An Optimal Two Transmit Antenna Space–Time Code and its Stacked Extensions

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***Abstract*—A space–time code is proposed that exhibits the highest coding gain among competing full-rate full transmit diversity space–time codes for the two transmit and receive antenna coherent quasi-static fading channel. The proposed code is derived from a layered architecture with real rotation of quadrature amplitude modulation (QAM) information symbols in two dimensions. The existing codes of similar architecture concentrate on application of complex full modulation diversity rotations or asymmetric real rotations. An analytic evaluation illustrates the significant improvement in coding gain achieved with the proposed code. Moreover, the coding gain of the proposed code is independent of its rate. This implies that the proposed code achieves the optimal diversity–multiplexing tradeoff curve for the two transmit antenna system. A stacked extension of the proposed code offers a reduced complexity capacity optimal alternative to the full diversity codes for larger number of transmit antennas. Performance enhancement in several scenarios is verified through simulations.**

***Index Terms*—Capacity optimal codes, coding gain, diversity–multiplexing tradeoff, fading channel, real rotation, space–time coding, transmit diversity.**

I. Introduction

The pioneering Alamouti space–time code [1] for the two transmit antenna coherent communication system remains unmatched in performance when only one receive antenna is employed. However, as the number of receive antennas is increased, the limitation of the Alamouti scheme due to loss of mutual information is well known [2].

In [3], a number-theoretic construction of a space–time code for two transmit antennas, called the $B_{2,\phi}$ code, provided a scheme that did not suffer from an information loss due to the space–time code constraints. The $B_{2,\phi}$ code exhibits full diversity and transmits at the full rate of two information symbols per channel use as opposed to one symbol per channel use with the Alamouti design. For more than one receive antenna, the superior performance of the $B_{2,\phi}$ code compared to the Alamouti scheme was shown by simulations. The $B_{2,\phi}$ code was subsumed as a special case of a wider class of full rate full diversity space–time codes for arbitrary number of transmit antennas called the Threaded Algebraic Space–Time (TAST) codes [4].

While this correspondence was under review, a code called the *tilted*-QAM code was proposed in [5] that not only achieves full rate and full diversity with two transmit antennas but is also optimal with respect to the diversity order attained when the rate of the code is increased with the signal-to-noise ratio. In other words, the *tilted*-QAM code achieves the optimal diversity–multiplexing tradeoff curve for the multiple-antenna channel derived in [6].

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In this correspondence, we propose a new information lossless or *capacity optimal* full diversity code that exhibits a significant improvement in a theoretical measure of *coding gain* compared to the $B_{2,\phi}$, the TAST, and the *tilted*-QAM codes for two transmit antennas. The proposed code, referred to as the $\mathcal{C}_r$ code, also achieves the optimal diversity–multiplexing tradeoff curve. Moreover, the higher coding gain of the proposed space–time code leads to a discernible improvement in error probability performance over the existing codes for high-rate wireless communications.

The concept of stacked extensions of a space–time code is introduced to design space–time codes for larger number of transmit antennas and time slots. The stacked extensions inherit several desirable properties of the constituent space–time code, such as capacity optimality and optimized coding gain, and can also be used for complexity reduction.

During the review of this correspondence, a manuscript on the "Golden Code" $\mathcal{C}_g$ [7] was submitted (the conference version of this paper also precedes that of [7]). However, the Golden code is equivalent to our $\mathcal{C}_r$ code obtained earlier in the sense that $\mathcal{C}_g = \boldsymbol{D}_1\mathcal{C}_r\boldsymbol{D}_2$ for two complex diagonal and unitary matrices $\boldsymbol{D}_1$ and $\boldsymbol{D}_2$. Hence, the performances of the these two codes are identical. There is a difference in the approach by which these two codes were obtained; the simpler and more direct construction of the $\mathcal{C}_r$ code is arrived at using the solution of a coding gain optimization problem, while $\mathcal{C}_g$ was obtained using the machinery of algebraic number theory. Also, the code structure of $\mathcal{C}_r$ involves only real rotation matrices and may be seen as being more insightful than the structure of the code $\mathcal{C}_g$.

The organization of this correspondence is as follows. The system model and a description of the existing space–time codes are given in Section II. Stacked extensions of a space–time code are described in Section III. The new two transmit antenna code $\mathcal{C}_r$ and the associated properties are presented in Section IV. Numerical results are presented in Section V. The conclusions are given in Section VI.

II. Preliminaries

The system model and space–time code design criterion are presented in the first subsection. The layered space–time code construction is described in the second subsection.

A. System Model and Definitions

An M transmit, N receive antenna quasi-static fading channel with coherence time T is referred to as an (M, N, T) channel. In the complex baseband representation, let $\mathcal{C}$ be a space–time codebook consisting of equiprobable $M \times T$ sized complex matrices. For the matrix channel input $\sqrt{\rho} \cdot \boldsymbol{X}$, with $\boldsymbol{X} \in \mathcal{C}$, the matrix of received statistics is given by

$$\boldsymbol{Y} = \sqrt{\rho}\boldsymbol{H}\boldsymbol{X} + \boldsymbol{N} \tag{1}$$

where $\boldsymbol{H}$ is the $N \times M$ matrix of fading coefficients and $\boldsymbol{N}$ is the $N \times T$ matrix of noise samples. The entries of $\boldsymbol{H}$ and $\boldsymbol{N}$ are independent zero-mean, unit-variance complex normal random variables. Let $E(\mathcal{C})$ denote the average energy $\mathrm{E}_{\boldsymbol{X}\in\mathcal{C}}[\|\boldsymbol{X}\|^2]$ of the space–time codebook $\mathcal{C}$. To operate at an average signal-to-noise ratio of SNR at each receive antenna, the constant ρ must be set to $\frac{SNR \times T}{E(\mathcal{C})}$. We consider the design of the space–time codebook $\mathcal{C}$ for the coherent scenario with optimum maximum-likelihood decoding which requires the fading matrix $\boldsymbol{H}$ to be perfectly known at the receiver. A Chernoff bound analysis of the pairwise error probability for this model was performed in [8] which led to the following guidelines for the design of $\mathcal{C}$.

- *Rank Criterion:* The transmit diversity $d(\mathcal{C})$ of $\mathcal{C}$, defined as
$$d(\mathcal{C}) = \min_{\substack{\boldsymbol{X}_1,\boldsymbol{X}_2\in\mathcal{C}\\ \boldsymbol{X}_1\neq\boldsymbol{X}_2}} \operatorname{rank}(\boldsymbol{X}_1 - \boldsymbol{X}_2) \tag{2}$$
should be maximized.
- *Determinant Criterion:* The coding gain $\delta(\mathcal{C})$ of a code $\mathcal{C}$ is defined as
$$\delta(\mathcal{C}) = \min_{\substack{\boldsymbol{X}_1,\boldsymbol{X}_2\in\mathcal{C}\\ \operatorname{rank}(\boldsymbol{X}_1-\boldsymbol{X}_2)=d(\mathcal{C})}} \prod_{i=1}^{d(\mathcal{C})} \lambda_i[(\boldsymbol{X}_1 - \boldsymbol{X}_2)(\boldsymbol{X}_1 - \boldsymbol{X}_2)^\dagger] \tag{3}$$
where $\lambda_i[\cdot]$ denotes the ith largest eigenvalue of the argument. The coding gain $\delta(\mathcal{C})$ should be maximized for a fixed value of $E(\mathcal{C})$.

