In the previous lecture, you learned the concept of **Maximum Likelihood Estimation (MLE)** and showed that the **sample proportion** is the maximum likelihood estimator of a **population proportion**.

In this lecture, you will learn how to use the **likelihood function** to **test a claim about a population proportion**.

Contents

Introduction to Likelihood Ratio and its Properties

Example:
A random sample of 100 students is selected from a college, and 60 of them use an iPhone. We want to use the sample data to determine whether the proportion of all students who use an iPhone is different from 50%.

The null and alternative hypotheses are:

$$H_0: p = 0.5$$

$$H_a: p \neq 0.5$$

To perform the test, we calculate how "likely" your observed data is under both hypotheses. Then compare these two likelihoods using a ratio (R):

$$R = \frac{Likelihood\ under\ Ho}{Maximum\ Likelihood\ under\ Ha}$$

The **Likelihood Ratio Test** asks the following question:

*Does the null hypothesis make your observed data nearly as likely to occur as the **maximum likelihood** under the alternative hypothesis?*

Properties of Likelihood Ratio

1. The likelihood ratio $\boldsymbol{R}$ is always **less than or equal to 1**, because the **maximum likelihood** under the alternative hypothesis is always **larger than or equal to** the likelihood under the null hypothesis.

2. If the null hypothesis makes the observed data **nearly as likely** as the **maximum likelihood** under the alternative hypothesis, then $\boldsymbol{R}$ **is close to 1**.

   This indicates that the observed data are **consistent with** H_0, and we **do not reject** H_0.

3. f the null hypothesis **does not** make the observed data nearly as likely as the **maximum likelihood** under the alternative hypothesis, then $\boldsymbol{R}$ **is close to 0**.

   This indicates that the observed data are **not consistent with** H_0, and we **reject** H_0.

4. As the likelihood ratio $\boldsymbol{R}$ gets **closer to 0**, the evidence against the null hypothesis becomes **stronger**.

Performing the Likelihood Ratio Test (LRT)

From the last lecture, we learned the likelihood function of the unknown population proportion is given by

$$L(p) = \ constant * \ p^x \ (1-p)^{n-x}$$

where

- p represents the unknown value of the population proportion
  (e.g. the proportion of all students who use an iPhone),

- x represents the number of individuals who fall into the category of interest
  (e.g. the number of students who use an iPhone),

- n represents the sample size.

Note: The $constant$ makes **no contribution** to the Likelihood Ratio Test because it **cancels out** when forming the likelihood ratio. Therefore, it can be safely omitted from the calculations.

In this example, we sampled **100 students**, and **60 of them use an iPhone**.
Then, $n = 100$ and $x = 60$. The likelihood function is:

$$L(p) =$$

Under the null hypothesis, $p = 0.5$. We substitute 0.5 into the likelihood function.

The likelihood under Ho is

$$L(0.5) =$$

Under the alternative hypothesis, $p \neq 0.5$. So, there are many possible choices of p.

We will choose the value of p that **maximizes** the likelihood function.

The value of p that **maximizes** the likelihood function is the **sample proportion, $\bar{\boldsymbol{p}}$.**

The sample proportion is: $\bar{p} = \frac{x}{n} = \frac{60}{100} = 0.6$

Under the alternative hypothesis, $p = 0.6$ We substitute 0.6 into the likelihood function.

The likelihood under Ha is

$$L(0.6) =$$

Therefore, the likelihood ratio is by:

$$R = \frac{L(0.5)}{L(0.6)} =$$

$$= \frac{0.5^{100}}{0.6^{60} * 40^{40}}$$

$$\approx 0.1335137$$

As we can see, the likelihood ratio is **very close to zero**. This indicates that the observed data are not consistent with H_0 and provide **strong evidence against the null hypothesis**.

To determine **how strong this evidence is**, we calculate the **p-value**.

