9

MIMO: Channel Capacity

Compared to a conventional single antenna system, the channel capacity of a multiple antenna system with N_T transmit and N_R receive antennas can be increased by the factor of $\min(N_T, N_R)$, without using additional transmit power or spectral bandwidth. Due to the ever increasing demand of faster data transmission speed in the recent or future telecommunication systems, the multiple antenna systems have been actively investigated [210, 211] and successfully deployed for the emerging broadband wireless access networks (e.g., Mobile WiMAX) [212].

Even when a wireless channel with high channel capacity is given, we still need to find good techniques to achieve high-speed data transmission or high reliability. Multiple antenna techniques can be broadly classified into two categories: diversity techniques and spatial-multiplexing techniques [213]. The diversity techniques intend to receive the same information-bearing signals in the multiple antennas or to transmit them from multiple antennas, thereby improving the transmission reliability [214, 215]. A basic idea of the diversity techniques is to convert Rayleigh fading wireless channel into more stable AWGN-like channel without any catastrophic signal fading. We will address the diversity techniques in Chapter 10. In the spatial–multiplexing techniques, on the other hand, the multiple independent data streams are simultaneously transmitted by the multiple transmit antennas, thereby achieving a higher transmission speed. We will address the spatial-multiplexing techniques in Chapter 11. When the spatial-multiplexing techniques are used, the maximum achievable transmission speed can be the same as the capacity of the MIMO channel; however, when the diversity techniques are used, the achievable transmission speed can be much lower than the capacity of the MIMO channel [216].

In this chapter, we discuss the capacity of the MIMO wireless channel. First, we address useful matrix identities that are frequently used in the expression of the corresponding capacity. In the subsequent sections, we derive the MIMO system capacities for deterministic and random channels.

9.1 Useful Matrix Theory

The matrix $\mathbf{H} \in \mathbb{C}^{N_R \times N_T}$ has a singular value decomposition (SVD), represented as

$$\mathbf{H} = \mathbf{U}\mathbf{\Sigma}\mathbf{V}^H \tag{9.1}$$

MIMO-OFDM Wireless Communications with MATLAB® Yong Soo Cho, Jaekwon Kim, Won Young Yang and Chung G. Kang
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where $\mathbf{U} \in \mathbb{C}^{N_R \times N_R}$ and $\mathbf{V} \in \mathbb{C}^{N_T \times N_T}$ are unitary matrices[1], and $\mathbf{\Sigma} \in \mathbb{C}^{N_R \times N_T}$ is a rectangular matrix, whose diagonal elements are non-negative real numbers and whose off-diagonal elements are zero. The diagonal elements of $\mathbf{\Sigma}$ are the singular values of the matrix $\mathbf{H}$, denoting them by $\sigma_1, \sigma_2, \cdots, \sigma_{N_{\min}}$, where $N_{\min} \triangleq \min(N_T, N_R)$. In fact, assume that $\sigma_1 \geq \sigma_2 \geq \ldots \geq \sigma_{N_{\min}}$, that is, the diagonal elements of $\mathbf{\Sigma}$, are the ordered singular values of the matrix $\mathbf{H}$. The rank of $\mathbf{H}$ corresponds to the number of non-zero singular values (i.e., $\operatorname{rank}(\mathbf{H}) \leq N_{\min}$). In case of $N_{\min} = N_T$, SVD in Equation (9.1) can also be expressed as

$$\begin{aligned}
\mathbf{H} &= \mathbf{U}\mathbf{\Sigma}\mathbf{V}^H \\
&= \underbrace{[\mathbf{U}_{N_{\min}}\ \mathbf{U}_{N_R - N_{\min}}]}_{\mathbf{U}} \underbrace{\begin{bmatrix} \mathbf{\Sigma}_{N_{\min}} \\ \mathbf{0}_{N_R - N_{\min}} \end{bmatrix}}_{\mathbf{\Sigma}} \mathbf{V}^H \\
&= \mathbf{U}_{N_{\min}} \mathbf{\Sigma}_{N_{\min}} \mathbf{V}^H
\end{aligned} \tag{9.2}$$

where $\mathbf{U}_{N_{\min}} \in \mathbb{C}^{N_R \times N_{\min}}$ is composed of $N_{\min}$ left-singular vectors corresponding to the maximum possible nonzero singular values, and $\mathbf{\Sigma}_{N_{\min}} \in \mathbb{C}^{N_{\min} \times N_{\min}}$ is now a square matrix. Since $N_{\min}$ singular vectors in $\mathbf{U}_{N_{\min}}$ are of length N_R, there always exist $(N_R - N_{\min})$ singular vectors such that $[\mathbf{U}_{N_{\min}}\ \mathbf{U}_{N_R - N_{\min}}]$ is unitary. In case of $N_{\min} = N_R$, SVD in Equation (9.1) can be expressed as

$$\begin{aligned}
\mathbf{H} &= \mathbf{U} \underbrace{[\mathbf{\Sigma}_{N_{\min}}\ \mathbf{0}_{N_T - N_{\min}}]}_{\mathbf{\Sigma}} \underbrace{\begin{bmatrix} \mathbf{V}^H_{N_{\min}} \\ \mathbf{V}^H_{N_T - N_{\min}} \end{bmatrix}}_{\mathbf{V}^H} \\
&= \mathbf{U}\mathbf{\Sigma}_{N_{\min}} \mathbf{V}^H_{N_{\min}}
\end{aligned} \tag{9.3}$$

where $\mathbf{V}_{N_{\min}} \in \mathbb{C}^{N_T \times N_{\min}}$ is composed of $N_{\min}$ right-singular vectors. Given SVD of $\mathbf{H}$, the following eigen-decomposition holds:

$$\mathbf{H}\mathbf{H}^H = \mathbf{U}\mathbf{\Sigma}\mathbf{\Sigma}^H\mathbf{U}^H = \mathbf{Q}\mathbf{\Lambda}\mathbf{Q}^H \tag{9.4}$$

where $\mathbf{Q} = \mathbf{U}$ such that $\mathbf{Q}^H\mathbf{Q} = \mathbf{I}_{N_R}$, and $\mathbf{\Lambda} \in \mathbb{C}^{N_R \times N_R}$ is a diagonal matrix with its diagonal elements given as

$$\lambda_i = \begin{cases} \sigma_i^2, & \text{if } i = 1, 2, \cdots, N_{\min} \\ 0, & \text{if } i = N_{\min} + 1, \cdots, N_R. \end{cases} \tag{9.5}$$

As the diagonal elements of $\mathbf{\Lambda}$ in Equation (9.4) are eigenvalues $\{\lambda_i\}_{i=1}^{N_R}$, Equation (9.5) indicates that the squared singular values $\{\sigma_i^2\}$ for $\mathbf{H}$ are the eigenvalues of the Hermitian symmetric matrix $\mathbf{H}\mathbf{H}^H$, or similarly, of $\mathbf{H}^H\mathbf{H}$.

