Stratified Approximations for the Pricing of Options on Average

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1 Introduction

This thesis is the addition to the new version of Privault and Yu (2016), which aims to compare the conditional lognormal and conditional gamma method to other arithmetic Asian option pricing strategies. The formulas in this thesis, except some part of the SLN Quad method and PDE, are all derived independent by myself, since the original authors only provided their ideas but not the implementation in details. The stock price in this thesis follows a geometric brownian motion $dS_t = rS_t dt + \sigma S_t dB_t$, where r is the risk-free rate and B_t is a standard brownian motion. T represents the maturity of the option and K is the fixed strike.

2 Preliminary results

2.1 Conditional expectation

Suppose X and Z are jointly normal distributed. If I condition the process X on some zero-mean Gaussian variable Z, it remains a Gaussian process. Here are some results about conditional expectation.

$$E(B_t|Z) = \frac{E(B_t Z)}{E(Z^2)} Z$$
$$\operatorname{Cov}(B_s, B_t|Z) = \operatorname{Cov}(B_s, B_t) - \frac{E(B_s Z)E(B_t Z)}{E(Z^2)}$$

Consider,

$$E\left(\left(B_t, \int_0^T B_s ds\right)' \left(B_t, \int_0^T B_s ds\right)\right) = \begin{bmatrix} t & t(T-\frac{1}{2}t) \\ t(T-\frac{1}{2}t) & \frac{1}{3}T^3 \end{bmatrix}$$

Thus,

$$E\left(B_t| \int_0^T B_s ds = Z\right) = \frac{3t(T-\frac{1}{2}t)}{T^3} Z$$
$$\operatorname{Var}\left(B_t| \int_0^T B_s ds = Z\right) = t - \frac{3t^2(T-\frac{1}{2}t)^2}{T^3}$$
$$E\left(\int_0^T B_s ds | B_t = Z\right) = (T - \frac{1}{2}t)Z$$
$$\operatorname{Var}\left(\int_0^T B_s ds | B_t = Z\right) = \frac{1}{3}T^3 - t(T-\frac{1}{2}t)^2$$

2.2 N-th moment of arithmetic stock price

Dufresne (1989) used Ito's lemma, time reversal and a recurrence argument to get the high-order moment of the arthimetic mean of the stock price. Geman and Yor (1993) used the Laplace transform and got the same result. I propose to use the mathematical induction to derive this formula.

Lemma 1 *Suppose $f(x)$ is a bounded function on interval $[a,b]$, then $\int_a^b \int_x^b f(x)f(y)dydx = \int_a^b \int_a^y f(x)f(y)dxdy$*

Proof. Consider

$$\Omega_1 = \{(x,y)|a \le x \le b, x \le y \le b\}$$
$$\Omega_2 = \{(x,y)|a \le y \le b, a \le x \le y\}$$

It is easy to prove that for any $(x,y) \in \Omega_1$, $(x,y) \in \Omega_2$.Thus, $\Omega_1 \subseteq \Omega_2$. Vice versa, $\Omega_2 \subseteq \Omega_1$. Thus $\Omega_1 = \Omega_2$. So Lemma 1 is proved.

Lemma 2 *Suppose $f(x)$ is a bounded function on interval $[a,b]$, then $\int_a^b \cdots \int_a^b f(x_1)\cdots f(x_n)dx_n \cdots dx_1 = n! \int_a^b \int_a^{x_1} \cdots \int_a^{x_{n-1}} f(x_1)\cdots f(x_n)dx_n \cdots dx_2 dx_1$*

Proof. Based on *Lemma 1*

$$\begin{aligned}\int_a^b \int_a^b f(x)f(y)dydx &= \int_a^b \int_x^b f(x)f(y)dydx + \int_a^b \int_a^x f(x)f(y)dxdy \\ &= 2! \int_a^b \int_a^x f(x)f(y)dxdy\end{aligned}$$

Suppose I have

$$\int_a^b \cdots \int_a^b f(x_1)\cdots f(x_{n-1})dx_{n-1}\cdots dx_1 = (n-1)! \int_a^b \int_a^{x_1} \cdots \int_a^{x_{n-2}} f(x_1)\cdots f(x_{n-1})dx_{n-1}\cdots dx_2 dx_1$$

