二. 换元积分法

$$\int \sin 3x \cdot 3 \cdot dx = -\cos 3x + C = \int \sin 3x d3x$$

(一) 第一类换元积分法

1. 基本公式

设$\int f(u)du = F(u) + C$，$u = \varphi(x)$可导，则

$$\int f[\varphi(x)]\varphi'(x)dx = F[\varphi(x)] + C = \int f(u)du\Big|_{u=\varphi(x)}$$

2. 凑微分

（1）凑系数

调整系数

把3x当作u,“d”后面凑成u

$$= \frac{1}{3}\int \sin u du = \frac{1}{3}(-\cos u + C_1)$$

1. 求$\int \sin 3x dx$

解：$\int \sin 3x dx = \frac{1}{3}\int \sin 3x d(3x) = -\frac{1}{3}\cos 3x + C$

2. $\int e^{4x} dx = \frac{1}{4}\int e^{4x} d(4x) = \frac{1}{4}e^{4x} + C$

3. $\int a^{2x} dx = \frac{1}{2}\int a^{2x} d(2x) = \frac{1}{2}\cdot\frac{a^{2x}}{\ln a} + C$

（2）凑线性式

调整系数时，只管**a**不管**b**. ∵**d(b)=0**

$$4.\int (ax+b)^5 dx = \frac{1}{a}\int (ax+b)^5 d(ax+b) = \frac{1}{6a}(ax+b)^6 + C$$

$$5.\int \sin(3x+2)dx = \frac{1}{3}\int \sin(3x+2)d\left(3x+2\right) = -\frac{1}{3}\cos(3x+2)+C$$

$$6.\int \sec^2(2x+1)dx = \frac{1}{2}\int \sec^2(2x+1)d(2x+1) = \frac{1}{2}\tan(2x+1)+C$$

$$7.\int \frac{1}{x^2-1}dx = \frac{1}{2}\int\left(\frac{1}{x-1}-\frac{1}{x+1}\right)dx = \frac{1}{2}\left[\int \frac{1}{x-1}d\left(x-1\right)-\int \frac{1}{x+1}d\left(x+1\right)\right]$$

$$=\frac{1}{2}\left[\ln|x-1|-\ln|x+1|\right]+C = \frac{1}{2}\ln\left|\frac{x-1}{x+1}\right|+C$$

（3）凑微分——逆向思维的程序化

$$\boxed{xdx}$$

例：$\int \boxed{x} \cdot e^{x^2} \boxed{dx} = \boxed{\frac{1}{2}} \int e^{x^2} \boxed{dx^2} = \frac{1}{2} e^{x^2} + C$

说明：

a ）凑，是一种逆向思维活动，一般构成教学上的难点，解决方法是使思维活动程序化。

b ）看被积函数由哪几个因式组成。

c ）把容易积分的因式先积分，积分结果放在微分号 “*d*” 的后面。如果有常数，则直接放在积分号前面。

d ）把 “*d* ”后面的表达式作为*u*,看能否积分。

e ）继续使用其它积分方法。

$$8.\int xe^{3x^2+2}dx = \frac{1}{2}\int e^{3x^2+2}dx^2 = \frac{1}{2}\cdot\frac{1}{3}\int e^{3x^2+2}d(3x^2+2)$$

$$= \frac{1}{6}e^{3x^2+2} + C$$

9. 求$\int x\sqrt{4-x^2}dx$

解：$\int \boxed{x}\sqrt{4-x^2}dx=\boxed{\frac{1}{2}}\int\sqrt{4-x^2}\boxed{dx^2}=\boxed{-}\frac{1}{2}\int\sqrt{4-x^2}d(\boxed{4-x^2})$

$$=-\frac{1}{2}\cdot\frac{2}{3}(4-x^2)^{\frac{3}{2}}+C$$

10. 求 $\int\frac{dx}{\cos^2 x(1+\tan x)}$

解：$\int\frac{dx}{\cos^2 x(1+\tan x)}=\int\boxed{\frac{1}{\cos^2 x}}\cdot\frac{1}{(1+\tan x)}\cdot dx=\int\frac{d(\boxed{\tan x})}{1+\tan x}$

$$=\int\frac{d(\boxed{1+\tan x})}{1+\tan x}=\ln|1+\tan x|+C$$

11. 求：$\int\frac{dx}{x(2+3\ln x)}$

解：$\int\frac{dx}{x(2+3\ln x)}=\int\boxed{\frac{1}{x}}\cdot\frac{1}{2+3\ln x}dx=\int\frac{d(\boxed{\ln x})}{2+3\ln x}=\boxed{\frac{1}{3}}\int\frac{d(\boxed{2+3\ln x})}{2+3\ln x}$

$$=\frac{1}{3}\ln|2+3\ln x|+C$$

12. 求 $\int \sec^4 x dx$.