[1] Recall that a unitary matrix $\mathbf{U}$ satisfies $\mathbf{U}^H\mathbf{U} = \mathbf{I}_{N_R}$ where $\mathbf{I}_{N_R}$ is an $N_R \times N_R$ identity matrix.

For a non-Hermitian square matrix $\mathbf{H} \in \mathbb{C}^{n\times n}$ (or non-symmetric real matrix), the eigen-decomposition is expressed as

$$\mathbf{H}\underbrace{[\mathbf{x}_1\ \mathbf{x}_2 \cdots \mathbf{x}_n]}_{\mathbf{X}} = \underbrace{[\mathbf{x}_1\ \mathbf{x}_2 \cdots\ \mathbf{x}_n]}_{\mathbf{X}}\boldsymbol{\Lambda}_{\text{non-}H} \tag{9.6}$$

or equivalently,

$$\mathbf{H} = \mathbf{X}\boldsymbol{\Lambda}_{\text{non-}H}\mathbf{X}^{-1} \tag{9.7}$$

where $\{\mathbf{x}_i\}_{i=1}^{n} \in \mathbb{C}^{n\times 1}$ are the right-side eigenvectors corresponding to eigenvalues in $\boldsymbol{\Lambda}_{\text{non-}H} \in \mathbb{C}^{n\times n}$. In Equation (9.7), linear independence of the eigenvectors is assumed. Comparing Equation (9.4) to Equation (9.7), it can be seen that the eigenvectors of a non-Hermitian matrix $\mathbf{H} \in \mathbb{C}^{n\times n}$ are not orthogonal, while those of a Hermitian matrix $\mathbf{H}\mathbf{H}^H$ are orthonormal (i.e., $\mathbf{Q}^{-1} = \mathbf{Q}^H$).

Meanwhile, the squared Frobenius norm of the MIMO channel is interpreted as a total power gain of the channel, that is,

$$\|\mathbf{H}\|_F^2 = \text{Tr}(\mathbf{H}\mathbf{H}^H) = \sum_{i=1}^{N_R}\sum_{j=1}^{N_T}|h_{i,j}|^2. \tag{9.8}$$

Using Equation (9.4), the squared Frobenius norm in Equation (9.8) can also be represented in various ways as follows:

$$\begin{aligned}\|\mathbf{H}\|_F^2 &= \left\|\mathbf{Q}^H\mathbf{H}\right\|_F^2 \\ &= \text{Tr}\left(\mathbf{Q}^H\mathbf{H}\mathbf{H}^H\mathbf{Q}\right) \\ &= \text{Tr}\left(\mathbf{Q}^H\mathbf{Q}\boldsymbol{\Lambda}\mathbf{Q}^H\mathbf{Q}\right) \\ &= \text{Tr}(\boldsymbol{\Lambda}) \\ &= \sum_{i=1}^{N_{\min}}\lambda_i \\ &= \sum_{i=1}^{N_{\min}}\sigma_i^2\end{aligned} \tag{9.9}$$

In deriving Equation (9.9), we have used the fact that the Frobenious norm of a matrix does not change by multiplication with a unitary matrix.

9.2 Deterministic MIMO Channel Capacity

For a MIMO system with N_T transmit and N_R receive antennas, as shown in Figure 9.1, a narrowband time-invariant wireless channel can be represented by $N_R \times N_T$ deterministic matrix $\mathbf{H} \in \mathbb{C}^{N_R\times N_T}$. Consider a transmitted symbol vector $\mathbf{x} \in \mathbb{C}^{N_T\times 1}$, which is composed of N_T independent input symbols $x_1, x_2, \cdots, x_{N_T}$. Then, the received signal $\mathbf{y} \in \mathbb{C}^{N_R\times 1}$ can be rewritten in a matrix form as follows:

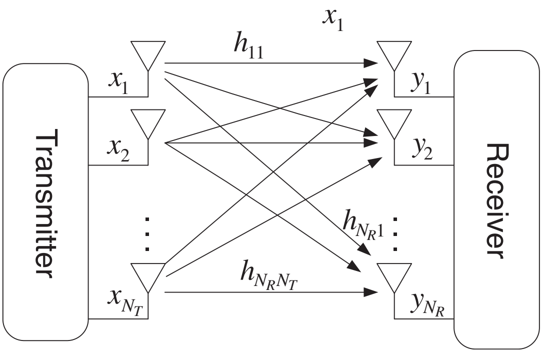

Figure 9.1 $N_R \times N_T$ MIMO system.

$$\mathbf{y} = \sqrt{\frac{\mathsf{E}_x}{N_T}}\mathbf{Hx} + \mathbf{z} \tag{9.10}$$

where $\mathbf{z} = (z_1, z_2, \cdots, z_{N_R})^T \in \mathbb{C}^{N_R \times 1}$ is a noise vector, which is assumed to be zero-mean *circular symmetric* complex Gaussian (ZMCSCG). Note that the noise vector $\mathbf{z}$ is referred to as *circular symmetric* when $e^{j\theta}\mathbf{z}$ has the same distribution as $\mathbf{z}$ for any θ. The autocorrelation of transmitted signal vector is defined as

$$\mathbf{R}_{xx} = E\{\mathbf{xx}^H\}. \tag{9.11}$$

Note that $\mathrm{Tr}(\mathbf{R}_{xx}) = N_T$ when the transmission power for each transmit antenna is assumed to be 1.