According to Fubini's theorem,

$$\begin{aligned}
& \int_a^b \cdots \int_a^b f(x_1)\cdots f(x_n)dx_n \cdots dx_1 \\
= \quad & (n-1)! \int_a^b \int_a^{x_1} \cdots \int_a^{x_{n-2}} f(x_1)\cdots f(x_{n-1})dx_{n-1}\cdots dx_2 dx_1 \int_a^b f(x_n)dx_n \\
= \quad & (n-1)! \int_a^b \int_a^{x_1} \cdots \int_a^{x_{n-2}} f(x_1)\cdots f(x_{n-1})dx_{n-1}\cdots dx_2 dx_1 \left(\int_a^{x_{n-1}} f(x_n)dx_n + \int_{x_{n-1}}^{x_{n-2}} f(x_n)dx_n + \cdots \int_{x_1}^b f(x_n)dx_n \right) \\
= \quad & (n-1)! \int_a^b \int_a^{x_1} \cdots \int_a^{x_{n-2}} \int_a^{x_{n-1}} f(x_1)\cdots f(x_n)dx_n \cdots dx_2 dx_1 \\
& +(n-1)! \int_a^b \int_a^{x_1} \cdots \int_a^{x_{n-2}} \int_{x_{n-1}}^{x_{n-2}} f(x_1)\cdots f(x_n)dx_n dx_{n-1} \cdots dx_2 dx_1 \\
& +(n-1)! \int_a^b \int_a^{x_1} \cdots \int_a^{x_{n-3}} \int_{x_{n-2}}^{x_{n-3}} \int_a^{x_{n-2}} f(x_1)\cdots f(x_n)dx_{n-1} dx_n dx_{n-2} \cdots dx_2 dx_1 \\
& + \qquad\qquad \vdots \\
& +(n-1)! \int_a^b \int_{x_1}^b \int_a^{x_1} \cdots \int_a^{x_{n-2}} f(x_1)\cdots f(x_{n-1})dx_{n-1}\cdots dx_2 dx_n dx_1 \\
= \quad & (n-1)! \int_a^b \int_a^{x_1} \cdots \int_a^{x_{n-2}} \int_a^{x_{n-1}} f(x_1)\cdots f(x_n)dx_n \cdots dx_2 dx_1 \\
& +(n-1)! \int_a^b \int_a^{x_1} \cdots \int_a^{x_{n-3}} \int_a^{x_n} \int_a^{x_{n-2}} f(x_1)\cdots f(x_n)dx_{n-1} dx_{n-2} dx_n \cdots dx_2 dx_1 \\
& + \qquad\qquad \vdots \\
& +(n-1)! \int_a^b \int_a^{x_n} \int_a^{x_1} \cdots \int_a^{x_{n-2}} f(x_1)\cdots f(x_{n-1})dx_{n-1}\cdots dx_2 dx_1 dx_n \\
= \quad & n! \int_a^b \int_a^{x_1} \cdots \int_a^{x_{n-1}} f(x_1)\cdots f(x_n)dx_n \cdots dx_2 dx_1
\end{aligned}$$

Based on *Lemma 2*, the n-th moment of arithmetic stock price can be calculated.

$$
\begin{aligned}
E\left[\left(\int_0^T S_t dt\right)^n\right] &= n!E\left[\int_0^T \int_0^{x_1} \cdots \int_0^{x_{n-1}} S_{x_1} \cdots S_{x_n} dx_n \cdots dx_1\right] \\
&= n! \int_0^T \int_0^{x_1} \cdots \int_0^{x_{n-1}} E\left[S_{x_1} \cdots S_{x_n}\right] dx_n \cdots dx_1 \\
&= n! \int_0^T \int_0^{x_1} \cdots \int_0^{x_{n-1}} S_0^n e^{(r-\frac{1}{2}\sigma^2)(x_1+\cdots+x_n)} E\left[\exp\left(B_{x_1} + \cdots + B_{x_n}\right)\right] dx_n \cdots dx_1 \\
&= n! \int_0^T \int_0^{x_1} \cdots \int_0^{x_{n-1}} S_0^n e^{(r-\frac{1}{2}\sigma^2)(x_1+\cdots+x_n)} \exp\left(\frac{1}{2}\sigma^2 \sum_{k=0}^{n} (2n-2k-1) x_{n-k}\right) dx_n \cdots dx_1 \\
&= n! \int_0^T \int_0^{x_1} \cdots \int_0^{x_{n-1}} S_0^n \exp\left(\sum_{k=1}^{n} (r+(k-1)\sigma^2) x_k\right) dx_n \cdots dx_1
\end{aligned}
$$

