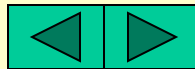


四、几种特殊类型函数的积分

(一)有理函数的积分

(二) 三角函数有理式的积分

(三)简单无理函数的积分:



(一)有理函数的积分

1. 有理函数

$$\frac{P(x)}{Q(x)} = \frac{a_0x^n + a_1x^{n-1} + \cdots + a_{n-1}x + a_n}{b_0x^m + b_1x^{m-1} + \cdots + b_{m-1}x + b_m} \quad (1)$$

(m 和 n 为非负整数, $a_0, a_1, \cdots, a_{n-1}, a_n$ 及 $b_0, b_1, \cdots, b_{m-1}, b_m$ 为实数, 且 $a_0 \neq 0, b_0 \neq 0$.)

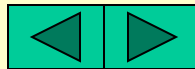
$$\text{例 } \frac{x^3 + 3}{x^2 - 5x + 6}, \frac{x^2 + 3}{x^2 - 5x + 6}, \frac{x + 3}{x^2 - 5x + 6}$$

假分式 ($n \geq m$)

真分式 ($n < m$)

2. 真分式、假分式

$$\begin{aligned} \frac{x^2 + 3}{x^2 - 5x + 6} &= \frac{(x^2 - 5x + 6) + 5x - 3}{x^2 - 5x + 6} = 1 + \frac{5x - 3}{x^2 - 5x + 6} \\ \frac{x^3 + 3}{x^2 - 5x + 6} &= x + 5 + \frac{19x - 27}{x^2 - 5x + 6} \end{aligned}$$



- 任何一个有理函数都可化为多项式与真分式之和。
- 真分式可以化为几个简单真分式的代数和。

例 $\frac{x+3}{x^2-5x+6} \quad \therefore \frac{x+3}{x^2-5x+6} = \frac{x+3}{(x-2)(x-3)}$

由代数理论知可分解为 $\frac{x+3}{x^2-5x+6} = \frac{A}{x-2} + \frac{B}{x-3} = \frac{A(x-3)+B(x-2)}{(x-2)(x-3)}$

法1. (比较系数法)

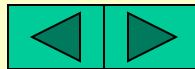
$$\because x+3 = A(x-3) + B(x-2) = (A+B)x - (3A+2B)$$

$$\therefore \begin{cases} A+B=1 \\ -(3A+2B)=3 \end{cases} \quad \therefore A=-5, B=6$$

法2. (赋值法) $\because x+3 = A(x-3) + B(x-2)$

令 $x=2$, 得 $A=-5$; 令 $x=3$, 得 $B=6$.

$$\therefore \frac{x+3}{x^2-5x+6} = -\frac{5}{x-2} + \frac{6}{x-3}$$



例 $\frac{1}{x(x-1)^2}$ 可分解为

$$\frac{1}{x(x-1)^2} = \frac{A}{x} + \frac{B}{(x-1)} + \frac{C}{(x-1)^2} = \frac{A(x-1)^2 + Bx(x-1) + Cx}{x(x-1)^2}$$

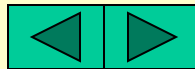
$$\therefore 1 = A(x-1)^2 + Bx(x-1) + Cx$$

令 $x = 0$, 得 $A = 1$; 令 $x = 1$, 得 $C = 1$; 令 $x = 2$, 得 $B = -1$.

$$\therefore \frac{1}{x(x-1)^2} = \frac{1}{x} - \frac{1}{x-1} + \frac{1}{(x-1)^2}$$

$$\text{又 } \frac{1}{x(x-1)^3} = \frac{A}{x} + \frac{B}{(x-1)} + \frac{C}{(x-1)^2} + \frac{D}{(x-1)^3}$$

$$\frac{1}{x^2(x-1)^3} = \frac{A}{x} + \frac{B}{x^2} + \frac{C}{(x-1)} + \frac{D}{(x-1)^2} + \frac{E}{(x-1)^3}$$



