# 第三节　泰勒公式

用简单函数（多项式）近似表示复杂函数

如：当 $|x|$很小时， $e^x \approx 1 + x$， $\sin x \approx x$， $\ln(1+x) \approx x$

$f(0) = P(0),\ f'(0) = P'(0)$

“神州10号”若用此公式进行近似计算，宇航员还能回到地球吗？

不足之处
- 精确度不高（仅仅是 $x$ 的
- 不能估算误差

**问题**：设函数$f(x)$在含有$x_0$的开区间内具有直到$(n+1)$阶导数，试找出一个关于$(x-x_0)$的$n$次多项式

$$f(x) \approx a_0 + a_1(x-x_0) + a_2(x-x_0)^2 + \cdots + a_n(x-x_0)^n \quad (1)$$

来近似表达$f(x)$， 要求：

① $|f(x) - p_n(x)| = o\left[(x-x_0)^n\right]$

② 给出误差的具体表达式。

$$f(x) \approx a_0 + a_1(x - x_0) + a_2(x - x_0)^2 + \dots + a_n(x - x_0)^n = p_n(x)$$

假设 $p_n(x)$ 在 $x_0$ 处的函数值及它的直到 $n$ 阶导数在 $x_0$ 处的值依次满足

$$f(x_0) = p_n(x_0), \quad f'(x_0) = p_n'(x_0), \quad f''(x_0) = p_n''(x_0),$$

$$\cdots, \quad f^{(n)}(x_0) = p_n^{(n)}(x_0)$$

$$\because \ p_n(x) = a_0 + a_1(x - x_0) + a_2(x - x_0)^2 + \cdots + a_n(x - x_0)^n$$

$$\boxed{a_0 = f(x_0),}$$

$$p_n'(x) = a_1 + 2a_2(x - x_0) + \cdots + na_n(x - x_0)^{n-1}$$

$$\boxed{a_1 = f'(x_0),}$$

$$p_n''(x) = 2 \cdot 1 \cdot a_2 + 3 \cdot 2a_3(x - x_0) + \cdots + n(n-1)a_n(x - x_0)^{n-2}$$

$$\boxed{a_2 = \frac{1}{2!} f''(x_0),}$$

$$\cdots \quad \cdots \quad \cdots \quad \cdots \quad \cdots \quad \cdots \quad \cdots \quad \cdots \quad \cdots \quad \cdots \quad \cdots \quad \cdots$$

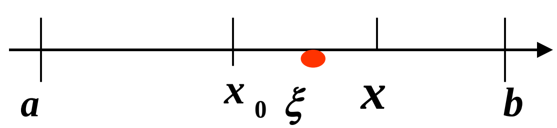
$$p_n^{(n)}(x) = n!a_n \qquad \boxed{a_n = \frac{1}{n!} f^{(n)}(x_0)}$$

**泰勒中值定理** 设函数 $f(x)$ 在含有 $x_0$ 的某个开区间 $(a,b)$ 内具有直到$(n+1)$阶导数，则当 $x\in(a,b)$ 时，$f(x)$ 可表示为 $(x-x_0)$ 的一个$n$ 次多项式与一个余项 $R_n(x)$之和：

$$\boxed{f(x)=f(x_0)+f'(x_0)(x-x_0)+\frac{1}{2!}f''(x_0)(x-x_0)^2+\cdots+\frac{1}{n!}f^{(n)}(x_0)(x-x_0)^n+R_n(x)} \qquad (2)$$

其中 $$\boxed{R_n(x)=\frac{1}{(n+1)!}f^{(n+1)}(\xi)(x-x_0)^{n+1}} \qquad (3)$$

这里 $\xi$ 是 $x_0$与 $x$ 之间的某个值。




**（2）式称为泰勒公式；（3）式称为拉格朗日余项。**

$$R_n(x)=\frac{f^{(n+1)}(\xi)}{(n+1)!}(x-x_0)^{n+1}\quad(\xi\text{在}x_0\text{与}x\text{之间})$$

拉格朗日形式的余项

$$|R_n(x)|=\left|\frac{f^{(n+1)}(\xi)}{(n+1)!}(x-x_0)^{n+1}\right|\le\frac{M}{(n+1)!}(x-x_0)^{n+1}$$

及 $\lim\limits_{x\to x_0}\dfrac{R_n(x)}{(x-x_0)^n}=0$

即 $R_n(x)=o[(x-x_0)^n]$. 皮亚诺形式的余项

$$\therefore\ f(x)=\sum_{k=0}^{n}\frac{f^{(k)}(x_0)}{k!}(x-x_0)^k+o[(x-x_0)^n]$$

# 注意:

1．当$n=0$时，泰勒公式变成拉氏中值公式

$$f(x)=f(x_0)+f'(\xi)(x-x_0)\quad(\xi在x_0与x之间)$$

2. 取$x_0=0$,

$\xi$在0与$x$之间，令$\xi=\theta x\quad(0<\theta<1)$

则余项 $$R_n(x)=\frac{f^{(n+1)}(\theta x)}{(n+1)!}x^{n+1}$$

3．当$x_0=0$时，取$\xi=\theta x\ (0<\theta<1)$，得麦可劳林公式

$$\boxed{f(x)=f(0)+f'(0)x+\frac{1}{2!}f''(0)x^2+\cdots+\frac{1}{n!}f^{(n)}(0)x^n+\frac{1}{(n+1)!}f^{(n+1)}(\theta x)x^{n+1}\quad(0<\theta<1)}$$

1. 写出 $f(x)=e^x$ 的 $n$ 阶麦可劳林公式

解 $f^{(n)}(x)=e^x$

$$f(0)=f'(0)=f''(0)=\cdots=f^{(n)}(0)=1$$

$$\boxed{e^x=1+x+\frac{x^2}{2!}+\cdots+\frac{x^n}{n!}+\frac{e^{\theta x}}{(n+1)!}x^{n+1}\quad(0<\theta<1)}$$

$$\boxed{\begin{aligned}f(x)&=f(0)+f'(0)x+\frac{1}{2!}f''(0)x^2+\cdots+\frac{1}{n!}f^{(n)}(0)x^n\\&+\frac{1}{(n+1)!}f^{(n+1)}(\theta x)x^{n+1}\end{aligned}}$$

**2.** 写出 $f(x)=\sin x$ 的 $n$ 阶麦可劳林公式

**解** $f^{(n)}(x)=\sin\left(x+n\cdot\frac{\pi}{2}\right)$

$$f(0)=0,\ f'(0)=1,\ f''(0)=0,\ f'''(0)=-1,\ f^{(4)}(0)=0,\cdots$$

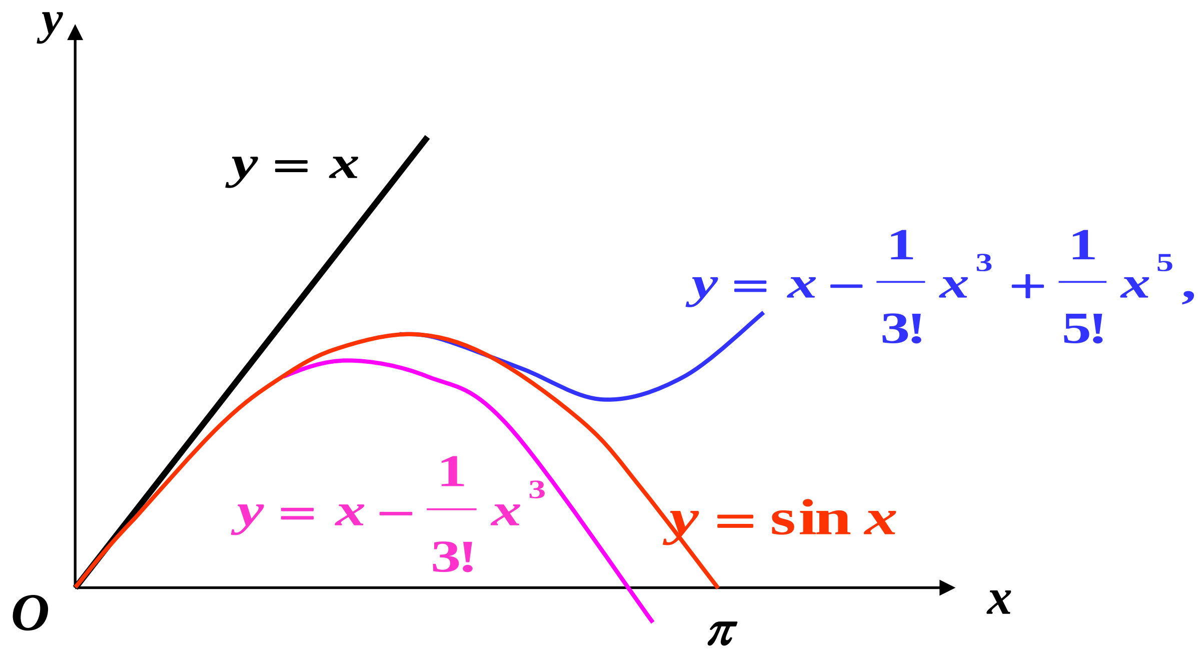
$$\boxed{\sin x = x-\frac{x^3}{3!}+\frac{x^5}{5!}+\cdots+(-1)^{m-1}\frac{x^{2m-1}}{(2m-1)!}+R_{2m}}$$

其中 $R_{2m}=\dfrac{\sin\left[\theta x+(2m+1)\cdot\frac{\pi}{2}\right]}{(2m+1)!}x^{2m+1}\quad(0<\theta<1)$

$$|R_{2m}|\le\frac{\left|x^{2m+1}\right|}{(2m+1)!}$$

$m = 2, 3$ 时，可得

$$\sin x \approx x - \frac{1}{3!}x^3$$ 和 $$\sin x \approx x - \frac{1}{3!}x^3 + \frac{1}{5!}x^5,$$



