

# Title

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\*funded by ...

# **1 Introduction**

Here is an introduction.

# **2 Formulation**

## **3 Results**

## **4 Conclusions**

## **Aknowledgements**

## **References**

- [1] (e.g.) Einstein, Albert 1905 *Annalen der Physik* **322**(10) 891-921.

## A Priors and Posteriors

These part show prior and conditional posterior  $X|\cdot$  for all variables  $X$

### A.1 $T_0$

Prior is multivariate normal

$$\sim N(\mu_0, \Sigma_0) \quad (1)$$

Posterior is multivariate normal

$$\sim N(\Psi_0 V_0, \Psi_0) \quad (2)$$

where

$$\mathbf{V}_0 = \Sigma^{-1} [\alpha \mathbf{T}_1 - \alpha(1 - \alpha)\mu \mathbf{1}] + \Sigma_0^{-1} \mu_0 \quad (3)$$

$$\Psi_0 = (\alpha^2 \Sigma^{-1} + \Sigma_0^{-1})^{-1} \quad (4)$$

### A.2 $T_k$

Posterior is multivariate normal (here  $0 < k < \kappa$ )

$$\sim N(\Psi_k V_k, \Psi_k) \quad (5)$$

where

$$\mathbf{V}_k = \mathbf{H}_k^T \Xi_k^{-1} (\mathbf{W}_k - \mathbf{B}_k) + \Sigma^{-1} [\alpha (\mathbf{T}_{k+1} + \mathbf{T}_{k-1}) + (1 - \alpha)^2 \mu \mathbf{1}] \quad (6)$$

$$\Psi_k = [\mathbf{H}_k^T \Xi_k^{-1} \mathbf{H}_k + (1 + \alpha^2) \Sigma^{-1}]^{-1} \quad (7)$$

### A.3 $T_\kappa$

Posterior is multivariate normal

$$\sim N(\Psi_\kappa V_\kappa, \Psi_\kappa) \quad (8)$$

where

$$\mathbf{V}_\kappa = \mathbf{H}_\kappa^T \Xi_\kappa^{-1} (\mathbf{W}_\kappa - \mathbf{B}_\kappa) + \Sigma^{-1} [\alpha \mathbf{T}_{\kappa-1} + (1 - \alpha)\mu \mathbf{1}] \quad (9)$$

$$\Psi_\kappa = (\mathbf{H}_\kappa^T \Xi_\kappa^{-1} \mathbf{H}_\kappa + \Sigma^{-1})^{-1} \quad (10)$$

#### A.4 $\alpha$

Prior is uniform

$$\sim U(a_0, a_1) \quad (11)$$

Posterior is truncated normal

$$\sim N_{[a_0, a_1]}(\Psi_\alpha V_\alpha, \Psi_\alpha) \quad (12)$$

where

$$V_\alpha = \sum_{k=1}^{\kappa} (\mathbf{T}_{k-1} - \mu \mathbf{1})^\top \Sigma^{-1} (\mathbf{T}_k - \mu \mathbf{1}) \quad (13)$$

$$\Psi_\alpha = \left[ \sum_{k=1}^{\kappa} (\mathbf{T}_{k-1} - \mu \mathbf{1})^\top \Sigma^{-1} (\mathbf{T}_{k-1} - \mu \mathbf{1}) \right]^{-1} \quad (14)$$

and boundaries  $a_0, a_1$  come from uniform prior

#### A.5 $\mu$

Prior is normal

$$\sim N(\mu_0, \sigma_0^2) \quad (15)$$

Posterior is normal

$$\sim N(\Psi_\mu V_\mu, \Psi_\mu) \quad (16)$$

where

$$V_\mu = \frac{\mu_0}{\sigma_0^2} + (1 - \alpha) \mathbf{1}^\top \Sigma^{-1} \left[ \sum_{k=1}^{\kappa} (\mathbf{T}_k - \alpha \mathbf{T}_{k-1}) \right] \quad (17)$$

$$\Psi_\mu = \left( \frac{1}{\sigma_0^2} + \kappa(1 - \alpha)^2 \mathbf{1}^\top \Sigma^{-1} \mathbf{1} \right)^{-1} \quad (18)$$

#### A.6 $\sigma^2$

Prior is inverse-gamma

$$\sim InvGamma(\lambda_T, \nu_T) \quad (19)$$

Posterior is inverse-gamma

$$\sim \text{InvGamma} \left( \lambda_T + \frac{N_A \kappa}{2}, \nu_T + \frac{1}{2} \sum_{k=1}^{\kappa} \Delta \mathbf{T}_{k,k-1}^T \mathbf{R}^{-1} \Delta \mathbf{T}_{k,k-1} \right) \quad (20)$$

where

$$\Delta \mathbf{T}_{k,k-1} = (\mathbf{T}_k - \alpha \mathbf{T}_{k-1} - (1 - \alpha) \mu \mathbf{1}) \quad (21)$$

