四、几种特殊类型函数的积分

(一)有理函数的积分

(二) 三角函数有理式的积分

(三) 简单无理函数的积分:

(一)有理函数的积分

1. 有理函数

$$\frac{P(x)}{Q(x)}=\frac{a_0x^n+a_1x^{n-1}+\cdots+a_{n-1}x+a_n}{b_0x^m+b_1x^{m-1}+\cdots+b_{m-1}x+b_m} \qquad (1)$$

(m和n为非负整数，$a_0,a_1,\cdots a_{n-1},a_n$及$b_0,b_1,\cdots,b_{m-1},b_m$为实数，且$a_0\neq 0,b_0\neq 0.$)

例 $\underbrace{\frac{x^3+3}{x^2-5x+6},\ \frac{x^2+3}{x^2-5x+6}}_{\text{假分式}(n\geq m)},\ \underline{\frac{x+3}{x^2-5x+6}}$

假分式$(n\geq m)$　　真分式 $(n<m)$

2. 真分式、假分式

$$\frac{x^2+3}{x^2-5x+6}=\frac{(x^2-5x+6)+5x-3}{x^2-5x+6}=1+\frac{5x-3}{x^2-5x+6}$$

$$\frac{x^3+3}{x^2-5x+6}=x+5+\frac{19x-27}{x^2-5x+6}$$

- **任何一个有理函数都可化为多项式与真分式之和。**
- **真分式可以化为几个简单真分式的代数和。**

例 $\dfrac{x+3}{x^2-5x+6}$ $\quad \because \dfrac{x+3}{x^2-5x+6}=\dfrac{x+3}{(x-2)(x-3)}$

由代数理论知可分解为 $\dfrac{x+3}{x^2-5x+6}=\dfrac{A}{x-2}+\dfrac{B}{x-3}=\dfrac{A(x-3)+B(x-2)}{(x-2)(x-3)}$

法1. （比较系数法）

$$\because x+3=A(x-3)+B(x-2)=(A+B)x-(3A+2B)$$

$$\therefore \begin{cases} A+B=1 \\ -(3A+2B)=3 \end{cases} \qquad \therefore A=-5, B=6$$

法2. （赋值法） $\because x+3=A(x-3)+B(x-2)$

令$x=2$，得$A=-5$；令$x=3$， 得$B=6$.

$$\therefore \frac{x+3}{x^2-5x+6}=-\frac{5}{x-2}+\frac{6}{x-3}$$

例 $\dfrac{1}{x(x-1)^2}$ 可分解为

$$\frac{1}{x(x-1)^2}=\frac{A}{x}+\frac{B}{(x-1)}+\frac{C}{(x-1)^2}=\frac{A(x-1)^2+Bx(x-1)+Cx}{x(x-1)^2}$$

$$\because 1=A(x-1)^2+Bx(x-1)+Cx$$

令$x=0$, 得$A=1$; 令$x=1$, 得$C=1$; 令$x=2$, 得$B=-1.$

$$\therefore \frac{1}{x(x-1)^2}=\frac{1}{x}-\frac{1}{x-1}+\frac{1}{(x-1)^2}$$

又 $$\frac{1}{x(x-1)^3}=\frac{A}{x}+\frac{B}{(x-1)}+\frac{C}{(x-1)^2}+\frac{D}{(x-1)^3}$$

$$\frac{1}{x^2(x-1)^3}=\frac{A}{x}+\frac{B}{x^2}+\frac{C}{(x-1)}+\frac{D}{(x-1)^2}+\frac{E}{(x-1)^3}$$

例 $\dfrac{1}{(1+2x)(1+x^2)}$ 可分解为

$$\frac{1}{(1+2x)(1+x^2)}=\frac{A}{1+2x}+\frac{Bx+C}{1+x^2}=\frac{A(1+x^2)+(Bx+C)(1+2x)}{(1+2x)(1+x^2)}$$

$$\because 1=A(1+x^2)+(Bx+C)(1+2x)$$

$$=(A+2B)x^2+(B+2C)x+(A+C)$$

令 $x=-\dfrac{1}{2}$, 得 $A=\dfrac{4}{5}$;