# 利用麦克劳林公式可以得到以下最常用的函数展开式

$$e^{x}=1+x+\frac{x^{2}}{2!}+\cdots+\frac{x^{n}}{n!}+o\left(x^{n}\right), \qquad x\in(-\infty,+\infty)$$

$$\sin x=x-\frac{x^{3}}{3!}+\frac{x^{5}}{5!}+\cdots+(-1)^{n}\frac{x^{2n+1}}{(2n+1)!}+o\left(x^{2n+2}\right), \qquad x\in(-\infty,+\infty)$$

$$\cos x=1-\frac{x^{2}}{2!}+\frac{x^{4}}{4!}+\cdots+(-1)^{n}\frac{x^{2n}}{(2n)!}+o\left(x^{2n+1}\right), \qquad x\in(-\infty,+\infty)$$

$$\ln(1+x)=x-\frac{x^{2}}{2}+\frac{x^{3}}{3}-\cdots+(-1)^{n}\frac{x^{n+1}}{n+1}+o\left(x^{n+1}\right), \qquad x\in(-1,1]$$

$$\frac{1}{1-x}=1+x+x^{2}+\cdots+x^{n}+o\left(x^{n}\right), \qquad x\in(-1,1)$$

$$(1+x)^{m}=1+mx+\frac{m(m-1)}{2!}x^{2}+\cdots+\frac{m(m-1)\cdots(m-n+1)}{m!}x^{n}+o\left(x^{n}\right), \quad x\in(-1,1)$$

# 利用泰勒公式求极限

**3.** 求 $\lim\limits_{x\to 0}\dfrac{x-\sin x}{x^2\left(e^x-1\right)}$

解 $\lim\limits_{x\to 0}\dfrac{x-\sin x}{x^2\left(e^x-1\right)}=\lim\limits_{x\to 0}\dfrac{x-\left(x-\dfrac{x^3}{3!}+o\left(x^4\right)\right)}{x^3}=\dfrac{1}{6}$

**4.** 求 $\lim\limits_{x\to 0}\dfrac{1}{x}\left(\dfrac{1}{x}-\cot x\right)$

解 $\lim\limits_{x\to 0}\dfrac{1}{x}\left(\dfrac{1}{x}-\cot x\right)=\lim\limits_{x\to 0}\dfrac{1}{x}\left(\dfrac{1}{x}-\dfrac{\cos x}{\sin x}\right)=\lim\limits_{x\to 0}\dfrac{\sin x-x\cos x}{x^2\sin x}$

$$=\lim_{x\to 0}\frac{x-\dfrac{x^3}{3!}+o\left(x^4\right)-x\left(1-\dfrac{x^2}{2!}+o\left(x^3\right)\right)}{x^3}$$

$$=\lim_{x\to 0}\frac{-\dfrac{x^3}{6}+\dfrac{x^3}{2}+o\left(x^4\right)}{x^3}=\frac{1}{3}$$

# 间接展开法

**1.** 写出 $f(x)=e^{-x}$ 的 $n$ 阶麦可劳林公式

解
$$\boxed{e^{x}=1+x+\frac{x^{2}}{2!}+\cdots+\frac{x^{n}}{n!}+\frac{e^{\theta x}}{(n+1)!}x^{n+1}\quad(0<\theta<1)}$$

$$e^{-x}=1-x+\frac{x^{2}}{2!}-\frac{x^{3}}{3!}+\cdots+(-1)^{n}\frac{x^{n}}{n!}+\frac{(-1)^{n+1}e^{\theta x}}{(n+1)!}x^{n+1}$$

$(0<\theta<1)$

**2.** 写出 $f(x)=\sin 2x$ 的 $n$ 阶麦可劳林公式

解
$$\boxed{\sin x=-\frac{x^{3}}{3!}+\frac{x^{5}}{5!}+\cdots+(-1)^{m-1}\frac{x^{2m-1}}{(2m-1)!}+R_{2m}}$$

$$\sin 2x=-\frac{2^{3}x^{3}}{3!}+\frac{2^{5}x^{5}}{5!}+\cdots++(-1)^{m-1}\frac{2^{2m-1}x^{2m-1}}{(2m-1)!}+R_{2m}$$

其中 $R_{2m}=\dfrac{\sin\left[2\theta x+(2m+1)\cdot\dfrac{\pi}{2}\right]}{(2m+1)!}2^{2m+1}x^{2m+1}\quad(0<\theta<1)$

**3.** 写出 $f(x)=\cos^2 x$ 的麦可劳林公式

**解** $\cos^2 x=\dfrac{1+\cos 2x}{2}=\dfrac{1}{2}+\dfrac{1}{2}\cos 2x$

$$\boxed{\cos x=1-\frac{x^2}{2!}+\frac{x^4}{4!}+\cdots+(-1)^n\frac{x^{2n}}{(2n)!}+o\left(x^{2n+1}\right),}$$

故 $\cos^2 x=\dfrac{1}{2}+\dfrac{1}{2}\cos 2x$

$$=\frac{1}{2}+\frac{1}{2}[1-\frac{(2x)^2}{2!}+\frac{(2x)^4}{4!}+\cdots+(-1)^n\frac{(2x)^{2n}}{(2n)!}+o\left((2x)^{2n+1}\right)]$$

$$=\frac{3}{2}-\frac{2x^2}{2!}+\frac{2^3x^4}{4!}+\cdots+(-1)^n\frac{2^{2n-1}x^{2n}}{(2n)!}+o\left(x^{2n+1}\right)]$$

**4.** 写出 $f(x)=\ln(1+3x)$的麦可劳林公式

**解**

$$\boxed{\ln(1+x)=x-\frac{x^2}{2}+\frac{x^3}{3}-\cdots+(-1)^n\frac{x^{n+1}}{n+1}+o\left(x^{n+1}\right)}$$

$$\ln(1+3x)=3x-\frac{(3x)^2}{2}+\frac{(3x)^3}{3}-\cdots+(-1)^n\frac{(3x)^{n+1}}{n+1}+o\left((3x)^{n+1}\right)$$

$$=3x-\frac{3^2x^2}{2}+\frac{3^3x^3}{3}-\cdots+(-1)^n\frac{3^{n+1}x^{n+1}}{n+1}+o\left((3x)^{n+1}\right)$$

5. 设 $f(x)$ 在$(a,b)$内二阶可导,且 $f''(x)\ge 0$,证明:对于$(a,b)$内任意两点$x_1,x_2$及$0\le t\le 1$, 有

$$f[(1-t)x_1+tx_2]\le(1-t)f(x_1)+tf(x_2).$$

证明 设$(1-t)x_1+tx_2=x_0$. $f(x)$ 在 $x_0$点的一阶泰勒展开式为

$$f(x)=f(x_0)+f'(x_0)(x-x_0)+\frac{f''(\xi)}{2!}(x-x_0)^2$$

（$\xi$介于$x$与$x_0$之间)

显然 $f(x)\ge f(x_0)+f'(x_0)(x-x_0)$,

于是 $f(x_1)\ge f(x_0)+f'(x_0)(x_1-x_0)$ (1)

$f(x_2)\ge f(x_0)+f'(x_0)(x_2-x_0)$ (2)

(1)×(1-$t$)+(2)$t$得

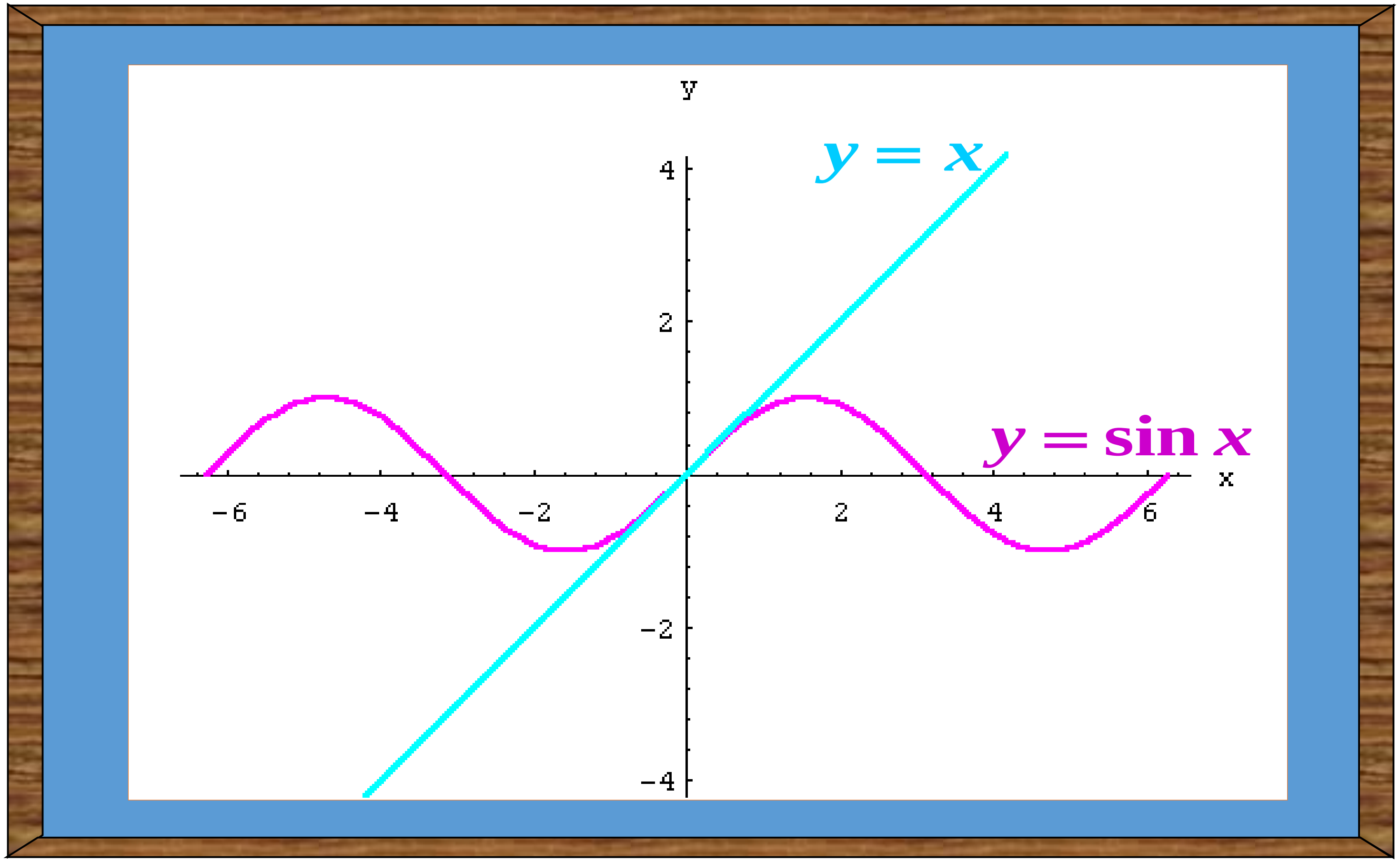
$$(1-t)f(x_1)+tf(x_2)\ge(1-t)f(x_0)+(1-t)f'(x_0)(x_1-x_0)$$