3 Upper bound

3.1 SLN Quad Upper bound

Let,

$$\alpha = r - \frac{1}{2}\sigma^2$$

Consider,

$$
\begin{aligned}
E\left[\left(\frac{1}{T}\int_0^T S\exp(\sigma B_t + \alpha t) dt - K\right)^+\right] &= E\left[\left(\int_0^T (\frac{1}{T} S\exp(\sigma B_t + \alpha t) - K f_t) dt\right)^+\right] \\
&\leq E\left[\int_0^T \left(\frac{1}{T} S\exp(\sigma B_t + \alpha t) - K f_t\right)^+ dt\right] \\
&= \int_0^T E\left[\left(\frac{1}{T} S\exp(\sigma B_t + \alpha t) - K f_t\right)^+\right] dt
\end{aligned}
$$

Let,

$$f_t = \mu_t + \widetilde{\sigma}\left(B_t - \frac{1}{T}\int_0^T B_s ds\right)$$

where μ_t is a deterministic function satisfying $\int_0^T \mu_t dt = 1$.

The Lagrangian condition to minimize the upper bound is that $P\left(\frac{1}{T} S\exp(\sigma B_t + \alpha t) \geq K f_t\right)$ must be independent of t. Thompson (2002) suggest that $\exp(\sigma B_t)$ could be approximated by $1 + \sigma B_t$. This leads to the condition $P(S\exp(\alpha t)(1 + \sigma B_t) \geq KT f_t)$. Using the facts of the preliminary results, I get:

$$\mu_t = \frac{1}{KT}\left(S e^{\alpha t} + \gamma\sqrt{\nu_t}\right)$$

where

$$
\begin{aligned}
\nu_t &= c_t^2 t + 2c_t(K\widetilde{\sigma}) t (T - \frac{1}{2}t) + \frac{1}{3}K^2\widetilde{\sigma}^2 T^3 \\
c_t &= \sigma S e^{\alpha t} - KT\widetilde{\sigma}
\end{aligned}
$$

Imposing $\int_0^T \mu_t dt = 1$ gives

$$\gamma = \frac{KT - \frac{S}{\alpha}\left(e^{\alpha T} - 1\right)}{\int_0^T \sqrt{\nu_t} dt}$$

Thus, I get the upper bound by conditioning on $B_t = x$.

$$e^{-rT}\int_0^T E\left[\left(\frac{1}{T}S\exp(\sigma B_t+\alpha t)-Kf_t\right)^+\right]dt = e^{-rT}\int_0^T\int_{-\infty}^{\infty}\frac{1}{\sqrt{t}}\phi\frac{x}{\sqrt{t}}E\left[(a(t,x)+b(t,x)N)^+\right]dxdt$$

where

$$\begin{aligned} a(t,x) &= \frac{1}{T}S\exp(\sigma B_t+\alpha t)-K(\mu_t+\widetilde{\sigma}x)+\frac{K\widetilde{\sigma}}{T}(T-\frac{1}{2}t)x \\ b(t,x) &= \frac{K\widetilde{\sigma}}{T^2}\sqrt{\frac{1}{3}T^3-t(T-\frac{1}{2}t)^2} \end{aligned}$$