比较 x^2 项得： $0=A+2B$ $\quad\therefore B=-\dfrac{2}{5}$

比较常数项得： $1=A+C$ $\quad\therefore C=\dfrac{1}{5}$

$$\therefore \frac{1}{(1+2x)(1+x^2)}=\frac{\frac{4}{5}}{1+2x}+\frac{-\frac{2}{5}x+\frac{1}{5}}{1+x^2}=\frac{4}{5}\cdot\frac{1}{1+2x}-\frac{1}{5}\cdot\frac{2x-1}{1+x^2}$$

再如：$$\frac{1}{(1+2x)(1+x^2)^3}=\frac{A}{1+2x}+\frac{Bx+C}{1+x^2}+\frac{Dx+E}{(1+x^2)^2}+\frac{Fx+G}{(1+x^2)^3}$$

由前面的讨论可知，任何一个有理函数都可化为：多项式、$\frac{A}{(x-a)^n}$、$\frac{Mx+N}{(x^2+px+q)^n}$ 的代数和。

例1 求 $\int\frac{x+3}{x^2-5x+6}dx$

解 $$\because \frac{x+3}{x^2-5x+6}=\frac{-5}{x-2}+\frac{6}{x-3}$$

$$\therefore \int\frac{x+3}{x^2-5x+6}dx=\int\left(\frac{-5}{x-2}+\frac{6}{x-3}\right)dx$$

$$=-5\ln|x-2|+6\ln|x-3|+C$$

例2 求$\int \frac{1}{x(x-1)^2}dx$

解 $\because \frac{1}{x(x-1)^2} = \frac{1}{x} + \frac{1}{(x-1)^2} - \frac{1}{x-1}$

$$\therefore \int \frac{1}{x(x-1)^2}dx = \int \left(\frac{1}{x} + \frac{1}{(x-1)^2} - \frac{1}{x-1} \right) dx$$

$$= \int \frac{1}{x}dx + \int \frac{1}{(x-1)^2}d(x-1) - \int \frac{1}{x-1}d(x-1)$$

$$= \ln|x| - \frac{1}{x-1} - \ln|x-1| + C = \ln\left|\frac{x}{x-1}\right| - \frac{1}{x-1} + C$$

例3 求$\int \frac{1}{(1+2x)(1+x^2)}dx$ $\int \frac{2x-1}{1+x^2}dx = \int \frac{2x}{1+x^2}dx - \int \frac{1}{1+x^2}dx$

解 $\because \frac{1}{(1+2x)(1+x^2)} = \frac{4}{5}\cdot\frac{1}{1+2x} - \frac{1}{5}\cdot\frac{2x-1}{1+x^2}$

$$\therefore \int \frac{1}{(1+2x)(1+x^2)}dx = \frac{4}{5}\int \frac{1}{1+2x}dx - \frac{1}{5}\int \frac{2x-1}{1+x^2}dx$$

$$= \frac{4}{5}\cdot\frac{1}{2}\int \frac{1}{1+2x}d(1+2x) - \frac{1}{5}\int \frac{1}{1+x^2}d(1+x^2) + \frac{1}{5}\int \frac{1}{1+x^2}dx$$

$$= \frac{2}{5}\ln|1+2x| - \frac{1}{5}\ln|1+x^2| + \frac{1}{5}\arctan x + C$$

$$= \frac{1}{5}\left[\ln\frac{(1+2x)^2}{1+x^2} + \arctan x\right] + C$$

注： 当被积函数容易分解时，也不必墨守成规非要用待定系数法或赋值法，只要直接分解就可以。如

例4 求 $\displaystyle\int \frac{1+x^3}{x(1-x^3)}dx$

解： $\boxed{法1}$ $\displaystyle\int \frac{1+x^3}{x(1-x^3)}dx = \int \frac{(1-x^3)+2x^3}{x(1-x^3)}dx = \int \frac{1}{x}dx + 2\int \frac{x^2}{1-x^3}dx$

$$= \ln|x| - \frac{2}{3}\int \frac{1}{1-x^3}d(1-x^3) = \ln|x| - \frac{2}{3}\ln|1-x^3| + C$$ **简单！**

$\boxed{法2}$ 令：$\displaystyle\frac{1+x^3}{x(1-x^3)} = \frac{A}{x} + \frac{B}{1-x} + \frac{Cx+D}{1+x+x^2}$

通分得： $1+x^3 = A(1-x^3) + Bx(1+x+x^2) + (Cx+D)x(1-x)$

比较x的系数得：$A=1, B=\dfrac{2}{3}, C=-\dfrac{4}{3}, D=-\dfrac{2}{3}$

繁！

$$\int \frac{1+x^3}{x(1-x^3)}dx = \int \frac{1}{x}dx + \frac{2}{3}\int \frac{1}{1-x}dx - \frac{2}{3}\int \frac{2x+2}{1+x+x^2}dx$$

