三、定积分的计算

(一)定积分的换元法

定理 (换元公式)

若$f(x)\in C[a,b]$, $x=\varphi(t)$满足条件:

$(1)\varphi(\alpha)=a,\varphi(\beta)=b;$

$(2)\varphi(t)$在$[\alpha,\beta]$(或(β,α))上具有连续导数，且$\varphi(t)\in[a,b]$,

则有

$$\boxed{\int_a^b f(x)dx=\int_\alpha^\beta f[\varphi(t)]\varphi'(t)dt}$$

换元必换限

1. 计算 $\int_0^a \sqrt{a^2 - x^2}\,dx \ (a > 0).$

解 设 $x = a\sin t$，则 $dx = a\cos t dt$，

$$x = 0, \Rightarrow t = 0; \quad x = a, \Rightarrow t = \frac{\pi}{2}.$$

$$\int_0^a \sqrt{a^2 - x^2}\,dx = a^2 \int_0^{\pi/2} \cos^2 t dt = \frac{a^2}{2} \int_0^{\pi/2} (1 + \cos 2t) dt$$

$$= \frac{a^2}{2} \left[t + \frac{1}{2}\sin 2t \right]_0^{\frac{\pi}{2}} = \frac{\pi a^2}{4}.$$

2. 计算 $\int_0^{\frac{\pi}{2}} \cos^5 x \sin x dx$

解 设 $t = \cos x$，则 $dt = -\sin x dx$，

$$x = 0, \Rightarrow t = 1, \qquad x = \frac{\pi}{2}, \Rightarrow t = 0.$$

$$\int_0^{\frac{\pi}{2}} \cos^5 x \sin x dx = -\int_1^0 t^5 dt = \int_0^1 t^5 dt = \left[\frac{t^6}{6} \right]_0^1 = \frac{1}{6}.$$

$$\int_0^{\frac{\pi}{2}} \cos^5 x \sin x dx = -\int_0^{\frac{\pi}{2}} \cos^5 x d(\cos x)$$

$$= -\left[\frac{\cos^6 x}{6}\right]_0^{\frac{\pi}{2}} = -\left(0 - \frac{1}{6}\right) = \frac{1}{6}$$

3. 计算 $\int_0^{\pi} \sqrt{\sin^3 x - \sin^5 x} dx$

解: $\because \sqrt{\sin^3 x - \sin^5 x} = \sqrt{\sin^3 x(1 - \sin^2 x)}$

$$= \sin^{\frac{3}{2}} x \cdot |\cos x|, x \in [0, \pi]$$

$$|\cos x| = \begin{cases} \cos x, x \in \left[0, \frac{\pi}{2}\right]; \\ -\cos x, x \in \left[\frac{\pi}{2}, \pi\right]. \end{cases}$$

$$\therefore \int_0^{\pi} \sqrt{\sin^3 x - \sin^5 x} dx$$

$$= \int_0^{\frac{\pi}{2}} \sin^{\frac{3}{2}} x \cos x dx + \int_{\frac{\pi}{2}}^{\pi} \sin^{\frac{3}{2}} x(-\cos x) dx$$

$$=\int_0^{\frac{\pi}{2}}\sin^{\frac{3}{2}}x\,d(\sin x)-\int_{\frac{\pi}{2}}^{\pi}\sin^{\frac{3}{2}}x\,d(\sin x)$$

$$=\left[\frac{2}{5}\sin^{\frac{5}{2}}x\right]_0^{\frac{\pi}{2}}-\left[\frac{2}{5}\sin^{\frac{5}{2}}x\right]_{\frac{\pi}{2}}^{\pi}=\frac{2}{5}-\left(-\frac{2}{5}\right)=\frac{4}{5}.$$

注意　如果忽略了 $|\cos x|=-\cos x,\ x\in\left[\frac{\pi}{2},\pi\right]$,

则下列计算是错误的:

$$\sqrt{\sin^3 x-\sin^5 x}=\sin^{\frac{3}{2}}x\cdot\cos x,\ x\in[0,\pi]$$

$$\Rightarrow\int_0^{\pi}\sqrt{\sin^3 x-\sin^5 x}\,dx$$

$$=\int_0^{\pi}\sin^{\frac{3}{2}}x\cos x\,dx\cancel{=\left[\frac{2}{5}\sin^{\frac{5}{2}}x\right]_0^{\pi}}=0$$