It can be easily proved that

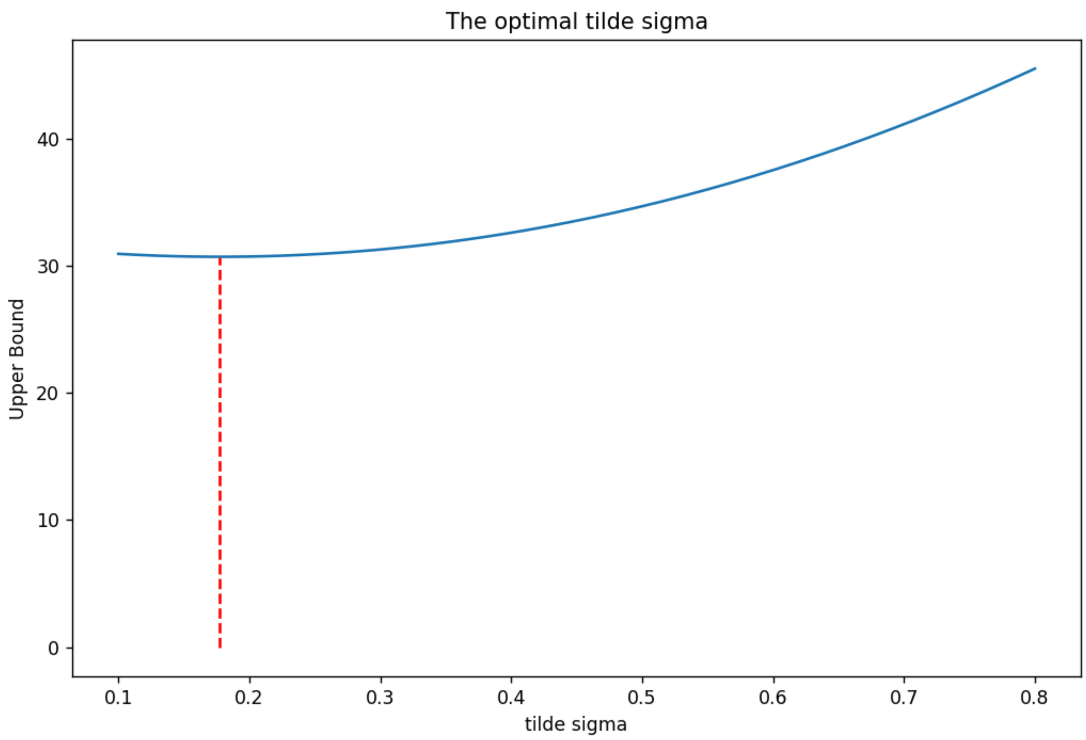
$$E\left[(a(t,x)+b(t,x)N)^+\right] = a(t,x)\Phi\left(\frac{a(t,x)}{b(t,x)}\right)+b(t,x)\phi\left(\frac{a(t,x)}{b(t,x)}\right)$$

To avoid the numerical problem, use the change of variable $v=\sqrt{t}$ and $w=x/\sqrt{t}$:

$$\begin{aligned} & e^{-rT}\int_0^T E\left[\left(\frac{1}{T}S\exp(\sigma B_t+\alpha t)-Kf_t\right)^+\right]dt \\ = \quad & e^{-rT}\int_0^{\sqrt{T}}\int_{-\infty}^{\infty}2v\phi(w)\left(a(t,x)\Phi\left(\frac{a(t,x)}{b(t,x)}\right)+b(t,x)\phi\left(\frac{a(t,x)}{b(t,x)}\right)\right)dwdv \qquad (1) \end{aligned}$$

$\widetilde{\sigma}$ is chosen arbitrarily. It could be shown that the upper bound has a quadratic form with respect to $\widetilde{\sigma}$.

Figure 1: Asian option Upper bound with respect to $\widetilde{\sigma}$



4 Partially exact approximation

Curran (1994) suggests a method based on conditioning on the geometric mean price for valuing Asian option. The basic idea comes from the constant inequality between arithmetic stock price and geometric stock price.

Consider,

$$\begin{aligned} G &= \exp\left(\frac{1}{T}\int_0^T S_u du\right) \\ &= S_0 \exp\left(\frac{T}{2}(r - \frac{\sigma^2}{2}) + \frac{\sigma}{T}\int_0^T B_u du\right) \\ \Lambda_T &= \int_0^T S_u du \end{aligned}$$

Consider the exponential function, it is easy to prove that $G > \frac{1}{T}\Lambda_T$. Thus, the arithmetic Asian option can be calculated as:

$$\begin{aligned} & e^{-rT} E\left[\left(\frac{1}{T}\int_0^T S_u du - K\right)^+\right] \\ = \quad & e^{-rT}\int_0^K E\left[\left(\frac{1}{T}\int_0^T S_u du - K\right)^+ |G = x\right] dP(G \leq x) \\ & + e^{-rT}\int_K^\infty E\left[\left(\frac{1}{T}\int_0^T S_u du - K\right)^+ |G = x\right] dP(G \leq x) \end{aligned}$$

Accordingly, denote

$$\begin{aligned} C_1 &= e^{-rT}\int_0^K E\left[\left(\frac{1}{T}\int_0^T S_u du - K\right)^+ |G = x\right] dP(G \leq x) \\ C_2 &= e^{-rT}\int_K^\infty E\left[\left(\frac{1}{T}\int_0^T S_u du - K\right)^+ |G = x\right] dP(G \leq x) \end{aligned}$$

4.1 Calculation of C_2

Since C_2 can be exactly calculated, I start from C_2.