If $T = M = d(\mathcal{C})$, then $\mathcal{C}$ is said to be a square full diversity space–time code and the coding gain expression becomes
$$\delta(\mathcal{C}) = \min_{\substack{\boldsymbol{X}_1,\boldsymbol{X}_2\in\mathcal{C}\\ \boldsymbol{X}_1\neq\boldsymbol{X}_2}} |\det(\boldsymbol{X}_1 - \boldsymbol{X}_2)|^2. \tag{4}$$
Full diversity square space–time codes lead to an overall diversity order of MN for the error probability. Moreover, if $\mathcal{C}_1$ and $\mathcal{C}_2$ are two square full diversity space–time codes with $E(\mathcal{C}_1) = E(\mathcal{C}_2)$, then $\frac{\delta(\mathcal{C}_1)}{\delta(\mathcal{C}_2)}$ is referred to as the coding gain of $\mathcal{C}_1$ over $\mathcal{C}_2$ and is an indicator of the relative performance of the two codes for asymptotically high SNR. Hence, the design procedure for a square full diversity code $\mathcal{C}$ is to ensure that (4) is nonzero and also that it is as large as possible.

A linear dispersion space–time code, as proposed in [2], comprises of codewords that are obtained as a linear combination of a fixed set of dispersion matrices with the coefficients of the linear combination being the information symbols drawn from a finite complex constellation. A linear dispersion code is said to be information lossless or capacity optimal [9] if the ergodic capacity of the effective channel seen by the information symbols is T times the ergodic capacity of the original multiple-input multiple-output (MIMO) channel. This means that the maximum achievable rate by employing an outer Gaussian codebook for the information symbols in an information lossless space–time code for the (M, N, T) channel is the same as the maximum achievable rate by employing an outer Gaussian codebook on the $(M, N, 1)$ channel. The linear dispersion codes also enable optimum decoding using the efficient *sphere decoding* algorithm [10].

While a fixed-rate full-diversity code provides robustness as the SNR of the channel increases, it does not utilize the corresponding increase in capacity of the channel. The concept of diversity–multiplexing tradeoff was proposed in [6] to characterize coding schemes that achieve both diversity gain and a fixed fraction of the capacity at high SNR. For a class of coding schemes whose rate scales with SNR as $r\log(SNR)$, with r known as the multiplexing gain, the best achievable diversity order $d^*(r)$ on a coherent multiple-antenna quasi-static fading channel was found in [6] and is known as the optimal diversity–multiplexing tradeoff curve. The optimal diversity–multiplexing tradeoff curve is a piecewise-linear function that passes through the points $(r, (M-r)(N-r))$, for integer r with $0 \le r \le \min(M, N)$, and was shown to be achieved by a Gaussian codebook when $T \ge M + N - 1$. While two different coding schemes may achieve the optimal diversity–multiplexing tradeoff, the actual performance is expected to be better for the one with a higher coding gain.

B. Layered Space–Time Codes

The space–time codes considered in this paper are derived from full modulation diversity (FMD) lattice constellations. A lattice constellation has the FMD property if any two distinct points in the constellation differ in all the coordinates. Let $i = \sqrt{-1}$. Let $\mathcal{I}_q$ denote the standard complex quadrature amplitude modulation (QAM) constellation with q points so that
$$\mathcal{I}_q = \{a + ib : -\sqrt{q} + 1 \le a, b \le \sqrt{q} - 1, a, b \text{ odd integers}\}$$
for q being a perfect square and $\mathcal{I}_q = \{-1, 1\}$ for $q = 2$. Let $\boldsymbol{G}$ be a full-rank square generator matrix of size n. Let $\Lambda_{\boldsymbol{G}}$ be the finite lattice constellation obtained from $\boldsymbol{G}$ as $\Lambda_{\boldsymbol{G}} = \{\boldsymbol{x} = \boldsymbol{G}\boldsymbol{u} : \boldsymbol{u} \in \mathcal{I}_q^n\}$. If the lattice constellation $\Lambda_{\boldsymbol{G}}$ exhibits full modulation diversity, then the minimum product distance $\psi_{\boldsymbol{G}}$ can be defined as in [11] to be
$$\psi_{\boldsymbol{G}} = \min_{\substack{\boldsymbol{x},\boldsymbol{x}'\in\Lambda_{\boldsymbol{G}}\\ \boldsymbol{x}\neq\boldsymbol{x}'}} |\prod_{l=1}^{n} (\boldsymbol{x} - \boldsymbol{x}')_l|^2. \tag{5}$$
On the single transmit antenna Rayleigh-fading channel, diversity is achieved by transmission of lattice points in $\Lambda_{\boldsymbol{G}}$ with the coordinates perfectly interleaved at the transmitter and de-interleaved at the receiver. Among two possible generator matrices $\boldsymbol{G}'$ and $\boldsymbol{G}''$ for this scheme, each resulting in equal average energies for $\Lambda_{\boldsymbol{G}'}$ and $\Lambda_{\boldsymbol{G}''}$, it is proposed in [11], [12] to choose the one that provides a higher minimum product distance. Hence, maximization of $\psi_{\boldsymbol{G}}$ is the design criterion for the lattice generator matrix $\boldsymbol{G}$ for the single-antenna interleaved channel.

The full diversity layered space–time codes for two transmit antennas proposed in [3], [4], [13] are all obtained by forming nonoverlapping *layers* of FMD lattice constellations of dimension $n = 2$. The one-layer code is obtained by placing the coordinates of an FMD lattice point on the main diagonal of the space–time codeword and provides a rate of one symbol per channel use. In the two-layer code, an additional second layer is added by placing the coordinates of another FMD lattice point on the off-diagonal positions of the codeword. The lattice coordinates of the second layer are also multiplied by $\phi^{\frac{1}{2}}$, where the constant ϕ is known as a Diophantine number and is chosen to ensure full transmit diversity for the resulting codebook [4]. The two-layer code leads to the full rate of two symbols per channel use and, additionally, is information lossless if the generator matrix of the FMD lattice for each layer is unitary and $|\phi| = 1$.

The coding gain, as defined in (4), for the layered codes depends on the choice of the component lattice generator matrices and the Diophantine number. For the one-layer code, the coding gain expression (4) coincides with the minimum product distance (5) of the FMD lattice generator matrix. Therefore, the code design problem for the one-layer case reduces to the extensively studied problem of minimum product distance maximization [11], [12], [14]–[16]. It is proved in [15] that the two-dimensional complex unitary matrix that maximizes the minimum product distance is the cyclotomic rotation matrix $\boldsymbol{G}_2$ mentioned in [14]. For the two-layer code, the expressions in (4) and (5) differ in general. In fact, the joint design of the lattice for each layer and the Diophantine number to maximize the coding gain of the two-layer code is an open problem. An *ad hoc* approach to obtain a two-layer information lossless code with high coding gain is to use the same complex FMD rotation matrix that is optimal for the one-layer case irrespective of the choice of the Diophantine number. However, in the light of the improved code proposed in this correspondence, it will become clear that the optimal generator matrix with respect to the minimum product distance does not give rise to the best two-layer space–time code. More specifically, it will be shown that a real generator matrix, which is worse than the optimal complex generator $\boldsymbol{G}_2$ for the one-layer code, actually leads to a higher coding gain compared to the complex generator matrix $\boldsymbol{G}_2$ for the two-layer code.

