

导数习题课4(与定积分有关的3个综合题)

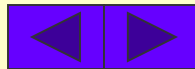
$$1. \text{ 设 } f(x) = \begin{cases} \frac{2}{x^2}(1 - \cos x), & x < 0 \\ 1, & x = 0 \\ \frac{1}{x} \int_0^x \cos t^2 dt, & x > 0 \end{cases},$$

试讨论 $f(x)$ 在 $x = 0$ 处的连续性和可导性.

$$2. \text{ 设 } f(x) \text{ 连续, } \varphi(x) = \int_0^1 f(xt) dt, \text{ 且 } \lim_{x \rightarrow 0} \frac{f(x)}{x} = A (A \text{ 为常数}),$$

求 $\varphi'(x)$, 并讨论 $\varphi'(x)$ 在 $x = 0$ 处的连续性.

$$3. \text{ 设 } \begin{cases} x = \cos(t^2) \\ y = t \cos(t^2) - \int_1^{t^2} \frac{1}{2\sqrt{u}} \cos u du \end{cases}, \text{ 求 } \frac{dy}{dx}, \frac{d^2 y}{dx^2} \text{ 在 } t = \sqrt{\frac{\pi}{2}} \text{ 的值.}$$



答案

1.解 (1) 由 $\lim_{x \rightarrow 0-0} \frac{2}{x^2} (1 - \cos x) = \lim_{x \rightarrow 0-0} \frac{\sin x}{x} = 1$

$$\lim_{x \rightarrow 0+0} \frac{1}{x} \int_0^x \cos t^2 dt = \lim_{x \rightarrow 0+0} \frac{\cos x^2}{1} = 1$$

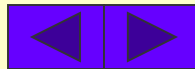
可知函数 $f(x)$ 在 $x = 0$ 处连续.

(2) 分别求 $f(x)$ 在 $x = 0$ 处的左、右导数

$$\begin{aligned} f'_-(0) &= \lim_{x \rightarrow 0-0} \frac{\frac{2}{x^2} (1 - \cos x) - 1}{x} = \lim_{x \rightarrow 0-0} \frac{2(1 - \cos x) - x^2}{x^3} = \lim_{x \rightarrow 0-0} \frac{2 \sin x - 2x}{3x^2} \\ &= \lim_{x \rightarrow 0-0} \frac{2 \cos x - 2}{6x} = \lim_{x \rightarrow 0-0} \frac{-\sin x}{3} = 0 \end{aligned}$$

$$\begin{aligned} f'_+(0) &= \lim_{x \rightarrow 0+0} \frac{\frac{1}{x} \int_0^x \cos t^2 dt}{x} - 1 = \lim_{x \rightarrow 0+0} \frac{\int_0^x \cos t^2 dt - x}{x^2} = \lim_{x \rightarrow 0+0} \frac{\cos x^2 - 1}{2x} \\ &= \lim_{x \rightarrow 0+0} \frac{-2x \sin x^2}{2} = 0 \end{aligned}$$

由于左、右导数都等于零, 可见 $f(x)$ 在 $x = 0$ 处可导. 且 $f'(0) = 0$.



2.由题设知 $f(0), \varphi(0) = 0$. 令 $u = xt$, 得 $\varphi(x) = \frac{\int_0^x f(u) \mathrm{d}u}{x}, (x \neq 0)$.

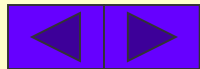
$$\text{从而 } \varphi'(x) = \frac{xf(x) - \int_0^x f(u) \mathrm{d}u}{x^2}, (x \neq 0).$$

由导数定义有

$$\varphi'(0) = \lim_{x \rightarrow 0} \frac{\int_0^x f(u) \mathrm{d}u}{x^2} = \lim_{x \rightarrow 0} \frac{f(x)}{2x} = \frac{A}{2}$$

$$\begin{aligned} \lim_{x \rightarrow 0} \varphi'(x) &= \lim_{x \rightarrow 0} \frac{xf(x) - \int_0^x f(u) \mathrm{d}u}{x^2} = \lim_{x \rightarrow 0} \frac{f(x)}{x} - \lim_{x \rightarrow 0} \frac{\int_0^x f(u) \mathrm{d}u}{x^2} \\ &= A - \frac{A}{2} = \frac{A}{2} = \varphi'(0). \end{aligned}$$

故 $\varphi'(x)$ 在 $x = 0$ 处连续.



$$3. \text{因 } \frac{dx}{dt} = -2t \sin(t^2), \quad \frac{dy}{dt} = -2t^2 \sin(t^2)$$

$$\text{所以 } \frac{dy}{dx} = t \quad \text{则} \quad \left. \frac{dy}{dx} \right|_{t=\sqrt{\frac{\pi}{2}}} = \frac{\pi}{2};$$

$$\frac{d^2 y}{dx^2} = \frac{\frac{d}{dt} \left(\frac{dy}{dx} \right)}{\frac{dx}{dt}} = \frac{\frac{d}{dt}(t)}{\frac{dx}{dt}} = -\frac{1}{2t \sin(t^2)}, \quad \text{则} \quad \left. \frac{d^2 y}{dx^2} \right|_{t=\sqrt{\frac{\pi}{2}}} = -\frac{1}{\sqrt{2\pi}}.$$