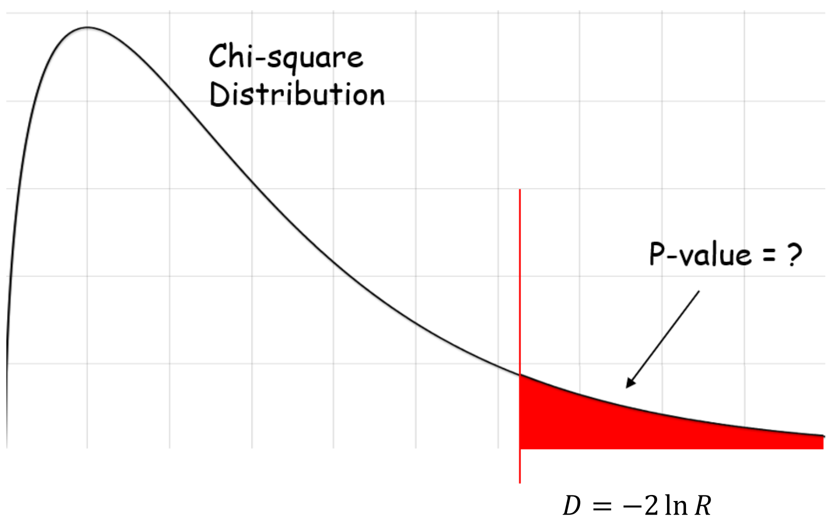
Calculate the P-value of the Likelihood Ratio Test (LRT)

To do this, we first convert the likelihood ratio Rinto a **Chi-Square test statistic** (also called the **deviance**):

$$D = -2\ln R \quad or \ -2\ln\left(\frac{Likelihood\ under\ Ho}{Maximum\ Likelihood\ under\ Ha}\right)$$

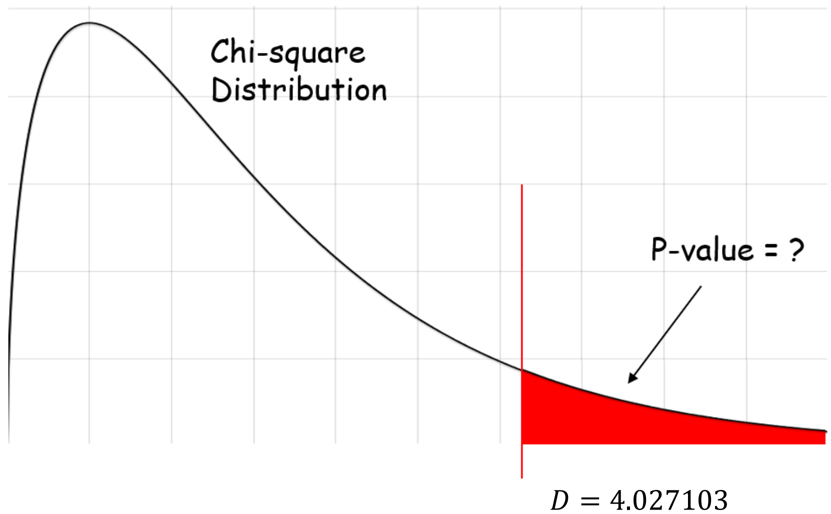
For a **large sample**, the statistic D follows a **Chi-Square distribution with 1 degree of freedom**.

The **p-value** is then calculated as the area on the right under the Chi-Square distribution (df = 1) **bounded by** D.



Let's get the p-value in our example.

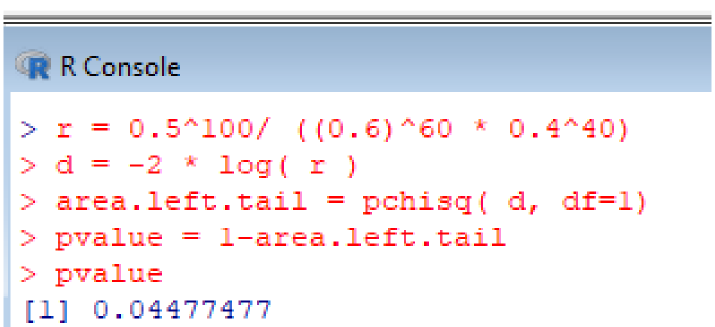
$$\begin{aligned} R &= \frac{0.5^{100}}{0.6^{60}\ 0.4^{40}} \\ \rightarrow\ D &= -2\ln\left(\frac{0.5^{100}}{0.6^{60}\ 0.4^{40}}\right) \\ &\approx 4.027103 \end{aligned}$$



The p-value of $D$is calculated from the **Chi-Square distribution with 1 degree of freedom**.
In R, you can obtain it using the following command:

```
r = 0.5^100/ ((0.6)^60 * 0.4^40)
d = -2 * log( r )

#Calculate the area on the left under the Chi-Square curve
#using pchisq(…) function.
#The pchisq function takes two inputs:
#   (1) the chi-square value and (2) the degree of freedom(df)
area.left.tail = pchisq( d, df=1)
pvalue = 1-area.left.tail
```



```
R Console
> r = 0.5^100/ ((0.6)^60 * 0.4^40)
> d = -2 * log( r )
> area.left.tail = pchisq( d, df=1)
> pvalue = 1-area.left.tail
> pvalue
[1] 0.04477477
```

Since the p-value (0.04477) is less than the significance level (0.05), we reject Ho.

