

导数习题课3(分段函数的导数、高阶导数)

1. 设 $f(x) = \begin{cases} g(x) \cos \frac{1}{x}, & x \neq 0 \\ 0, & x = 0 \end{cases}$, $g(0) = g'(0) = 0$, 求 $f'(0)$.

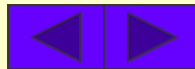
2. 设 $f(x) = \begin{cases} \frac{g(x) - e^{-x}}{x}, & x \neq 0 \\ 0, & x = 0 \end{cases}$, 其中 $g(x)$ 有二阶连续导数,

且 $g(0) = 1, g'(0) = -1$.

(1) 求 $f'(x)$; (2) 讨论 $f'(x)$ 在 $(-\infty, +\infty)$ 上的连续性.

3. 设 $F(x) = \begin{cases} e^x \cos x, & x \leq 0 \\ ax^2 + bx + c, & x > 0 \end{cases}$,

试确定 a, b, c , 使 $F(x)$ 在 $(-\infty, +\infty)$ 上二阶可导.



4.已知函数 $f(x)$ 具有任意阶导数，且 $f'(x) = [f(x)]^2$ ，
则 $f^{(n)}(x) = \underline{\hspace{2cm}}$. ($n \geq 2$)

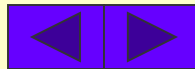
5.设 $y = \sin^4 x - \cos^4 x$ ，则 $y^{(n)} = \underline{\hspace{2cm}}$.

6.设 $y = \sin[f(x^2)]$ ，其中 f 具有二阶导数，求 $\frac{d^2 y}{dx^2}$.

7.设 $f(x) = (x - a)^n \varphi(x)$ ，其中 $\varphi(x)$ 在 a 点的一个邻域内有
($n - 1$)阶连续导数，则 $f^{(n)}(x) = \underline{\hspace{2cm}}$.

8.若 $y = [f(x^2)]^{\frac{1}{x}}$ ，其中 f 为可微正值函数，求 dy .

9.求函数 $f(x) = x^2 \ln(1 + x)$ 在 $x = 0$ 处的 n 阶导数 $f^{(n)}(0)$ ($n \geq 3$).

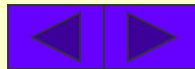


10. 设曲线 $y = f(x)$ 与 $y = \sin x$ 在原点相切, 求 $\lim_{n \rightarrow \infty} \sqrt{nf\left(\frac{2}{n}\right)}$.

11. 设函数 $y = y(x)$ 由方程组 $\begin{cases} x = 3t^2 + 2t + 3 \\ e^y \sin t - y + 1 = 0 \end{cases}$ 所确定,

试求: $\frac{d^2 y}{dx^2} \Big|_{t=0}$.

12. 曲线 $y = \frac{1}{\sqrt{x}}$ 的切线与 x 轴和 y 轴围成一个三角形, 记切点的横坐标为 a , 试求切线方程和此三角形面积. 又问当切点沿曲线趋于无穷远时, 该面积变化趋势如何?



答案

$$1. f'(0) = \lim_{x \rightarrow 0} \frac{f(x) - f(0)}{x} = \lim_{x \rightarrow 0} \frac{g(x) \cos \frac{1}{x}}{x}.$$

因为 $g'(0) = 0$. 所以 $x \rightarrow 0$ 时, $\frac{g(x) - g(0)}{x} \rightarrow 0$.

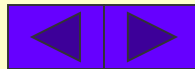
根据无穷小与有界量乘积仍为无穷小量知 $f'(0) = 0$

2. (1) 当 $x \neq 0$ 时, 有

$$f'(x) = \frac{x[g'(x) + e^{-x}] - g(x) + e^{-x}}{x^2} = \frac{xg'(x) - g(x) + (x+1)e^{-x}}{x^2}$$

当 $x = 0$ 时, 由导数定义有

$$\begin{aligned} f'(0) &= \lim_{x \rightarrow 0} \frac{g(x) - e^{-x}}{x^2} = \lim_{x \rightarrow 0} \frac{g'(x) + e^{-x}}{2x} = \lim_{x \rightarrow 0} \frac{g''(x) - e^{-x}}{2} \\ &= \frac{g''(0) - 1}{2} \end{aligned}$$

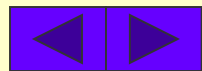


$$\text{所以 } f'(x) = \begin{cases} \frac{xg'(x) - g(x) + (x+1)e^{-x}}{x^2}, & x \neq 0 \\ \frac{g''(0) - 1}{2}, & x = 0 \end{cases}$$

(2) 因为在 $x = 0$ 处, 有

$$\begin{aligned} \lim_{x \rightarrow 0} f'(x) &= \lim_{x \rightarrow 0} \frac{g'(x) + xg''(x) - g'(x) + e^{-x} - (x+1)e^{-x}}{2x} \\ &= \lim_{x \rightarrow 0} \frac{g''(x) - e^{-x}}{2} = \frac{g''(0) - 1}{2} = f'(0) \end{aligned}$$

而 $f'(x)$ 在 $x \neq 0$ 处是连续函数, 所以 $f'(x)$ 在 $(-\infty, +\infty)$ 上为连续函数.