例5 求$\int \frac{x-2}{x^2+2x+3}dx$

解 x^2+2x+3是二次质因式，但$(x^2+2x+3)'=2x+2$

$$x-2=\left[\frac{1}{2}(2x+2)-1\right]-2=\frac{1}{2}(2x+2)-3$$

$$\int \frac{x-2}{x^2+2x+3}dx=\int \frac{\frac{1}{2}(2x+2)-3}{x^2+2x+3}dx$$

$$=\frac{1}{2}\int \frac{2x+2}{x^2+2x+3}dx-3\int \frac{1}{x^2+2x+3}dx$$

$$=\frac{1}{2}\int \frac{d\left(x^2+2x+3\right)}{x^2+2x+3}-3\int \frac{d(x+1)}{(x+1)^2+\left(\sqrt{2}\right)^2}$$

$$=\frac{1}{2}\ln\left(x^2+2x+3\right)-\frac{3}{\sqrt{2}}\arctan\frac{x+1}{\sqrt{2}}+C$$

若分解后的有理分式出现 $\dfrac{A}{(x-a)^n}$、$\dfrac{Ax+B}{\left(x^2+a^2\right)^n}$ 这种部分分式，

前面已经解决。若出现 $\dfrac{Mx+N}{(x^2+px+q)^n}$ 如何解决？

讨论 $\displaystyle\int \frac{Mx+N}{(x^2+px+q)^n}dx.$ $\quad x^2+px+q=\left(x+\frac{p}{2}\right)^2+q-\frac{p^2}{4},$

令 $x+\dfrac{p}{2}=t,$ $\because \dfrac{p^2}{4}-q=\dfrac{1}{4}(p^2-4q)<0,$ 即 $q-\dfrac{p^2}{4}>0,$

记 $\boxed{x^2+px+q=t^2+a^2,}$

$\boxed{Mx+N=Mt+b,}$

$$\because Mx+N=M\left(t-\frac{p}{2}\right)+N$$

$$=Mt+N-\frac{Mp}{2}$$

则 $a^2=q-\dfrac{p^2}{4},$ $\quad b=N-\dfrac{Mp}{2},$

$$\therefore \int \frac{Mx+N}{\left(x^2+px+q\right)^n}dx=\int \frac{Mt+b}{\left(t^2+a^2\right)^n}dt=\int \frac{Mt}{\left(t^2+a^2\right)^n}dt+\int \frac{b}{\left(t^2+a^2\right)^n}dt$$

$$\int \frac{Mt}{\left(t^2+a^2\right)^n}dt+\int \frac{b}{\left(t^2+a^2\right)^n}dt$$

当 $n=1$时，如例4。

当 $n>1$时，

$$\int \frac{Mx+N}{\left(x^2+px+q\right)^n}dx=\frac{M}{2}\int \frac{1}{\left(t^2+a^2\right)^n}d\left(t^2+a^2\right)+b\int \frac{dt}{\left(t^2+a^2\right)^n}$$

$$=-\frac{M}{2(n-1)\left(t^2+a^2\right)^{n-1}}+b\int \frac{dt}{\underline{\left(t^2+a^2\right)^n}}$$

有理函数的原函数都是初等函数。

注：对于有理函数的积分要灵活运用各种方法。如：

例6 求$\int \frac{x}{x^8-1}dx$

解：
$$\int \frac{x}{x^8-1}dx = \int \frac{x}{(x^4-1)(x^4+1)}dx = \frac{1}{2}\int\left(\frac{x}{x^4-1}-\frac{x}{x^4+1}\right)dx$$

$$= \frac{1}{4}\int \frac{1}{(x^2)^2-1}dx^2 - \frac{1}{4}\int \frac{1}{(x^2)^2+1^2}dx^2$$

$$= \frac{1}{8}\ln\left|\frac{x^2-1}{x^2+1}\right| - \frac{1}{4}\arctan x^2 + C$$

例7 求$\int \frac{1}{x(x^{10}-1)}dx$

解：
$$\int \frac{1}{x(x^{10}-1)}dx = \int \frac{x^9}{x^{10}(x^{10}-1)}dx = \frac{1}{10}\int\left(\frac{1}{x^{10}-1}-\frac{1}{x^{10}}\right)dx^{10}$$

$$= \frac{1}{10}\ln\left|\frac{x^{10}-1}{x^{10}}\right| + C = \frac{1}{10}\ln\left|1-\frac{1}{x^{10}}\right| + C$$

(二) 三角函数有理式的积分

常数和三角函数经过有限次四则运算所构成的函数.