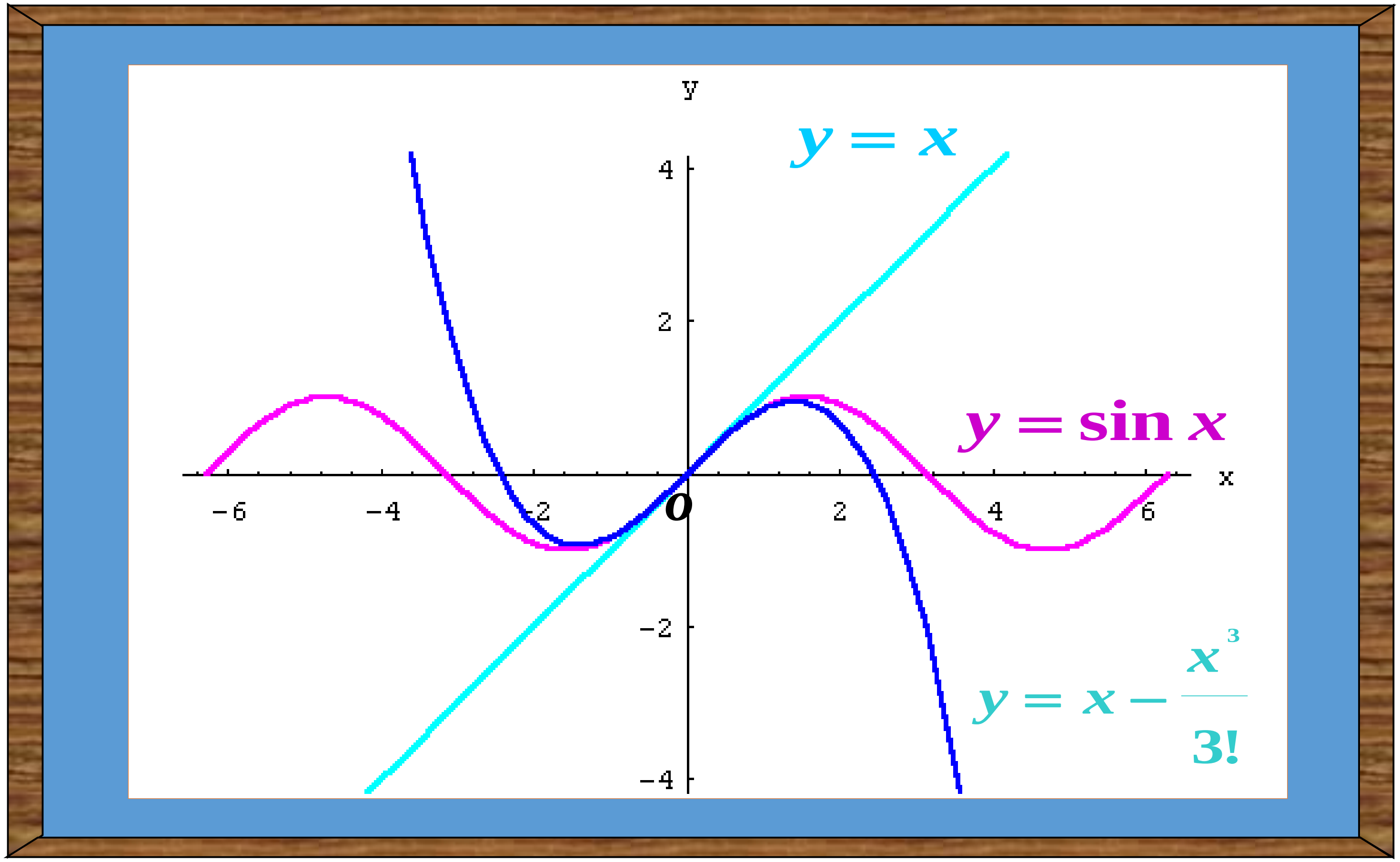
$$+tf(x_0)+tf'(x_0)(x_2-x_0)$$

$$=f(x_0)+f'(x_0)[(1-t)(x_1-x_0)+t(x_2-x_0)]$$

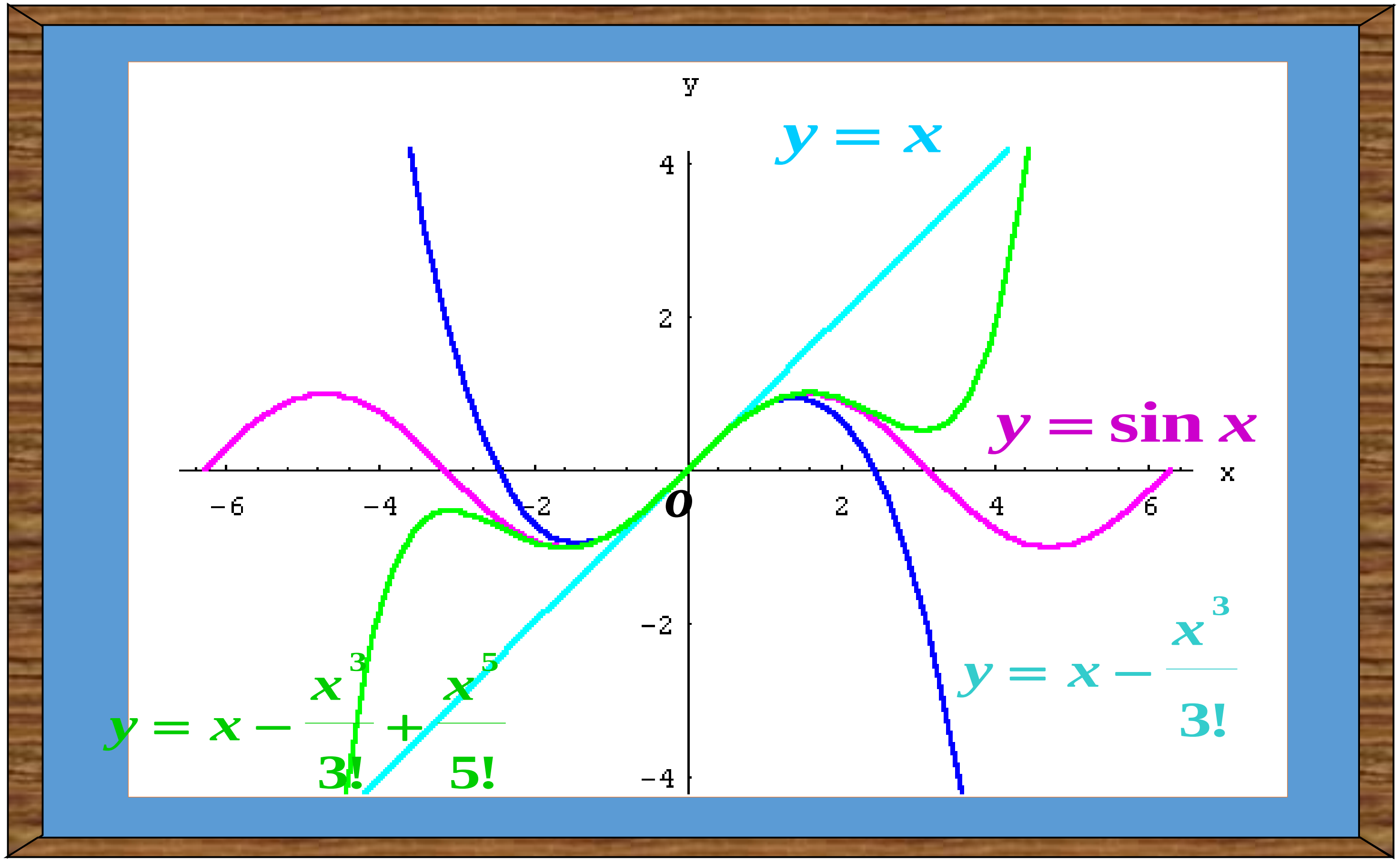
$$=f(x_0)+f'(x_0)[(1-t)x_1-x_0+tx_2]=f(x_0).$$

于是命题得证.