解: $\int \sec^4 x dx = \int \boxed{\sec^2 x} \cdot \boxed{\sec^2 x} dx = \int (\boxed{\tan^2 x + 1}) d(\boxed{\tan x})$

$$= \frac{1}{3}\tan^3 x + \tan x + C$$

13. 求 $\int \sin^3 x dx$.

解: $\int \sin^3 x dx = \int \boxed{\sin^2 x} \cdot \boxed{\sin x} dx = \boxed{-}\int (\boxed{1-\cos^2 x}) d(\boxed{\cos x})$

$$= \int (\cos^2 x - 1) d(\cos x) = \frac{1}{3}\cos^3 x - \cos x + C$$

说明:(1) **凡是sin$x$、cos$x$的奇次幂，都可以采用这种分出一次因式、将剩余部分用平方关系变形的方法。**

(2) 类似的: $\int \tan^m x \sec^{2n} x dx$ 则可以先分出 $\sec^2 x$ 凑微分。

$$\int \tan^m x \sec^{2n} x dx = \int \tan^m x \sec^{2n-2} x \cdot \sec^2 x dx$$

$$= \int \tan^m x \, (1+\tan^2 x)^{n-1} d(\tan x) = \cdots$$

14. 证明$\int \frac{dx}{\sqrt{a^2-x^2}} = \arcsin\frac{x}{a} + C$（$a>0$）

证：$\int \frac{dx}{\sqrt{a^2-x^2}} = \int \frac{dx}{a\sqrt{1-\left(\frac{x}{a}\right)^2}} = \int \frac{d\left(\frac{x}{a}\right)}{\sqrt{1-\left(\frac{x}{a}\right)^2}} = \arcsin\frac{x}{a} + C$

常用的凑微分公式:

$$\int f(ax+b)dx = \frac{1}{a}\int f(ax+b)d(ax+b) \qquad a \neq 0$$

$$\int f(ax^2+b)xdx = \frac{1}{2a}\int f(ax^2+b)d(ax^2+b) \qquad a \neq 0$$

$$\int f(ax^\alpha+b)x^{\alpha-1}dx = \frac{1}{\alpha\cdot a}\int f(ax^\alpha+b)d(ax^\alpha+b) \qquad a \neq 0$$

$$\int f\left(\frac{1}{x}\right)\frac{1}{x^2}dx = -\int f\left(\frac{1}{x}\right)d\left(\frac{1}{x}\right)$$

$$\int f(\ln x)\frac{1}{x}dx = \int f(\ln x)d(\ln x)$$

$$\int f\left(e^{\alpha x}\right)e^{\alpha x}dx=\frac{1}{\alpha}\int f\left(e^{\alpha x}\right)d\left(e^{\alpha x}\right)\qquad \alpha\neq 0$$

$$\int f(\sin x)\cos x\,dx=\int f(\sin x)d(\sin x)$$

$$\int f(\cos x)\sin x\,dx=-\int f(\cos x)d(\cos x)$$

$$\int f(\tan x)\sec^2 x\,dx=\int f(\tan x)\frac{1}{\cos^2 x}dx=\int f(\tan x)d(\tan x)$$

$$\int f(\cot x)\csc^2 x\,dx=\int f(\cot x)\frac{1}{\sin^2 x}dx=-\int f(\cot x)d(\cot x)$$

$$\int f(\sec x)\sec x\tan x\,dx=\int f(\sec x)d(\sec x)$$

$$\int f(\arcsin x)\frac{1}{\sqrt{1-x^2}}dx=\int f(\arcsin x)d(\arcsin x)$$

$$\int f(\arctan x)\frac{1}{1+x^2}dx=\int f(\arctan x)d(\arctan x)$$

练习：

$$(1)\int \frac{1}{\sqrt{1-3x}}dx$$

$$(2)\int x^2\sqrt{x^3+1}dx$$

$$(3)\int \frac{2x-6}{x^2-6x+13}dx$$

$$(4)\int \frac{x^3}{\sqrt{1+x^2}}dx$$

$$(5)\int \frac{\cos\sqrt{t}}{\sqrt{t}}dt$$

$$(6)\int \frac{1}{1+e^x}dx$$

答案:

$$(1)\int \frac{1}{\sqrt{1-3x}}dx=-\frac{1}{3}\int \frac{1}{\sqrt{1-3x}}d(1-3x)=-\frac{1}{3}\cdot 2\sqrt{1-3x}+C$$

$$(2)\int x^2\sqrt{x^3+1}dx=\frac{1}{3}\int \sqrt{x^3+1}d(x^3+1)=\frac{1}{3}\sqrt{x^3+1}+C$$