From an information-theoretic point of view, the layered space–time coding scheme with QAM symbols can also achieve the optimal diversity–multiplexing tradeoff curve. A Gaussian codebook argument was shown in [6] to be insufficient in proving the achievability of the optimal diversity–multiplexing tradeoff curve for all multiplexing gains when $T < M + N - 1$. However, for $M = 2, N \geq 2$, and any $T \geq 2$, it was proved in [5] that for a coding scheme $\mathcal{C}$ whose rate R scales such that the average and peak energies satisfy[1]

$$E(\mathcal{C}) \dot{=} \max_{\boldsymbol{X} \in \mathcal{C}} \|\boldsymbol{X}\|^2 \dot{=} 2^{R/2}$$

a sufficient condition for achieving the entire optimal diversity–multiplexing tradeoff curve is

$$\min_{\boldsymbol{X}_1 \neq \boldsymbol{X}_2} |\det(\boldsymbol{X}_1 - \boldsymbol{X}_2)| \dot{\geq} 1.$$

This condition is met by the layered space–time coding schemes based on QAM symbols if the coding gain does not change with the size of the QAM constellation. The *tilted*-QAM space–time code proposed in [5] is a layered space–time code designed to meet this condition and therefore achieves the optimal diversity–multiplexing tradeoff curve. It turns out that the solution of the coding gain optimization problem proposed in this correspondence leads to a new code that also satisfies this sufficient condition for achieving the optimal diversity–multiplexing tradeoff curve. However, the new code has a higher coding gain than the *tilted*-QAM code and leads to a performance improvement in terms of symbol error rate.

III. Stacked Extensions of a Space–Time Code

In this correspondence, we introduce simple extensions of space–time codes that inherit the desirable properties of the constituent code. Any space–time code $\mathcal{C}$ can be extended by horizontal and vertical stacking to the codes $H_b(\mathcal{C})$ and $V_b(\mathcal{C})$, given by

$$\begin{aligned} H_b(\mathcal{C}) &= \{[\boldsymbol{X}_1, \dots, \boldsymbol{X}_b] : \boldsymbol{X}_j \in \mathcal{C}, 1 \leq j \leq b\} \\ V_b(\mathcal{C}) &= \{[\boldsymbol{X}_1^\top, \dots, \boldsymbol{X}_b^\top]^\top : \boldsymbol{X}_j \in \mathcal{C}, 1 \leq j \leq b\} \end{aligned} \tag{6}$$

which can be used on the (M, N, bT) and the (bM, N, T) channels, respectively. Some useful properties of the extended codes are summarized in the following proposition.

Proposition 1: For each $b \geq 1$, the stacked extensions $H_b(\mathcal{C})$ and $V_b(\mathcal{C})$ of a space–time code $\mathcal{C}$ satisfy the following properties:

- if the component code $\mathcal{C}$ is capacity optimal for the (M, N, T) channel, then the codes $H_b(\mathcal{C})$ and $V_b(\mathcal{C})$ are also capacity optimal for the (M, N, bT) and (bM, N, T) channels, respectively;
- $d(H_b(\mathcal{C})) = d(V_b(\mathcal{C})) = d(\mathcal{C})$;
- $\delta(H_b(\mathcal{C})) = \delta(V_b(\mathcal{C})) = \delta(\mathcal{C})$.

The capacity optimality property of the stacked extensions is an easy consequence of the results in [9]. Also, from (2) and (3), it follows easily that the transmit diversity and coding gain of the extended codes are equal to those of the constituent code. Hence, the stacked extensions $H_b(\mathcal{C})$ and $V_b(\mathcal{C})$ directly benefit from the improvement in the design of the component code $\mathcal{C}$. In particular, maximizing the coding gain of the stacked extensions $H_b(\mathcal{C})$ and $V_b(\mathcal{C})$, when $\mathcal{C}$ is drawn from a family of codes $\mathcal{F}$, reduces to choosing the component code $\mathcal{C}$ as the one which has the maximum coding gain in the given family $\mathcal{F}$. In Section IV, we focus on such a judicious choice of the component code $\mathcal{C}$ from a family of capacity optimal 2×2 space–time codes.

The utility of the $V_b(\mathcal{C})$ code is to provide complexity reduction for large number of transmit antennas. When $M = N = 2b, b \geq 2$, the full layer full diversity TAST code proposed in [4] requires the extreme complexity of a sphere decoder of size M^2 for optimum decoding. One can employ the L-layer TAST code with $L < M$ to reduce the sphere decoder size to LM. However, this leads to a decrease in rate and loss of the capacity optimality property. On the other hand, the proposed code $V_b(\mathcal{C})$, for a full diversity capacity optimal code $\mathcal{C}$ of size 2×2, reduces the complexity to that of a sphere decoder of size $2M$ by requiring reduced diversity but maintains the full rate and the capacity optimality property. A simulation example to show the performance gain of the vertical extension of the new code proposed in this correspondence over the reduced layer TAST codes will be presented in Section V.

IV. An Optimal Space–Time Code Construction

Following the code design criterion of maximization of the coding gain expression in (4), we pose the following problem for selecting the best among a class of information lossless space–time codes for $M = 2$.

Problem Statement: Let $\theta \in [0, 2\pi]$ and let ϕ be a complex number with $|\phi| = 1$. Let $\boldsymbol{M}_\theta$ be the two-dimensional real rotation given by

$$\boldsymbol{M}_\theta = \begin{bmatrix} \cos(\theta) & \sin(\theta) \\ -\sin(\theta) & \cos(\theta) \end{bmatrix}.$$

Consider the uncountable family $\mathcal{F}$ of space–time codes parametrized by (q, θ, ϕ) such that any code $\mathcal{C}(q, \theta, \phi) \in \mathcal{F}$ is given by

$$\mathcal{C}(q, \theta, \phi) = \left\{ \boldsymbol{X} = \begin{bmatrix} \boldsymbol{x}_1 & \phi^{\frac{1}{2}} \boldsymbol{y}_1 \\ \phi^{\frac{1}{2}} \boldsymbol{y}_2 & \boldsymbol{x}_2 \end{bmatrix} : [\boldsymbol{x}_1, \boldsymbol{x}_2]^\top = \boldsymbol{M}_\theta \boldsymbol{u}, [\boldsymbol{y}_1, \boldsymbol{y}_2]^\top = \boldsymbol{M}_\theta \boldsymbol{v}, \boldsymbol{u} \in \mathcal{I}_q^2, \boldsymbol{v} \in \mathcal{I}_q^2 \right\}. \tag{7}$$

For a fixed q, find a pair $(\hat{\theta}_q, \hat{\phi}_q)$ given by

$$(\hat{\theta}_q, \hat{\phi}_q) = \arg \sup_{\substack{(\theta,\phi) \\ d(\mathcal{C}(q,\theta,\phi))=2}} \delta(\mathcal{C}(q, \theta, \phi)), \tag{8}$$

if such a pair exists.

