课前练习

1. $\lim\limits_{x\to+\infty}\dfrac{x^{1999}}{x^n-(x-1)^n}=\dfrac{1}{a}$，则 $n=$ ______，$a=$ ______.

2. $\lim\limits_{x\to\infty}\left(\dfrac{x^2}{x+1}-ax-b\right)=0$， 求 a，b.

3. $\lim\limits_{x\to+\infty}\left(x-\sqrt{ax^2-bx}\right)=-1$. 求 a，b.

4. $\lim\limits_{x\to 2}\dfrac{x-2}{2x^2-3x+k}=a \quad (a\neq 0)$. 求 k，a

5. $f(x)=\lim\limits_{n\to\infty}\dfrac{x^n}{1+x+\cdots+x^{n-1}} \quad (x>0)$， 求 $f(x)$.

6. $f(x)=\lim\limits_{n\to\infty}\dfrac{1-x^{2n}}{1+x^{2n}}x$， 求 $f(x)$.

答案

1. $n = 2000 \quad a = 2000$

2. $a = 1, \ b = -1$

3. $a = 1, \ b = -2$

4. $k = -2, \ a = \dfrac{1}{5}$

5. $f(x) = \begin{cases} 0 & , 0 < x \le 1 \\ x - 1 & , x > 1 \end{cases}$

6. $f(x) = \begin{cases} x, & |x| < 1 \\ 0, & |x| = 1 \\ -x, & |x| > 1 \end{cases}$

第三节 极限存在准则
两个重要极限

上一节我们讨论了极限的四则运算法则，其中有"$\frac{0}{0}$"型未定式的题目：

$$1.\lim_{x\to 1}\frac{x^2-3x+2}{x^2+2x-3}=\lim_{x\to 1}\frac{(x-1)(x-2)}{(x-1)(x+3)}=\lim_{x\to 1}\frac{x-2}{x+3}=-\frac{1}{4}$$

$$2.\lim_{x\to 1}\frac{\sqrt[3]{x}-1}{\sqrt{x}-1}=\lim_{x\to 1}\frac{(x-1)(\sqrt{x}+1)}{(x-1)(\sqrt[3]{x^2}+\sqrt[3]{x}+1)}=\lim_{x\to 1}\frac{\sqrt{x}+1}{\sqrt[3]{x^2}+\sqrt[3]{x}+1}=\frac{2}{3}$$

$$\lim_{x\to 0}\frac{\sin x}{x}=\ ? \qquad \lim_{x\to 0}\frac{e^x-1}{x}=\ ?$$

一、夹逼准则及第一重要极限

准则Ⅰ　如果数列 $\{x_n\}$、$\{y_n\}$ 及 $\{z_n\}$ 满足下列条件：

(1) $y_n \le x_n \le z_n,\quad n = 1, 2, \cdots,$　(2) $\lim\limits_{n\to\infty} y_n = \lim\limits_{n\to\infty} z_n = a,$

则数列 $\{x_n\}$的极限存在，且 $\lim\limits_{n\to\infty} x_n = a$

准则Ⅰ'　(1) 当 $x \in \{x \mid 0 <|x - x_0| < h\}$ (或 $|x| > M$)时，有

$$g(x) \le f(x) \le h(x)$$

(2) $\lim\limits_{\substack{x\to x_0\\(x\to\infty)}} g(x) = \lim\limits_{\substack{x\to x_0\\(x\to\infty)}} h(x) = a,$

则 $\lim\limits_{\substack{x\to x_0\\(x\to\infty)}} f(x)$ 存在，且 $\lim\limits_{\substack{x\to x_0\\(x\to\infty)}} f(x) = a.$

例1. $\lim\limits_{n\to\infty}\left(\frac{1}{\sqrt{n^2+1}}+\frac{1}{\sqrt{n^2+2}}+\cdots+\frac{1}{\sqrt{n^2+n}}\right)$

