习题课4(变上限函数的求导)15题

1. 已知$x \geq 0$时，$f(x)$连续，且$\int_0^{x^2(1+x)} f(t)\mathrm{d}t = x$，则$f(2) =$ ____

2. $\dfrac{\mathrm{d}}{\mathrm{d}x}\left(\int_{x^2}^{0} x\cos t^2 \mathrm{d}t\right)$

3. 已知$F(x) = \int_0^x \dfrac{\sqrt{x}-\sqrt{t}}{1+t}\mathrm{d}t$，求$F'(x)$.

4. 求 $\dfrac{d}{dx}\int_{\sin x}^{\cos x} \cos(\pi t^2)dt$

5. 设$f(x)$在$(-\infty,+\infty)$内可导，$f'(1) = f(1) = 1$，$g(x) = \int_{1+x}^{\mathrm{e}^x} f(t)\mathrm{d}t$，

则$g''(0) =$ ____.

6.已知当$x \to 0$时，$F(x)=\int_0^x (x^2-t^2)f''(t)\mathrm{d}t$ 的导数与x^2是等价无穷小，则$f''(0)$的值为

(A) 0 (B) $\frac{1}{2}$ (C) 1 (D) ∞

7.$\int_0^y \mathrm{e}^{t^2}\mathrm{d}t=\int_0^{3x^2}\ln\sqrt{t+x^2}\,\mathrm{d}t \quad (x>0)$，求$\frac{\mathrm{d}y}{\mathrm{d}x}$.

8.设 $\begin{cases} x=\int_1^t u\ln u\,\mathrm{d}u \\ y=\int_1^{t^2} u^2\ln u\,\mathrm{d}u \end{cases}$，求$\frac{\mathrm{d}y}{\mathrm{d}x},\frac{\mathrm{d}^2y}{\mathrm{d}x^2}$.(其中$t>0$)

9. $\dfrac{\mathrm{d}}{\mathrm{d}x}\int_0^x \sin(x-t)^2\,\mathrm{d}t=$ ________.

10. 设$f(x)$连续，则 $\dfrac{\mathrm{d}}{\mathrm{d}x}\left[\int_0^x tf(x^2-t^2)\mathrm{d}t\right]=$ ________.

11. 设$f(x)\in C[0,+\infty)$, $f(x)>0$, 求证

$$F(x)=\frac{\int_0^x tf(t)dt}{\int_0^x f(t)dt}$$

在 $(0,+\infty)$ 内是单调增加函数。

12. 求 $\lim\limits_{x\to 0}\dfrac{\int_{2x}^0 \sin t^2 dt}{x^3}$

13. 求 $\lim\limits_{x\to 0}\dfrac{\int_{\cos x}^{1} e^{-t^2}dt}{x^2}$

14.设$g(x)$在$(-\infty,+\infty)$上连续，$g(1)=1,\int_0^1 g(x)\mathrm{d}x=\dfrac{1}{2}$.

令$f(x)=\int_0^x g(x-t)t^2\mathrm{d}t$，求$f''(1), f'''(1)$.

15. 对一切实数 t, 函数$f(t)$是连续的正函数，函数

$$g(x)=\int_{-a}^{a}|x-t|f(t)dt,\quad -a\le x\le a(a>0).$$

证明$g'(x)$ 是单调增加的。

答案

1.解 等式两端同时求导得:$(2x+3x^2)f(x^2(1+x))=1$

令$x=1$得:$5f(2)=1\Rightarrow f(2)=\dfrac{1}{5}$

2. 解 $\dfrac{\mathrm{d}}{\mathrm{d}x}(\int_{x^2}^{0}x\cos t^2\mathrm{d}t)=\dfrac{\mathrm{d}}{\mathrm{d}x}(x\int_{x^2}^{0}\cos t^2\mathrm{d}t)$

$=\int_{x^2}^{0}\cos t^2\mathrm{d}t+x\cdot(-1)\cos(x^2)^2\cdot 2x$

3.解 $F(x)=\sqrt{x}\int_0^x\dfrac{1}{1+t}\mathrm{d}t-\int_0^x\dfrac{\sqrt{t}}{1+t}\mathrm{d}t$