9.2.1 Channel Capacity when CSI is Known to the Transmitter Side

The capacity of a deterministic channel is defined as

$$C = \max_{f(\mathbf{x})} I(\mathbf{x}; \mathbf{y}) \text{ bits/channel use} \tag{9.12}$$

in which $f(\mathbf{x})$ is the probability density function (PDF) of the transmit signal vector $\mathbf{x}$, and $I(\mathbf{x}; \mathbf{y})$ is the mutual information of random vectors $\mathbf{x}$ and $\mathbf{y}$. Namely, the channel capacity is the maximum mutual information that can be achieved by varying the PDF of the transmit signal vector. From the fundamental principle of the information theory, the mutual information of the two continuous random vectors, $\mathbf{x}$ and $\mathbf{y}$, is given as

$$I(\mathbf{x}; \mathbf{y}) = H(\mathbf{y}) - H(\mathbf{y}|\mathbf{x}) \tag{9.13}$$

in which $H(\mathbf{y})$ is the differential entropy of $\mathbf{y}$ and $H(\mathbf{y}|\mathbf{x})$ is the conditional differential entropy of $\mathbf{y}$ when $\mathbf{x}$ is given. Using the statistical independence of the two random vectors $\mathbf{z}$ and $\mathbf{x}$ in Equation (9.10), we can show the following relationship:

$$H(\mathbf{y}|\mathbf{x}) = H(\mathbf{z}) \tag{9.14}$$

Using Equation (9.14), we can express Equation (9.13) as

$$I(\mathbf{x};\mathbf{y}) = H(\mathbf{y}) - H(\mathbf{z}) \tag{9.15}$$

From Equation (9.15), given that $H(\mathbf{z})$ is a constant, we can see that the mutual information is maximized when $H(\mathbf{y})$ is maximized. Using Equation (9.10), meanwhile, the auto-correlation matrix of $\mathbf{y}$ is given as

$$\begin{aligned}
\mathbf{R}_{yy} &= E\{\mathbf{y}\mathbf{y}^H\} = E\left\{\left(\sqrt{\frac{\mathsf{E}_x}{N_T}}\mathbf{H}\mathbf{x} + \mathbf{z}\right)\left(\sqrt{\frac{\mathsf{E}_x}{N_T}}\mathbf{x}^H\mathbf{H}^H + \mathbf{z}^H\right)\right\} \\
&= E\left\{\left(\frac{\mathsf{E}_x}{N_T}\mathbf{H}\mathbf{x}\mathbf{x}^H\mathbf{H}^H + \mathbf{z}\mathbf{z}^H\right)\right\} \\
&= \frac{\mathsf{E}_x}{N_T}E\{\mathbf{H}\mathbf{x}\mathbf{x}^H\mathbf{H}^H + \mathbf{z}\mathbf{z}^H\} \\
&= \frac{\mathsf{E}_x}{N_T}\mathbf{H}E\{\mathbf{x}\mathbf{x}^H\}\mathbf{H}^H + E\{\mathbf{z}\mathbf{z}^H\} \\
&= \frac{\mathsf{E}_x}{N_T}\mathbf{H}\mathbf{R}_{xx}\mathbf{H}^H + \mathsf{N}_0\mathbf{I}_{N_R}
\end{aligned} \tag{9.16}$$

where E_x is the energy of the transmitted signals, and N_0 is the power spectral density of the additive noise $\{z_i\}_{i=1}^{N_R}$. The differential entropy $H(\mathbf{y})$ is maximized when $\mathbf{y}$ is ZMCSCG, which consequently requires $\mathbf{x}$ to be ZMCSCG as well. Then, the mutual information of $\mathbf{y}$ and $\mathbf{z}$ is respectively given as

$$\begin{aligned}
H(\mathbf{y}) &= \log_2\{\det(\pi e \mathbf{R}_{yy})\} \\
H(\mathbf{z}) &= \log_2\{\det(\pi e \mathsf{N}_0 \mathbf{I}_{N_R})\}
\end{aligned} \tag{9.17}$$

In [217], it has been shown that using Equation (9.17), the mutual information of Equation (9.15) is expressed as

$$I(\mathbf{x};\mathbf{y}) = \log_2 \det\left(\mathbf{I}_{N_R} + \frac{\mathsf{E}_x}{N_T\mathsf{N}_0}\mathbf{H}\mathbf{R}_{xx}\mathbf{H}^H\right) \text{ bps/Hz.} \tag{9.18}$$

Then, the channel capacity of deterministic MIMO channel is expressed as

$$C = \max_{Tr(\mathbf{R}_{xx})=N_T} \log_2 \det\left(\mathbf{I}_{N_R} + \frac{\mathsf{E}_x}{N_T \mathsf{N}_0}\mathbf{H}\mathbf{R}_{xx}\mathbf{H}^H\right) \text{ bps/Hz.} \tag{9.19}$$

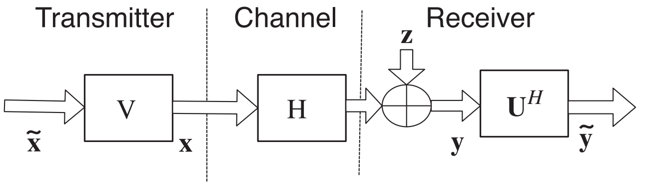

Figure 9.2 Modal decomposition when CSI is available at the transmitter side.

When channel state information (CSI) is available at the transmitter side, modal decomposition can be performed as shown in Figure 9.2, in which a transmitted signal is pre-processed with $\mathbf{V}$ in the transmitter and then, a received signal is post-processed with $\mathbf{U}^H$ in the receiver. Referring to the notations in Figure 9.2, the output signal in the receiver can be written as

$$\tilde{\mathbf{y}} = \sqrt{\frac{\mathsf{E}_x}{N_T}}\mathbf{U}^H\mathbf{H}\mathbf{V}\tilde{\mathbf{x}} + \tilde{\mathbf{z}} \tag{9.20}$$

where $\tilde{\mathbf{z}} = \mathbf{U}^H\mathbf{z}$. Using the singular value decomposition in Equation (9.1), we can rewrite Equation (9.20) as

$$\tilde{\mathbf{y}} = \sqrt{\frac{\mathsf{E}_x}{N_T}}\mathbf{\Sigma}\tilde{\mathbf{x}} + \tilde{\mathbf{z}}$$

which is equivalent to the following r virtual SISO channels, that is,

$$\tilde{y}_i = \sqrt{\frac{\mathsf{E}_x}{N_T}}\sqrt{\lambda_i}\tilde{x}_i + \tilde{z}_i, \quad i = 1, 2, \cdots, r. \tag{9.21}$$

