

三、定积分的计算

(一) 定积分的换元法

定理 (换元公式)

若 $f(x) \in C[a, b]$, $x = \varphi(t)$ 满足条件:

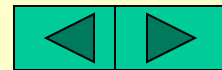
(1) $\varphi(\alpha) = a, \varphi(\beta) = b$;

(2) $\varphi(t)$ 在 $[\alpha, \beta]$ (或 (β, α)) 上具有连续导数, 且 $\varphi(t) \in [a, b]$,

则有

$$\int_a^b f(x) dx = \int_{\alpha}^{\beta} f[\varphi(t)] \varphi'(t) dt$$

换元必换限



1. 计算 $\int_0^a \sqrt{a^2 - x^2} dx \quad (a > 0).$

解 设 $x = a \sin t$, 则 $dx = a \cos t dt$,

$$x = 0, \Rightarrow t = 0; \quad x = a, \Rightarrow t = \frac{\pi}{2}.$$

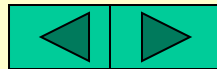
$$\begin{aligned} \int_0^a \sqrt{a^2 - x^2} dx &= a^2 \int_0^{\pi/2} \cos^2 t dt = \frac{a^2}{2} \int_0^{\pi/2} (1 + \cos 2t) dt \\ &= \frac{a^2}{2} \left[t + \frac{1}{2} \sin 2t \right]_0^{\pi/2} = \frac{\pi a^2}{4}. \end{aligned}$$

2. 计算 $\int_0^{\pi/2} \cos^5 x \sin x dx$

解 设 $t = \cos x$, 则 $dt = -\sin x dx$,

$$x = 0, \Rightarrow t = 1, \quad x = \frac{\pi}{2}, \Rightarrow t = 0.$$

$$\int_0^{\pi/2} \cos^5 x \sin x dx = -\int_1^0 t^5 dt = \int_0^1 t^5 dt = \left[\frac{t^6}{6} \right]_0^1 = \frac{1}{6}.$$



$$\begin{aligned}\int_0^{\frac{\pi}{2}} \cos^5 x \sin x dx &= -\int_0^{\frac{\pi}{2}} \cos^5 x d(\cos x) \\ &= -\left[\frac{\cos^6 x}{6} \right]_0^{\frac{\pi}{2}} = -\left(0 - \frac{1}{6} \right) = \frac{1}{6}\end{aligned}$$

3. 计算 $\int_0^{\pi} \sqrt{\sin^3 x - \sin^5 x} dx$

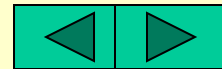
解: $\because \sqrt{\sin^3 x - \sin^5 x} = \sqrt{\sin^3 x (1 - \sin^2 x)}$

$$= \sin^{\frac{3}{2}} x \cdot |\cos x|, x \in [0, \pi]$$

$$\therefore \int_0^{\pi} \sqrt{\sin^3 x - \sin^5 x} dx$$

$$= \int_0^{\frac{\pi}{2}} \sin^{\frac{3}{2}} x \cos x dx + \int_{\frac{\pi}{2}}^{\pi} \sin^{\frac{3}{2}} x (-\cos x) dx$$

$$|\cos x| = \begin{cases} \cos x, x \in \left[0, \frac{\pi}{2}\right]; \\ -\cos x, x \in \left[\frac{\pi}{2}, \pi\right]. \end{cases}$$



$$\begin{aligned}
 &= \int_0^{\frac{\pi}{2}} \sin^{\frac{3}{2}} x d(\sin x) - \int_{\frac{\pi}{2}}^{\pi} \sin^{\frac{3}{2}} x d(\sin x) \\
 &= \left[\frac{2}{5} \sin^{\frac{5}{2}} x \right]_0^{\frac{\pi}{2}} - \left[\frac{2}{5} \sin^{\frac{5}{2}} x \right]_{\frac{\pi}{2}}^{\pi} = \frac{2}{5} - \left(-\frac{2}{5} \right) = \frac{4}{5}.
 \end{aligned}$$

注意 如果忽略了 $|\cos x| = -\cos x, x \in \left[\frac{\pi}{2}, \pi \right]$,
则下列计算是错误的:

$$\begin{aligned}
 &\sqrt{\sin^3 x - \sin^5 x} = \sin^{\frac{3}{2}} x \cdot \cos x, x \in [0, \pi] \\
 \Rightarrow &\int_0^{\pi} \sqrt{\sin^3 x - \sin^5 x} dx \\
 &= \int_0^{\pi} \sin^{\frac{3}{2}} x \cos x dx = \left[\frac{2}{5} \sin^{\frac{5}{2}} x \right]_0^{\pi} = 0
 \end{aligned}$$

4. 计算 $\int_0^4 \frac{x+2}{\sqrt{2x+1}} dx$.