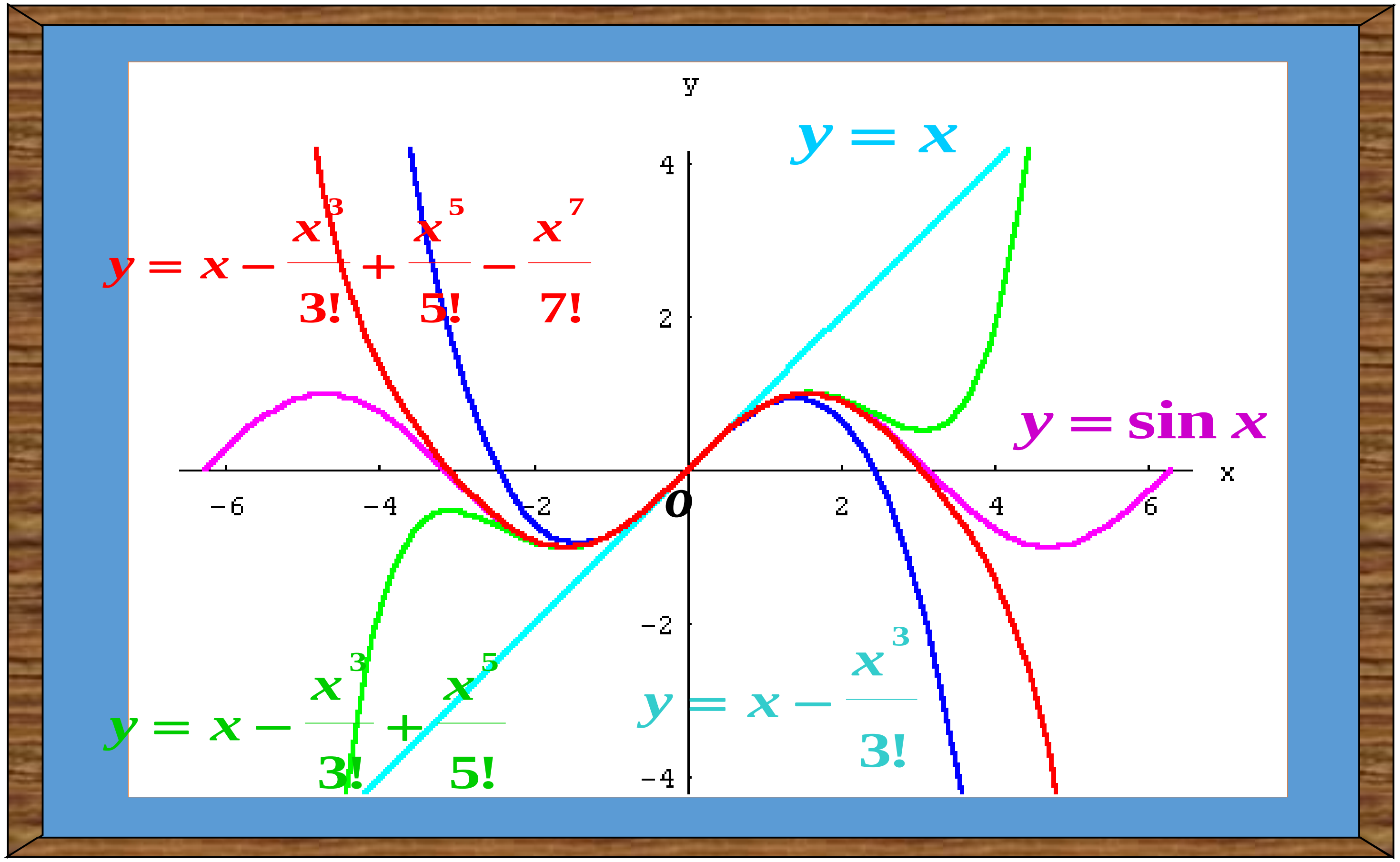

y
$y = x$
4
2
$y = \sin x$
x
-6
-4
-2
2
4
6
-2
-4



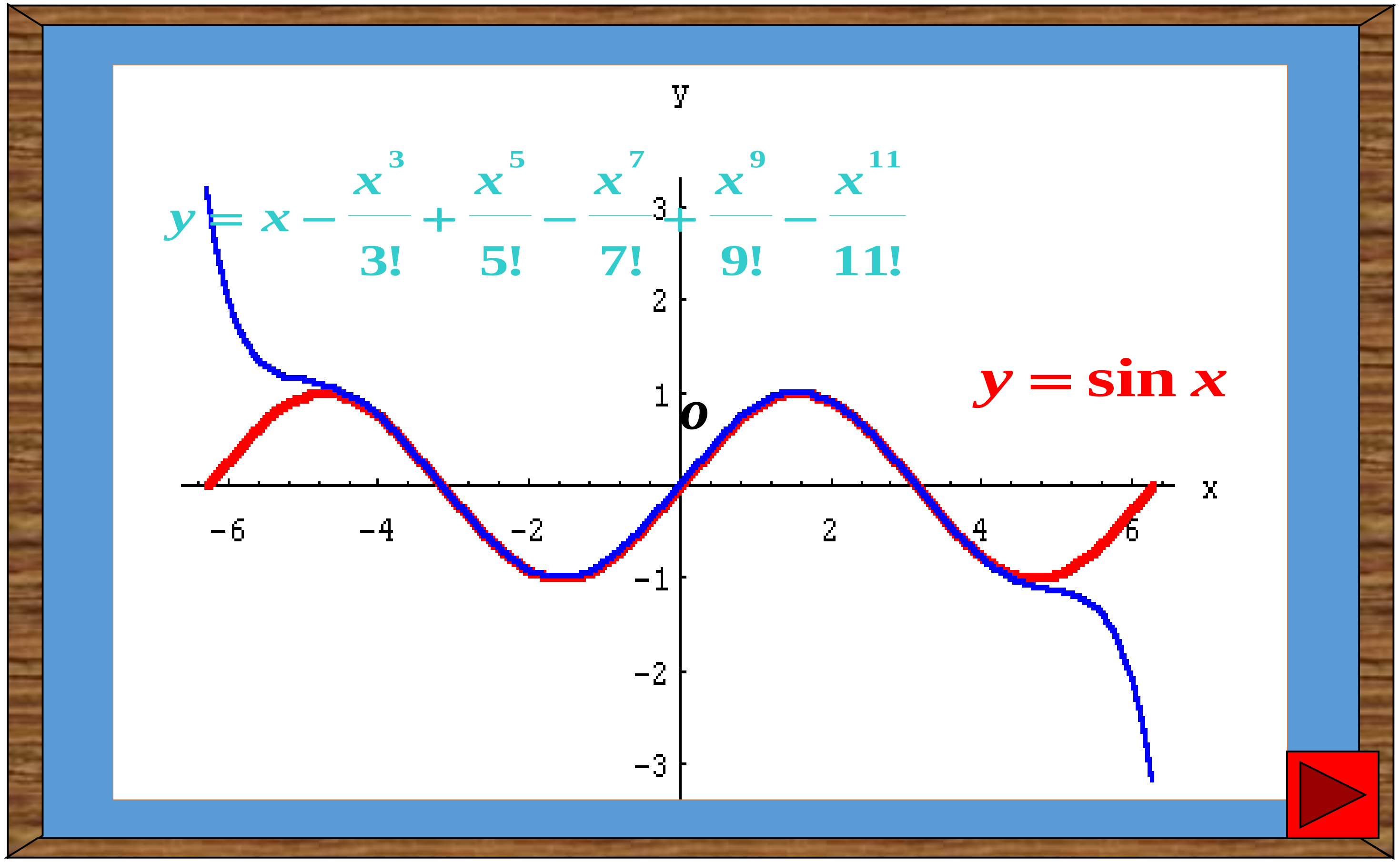
y
$y = x$
4
2
$y = \sin x$
x
−6
−4
−2
$o$
2
4
6
−2
−4
$y = x - \frac{x^3}{3!}$