分析：不能直接使用求极限的四则运算法则

$$\lim_{n\to\infty}\left(\frac{1}{n^2}+\frac{2}{n^2}+\cdots+\frac{n}{n^2}\right)=\lim_{n\to\infty}\frac{1+2+\cdots+n}{n^2}$$

$$=\lim_{n\to\infty}\frac{\frac{n(n+1)}{2}}{n^2}=\frac{1}{2}\lim_{n\to\infty}\left(1+\frac{1}{n}\right)=\frac{1}{2}$$

分母不同，如何求和？

解 记 $x_n=\frac{1}{\sqrt{n^2+1}}+\frac{1}{\sqrt{n^2+2}}+\cdots+\frac{1}{\sqrt{n^2+n}}$

每一项缩小

$$x_n \ge \frac{1}{\sqrt{n^2+n}}+\frac{1}{\sqrt{n^2+n}}+\cdots+\frac{1}{\sqrt{n^2+n}}=\frac{n}{\sqrt{n^2+n}}$$

每一项放大

$$x_n \le \frac{1}{\sqrt{n^2+1}}+\frac{1}{\sqrt{n^2+1}}+\cdots+\frac{1}{\sqrt{n^2+1}}=\frac{n}{\sqrt{n^2+1}}$$

$$\lim_{n\to\infty}\frac{n}{\sqrt{n^2+n}}=\lim_{n\to\infty}\frac{1}{\sqrt{1+\frac{1}{n}}}=1 \qquad \lim_{n\to\infty}\frac{n}{\sqrt{n^2+1}}=\lim_{n\to\infty}\frac{1}{\sqrt{1+\frac{1}{n^2}}}=1$$

$$\therefore \lim_{n\to\infty}\left(\frac{1}{\sqrt{n^2+1}}+\frac{1}{\sqrt{n^2+2}}+\cdots+\frac{1}{\sqrt{n^2+n}}\right)=1$$

例 2. $\lim\limits_{n \to \infty} \sqrt[n]{a^n + b^n} \quad (b > a > 0).$

解 $b = \sqrt[n]{b^n} \le \sqrt[n]{a^n + b^n} \le \sqrt[n]{b^n + b^n} = \sqrt[n]{2}\, b$

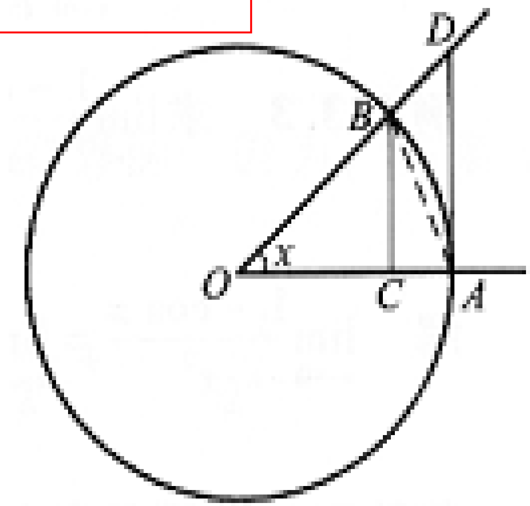
$\lim\limits_{n \to \infty} b = b$ $\quad \lim\limits_{n \to \infty} \sqrt[n]{2}\, b = b$ $\quad$ 公式：$\lim\limits_{n \to \infty} \sqrt[n]{a} = 1$

$\therefore \lim\limits_{n \to \infty} \sqrt[n]{a^n + b^n} = b$

夹逼准则推导出第一个重要极限

$$\lim_{x \to 0} \frac{\sin x}{x} = 1$$

证： 如图在单位圆中，此时 $0 < x < \dfrac{\pi}{2}$



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即：$\dfrac{1}{2}\sin x < \dfrac{x}{2} < \dfrac{1}{2}\tan x$

$\Rightarrow \sin x < x < \tan x$

以 $\sin x(\sin x \neq 0)$去除上式得：$1 < \dfrac{x}{\sin x} < \dfrac{1}{\cos x}$ 即 $\cos x < \dfrac{\sin x}{x} < 1$

由夹逼准则可得：$\lim\limits_{x \to 0^+} \dfrac{\sin x}{x} = 1.$ 当 $-\dfrac{\pi}{2} < x < 0$时，设 $y = -x$，