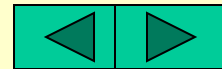
解 $\sqrt{2x+1} = t, \Rightarrow x = \frac{t^2-1}{2}$, 则 $dx = t dt$,

$x = 0, \Rightarrow t = 1; x = 4, \Rightarrow t = 3$.

$$\begin{aligned} \int_0^4 \frac{x+2}{\sqrt{2x+1}} dx &= \int_1^3 \frac{\frac{t^2-1}{2} + 2}{t} t dt = \frac{1}{2} \int_1^3 (t^2 + 3) dt \\ &= \frac{1}{2} \left[\frac{t^3}{3} + 3t \right]_1^3 = \frac{22}{3} \end{aligned}$$

5. 证明: 若 $f(x) \in C[-a, a], \Rightarrow$

$$\int_{-a}^a f(x) dx = \begin{cases} 2 \int_0^a f(x) dx, & f(x) \text{ 为偶函数} \\ 0, & f(x) \text{ 为奇函数} \end{cases}$$



证 $\because \int_{-a}^a f(x)dx = \int_{-a}^0 f(x)dx + \int_0^a f(x)dx$

$$\int_{-a}^0 f(x)dx \xrightarrow{x=-t} - \int_a^0 f(-t)dt$$

$$= \int_0^a f(-t)dt = \int_0^a f(-x)dx$$

$$\Rightarrow \int_{-a}^a f(x)dx = \int_0^a f(-x)dx + \int_0^a f(x)dx$$

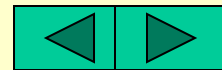
$$= \int_0^a [f(-x) + f(x)]dx$$

(1) 若 $f(x)$ 为偶函数, 则

$$f(x) + f(-x) = 2f(x), \Rightarrow \int_{-a}^a f(x)dx = 2 \int_0^a f(x)dx.$$

(2) 若 $f(x)$ 为奇函数, 则

$$f(x) + f(-x) = 0, \Rightarrow \int_{-a}^a f(x)dx = 0.$$



6. 若 $f(x) \in C[0,1]$, 证明

$$(1) \int_0^{\frac{\pi}{2}} f(\sin x) dx = \int_0^{\frac{\pi}{2}} f(\cos x) dx;$$

$$(2) \int_0^{\pi} x f(\sin x) dx = \frac{\pi}{2} \int_0^{\pi} f(\sin x) dx,$$

并由此计算 $\int_0^{\pi} \frac{x \sin x}{1 + \cos^2 x} dx$.

证 (1) 设 $x = \frac{\pi}{2} - t$, 则 $dx = -dt$,

$$x = 0, \Rightarrow t = \frac{\pi}{2}; x = \frac{\pi}{2}, \Rightarrow t = 0.$$

$$\Rightarrow \int_0^{\frac{\pi}{2}} f(\sin x) dx = - \int_{\frac{\pi}{2}}^0 f\left[\sin\left(\frac{\pi}{2} - t\right)\right] dt$$

$$= - \int_{\frac{\pi}{2}}^0 f(\cos t) dt = \int_0^{\frac{\pi}{2}} f(\cos x) dx.$$

$$\boxed{\int_0^{\frac{\pi}{2}} f(\sin x) dx = \int_0^{\frac{\pi}{2}} f(\cos x) dx;}$$

$$\text{证} \quad \int_0^{\pi} x f(\sin x) dx = \frac{\pi}{2} \int_0^{\pi} f(\sin x) dx$$

(2) 设 $x = \pi - t$, 则 $dx = -dt$, $x = 0, \Rightarrow t = \pi; x = \pi, \Rightarrow t = 0$.

$$\Rightarrow \int_0^{\pi} x f(\sin x) dx = - \int_{\pi}^0 (\pi - t) f[\sin(\pi - t)] dt$$