The following proposition summarizes the primary result of this correspondence.

Proposition 2: For all $q \geq 2$, the solution to (8) is given by

$$(\hat{\theta}_q, \hat{\phi}_q) = \left(\frac{1}{2} \tan^{-1}(2), -i \right). \tag{9}$$

The proof of Proposition 2 is divided into three parts. First, the optimality of a particular real rotation in terms of maximization of the minimum product distance is proved. This optimal real rotation is then applied in the construction of a full diversity space–time code in $\mathcal{F}$. The coding gain of any other code in $\mathcal{F}$ is then shown to be smaller than the one obtained with the optimal real rotation. Each of the three steps is described in detail in the subsections to follow.

Let the set of all possible differences of elements in $\mathcal{I}_q$ be denoted by $\mathcal{I}_{q,d}$. All the elements of $\mathcal{I}_{q,d}$ are Gaussian integers divisible by 2. Moreover, $0, 2 \in \mathcal{I}_{q,d}$ for all $q \geq 2$.

A. Optimum Real Rotation

In this subsection, we derive the optimum real rotation $\boldsymbol{M}_\theta$ that leads to the largest value of the minimum product distance $\psi_{\boldsymbol{M}_\theta}$ among all θ.

Lemma 1: The maximum value of the minimum product distance $\psi_{\boldsymbol{M}_\theta}$ is achieved with $\theta = \theta_g \triangleq \frac{1}{2} \tan^{-1}(2)$ for each $q \geq 2$. The optimum minimum product distance is $\psi_{\boldsymbol{M}_{\theta_g}} = 16|\cos(2\theta_g)|^2 = 3.2$ and is independent of q.

[1] $f \dot{=} g$ means that

$$\lim_{SNR \to \infty} \frac{\log(f)}{\log(SNR)} = \lim_{SNR \to \infty} \frac{\log(g)}{\log(SNR)}$$

and $\dot{\geq}$ is defined analogously.

Proof: The minimum product distance is given by

$$\psi_{\boldsymbol{M}_\theta} \triangleq \min_{\substack{\boldsymbol{u},\boldsymbol{u}'\in\mathcal{I}_q^2\\ \boldsymbol{u}\neq\boldsymbol{u}'}} \left|[\boldsymbol{M}_\theta(\boldsymbol{u}-\boldsymbol{u}')]_1 \times [\boldsymbol{M}_\theta(\boldsymbol{u}-\boldsymbol{u}')]_2\right|^2 \tag{10}$$

$$= \min_{\substack{u_1,u_2\in\mathcal{I}_{q,d}\\ (u_1,u_2)\neq(0,0)}} |(u_1\cos(\theta)+u_2\sin(\theta)) \times (-u_1\sin(\theta)+u_2\cos(\theta))|^2$$

$$= \min_{\substack{u_1,u_2\in\mathcal{I}_{q,d}\\ (u_1,u_2)\neq(0,0)}} |(u_2^2-u_1^2)\frac{\sin(2\theta)}{2}+u_1u_2\cos(2\theta)|^2. \tag{11}$$

Now, two upper bounds for the expression in (11) are obtained by substituting specific values for u_1 and u_2. Since $0, 2 \in \mathcal{I}_{q,d}$, one can set $u_1 = 0$, $u_2 = 2$ in (11) to get that

$$\psi_{\boldsymbol{M}_\theta} \leq |2\sin(2\theta)|^2.$$

Also, setting $u_1 = u_2 = 2$, we get that

$$\psi_{\boldsymbol{M}_\theta} \leq |4\cos(2\theta)|^2.$$

Hence, combining the two upper bounds given above, we get that

$$\psi_{\boldsymbol{M}_\theta} \leq \min\{|4\cos(2\theta)|^2, |2\sin(2\theta)|^2\} = 16 \times \min\left\{|\cos(2\theta)|^2, \frac{|\sin(2\theta)|^2}{4}\right\}. \tag{12}$$

If we define

$$f_1(\theta) = |\cos(2\theta)|^2 = \frac{1+\cos(4\theta)}{2}$$

and

$$f_2(\theta) = \frac{|\sin(2\theta)|^2}{4} = \frac{1-\cos(4\theta)}{8}$$

then both f_1 and f_2 are periodic in θ with a period of $\frac{\pi}{2}$. For $\theta \in [0, \frac{\pi}{4}]$, the maximum value of $\min(f_1(\theta), f_2(\theta))$ is obtained when $f_1(\theta) = f_2(\theta)$. The required solution $\theta_g \in [0, \frac{\pi}{4}]$ to $f_1(\theta) = f_2(\theta)$ satisfies $\cos(2\theta_g) = \frac{1}{\sqrt{5}}$ or, equivalently, $\tan(2\theta_g) = 2$. Similarly, one obtains that the maximum of $\min(f_1(\theta), f_2(\theta))$ in the interval $[\frac{\pi}{4}, \frac{\pi}{2}]$ occurs at $\theta = \frac{\pi}{2} - \theta_g$ and the maximum value is the same as that with $\theta = \theta_g$, namely, $f_1(\theta_g)$. Hence, an upper bound on (12) independent of θ is given by

$$\psi_{\boldsymbol{M}_\theta} \leq 16|\cos(2\theta_g)|^2. \tag{13}$$

The above inequality imposes a fundamental limit on the maximum achievable minimum product distance with a real rotation of the form $\boldsymbol{M}_\theta$. We now show that this limit is achieved by the rotation $\boldsymbol{M}_{\theta_g}$. From (11), we get that

$$\psi_{\boldsymbol{M}_{\theta_g}} = \min_{\substack{u_1,u_2\in\mathcal{I}_{q,d},\\ (u_1,u_2)\neq(0,0)}} \left|(u_2^2-u_1^2)\frac{\sin(2\theta_g)}{2}+u_1u_2\cos(2\theta_g)\right|^2$$

$$= \min_{\substack{u_1,u_2\in\mathcal{I}_{q,d},\\ (u_1,u_2)\neq(0,0)}} |u_2^2-u_1^2+u_1u_2|^2|\cos(2\theta_g)|^2 \tag{14}$$

where the last equality follows from the fact that $\tan(2\theta_g) = 2$. Now, for any $u_1, u_2 \in Z[i]$, if $u_2^2 - u_1^2 + u_1u_2 = 0$, then

$$u_2 = \frac{-u_1 \pm \sqrt{u_1^2+4u_1^2}}{2} = u_1\left(\frac{-1\pm\sqrt{5}}{2}\right)$$

which is impossible unless $u_1 = u_2 = 0$. Hence, it is clear from (14) that $\psi_{\boldsymbol{M}_{\theta_g}} \neq 0$. Also, each of u_1 and u_2 are Gaussian integer multiples of 2 and therefore $\psi_{\boldsymbol{M}_{\theta_g}} \geq 16|\cos(2\theta_g)|^2$. Since (13) implies that $\psi_{\boldsymbol{M}_{\theta_g}} \leq 16|\cos(2\theta_g)|^2$, we conclude that $\psi_{\boldsymbol{M}_{\theta_g}} = 16|\cos(2\theta_g)|^2$. The real rotation matrix $\boldsymbol{M}_{\theta_g}$, therefore, maximizes the minimum product distance with QAM information symbols for any $q \geq 2$. □

B. A Full Diversity Space–Time Code in $\mathcal{F}$

The optimum real rotation matrix given by Lemma 1 is now utilized in the construction of a full diversity 2×2 space–time code. The proposed space–time code is obtained by setting $\theta = \theta_g$ and $\phi = -i$ and is denoted by $\mathcal{C}_r$ so that

$$\mathcal{C}_r \triangleq \mathcal{C}(q, \theta_g, -i) \in \mathcal{F}.$$

In the following proposition, we establish the full transmit diversity of $\mathcal{C}_r$ and also compute its coding gain.

