

习题课4(变上限函数的求导) 15题

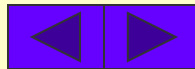
1. 已知 $x \geq 0$ 时, $f(x)$ 连续, 且 $\int_0^{x^2(1+x)} f(t)dt = x$, 则 $f(2) = \underline{\hspace{2cm}}$

2. $\frac{d}{dx} \left(\int_{x^2}^0 x \cos t^2 dt \right)$

3. 已知 $F(x) = \int_0^x \frac{\sqrt{x} - \sqrt{t}}{1+t} dt$, 求 $F'(x)$.

4. 求 $\frac{d}{dx} \int_{\sin x}^{\cos x} \cos(\pi t^2) dt$

5. 设 $f(x)$ 在 $(-\infty, +\infty)$ 内可导, $f'(1) = f(1) = 1$, $g(x) = \int_{1+x}^{e^x} f(t)dt$, 则 $g''(0) = \underline{\hspace{2cm}}$.

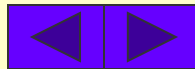


6. 已知当 $x \rightarrow 0$ 时, $F(x) = \int_0^x (x^2 - t^2) f''(t) dt$ 的导数与 x^2 是等价无穷小, 则 $f''(0)$ 的值为

(A) 0 (B) $\frac{1}{2}$ (C) 1 (D) ∞

7. $\int_0^y e^{t^2} dt = \int_0^{3x^2} \ln \sqrt{t + x^2} dt \quad (x > 0)$, 求 $\frac{dy}{dx}$.

8. 设
$$\begin{cases} x = \int_1^t u \ln u du \\ y = \int_1^{t^2} u^2 \ln u du \end{cases}, \text{ 求 } \frac{dy}{dx}, \frac{d^2 y}{dx^2}. (\text{其中 } t > 0)$$



9. $\frac{d}{dx} \int_0^x \sin(x-t)^2 dt = \underline{\hspace{2cm}}.$

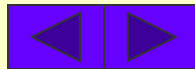
10. 设 $f(x)$ 连续, 则 $\frac{d}{dx} \left[\int_0^x t f(x^2 - t^2) dt \right] = \underline{\hspace{2cm}}.$

11. 设 $f(x) \in C[0, +\infty)$, $f(x) > 0$, 求证

$$F(x) = \frac{\int_0^x t f(t) dt}{\int_0^x f(t) dt}$$

在 $(0, +\infty)$ 内是单调增加函数。

12. 求 $\lim_{x \rightarrow 0} \frac{\int_{2x}^0 \sin t^2 dt}{x^3}$



13. 求 $\lim_{x \rightarrow 0} \frac{\int_{\cos x}^1 e^{-t^2} dt}{x^2}$

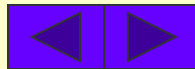
14. 设 $g(x)$ 在 $(-\infty, +\infty)$ 上连续, $g(1) = 1, \int_0^1 g(x) dx = \frac{1}{2}$.

令 $f(x) = \int_0^x g(x-t)t^2 dt$, 求 $f''(1), f'''(1)$.

15. 对一切实数 t , 函数 $f(t)$ 是连续的正函数, 函数

$$g(x) = \int_{-a}^a |x-t| f(t) dt, \quad -a \leq x \leq a (a > 0).$$

证明 $g'(x)$ 是单调增加的。



答案

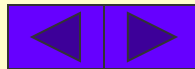
1.解 等式两端同时求导得: $(2x + 3x^2)f(x^2(1+x)) = 1$

$$\text{令 } x=1 \text{ 得: } 5f(2) = 1 \Rightarrow f(2) = \frac{1}{5}$$

$$\begin{aligned} 2. \text{ 解 } \frac{d}{dx} \left(\int_{x^2}^0 x \cos t^2 dt \right) &= \frac{d}{dx} \left(x \int_{x^2}^0 \cos t^2 dt \right) \\ &= \int_{x^2}^0 \cos t^2 dt + x \cdot (-1) \cos(x^2)^2 \cdot 2x \end{aligned}$$

$$3. \text{ 解 } F(x) = \sqrt{x} \int_0^x \frac{1}{1+t} dt - \int_0^x \frac{\sqrt{t}}{1+t} dt$$