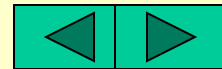
$$= \int_0^{\pi} (\pi - t) f(\sin t) dt = \int_0^{\pi} \pi f(\sin t) dt - \int_0^{\pi} t f(\sin t) dt$$

$$= \int_0^{\pi} \pi f(\sin x) dx - \int_0^{\pi} x f(\sin x) dx.$$

$$\Rightarrow \int_0^{\pi} x f[\sin x] dx = \frac{\pi}{2} \int_0^{\pi} f[\sin x] dx$$

$$\Rightarrow \int_0^{\pi} \frac{x \sin x}{1 + \cos^2 x} dx = \frac{\pi}{2} \int_0^{\pi} \frac{\sin x}{1 + \cos^2 x} dx.$$

$$= -\frac{\pi}{2} \int_0^{\pi} \frac{d(\cos x)}{1 + \cos^2 x} = -\frac{\pi}{2} [\arctan(\cos x)]_0^{\pi} = \frac{\pi^2}{4}.$$



7. 设函数

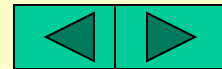
$$f(x) = \begin{cases} xe^{-x^2}, & x \geq 0, \\ \frac{1}{1 + \cos x}, & -1 < x < 0, \end{cases}$$

计算 $\int_1^4 f(x-2)dx$.

解 设 $x-2=t$, 则 $dx=dt$,

$$x=1, \Rightarrow t=-1; x=4, \Rightarrow t=2.$$

$$\begin{aligned} \int_1^4 f(x-2)dx &= \int_{-1}^2 f(t)dt = \int_{-1}^0 \frac{dt}{1 + \cos t} + \int_0^2 te^{-t^2} dt \\ &= \int_{-1}^0 \frac{1}{2\cos^2 \frac{t}{2}} dt - \frac{1}{2} \cdot \int_0^2 e^{-t^2} d(-t^2) \\ &= \left[\tan \frac{t}{2} \right]_{-1}^0 - \left[\frac{1}{2} e^{-t^2} \right]_0^2 = \tan \frac{1}{2} - \frac{1}{2} e^{-4} + \frac{1}{2}. \end{aligned}$$



8. $f(x) \in C(-\infty, \infty), f(x+T) = f(x), \forall x \in (-\infty, \infty).$

证明 $\int_a^{a+T} f(x)dx = \int_0^T f(x)dx, \quad \forall a \in (-\infty, \infty).$

证 $\int_a^{a+T} f(x)dx = \int_a^T f(x)dx + \int_T^{a+T} f(x)dx. \quad \text{引进变量: } t = x - T. = \int_0^a f(x)dx?$

令 $x = t + T$, 则 $dx = dt; x = T, \Rightarrow t = 0; x = a + T, \Rightarrow t = a.$

$$\Rightarrow \int_T^{a+T} f(x)dx = \int_0^a f(t+T)dt = \int_0^a f(t)dt$$

$$\therefore \int_a^{a+T} f(x)dx = \int_a^T f(x)dx + \int_0^a f(x)dx = \int_0^T f(x)dx$$

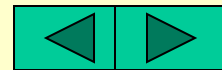
(二) 定积分的分部积分法

设函数 $u(x)$ 、 $v(x)$ 在 $[a, b]$ 上具有连续的导函数 $u'(x)$ 、 $v'(x)$,
则

$$(uv)' = u'v + uv'.$$

$$\int_a^b u dv = [uv]_a^b - \int_a^b v du.$$

—— 定积分的分部积分法公式



1. 计算 $\int_0^1 \arctan x dx$.

解 设 $u = \arctan x, dv = dx$, 则 $du = \frac{dx}{1+x^2}, v = x$,

$$\begin{aligned}\int_0^1 \arctan x dx &= [x \arctan x]_0^1 - \int_0^1 \frac{x}{1+x^2} dx \\ &= \frac{\pi}{4} - \frac{1}{2} [\ln(1+x^2)]_0^1 = \frac{\pi}{4} - \frac{1}{2} \ln 2.\end{aligned}$$

2. 计算 $\int_0^1 e^{\sqrt{x}} dx$.

解 令 $\sqrt{x} = t, \Rightarrow x = t^2$, 则 $dx = 2t dt$,
 $x = 0, \Rightarrow t = 0; x = 1, \Rightarrow t = 1$.