——— 三角函数有理式

$\sin x$、$\cos x$的有理式,记为$R(\sin x, \cos x)$,

其中$R(u,v)$表示u、v两个变量的有理式.

例$\dfrac{1+\sin x}{\sin x(1+\cos x)}$、$\dfrac{\tan x+\sec x}{\sin x(1+\cos x)}$是三角函数有理式,

而$\dfrac{x+\sin x}{\sin x(1+\cos x)}$不是三角函数有理式

例1 求 $\int \frac{1+\sin x}{\sin x(1+\cos x)}dx$

$$\sin x = \frac{2\tan\frac{x}{2}}{1+\tan^2\frac{x}{2}}, \quad \cos x = \frac{1-\tan^2\frac{x}{2}}{1+\tan^2\frac{x}{2}}$$

解 令 $u = \tan\frac{x}{2}$, 则 $\sin x = \frac{2u}{1+u^2}$, $\cos x = \frac{1-u^2}{1+u^2}$.

而 $x = 2\arctan u$ $\quad\therefore dx = \frac{2}{1+u^2}du$

$$\int \frac{1+\sin x}{\sin x(1+\cos x)}dx = \int \frac{\left(1+\frac{2u}{1+u^2}\right)\frac{2du}{1+u^2}}{\frac{2u}{1+u^2}\left(1+\frac{1-u^2}{1+u^2}\right)} = \frac{1}{2}\int\left(u+2+\frac{1}{u}\right)du$$

$$= \frac{1}{2}\left(\frac{u^2}{2}+2u+\ln|u|\right)+C \quad = \frac{1}{4}\tan^2\frac{x}{2}+\tan\frac{x}{2}+\frac{1}{2}\ln\left|\tan\frac{x}{2}\right|+C$$

$$\int R(\sin x,\cos x)dx \overset{\text{令}\tan\frac{x}{2}=u}{=\!=\!=\!=\!=} \int R\left(\frac{2u}{1+u^2},\frac{1-u^2}{1+u^2}\right)\frac{2}{1+u^2}du$$ 有理分式！

例2 求 $\int \frac{dx}{2\sin x-\cos x+5}$

解 令$u=\tan\frac{x}{2}$，则$\sin x=\frac{2u}{1+u^2}$，$\cos x=\frac{1-u^2}{1+u^2}$.

而$x=2\arctan u$ $\quad \therefore dx=\frac{2}{1+u^2}du$

$$\int \frac{dx}{2\sin x-\cos x+5}=\int \frac{\frac{2}{1+u^2}du}{2\cdot\frac{2u}{1+u^2}-\frac{1-u^2}{1+u^2}+5}$$

$$=\int\frac{du}{3u^2+2u+2}=\frac{1}{3}\int\frac{du}{\left(u+\frac{1}{3}\right)^2+\frac{5}{9}}=\frac{1}{3}\int\frac{d\left(u+\frac{1}{3}\right)}{\left(u+\frac{1}{3}\right)^2+\left(\frac{\sqrt{5}}{3}\right)^2}$$

$$=\frac{1}{3}\cdot\frac{3}{\sqrt{5}}\arctan\frac{u+\frac{1}{3}}{\frac{\sqrt{5}}{3}}+C=\frac{1}{\sqrt{5}}\arctan\frac{3\tan\frac{x}{2}+1}{\sqrt{5}}+C$$

此法往往较繁，能用简单方法的尽量用简单方法。

例3 求$\int\frac{\cos x}{1+\sin x}dx$

解 $\int\frac{\cos x}{1+\sin x}dx=\int\frac{d(1+\sin x)}{1+\sin x}=\ln(1+\sin x)+C$

（三）简单无理函数的积分：

只讨论$R\left(x,\sqrt[n]{ax+b}\right)$及$R\left(x,\sqrt[n]{\dfrac{ax+b}{cx+d}}\right)$