$$(3)\int \frac{2x-6}{x^2-6x+13}dx=\int \frac{1}{x^2-6x+13}d(x^2-6x+13)=\ln(x^2-6x+13)+C$$

$$(4)\int \frac{x^3}{\sqrt{1+x^2}}dx=\int \frac{x^2+1-1}{\sqrt{1+x^2}}d(x^2+1)=\int\left(\sqrt{x^2+1}-\frac{1}{\sqrt{x^2+1}}\right)d(x^2+1)$$

$$(5)\int \frac{\cos\sqrt{t}}{\sqrt{t}}dt=2\int \cos\sqrt{t}\,d\sqrt{t}=2\sin\sqrt{t}+C$$

$$(6)\int \frac{1}{1+e^x}dx=\int \frac{1+e^x-e^x}{1+e^x}dx=\int\left(1-\frac{e^x}{1+e^x}\right)dx=x-\int \frac{1}{1+e^x}d(e^x+1)$$

例题:

15. 求$\int \tan x dx$

$$\int \cot x dx = \ln|\sin x| + C$$

解 $\int \tan x dx = \int \frac{\sin x}{\cos x} dx = -\int \frac{1}{\cos x} d\cos x = -\ln|\cos x| + C$

16. 求$\int \frac{1}{a^2 + x^2} dx$

解 $\int \frac{1}{a^2 + x^2} dx = \int \frac{1}{a^2} \cdot \frac{1}{1 + \left(\frac{x}{a}\right)^2} dx = \int \frac{1}{a} \cdot \frac{1}{1 + \left(\frac{x}{a}\right)^2} d\left(\frac{x}{a}\right)$

$= \frac{1}{a} \arctan \frac{x}{a} + C$

17. 求 $\int \frac{1}{\sqrt{a^2 - x^2}} dx \quad (a > 0)$

解 $\int \frac{1}{\sqrt{a^2 - x^2}} dx$

$$= \int \frac{1}{a} \frac{1}{\sqrt{1 - \left(\frac{x}{a}\right)^2}} dx$$

$$= \int \frac{1}{\sqrt{1 - \left(\frac{x}{a}\right)^2}} d\frac{x}{a}$$

$$= \arcsin\left(\frac{x}{a}\right) + C$$

18. 求 $\int \frac{1}{x^2 - a^2} dx$

解 $\int \frac{1}{x^2 - a^2} dx$

$$= \int \frac{1}{(x - a)(x + a)} dx$$

$$= \int \frac{1}{2a} \left(\frac{1}{x - a} - \frac{1}{x + a} \right) dx$$

$$= \frac{1}{2a} \left(\int \frac{1}{x - a} dx - \int \frac{1}{x + a} dx \right)$$

$$= \frac{1}{2a} \left(\ln|x - a| - \ln|x + a| \right) + C$$

$$= \frac{1}{2a} \left(\ln\left|\frac{x - a}{x + a}\right| \right) + C$$

19. 求$\int \frac{e^{3\sqrt{x}}}{\sqrt{x}}dx$

解 $\int \frac{e^{3\sqrt{x}}}{\sqrt{x}}dx$

$$= 2\int e^{3\sqrt{x}}d\sqrt{x}$$

$$= \frac{2}{3}\int e^{3\sqrt{x}}d3\sqrt{x}$$

$$= \frac{2}{3}e^{3\sqrt{x}} + C$$

20. 求$\int \sin^2 x\cos^5 xdx$

解 $\int \sin^2 x\cos^5 xdx$

$$= \int \sin^2 x\cos^4 x\cos xdx$$

$$= \int \sin^2 x\left(1-\sin^2 x\right)^2 d\sin x$$

$$= \int \sin^2 x\left(1-2\sin^2 x+\sin^4 x\right)d\sin x$$

$$= \int \left(\sin^2 x-2\sin^4 x+\sin^6 x\right)d\sin x$$

$$= \frac{1}{3}\sin^3 x-\frac{2}{5}\sin^5 x+\frac{1}{7}\sin^7 x+C$$

21. 求$\int \cos^2 x dx$

解 $\int \cos^2 x dx$

$$= \int \frac{1+\cos 2x}{2} dx$$

$$= \frac{1}{2}\left(\int dx + \int \cos 2x dx\right)$$

$$= \frac{1}{2}\left(\int dx + \frac{1}{2}\int \cos 2x d2x\right)$$

$$= \frac{1}{2}\left(x + \frac{1}{2}\sin 2x\right) + C$$

$$= \frac{1}{2}x + \frac{1}{4}\sin 2x + C$$

22. 求$\int \cos^4 x dx$

解 $\int \cos^4 x dx = \int \left(\frac{1+\cos 2x}{2}\right)^2 dx$

$$= \frac{1}{4}\int \left(1 + 2\cos 2x + \cos^2 2x\right) dx$$

$$= \frac{1}{4}\int \left(1 + 2\cos 2x + \frac{1+\cos 4x}{2}\right) dx$$

$$= \frac{1}{4}\int \left(\frac{3}{2} + 2\cos 2x + \frac{\cos 4x}{2}\right) dx$$

$$= \frac{1}{4}\left(\frac{3}{2}x + \sin 2x + \frac{1}{2}\cdot\frac{1}{4}\sin 4x\right) + C$$

$$= \frac{3}{8}x + \frac{1}{4}\sin 2x + \frac{1}{32}\sin 4x + C$$

结论: 被积函数是正弦或余弦的偶次幂, 用余弦半角公式降幂.