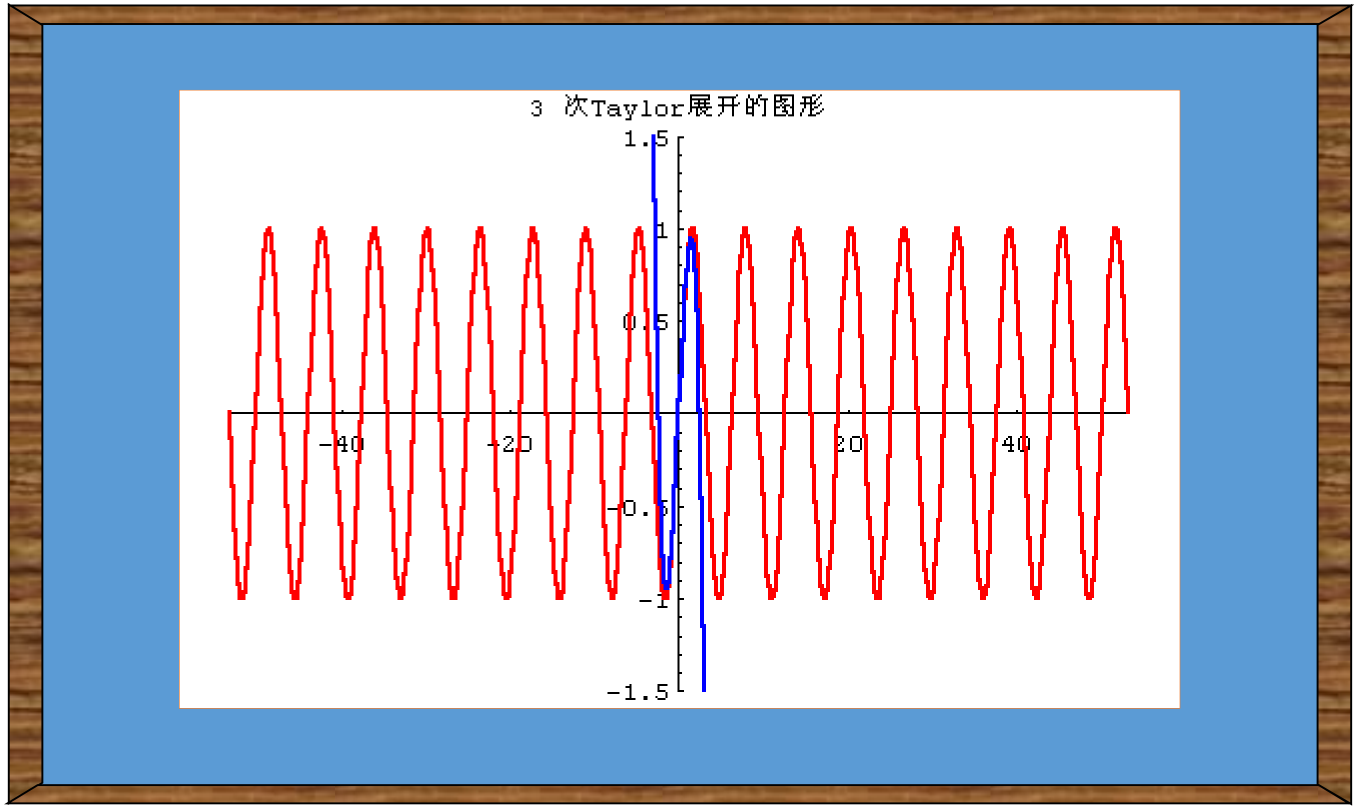
y
x
4
2
-2
-4
-6
-4
-2
2
4
6
O
$y = x$
$y = \sin x$
$y = x - \frac{x^3}{3!}$
$y = x - \frac{x^3}{3!} + \frac{x^5}{5!}$



y
x
4
2
−2
−4
−6
−4
−2
2
4
6
O
$y = x$
$y = \sin x$
$y = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!}$
$y = x - \frac{x^3}{3!} + \frac{x^5}{5!}$
$y = x - \frac{x^3}{3!}$



y
$y = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \frac{x^9}{9!} - \frac{x^{11}}{11!}$
$y = \sin x$
o
x
-6
-4
-2
2
4
6
3
2
1
-1
-2
-3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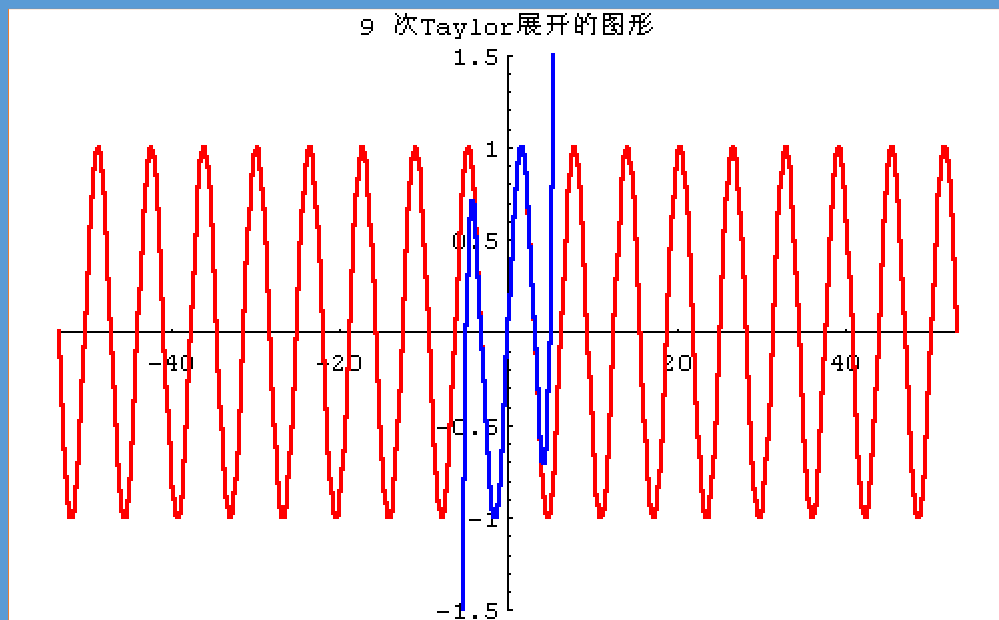
1
0.5
-0.5
-1
-1.5
-40
-20
20
40



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0.5
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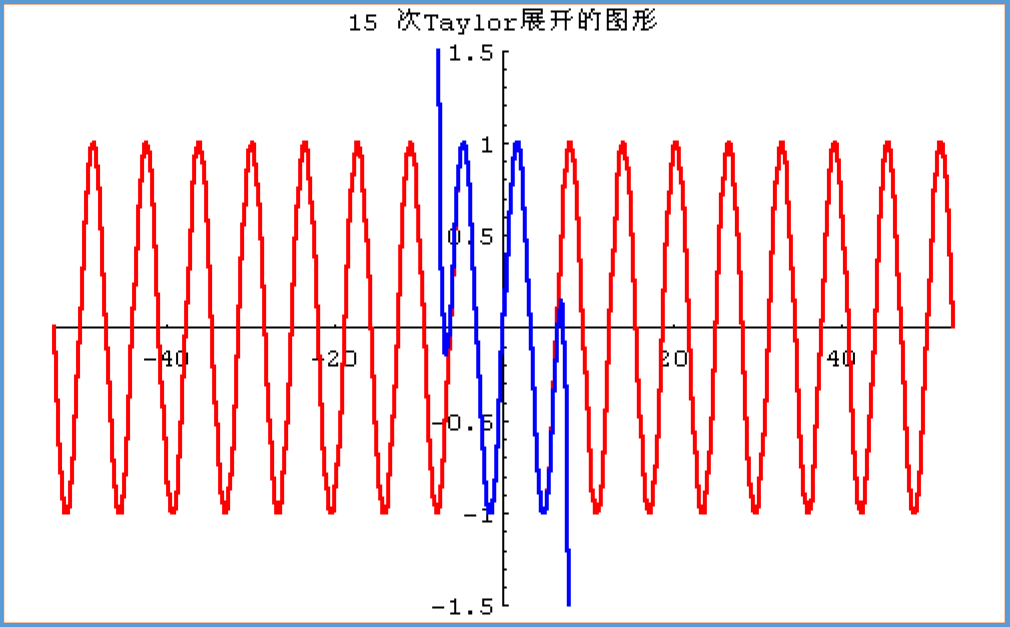
-1
-1.5
-40
-20
20
40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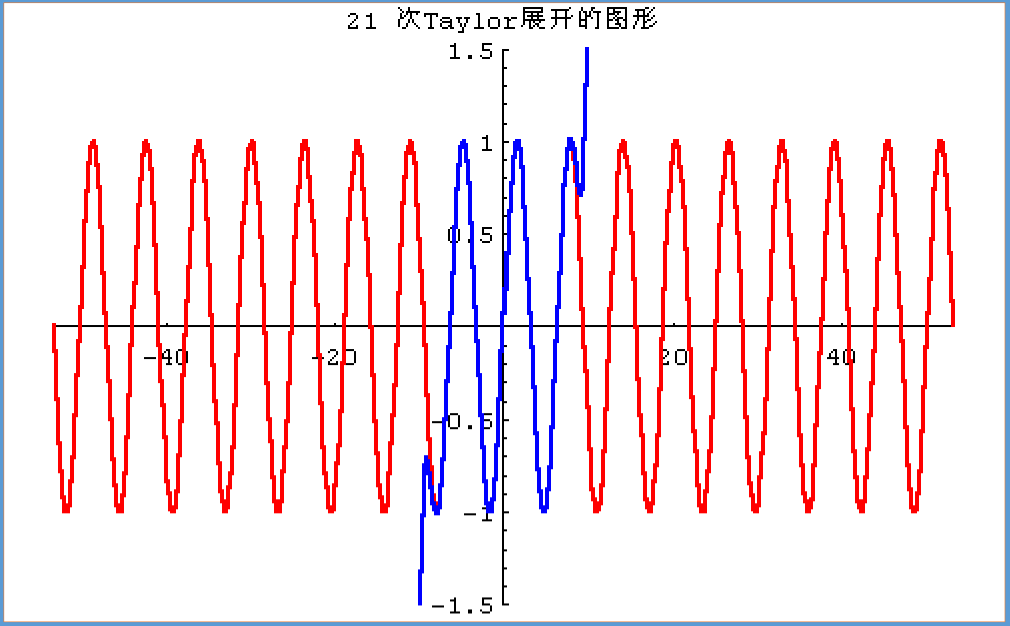
1
0.5
-0.5
-1
-1.5
-40
-20
20
40



