6(c)}$

	yz	yz'	$y'z'$	$y'z$
wx	$+$	$+$		$+$
wx'	$+$	$+$	$+$	$+$
$w'x'$			$+$	$+$
$w'x$	$+$			$+$

$f = wy + x'y' + xz$

Note: trying to use wx' or $y'z$ doesn't give as good a solution

Supplementary Exercise

Exercise

$\boxed{10.6.6(c)}$

	yz	yz'	$y'z'$	$y'z$
wx	$+$	$+$		$+$
wx'	$+$	$+$	$+$	$+$
$w'x'$			$+$	$+$
$w'x$	$+$			$+$

$f = wy + x'y' + xz$

Note: trying to use wx' or $y'z$ doesn't give as good a solution

Boolean Algebras in Computer Science

Several data structures have natural operations following essentially the same rules as logical $\wedge$, $\vee$ and $\neg$.

- n-tuples of 0's and 1's with Boolean operations, e.g.

$$\begin{aligned} \textit{join:} \quad & (1,0,0,1) + (1,1,0,0) = (1,1,0,1) \\ \textit{meet:} \quad & (1,0,0,1) \cdot (1,1,0,0) = (1,0,0,0) \\ \textit{complementation:} \quad & (1,0,0,1)' = (0,1,1,0) \end{aligned}$$

- $\mathrm{Pow}(S)$ — subsets of S

$$\textit{join: } A \cup B, \quad \textit{meet: } A \cap B, \quad \textit{complement: } A^{c} = S \setminus A$$

Example

Exercise

$\boxed{\text{Example 10.1.1}}$ Define a Boolean algebra for the power set $\mathrm{Pow}(S)$ of $S = \{a, b, c\}$

$0 \stackrel{\text{def}}{=} \emptyset$
$1 \stackrel{\text{def}}{=} \{a, b, c\}$
join: $X, Y \mapsto X \cup Y$
meet: $X, Y \mapsto X \cap Y$
complementation: $X \mapsto \{a, b, c\} \setminus X$

Additional exercise:
Verify that all Boolean algebra laws (cf. slide 39) hold for $X, Y, Z \in \mathrm{Pow}(\{a, b, c\})$

Example

Exercise

$\boxed{\text{Example 10.1.1}}$ Define a Boolean algebra for the power set $\mathrm{Pow}(S)$ of $S = \{a, b, c\}$

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Additional exercise:
Verify that all Boolean algebra laws (cf. slide 39) hold for $X, Y, Z \in \mathrm{Pow}(\{a, b, c\})$

More Examples of Boolean Algebras in CS

- Functions from any set S to $\mathbb{B}$; their set is denoted $\mathrm{Map}(S, \mathbb{B})$

  If $f, g : S \longrightarrow \mathbb{B}$ then
  - $(f+g) : S \longrightarrow \mathbb{B}$ is defined by $s \mapsto f(s) + g(s)$
  - $(f \cdot g) : S \longrightarrow \mathbb{B}$ is defined by $s \mapsto f(s) \cdot g(s)$
  - $f' : S \longrightarrow \mathbb{B}$ is defined by $s \mapsto (f(s))'$

  There are 2^n such functions for $|S| = n$

- All Boolean functions of n variables, e.g.

$$(p_1, p_2, p_3) \mapsto (p_1 + p_2{}') \cdot (p_1 + p_3) \cdot p_2 + p_3{}'$$

  There are 2^{2^n} of them; their collection is denoted $\textsc{Bool}(n)$

Fact

Every Boolean algebra with finite set of elements T satisfies: $|T| = 2^k$ *for some* k.

Definition

Consider

- Boolean algebra B_1 over a set S with distinct elements $0_S, 1_S$
- Boolean algebra B_2 over a set T with distinct elements $0_T, 1_T$

They are **isomorphic**, written $B_1 \simeq B_2$, if and only if there is a one-to-one correspondence $\iota : S \mapsto T$ such that

1. $\iota(0_S) = 0_T$
2. $\iota(1_S) = 1_T$
3. $\iota(s_1 + s_2) = \iota(s_1) + \iota(s_2)$ for all $s_1, s_2 \in S$
4. $\iota(s_1 \cdot s_2) = \iota(s_1) \cdot \iota(s_2)$ for all $s_1, s_2 \in S$
5. $\iota(s') = \iota(s)'$ for all $s \in S$

Fact

All algebras with the same number of elements are **isomorphic**, *i.e. "structurally similar". Therefore, studying one such algebra describes properties of all.*

A cartesian product of Boolean algebras is again a Boolean algebra. We write

$$\mathbb{B}^k = \mathbb{B} \times \ldots \times \mathbb{B}$$

The algebras mentioned above are all of this form

- n-tuples $\simeq \mathbb{B}^n$
- $\mathrm{Pow}(S) \simeq \mathbb{B}^{|S|}$
- $\mathrm{Map}(S, \mathbb{B}) \simeq \mathbb{B}^{|S|}$
- $\mathrm{BOOL}(n) \simeq \mathbb{B}^{2^n}$

NB

Boolean algebra as the calculus of two values is fundamental to computer circuits and computer programming.
Example: Encoding subsets as bit vectors.

Summary

- equivalence $\equiv$, some well-known equivalences (slides 3–4)
- satisfiable formulae, valid formulae (tautologies)
- logical entailment $\models$
- Proof methods: contrapositive, by contradiction, by cases
- Boolean algebra, CNF, DNF, canonical form, Karnaugh maps

Supplementary reading [LLM]

- Ch. 1, Sec. 1.5-1.9 (more about good proofs)
- Ch. 3, Sec. 3.3 (more about proving equivalences of formulae)

Coming up ...

- Ch. 1, Sec. 1.7 (Functions)
- Ch. 3, Sec. 3.1, 3.3 (Relations)