作代换$\sqrt[n]{ax+b}=t$及$\sqrt[n]{\dfrac{ax+b}{cx+d}}=t$。

例1　求$\displaystyle\int\frac{\sqrt{x-1}}{x}dx$

解　令$\sqrt{x-1}=u$，　则$x=u^2+1$，　$dx=2udu$

$$\int\frac{\sqrt{x-1}}{x}dx=\int\frac{u}{u^2+1}2udu=2\int\frac{u^2}{u^2+1}du=2\int\left(1-\frac{1}{u^2+1}\right)du$$

$$=2\left(u-\arctan u\right)+C=2\left(\sqrt{x-1}-\arctan\sqrt{x-1}\right)+C$$

例2 求$\int \frac{dx}{1+\sqrt[3]{x+2}}$

解 令$\sqrt[3]{x+2}=u$，则$x=u^3-2$，$dx=3u^2du$

$$\int \frac{dx}{1+\sqrt[3]{x+2}}=\int \frac{3u^2du}{1+u}=3\int \frac{u^2-1+1}{1+u}du=3\int\left(u-1+\frac{1}{1+u}\right)du$$

$$=3\left(\frac{u^2}{2}-u+\ln|1+u|+C\right)=3\left(\frac{\sqrt[3]{(x+2)^2}}{2}-\sqrt[3]{x+2}+\ln\left|1+\sqrt[3]{x+2}\right|+C\right)$$

例3 求$\int \frac{dx}{\left(1+\sqrt[3]{x}\right)\sqrt{x}}$

解 令$\sqrt[6]{x}=t$，则$x=t^6$，$dx=6t^5dt$

$$\int \frac{dx}{\left(1+\sqrt[3]{x}\right)\sqrt{x}}=\int \frac{6t^5dt}{\left(1+t^2\right)t^3}=\int \frac{6t^2dt}{\left(1+t^2\right)}=6\int\left(1-\frac{1}{1+t^2}\right)dt$$

$$=6(t-\arctan t)+C=6\left(\sqrt[6]{x}-\arctan\sqrt[6]{x}\right)+C$$

例4 求$\int \frac{1}{x}\sqrt{\frac{1+x}{x}}dx$

解 令$\sqrt{\frac{1+x}{x}}=t$, 则$x=\frac{1}{t^2-1}$, $dx=-\frac{2tdt}{\left(t^2-1\right)^2}$

$$\int \frac{1}{x}\sqrt{\frac{1+x}{x}}dx=\int\left(t^2-1\right)t\cdot\frac{-2t}{\left(t^2-1\right)^2}dt=-2\int\frac{t^2}{t^2-1}dt$$

$$=-2\int\left(1+\frac{1}{t^2-1}\right)dt \quad =-2t-\ln\left|\frac{t-1}{t+1}\right|+C \quad =-2t-\ln\left|\frac{(t-1)(t+1)}{(t+1)^2}\right|+C$$

$$=-2t+2\ln(t+1)-\ln\left|t^2-1\right|+C$$

$$-\ln\left|t^2-1\right|=\ln\left|\frac{1}{t^2-1}\right|$$

$$=-2\sqrt{\frac{1+x}{x}}+2\ln\left(\sqrt{\frac{1+x}{x}}+1\right)+\ln|x|+C$$

例5 求$\int \frac{1}{\sqrt[3]{(x+1)^2(x-1)^4}}dx$

解 $\int \frac{dx}{\sqrt[3]{(x+1)^2(x-1)^4}}dx = \int \frac{1}{(x+1)(x-1)}\sqrt[3]{\frac{x+1}{x-1}}dx$

令$\sqrt[3]{\frac{x+1}{x-1}} = t$，则$x = \frac{t^3+1}{t^3-1}$，$dx = \frac{-6t^2}{(t^3-1)^2}dt$。

$$\int \frac{dx}{\sqrt[3]{(x+1)^2(x-1)^4}}dx = \int \frac{t}{\frac{4t^3}{(t^3-1)^2}} \cdot \frac{-6t^2}{(t^3-1)^2}dt$$

$$= -\frac{3}{2}\int dt = -\frac{3}{2}t + C = -\frac{3}{2}\sqrt[3]{\frac{x+1}{x-1}} + C$$

初等函数在其定义区间上的原函数一定存在， 但原函数不一定都是初等函数.

例 $\int e^{-x^2}dx$、$\int \frac{\sin x}{x}dx$、$\int \frac{dx}{\ln x}$、$\int \frac{dx}{\sqrt{1+x^4}}$

都不是初等函数。

不可积函数类