Proposition 3: The space–time code $\mathcal{C}_r$ exhibits full transmit diversity for any $q \geq 2$. The coding gain is given by $\delta(\mathcal{C}_r) = 3.2$ and is, therefore, independent of q.

Proof: Let two distinct codewords $\boldsymbol{X}$ and $\boldsymbol{X}'$ in $\mathcal{C}_r$ correspond to the two distinct sets $(\boldsymbol{u}, \boldsymbol{v})$ and $(\boldsymbol{u}', \boldsymbol{v}')$ of information symbol vectors from $\mathcal{I}_q^2$. If $[a_1, a_2]^\top = \boldsymbol{u} - \boldsymbol{u}'$ and $[b_1, b_2]^\top = \boldsymbol{v} - \boldsymbol{v}'$, then each of a_1, a_2, b_1, b_2 are elements in $\mathcal{I}_{q,d}$ and at least one of them is nonzero. Let $\boldsymbol{x} = \boldsymbol{M}\boldsymbol{u}, \boldsymbol{x}' = \boldsymbol{M}\boldsymbol{u}', \boldsymbol{y} = \boldsymbol{M}\boldsymbol{v}$, and $\boldsymbol{y}' = \boldsymbol{M}\boldsymbol{v}'$. Using the fact that $\theta = \theta_g$ and $\phi = -i$, we get that

$$\begin{aligned}
&|\det(\boldsymbol{X}-\boldsymbol{X}')|^2\\
&= |(\boldsymbol{x}_1-\boldsymbol{x}_1')(\boldsymbol{x}_2-\boldsymbol{x}_2') - \phi(\boldsymbol{y}_1-\boldsymbol{y}_1')(\boldsymbol{y}_2-\boldsymbol{y}_2')|^2\\
&= \left|\left\{(a_2^2-a_1^2)\frac{\sin(2\theta_g)}{2}+a_1a_2\cos(2\theta_g)\right\}\right.\\
&\quad \left. +i\left\{(b_2^2-b_1^2)\frac{\sin(2\theta_g)}{2}+b_1b_2\cos(2\theta_g)\right\}\right|^2\\
&= |(a_2^2-a_1^2+a_1a_2)+i(b_2^2-b_1^2+b_1b_2)|^2|\cos(2\theta_g)|^2.
\end{aligned}$$

Define the integer valued function

$$h(a_1,a_2,b_1,b_2) = |(a_2^2-a_1^2+a_1a_2)+i(b_2^2-b_1^2+b_1b_2)|^2.$$

The minimum value of the determinant given above is

$$\min_{\substack{\boldsymbol{X},\boldsymbol{X}'\in\mathcal{C}_r\\ \boldsymbol{X}\neq\boldsymbol{X}'}} |\det(\boldsymbol{X}-\boldsymbol{X}')|^2 = \left(\min_{\substack{a_1,a_2,b_1,b_2\in\mathcal{I}_{q,d}\\ (a_1,a_2,b_1,b_2)\neq(0,0,0,0)}} h(a_1,a_2,b_1,b_2)\right) \times |\cos(2\theta_g)|^2. \tag{15}$$

Substituting the specific values $a_1 = 2, a_2 = b_1 = b_2 = 0$, one obtains that $16|\cos(2\theta_g)|^2$ is an upper bound on the expression in (15). On the other hand, each of a_1, a_2, b_1, b_2 are multiples of 2 and thus, the function $h(a_1, a_2, b_1, b_2)$ is always a multiple of 16. The result of Lemma 2 in the Appendix implies that $h(a_1, a_2, b_1, b_2) \neq 0$ for any value of $q \geq 2$. Therefore, the code $\mathcal{C}_r$ has full diversity for any q and the expression in (15) is equal to $16|\cos(2\theta_g)|^2 = 3.2$. □

The result of Proposition 3 implies that the coding gain of the proposed space time code is, in fact, given by

$$\delta(\mathcal{C}_r) = \psi_{\boldsymbol{M}_{\theta_g}}, \tag{16}$$

where $\psi_{\boldsymbol{M}_{\theta_g}}$ is the minimum product distance of the lattice constellation generated by $\boldsymbol{M}_{\theta_g}$.

Proposition 3 also implies that the coding gain of the proposed code $\mathcal{C}_r$ remains unchanged as the size q of the QAM constellation increases. Therefore, from the discussion in Section II-B and Proposition 1, we can conclude that the new code and its horizontal extensions are optimal from the diversity–multiplexing tradeoff point of view.

Proposition 4: The code $\mathcal{C}_r$ achieves the optimal diversity–multiplexing tradeoff curve of the $M = T = 2$, $N \geq 2$ channel with QAM symbols for all multiplexing gains $r \in [0, 2]$. Furthermore, for each $b \geq 1$, the code $H_b(\mathcal{C}_r)$ achieves the optimal diversity–multiplexing tradeoff curve of the $M = 2, T = 2b, N \geq 2$ channel for all multiplexing gains $r \in [0, 2]$.

Both the *tilted*-QAM code proposed in [5] and the $\mathcal{C}_r$ proposed in this paper achieve the optimal diversity–multiplexing tradeoff curve. However, the actual value of the coding gain is higher with the proposed $\mathcal{C}_r$ code. This comparison is further elaborated on in Section V. It must be noted that achieving the optimal tradeoff curve with QAM symbols and the $\mathcal{C}_r$ code is a stronger property than the information lossless condition which is defined with respect to an outer Gaussian codebook assumption.