23. 求$\int \csc x dx$

解 $\int \csc x dx = \int \frac{1}{\sin x} dx = \int \frac{1}{2\sin\frac{x}{2}\cos\frac{x}{2}} dx = \int \frac{\cos\frac{x}{2}}{\sin\frac{x}{2}\cos^2\frac{x}{2}} d\left(\frac{x}{2}\right)$

$$= \int \frac{1}{\tan\frac{x}{2}\cos^2\frac{x}{2}} d\left(\frac{x}{2}\right) = \int \frac{1}{\tan\frac{x}{2}} d\left(\tan\frac{x}{2}\right) = \ln\left|\tan\frac{x}{2}\right| + C$$

$$\because \tan\frac{x}{2} = \frac{1-\cos x}{\sin x} = \csc x - \cot x$$

$$\boxed{\therefore \int \csc x dx = \ln\left|\csc x - \cot x\right| + C}$$

$$\left[= \int \frac{\sin^2\frac{x}{2} + \cos^2\frac{x}{2}}{\sin\frac{x}{2}\cos\frac{x}{2}} d\left(\frac{x}{2}\right) = \int (\tan\frac{x}{2} + \cot\frac{x}{2}) d\left(\frac{x}{2}\right) \right.$$

$$\left. = -\ln\left|\cos\frac{x}{2}\right| + \ln\left|\sin\frac{x}{2}\right| + C \right]$$

24. 求$\int \sec x dx$.

解 $\int \sec x dx = \int \frac{1}{\cos x} dx = \int \frac{1}{\sin\left(x + \frac{\pi}{2}\right)} dx = \int \csc\left(x + \frac{\pi}{2}\right) d(x + \frac{\pi}{2})$

$= \ln\left|\csc\left(x + \frac{\pi}{2}\right) - \cot\left(x + \frac{\pi}{2}\right)\right| + C \quad = \ln\left|\sec x + \tan x\right| + C$

例16 求$\int \sec^6 x dx$.

解 $\int \sec^6 x dx = \int (\sec^2 x)^2 \sec^2 x dx = \int (1 + \tan^2 x)^2 d\tan x$

$= \int (1 + 2\tan^2 x + \tan^4 x) d\tan x = \tan x + \frac{2}{3}\tan^3 x + \frac{1}{5}\tan^5 x + C$

25. 求$\int \tan^5 x \sec^3 x dx$

解 $\int \tan^5 x \sec^3 x dx = \int \tan^4 x \sec^2 x \cdot \tan x \sec x dx$

$$= \int \left(\sec^2 x - 1\right)^2 \sec^2 x d \sec x = \int \left(\sec^6 x - 2\sec^4 x + \sec^2 x\right) d \sec x$$

$$= \frac{1}{7}\sec^7 x - \frac{2}{5}\sec^5 x + \frac{1}{3}\sec^3 x + C$$

26. 求$\int \cos 3x \cos 2x dx$

解 $\int \cos 3x \cos 2x dx = \frac{1}{2} \cdot \int (\cos x + \cos 5x) dx$

$$= \frac{1}{2}\sin x + \frac{1}{10}\sin 5x + C$$

练习：

$$(7)\int\frac{1}{1+\sin x}dx \qquad (8)\int\frac{\sin 2x}{\sqrt{3-\cos^4 x}}dx$$

$$(9)\int\tan^3 x dx \qquad (10)\int\tan^{10} x\sec^2 x dx$$

$$(11)\int\frac{7\cos x-3\sin x}{5\cos x+2\sin x}dx \qquad (12)\int\frac{1}{\sin^2 x+2\cos^2 x}dx$$

$$(13)\int\frac{\tan x}{\sqrt{\cos x}}dx$$

答案:

$$(7)\int\frac{1}{1+\sin x}dx=\int\frac{1-\sin x}{1-\sin^2 x}dx=\int\sec^2 x dx-\int\frac{1}{\cos^2 x}d(\cos x)$$

