习题课1(不定积分)21题

1. 求 $\int \frac{2^{x+1}-5^{x-1}}{10^x}dx$

2. 求 $\int \frac{\sqrt{1+\cos x}}{\sin x}dx$

3. $\int \frac{\tan x}{\sqrt{\cos x}}dx$

4. $\int \frac{\sin x}{1+\sin x}dx$

5. $\int \frac{\arctan\sqrt{x}}{\sqrt{x}(1+x)}\mathrm{d}x$

6. $\int \frac{\arctan\frac{1}{x}}{1+x^2}\mathrm{d}x$

7. $\int \frac{x^3}{\sqrt{1+x^2}}dx$

8. $\int \frac{\ln\left(1+\frac{1}{x}\right)}{x(x+1)}dx$

$$9. \int \frac{\sin x \cos x}{\sin^4 x + \cos^4 x} dx \qquad 10. \int \frac{1 + \cos x}{1 + \sin^2 x} dx$$

$$11. \int \frac{x^3}{\left(1 + x^8\right)^2} dx \qquad 12. \int \frac{1}{x^2 \sqrt{x^2 - 1}} dx$$

$$13. \int \frac{x + \sin x}{1 + \cos x} dx \qquad 14. \int \left(\tan x + \sec^2 x\right) e^x dx$$

$$15. \int \frac{xe^x}{\left(e^x + 1\right)^2} dx \qquad 16. \int \sqrt{1 - x^2} \arcsin x dx$$

17. 设 $\int xf(x)dx = \arcsin x + C$， 求$\int \frac{1}{f(x)}dx$.

18. 已知 $f'(\sin^2 x) = \cos 2x + \tan^2 x$， 当 $0 < x < 1$ 时，求 $f(x)$.

19.设 $f(x)$的一个原函数为 $\frac{\sin x}{x}$， 求 $\int xf'(2x)dx$.

20. 设 $f(2+x^4) = \ln \frac{5+2x^4}{x^4-1}$，且 $f[\varphi(x)] = \ln(x+1)$，求$\int \varphi(x)dx$.

21. 求 $\int \frac{1}{x^4+1}dx$

答案

$$1.\int \frac{2^{x+1}-5^{x-1}}{10^x}dx = \int 2\cdot\left(\frac{1}{5}\right)^x dx - \int \frac{1}{5}\left(\frac{1}{2}\right)^x dx$$

$$= \frac{1}{5\ln 2}\left(\frac{1}{2}\right)^x - \frac{2}{\ln 5}\left(\frac{1}{5}\right)^x + C$$

$$2.\int \frac{\sqrt{1+\cos x}}{\sin x}dx = \int \frac{\sqrt{2\cos^2\frac{x}{2}}}{2\sin\frac{x}{2}\cos\frac{x}{2}}dx = \frac{\sqrt{2}}{2}\int \frac{\cos\frac{x}{2}}{\sin\frac{x}{2}\cos\frac{x}{2}}dx$$

$$= \frac{\sqrt{2}}{2}\int \frac{1}{\sin\frac{x}{2}}dx = \sqrt{2}\ln\left|\csc\frac{x}{2} - \cot\frac{x}{2}\right| + C$$

$$3.\int \frac{\tan x}{\sqrt{\cos x}}dx = \int \frac{\sin x}{\cos^{\frac{3}{2}} x}dx = -\int \frac{1}{\cos^{\frac{3}{2}} x}d\cos x = \frac{2}{\sqrt{\cos x}} + C$$

$$4.\int \frac{\sin x}{1+\sin x}dx = \int \frac{\sin x\left(1-\sin x\right)}{1-\sin^2 x}dx$$

$$= \int \left(\tan x \sec x - \tan^2 x\right)dx = \int \left(\tan x \sec x - \sec^2 x + 1\right)dx$$

$$= \sec x - \tan x + x + C$$

$$5.\int \frac{\arctan\sqrt{x}}{\sqrt{x}(1+x)}\mathrm{d}x = 2\int \frac{\arctan\sqrt{x}}{1+(\sqrt{x})^2}\mathrm{d}\sqrt{x}$$

$$= 2\int \arctan\sqrt{x}\,\mathrm{d}\arctan\sqrt{x} = (\arctan\sqrt{x})^2 + \mathrm{C}$$

$$6.\int \frac{\arctan\frac{1}{x}}{1+x^2}\mathrm{d}x = -\int \frac{\arg\tan\frac{1}{x}}{1+(\frac{1}{x})^2}\mathrm{d}\frac{1}{x} = -\frac{1}{2}[\arctan(\frac{1}{x})]^2 + \mathrm{C}$$