例 $\frac{1}{(1+2x)(1+x^2)}$ 可分解为

$$\frac{1}{(1+2x)(1+x^2)} = \frac{A}{1+2x} + \frac{Bx+C}{1+x^2} = \frac{A(1+x^2) + (Bx+C)(1+2x)}{(1+2x)(1+x^2)}$$

$$\therefore 1 = A(1+x^2) + (Bx+C)(1+2x)$$

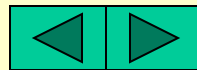
$$= (A+2B)x^2 + (B+2C)x + (A+C)$$

$$\text{令 } x = -\frac{1}{2}, \text{ 得 } A = \frac{4}{5};$$

$$\text{比较 } x^2 \text{ 项得: } 0 = A + 2B \quad \therefore B = -\frac{2}{5}$$

$$\text{比较常数项得: } 1 = A + C \quad \therefore C = \frac{1}{5}$$

$$\therefore \frac{1}{(1+2x)(1+x^2)} = \frac{\frac{4}{5}}{1+2x} + \frac{-\frac{2}{5}x + \frac{1}{5}}{1+x^2} = \frac{4}{5} \cdot \frac{1}{1+2x} - \frac{1}{5} \cdot \frac{2x-1}{1+x^2}$$



再如:
$$\frac{1}{(1+2x)(1+x^2)^3} = \frac{A}{1+2x} + \frac{Bx+C}{1+x^2} + \frac{Dx+E}{(1+x^2)^2} + \frac{Fx+G}{(1+x^2)^3}$$

由前面的讨论可知, 任何一个有理函数都可化为: 多项式、

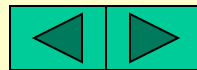
$\frac{A}{(x-a)^n}$ 、 $\frac{Mx+N}{(x^2+px+q)^n}$ 的代数和。

例1 求 $\int \frac{x+3}{x^2-5x+6} dx$

解
$$\because \frac{x+3}{x^2-5x+6} = \frac{-5}{x-2} + \frac{6}{x-3}$$

$$\therefore \int \frac{x+3}{x^2-5x+6} dx = \int \left(\frac{-5}{x-2} + \frac{6}{x-3} \right) dx$$

$$= -5\ln|x-2| + 6\ln|x-3| + C$$



例2 求 $\int \frac{1}{x(x-1)^2} dx$

解 $\because \frac{1}{x(x-1)^2} = \frac{1}{x} + \frac{1}{(x-1)^2} - \frac{1}{x-1}$

$$\therefore \int \frac{1}{x(x-1)^2} dx = \int \left(\frac{1}{x} + \frac{1}{(x-1)^2} - \frac{1}{x-1} \right) dx$$

$$= \int \frac{1}{x} dx + \int \frac{1}{(x-1)^2} d(x-1) - \int \frac{1}{x-1} d(x-1)$$

$$= \ln|x| - \frac{1}{x-1} - \ln|x-1| + C = \ln \left| \frac{x}{x-1} \right| - \frac{1}{x-1} + C$$

例3 求 $\int \frac{1}{(1+2x)(1+x^2)} dx$

$$\int \frac{2x-1}{1+x^2} dx = \int \frac{2x}{1+x^2} dx - \int \frac{1}{1+x^2} dx$$

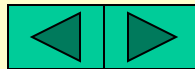
解 $\because \frac{1}{(1+2x)(1+x^2)} = \frac{4}{5} \cdot \frac{1}{1+2x} - \frac{1}{5} \cdot \frac{2x-1}{1+x^2}$

$$\therefore \int \frac{1}{(1+2x)(1+x^2)} dx = \frac{4}{5} \int \frac{1}{1+2x} dx - \frac{1}{5} \int \frac{2x-1}{1+x^2} dx$$

$$= \frac{4}{5} \cdot \frac{1}{2} \int \frac{1}{1+2x} d(1+2x) - \frac{1}{5} \int \frac{1}{1+x^2} d(1+x^2) + \frac{1}{5} \int \frac{1}{1+x^2} dx$$