$$(8)\int\frac{\sin 2x}{\sqrt{3-\cos^4 x}}dx=\int\frac{2\sin x\cos x}{\sqrt{3-\cos^4 x}}dx=-\int\frac{d(\cos^2 x)}{\sqrt{3-(\cos^2 x)^2}}$$

$$(9)\int\tan^3 x dx=\int[\tan x(\tan^2 x+1)-\tan x]dx=\int\tan x d(\tan x)-\int\tan x dx$$

$$(10)\int\tan^{10} x\sec^2 x dx=\int\tan^{10} x d(\tan x)=\frac{1}{11}\tan^{11} x+C$$

$$(11)\int\frac{7\cos x-3\sin x}{5\cos x+2\sin x}dx=\int\frac{(2\cos x-5\sin x)+(5\cos x+2\sin x)}{5\cos x+2\sin x}dx$$

$$(12)\int\frac{1}{\sin^2 x+2\cos^2 x}dx=\int\frac{1}{\cos^2 x(\tan^2 x+2)}dx=\int\frac{1}{\tan^2 x+2}d(\tan x)$$

$$(13)\int\frac{\tan x}{\sqrt{\cos x}}dx=\int\frac{\sin x}{\cos x\sqrt{\cos x}}dx=-\int\cos^{-\frac{3}{2}} x d(\cos x)$$

课后思考与练习

1.若$\int f(x)dx = F(x)+C$，则$\int e^{-x} f(e^{-x})dx =$ ________

2.若$f'(x^2)=\frac{1}{x}(x>0)$，则$f(x)=$ ________

3.$\int \frac{f'(x)}{1+f^2(x)}dx =$ ________

4.求$\int \frac{dx}{x(x^{10}+2)}$.

5.求$\int \frac{f'(\ln x)}{x\sqrt{f(\ln x)}}dx$.

6.设$\int xf(x)dx = \arcsin x + C$,求$\int \frac{dx}{f(x)}$.

二、第二类换元法

$$\int x\sqrt{1-x^2}dx=-\frac{1}{2}\int\sqrt{1-x^2}d(1-x^2)$$

$$\int\sqrt{1-x^2}dx=?\qquad \int x^2\sqrt{1-x^2}dx=?$$

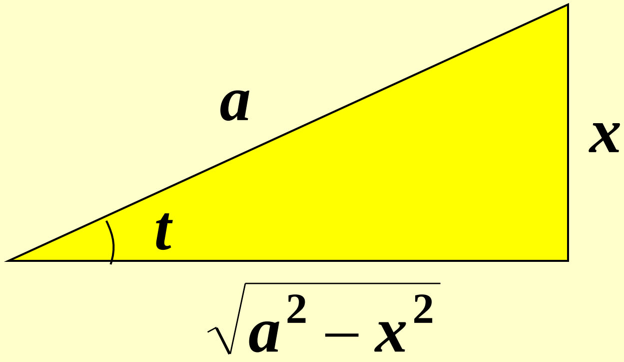
$$\int x\sqrt{1-x}dx=?\qquad \int\frac{\sqrt{1-x}}{x}dx=?$$

定理2 设$x=\varphi(t)$是单调的、可导的，并且$\varphi'(t)\neq 0$,并设

$f[\varphi(t)]\varphi'(t)$ 具有原函数，则

$$\int f(x)dx \underline{\underline{\;x=\varphi(t)\;}} \int f[\varphi(t)]\varphi'(t)dt$$

1. 求$\int\sqrt{a^2-x^2}dx\quad(a>0)$



解 设$x=a\sin t\left(-\dfrac{\pi}{2}<t<\dfrac{\pi}{2}\right)$,则

$$dx=a\cos t\,dt$$

$$\sqrt{a^2-x^2}=\sqrt{a^2-a^2\sin^2 t}=a\cos t$$

$$\sin t=\frac{x}{a}$$

$$\therefore\int\sqrt{a^2-x^2}dx=\int a\cos t\cdot a\cos t\,dt$$

$$t=\arcsin\frac{x}{a}$$

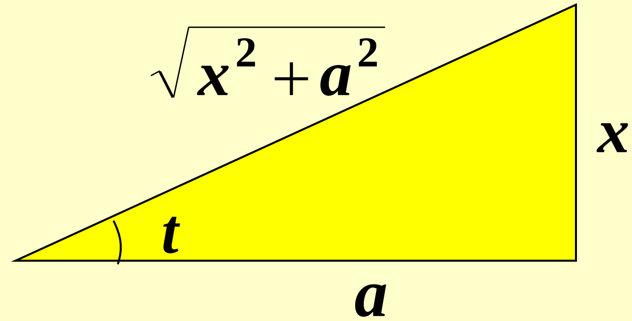
$$=\int a^2\cos^2 t\,dt=\frac{a^2}{2}t+\frac{a^2}{4}\sin 2t+C$$