$$7.\int \frac{x^3}{\sqrt{1+x^2}}dx = \frac{1}{2}\int \frac{x^2}{\sqrt{1+x^2}}dx^2 = \frac{1}{2}\int \frac{x^2+1-1}{\sqrt{1+x^2}}dx^2$$

$$= \frac{1}{2}\int \left(\sqrt{1+x^2} - \frac{1}{\sqrt{1+x^2}}\right)d\left(1+x^2\right) = \frac{1}{3}\left(1+x^2\right)^{\frac{3}{2}} - \sqrt{1+x^2} + C$$

$$8.\int \frac{\ln\left(1+\frac{1}{x}\right)}{x(x+1)}dx = \int \frac{\ln\left(1+\frac{1}{x}\right)}{\left(1+\frac{1}{x}\right)}\cdot\frac{1}{x^2}dx = -\int \frac{\ln\left(1+\frac{1}{x}\right)}{\left(1+\frac{1}{x}\right)}d\left(1+\frac{1}{x}\right)$$

$$= -\int \ln\left(1+\frac{1}{x}\right)d\ln\left(1+\frac{1}{x}\right) = -\frac{1}{2}\ln^2\left(1+\frac{1}{x}\right)+C$$

$$9.\int \frac{\sin x\cos x}{\sin^4 x+\cos^4 x}dx$$

$$= \int \frac{\sin 2x}{2\left(\sin^2 x+\cos^2 x\right)^2 - 4\sin^2 x\cos^2 x}dx$$

$$= \int \frac{\sin 2x}{2-\sin^2 2x}dx = -\frac{1}{2}\int \frac{1}{1+\cos^2 2x}d\cos 2x$$

$$= -\frac{1}{2}\arctan\ \cos 2x + C$$

$$10.\int \frac{1+\cos x}{1+\sin^2 x}dx = \int \frac{1}{1+\sin^2 x}dx + \int \frac{\cos x}{1+\sin^2 x}dx$$

$$= \int \frac{1}{\cos^2 x + 2\sin^2 x}dx + \int \frac{1}{1+\sin^2 x}d\sin x$$

$$= \int \frac{\sec^2 x}{1+2\tan^2 x}dx + \arctan \sin x$$

$$= \frac{1}{\sqrt{2}}\int \frac{1}{1+2\tan^2 x}d\left(\sqrt{2}\tan x\right) + \arctan \sin x$$

$$= \frac{1}{\sqrt{2}}\arctan\left(\sqrt{2}\tan x\right) + \arctan \sin x + C$$

$$11.\int \frac{x^3}{(1+x^8)^2}dx = \frac{1}{4}\int \frac{1}{\left[1+(x^4)^2\right]^2}dx^4$$

令$x^4 = \tan u \quad \left(0 < u < \frac{\pi}{2}\right)$，则$dx^4 = \sec^2 u du$，

$$I = \frac{1}{4}\int \frac{1}{(1+x^8)^2}dx^4 = \frac{1}{4}\int \frac{\sec^2 u}{(1+\tan^2 u)^2}du$$

$$= \frac{1}{4}\int \cos^2 u du = \frac{1}{8}\left(u + \frac{1}{2}\sin 2u\right) + C$$

$$= \frac{1}{8}(u + \sin u \cos u) + C = \frac{1}{8}\left(\arctan x^4 + \frac{x^4}{x^8+1}\right) + C$$

$$u = \arctan x^4$$

$$\sin u = \frac{x^4}{\sqrt{x^8+1}}$$

$$\cos u = \frac{1}{\sqrt{x^8+1}}$$

12.令 $x=\frac{1}{t}$，则 $dx=-\frac{1}{t^2}dt$，

$$\int\frac{1}{x^2\sqrt{x^2-1}}dx=\int\frac{-\frac{1}{t^2}dt}{\frac{1}{t^2}\sqrt{\frac{1}{t^2}-1}}=-\int\frac{|t|}{\sqrt{1-t^2}}dt$$

当 $t>0$ 时，

$$原式=-\int\frac{t}{\sqrt{1-t^2}}dt=\frac{1}{2}\int\frac{1}{\sqrt{1-t^2}}d\left(1-t^2\right)=\sqrt{1-t^2}+C=\frac{\sqrt{x^2-1}}{x}+C$$