4. 计算 $\int_0^4 \frac{x+2}{\sqrt{2x+1}}dx.$

解 $\sqrt{2x+1}=t, \Rightarrow x=\frac{t^2-1}{2}$, 则 $dx=tdt,$

$x=0, \Rightarrow t=1; x=4, \Rightarrow t=3.$

$$\int_0^4 \frac{x+2}{\sqrt{2x+1}}dx = \int_1^3 \frac{\frac{t^2-1}{2}+2}{t}tdt = \frac{1}{2}\int_1^3 (t^2+3)dt$$

$$= \frac{1}{2}\left[\frac{t^3}{3}+3t\right]_1^3 = \frac{22}{3}$$

5. 证明: 若 $f(x) \in C[-a,a], \Rightarrow$

$$\int_{-a}^{a} f(x)dx = \begin{cases} 2\int_0^a f(x)dx, & f(x) \text{ 为偶函数} \\ 0, & f(x) \text{ 为奇函数} \end{cases}$$

证 $\because \int_{-a}^{a} f(x)dx = \int_{-a}^{0} f(x)dx + \int_{0}^{a} f(x)dx$

$$\int_{-a}^{0} f(x)dx \overset{x=-t}{=\!=\!=} -\int_{a}^{0} f(-t)dt$$

$$= \int_{0}^{a} f(-t)dt = \int_{0}^{a} f(-x)dx$$

$$\Rightarrow \int_{-a}^{a} f(x)dx = \int_{0}^{a} f(-x)dx + \int_{0}^{a} f(x)dx$$

$$= \int_{0}^{a} [f(-x) + f(x)]dx$$

(1) 若 $f(x)$ 为偶函数, 则

$$f(x) + f(-x) = 2f(x), \Rightarrow \int_{-a}^{a} f(x)dx = 2\int_{0}^{a} f(x)dx.$$

(2) 若 $f(x)$为奇函数, 则

$$f(x) + f(-x) = 0, \quad \Rightarrow \int_{-a}^{a} f(x)dx = 0.$$

6. 若 $f(x)\in C[0,1]$, 证明

$$(1)\int_0^{\frac{\pi}{2}} f(\sin x)dx=\int_0^{\frac{\pi}{2}} f(\cos x)dx;$$

$$(2)\int_0^{\pi} xf(\sin x)dx=\frac{\pi}{2}\int_0^{\pi} f(\sin x)dx,$$

并由此计算 $\int_0^{\pi}\frac{x\sin x}{1+\cos^2 x}dx.$

证 (1) 设 $x=\frac{\pi}{2}-t$, 则 $dx=-dt$,

$$x=0,\Rightarrow t=\frac{\pi}{2};\ x=\frac{\pi}{2},\Rightarrow t=0.$$

$$\Rightarrow\int_0^{\frac{\pi}{2}} f(\sin x)dx=-\int_{\frac{\pi}{2}}^{0} f\left[\sin\left(\frac{\pi}{2}-t\right)\right]dt$$

$$=-\int_{\frac{\pi}{2}}^{0} f(\cos t)dt=\int_0^{\frac{\pi}{2}} f(\cos x)dx.$$

$$\boxed{\int_0^{\frac{\pi}{2}} f(\sin x)dx=\int_0^{\frac{\pi}{2}} f(\cos x)dx;}$$

证 $\int_0^{\pi} xf(\sin x)dx = \frac{\pi}{2}\int_0^{\pi} f(\sin x)dx$

(2) 设 $x=\pi-t$,则$dx=-dt$, $x=0,\Rightarrow t=\pi; x=\pi,\Rightarrow t=0.$

$$\Rightarrow \boxed{\int_0^{\pi} xf(\sin x)dx} = -\int_{\pi}^{0}(\pi-t)f[\sin(\pi-t)]dt$$

$$= \int_0^{\pi}(\pi-t)f(\sin t)dt = \int_0^{\pi}\pi f(\sin t)dt - \int_0^{\pi} tf(\sin t)dt$$

$$\boxed{= \int_0^{\pi}\pi f(\sin x)dx - \int_0^{\pi} xf(\sin x)dx}.$$

$$\Rightarrow \boxed{\int_0^{\pi} xf[\sin x]dx = \frac{\pi}{2}\int_0^{\pi} f[\sin x]dx}$$

$$\Rightarrow \int_0^{\pi}\frac{x\sin x}{1+\cos^2 x}dx = \frac{\pi}{2}\int_0^{\pi}\frac{\sin x}{1+\cos^2 x}dx.$$

$$= -\frac{\pi}{2}\int_0^{\pi}\frac{d(\cos x)}{1+\cos^2 x} = -\frac{\pi}{2}\left[\arctan(\cos x)\right]_0^{\pi} = \frac{\pi^2}{4}.$$