The above equivalent representation can be illustrated as in Figure 9.3. If the transmit power for the ith transmit antenna is given by $\gamma_i = E\{|x_i|^2\}$, the capacity of the ith virtual SISO channel is

$$C_i(\gamma_i) = \log_2\left(1 + \frac{\mathsf{E}_x\gamma_i}{N_T\mathsf{N}_0}\lambda_i\right), \quad i = 1, 2, \cdots, r. \tag{9.22}$$

Assume that total available power at the transmitter is limited to

$$E\{\mathbf{x}^H\mathbf{x}\} = \sum_{i=1}^{N_T} E\left\{|x_i|^2\right\} = N_T. \tag{9.23}$$

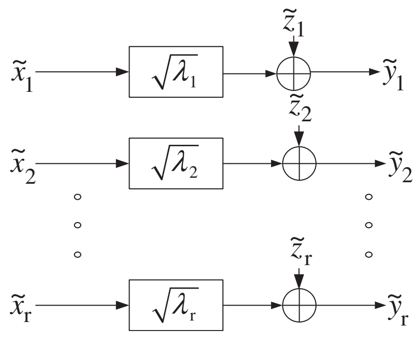

Figure 9.3 The r virtual SISO channels obtained from the modal decomposition of a MIMO channel.

The MIMO channel capacity is now given by a sum of the capacities of the virtual SISO channels, that is,

$$C = \sum_{i=1}^{r} C_i(\gamma_i) = \sum_{i=1}^{r} \log_2\left(1 + \frac{\mathsf{E}_x \gamma_i}{N_T \mathsf{N}_0}\lambda_i\right) \tag{9.24}$$

where the total power constraint in Equation (9.23) must be satisfied. The capacity in Equation (9.24) can be maximized by solving the following power allocation problem:

$$C = \max_{\{\gamma_i\}} \sum_{i=1}^{r} \log_2\left(1 + \frac{\mathsf{E}_x \gamma_i}{N_T \mathsf{N}_0}\lambda_i\right) \tag{9.25}$$

subject to $\sum_{i=1}^{r} \gamma_i = N_T$.

It can be shown that a solution to the optimization problem in Equation (9.25) is given as

$$\gamma_i^{opt} = \left(\mu - \frac{N_T \mathsf{N}_0}{\mathsf{E}_x \lambda_i}\right)^+, \quad i = 1, \cdots, r \tag{9.26}$$

$$\sum_{i=1}^{r} \gamma_i^{opt} = N_T. \tag{9.27}$$

where μ is a constant and $(x)^+$ is defined as

$$(x)^+ = \begin{cases} x & \text{if } x \geq 0 \\ 0 & \text{if } x < 0 \end{cases}. \tag{9.28}$$

The above solution in Equation (9.26) satisfying the constraint in Equation (9.27) is the well-known *water-pouring* power allocation algorithm, which is illustrated in Figure 9.4 (also, refer to Section 4.2.5). It addresses the fact that more power must be allocated to the mode with

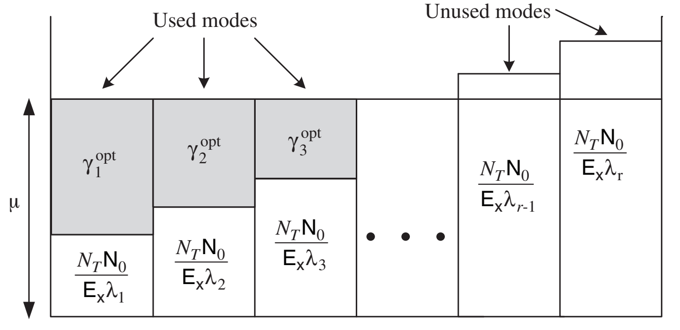

Figure 9.4 Water-pouring power allocation algorithm.

higher SNR. Furthermore, if an SNR is below the threshold given in terms of μ, the corresponding modes must not be used, that is, no power allocated to them.

9.2.2 Channel Capacity when CSI is Not Available at the Transmitter Side

When $\mathbf{H}$ is not known at the transmitter side, one can spread the energy equally among all the transmit antennas, that is, the autocorrelation function of the transmit signal vector $\mathbf{x}$ is given as

$$\mathbf{R}_{xx} = \mathbf{I}_{N_T} \tag{9.29}$$

In this case, the channel capacity is given as

$$C = \log_2 \det\left(\mathbf{I}_{N_R} + \frac{E_x}{N_T N_0}\mathbf{H}\mathbf{H}^H\right). \tag{9.30}$$

Using the eigen-decomposition $\mathbf{H}\mathbf{H}^H = \mathbf{Q}\boldsymbol{\Lambda}\mathbf{Q}^H$ and the identity $\det(\mathbf{I}_m + \mathbf{AB}) = \det(\mathbf{I}_n + \mathbf{BA})$, where $\mathbf{A} \in \mathbb{C}^{m\times n}$ and $\mathbf{B} \in \mathbb{C}^{n\times m}$, the channel capacity in Equation (9.30) is expressed as

$$\begin{aligned} C &= \log_2 \det\left(\mathbf{I}_{N_R} + \frac{E_x}{N_T N_0}\mathbf{Q}\boldsymbol{\Lambda}\mathbf{Q}^H\right) = \log_2 \det\left(\mathbf{I}_{N_R} + \frac{E_x}{N_T N_0}\boldsymbol{\Lambda}\right) \\ &= \sum_{i=1}^{r} \log_2\left(1 + \frac{E_x}{N_T N_0}\lambda_i\right) \end{aligned} \tag{9.31}$$

where r denotes the rank of $\mathbf{H}$, that is, $r = N_{\min} \triangleq \min(N_T, N_R)$. From Equation (9.31), we can see that a MIMO channel is converted into r virtual SISO channels with the transmit power E_x/N_T for each channel and the channel gain of λ_i for the ith SISO channel. Note that the result in Equation (9.31) is a special case of Equation (9.23) with $\gamma_i = 1$, $i = 1, 2, \ldots, r$, when CSI is not available at the transmitter and thus, the total power is equally allocated to all transmit antennas.