则 $\lim\limits_{x \to 0^-} \dfrac{\sin x}{x} = \lim\limits_{y \to 0^+} \dfrac{\sin(-y)}{-y} = \lim\limits_{y \to 0^+} \dfrac{\sin y}{y} = 1$

第一重要极限

$$\boxed{\lim_{x \to 0} \frac{\sin x}{x} = 1} \qquad \lim_{x \to 0} \frac{x}{\sin x} = 1$$

等价形式

注：(1) $\lim_{u \to 0} \frac{\sin u}{u} = 1$ $\qquad \lim_{\varphi(x) \to 0} \frac{\sin \varphi(x)}{\varphi(x)} = 1$

(2) $\lim_{x \to \infty} \frac{\sin x}{x} = 0$

$x \to \infty$时，$\frac{1}{x}$为无穷小，$|\sin x| \le 1$

(3) $\lim_{x \to x_0} \frac{\sin x}{x} = \frac{\sin x_0}{x_0} \quad (x_0 \neq 0)$ 例：$\lim_{x \to \frac{\pi}{2}} \frac{\sin x}{x} = \frac{2}{\pi}$

例题

$$1.\ \lim_{x \to 0} \frac{\tan x}{x} = \lim_{x \to 0} \frac{\sin x}{x} \cdot \frac{1}{\cos x} = 1$$

$$2.\ \lim_{x \to 0} \frac{1 - \cos x}{x^2} = \lim_{x \to 0} \frac{2 \sin^2 \frac{x}{2}}{x^2} = \frac{1}{2}$$

$$3.\ \lim_{x \to 0} \frac{x - \sin 3x}{x + \sin 3x} = \lim_{x \to 0} \frac{1 - \frac{\sin 3x}{x}}{1 + \frac{\sin 3x}{x}} = \frac{1 - \lim\limits_{x \to 0} \frac{\sin 3x}{x}}{1 + \lim\limits_{x \to 0} \frac{\sin 3x}{x}}$$

$$= \frac{1 - 3}{1 + 3} = -\frac{1}{2}$$

4. $\lim\limits_{x\to 0}\dfrac{x^2\sin\dfrac{1}{x}}{\tan x}$

$$= \lim_{x\to 0}\frac{x}{\sin x}\cdot\lim_{x\to 0}\cos x\cdot\lim_{x\to 0}x\sin\frac{1}{x} = 0$$

5. $\lim\limits_{x\to\infty}\dfrac{3x^2+5}{5x+3}\sin\dfrac{2}{x} \overset{t=\frac{1}{x}}{=} \lim\limits_{t\to 0}\dfrac{3+5t^2}{5+3t}\cdot\dfrac{\sin 2t}{t} = \dfrac{6}{5}.$

或 $I = \lim\limits_{x\to\infty}\dfrac{3x+\dfrac{5}{x}}{5x+3}\cdot\dfrac{\sin\dfrac{2}{x}}{\dfrac{1}{x}}$

思考题 求极限：$\lim\limits_{n\to\infty}\sin(\pi\sqrt{n^2+1})$

【116】极限 $\lim\limits_{n\to\infty}\sin(\pi\sqrt{n^2+1})=$ ________.

解 本题似乎无法下手，我们先将原式恒等变形后再利用无穷小量的性质来求.

$$\sin(\pi\sqrt{n^2+1})=\sin[n\pi+\pi(\sqrt{n^2+1}-n)]=(-1)^n\sin(\pi\sqrt{n^2+1}-\pi n),$$

$\{(-1)^n\}$是个有界量，而 $0<\pi(\sqrt{n^2+1}-n)=\dfrac{\pi}{\sqrt{n^2+1}+n}<\dfrac{2}{n}$.

因 $0<\sin(\pi\sqrt{n^2+1}-\pi n)<\sin\dfrac{2}{n}<\dfrac{2}{n}$，且 $\lim\limits_{n\to\infty}\dfrac{2}{n}=0$，由夹逼准则知

$$\lim_{n\to\infty}\sin(\pi\sqrt{n^2+1}-\pi n)=0,\quad \text{所以}\quad \lim_{n\to\infty}\sin(\pi\sqrt{n^2+1})=0.$$