7. 设函数

$$f(x)=\begin{cases} xe^{-x^2}, & x\ge 0, \\ \dfrac{1}{1+\cos x}, & -1< x<0, \end{cases}$$

计算 $\int_1^4 f(x-2)dx.$

解 设$x-2=t$,则$dx=dt$,

$$x=1,\Rightarrow t=-1;\ x=4,\Rightarrow t=2.$$

$$\int_1^4 f(x-2)dx=\int_{-1}^2 f(t)dt=\int_{-1}^0\frac{dt}{1+\cos t}+\int_0^2 te^{-t^2}dt$$

$$=\int_{-1}^0\frac{1}{2\cos^2\frac{t}{2}}dt-\frac{1}{2}\cdot\int_0^2 e^{-t^2}d(-t^2)$$

$$=\left[\tan\frac{t}{2}\right]_{-1}^0-\left[\frac{1}{2}e^{-t^2}\right]_0^2=\tan\frac{1}{2}-\frac{1}{2}e^{-4}+\frac{1}{2}.$$

8. $f(x)\in C(-\infty,\infty),\ f(x+T)=f(x),\ \forall x\in(-\infty,\infty).$

证明 $\int_a^{a+T} f(x)dx=\int_0^T f(x)dx,\quad \forall a\in(-\infty,\infty).$

引进变量：$t=x-T.$

证 $\int_a^{a+T} f(x)dx=\int_a^T f(x)dx+\int_T^{a+T} f(x)dx.=\int_0^a f(x)dx?$

令 $x=t+T$，则 $dx=dt$; $x=T,\Rightarrow t=0;\ x=a+T,\Rightarrow t=a.$

$$\Rightarrow \boxed{\int_T^{a+T} f(x)dx}=\int_0^a f(t+T)dt=\boxed{\int_0^a f(t)dt}$$

$$\therefore \int_a^{a+T} f(x)dx=\int_a^T f(x)dx+\int_0^a f(x)dx=\int_0^T f(x)dx$$

（二）定积分的分部积分法

设函数 $u(x)$、$v(x)$ 在 $[a,b]$ 上具有连续的导函数 $u'(x)$、$v'(x)$，

则
$$(uv)' = u'v + uv'.$$

$$\boxed{\int_a^b u\,dv = [uv]_a^b - \int_a^b v\,du.}$$

—— 定积分的分部积分法公式

1. 计算 $\int_0^1 \arctan x dx$.

解 设$u = \arctan x, dv = dx$, 则$du = \dfrac{dx}{1+x^2}, v = x$,

$$\int_0^1 \arctan x dx = \left[x \arctan x\right]_0^1 - \int_0^1 \frac{x}{1+x^2} dx$$

$$= \frac{\pi}{4} - \frac{1}{2}\left[\ln(1+x^2)\right]_0^1 = \frac{\pi}{4} - \frac{1}{2}\ln 2.$$

2. 计算 $\int_0^1 e^{\sqrt{x}} dx$.

解 令$\sqrt{x} = t, \Rightarrow x = t^2$, 则$dx = 2tdt$,

$x = 0, \Rightarrow t = 0; x = 1, \Rightarrow t = 1.$

$$\int_0^1 e^{\sqrt{x}} dx = 2\int_0^1 te^t dt = 2\int_0^1 td(e^t)$$

$$= 2\left[te^t\right]_0^1 - 2\int_0^1 e^t dt = 2e - 2\left[e^t\right]_0^1 = 2.$$