当 $t<0$ 时，

$$原式=\int\frac{t}{\sqrt{1-t^2}}dt=-\frac{1}{2}\int\frac{1}{\sqrt{1-t^2}}d\left(1-t^2\right)=-\sqrt{1-t^2}+C$$

$$=-\sqrt{1-\frac{1}{x^2}}+C=\frac{\sqrt{x^2-1}}{x}+C$$

所以 $\int\frac{1}{x^2\sqrt{x^2-1}}dx=\frac{\sqrt{x^2-1}}{x}+C$

$$13.\int \frac{x+\sin x}{1+\cos x}dx=\frac{1}{2}\int (x+\sin x)\sec^2\frac{x}{2}dx$$

$$=\int (x+\sin x)\sec^2\frac{x}{2}d\left(\frac{x}{2}\right)=\int (x+\sin x)d\left(\tan\frac{x}{2}\right)$$

$$=(x+\sin x)\tan\frac{x}{2}-\int \tan\frac{x}{2}(1+\cos x)dx$$

$$=(x+\sin x)\tan\frac{x}{2}-\int \tan\frac{x}{2}\cdot 2\cos^2\frac{x}{2}dx$$

$$=(x+\sin x)\tan\frac{x}{2}-\int \sin x dx$$

$$=(x+\sin x)\cdot\frac{1-\cos x}{\sin x}+\cos x+C_1$$

$$=x(\csc x-\cot x)+C$$

$$14.\int\left(\tan x+\sec^2 x\right)e^x dx=\int \tan xe^x dx+\int \sec^2 xe^x dx$$

$$=\int \tan xe^x dx+\int e^x d\tan x$$

$$=\int \tan xe^x dx+e^x\tan x-\int \tan xe^x dx=e^x\tan x+C$$

$$15.\int\frac{xe^x}{\left(e^x+1\right)^2}dx=\int\frac{x}{\left(e^x+1\right)^2}d\left(e^x+1\right)=-\int xd\frac{1}{e^x+1}$$

$$=-\frac{x}{e^x+1}+\int\frac{1}{e^x+1}dx=-\frac{x}{e^x+1}+\int\frac{1+e^x-e^x}{e^x+1}dx$$

$$=-\frac{x}{e^x+1}+x-\int\frac{1}{e^x+1}d\left(e^x+1\right)=\frac{xe^x}{e^x+1}-\ln\left(e^x+1\right)+C$$

16. $x=\sin t,\left(-\frac{\pi}{2}\le t\le\frac{\pi}{2}\right),\ \arcsin x=t, dx=\cos dt,$

$$\int\sqrt{1-x^2}\arcsin x\,dx=\int\cos t\cdot t\cdot\cos t\,dt=\int t\cos^2 t\,dt$$

$$=\frac{1}{2}\int t(1+\cos 2t)dt=\frac{1}{4}t^2+\frac{1}{4}\int t\cos 2t\,d(2t)$$

$$=\frac{1}{4}t^2+\frac{1}{4}\int t\,d(\sin 2t)=\frac{1}{4}t^2+\frac{1}{4}\left(t\sin 2t-\int\sin 2t\,dt\right)$$

$$=\frac{1}{4}t^2+\frac{1}{4}t\sin 2t+\frac{1}{8}\cos 2t+C_1$$

$$=\frac{1}{4}t^2+\frac{1}{2}t\sin t\cos t+\frac{1}{8}\left(1-2\sin^2 t\right)+C_1$$

$$=\frac{1}{4}(\arcsin x)^2+\frac{1}{2}x\sqrt{1-x^2}\cdot\arcsin x-\frac{1}{4}x^2+C.$$

17.解 $xf(x)=(\arcsin x+C)'=\dfrac{1}{\sqrt{1-x^2}}$ $\quad\therefore\ f(x)=\dfrac{1}{x\sqrt{1-x^2}}$

$$\int\frac{1}{f(x)}dx=\int x\sqrt{1-x^2}dx=-\frac{1}{2}\int\sqrt{1-x^2}d\left(1-x^2\right)$$

$$=-\frac{1}{3}\left(1-x^2\right)^{\frac{3}{2}}+C$$

18. $f'\left(\sin^2 x\right)=\cos 2x+\tan^2 x=1-2\sin^2 x+\dfrac{\sin^2 x}{1-\sin^2 x}$

$$\therefore\ f'(x)=1-2x+\frac{x}{1-x}=\frac{1}{1-x}-2x$$

$$\therefore\ f(x)=\int\left(\frac{1}{1-x}-2x\right)dx=-\ln(1-x)-x^2+C$$

19. 解 $\int xf'(2x)\mathrm{d}x \xlongequal{t=2x} \frac{1}{4}\int tf'(t)\mathrm{d}t = \frac{1}{4}\int t\mathrm{d}f(t)$