If we assume that the total channel gain is fixed, for example, $\|\mathbf{H}\|_F^2 = \sum_{i=1}^{r} \lambda_i = \zeta$, $\mathbf{H}$ has a full rank, $N_T = N_R = N$, and $r = N$, then the channel capacity Equation (9.31) is maximized when the singular values of $\mathbf{H}$ are the same for all (SISO) parallel channels, that is,

$$\lambda_i = \frac{\zeta}{N}, \quad i = 1, 2, \cdots, N. \tag{9.32}$$

Equation (9.32) implies that the MIMO capacity is maximized when the channel is orthogonal, that is,

$$\mathbf{H}\mathbf{H}^H = \mathbf{H}^H\mathbf{H} = \frac{\zeta}{N}\mathbf{I}_N \tag{9.33}$$

which leads its capacity to N times that of each parallel channel, that is,

$$C = N \log_2\left(1 + \frac{\zeta \mathsf{E}_x}{\mathsf{N}_0 N}\right). \tag{9.34}$$

9.2.3 Channel Capacity of SIMO and MISO Channels

For the case of a SIMO channel with one transmit antenna and N_R receive antennas, the channel gain is given as $\mathbf{h} \in \mathbb{C}^{N_R \times 1}$, and thus $r = 1$ and $\lambda_1 = \|\mathbf{h}\|_F^2$. Consequently, regardless of the availability of CSI at the transmitter side, the channel capacity is given as

$$C_{SIMO} = \log_2\left(1 + \frac{\mathsf{E}_x}{\mathsf{N}_0}\|\mathbf{h}\|_F^2\right). \tag{9.35}$$

If $|h_i|^2 = 1$, $i = 1, 2, \cdots, N_R$, and consequently $\|\mathbf{h}\|_F^2 = N_R$, the capacity is given as

$$C_{SIMO} = \log_2\left(1 + \frac{\mathsf{E}_x}{\mathsf{N}_0} N_R\right). \tag{9.36}$$

From Equation (9.36), we can see that the channel capacity increases logarithmically as the number of antennas increases. We can also see that only a single data stream can be transmitted and that the availability of CSI at the transmitter side does not improve the channel capacity at all.

For the case of a MISO channel, the channel gain is given as $\mathbf{h} \in \mathbb{C}^{1 \times N_T}$, thus $r = 1$ and $\lambda_1 = \|\mathbf{h}\|_F^2$. When CSI is not available at the transmitter side, the channel capacity is given as

$$C_{MISO} = \log_2\left(1 + \frac{\mathsf{E}_x}{N_T \mathsf{N}_0}\|\mathbf{h}\|_F^2\right). \tag{9.37}$$

If $|h_i|^2 = 1$, $i = 1, 2, \cdots, N_T$, and consequently $\|\mathbf{h}\|_F^2 = N_T$, Equation (9.37) reduces to

$$C_{MISO} = \log_2\left(1 + \frac{\mathsf{E}_x}{\mathsf{N}_0}\right). \tag{9.38}$$

From Equation (9.38), we can see that the capacity is the same as that of a SISO channel. One might ask what the benefit of multiple transmit antennas is when the capacity is the same as that of a single transmit antenna system. Although the maximum achievable transmission speeds of the two systems are the same, there are various ways to utilize the multiple antennas, for example, the space-time coding technique, which improves the transmission reliability as will be addressed in Chapter 10.

When CSI is available at the transmitter side (i.e., $\mathbf{h}$ is known), the transmit power can be concentrated on that particular mode of the current channel. In other words, $(\mathbf{h}^H/||\mathbf{h}||)x$ is transmitted instead of x directly. Then the received signal can be expressed as

$$y = \sqrt{\mathsf{E}_x}\mathbf{h} \cdot \frac{\mathbf{h}^H}{||\mathbf{h}||}x + z = \sqrt{\mathsf{E}_x}||\mathbf{h}||x + z \tag{9.39}$$

Note that the received signal power has been increased by N_T times in Equation (9.39) and thus, the channel capacity is given as

$$C_{MISO} = \log_2\left(1 + \frac{\mathsf{E}_x}{\mathsf{N}_0}\|\mathbf{h}\|_F^2\right) = \log_2\left(1 + \frac{\mathsf{E}_x}{\mathsf{N}_0}N_T\right). \tag{9.40}$$

9.3 Channel Capacity of Random MIMO Channels

In Section 9.2, we have assumed that MIMO channels are deterministic. In general, however, MIMO channels change randomly. Therefore, $\mathbf{H}$ is a random matrix, which means that its channel capacity is also randomly time-varying. In other words, the MIMO channel capacity can be given by its time average. In practice, we assume that the random channel is an *ergodic*[2] process. Then, we should consider the following statistical notion of the MIMO channel capacity:

$$\overline{C} = E\{C(\mathbf{H})\} = E\left\{\max_{\mathrm{Tr}(\mathbf{R}_{xx})=N_T} \log_2 \det\left(\mathbf{I}_{N_R} + \frac{\mathsf{E}_x}{N_T \mathsf{N}_0}\mathbf{H}\mathbf{R}_{xx}\mathbf{H}^H\right)\right\} \tag{9.41}$$

which is frequently known as an ergodic channel capacity. For example, the ergodic channel capacity for the open-loop system without using CSI at the transmitter side, from Equation (9.31), is given as

[2] A random process is ergodic if its time average converges to the same limit for almost all realizations of the process, for example, for a discrete random process $X[n]$, $1N\sum n = 1NX[n] \rightarrow EX[n]$ as $N \rightarrow \infty$.

$$\overline{C_{OL}} = E\left\{\sum_{i=1}^{r}\log_2\left(1+\frac{\mathsf{E}_x}{N_T\mathsf{N}_0}\lambda_i\right)\right\}. \tag{9.42}$$

Similarly, the ergodic channel capacity for the closed-loop (CL) system using CSI at the transmitter side, from Equation (9.24), is given as

$$\overline{C}_{CL} = E\left\{\max_{\sum_{i=1}^{r}\gamma_i=N_T}\sum_{i=1}^{r}\log_2\left(1+\frac{\mathsf{E}_x}{N_T\mathsf{N}_0}\gamma_i\lambda_i\right)\right\} \tag{9.43}$$

$$= E\left\{\sum_{i=1}^{r}\log_2\left(1+\frac{\mathsf{E}_x}{N_T\mathsf{N}_0}\gamma_i^{opt}\lambda_i\right)\right\}. \tag{9.44}$$

Another statistical notion of the channel capacity is the outage channel capacity. Define the outage probability as

$$P_{out}(R) = \Pr(C(\mathbf{H})<R) \tag{9.45}$$

In other words, the system is said to be in outage if the decoding error probability cannot be made arbitrarily small with the transmission rate of R bps/Hz. Then, the ε-outage channel capacity is defined as the largest possible data rate such that the outage probability in Equation (9.45) is less than ε. In other words, it is corresponding to C_ε such that $P(C(\mathbf{H}) \leq C_\varepsilon) = \varepsilon$.