$$F'(x) = \frac{1}{2\sqrt{x}} \int_0^x \frac{1}{1+t} dt + \sqrt{x} \cdot \frac{1}{1+x} - \frac{\sqrt{x}}{1+x} = \frac{1}{2\sqrt{x}} \int_0^x \frac{1}{1+t} dt$$



$$4. \text{解} \quad \frac{d}{dx} \int_{\sin x}^{\cos x} \cos(\pi t^2) dt$$

$$= \cos(\pi \cos^2 x)(\cos x)' - \cos(\pi \sin^2 x)(\sin x)'$$

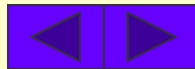
$$= -\cos(\pi \cos^2 x) \sin x - \cos(\pi \sin^2 x) \cos x$$

$$= -\cos[\pi(1 - \sin^2 x)] \sin x - \cos(\pi \sin^2 x) \cos x$$

$$= -\cos(\pi - \pi \sin^2 x) \sin x - \cos(\pi \sin^2 x) \cos x$$

$$= \cos(\pi \sin^2 x) \sin x - \cos(\pi \sin^2 x) \cos x$$

$$= (\sin x - \cos x) \cos(\pi \sin^2 x)$$



5. 设 $f(x)$ 在 $(-\infty, +\infty)$ 内可导, $f'(1) = f(1) = 1$, $g(x) = \int_{1+x}^{e^x} f(t)dt$, 则 $g''(0) = \underline{\hspace{2cm}}$.

解 $g'(x) = f(e^x) \cdot e^x - f(1+x)$

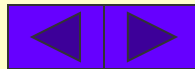
$$g''(x) = f'(e^x) \cdot e^{2x} + f(e^x) \cdot e^x - f'(1+x)$$

令 $x=0$ 得: $g''(0) = f'(1) + f(1) - f'(1) = f(1) = 1$

6. 解 $F'(x) = (x^2 \int_0^x f''(t)dt)' - (\int_0^x t^2 f''(t)dt)'$

$$= x^2 f''(x) + 2x \int_0^x f''(t)dt - x^2 f''(x) = 2x[f'(x) - f'(0)]$$

因为 $f''(0) = \lim_{x \rightarrow 0} \frac{f'(x) - f'(0)}{x - 0} = \lim_{x \rightarrow 0} \frac{F'(x)}{2x^2} = \frac{1}{2}.$



7.解 两边同时对 x 求导.其中 y 是 x 的函数.

$$e^{y^2} \cdot \frac{dy}{dx} = 8x \ln 2 + 6x \ln x \quad \text{故} \quad \frac{dy}{dx} = \frac{8x \ln 2 + 6x \ln x}{e^{y^2}}$$

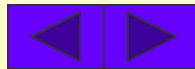
$$8. \frac{dy}{dx} = \frac{2t \cdot t^4 \ln t^2}{t \ln t} = 4t^4, \quad \frac{d^2 y}{dx^2} = \frac{16t^2}{t \ln t}$$

$$9. \text{解} \quad \int_0^x \sin(x-t)^2 dt \quad \underline{x-t=u} - \int_x^0 \sin u^2 du = \int_0^x \sin u^2 du$$

$$\frac{d}{dx} \int_0^x \sin(x-t)^2 dt = \sin x^2$$

$$10. \text{解} \quad \int_0^x t f(x^2 - t^2) dt \stackrel{x^2 - t^2 = u}{=} \int_{x^2}^0 -\frac{1}{2} f(u) du = \frac{1}{2} \int_0^{x^2} f(u) du$$

$$\frac{d}{dx} \left[\int_0^x t f(x^2 - t^2) dt \right] = \frac{1}{2} f(x^2) \cdot 2x = x f(x^2).$$



$$11. \text{证 } F'(x) = \frac{xf(x)\int_0^x f(t)dt - f(x)\int_0^x tf(t)dt}{\left(\int_0^x f(t)dt\right)^2}$$