3. 证明定积分公式

$$I_n=\int_0^{\frac{\pi}{2}}\sin^n x\,dx=\int_0^{\frac{\pi}{2}}\cos^n x\,dx$$

$$=\begin{cases}\dfrac{n-1}{n}\cdot\dfrac{n-3}{n-2}\cdots\cdots\dfrac{3}{4}\cdot\dfrac{1}{2}\cdot\dfrac{\pi}{2}, & n\text{为正偶数；}\\[2ex] \dfrac{n-1}{n}\cdot\dfrac{n-3}{n-2}\cdots\cdots\dfrac{4}{5}\cdot\dfrac{2}{3}, & n\text{为大于1的正奇数。}\end{cases}$$

证 令 $u=\sin^{n-1}x, dv=\sin x\,dx,$

则 $du=(n-1)\sin^{n-2}x\cos x\,dx, v=-\cos x.$

$$I_n=\int_0^{\frac{\pi}{2}}\sin^n x\,dx=\int_0^{\frac{\pi}{2}}\sin^{n-1}x\,d(-\cos x)$$

$$=\left[-\cos x\sin^{n-1}x\right]_0^{\frac{\pi}{2}}+(n-1)\int_0^{\frac{\pi}{2}}\sin^{n-2}x\cos^2 x\,dx$$

$$=(n-1)\int_0^{\frac{\pi}{2}}\sin^{n-2}x(1-\sin^2 x)\,dx$$

$$= (n-1)\int_0^{\frac{\pi}{2}} \sin^{n-2} x dx - (n-1)\int_0^{\frac{\pi}{2}} \sin^n x dx$$

$$I_n = (n-1)I_{n-2} - (n-1)I_n$$

$$\boxed{I_n = \frac{n-1}{n} I_{n-2}.}$$ —— I_n 关于下标的递推公式

$$I_{n-2} = \frac{n-3}{n-2} I_{n-4}.$$

依次进行下去可得

$$I_{2m} = \frac{2m-1}{2m}\cdot\frac{2m-3}{2m-2}\cdot\frac{2m-5}{2m-4}\cdots\cdots\frac{5}{6}\cdot\frac{3}{4}\cdot\frac{1}{2}I_0,$$

$$I_{2m+1} = \frac{2m}{2m+1}\cdot\frac{2m-2}{2m-1}\cdot\frac{2m-4}{2m-3}\cdots\cdots\frac{6}{7}\cdot\frac{4}{5}\cdot\frac{2}{3}I_1$$

$$(m = 1,2,\cdots).$$

$$I_0 = \int_0^{\frac{\pi}{2}} dx = \frac{\pi}{2}, \qquad I_1 = \int_0^{\frac{\pi}{2}} \sin x dx = 1,$$

$$I_{2m}=\int_0^{\frac{\pi}{2}}\sin^{2m}x\,dx$$

$$=\frac{2m-1}{2m}\cdot\frac{2m-3}{2m-2}\cdot\frac{2m-5}{2m-4}\cdots\cdots\frac{5}{6}\cdot\frac{3}{4}\cdot\frac{1}{2}\cdot\frac{\pi}{2},$$

$$(m=1,2,\cdots).$$

$$I_{2m+1}=\int_0^{\frac{\pi}{2}}\sin^{2m+1}x\,dx$$

$$=\frac{2m}{2m+1}\cdot\frac{2m-2}{2m-1}\cdot\frac{2m-4}{2m-3}\cdots\cdots\frac{6}{7}\cdot\frac{4}{5}\cdot\frac{2}{3}$$

$$\int_0^{\frac{\pi}{2}}\sin^n x\,dx=\int_0^{\frac{\pi}{2}}\cos^n x\,dx.$$

证毕

4.设$f(x)$为连续函数，证明：

$$\int_0^x f(t)(x-t)\mathrm{d}t=\int_0^x\left(\int_0^t f(u)\mathrm{d}u\right)\mathrm{d}t.$$

证 $$\int_0^x\left(\int_0^t f(u)\mathrm{d}u\right)\mathrm{d}t=t\int_0^t f(u)\mathrm{d}u\Big|_0^x-\int_0^x tf(t)\mathrm{d}t$$

$$=x\int_0^x tf(t)\mathrm{d}t-\int_0^x tf(t)\mathrm{d}t$$

$$=\int_0^x f(t)(x-t)\mathrm{d}t$$

即 $$\int_0^x f(t)(x-t)\mathrm{d}t=\int_0^x\left(\int_0^t f(u)\mathrm{d}u\right)\mathrm{d}t$$