C. Proof of Proposition 2

It is now shown that $\theta = \theta_g$ and $\phi = -i$ is an optimal choice with respect to the coding gain among all space–time codes in $\mathcal{F}$. Consider the particular instance of a code $\mathcal{C}_0 = \mathcal{C}(q, \theta_0, \phi_0) \in \mathcal{F}$. An upper bound on $\delta(\mathcal{C}_0)$ can be obtained by considering a specific choice of two distinct codewords in $\mathcal{C}_0$. Two distinct codewords $\boldsymbol{X}$ and $\boldsymbol{X}'$ in $\mathcal{C}_0$ correspond to the two distinct sets $(\boldsymbol{u}, \boldsymbol{v})$ and $(\boldsymbol{u}', \boldsymbol{v}')$ of information symbol vectors from $\mathcal{I}_q^2$. Set $\boldsymbol{v} = \boldsymbol{v}'$ and choose $\boldsymbol{u}$ and $\boldsymbol{u}'$ such that $|\det(\boldsymbol{X} - \boldsymbol{X}')|^2$ is the unnormalized minimum product distance $\psi_{\boldsymbol{M}_{\theta_0}}$. Hence, $\delta(\mathcal{C}_0)$ cannot be greater than $\psi_{\boldsymbol{M}_{\theta_0}}$ in the presence of two layers in the space–time code. Combining this upper bound on $\delta(\mathcal{C}_0)$ with Lemma 1 and (16), we get that

$$\delta(\mathcal{C}_0) \leq \psi_{\boldsymbol{M}_{\theta_0}} \leq \psi_{\boldsymbol{M}_{\theta_g}} = \delta(\mathcal{C}_r).$$

Thus, $(\hat{\theta}_q, \hat{\phi}_q) = (\theta_g, -i)$ and the code $\mathcal{C}_r$ is optimal among all codes in $\mathcal{F}$ for each $q \geq 2$. Note that, for the code $\mathcal{C}_r$, it is the special choice of $\phi = -i$ that results in no decrease in the coding gain below $\psi_{\boldsymbol{M}_{\theta_g}}$ even with the addition of the second layer.

Having solved the coding gain optimization problem for the family of codes $\mathcal{F}$ proposed in (7), we can now conclude from Proposition 1 that, for each $b \geq 1$ and $q \geq 2$, the extended codes $H_b(\mathcal{C}_r)$ and $V_b(\mathcal{C}_r)$ are optimal in terms of coding gain among all possible codes $H_b(\mathcal{C})$ and $V_b(\mathcal{C})$, respectively, with $\mathcal{C} \in \mathcal{F}$.

V. Coding Gain and Performance Comparisons

The proposed code $\mathcal{C}_r$ is now compared with the $B_{2,\phi}$ code, the TAST code for $M = 2$, and the *tilted*-QAM code. A brief description of the construction of these codes is summarized as follows.

- The $B_{2,\phi}$ code is obtained from (7) by replacing $\boldsymbol{M}_\theta$ by

$$\boldsymbol{M} = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 & e^{i\lambda} \\ 1 & -e^{i\lambda} \end{bmatrix}$$

  and setting $\phi = e^{i\lambda}$ for $\lambda \in [0, 2\pi]$. Full diversity is ensured by construction as shown in [3], wherein, the optimum values of λ that lead to the maximum coding gain of the space–time code were found by exhaustive search to be $\lambda = 0.5$ for 4-QAM and $\lambda = 0.521$ for 16-QAM.
- The TAST code for $M = 2$ and full rate of two symbols per channel use is denoted by $\mathcal{T}_{2,2,N}$. The $\mathcal{T}_{2,2,N}$ code is obtained from (7) by replacing $\boldsymbol{M}_\theta$ by

$$\boldsymbol{G}_2 = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 & e^{i\frac{\pi}{4}} \\ 1 & -e^{i\frac{\pi}{4}} \end{bmatrix}.$$

  To guarantee full transmit diversity, it was suggested in [4] to choose the parameter ϕ to be an algebraic number of degree 2 over $Q(i)$. The number $\phi = e^{i\frac{\pi}{6}}$ was shown to be optimal in terms of maximizing the coding gain of $\mathcal{T}_{2,2,N}$ with $q = 4$. For $q = 16$, we shall use $\phi = e^{i\frac{\pi}{4}}$.
- The *tilted*-QAM code is obtained by choosing two different real rotations, namely, $\boldsymbol{M}_{\theta_g}$ and $\boldsymbol{M}_{\frac{\pi}{4}-\theta_g}$, for the two layers in (7) and setting $\phi = 1$. This code will be denoted by t-Q.

TABLE I
Coding Gain With 4-QAM

Code $\mathcal{C}$	$\delta(\mathcal{C})$
$B_{2,\phi}$	0.896
$\mathcal{T}_{2,2,N}$	1.088
t-Q	0.800
$\mathcal{C}_r$	3.200

TABLE II
Coding Gain With 16-QAM

Code $\mathcal{C}$	$\delta(\mathcal{C})$
$B_{2,\phi}$	0.0127
$\mathcal{T}_{2,2,N}$	0.0118
t-Q	0.8000
$\mathcal{C}_r$	3.2000

The coding gain of these competing codes with 4-QAM and 16-QAM information symbols are shown in Tables I and II, respectively. The asymptotic coding gain of $\mathcal{C}_r$ with respect to the better of the $B_{2,\phi}$ and the $\mathcal{T}_{2,2,N}$ codes is seen to be 4.7 dB for 4-QAM and 24 dB for 16-QAM. It can be shown using Proposition 3 and [5, Theorem 2] that the asymptotic coding gain of $\mathcal{C}_r$ with respect to the t-Q code is 6 dB at any rate fixed for the two codes. The simulated performance of these codes are shown in Fig. 1 for 4-QAM symbols. The higher coding gain of the proposed $\mathcal{C}_r$ code manifests an improved performance of about 1 dB relative to the best of the $B_{2,\phi}$, $\mathcal{T}_{2,2,2}$, and t-Q codes at the highest SNR shown. Similar curves at a higher rate with 16-QAM symbols are shown in Fig. 2. For this rate, the proposed code $\mathcal{C}_r$ performs noticeably better compared to the existing codes with a gain of almost 1 dB in the same range of error probabilities as in Fig. 1.

If the coherence time of fading is larger, a longer frame consisting of independent blocks of the $\mathcal{C}_r$ code, namely, the horizontal extension $H_b(\mathcal{C}_r)$, also results in full transmit diversity for the frame error probability. The simulated performance of such a scheme with a frame length of 120 time slots is shown in Fig. 3. Once again, the $\mathcal{C}_r$ code offers an improvement of up to 1 dB compared to the best of the $B_{2,\phi}$, $\mathcal{T}_{2,2,2}$, and t-Q codes in the range of SNR shown.

In Fig. 4, the performance of the $V_2(\mathcal{C}_r)$ code for $M = N = 4$ is shown. Compared to the full layer TAST code, the $V_2(\mathcal{C}_r)$ code is 2 dB away in performance but requires only half the size of the sphere decoder. Moreover, the $V_2(\mathcal{C}_r)$ code leads to a significant improvement in performance compared to the optimum decoding of the uncoded Bell Labs layered space–time (BLAST) scheme [2]. The two-layer TAST code requires the same complexity as the $V_2(\mathcal{C}_r)$ code and has a higher diversity order but its performance is even worse than that of the uncoded scheme in the given range of SNR. In fact, the capacity optimal $V_2(\mathcal{C}_r)$ code exhibits more than 6-dB improvement in performance compared to the two-layer TAST code and is therefore a better option for complexity reduction with respect to the full layer TAST code.