$$\cos t=\frac{\sqrt{a^2-x^2}}{a}$$

$$=\frac{a^2}{2}t+\frac{a^2}{2}\sin t\cos t+C=\frac{a^2}{2}\arcsin\frac{x}{a}+\frac{a^2}{2}\frac{x}{a}\cdot\frac{\sqrt{a^2-x^2}}{a}+C$$

$$=\frac{a^2}{2}\arcsin\frac{x}{a}+\frac{1}{2}x\cdot\sqrt{a^2-x^2}+C$$

2. 求$\int \frac{dx}{\sqrt{x^2+a^2}}\ (a>0)$



解 设$x = a\tan t\left(-\frac{\pi}{2}<t<\frac{\pi}{2}\right)$,则

$$dx = a\sec^2 t dt$$

$$\sqrt{x^2+a^2} = \sqrt{a^2\tan^2 t + a^2} = a\sec t$$

$$\tan t = \frac{x}{a}$$

$$\therefore \int \frac{dx}{\sqrt{x^2+a^2}} = \int \frac{a\sec^2 t}{a\sec t}dt$$

$$\sec t = \frac{\sqrt{x^2+a^2}}{a}$$

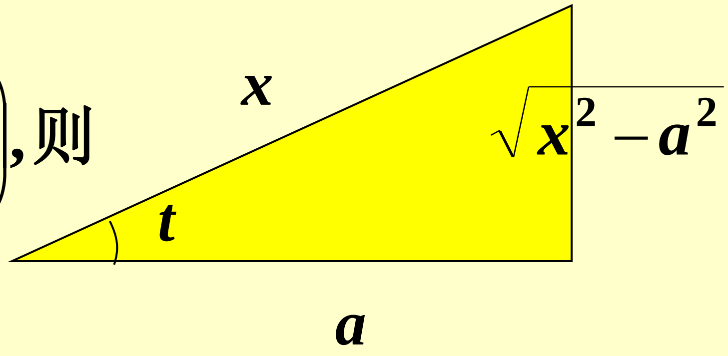
$$= \int \sec t dt = \ln\left|\sec t + \tan t\right| + C_1$$

$$= \ln\left|\frac{\sqrt{x^2+a^2}}{a} + \frac{x}{a}\right| + C_1 = \ln\left|\sqrt{x^2+a^2} + x\right| - \ln a + C_1$$

$$= \ln(x + \sqrt{x^2+a^2}) + C$$

3. 求$\int \frac{dx}{\sqrt{x^2-a^2}} \quad (a>0)$

解 (i)$x>a$时，设$x = a\sec t\left(0<t<\frac{\pi}{2}\right)$,则



$$dx = a\sec t\tan t dt$$

$$\sqrt{x^2-a^2} = \sqrt{a^2\sec^2 t - a^2} = a\tan t$$

$$\sec t = \frac{x}{a}$$

$$\therefore \int \frac{dx}{\sqrt{x^2-a^2}} = \int \frac{a\sec t\tan t dt}{a\tan t}$$

$$= \int \sec t dt \qquad = \ln(\sec t + \tan t) + C_1$$

$$\tan t = \frac{\sqrt{x^2-a^2}}{a}$$

$$= \ln\left(\frac{x}{a} + \frac{\sqrt{x^2-a^2}}{a}\right) + C_1 = \ln\left(x+\sqrt{x^2-a^2}\right) + C$$

$(\text{ii})\ x < -a$时，设$x = -u$,则

$$\int \frac{dx}{\sqrt{x^2 - a^2}} = -\int \frac{du}{\sqrt{u^2 - a^2}}$$

$$= -\ln\left(u + \sqrt{u^2 - a^2}\right) + C_1 = -\ln\left(-x + \sqrt{x^2 - a^2}\right) + C_1$$

$$= \ln \frac{1}{-x + \sqrt{x^2 - a^2}} + C_1 = \ln \frac{-x - \sqrt{x^2 - a^2}}{a^2} + C_1$$

$$= \ln\left(-x - \sqrt{x^2 - a^2}\right) + C$$

$\because x < -a < 0, \qquad \therefore -x - \sqrt{x^2 - a^2} = \left|x + \sqrt{x^2 - a^2}\right|$

由(i)(ii)得: $\boxed{\int \frac{dx}{\sqrt{x^2 - a^2}} = \ln\left|x + \sqrt{x^2 - a^2}\right| + C}$