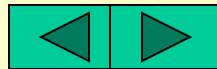
$$\begin{aligned}\int_0^1 e^{\sqrt{x}} dx &= 2 \int_0^1 t e^t dt = 2 \int_0^1 t d(e^t) \\ &= 2 [t e^t]_0^1 - 2 \int_0^1 e^t dt = 2e - 2[e^t]_0^1 = 2.\end{aligned}$$

3. 证明定积分公式

$$\begin{aligned} I_n &= \int_0^{\frac{\pi}{2}} \sin^n x dx = \int_0^{\frac{\pi}{2}} \cos^n x dx \\ &= \begin{cases} \frac{n-1}{n} \cdot \frac{n-3}{n-2} \cdots \frac{3}{4} \cdot \frac{1}{2} \cdot \frac{\pi}{2}, & n \text{ 为正偶数;} \\ \frac{n-1}{n} \cdot \frac{n-3}{n-2} \cdots \frac{4}{5} \cdot \frac{2}{3}, & n \text{ 为大于1的正奇数。} \end{cases} \end{aligned}$$

证 令 $u = \sin^{n-1} x, dv = \sin x dx$,
则 $du = (n-1) \sin^{n-2} x \cos x dx, v = -\cos x$.

$$\begin{aligned} I_n &= \int_0^{\frac{\pi}{2}} \sin^n x dx = \int_0^{\frac{\pi}{2}} \sin^{n-1} x d(-\cos x) \\ &= \left[-\cos x \sin^{n-1} x \right]_0^{\frac{\pi}{2}} + (n-1) \int_0^{\frac{\pi}{2}} \sin^{n-2} x \cos^2 x dx \\ &= (n-1) \int_0^{\frac{\pi}{2}} \sin^{n-2} x (1 - \sin^2 x) dx \end{aligned}$$



$$= (n-1) \int_0^{\frac{\pi}{2}} \sin^{n-2} x dx - (n-1) \int_0^{\frac{\pi}{2}} \sin^n x dx$$

$$I_n = (n-1)I_{n-2} - (n-1)I_n$$

$$I_n = \frac{n-1}{n} I_{n-2} \quad \text{———} I_n \text{ 关于下标的递推公式}$$

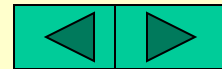
$$I_{n-2} = \frac{n-3}{n-2} I_{n-4}.$$

依次进行下去可得

$$I_{2m} = \frac{2m-1}{2m} \cdot \frac{2m-3}{2m-2} \cdot \frac{2m-5}{2m-4} \cdots \frac{5}{6} \cdot \frac{3}{4} \cdot \frac{1}{2} I_0,$$

$$I_{2m+1} = \frac{2m}{2m+1} \cdot \frac{2m-2}{2m-1} \cdot \frac{2m-4}{2m-3} \cdots \frac{6}{7} \cdot \frac{4}{5} \cdot \frac{2}{3} I_1 \quad (m=1,2,\cdots).$$

$$I_0 = \int_0^{\frac{\pi}{2}} dx = \frac{\pi}{2}, \quad I_1 = \int_0^{\frac{\pi}{2}} \sin x dx = 1,$$



$$I_{2m} = \int_0^{\frac{\pi}{2}} \sin^{2m} x dx$$

$$= \frac{2m-1}{2m} \cdot \frac{2m-3}{2m-2} \cdot \frac{2m-5}{2m-4} \cdots \frac{5}{6} \cdot \frac{3}{4} \cdot \frac{1}{2} \cdot \frac{\pi}{2},$$

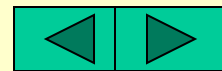
$(m = 1, 2, \cdots).$

$$I_{2m+1} = \int_0^{\frac{\pi}{2}} \sin^{2m+1} x dx$$

$$= \frac{2m}{2m+1} \cdot \frac{2m-2}{2m-1} \cdot \frac{2m-4}{2m-3} \cdots \frac{6}{7} \cdot \frac{4}{5} \cdot \frac{2}{3}$$

$$\int_0^{\frac{\pi}{2}} \sin^n x dx = \int_0^{\frac{\pi}{2}} \cos^n x dx.$$

证毕



4. 设 $f(x)$ 为连续函数, 证明:

$$\int_0^x f(t)(x-t)dt = \int_0^x \left(\int_0^t f(u)du \right) dt.$$

证 $\int_0^x \left(\int_0^t f(u)du \right) dt = t \int_0^t f(u)du \Big|_0^x - \int_0^x tf(t)dt$

$$= x \int_0^x tf(t)dt - \int_0^x tf(t)dt$$

$$= \int_0^x f(t)(x-t)dt$$

即 $\int_0^x f(t)(x-t)dt = \int_0^x \left(\int_0^t f(u)du \right) dt$