3. 显然当 $x \in (-\infty, 0)$ 及 $x \in (0, +\infty)$ 时, $F(x)$ 均为二阶可导函数

由 $F(x)$ 在 $x = 0$ 点二阶可导. 故 $F(x)$ 及 $F'(x)$ 在 $x = 0$ 点必连续

$$F(0-0) = \lim_{x \rightarrow 0^-} e^x \cos x = 1 \quad F(0+0) = \lim_{x \rightarrow 0^+} (ax^2 + bx + c) = c$$

故 $c = 1$

$$F'_-(0) = \lim_{x \rightarrow 0^-} \frac{F(x) - F(0)}{x} = \lim_{x \rightarrow 0^-} \frac{e^x \cos x - 1}{x} \cdot \underbrace{\frac{0}{0}}_{\lim_{x \rightarrow 0^-} e^x (\cos x - \sin x) = 1} = 1.$$

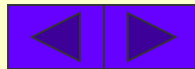
$$F'_+(0) = \lim_{x \rightarrow 0^+} \frac{F(x) - F(0)}{x} = \lim_{x \rightarrow 0^+} \frac{ax^2 + bx + 1 - 1}{x} = b \quad \text{故} \quad b = 1.$$

$$F'(x) = \begin{cases} e^x (\cos x - \sin x) & x \leq 0 \\ 2ax + b & x > 0 \end{cases}$$

$$\bullet F''_- = \lim_{x \rightarrow 0^-} \frac{F'(x) - F'(0)}{x} = \lim_{x \rightarrow 0^-} \frac{e^x (\cos x - \sin x) - 1}{x} = 0$$

$$F''_+(0) = \lim_{x \rightarrow 0^+} \frac{F'(x) - F'(0)}{x} = \lim_{x \rightarrow 0^+} \frac{2ax + 1 - 1}{x} = 2a \quad \text{故} \quad a = 0$$

综上所述: 当 $a = 0, b = c = 1$ 时, 可使 $F(x)$ 在 $(-\infty, +\infty)$ 上二阶可导.



$$4. \quad f''(x) = 2f(x) \cdot f'(x) = 2[f(x)]^3$$

$$f'''(x) = 3![f(x)]^2 \cdot f'(x) = 3![f(x)]^4$$

假设 $f^{(n-1)}(x) = (n-1)![f(x)]^n$,

则 $f^{(n)}(x) = (n-1)!n[f(x)]^{n-1} \cdot f'(x) = n![f(x)]^{n+1}$

$$5. y = \sin^4 x - \cos^4 x = (\sin^2 x - \cos^2 x)(\sin^2 x + \cos^2 x) \\ = \sin^2 x - \cos^2 x = -\cos 2x, y' = 2 \sin 2x,$$

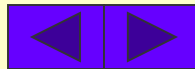
$$y^{(n)} = 2 \cdot 2^{n-1} \sin(2x + \frac{n-1}{2}\pi) = 2^n \sin(2x + \frac{n-1}{2}\pi)$$

故应填 $2^n \sin(2x + \frac{n-1}{2}\pi)$.

$$6. \quad \frac{dy}{dx} = 2xf'(x^2) \cos[f(x^2)]$$

$$\frac{d^2 y}{dx^2} = 2 \{ f'(x^2) \cos[f(x^2)] + 2x^2 f''(x^2) \cos[f(x^2)] - 2x^2 [f'(x^2)]^2 \sin[f(x^2)] \}$$

$$= 2f'(x^2) \cos[f(x^2)] + 4x^2 \{ f''(x^2) \cos[f(x^2)] - [f'(x^2)]^2 \sin[f(x^2)] \}$$



7. 由莱布尼兹公式有

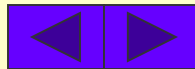
$$\begin{aligned} f^{(n-1)}(x) &= [(x-a)^n \varphi(x)]^{(n-1)} \\ &= (x-a)^n \varphi^{(n-1)}(x) + C_{n-1}^1 n(x-a)^{n-1} \varphi^{(n-2)}(x) + \cdots \\ &\quad + C_{n-1}^{n-2} n(n-1) \cdots 3(x-a)^2 \varphi'(x) + n!(x-a) \varphi(x) \end{aligned}$$

由此可知, $f^{(n-1)}(a) = 0$, 再由导数定义得

$$\begin{aligned} f^{(n)}(a) &= \lim_{x \rightarrow a} \frac{f^{(n-1)}(x) - f^{(n-1)}(a)}{x - a} \\ &= \lim_{x \rightarrow a} [(x-a)^{n-1} \varphi^{(n-1)}(x) + C_{n-1}^1 n(x-a)^{n-2} \varphi^{(n-2)}(x) + \cdots \\ &\quad + C_{n-1}^{n-2} n(n-1) \cdots 3(x-a) \varphi'(x) + n! \varphi(x)] = n! \varphi(a) \end{aligned}$$