$$= \frac{2}{5} \ln|1+2x| - \frac{1}{5} \ln|1+x^2| + \frac{1}{5} \arctan x + C$$

$$= \frac{1}{5} \left[\ln \frac{(1+2x)^2}{1+x^2} + \arctan x \right] + C$$



注：当被积函数容易分解时，也不必墨守成规非要用待定系数法或赋值法，只要直接分解就可以。如

例4 求 $\int \frac{1+x^3}{x(1-x^3)} dx$

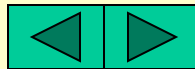
解： **法1** $\int \frac{1+x^3}{x(1-x^3)} dx = \int \frac{(1-x^3)+2x^3}{x(1-x^3)} dx = \int \frac{1}{x} dx + 2 \int \frac{x^2}{1-x^3} dx$
 $= \ln|x| - \frac{2}{3} \int \frac{1}{1-x^3} d(1-x^3) = \ln|x| - \frac{2}{3} \ln|1-x^3| + C$ **简单！**

法2 令： $\frac{1+x^3}{x(1-x^3)} = \frac{A}{x} + \frac{B}{1-x} + \frac{Cx+D}{1+x+x^2}$

通分得： $1+x^3 = A(1-x^3) + Bx(1+x+x^2) + (Cx+D)x(1-x)$

比较x的系数得： $A=1, B=\frac{2}{3}, C=-\frac{4}{3}, D=-\frac{2}{3}$

$\int \frac{1+x^3}{x(1-x^3)} dx = \int \frac{1}{x} dx + \frac{2}{3} \int \frac{1}{1-x} dx - \frac{2}{3} \int \frac{2x+2}{1+x+x^2} dx$ **繁！**

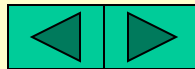


例5 求 $\int \frac{x-2}{x^2+2x+3} dx$

解 x^2+2x+3 是二次质因式, 但 $(x^2+2x+3)' = 2x+2$

$$x-2 = \left[\frac{1}{2}(2x+2) - 1 \right] - 2 = \frac{1}{2}(2x+2) - 3$$

$$\begin{aligned} \int \frac{x-2}{x^2+2x+3} dx &= \int \frac{\frac{1}{2}(2x+2) - 3}{x^2+2x+3} dx \\ &= \frac{1}{2} \int \frac{2x+2}{x^2+2x+3} dx - 3 \int \frac{1}{x^2+2x+3} dx \\ &= \frac{1}{2} \int \frac{d(x^2+2x+3)}{x^2+2x+3} - 3 \int \frac{d(x+1)}{(x+1)^2 + (\sqrt{2})^2} \\ &= \frac{1}{2} \ln(x^2+2x+3) - \frac{3}{\sqrt{2}} \arctan \frac{x+1}{\sqrt{2}} + C \end{aligned}$$



若分解后的有理分式出现 $\frac{A}{(x-a)^n}$ 、 $\frac{Ax+B}{(x^2+a^2)^n}$ 这种部分分式，

前面已经解决。若出现 $\frac{Mx+N}{(x^2+px+q)^n}$ 如何解决？

讨论 $\int \frac{Mx+N}{(x^2+px+q)^n} dx$. $x^2+px+q = \left(x + \frac{p}{2}\right)^2 + q - \frac{p^2}{4}$,

令 $x + \frac{p}{2} = t$, $\because \frac{p^2}{4} - q = \frac{1}{4}(p^2 - 4q) < 0$, 即 $q - \frac{p^2}{4} > 0$,

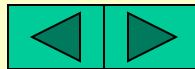
记 $x^2 + px + q = t^2 + a^2$,

$Mx + N = Mt + b$,

则 $a^2 = q - \frac{p^2}{4}$, $b = N - \frac{Mp}{2}$,

$$\therefore \int \frac{Mx+N}{(x^2+px+q)^n} dx = \int \frac{Mt+b}{(t^2+a^2)^n} dt = \int \frac{Mt}{(t^2+a^2)^n} dt + \int \frac{b}{(t^2+a^2)^n} dt$$

$$\begin{aligned} \because Mx + N &= M\left(t - \frac{p}{2}\right) + N \\ &= Mt + N - \frac{Mp}{2} \end{aligned}$$