Consider,

$$\log G = \log S_0 + \frac{T}{2}(r - \frac{\sigma^2}{2}) + \frac{\sigma}{T}\int_0^T B_u du$$

Thus,

$$\begin{aligned} E(G) &= S_0 e^{\frac{T}{2}(r - \frac{\sigma^2}{6})} \\ E(G^2) &= S_0 e^{T(r + \frac{\sigma^2}{6})} \\ E(\log G) &= \frac{T}{2}(r - \frac{\sigma^2}{2}) \\ \mathrm{Var}(\log G) &= \frac{1}{3}\sigma^2 T \end{aligned}$$

I also calculate the first and the second moment of Λ_T:

$$E(\Lambda_T) = S_0\frac{e^{rT}-1}{r}$$
$$E\left[(\Lambda_T)^2\right] = 2S_0^2\frac{re^{(\sigma^2+2r)T}-(\sigma^2+2r)e^{rT}+(\sigma^2+r)}{(\sigma^2+r)(\sigma^2+2r)r}$$

Use conditional expectation to calculate C_2 in the Curran approximation:

$$\begin{aligned}
E\left[\int_0^T S_u du|G=x\right] &= \int_0^T E(S_u|G=x)du \\
&= \int_0^T E\left(S_u|\log(G)=\log(x)\right)du \\
&= \int_0^T E\left(S_u|\frac{\sigma}{T}\int_0^T B_u du = \log(x)-\log(S_0)-\frac{T}{2}(r-\frac{\sigma^2}{2})\right)du
\end{aligned}$$

Let $y=\log(x)-\log(S_0)-\frac{T}{2}(r-\frac{\sigma^2}{2})$, So y is a zero-mean Gaussian variable. Consider:

$$\begin{aligned}
E\left[B_u|Y\right] &= \mu_{B_u|Y} \\
&= \frac{3u(T-\frac{1}{2}u)}{\sigma T^2}Y \\
\mathrm{Var}\left(B_u|Y\right) &= \sigma^2_{B_u|Y} \\
&= u-\frac{3u^2(T-\frac{1}{2}u)^2}{T^3}
\end{aligned}$$

$B_u|Y$ is still a Gaussian variable with mean $\mu_{B_u|Y}$ and variance $\sigma^2_{B_u|Y}$.

Thus,

$$\begin{aligned}
E\left(S_u|Y\right) &= S_0e^{(r-\frac{\sigma^2}{2})u}E\left(e^{\sigma B_u}|Y\right) \\
&= S_0e^{(r-\frac{\sigma^2}{2})u}\exp\left(\sigma\mu_{B_u|Y}+\frac{1}{2}\sigma^2\sigma^2_{B_u|Y}\right)
\end{aligned}$$

Also, I calculate $P(G\geq K)$:

$$\begin{aligned}
P(G\geq K) &= P\left(\log(G)\geq\log(K)\right) \\
&= \Phi\left(\frac{\sqrt{3}}{\sigma\sqrt{T}}\left(\frac{T}{2}(r-\frac{\sigma^2}{2})+\log\frac{S_0}{K}\right)\right)
\end{aligned}$$

Finally, C_2 can be calculated:

$$\begin{aligned}
C_2 &= \frac{e^{-rT}}{T}\int_K^\infty E\left[\int_0^T S_u du|G=x\right]dP(G\leq x)-Ke^{-rT}\int_K^\infty dP(G\leq x) \\
&= \frac{e^{-rT}}{T}\int_{\log\frac{K}{S_0}-\frac{T}{2}(r-\frac{\sigma^2}{2})}^\infty\int_0^T S_0e^{(r-\frac{\sigma^2}{2})u}\exp\left(\sigma\mu_{B_u|Y}+\frac{1}{2}\sigma^2\sigma^2_{B_u|Y}\right)dudP(Y\leq y) \\
&-Ke^{-rT}\Phi\left(\frac{\sqrt{3}}{\sigma\sqrt{T}}\left(\frac{T}{2}(r-\frac{\sigma^2}{2})+\log\frac{S_0}{K}\right)\right)
\end{aligned} \tag{2}$$

4.2 Curran 2M approximation

Curran 2M approximation proposes to use a lognormal distribution to approximate C_1.

