导数应用习题课2(洛必达法则)

$$1.\lim_{x\to 0}\frac{\int_0^x (3\sin t+t^2\cos\frac{1}{t})\mathrm{d}t}{(1+\cos x)\int_0^x \ln(1+t)\mathrm{d}t}$$

$$2.\lim_{x\to +\infty}\frac{\int_0^x (1+t^2)\mathrm{e}^{t^2}\mathrm{d}t}{x\mathrm{e}^{x^2}}$$

$$3.\lim_{x\to \infty}(\sin\frac{2}{x}+\cos\frac{1}{x})^x$$

4.求 $\lim_{n\to\infty}(n\cdot\tan\frac{1}{n})^{n^2}$ (n为自然数).

5. 设函数$f(x)$在点$x=a$处具有二阶导数，并且$f'(a)\neq 0$

求$\lim\limits_{x\to a}\left[\dfrac{1}{f(x)-f(a)}-\dfrac{1}{(x-a)f'(a)}\right]$

6.设$f''(x_0)$存在，求：

$$\lim_{h\to 0}\frac{f(x_0+h)-2f(x_0)+f(x_0-h)}{h^2}.$$

7.设$f(x)=\begin{cases}\dfrac{g(x)}{x}, & x\neq 0\\ 0, & x=0\end{cases}$，已知$g(x)$在$(-\infty,+\infty)$内有二阶

连续导数，且$g(0)=g'(0)=0$，讨论$f'(x)$的连续性.

8.设 $f(x)$有二阶导数，当 $x\neq 0$时，$f(x)\neq 0$. $\lim\limits_{x\to 0}\dfrac{f(x)}{x}=0$，$f''(0)=4$，求：$\lim\limits_{x\to 0}\left[1+\dfrac{f(x)}{x}\right]^{\frac{1}{x}}$.

9.设 $f(x)$在 $(-\delta,\delta)$内二阶可导，$\lim\limits_{x\to 0}\left[1+x+\dfrac{f(x)}{x}\right]^{\frac{1}{x}}=e^{3}$，求：$\lim\limits_{x\to 0}\left[1+\dfrac{f(x)}{x}\right]^{\frac{1}{x}}$ 及 $f(0)$、$f'(0)$、$f''(0)$.

答案

1. 解 $$\lim_{x\to 0}\frac{\int_0^x (3\sin t + t^2\cos\frac{1}{t})\mathrm{d}t}{(1+\cos x)\int_0^x \ln(1+t)\mathrm{d}t} = \frac{1}{2}\lim_{x\to 0}\frac{\int_0^x (3\sin t + t^2\cos\frac{1}{t})}{\int_0^x \ln(1+t)dt}$$

$$= \frac{1}{2}\lim_{x\to 0}\frac{3\sin x + x^2\cos\frac{1}{x}}{\ln(1+x)} = \frac{1}{2}\lim_{x\to 0}\frac{3\sin x + x^2\cos\frac{1}{x}}{x} = \frac{3}{2}$$

2. 解 $$\lim_{x\to +\infty}\frac{\int_0^x (1+t^2)\mathrm{e}^{t^2}\mathrm{d}t}{x\mathrm{e}^{x^2}} = \lim_{x\to +\infty}\frac{(1+x^2)\mathrm{e}^{x^2}}{\mathrm{e}^{x^2}+2x^2\mathrm{e}^{x^2}} = \lim_{x\to +\infty}\frac{1+x^2}{1+2x^2} = \frac{1}{2}$$

3. 解：因为 $$\lim_{x\to\infty} x\ln(\sin\frac{2}{x}+\cos\frac{1}{x}) \underline{\underline{x=1/t}} \lim_{t\to 0}\frac{\ln(\sin 2t+\cos t)}{t}$$

$$= \lim_{t\to 0}\frac{2\cos 2t - \sin t}{\sin 2t + \cos t} = 2$$

所以，$$\lim_{x\to\infty}(\sin\frac{2}{x}+\cos\frac{1}{x})^x = \mathrm{e}^2$$

4. **解** 因为 $\lim\limits_{x\to 0^+}\left(\frac{\tan x}{x}\right)^{\frac{1}{x^2}}=\lim\limits_{x\to 0^+}\left[\left(1+\frac{\tan x-x}{x}\right)^{\frac{x}{\tan x-x}}\right]^{\frac{\tan x-x}{x^3}}$，其中

$$\lim_{x\to 0^+}\frac{\tan x-x}{x^3}\overset{\frac{0}{0}}{=\!=}\lim_{x\to 0^+}\frac{\sec^2 x-1}{3x^2}=\frac{1}{3}.$$

取 $x=\frac{1}{n}$，则原式 $=\mathrm{e}^{\frac{1}{3}}$.