$$\mathbf{R}_{ij} = \exp(-\phi |\mathbf{x}_i - \mathbf{x}_j|) \quad (22)$$

## A.7 $\phi$

Prior is log normal

$$\sim \text{logNormal}(\mu_\phi, \sigma_\phi^2) \quad (23)$$

Posterior is not analytically distributed. Its probability density function is

$$P(\phi | \cdot) \propto P(\phi) \cdot |\mathbf{R}|^{-\kappa/2} \cdot \exp \left( -\frac{1}{2\sigma^2} \sum_{k=1}^{\kappa} \Delta \mathbf{T}_{k,k-1}^T \mathbf{R}^{-1} \Delta \mathbf{T}_{k,k-1} \right) \quad (24)$$

Since the prior of  $\phi$  is log normal, it is more convenient to sample from  $\log(\phi)$ . We use transformation  $\Phi \equiv \log(\phi)$ , and posterior for  $\Phi$  is

$$P(\Phi | \cdot) \propto |\mathbf{R}|^{-\kappa/2} \cdot \exp \left( \frac{-(\Phi - \mu_\phi)^2}{2\sigma_\phi^2} - \frac{1}{2\sigma^2} \sum_{k=1}^{\kappa} \Delta \mathbf{T}_{k,k-1}^T \mathbf{R}^{-1} \Delta \mathbf{T}_{k,k-1} \right) \quad (25)$$

We use Metropolis sampling in this step, and trial values are normal distributions

$$\Phi^* | \Phi_{t-1} \sim N(\Phi_{t-1}, \sigma_{\Phi, MH}^2) \quad (26)$$

## A.8 $\tau_I^2$

Prior is inverse-gamma

$$\sim \text{InvGamma}(\lambda_I, \nu_I) \quad (27)$$

Posterior is inverse-gamma

$$\sim \text{InvGamma} \left( \lambda_I + \frac{M_I}{2}, \nu_I + \frac{1}{2} \sum_{k=1}^k \mathbf{r}_{I,k}^T \mathbf{r}_{I,k} \right) \quad (28)$$

where

$$\mathbf{r}_{I,k} = \mathbf{W}_{I,k} - [\mathbf{H}_k \mathbf{T}_k + \mathbf{B}_k]_I \quad (29)$$

### A.9 $\tau_P^2$

Prior is inverse-gamma

$$\sim \text{InvGamma}(\lambda_P, \nu_P) \quad (30)$$

Posterior is inverse-gamma

$$\sim \text{invGamma}\left(\lambda_P + \frac{M_P}{2}, \nu_P + \frac{1}{2} \sum_{k=1}^{\kappa} \mathbf{r}_{P,k}^T \mathbf{r}_{P,k}\right) \quad (31)$$

where

$$\mathbf{r}_{P,k} = \mathbf{W}_{P,k} - [\mathbf{H}_k \mathbf{T}_k + \mathbf{B}_k]_P \quad (32)$$

### A.10 $\beta_1$

Prior is normal

$$\sim N(\eta_1, \delta_1^2) \quad (33)$$

where

$$\eta_1 = \left[ \frac{(1 - \tau_P^2)(1 - \alpha^2)}{\sigma^2} \right]^{-1/2} \quad (34)$$

Posterior is normal

$$\sim N(\Psi_{\beta_1} V_{\beta_0}, \Psi_{\beta_1}) \quad (35)$$

where

$$V_{\beta_1} = \frac{\eta_1}{\delta_1^2} + \frac{1}{\tau_P^2} \sum_{k=1}^{\kappa} \mathbf{T}_k^T (\mathbf{W}_{P,k} - \beta_0 \mathbf{1}_{N_{P,k}}) \quad (36)$$

$$\Psi_{\beta_1} = \left( \frac{1}{\delta_1^2} + \frac{1}{\tau_P^2} \sum_{k=1}^{\kappa} \mathbf{T}_{P,k}^T \mathbf{T}_{P,k} \right)^{-1} \quad (37)$$

### A.11 $\beta_0$

Prior is normal

$$\sim N(\eta_0, \delta_0^2) \quad (38)$$

where  $\eta_0 = -\mu_0 \eta_1$  is the negative of the product of prior means for  $\mu$  and  $\beta_1$

Posterior is normal

$$\sim N(\Psi_{\beta_0} V_{\beta_0}, \Psi_{\beta_0}) \quad (39)$$

where

$$V_{\beta_0} = \frac{\eta_0}{\delta_0^2} + \frac{1}{\tau_P^2} \sum_{k=1}^{\kappa} \mathbf{1}_{N_{P,k}}^T (\mathbf{W}_{P,k} - \beta_1 \mathbf{H}_{P,k} \mathbf{T}_k) \quad (40)$$

$$\Psi_{\beta_0} = \left( \frac{1}{\delta_0^2} + \frac{M_P}{\tau_P^2} \right)^{-1} \quad (41)$$