Let

$$\Lambda_T = G_T+S_0\exp(\alpha+\beta Z) \text{ ,where Z is an independent standard normal variable}$$

Use the conditional first moment and the conditional second moment matching to calibrate α and β (conditioning on $G_T = K$):

$$E\left[G_T + S_0\text{exp}(\alpha + \beta Z)|G_T = K\right] = K + S_0\text{exp}(\alpha + \frac{1}{2}\beta^2)$$
$$E\left[(G_T + S_0\text{exp}(\alpha + \beta Z))^2\,|G_T = K\right] = K^2 + 2S_0K\text{exp}(\alpha + \frac{1}{2}\beta^2) + S_0^2\text{exp}(2\alpha + 2\beta^2)$$

By equating the conditional first and the conditional second moment, I can get α and β.

$$E\left[G_T + S_0\text{exp}(\alpha + \beta Z)|G_T = K\right] = E(\Lambda_T|G_T = K) = E(\Lambda_T|G_T = x)$$
$$E\left[(G_T + S_0\text{exp}(\alpha + \beta Z))^2\,|G_T = K\right] = E\left[(\Lambda_T)^2|G_T = K\right] = E\left[(\Lambda_T)^2|G_T = x\right]$$

Calculate the first conditional moment:

$$\begin{aligned} E(\Lambda_T|G_T = K) &= E\left[\Lambda_T\Big|y = \log\left(\frac{K}{S_0}\right) - \frac{T}{2}(r - \frac{\sigma^2}{2})\right] \\ &= \int_0^T S_0 e^{(r-\frac{\sigma^2}{2})u}\text{exp}\left(\sigma\mu_{B_u|y} + \frac{1}{2}\sigma^2\sigma^2_{B_u|y}\right)du \end{aligned}$$

Calculate the second conditional moment:

$$\begin{aligned} E({\Lambda_T}^2|G_T = K) &= E\left[{\Lambda_T}^2\Big|Y = \log\left(\frac{K}{S_0}\right) - \frac{T}{2}(r - \frac{\sigma^2}{2})\right]dudv \\ &= E\left[\int_0^T\int_0^T S_uS_v dudv\Big|\frac{\sigma}{T}\int_0^T B_u du\right]dudv \\ &= 2{S_0}^2\int_0^T\int_0^v \text{exp}\left((r + \frac{1}{2}\sigma^2)(u+v)\right)E\left[e^{\sigma(B_u+B_v)}\Big|\frac{\sigma}{T}\int_0^T B_u du\right]dudv \end{aligned}$$

Since,

$$\begin{aligned} E\left[B_u + B_v\Big|\frac{\sigma}{T}\int_0^T B_u du\right] &= \mu_{B_u+B_v|Y} \\ &= \frac{E[(B_u+B_v)Y]}{E[Y^2]}Y \\ &= \frac{3\left(u(T-\frac{1}{2}u) + v(T-\frac{1}{2}v)\right)}{\sigma T^2}Y \end{aligned}$$

$$\begin{aligned} \text{Var}\,(B_u + B_v) &= E\left[(B_u+B_v)^2\right] \\ &= E\left[{B_u}^2 + 2B_uB_v + {B_v}^2\right] \\ &= 3u + v \end{aligned}$$

$$\begin{aligned} \text{Var}\left(B_u + B_v\Big|\frac{\sigma}{T}\int_0^T B_u du\right) &= \sigma^2_{B_u+B_v|Y} \\ &= \text{Var}\,(B_u+B_v) - \frac{E[(B_u+B_v)Y]^2}{E[Y^2]} \\ &= 3u + v - \frac{3\big(u(T-\frac{1}{2}u) + v(T-\frac{1}{2}v)\big)^2}{T^3} \end{aligned}$$