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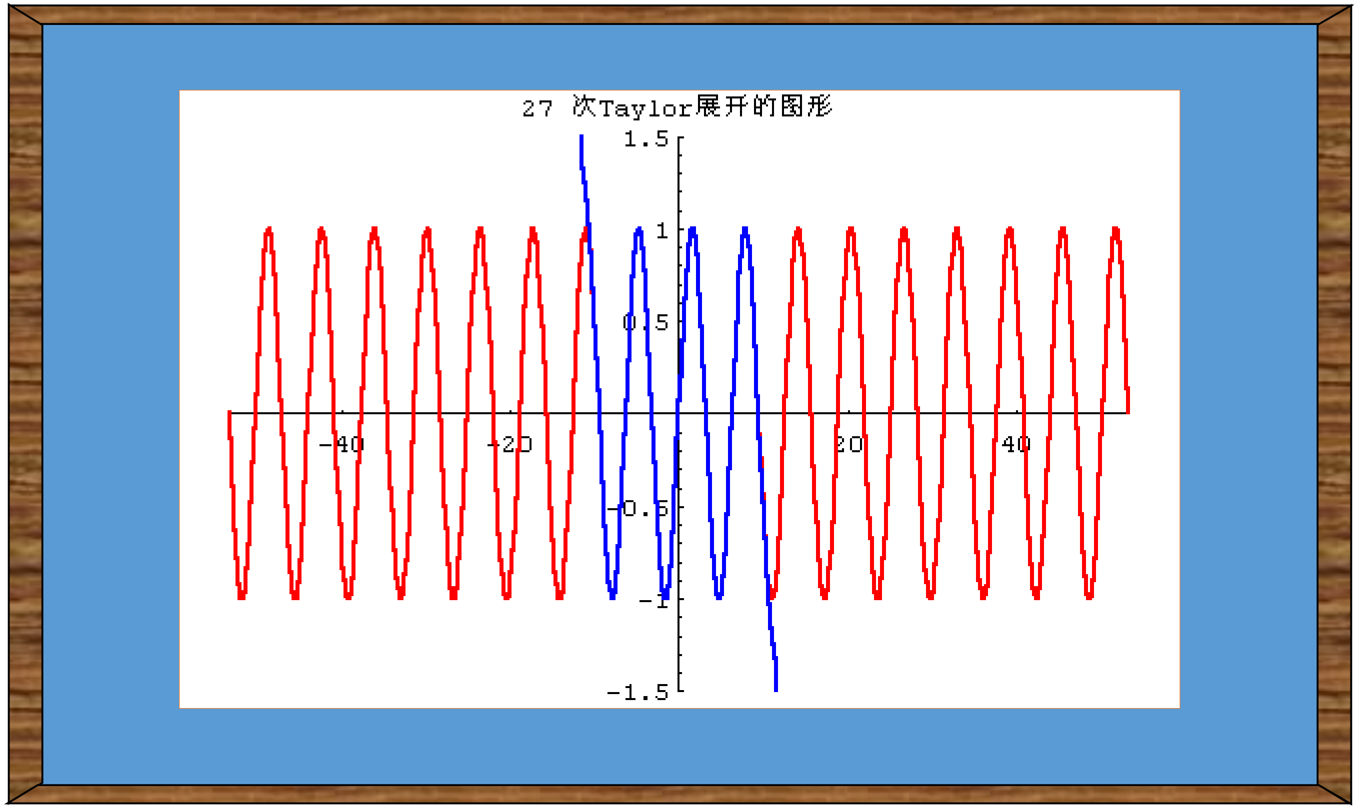
1.5
1
0.5
-0.5
-1
-1.5
-40
-20
20
40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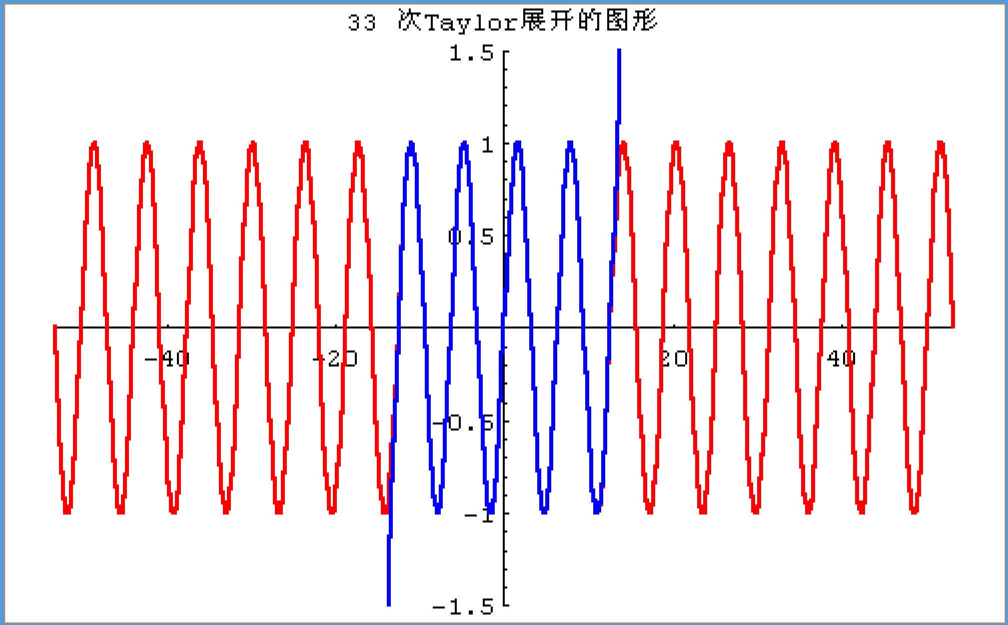
1.5
1
0.5
-0.5
-1
-1.5
-40
-20
20
40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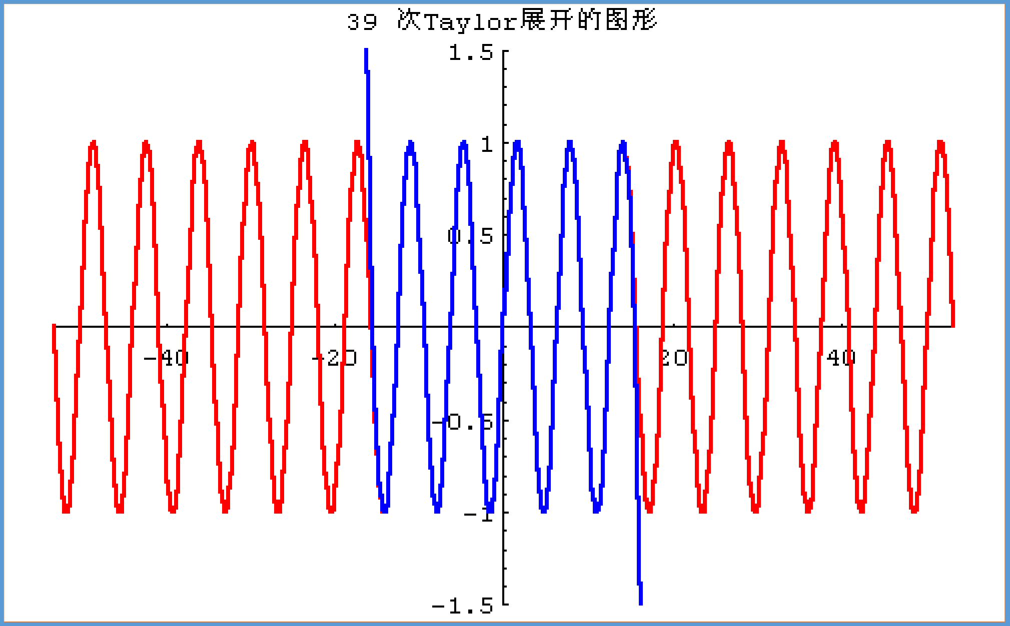
1.5
1
0.5
-0.5
-1
-1.5
-40
-20
20
40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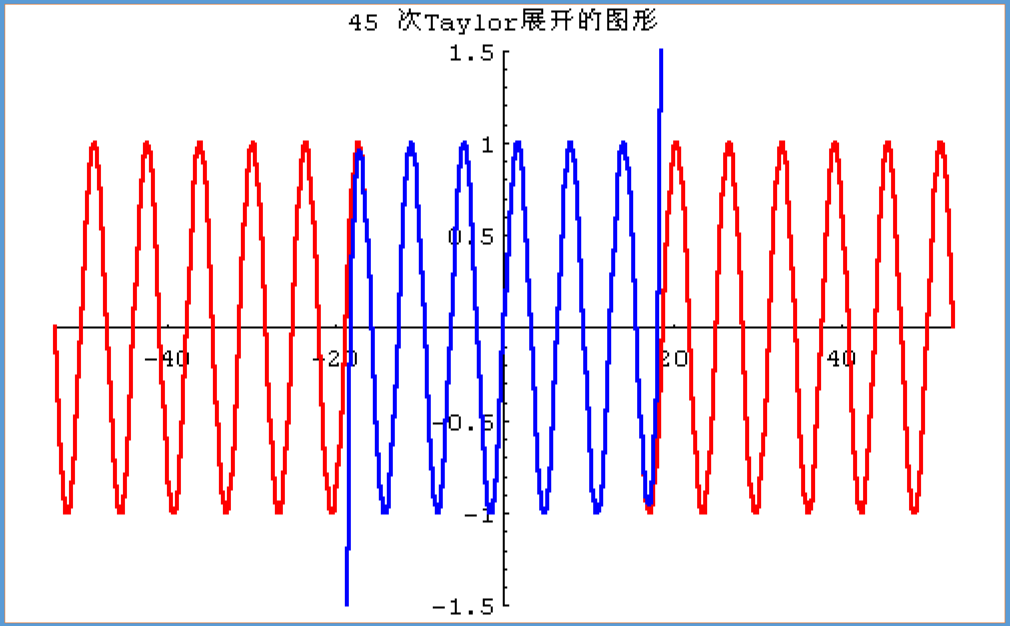
1.5
1
0.5
-0.5
-1
-1.5
-40
-20
20