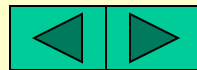
$$\int \frac{Mt}{(t^2 + a^2)^n} dt + \int \frac{b}{(t^2 + a^2)^n} dt$$

当 $n = 1$ 时，如例4。

当 $n > 1$ 时，

$$\begin{aligned} \int \frac{Mx + N}{(x^2 + px + q)^n} dx &= \frac{M}{2} \int \frac{1}{(t^2 + a^2)^n} d(t^2 + a^2) + b \int \frac{dt}{(t^2 + a^2)^n} \\ &= -\frac{M}{2(n-1)(t^2 + a^2)^{n-1}} + b \int \frac{dt}{\underline{(t^2 + a^2)^n}} \end{aligned}$$

有理函数的原函数都是初等函数。



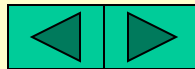
注：对于有理函数的积分要灵活运用各种方法。如：

例6 求 $\int \frac{x}{x^8 - 1} dx$

解：
$$\begin{aligned}\int \frac{x}{x^8 - 1} dx &= \int \frac{x}{(x^4 - 1)(x^4 + 1)} dx = \frac{1}{2} \int \left(\frac{x}{x^4 - 1} - \frac{x}{x^4 + 1} \right) dx \\ &= \frac{1}{4} \int \frac{1}{(x^2)^2 - 1} dx^2 - \frac{1}{4} \int \frac{1}{(x^2)^2 + 1^2} dx^2 \\ &= \frac{1}{8} \ln \left| \frac{x^2 - 1}{x^2 + 1} \right| - \frac{1}{4} \arctan x^2 + C\end{aligned}$$

例7 求 $\int \frac{1}{x(x^{10} - 1)} dx$

解：
$$\begin{aligned}\int \frac{1}{x(x^{10} - 1)} dx &= \int \frac{x^9}{x^{10}(x^{10} - 1)} dx = \frac{1}{10} \int \left(\frac{1}{x^{10} - 1} - \frac{1}{x^{10}} \right) dx^{10} \\ &= \frac{1}{10} \ln \left| \frac{x^{10} - 1}{x^{10}} \right| + C = \frac{1}{10} \ln \left| 1 - \frac{1}{x^{10}} \right| + C\end{aligned}$$



(二) 三角函数有理式的积分

常数和三角函数经过有限次四则运算所构成的函数.

———— 三角函数有理式

$\sin x$ 、 $\cos x$ 的有理式, 记为 $R(\sin x, \cos x)$,

其中 $R(u, v)$ 表示 u 、 v 两个变量的有理式.

例 $\frac{1 + \sin x}{\sin x(1 + \cos x)}$ 、 $\frac{\tan x + \sec x}{\sin x(1 + \cos x)}$ 是三角函数有理式,

而 $\frac{x + \sin x}{\sin x(1 + \cos x)}$ 不是三角函数有理式



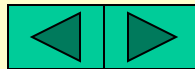
例1 求 $\int \frac{1 + \sin x}{\sin x(1 + \cos x)} dx$

$$\sin x = \frac{2 \tan \frac{x}{2}}{1 + \tan^2 \frac{x}{2}}, \quad \cos x = \frac{1 - \tan^2 \frac{x}{2}}{1 + \tan^2 \frac{x}{2}}$$

解 令 $u = \tan \frac{x}{2}$, 则 $\sin x = \frac{2u}{1 + u^2}$, $\cos x = \frac{1 - u^2}{1 + u^2}$.