Using Program 9.1 (“Ergodic_Capacity_CDF.m”), we can produce the cumulative distribution function (CDF) of the capacity for the random MIMO channel when CSI is not available at the transmitter side. Figure 9.5 shows the CDFs of the random 2×2 and 4×4 MIMO channel capacities when SNR is 10dB, in which $\varepsilon = 0.01$-outage capacity is indicated. It is clear

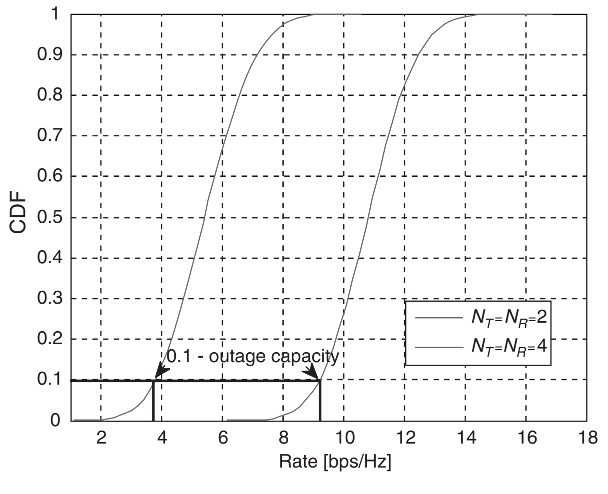

Figure 9.5 Distribution of MIMO channel capacity (SNR = 10dB; CSI is not available at the transmitter side).

from Figure 9.5 that the MIMO channel capacity improves with increasing the number of transmit and receive antennas.

MATLAB® Program: Ergodic Channel Capacity

Program 9.1 "Ergodic_Capacity_CDF.m" for ergodic capacity of MIMO channel

```
% Ergodic_Capacity_CDF.m
clear all, close all
SNR_dB=10; SNR_linear=10.^(SNR_dB/10.);
N_iter=50000; sq2=sqrt(0.5); grps = ['b:'; 'b-'];
for Icase=1:2
   if Icase==1, nT=2; nR=2; % 2x2
   else nT=4; nR=4; % 4x4
   end
   n=min(nT,nR); I = eye(n);
   for iter=1:N_iter
      H = sq2*(randn(nR,nT)+j*randn(nR,nT));
      C(iter) = log2(real(det(I+SNR_linear/nT*H'*H)));
   end
   [PDF,Rate] = hist(C,50);
   PDF = PDF/N_iter;
   for i=1:50
     CDF(Icase,i) = sum(PDF([1:i]));
   end
   plot(Rate,CDF(Icase,:),grps(Icase,:)); hold on
end
xlabel('Rate[bps/Hz]'); ylabel('CDF')
axis([1 18 0 1]); grid on; set(gca,'fontsize',10);
legend('{\it N_T}={\it N_R}=2','{\it N_T}={\it N_R}=4');
```

Using Program 9.2 ("Ergodic_Capacity_vs_SNR.m"), we can compute the ergodic capacity of the MIMO channel as SNR is varied, when CSI is not known at the transmitter side. Figure 9.6 shows the ergodic channel capacity as varying the number of antennas, under the same conditions as for Figure 9.5.

MATLAB® Program: Ergodic Channel Capacity for Various Antenna Configurations

Program 9.2 "Ergodic_Capacity_vs_SNR.m" for ergodic channel capacity vs. SNR in Figure 9.6.

```
% Ergodic_Capacity_vs_SNR.m
clear all, close all
SNR_dB=[0:5:20]; SNR_linear=10.^(SNR_dB/10);
N_iter=1000; sq2 = sqrt(0.5);
for Icase=1:5
   if Icase==1, nT=1; nR=1; % 1x1
```

```
    elseif Icase==2, nT=1; nR=2; % 1x2
    elseif Icase==3, nT=2; nR=1; % 2x1
    elseif Icase==4, nT=2; nR=2; % 2x2
    else nT=4; nR=4; % 4x4
 end
 n=min(nT,nR); I = eye(n);
 C(Icase,:) = zeros(1,length(SNR_dB));
 for iter=1:N_iter
    H = sq2*(randn(nR,nT)+j*randn(nR,nT));
    if nR>=nT, HH = H'*H; else HH = H*H'; end
    for i=1:length(SNR_dB) % Random channel generation
       C(Icase,i) = C(Icase,i)+log2(real(det(I+SNR_linear(i)/nT*HH)));
    end
  end
end
C = C/N_iter;
plot(SNR_dB,C(1,:),'b-o', SNR_dB,C(2,:),'b-', SNR_dB,C(3,:),'b-s');
hold on, plot(SNR_dB,C(4,:),'b->', SNR_dB,C(5,:),'b-^');
xlabel('SNR[dB]'); ylabel('bps/Hz');
```

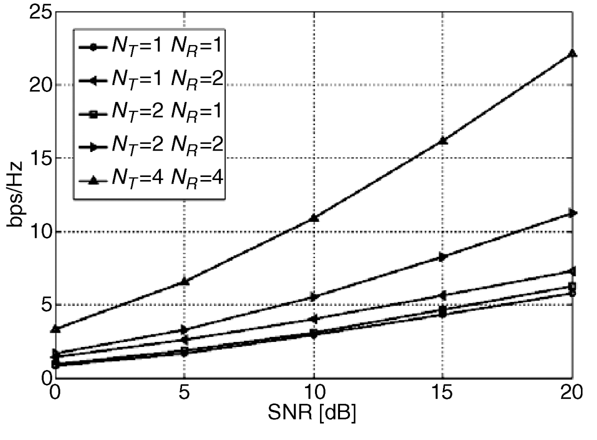

Figure 9.6 Ergodic MIMO channel capacity when CSI is not available at the transmitter.