40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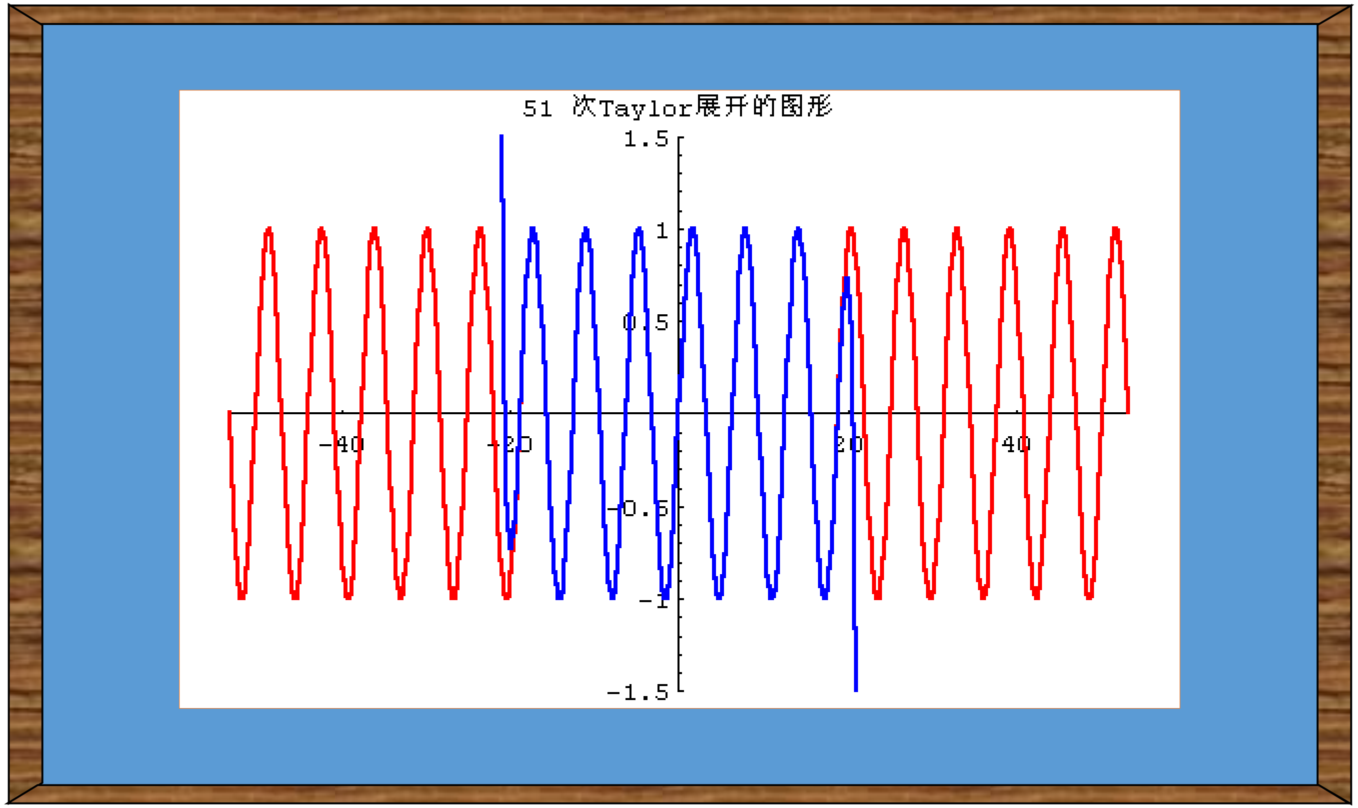
1.5
1
0.5
-0.5
-1
-1.5
-40
-20
20
40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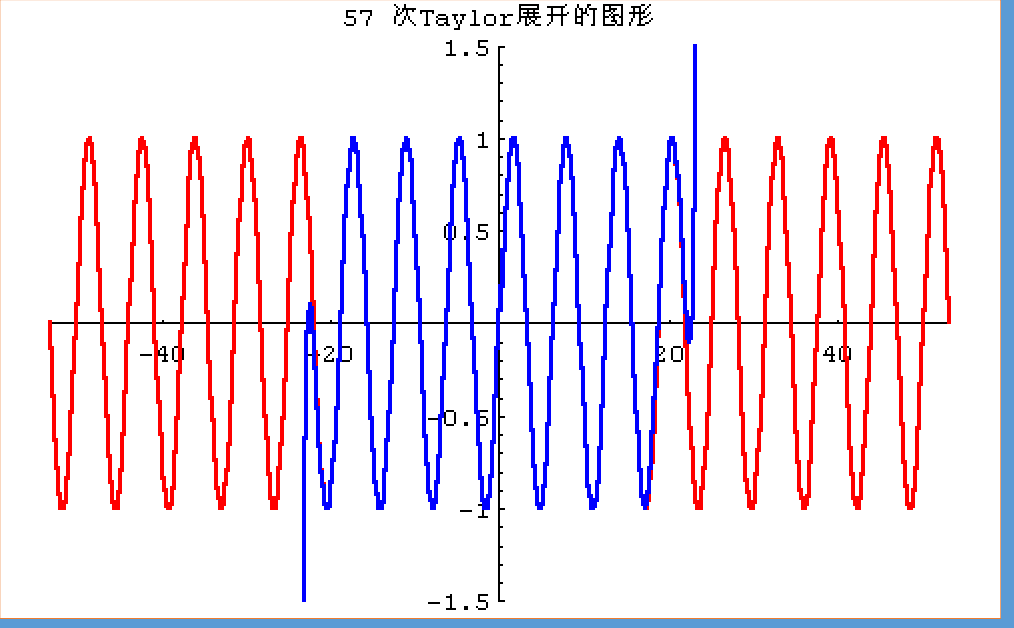
1.5
1
0.5
-40
-20
20
40
-0.5
-1
-1.5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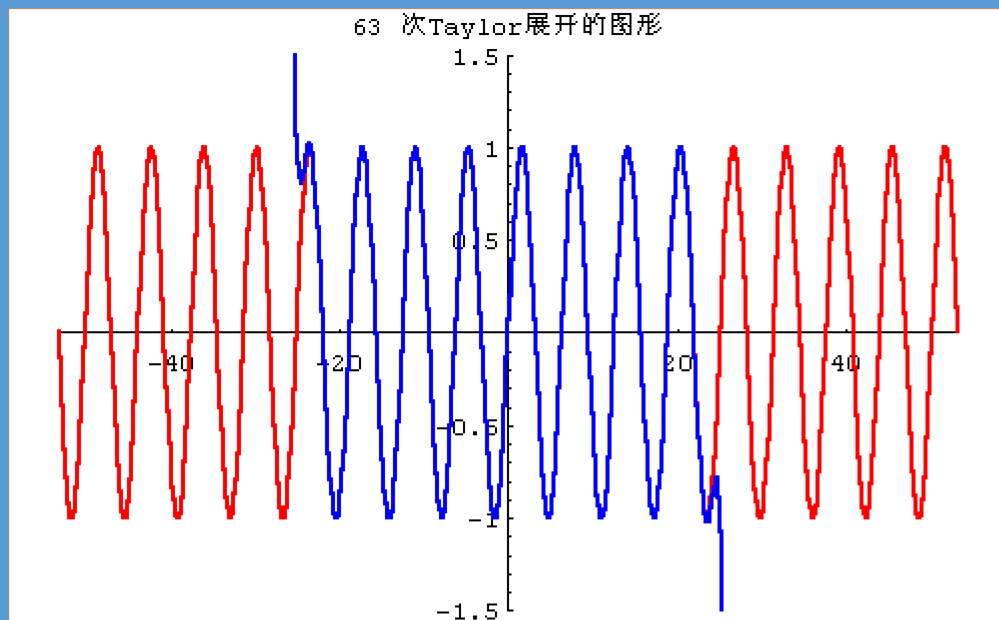
1.5
1
0.5
-0.5
-1
-1.5
-40
-20
20
40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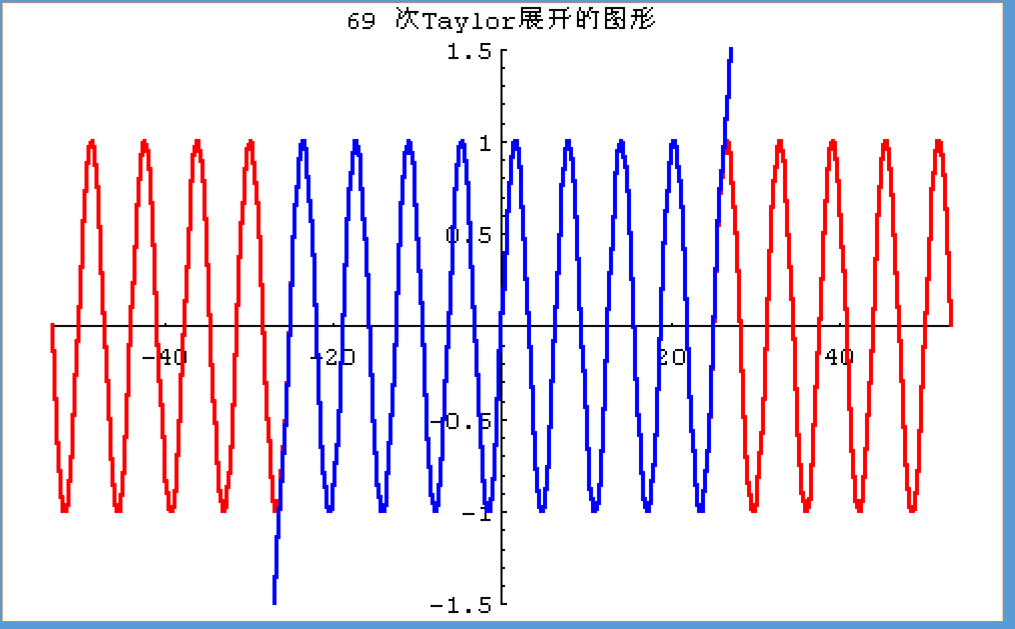
1
0.5
-0.5
-1
-1.5
-40
-20
20
40