被积函数　　　　三角代换

含 $\sqrt{a^2-x^2}$ (a^2-x^2) 　　$x=a\sin t$

$\sqrt{x^2+a^2}$ (x^2+a^2) 　　$x=a\tan t$

$\sqrt{x^2-a^2}$ (x^2-a^2) 　　$x=a\sec t$

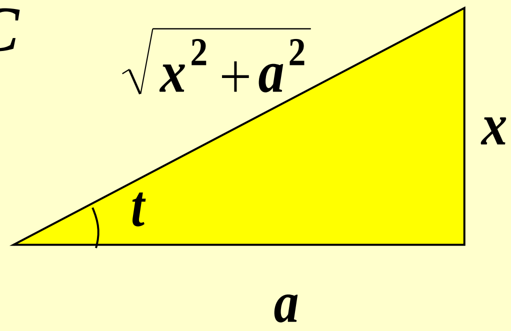
如求 $\int\frac{dx}{(x^2+a^2)^2}(a>0)$

解 令 $x=a\tan t$, 则 $dx=a\sec^2 t dt$

$$\int\frac{dx}{(x^2+a^2)^2}=\int\frac{1}{(a^2\tan^2 t+a^2)^2}\cdot a\sec^2 t dt$$

$$=\int\frac{a\sec^2 t}{a^4\sec^4 t}dt=\frac{1}{a^3}\int\cos^2 t dt$$

$$= \frac{1}{2a^3}\int(1+\cos 2t)dt \quad = \frac{1}{2a^3}\left(t+\frac{1}{2}\sin 2t\right)+C$$



$$= \frac{1}{2a^3}(t+\sin t\cdot\cos t)+C$$

$$= \frac{1}{2a^3}\left(\arcsin\frac{x}{a}+\frac{ax}{x^2+a^2}\right)+C$$

$$\tan t = \frac{x}{a} \qquad \sin t = \frac{x}{\sqrt{x^2+a^2}}$$

$$t = \arctan\frac{x}{a} \qquad \cos t = \frac{a}{\sqrt{x^2+a^2}}$$

4. $\boxed{\int x\sqrt{1-x}dx = ?}$

解　令 $\sqrt{1-x} = t$,　得 $x = 1-t^2$,　$dx = -2tdt$,

$$\int x\sqrt{1-x}dx = \int (1-t^2)t(-2t)dt = 2\int (t^4 - t^2)dt$$

$$= \frac{2}{5}t^5 - \frac{2}{3}t^3 + C \quad = \frac{2}{5}(1-x)^{\frac{5}{2}} + \frac{2}{3}(1-x)^{\frac{3}{2}} + C$$

5. 求 $\int \frac{1}{\sqrt{x}+\sqrt[3]{x}}dx$

解 令 $\sqrt[6]{x}=t$, 则 $x=t^6$, $dx=6t^5dt$.

$$\int \frac{1}{\sqrt{x}+\sqrt[3]{x}}dx = \int \frac{1}{t^3+t^2}\cdot 6t^5dt = 6\int \frac{t^3}{1+t}dt$$

$$= 6\int \frac{t^3+1-1}{1+t}dt = 6\int \left(t^2-t+1-\frac{1}{1+t}\right)dt$$

$$= 2t^3-3t^2+6t-6\ln(1+t)+C$$

$$= 2\sqrt{x}-3\sqrt[3]{x}+6\sqrt[6]{x}-6\ln(1+\sqrt[6]{x})+C$$

倒代换——消去分母中的变量因子 x

6. 求$\int \frac{\sqrt{a^2-x^2}}{x^4}dx$

解 令$x=\frac{1}{t}$， 则$dx=-\frac{1}{t^2}dt$

$$\int \frac{\sqrt{a^2-x^2}}{x^4}dx = \int \frac{\sqrt{a^2-\frac{1}{t^2}}}{\frac{1}{t^4}}\left(-\frac{1}{t^2}\right)dt = -\int \left(a^2t^2-1\right)^{\frac{1}{2}}|t|dt$$

$x>0$时，$t>0$

$$\int \frac{\sqrt{a^2-x^2}}{x^4}dx = -\int \left(a^2t^2-1\right)^{\frac{1}{2}}tdt = -\frac{1}{2a^2}\int \left(a^2t^2-1\right)^{\frac{1}{2}}d\left(a^2t^2-1\right)$$

$$= -\frac{\left(a^2t^2-1\right)^{\frac{3}{2}}}{3a^2}+C = -\frac{\left(\frac{a^2}{x^2}-1\right)^{\frac{3}{2}}}{3a^2}+C = -\frac{\left(a^2-x^2\right)^{\frac{3}{2}}}{3a^2x^3}+C$$