VI. Conclusion

A construction of a space–time code from an optimum real rotation of QAM symbols was shown to offer significant coding gain over other known space–time codes for the two transmit antenna coherent quasi-

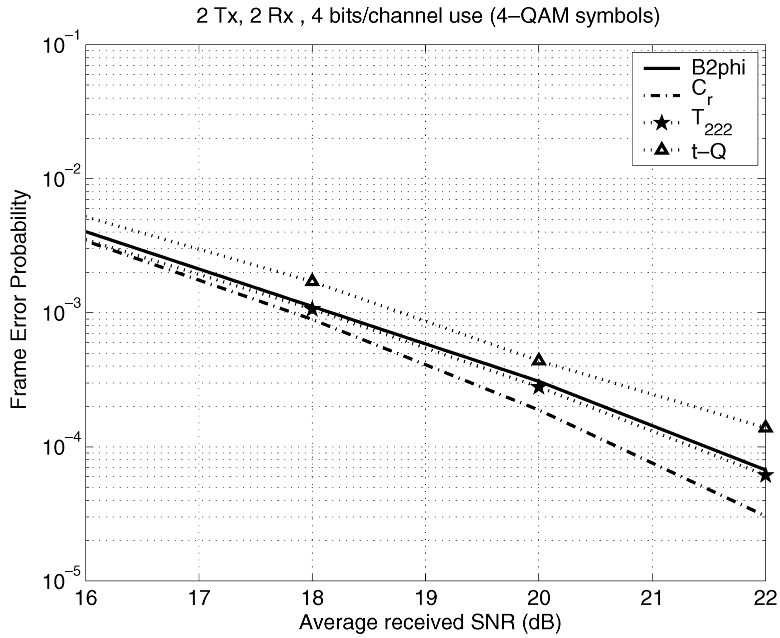

Fig. 1. Performance of $\mathcal{C}_r$ at 4 bits per channel use (bpcu).

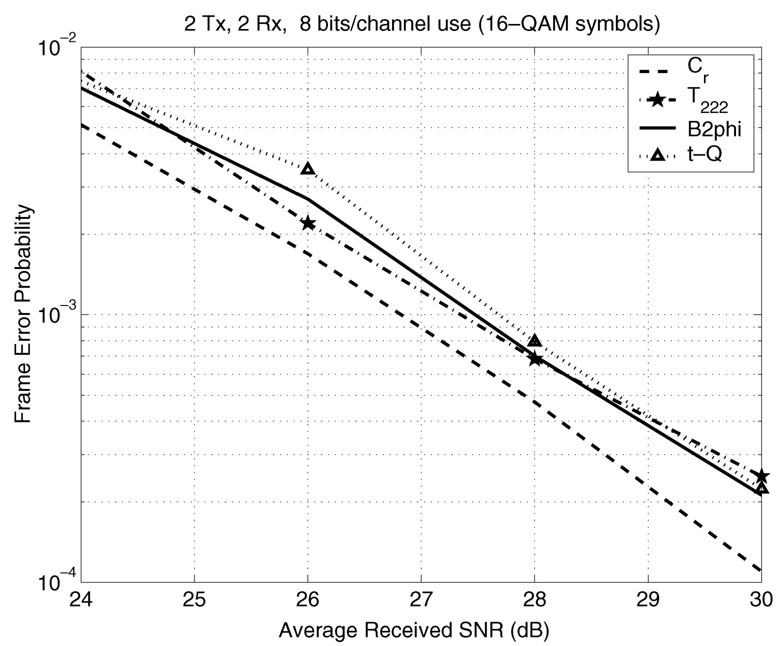

Fig. 2. Performance of $\mathcal{C}_r$ at 8 bpcu.

static fading channel. The proposed code is obtained from the solution of a coding gain optimization problem over a restricted class of layered space–time codes. It has a coding gain that is independent of the rate which allows us to conclude that it achieves the optimal diversity–multiplexing tradeoff curve for the two transmit and receive antenna channel. A horizontal extension of the new code is proposed for performance enhancement for larger coherence times. A vertical extension of the new code is also proposed for complexity reduction with larger number of transmit antennas.

APPENDIX

The following lemma is proved using a technique similar to the one in [5].

Lemma 2: If

$$h(a_1, a_2, b_1, b_2) = (a_2^2 - a_1^2 + a_1 a_2) + i(b_2^2 - b_1^2 + b_1 b_2) = 0$$

for some $a_1, a_2, b_1, b_2 \in Z[i]$, then $a_1 = a_2 = b_1 = b_2 = 0$.

Proof: Make a change of variables as

$$A_1 = 2a_1 - a_2, \quad A_2 = a_2$$
$$B_1 = 2b_1 - b_2, \quad B_2 = b_2$$

so that

$$4h(a_1, a_2, b_1, b_2) = (5A_2^2 - A_1^2) + i(5B_2^2 - B_1^2). \qquad (17)$$

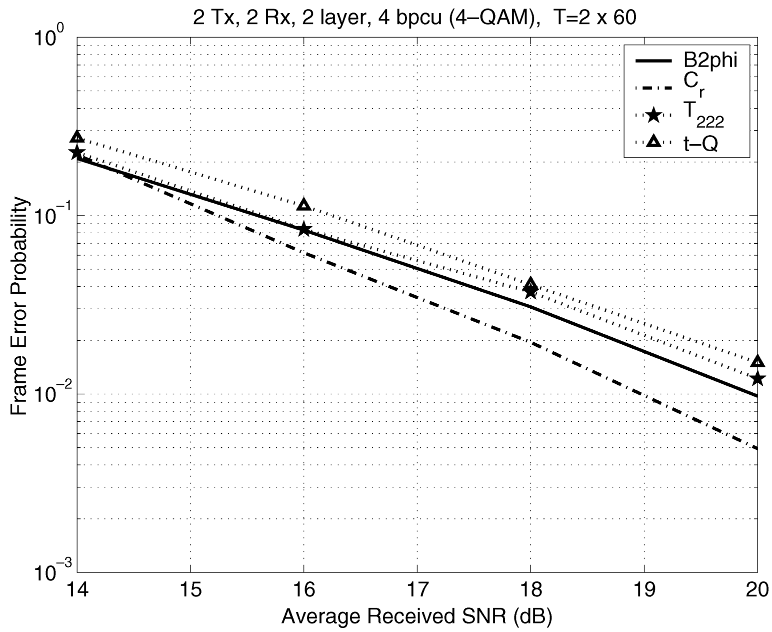

Fig. 3. Performance of $\mathcal{C}_r$ for frame length 120 slots, 4 bpcu.