At the 5% significance level, the sample data provide sufficient evidence to conclude that the proportion of all students who use an iPhone is different from 50%.

Conditions for a Valid Likelihood Ratio Test of a Population Proportion

As mentioned earlier, the sample size must be sufficiently large. But how large is enough?

For the likelihood ratio test of a population proportion, it is recommended that:

- The expected number of individuals who fall into the category of interest and
- The expected number of individuals who **do not** fall into the category of interest

are both **at least 5** in the sample. The expected numbers are calculated based on the null hypothesis.

In our example, under Ho, we expect 50% of 100 sampled students use an iPhone,

So, the expected numbers are:

	Expected Number
Using an iPhone	50% of 100 = 50 $\geq \mathbf{5}$
Not using an iPhone	$100 - 50$ = 50 $\geq \mathbf{5}$

Since both numbers are at least 5, the sample is sufficiently large.

The conclusion drawn from the test should be valid.

Likelihood Ratio Test vs One-Sample Z Test

We can also perform the Z-test to determine whether the proportion of all students who use an iPhone is different from 50%. The null and alternative hypotheses are shown:

$$H_0: p = 0.5$$

$$H_a: p \neq 0.5$$

The z-statistic and p-value are shown below.

p_0 is the population proportion of iPhone users under Ho $\rightarrow 0.5$

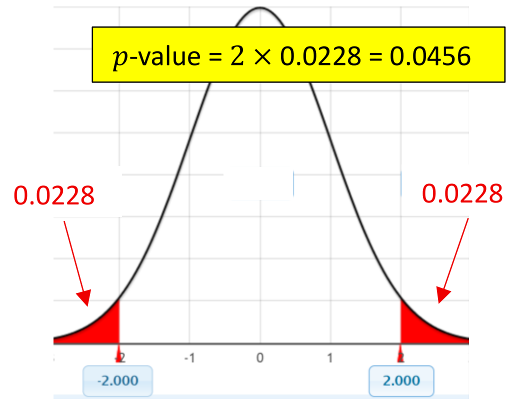
$\bar{p}$ is the sample proportion of iPhone users $\rightarrow 0.6$

n is the sample size $\rightarrow 100$

$$z = \frac{\bar{p} - p_0}{\sqrt{\frac{p_0(1-p_0)}{n}}}$$

$$= \frac{0.6 - 0.5}{\sqrt{\frac{0.5(1-0.5)}{100}}}$$

$$= 2$$



The results of the Likelihood Ratio Test and the Z-Test are summarized below:

	Likelihood Ratio Test	Z-Test
p-value	0.0448	0.0456

The p-values obtained from the two tests are nearly identical. This indicates a strong agreement between the Likelihood Ratio Test and the Z-Test.

In fact, the chi-square statistic (often referred to as the deviance) used in the Likelihood Ratio Test can be approximated by the following expression.

Likelihood Ratio Test

$$-2\ln\left(\frac{Likelihood\ under\ Ho}{Maximum\ Likelihood\ under\ Ha}\right) \quad \approx \quad \left(\frac{\bar{p}-p_0}{\sqrt{\frac{\bar{p}(1-\bar{p}\,)}{n}}}\right)^2$$

The quantity inside the square, $\dfrac{\bar{p}-p_0}{\sqrt{\frac{\bar{p}(1-\bar{p}\,)}{n}}}$,

is similar the z-statistic under the Z-test $\dfrac{\bar{p}-p_0}{\sqrt{\frac{p_0(1-p_0\,)}{n}}}$.

Mathematically, both two quantities follow a standard normal distribution (with mean of 0 and SD of 1).

As a result, the Likelihood Ratio Test and the Z-test produce very similar p-values.