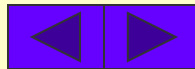
$$F'(x) = \frac{f(x)\left(x\int_0^x f(t)dt - \int_0^x tf(t)dt\right)}{\left(\int_0^x f(t)dt\right)^2} = \frac{f(x)\left(\int_0^x xf(t)dt - \int_0^x tf(t)dt\right)}{\left(\int_0^x f(t)dt\right)^2}$$

$$= \frac{f(x)\int_0^x (x-t)f(t)dt}{\left(\int_0^x f(t)dt\right)^2}$$

$$\left. \begin{array}{l} \because t \in [0, x], f(t) > 0, (x-t)f(t) \geq 0, \\ \text{且 } (x-t)f(t) \not\equiv 0, \end{array} \right\} \Rightarrow$$

$$\int_0^x f(t)dt > 0, \int_0^x (x-t)f(t)dt > 0, \quad \therefore F'(x) > 0,$$

$\therefore F(x)$ 在 $(0, +\infty)$ 内是单调增加函数。



12. 求 $\lim_{x \rightarrow 0} \frac{\int_{2x}^0 \sin t^2 dt}{x^3}$

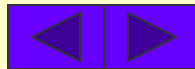
解 $\lim_{x \rightarrow 0} \frac{\int_{2x}^0 \sin t^2 dt}{x^3} \left(\frac{0}{0} \text{型} \right) = \lim_{x \rightarrow 0} \frac{\left(\int_{2x}^0 \sin t^2 dt \right)'}{(x^3)'} =$

$$= \lim_{x \rightarrow 0} \frac{-\sin(2x)^2 \cdot (2x)'}{3x^2} = - \lim_{x \rightarrow 0} \frac{4x^2 \times 2}{3x^2} = -\frac{8}{3}$$

13. 求 $\lim_{x \rightarrow 0} \frac{\int_{\cos x}^1 e^{-t^2} dt}{x^2}$

解 $\lim_{x \rightarrow 0} \frac{\int_{\cos x}^1 e^{-t^2} dt}{x^2} \left(\frac{0}{0} \text{型} \right) = \lim_{x \rightarrow 0} \frac{\left(\int_{\cos x}^1 e^{-t^2} dt \right)'}{(x^2)'}$

$$= \lim_{x \rightarrow 0} \frac{-e^{-\cos^2 x} \cdot (-\sin x)}{2x} = \lim_{x \rightarrow 0} \frac{e^{-\cos^2 x}}{2} \cdot \frac{\sin x}{x} = \frac{1}{2e}$$



$$14. \text{ 解 } f(x) = \int_0^x g(x-t)t^2 dt \quad \underline{\underline{x-t=u}} - \int_x^0 g(u)(x-u)^2 du$$

$$= \int_0^x g(u)(x^2 - 2xu + u^2) du$$

$$= x^2 \int_0^x g(u) du - 2x \int_0^x ug(u) du + \int_0^x u^2 g(u) du$$

$$f'(x) = 2x \int_0^x g(u) du + x^2 g(x) - 2 \int_0^x ug(u) du - 2x^2 g(x) + x^2 g(x)$$

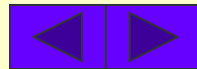
$$= 2x \int_0^x g(u) du - 2 \int_0^x ug(u) du$$

$$f''(x) = 2 \int_0^x g(u) du + 2xg(x) - 2xg(x) = 2 \int_0^x g(u) du$$

$$\text{所以 } f''(1) = 2 \int_0^1 g(u) du = 2 \times \frac{1}{2} = 1$$

$$f'''(x) = 2g(x).$$

$$f'''(1) = 2g(1) = 2$$



15. 对一切实数 t , 函数 $f(t)$ 是连续的正函数, 函数

$$g(x) = \int_{-a}^a |x - t| f(t) dt, \quad -a \leq x \leq a (a > 0).$$

证明 $g'(x)$ 是单调增加的。

$$t \in (-a, x)$$

$$t \in (x, a)$$

证
$$g(x) = \int_{-a}^a |x - t| f(t) dt = \int_{-a}^x (x - t) f(t) dt + \int_x^a (t - x) f(t) dt$$

$$= x \int_{-a}^x f(t) dt - \int_{-a}^x t f(t) dt + \int_x^a t f(t) dt - x \int_x^a f(t) dt$$

$$g'(x) = \int_{-a}^x f(t) dt + x f(x) - x f(x) - x f(x) - \int_x^a f(t) dt + x f(x)$$

$$= \int_{-a}^x f(t) dt - \int_x^a f(t) dt = \int_{-a}^x f(t) dt + \int_a^x f(t) dt$$

$$g''(x) = f(x) + f(x) = 2f(x) > 0$$

$\therefore g'(x)$ 是单调增加的。