而 $x = 2 \arctan u \quad \therefore dx = \frac{2}{1 + u^2} du$

$$\begin{aligned} \int \frac{1 + \sin x}{\sin x(1 + \cos x)} dx &= \int \frac{\left(1 + \frac{2u}{1 + u^2}\right) \frac{2du}{1 + u^2}}{\frac{2u}{1 + u^2} \left(1 + \frac{1 - u^2}{1 + u^2}\right)} = \frac{1}{2} \int \left(u + 2 + \frac{1}{u}\right) du \\ &= \frac{1}{2} \left(\frac{u^2}{2} + 2u + \ln|u|\right) + C = \frac{1}{4} \tan^2 \frac{x}{2} + \tan \frac{x}{2} + \frac{1}{2} \ln \left| \tan \frac{x}{2} \right| + C \end{aligned}$$



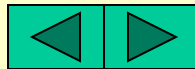
$$\int R(\sin x, \cos x) dx \quad \xrightarrow{\text{令 } \tan \frac{x}{2} = u} \int R\left(\frac{2u}{1+u^2}, \frac{1-u^2}{1+u^2}\right) \frac{2}{1+u^2} du \quad \text{有理分式!}$$

例2 求 $\int \frac{dx}{2\sin x - \cos x + 5}$

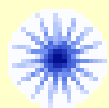
解 令 $u = \tan \frac{x}{2}$, 则 $\sin x = \frac{2u}{1+u^2}$, $\cos x = \frac{1-u^2}{1+u^2}$.

而 $x = 2\arctan u \quad \therefore dx = \frac{2}{1+u^2} du$

$$\int \frac{dx}{2\sin x - \cos x + 5} = \int \frac{\frac{2}{1+u^2} du}{2 \cdot \frac{2u}{1+u^2} - \frac{1-u^2}{1+u^2} + 5}$$



$$\begin{aligned}
 &= \int \frac{du}{3u^2 + 2u + 2} = \frac{1}{3} \int \frac{du}{\left(u + \frac{1}{3}\right)^2 + \frac{5}{9}} = \frac{1}{3} \int \frac{d\left(u + \frac{1}{3}\right)}{\left(u + \frac{1}{3}\right)^2 + \left(\frac{\sqrt{5}}{3}\right)^2} \\
 &= \frac{1}{3} \cdot \frac{3}{\sqrt{5}} \arctan \frac{u + \frac{1}{3}}{\frac{\sqrt{5}}{3}} + C = \frac{1}{\sqrt{5}} \arctan \frac{3 \tan \frac{x}{2} + 1}{\sqrt{5}} + C
 \end{aligned}$$



此法往往较繁，能用简单方法的尽量用简单方法。

例3 求 $\int \frac{\cos x}{1 + \sin x} dx$

解 $\int \frac{\cos x}{1 + \sin x} dx = \int \frac{d(1 + \sin x)}{1 + \sin x} = \ln(1 + \sin x) + C$

(三) 简单无理函数的积分:

只讨论 $R(x, \sqrt[n]{ax+b})$ 及 $R\left(x, \sqrt[n]{\frac{ax+b}{cx+d}}\right)$

作代换 $\sqrt[n]{ax+b} = t$ 及 $\sqrt[n]{\frac{ax+b}{cx+d}} = t$ 。

例1 求 $\int \frac{\sqrt{x-1}}{x} dx$

解 令 $\sqrt{x-1} = u$, 则 $x = u^2 + 1$, $dx = 2u du$

$$\begin{aligned}\int \frac{\sqrt{x-1}}{x} dx &= \int \frac{u}{u^2+1} 2u du = 2 \int \frac{u^2}{u^2+1} du = 2 \int \left(1 - \frac{1}{u^2+1}\right) du \\ &= 2(u - \arctan u) + C = 2(\sqrt{x-1} - \arctan \sqrt{x-1}) + C\end{aligned}$$