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1
0.5
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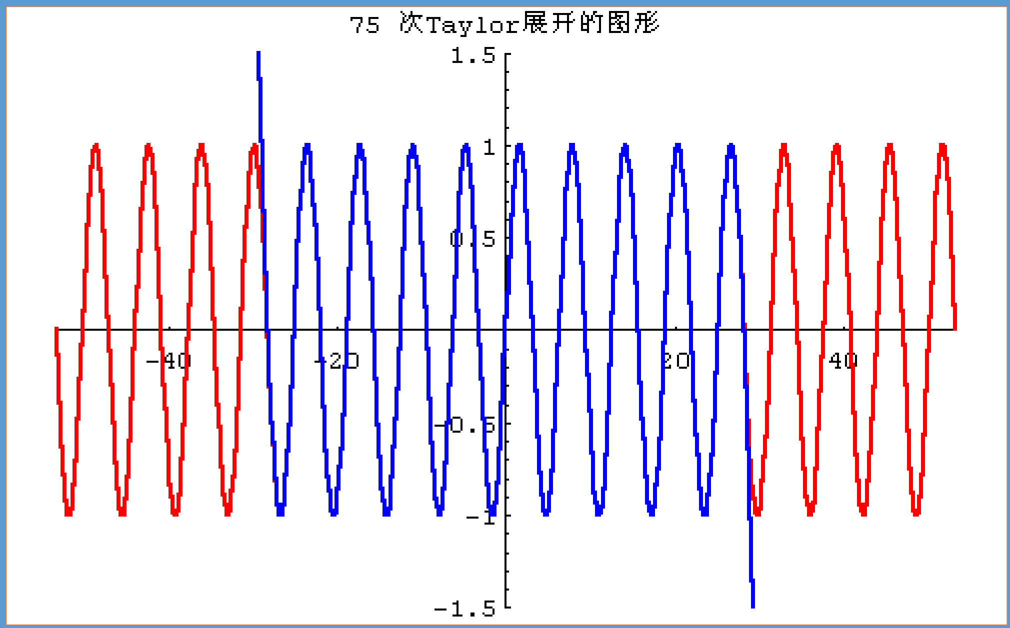
-1
-1.5
-40
-20
20
40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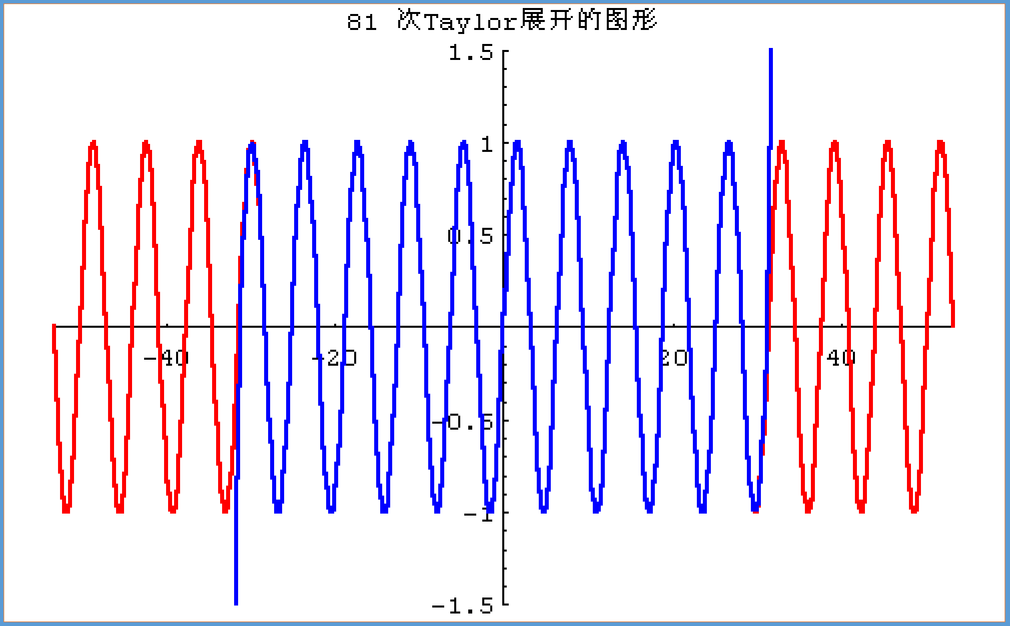
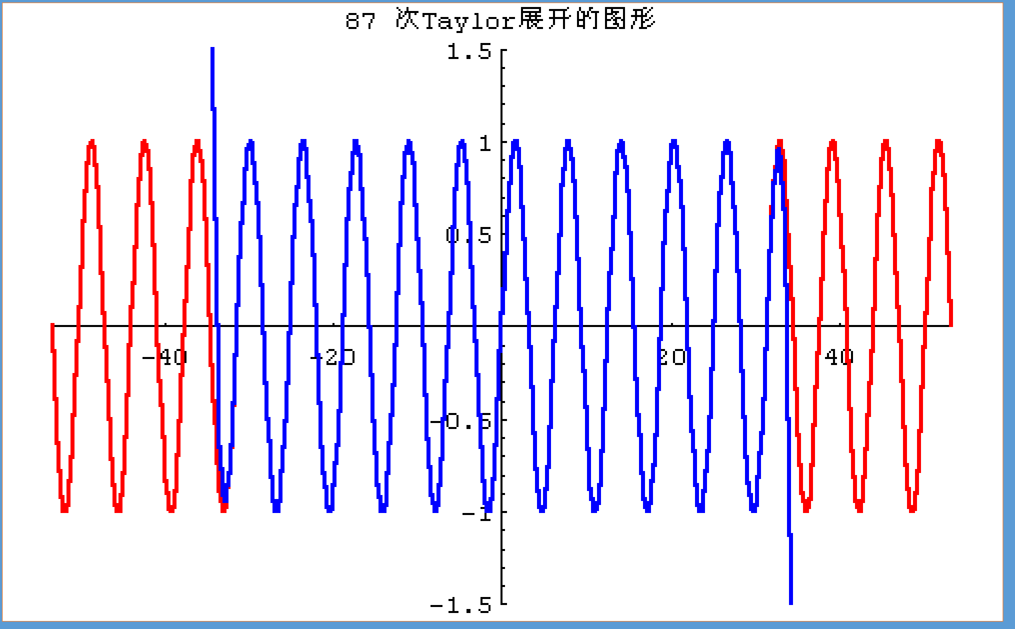
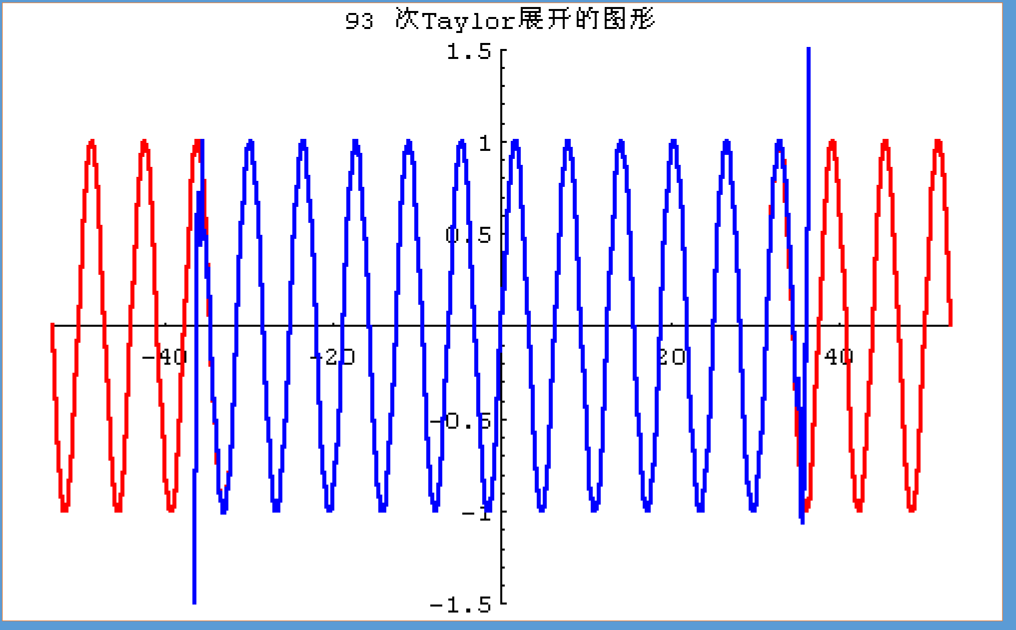
0.5
-0.5
-1
-1.5
-40
-20
20
40



81 次Taylor展开的图形
1.5
1
0.5
-0.5
-1
-1.5
-40
-20
20
40



87 次Taylor展开的图形
1.5
1
0.5
-0.5
-1
-1.5
-40
-20
20
40


93 次Taylor展开的图形
1.5
1
0.5
-0.5
-1
-1.5
-40
-20
20
40