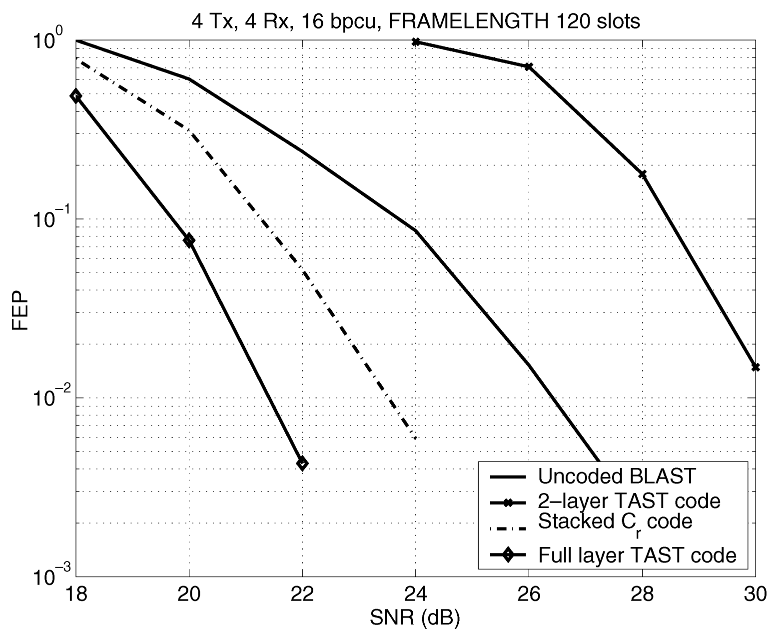

Fig. 4. Performance of $V_2(\mathcal{C}_r)$ at 4 bpcu.

If $h(a_1, a_2, b_1, b_2) = 0$, then (17) implies that

$$5(A_2^2 + iB_2^2) = A_1^2 + iB_1^2. \tag{18}$$

For any $x, y \in Z[i]$, it can be shown by considering the finite number of possibilities for $x, y \bmod 5$ that $5/(x^2 + iy^2) \Leftrightarrow 5/x \text{ and } 5/y$. Hence, (18) implies that $5/A_1$ and $5/B_1$. Thus, $5^2/(A_1^2 + iB_1^2)$ and so $5/(A_2^2 + iB_2^2)$. But this again means that $5/A_2$ and $5/B_2$. Dividing each of A_1, A_2, B_1, B_2 by 5, we obtain $A_1', A_2', B_1', B_2' \in Z[i]$ such that $5(A_2'^2 + iB_2'^2) = A_1'^2 + iB_1'^2$. Since this process can be repeated inductively, we have a contradiction unless $A_1 = A_2 = B_1 = B_2 = 0$ which implies that $a_1 = a_2 = b_1 = b_2 = 0$. $\square$

Lemma 2 says that $h(a_1, a_2, b_1, b_2) \neq 0$ over the infinite set $Z[i]^4 \backslash \{0,0,0,0\}$. Since $\mathcal{I}_{q,d} \subset Z[i]$, the function $h(a_1, a_2, b_1, b_2)$ is nonzero over $\mathcal{I}_{q,d}^4 \backslash \{0,0,0,0\}$ for all values of $q \geq 2$.

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Space–Time Codes With AM-PSK Constellations

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***Abstract*—This correspondence presents a new signal mapper that maps the maximal rank distance codes to space–time (ST) codes with amplitude modulation phase-shift keying (AM-PSK) constellations. It is shown that this new mapper is rank-distance preserving. Comparing to the multiradii construction proposed by Hammons, this new mapper has linear increase in the radii and the resulting signal constellations have larger minimum distance and lower peak-to-average power ratio. Variations of this new mapper are also given to provide ST codes with rotated AM-PSK constellations.**

***Index Terms*—Amplitude modulation phase-shift keying (AM-PSK) constellation, diversity gain advantage, multiple antennas, multiple input multiple output (MIMO), rate-diversity tradeoff, space–time (ST) codes, unified construction.**

I. Introduction

Multiple-input multiple-output (MIMO) communication has recently attracted much attention in wireless communication. Channel codes dedicated to MIMO communication are specifically termed *space–time* (ST) codes when the coding is applied in the time domain.

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To elaborate more, consider a quasi-static, Rayleigh block-fading MIMO channel with quasi-static interval T, n_t transmit and n_r receive antennas. Let $\mathcal{S}$ be an $(n_t \times T)$ ST code. Codewords of $\mathcal{S}$ are arranged as $(n_t \times T)$ matrices whose entries are drawn from a transmission symbol alphabet set (or signal constellation) $\mathcal{A} \subseteq \mathbb{C}$. Given the transmitted code matrix $S \in \mathcal{S}$, the corresponding $(n_r \times T)$ received signal matrix Y is

$$Y = \rho H S + W$$

where H is the $(n_r \times n_t)$ channel matrix and W is the $(n_r \times T)$ noise matrix. The entries of H and W are assumed to be i.i.d., circularly symmetric, complex-Gaussian, $\mathbb{C}\mathcal{N}(0,1)$ random variables. The scale factor ρ is chosen to ensure that

$$\mathbb{E}(\|\rho S\|_F^2) = T\,\mathrm{SNR}$$

where the notation $\|\cdot\|_F$ is the Frobenius norm of a matrix.

In designing ST codes, by making use of the maximal rank distance (MRD) codes [1], the author has developed a signal modulation map, termed *unified map* [2], that can map the MRD codes over $\mathbb{F}_p$, a finite field with p elements, p a prime, to the complex number field $\mathbb{C}$ and preserve the rank distance property at the same time. In other words, if the MRD code $\mathcal{C}$ has minimum rank distance d, then the ST code $\mathcal{S}$ resulting from the unified map has minimum rank distance exactly d, i.e., it achieves transmit diversity gain d [3]. Almost all of the commonly used constellations are included in the unified map, for instance, the PAM, QAM, and PSK constellations. ST codes derived from unified map are optimal in terms of the rate-diversity tradeoff [4].

Motivated by the potential advantages of dual-radii amplitude modulation phase-shift keying (AM-PSK) constellations over conventional PSK constellations [5], [6], Hammons proposed in [7], [8] a new generalization of the unified mapper over 2^K-ary PSK constellations, termed multiradii unified construction, for constructing ST codes with AM-PSK constellations. We outline Hammons' construction in the following theorem.

Theorem 1 (Multiradii Unified Construction [8]): Let $\{\mathcal{C}_1, \mathcal{C}_2, \ldots, \mathcal{C}_U, \mathcal{D}_0, \mathcal{D}_1, \ldots, \mathcal{D}_{K-1}\}$ be a set of $(U+K)$ rank-d, $(n_t \times T)$ binary MRD codes. Let ω be a complex, primitive 2^Kth root of unity and choose $0 \le \epsilon_\ell \in (1-\omega)\mathbb{Z}[\omega]$, $1 \le \ell \le U$, where

$$\mathbb{Z}[\omega] := \left\{ \sum_{k=0}^{2^K-1} z_k \omega^k : z_k \in \mathbb{Z} \right\}$$

is the ring of algebraic integers. Let

$$\gamma_\ell := 2\prod_{i=1}^{\ell} \epsilon_i + 1. \tag{1}$$

Then the modulated ST code $\mathcal{S}$

$$\mathcal{S} = \left\{ S = \left(\bigodot_{\ell=1}^{U} \gamma_\ell^{C_\ell} \right) \odot \omega^{\sum_{k=0}^{K-1} 2^k D_k} : C_\ell \in \mathcal{C}_\ell, D_k \in \mathcal{D}_k \right\} \tag{2}$$

achieves transmit diversity gain at least d, where $\odot$ denotes the Hadamard (componentwise) product for matrices. If the underlying constellation is nondegenerate, then the code $\mathcal{S}$ achieves the rate-diversity tradeoff. □