5．解 $\lim\limits_{x\to a}\left[\frac{1}{f(x)-f(a)}-\frac{1}{(x-a)f'(a)}\right]=\lim\limits_{x\to a}\frac{(x-a)f'(a)-f(x)+f(a)}{(x-a)f'(a)[f(x)-f(a)]}$

$$\overset{\frac{0}{0}}{=}\frac{1}{f'(a)}\lim_{x\to a}\frac{f'(a)-f'(x)}{f(x)-f(a)+(x-a)f'(x)}=-\frac{1}{f'(a)}\lim_{x\to a}\frac{\frac{f'(x)-f'(a)}{x-a}}{\frac{f(x)-f(a)}{x-a}+f'(x)}$$

$$=-\frac{f''(a)}{2[f'(a)]^2}$$

6. $\lim\limits_{h\to 0}\dfrac{f(x_0+h)-2f(x_0)+f(x_0-h)}{h^2}$

$$\overset{\frac{0}{0}}{=}\lim_{h\to 0}\frac{f'(x_0+h)-f'(x_0-h)}{2h}$$

$$=\frac{1}{2}\left[\lim_{h\to 0}\frac{f'(x_0+h)-f'(x_0)}{h}+\lim_{h\to 0}\frac{f'(x_0-h)-f'(x_0)}{-h}\right]=f''(x_0).$$

7.

解: $f'(0)=\lim\limits_{x\to 0}\dfrac{\frac{g(x)}{x}-0}{x}=\lim\limits_{x\to 0}\dfrac{g(x)}{x^2}\overset{\frac{0}{0}}{=}\lim\limits_{x\to 0}\dfrac{g'(x)}{2x}\overset{\frac{0}{0}}{=}\dfrac{1}{2}\lim\limits_{x\to 0}\dfrac{g''(x)}{1}=\dfrac{1}{2}g''(0)$

$$f'(x)=\begin{cases}\dfrac{xg'(x)-g(x)}{x^2}, & x\neq 0\\[2ex] \dfrac{1}{2}g''(0), & x=0\end{cases}$$

$$\lim_{x\to 0}f'(x)=\lim_{x\to 0}\frac{xg'(x)-g(x)}{x^2}\overset{\frac{0}{0}}{=}\lim_{x\to 0}\frac{g'(x)+xg''(x)-g'(x)}{2x}=\frac{1}{2}g''(0)=f'(0)$$

$\therefore$ $(-\infty,+\infty)$ 连续.

8.

解： $\lim\limits_{x\to 0}\ln y = \lim\limits_{x\to 0}\dfrac{\ln\left[1+\frac{f(x)}{x}\right]}{x} = \lim\limits_{x\to 0}\dfrac{f(x)}{x^2} \overset{\frac{0}{0}}{=} \lim\limits_{x\to 0}\dfrac{f'(x)}{2x}$ $\quad f'(0)=0$

$= \dfrac{1}{2}\lim\limits_{x\to 0}\dfrac{f'(x)-f'(0)}{x} = \dfrac{1}{2}f''(0) = 2$

$\therefore$ 原式 $= e^2$

9.

解: $3=\lim\limits_{x\to 0}\dfrac{\ln\left[1+x+\frac{f(x)}{x}\right]}{x}=\lim\limits_{x\to 0}\dfrac{x+\frac{f(x)}{x}}{x}=1+\lim\limits_{x\to 0}\dfrac{f(x)}{x^2}$

$\Rightarrow \lim\limits_{x\to 0}\dfrac{f(x)}{x^2}=2 \qquad f(0)=0$

$\lim\limits_{x\to 0}\left[1+\dfrac{f(x)}{x}\right]^{\frac{1}{x}}=e^{\lim\limits_{x\to 0}\frac{\ln\left[1+\frac{f(x)}{x}\right]}{x}}=e^{\lim\limits_{x\to 0}\frac{f(x)}{x^2}}=e^2$

$f'(0)=\lim\limits_{x\to 0}\dfrac{f(x)-f(0)}{x}=\lim\limits_{x\to 0}\dfrac{f(x)}{x}=\lim\limits_{x\to 0}\dfrac{f(x)}{x^2}\cdot x=0$

$f''(0)=\lim\limits_{x\to 0}\dfrac{f'(x)-f'(0)}{x}=\lim\limits_{x\to 0}\dfrac{f'(x)}{x}$

而 $2=\lim\limits_{x\to 0}\dfrac{f(x)}{x^2}\overset{\frac{0}{0}}{=}\lim\limits_{x\to 0}\dfrac{f'(x)}{2x}=\dfrac{1}{2}f''(0)\Rightarrow f''(0)=4$