$x<0$时，$t<0$

$$\int\frac{\sqrt{a^2-x^2}}{x^4}dx=\int\left(a^2t^2-1\right)^{\frac{1}{2}}tdt=\frac{1}{2a^2}\int\left(a^2t^2-1\right)^{\frac{1}{2}}d\left(a^2t^2-1\right)$$

$$=\frac{\left(a^2t^2-1\right)^{\frac{3}{2}}}{3a^2}+C=\frac{\left(\dfrac{a^2}{x^2}-1\right)^{\frac{3}{2}}}{3a^2}+C=-\frac{\left(a^2-x^2\right)^{\frac{3}{2}}}{3a^2x^3}+C$$

$$\therefore\int\frac{\sqrt{a^2-x^2}}{x^4}dx=-\frac{\left(a^2-x^2\right)^{\frac{3}{2}}}{3a^2x^3}+C$$

第二类换元积分法常用代换：

去根号

1. 三角代换

含$\sqrt{a^2-x^2}$或 $\left(a^2-x^2\right)$ 令$x=a\sin t$

含$\sqrt{x^2+a^2}$或 $\left(x^2+a^2\right)$ 令$x=a\tan t$

含$\sqrt{x^2-a^2}$或 $\left(x^2-a^2\right)$ 令$x=\pm a\sec t$

2. 含$\sqrt[n]{ax+b}$ 令 $\sqrt[n]{ax+b}=t$

3、含$\sqrt[n]{\dfrac{cx+d}{ax+b}}$， 令$\sqrt[n]{\dfrac{cx+d}{ax+b}}=t$

4、含$\sqrt{x},\sqrt[3]{x},\cdots\sqrt[m]{x}$ 令$t=\sqrt[k]{x}$

（$k=2,3,\cdots m$的最小公倍数）

5. 倒代换 $x=\dfrac{1}{t}$消去分母的变量因子$\dfrac{1}{x^n}$

$$(16)\int \tan x dx = -\ln|\cos x| + C$$

$$(17)\int \cot x dx = \ln|\sin x| + C$$

$$(18)\int \sec x dx = \ln|\sec x + \tan x| + C$$

$$(19)\int \csc x dx = \ln|\csc x - \cot x| + C$$

$$(20)\int \frac{dx}{a^2 + x^2} = \frac{1}{a}\arctan\frac{x}{a} + C$$

$$(21)\int \frac{dx}{x^2 - a^2} = \frac{1}{2a}\ln\left|\frac{x-a}{x+a}\right| + C$$

$$(22)\int \frac{dx}{\sqrt{a^2 - x^2}} = \arcsin\frac{x}{a} + C$$

$$(23)\int \frac{dx}{\sqrt{x^2 + a^2}} = \ln(x + \sqrt{x^2 + a^2}) + C$$

$$(24)\int \frac{dx}{\sqrt{x^2 - a^2}} = \ln\left|x + \sqrt{x^2 - a^2}\right| + C$$

使用上述公式计算的例题

7. 求$\int \frac{1}{x^2+2x+3}dx$

解 $$\int \frac{1}{x^2+2x+3}dx = \int \frac{1}{(x+1)^2+(\sqrt{2})^2}d(x+1) = \frac{1}{\sqrt{2}}\arctan\frac{x+1}{\sqrt{2}}+C$$

8. 求$\int \frac{dx}{\sqrt{4x^2+9}}$

解 $$\int \frac{dx}{\sqrt{4x^2+9}} = \int \frac{dx}{\sqrt{(2x)^2+3^2}} = \frac{1}{2}\int \frac{d(2x)}{\sqrt{(2x)^2+3^2}}$$

$$= \frac{1}{2}\ln\left(2x+\sqrt{4x^2+9}\right)+C$$

9. 求$\int \frac{dx}{\sqrt{1+x-x^2}}$

解 $$\int \frac{dx}{\sqrt{1+x-x^2}} = \int \frac{d\left(x-\frac{1}{2}\right)}{\sqrt{\left(\frac{\sqrt{5}}{2}\right)^2-\left(x-\frac{1}{2}\right)^2}} = \arcsin\frac{2x-1}{\sqrt{5}}+C$$

10. 求$\int \frac{dx}{x(x+2)}$.

解一：$\int \frac{dx}{x(x+2)} = \int \frac{d(x+1)}{(x+1)^2-1} = \frac{1}{2}\ln\left|\frac{(x+1)-1}{(x+1)+1}\right| + C = \frac{1}{2}\ln\left|\frac{x}{x+2}\right| + C$

公式$\int \frac{dx}{x^2-1} = \frac{1}{2}\ln\left|\frac{x-1}{x+1}\right| + C$

解二：利用有理分式函数的积分法

$$\because \frac{1}{x(x+2)} = \frac{1}{2}\left(\frac{1}{x} - \frac{1}{x+2}\right)$$

$$\therefore \int \frac{dx}{x(x+2)} = \frac{1}{2}\int\left(\frac{1}{x} - \frac{1}{x+2}\right)dx = \frac{1}{2}\left(\ln|x| - \ln|x+2|\right) + C = \frac{1}{2}\ln\left|\frac{x}{x+2}\right| + C$$