Using Programs 9.3 ("OL_CL_Comparison.m") and Program 9.4 ("Water_Pouring"), the ergodic capacities for the closed-loop and open-loop systems are computed and compared. Figure 9.7 compares the ergodic capacities for 4×4 MIMO channels with and without using CSI at the transmitter side. It shows that the closed-loop system provides more capacity than the

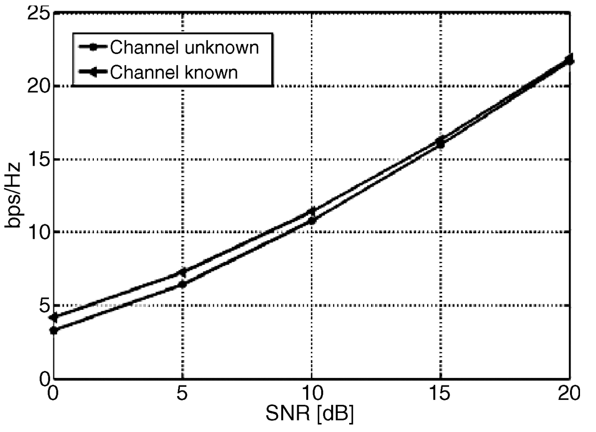

Figure 9.7 Ergodic channel capacity: $N_T = N_R = 4$.

open-loop system. However, we can see that the CSI availability does not help to improve the channel capacity when the average SNR is extremely high. It implies that even the lowest SNR mode is good enough to get almost the same transmit power allocated as the highest SNR mode, when the average SNR is extremely high.

MATLAB® Programs: Open-Loop vs. Closed-Loop MIMO Channel Capacity

Program 9.3 "OL_CL_Comparison.m" for Ergodic channel capacity: open-loop vs. closed-loop

```
%OL_CL_Comparison.m
clear all, close all;
SNR_dB=[0:5:20]; SNR_linear=10.^(SNR_dB/10.);
rho=0.2;
Rtx=[1      rho     rho^2   rho^3;
     rho     1       rho     rho^2;
     rho^2   rho     1       rho;
     rho^3   rho^2   rho     1];
Rrx=[1      rho     rho^2   rho^3;
     rho     1       rho     rho^2;
     rho^2   rho     1       rho;
     rho^3   rho^2   rho     1];
N_iter=1000;
nT=4; nR=4; n=min(nT,nR);
I = eye(n); % 4x4
```

```
sq2 = sqrt(0.5);
C_44_OL=zeros(1,length(SNR_dB));
C_44_CL=zeros(1,length(SNR_dB));
for iter=1:N_iter
    Hw = sq2*(randn(4,4) + j*randn(4,4));
    H = Rrx^(1/2)*Hw*Rtx^(1/2);
    tmp = H'*H/nT;
    SV = svd(H'*H);
    for i=1:length(SNR_dB) %random channel generation
      C_44_OL(i) = C_44_OL(i)+log2(det(I+SNR_linear(i)*tmp)); %Eq.(9.41)
      Gamma = Water_Pouring(SV,SNR_linear(i),nT);
      C_44_CL(i) = C_44_CL(i) + log2(det(I+SNR_linear(i)/nT*diag(Gamma)
                               *diag(SV))); %Eq.(9.44)
   end
end
C_44_OL = real(C_44_OL)/N_iter;
C_44_CL = real(C_44_CL)/N_iter;
plot(SNR_dB, C_44_OL,'-o', SNR_dB, C_44_CL,'-');
```

Program 9.4 “Water_Pouring” for water-pouring algorithm

```
function [Gamma]=Water_Pouring(Lamda,SNR,nT)
Gamma=zeros(1,length(Lamda));
r=length(Lamda); index=[1:r]; index_temp=index;
p=1;
while p<r
   irp=1:r-p+1; temp = sum(1./Lamda(index_temp(irp)));
   mu = nT/(r-p+1)*(1+1/SNR*temp);
   Gamma(index_temp(irp)) = mu - nT./(SNR*Lamda(index_temp(irp)));
   if min(Gamma(index_temp))<0
     i=find(Gamma==min(Gamma));
     ii=find(index_temp==i);
     index_temp2=[index_temp([1:ii-1]) index_temp([ii+1:end])];
     clear index_temp;
     index_temp=index_temp2;
     p=p+1;
     clear Gamma;
   else
     p=r;
   end
end
Gamma_t=zeros(1,length(Lamda));
Gamma_t(index_temp)=Gamma(index_temp);
Gamma=Gamma_t;
```

In general, the MIMO channel gains are not independent and identically distributed (i.i.d.). The channel correlation is closely related to the capacity of the MIMO channel. In the sequel, we consider the capacity of the MIMO channel when the channel gains between transmit and received antennas are correlated. When the SNR is high, the deterministic channel capacity can be approximated as

$$C \approx \max_{\mathrm{Tr}(\mathbf{R}_{xx})=N} \log_2 \det(\mathbf{R}_{xx}) + \log_2 \det\left(\frac{\mathsf{E}_x}{N\mathsf{N}_0}\mathbf{H}_w\mathbf{H}_w^H\right) \tag{9.46}$$

From Equation (9.46), we can see that the second term is constant, while the first term involving $\det(\mathbf{R}_{xx})$ is maximized when $\mathbf{R}_{xx} = I_N$. Consider the following correlated channel model:

$$\mathbf{H} = \mathbf{R}_r^{1/2}\mathbf{H}_w\mathbf{R}_t^{1/2} \tag{9.47}$$

where $\mathbf{R}_t$ is the correlation matrix, reflecting the correlations between the transmit antennas (i.e., the correlations between the column vectors of $\mathbf{H}$), $\mathbf{R}_r$ is the correlation matrix reflecting the correlations between the receive antennas (i.e., the correlations between the row vectors of $\mathbf{H}$), and $\mathbf{H}_w$ denotes the i.i.d. Rayleigh fading channel gain matrix. The diagonal entries of $\mathbf{R}_t$ and $\mathbf{R}_r$ are constrained to be a unity. From Equation (9.30), then, the MIMO channel is given as

$$C = \log_2 \det\left(\mathbf{I}_{N_R} + \frac{\mathsf{E}_x}{N_T\mathsf{N}_0}\mathbf{R}_r^{1/2}\mathbf{H}_w\mathbf{R}_t\mathbf{H}_w^H\mathbf{R}_r^{H/2}\right). \tag{9.48}$$