$F'(x)=\dfrac{1}{2\sqrt{x}}\int_0^x\dfrac{1}{1+t}\mathrm{d}t+\sqrt{x}\cdot\dfrac{1}{1+x}-\dfrac{\sqrt{x}}{1+x}\ =\dfrac{1}{2\sqrt{x}}\int_0^x\dfrac{1}{1+t}\mathrm{d}t$

4.解 $\dfrac{d}{dx}\int_{\sin x}^{\cos x}\cos(\pi t^2)dt$

$$=\cos(\pi\cos^2 x)(\cos x)'-\cos(\pi\sin^2 x)(\sin x)'$$

$$=-\cos(\pi\cos^2 x)\sin x-\cos(\pi\sin^2 x)\cos x$$

$$=-\cos\left[\pi\left(1-\sin^2 x\right)\right]\sin x-\cos(\pi\sin^2 x)\cos x$$

$$=-\cos\left(\pi-\pi\sin^2 x\right)\sin x-\cos(\pi\sin^2 x)\cos x$$

$$=\cos(\pi\sin^2 x)\sin x-\cos(\pi\sin^2 x)\cos x$$

$$=(\sin x-\cos x)\cos(\pi\sin^2 x)$$

5. 设$f(x)$在$(-\infty,+\infty)$内可导，$f'(1)=f(1)=1$，$g(x)=\int_{1+x}^{e^x} f(t)\mathrm{d}t$，

则$g''(0)=$ ____.

解 $g'(x)=f(\mathrm{e}^x)\cdot\mathrm{e}^x-f(1+x)$

$$g''(x)=f'(\mathrm{e}^x)\cdot\mathrm{e}^{2x}+f(\mathrm{e}^x)\cdot\mathrm{e}^x-f'(1+x)$$

令$x=0$得：$g''(0)=f'(1)+f(1)-f'(1)=f(1)=1$

6. 解 $F'(x)=(x^2\int_0^x f''(t)\mathrm{d}t)'-(\int_0^x t^2 f''(t)\mathrm{d}t)'$

$$=x^2 f''(x)+2x\int_0^x f''(t)\mathrm{d}t-x^2 f''(x)=2x\left[f'(x)-f'(0)\right]$$

因为 $f''(0)=\lim\limits_{x\to 0}\dfrac{f'(x)-f'(0)}{x-0}=\lim\limits_{x\to 0}\dfrac{F'(x)}{2x^2}=\dfrac{1}{2}$.

7.解 两边同时对x求导.其中y是x的函数.

$$e^{y^2}\cdot\frac{dy}{dx}=8x\ln 2+6x\ln x \quad 故\frac{dy}{dx}=\frac{8x\ln 2+6x\ln x}{e^{y^2}}$$

$$8.\frac{dy}{dx}=\frac{2t\cdot t^4\ln t^2}{t\ln t}=4t^4,\quad \frac{d^2y}{dx^2}=\frac{16t^2}{t\ln t}$$

9. 解 $$\int_0^x \sin(x-t)^2 dt \underline{\underline{x-t=u}} -\int_x^0 \sin u^2 du=\int_0^x \sin u^2 du$$

$$\frac{d}{dx}\int_0^x \sin(x-t)^2 dt=\sin x^2$$

10.解 $$\int_0^x tf(x^2-t^2)dt \overset{x^2-t^2=u}{=\!=} \int_{x^2}^0 -\frac{1}{2}f(u)du=\frac{1}{2}\int_0^{x^2} f(u)du$$

$$\frac{d}{dx}[\int_0^x tf(x^2-t^2)dt]=\frac{1}{2}f(x^2)\cdot 2x=xf(x^2).$$

11.证 $F'(x)=\dfrac{xf(x)\int_0^x f(t)dt-f(x)\int_0^x tf(t)dt}{\left(\int_0^x f(t)dt\right)^2}$

$$F'(x)=\frac{f(x)\left(x\int_0^x f(t)dt-\int_0^x tf(t)dt\right)}{\left(\int_0^x f(t)dt\right)^2}=\frac{f(x)\left(\int_0^x xf(t)dt-\int_0^x tf(t)dt\right)}{\left(\int_0^x f(t)dt\right)^2}$$