Thus,

$$E({\Lambda_T}^2|G_T = K) = 2{S_0}^2\int_0^T\int_0^v \text{exp}\left((r + \frac{1}{2}\sigma^2)(u+v)\right)\text{exp}\left(\sigma\mu_{B_u+B_v|Y} + \frac{1}{2}\sigma^2\sigma^2_{B_u+B_v|Y}\right)dudv$$

I get,

$$\alpha + \frac{1}{2}\beta^2 = \log\left(\frac{1}{S_0}(E[\Lambda_T|Y] - K)\right)$$
$$\alpha + \beta^2 = \frac{1}{2}\log\left(E[\Lambda_T^2|Y] - K^2 - 2K(E[\Lambda_T|Y] - K)\right)$$

Thus, α and β can be solved.

Finally, I can calculate C_1:

$$\begin{aligned}
C_1 &= e^{-rT}\int_0^K E\left[\left(\frac{1}{T}\int_0^T S_u du - K\right)^+ \Big| G = x\right] dP(G \le x) \\
&= e^{-rT}\int_0^K \int_{\frac{1}{\beta}\log\left(\frac{TK-x}{S_0}-\alpha\right)}^{\infty} \left(\frac{1}{T}\left(x + S_0 e^{\alpha+\beta z}\right) - K\right) dP(Z = z)dP(G \le x) \\
&= e^{-rT}\int_{-\infty}^{\log(\frac{K}{S_0}) - \frac{T}{2}(r - \frac{\sigma^2}{2})} \int_{\frac{1}{\beta}\log\left(\frac{TK-x}{S_0}-\alpha\right)}^{\infty} \left(\frac{1}{T}\left(x + S_0 e^{\alpha+\beta z}\right) - K\right) dP(Z \le z)dP(Y \le y) \qquad (3)
\end{aligned}$$

4.3 Curran 2M+ approximation

Curran 2M+ approximation still uses a lognormal distribution to approximate C_1, but globally, while Curran 2M approximation is conditioned on $G_T = K$. By equating the conditional first and the conditional second moment,

$$E\left[G_T + S_0\exp(\alpha + \beta Z)|G_T = x\right] = E(\Lambda_T|G_T = x)$$
$$E\left[(G_T + S_0\exp(\alpha + \beta Z))^2\,|G_T = x\right] = E\left[(\Lambda_T)^2|G_T = x\right]$$

For every x and every y, I get,

$$\alpha_y + \frac{1}{2}\beta_y^2 = \log\left(\frac{1}{S_0}(E[\Lambda_T|Y] - K)\right)$$
$$\alpha_y + \beta_y^2 = \frac{1}{2}\log\left(E[\Lambda_T^2|Y] - K^2 - 2K(E[\Lambda_T|Y] - K)\right)$$

Thus,

$$C_1 = e^{-rT}\int_{-\infty}^{\log(\frac{K}{S_0}) - \frac{T}{2}(r - \frac{\sigma^2}{2})} \int_{\frac{1}{\beta}\log\left(\frac{TK-x}{S_0}-\alpha\right)}^{\infty} \left(\frac{1}{T}\left(x + S_0 e^{\alpha_y+\beta_y z}\right) - K\right) dP(Z \le z)dP(Y \le y) \qquad (4)$$

4.4 Curran 2M+Uniform approximation

Curran 2M+Uniform approximation uses the geometric stock price plus a uniform random variable to approximate the arithmetic stock price.
Let

$$\Lambda_T = G_T + S_0\eta \text{ ,where } \eta \sim U(a, b)$$

For every x and y, solve the following equation system,

$$x + S_0\frac{a_y + b_y}{2} = E(\Lambda_T|Y = y)$$
$$x^2 + 2xS_0\frac{a_y + b_y}{2} + S_0^2\frac{a_y^2 + b_y^2 + a_y b_y}{3} = E({\Lambda_T}^2|Y = y)$$

Finally, I can calculate C_1:

$$
\begin{aligned}
C_1 &= e^{-rT}\int_0^K E\left[\left(\frac{1}{T}\int_0^T S_u du - K\right)^+ |G=x\right] dP(G\le x) \\
&= e^{-rT}\int_0^K \int_{\frac{TK-x}{S_0}}^{b_y}\left(\frac{1}{T}\left(x+S_0 z\right)-K\right) dP(\eta=z)dP(G\le x) \\
&= e^{-rT}\int_{-\infty}^{\log(\frac{K}{S_0})-\frac{T}{2}(r-\frac{\sigma^2}{2})} \int_{\frac{TK-x}{S_0}}^{b_y}\left(\frac{1}{T}\left(x+S_0 z\right)-K\right) dP(\eta\le z)dP(Y\le y) \qquad (5)
\end{aligned}
$$

4.5 Curran 3M+ approximation

Curran 3M+ approximation aims to use a shifted lognormal distribution to approximate C_1.