例2 求 $\int \frac{dx}{1+\sqrt[3]{x+2}}$

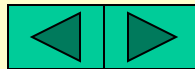
解 令 $\sqrt[3]{x+2} = u$, 则 $x = u^3 - 2$, $dx = 3u^2 du$

$$\begin{aligned}\int \frac{dx}{1+\sqrt[3]{x+2}} &= \int \frac{3u^2 du}{1+u} = 3 \int \frac{u^2 - 1 + 1}{1+u} du = 3 \int \left(u - 1 + \frac{1}{1+u} \right) du \\ &= 3 \left(\frac{u^2}{2} - u + \ln|1+u| + C \right) = 3 \left(\frac{\sqrt[3]{(x+2)^2}}{2} - \sqrt[3]{x+2} + \ln|1+\sqrt[3]{x+2}| + C \right)\end{aligned}$$

例3 求 $\int \frac{dx}{(1+\sqrt[3]{x})\sqrt{x}}$

解 令 $\sqrt[6]{x} = t$, 则 $x = t^6$, $dx = 6t^5 dt$

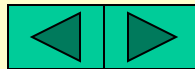
$$\begin{aligned}\int \frac{dx}{(1+\sqrt[3]{x})\sqrt{x}} &= \int \frac{6t^5 dt}{(1+t^2)t^3} = \int \frac{6t^2 dt}{(1+t^2)} = 6 \int \left(1 - \frac{1}{1+t^2} \right) dt \\ &= 6(t - \arctan t) + C = 6(\sqrt[6]{x} - \arctan \sqrt[6]{x}) + C\end{aligned}$$



例4 求 $\int \frac{1}{x} \sqrt{\frac{1+x}{x}} dx$

解 令 $\sqrt{\frac{1+x}{x}} = t$, 则 $x = \frac{1}{t^2 - 1}$, $dx = -\frac{2t dt}{(t^2 - 1)^2}$

$$\begin{aligned} \int \frac{1}{x} \sqrt{\frac{1+x}{x}} dx &= \int (t^2 - 1)t \cdot \frac{-2t}{(t^2 - 1)^2} dt = -2 \int \frac{t^2}{t^2 - 1} dt \\ &= -2 \int \left(1 + \frac{1}{t^2 - 1} \right) dt = -2t - \ln \left| \frac{t-1}{t+1} \right| + C = -2t - \ln \left| \frac{(t-1)(t+1)}{(t+1)^2} \right| + C \\ &= -2t + 2\ln(t+1) - \ln|t^2 - 1| + C \quad -\ln|t^2 - 1| = \ln \left| \frac{1}{t^2 - 1} \right| \\ &= -2\sqrt{\frac{1+x}{x}} + 2\ln \left(\sqrt{\frac{1+x}{x}} + 1 \right) + \ln|x| + C \end{aligned}$$



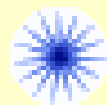
例5 求 $\int \frac{1}{\sqrt[3]{(x+1)^2(x-1)^4}} dx$

解 $\int \frac{dx}{\sqrt[3]{(x+1)^2(x-1)^4}} = \int \frac{1}{(x+1)(x-1)} \sqrt[3]{\frac{x+1}{x-1}} dx$

令 $\sqrt[3]{\frac{x+1}{x-1}} = t$, 则 $x = \frac{t^3+1}{t^3-1}$, $dx = \frac{-6t^2}{(t^3-1)^2} dt$ 。

$$\int \frac{dx}{\sqrt[3]{(x+1)^2(x-1)^4}} = \int \frac{t}{\frac{4t^3}{(t^3-1)^2}} \cdot \frac{-6t^2}{(t^3-1)^2} dt$$

$$= -\frac{3}{2} \int dt = -\frac{3}{2} t + C = -\frac{3}{2} \sqrt[3]{\frac{x+1}{x-1}} + C$$



初等函数在其定义区间上的原函数一定存在， 但原函数
不一定是初等函数.

$$\text{例 } \int e^{-x^2} dx, \int \frac{\sin x}{x} dx, \int \frac{dx}{\ln x}, \int \frac{dx}{\sqrt{1+x^4}}$$

都不是初等函数。

不可积函数类