$$=\frac{f(x)\int_0^x (x-t)f(t)dt}{\left(\int_0^x f(t)dt\right)^2}$$

$$\left.\begin{array}{r}\because t\in[0,x],\ f(t)>0,(x-t)f(t)\geq 0,\\ \text{且}(x-t)f(t)\not\equiv 0,\end{array}\right\}\Rightarrow$$

$$\int_0^x f(t)dt>0,\int_0^x (x-t)f(t)dt>0,\quad \therefore F'(x)>0,$$

$\therefore F(x)$在$(0,+\infty)$内是单调增加函数。

12. 求 $\lim\limits_{x\to 0}\dfrac{\int_{2x}^{0}\sin t^2 dt}{x^3}$

解 $\lim\limits_{x\to 0}\dfrac{\int_{2x}^{0}\sin t^2 dt}{x^3}\left(\dfrac{0}{0}\text{型}\right)=\lim\limits_{x\to 0}\dfrac{\left(\int_{2x}^{0}\sin t^2 dt\right)'}{(x^3)'}$

$$=\lim_{x\to 0}\frac{-\sin(2x)^2\cdot(2x)'}{3x^2}=-\lim_{x\to 0}\frac{4x^2\times 2}{3x^2}=-\frac{8}{3}$$

13. 求 $\lim\limits_{x\to 0}\dfrac{\int_{\cos x}^{1}e^{-t^2}dt}{x^2}$

解 $\lim\limits_{x\to 0}\dfrac{\int_{\cos x}^{1}e^{-t^2}dt}{x^2}\left(\dfrac{0}{0}\text{型}\right)=\lim\limits_{x\to 0}\dfrac{\left(\int_{\cos x}^{1}e^{-t^2}dt\right)'}{\left(x^2\right)'}$

$$=\lim_{x\to 0}\frac{-e^{-\cos^2 x}\cdot(-\sin x)}{2x}=\lim_{x\to 0}\frac{e^{-\cos^2 x}}{2}\cdot\frac{\sin x}{x}=\frac{1}{2e}$$

14. 解 $f(x)=\int_0^x g(x-t)t^2\mathrm{d}t \underline{\underline{x-t=u}} -\int_x^0 g(u)(x-u)^2\mathrm{d}u$

$$=\int_0^x g(u)(x^2-2xu+u^2)\mathrm{d}u$$

$$=x^2\int_0^x g(u)\mathrm{d}u-2x\int_0^x ug(u)\mathrm{d}u+\int_0^x u^2 g(u)\mathrm{d}u$$

$$f'(x)=2x\int_0^x g(u)\mathrm{d}u+x^2g(x)-2\int_0^x ug(u)\mathrm{d}u-2x^2g(x)+x^2g(x)$$

$$=2x\int_0^x g(u)\mathrm{d}u-2\int_0^x ug(u)\mathrm{d}u$$

$$f''(x)=2\int_0^x g(u)\mathrm{d}u+2xg(x)-2xg(x)=2\int_0^x g(u)\mathrm{d}u$$

所以 $f''(1)=2\int_0^1 g(u)\mathrm{d}u=2\times\frac{1}{2}=1$

$$f'''(x)=2g(x).$$

$$f'''(1)=2g(1)=2$$

15. 对一切实数 t, 函数 $f(t)$ 是连续的正函数，函数

$$g(x)=\int_{-a}^{a}|x-t|f(t)dt,\quad -a\le x\le a(a>0).$$

证明 $g'(x)$ 是单调增加的。

$t\in(-a,x)$ $t\in(x,a)$

证 $$g(x)=\int_{-a}^{a}|x-t|f(t)dt=\int_{-a}^{x}(x-t)f(t)dt+\int_{x}^{a}(t-x)f(t)dt$$

$$=x\int_{-a}^{x}f(t)dt-\int_{-a}^{x}tf(t)dt+\int_{x}^{a}tf(t)dt-x\int_{x}^{a}f(t)dt$$

$$g'(x)=\int_{-a}^{x}f(t)dt+xf(x)-xf(x)-xf(x)-\int_{x}^{a}f(t)dt+xf(x)$$

$$=\int_{-a}^{x}f(t)dt-\int_{x}^{a}f(t)dt=\int_{-a}^{x}f(t)dt+\int_{a}^{x}f(t)dt$$

$$g''(x)=f(x)+f(x)=2f(x)>0$$

$\therefore g'(x)$ 是单调增加的。