8. 解 $y = [f(x^2)]^{\frac{1}{x}} \quad \ln y = \frac{\ln f(x^2)}{x}$

$$\begin{aligned} \frac{1}{y} y' &= \frac{x \cdot \frac{1}{f(x^2)} f'(x^2) \cdot 2x - \ln f(x^2)}{x^2} = \frac{2x^2 f'(x^2) - f(x^2) \ln f(x^2)}{x^2 f(x^2)} \\ &= \frac{2f'(x^2)}{f(x^2)} - \frac{\ln f(x^2)}{x^2} \quad \text{故} \quad dy = [f(x^2)]^{\frac{1}{x}} \left[\frac{2f'(x^2)}{f(x^2)} - \frac{\ln f(x^2)}{x^2} \right] dx. \end{aligned}$$



9. 由莱布尼兹公式

$$(uv)^{(n)} = u^{(n)}v^{(0)} + C_n^1 u^{(n-1)}v^{(1)} + C_n^2 u^{(n-2)}v^{(2)} + \cdots + u^{(0)}v^{(n)}$$

$$\text{及 } [\ln(1+x)]^{(k)} = \frac{(-1)^{k-1}(k-1)!}{(1+x)^k} \quad (k \text{ 为整数})$$

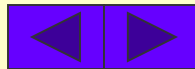
$$\text{得 } f^{(n)}(x) = x^2 \frac{(-1)^{n-1}(n-1)!}{(1+x)^n} + 2nx \frac{(-1)^{n-2}(n-2)!}{(1+x)^{n-1}} + n(n-1) \frac{(-1)^{n-3}(n-3)!}{(1+x)^{n-2}}$$

$$\text{所以 } f^{(n)}(0) = (-1)^{n-3}n(n-1)(n-3)! = \frac{(-1)^{n-1}n!}{n-2}.$$

10. 解 $f'(x) = (\sin x)' = \cos x$, 因 $(0,0)$ 为切点.

$$\text{故 } f(0) = \sin 0 = 0 \quad f'(0) = (\sin x)' \Big|_{x=0} = 1$$

$$\text{所以 } \lim_{n \rightarrow \infty} \sqrt{nf\left(\frac{2}{n}\right)} = \lim_{n \rightarrow \infty} \sqrt{2 \cdot \frac{f\left(\frac{2}{n}\right) - f(0)}{\frac{2}{n}}} = \lim_{n \rightarrow \infty} \sqrt{2f'(0)} = \sqrt{2}.$$



11.解 对方程组两边分别取微分, 得

$$\begin{cases} dx = (6t + 2)dt \\ e^y \sin t dy + e^y \cos t dt - dy = 0 \end{cases}$$

$$\text{则 } \frac{dx}{dt} = 6t + 2, \frac{dy}{dt} = \frac{e^y \cos t}{1 - e^y \sin t} \text{ 且 } y = e^y \sin t + 1$$

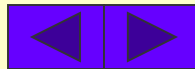
$$\frac{dy}{dx} = \frac{dy}{dt} \cdot \frac{dt}{dx} = \frac{e^y \cos t}{(1 - e^y \sin t)(6t + 2)} = \frac{e^y \cos t}{(2 - y)(6t + 2)}$$

$$\frac{d^2 y}{dx^2} = \frac{d}{dx} \left(\frac{dy}{dx} \right) = \frac{d}{dt} \left[\frac{e^y \cos t}{(2 - y)(6t + 2)} \right] \frac{dt}{dx}$$

$$= \frac{(2 - y)(6t + 2)(e^y \cos t \frac{dy}{dt} - e^y \sin t)}{(2 - y)^2 (6t + 2)^3} - \frac{e^y \cos t \left[(2 - y)6 - \frac{dy}{dt}(6t + 2) \right]}{(2 - y)^2 (6t + 2)^3}$$

由于 $\frac{dy}{dt} \Big|_{t=0} = e$, $y \Big|_{t=0} = 1$, 代入上式得

$$\frac{d^2 y}{dx^2} \Big|_{t=0} = \frac{e(2e - 3)}{4}$$



12. 由 $y = \frac{1}{\sqrt{x}}$, 得 $y' = -\frac{1}{2}x^{-\frac{3}{2}}$, 则切点 $P(a, \frac{1}{\sqrt{a}})$ 处的切线方程为

$$y - \frac{1}{\sqrt{a}} = -\frac{1}{2\sqrt{a^3}}(x - a)$$

切线与 x 轴和 y 轴的交点分别为 $Q(3a, 0)$ 和 $R(0, \frac{3}{2\sqrt{a}})$

于是 $\triangle ORQ$ 的面积 $S = \frac{1}{2} \cdot 3a \cdot \frac{3}{2\sqrt{a}} = \frac{9}{4\sqrt{a}}$

当切点按 x 轴正方向趋于无穷远时, 有 $\lim_{x \rightarrow +\infty} S = +\infty$

当切点按 y 轴的方向趋于无穷远时, 有 $\lim_{a \rightarrow 0^+} S = 0$.