$= \frac{1}{4}tf(t) - \frac{1}{4}\int f(t)\mathrm{d}t = \frac{1}{4}tf(t) - \frac{1}{4}\frac{\sin t}{t} + \mathrm{C} = \frac{2x}{4}f(2x) - \frac{1}{4}\cdot\frac{\sin 2x}{2x} + \mathrm{C}$

由于 $f(x) = \left(\frac{\sin x}{x}\right)' = \frac{x\cos x - \sin x}{x^2}$ 故 原式 $= \frac{\cos 2x}{4} - \frac{\sin 2x}{4x} + \mathrm{C}$

20. 令 $2 + x^4 = u$, 则 $x^4 = u - 2$.

$$\therefore\ f(u) = \ln\frac{5 + 2(u-2)}{u - 2 - 1} = \ln\frac{2u+1}{u-3}$$

$$\therefore\ f[\varphi(x)] = \ln\left(\frac{2\varphi(x)+1}{\varphi(x)-3}\right) = \ln(x+1)$$

即 $\frac{2\varphi(x)+1}{\varphi(x)-3} = x + 1$, 解得: $\varphi(x) = \frac{3x+4}{x-1}$.

$$\int \varphi(x)dx = \int\frac{3x+4}{x-1}dx = \int\left(3 + \frac{7}{x-1}\right)dx = 3x + 7\ln|x-1| + C$$

21. 求 $\int \frac{1}{x^4+1}dx$

解 $\frac{1}{x^4+1}=\frac{1}{(x^2+1)^2-2x^2}=\frac{1}{(x^2+\sqrt{2}x+1)(x^2-\sqrt{2}x+1)}$

令 $\frac{1}{(x^2+\sqrt{2}x+1)(x^2-\sqrt{2}x+1)}=\frac{Ax+B}{x^2+\sqrt{2}x+1}+\frac{Cx+D}{x^2-\sqrt{2}x+1}$

得 $1=(Ax+B)(x^2-\sqrt{2}x+1)+(Cx+D)(x^2+\sqrt{2}x+1)$

比较系数得 $\begin{cases} A+C=0 \\ B-\sqrt{2}A+D+\sqrt{2}C=0 \\ A-\sqrt{2}B+C+\sqrt{2}D=0 \\ B+D=1 \end{cases}$ 解得 $A=\frac{\sqrt{2}}{4}, B=\frac{1}{2},$ $C=-\frac{\sqrt{2}}{4}, D=\frac{1}{2}$

所以 $\frac{1}{x^4+1}=\frac{\frac{\sqrt{2}}{4}x+\frac{1}{2}}{x^2+\sqrt{2}x+1}+\frac{-\frac{\sqrt{2}}{4}x+\frac{1}{2}}{x^2-\sqrt{2}x+1}$

由于 $\left(x^{2}+\sqrt{2}x+1\right)'=2x+\sqrt{2}$; $\left(x^{2}-\sqrt{2}x+1\right)'=2x-\sqrt{2}$

且 $\frac{\sqrt{2}}{4}x+\frac{1}{2}=\frac{\sqrt{2}}{8}\left(2x+\sqrt{2}\right)+\frac{1}{4}$; $-\frac{\sqrt{2}}{4}x+\frac{1}{2}=-\frac{\sqrt{2}}{8}\left(2x-\sqrt{2}\right)+\frac{1}{4}$,

则 $$\int\frac{1}{x^{4}+1}dx=\int\left(\frac{\frac{\sqrt{2}}{8}\left(2x+\sqrt{2}\right)+\frac{1}{4}}{x^{2}+\sqrt{2}x+1}+\frac{-\frac{\sqrt{2}}{8}\left(2x+\sqrt{2}\right)+\frac{1}{4}}{x^{2}-\sqrt{2}x+1}\right)dx$$

$$=\frac{\sqrt{2}}{8}\int\frac{d\left(x^{2}+\sqrt{2}x+1\right)}{x^{2}+\sqrt{2}x+1}+\frac{1}{4}\int\frac{d\left(x+\frac{\sqrt{2}}{2}\right)}{\left(x+\frac{\sqrt{2}}{2}\right)^{2}+\frac{1}{2}}$$

$$-\frac{\sqrt{2}}{8}\int\frac{d\left(x^{2}-\sqrt{2}x+1\right)}{x^{2}-\sqrt{2}x+1}+\frac{1}{4}\int\frac{d\left(x-\frac{\sqrt{2}}{2}\right)}{\left(x-\frac{\sqrt{2}}{2}\right)^{2}+\frac{1}{2}}$$