If $N_T = N_R = N$, $\mathbf{R}_r$ and $\mathbf{R}_t$ are of full rank, and SNR is high, Equation (9.48) can be approximated as

$$C \approx \log_2 \det\left(\frac{\mathsf{E}_x}{N_T\mathsf{N}_0}\mathbf{H}_w\mathbf{H}_w^H\right) + \log_2 \det(\mathbf{R}_r) + \log_2 \det(\mathbf{R}_t). \tag{9.49}$$

We find from Equation (9.49) that the MIMO channel capacity has been reduced, and the amount of capacity reduction (in bps) due to the correlation between the transmit and receive antennas is

$$\log_2 \det(\mathbf{R}_r) + \log_2 \det(\mathbf{R}_t). \tag{9.50}$$

In the sequel, it is shown that the value in Equation (9.50) is always negative by the fact that $\log_2 \det(\mathbf{R}) \le 0$ for any correlation matrix $\mathbf{R}$. Since $\mathbf{R}$ is a symmetric matrix, eigen-decomposition in Equation (9.2) is applicable, that is, $\mathbf{R} = \mathbf{Q}\boldsymbol{\Lambda}\mathbf{Q}^H$. Since the determinant of a unitary matrix is unity, the determinant of a correlation matrix can be expressed as

$$\det(\mathbf{R}) = \prod_{i=1}^{N} \lambda_i. \tag{9.51}$$

Note that the geometric mean is bounded by the arithmetic mean, that is,

$$\left(\Pi_{i=1}^{N}\lambda_i\right)^{\frac{1}{N}} \leq \frac{1}{N}\sum_{i=1}^{N}\lambda_i = 1. \tag{9.52}$$

From Equations (9.51) and (9.52), it is obvious that

$$\log_2 \det(\mathbf{R}) \leq 0 \tag{9.53}$$

The equality in Equation (9.53) holds when the correlation matrix is the identity matrix. Therefore, the quantities in Equation (9.50) are all negative.

Program 9.5 computes the ergodic MIMO channel capacity when there exists a correlation between the transmit and receive antennas, with the following channel correlation matrices: $\mathbf{R}_r = \mathbf{I}_4$ and

$$\mathbf{R}_t = \begin{bmatrix} 1 & 0.76e^{j0.17\pi} & 0.43e^{j0.35\pi} & 0.25e^{j0.53\pi} \\ 0.76e^{-j0.17\pi} & 1 & 0.76e^{j0.17\pi} & 0.43e^{j0.35\pi} \\ 0.43e^{-j0.35\pi} & 0.76e^{-j0.17\pi} & 1 & 0.76e^{j0.17\pi} \\ 0.25e^{-j0.53\pi} & 0.43e^{-j0.35\pi} & 0.76e^{-j0.17\pi} & 1 \end{bmatrix} \tag{9.54}$$

$\mathbf{R}_r = \mathbf{I}_4$ states that no correlation exists between the receive antennas. Figure 9.8 has been generated by Program 9.5, from which it can be shown that a capacity of 3.3 bps/Hz is lost due to the channel correlation when SNR is 18dB.

MATLAB® Program: Ergodic MIMO Capacity for Correlated Channel

Program 9.5 "Ergodic_Capacity_Correlation.m:" Channel capacity reduction due to correlation

```
% Ergodic_Capacity_Correlation.m
% Capacity reduction due to correlation of MIMO channels (Fig. 9.8)
clear all, close all;
SNR_dB=[0:5:20]; SNR_linear=10.^(SNR_dB/10.);
N_iter=1000; N_SNR=length(SNR_dB);
nT=4; nR=4; n=min(nT,nR); I = eye(n); sq2=sqrt(0.5); % 4x
R=[1 0.76*exp(0.17j*pi) 0.43*exp(0.35j*pi) 0.25*exp(0.53j*pi);
    0.76*exp(-0.17j*pi) 1 0.76*exp(0.17j*pi) 0.43*exp(0.35j*pi);
    0.43*exp(-0.35j*pi) 0.76*exp(-0.17j*pi) 1 0.76*exp(0.17j*pi);
    0.25*exp(-0.53j*pi) 0.43*exp(-0.35j*pi) 0.76*exp(-0.17j*pi) 1]; %(9.54)
C_44_iid=zeros(1,N_SNR); C_44_corr=zeros(1,N_SNR);
```

```
for iter=1:N_iter
   H_iid = sq2*(randn(nR,nT)+j*randn(nR,nT));
   H_corr = H_iid*R^(1/2);
   tmp1 = H_iid'*H_iid/nT; tmp2 = H_corr'*H_corr/nT;
   for i=1:N_SNR % Eq.(9.48)
      C_44_iid(i) = C_44_iid(i) + log2(det(I+SNR_linear(i)*tmp1));
      C_44_corr(i) = C_44_corr(i) + log2(det(I+SNR_linear(i)*tmp2));
   end
end
C_44_iid = real(C_44_iid)/N_iter;
C_44_corr = real(C_44_corr)/N_iter;
plot(SNR_dB,C_44_iid, SNR_dB,C_44_corr,':');
```

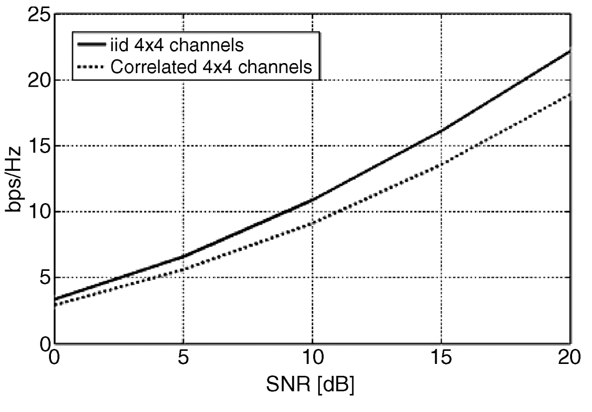

Figure 9.8 Capacity reduction due to the channel correlation.