

## 极限习题课2(两个准则)(5题)

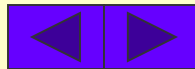
1. 设  $a_1 = 2, \dots, a_{n+1} = \frac{1}{2}(a_n + \frac{1}{a_n})$  ( $n = 1, 2, \dots$ ), 证明  $\lim_{n \rightarrow \infty} a_n$  存在, 并求之

$$2. \lim_{n \rightarrow \infty} \left( \frac{1}{n^2 + n + 1} + \frac{2}{n^2 + n + 2} + \dots + \frac{n}{n^2 + n + n} \right)$$

$$3. \text{求 } \lim_{n \rightarrow +\infty} \sqrt[n]{1 + \frac{1}{2} + \frac{1}{3} + \dots + \frac{1}{n}} \quad \text{提示: } \lim_{n \rightarrow \infty} \sqrt[n]{n} = 1$$

$$4. \lim_{n \rightarrow \infty} \sin(\pi \sqrt{n^2 + 1})$$

$$5. \text{证明: } \lim_{n \rightarrow \infty} \left( \frac{1}{2} \cdot \frac{3}{4} \cdot \frac{5}{6} \cdot \dots \cdot \frac{2n-1}{2n} \right) = 0$$



# 答案

1. 解 因为  $a_{n+1} = \frac{1}{2}(a_n + \frac{1}{a_n}) \geq \sqrt{a_n \cdot \frac{1}{a_n}} = 1$ , 所以  $\{a_n\}$  下方有界, 又因

为  $\frac{a_{n+1}}{a_n} = \frac{1}{2}(1 + \frac{1}{a_n^2}) \leq \frac{1}{2}(1 + 1) = 1$ , 所以  $\{a_n\}$  单调减少, 故  $\lim_{n \rightarrow \infty} a_n$  存在,

令  $\lim_{n \rightarrow \infty} a_n = l$ , 则  $\lim_{n \rightarrow \infty} a_{n+1} = \lim_{n \rightarrow \infty} \frac{1}{2}(a_n + \frac{1}{a_n})$ , 即  $l = \frac{1}{2}(l + \frac{1}{l})$ , 所以  $l = 1$ ,

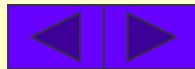
也即  $\lim_{n \rightarrow \infty} a_n = 1$ .

2. 解  $\frac{1 + 2 + \cdots + n}{n^2 + n + n} \leq \frac{1}{n^2 + n + 2} + \frac{2}{n^2 + n + 2} + \cdots + \frac{n}{n^2 + n + n} \leq \frac{1 + 2 + \cdots + n}{n^2 + n + 1}$

而  $\lim_{n \rightarrow \infty} \frac{1 + 2 + \cdots + n}{n^2 + n + n} = \lim_{n \rightarrow \infty} \frac{\frac{n(n+1)}{2}}{n^2 + n + n} = \frac{1}{2}$

$\lim_{n \rightarrow \infty} \frac{1 + 2 + \cdots + n}{n^2 + n + 1} = \lim_{n \rightarrow \infty} \frac{\frac{n(n+1)}{2}}{n^2 + n + 1} = \frac{1}{2}$

所以由夹逼定理得 : 原式 =  $\frac{1}{2}$



$$3. \text{ 解 } 1 < \sqrt[n]{1 + \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{n}} < \sqrt[n]{\underbrace{1+1+\cdots+1}_{n\uparrow}} = \sqrt[n]{n}$$

$$\text{而 } \lim_{n \rightarrow \infty} \sqrt[n]{n} = 1$$

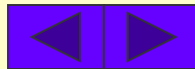
$$\text{故由夹逼准则知 } \lim_{n \rightarrow \infty} \sqrt[n]{1 + \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{n}} = 1.$$

$$4. \text{ 解 } \lim_{n \rightarrow \infty} \sin(\pi \sqrt{n^2 + 1}) = \lim_{n \rightarrow \infty} \sin[n\pi + \pi(\sqrt{n^2 + 1} - n)]$$

$$= \lim_{n \rightarrow \infty} (-1)^n \sin \frac{\pi}{\sqrt{n^2 + 1} + n}$$

$$\because 0 < \sin \frac{\pi}{\sqrt{n^2 + 1} + n} < \sin \frac{4}{2n} = \sin \frac{2}{n}$$

由夹逼定理得 原式 = 0



5. 证明： $\lim_{n \rightarrow \infty} \left( \frac{1}{2} \cdot \frac{3}{4} \cdot \frac{5}{6} \cdots \frac{2n-1}{2n} \right) = 0$

5. 证 令  $x_n = \frac{1}{2} \cdot \frac{3}{4} \cdot \frac{5}{6} \cdots \frac{2n-1}{2n}$

$$y_n = \frac{2}{3} \cdot \frac{4}{5} \cdot \frac{6}{7} \cdots \frac{2n}{2n+1}$$

则有  $0 < x_n < y_n$   $0 < x_n^2 < x_n y_n = \frac{1}{2n+1}$

得  $0 < x_n < \frac{1}{\sqrt{2n+1}}$  , 而  $\lim_{n \rightarrow \infty} \frac{1}{\sqrt{2n+1}} = 0$ ,

由夹逼定理得 原式 = 0