Consider the third conditional moment of Λ_T,

$$
\begin{aligned}
E({\Lambda_T}^3|G_T=x) &= E\left[{\Lambda_T}^3\Big|Y=\log\left(\frac{x}{S_0}\right)-\frac{T}{2}(r-\frac{\sigma^2}{2})\right] \\
&= E\left[\int_0^T\int_0^T\int_0^T S_w S_v S_u dwdvdu\Big|\frac{\sigma}{T}\int_0^T B_u du\right] \\
&= 6{S_0}^3\int_0^T\int_0^u\int_0^v \exp\left((r+\frac{1}{2}\sigma^2)(u+v+w)\right) E\left[e^{\sigma(B_u+B_v+B_w)}\Big|\frac{\sigma}{T}\int_0^T B_u du\right] dwdvdu
\end{aligned}
$$

Since,

$$
\begin{aligned}
E\left[B_u+B_v+B_w\Big|\frac{\sigma}{T}\int_0^T B_u du\right] &= \mu_{B_u+B_v B_w|Y} \\
&= \frac{E[(B_u+B_v+B_w)Y]}{E[Y^2]}Y \\
&= \frac{3\left(u(T-\frac{1}{2}u)+v(T-\frac{1}{2}v)+w(T-\frac{1}{2}w)\right)}{\sigma T^2}Y
\end{aligned}
$$

$$
\begin{aligned}
\mathrm{Var}\left(B_u+B_v+B_w\right) &= E\left[(B_u+B_v+B_w)^2\right] \\
&= E\left[{B_u}^2+{B_v}^2+{B_w}^2+2B_uB_v+2B_uB_w+2B_vB_w\right] \\
&= 5w+3v+u
\end{aligned}
$$

$$
\begin{aligned}
\mathrm{Var}\left(B_u+B_v+B_w\Big|\frac{\sigma}{T}\int_0^T B_u du\right) &= \sigma^2_{B_u+B_v|Y} \\
&= \mathrm{Var}\left(B_u+B_v+B_w\right)-\frac{E[(B_u+B_v+B_w)Y]^2}{E[Y^2]} \\
&= 5w+3v+u-\frac{3\left(u(T-\frac{1}{2}u)+v(T-\frac{1}{2}v)+w(T-\frac{1}{2}w)\right)^2}{T^3}
\end{aligned}
$$

Thus,

$$
E({\Lambda_T}^3|G_T=x) = 6{S_0}^3\int_0^T\int_0^u\int_0^v \exp\left((r+\frac{1}{2}\sigma^2)(u+v+w)\right)\exp\left(\sigma\mu_{B_u+B_v+B_w|Y}+\frac{1}{2}\sigma^2\sigma^2_{B_u+B_v+B_w|Y}\right) dwdvdu
$$

Let

$$
\Lambda_T = S_0\alpha + S_0\exp(\beta+\gamma Z) \text{ ,where } Z \text{ is an independent standard normal variable}
$$

Use the conditional first moment, the conditional second moment and the conditional third moment globally matching to calibrate α, β and γ:

$$
\begin{aligned}
E\left[S_0\alpha + S_0\exp(\beta+\gamma Z)|G_T = x\right] &= S_0\alpha + S_0 e^{\beta+\frac{1}{2}\gamma^2}\\
E\left[\left(S_0\alpha + S_0\exp(\beta+\gamma Z)\right)^2|G_T = x\right] &= S_0^2\alpha^2 + 2S_0^2\alpha e^{\beta+\frac{1}{2}\gamma^2} + S_0^2 e^{2\beta+2\gamma^2}\\
E\left[\left(S_0\alpha + S_0\exp(\beta+\gamma Z)\right)^3|G_T = x\right] &= S_0^3\alpha^3 + S_0^3 e^{3\beta+\frac{9}{2}\gamma^2} + 3S_0^3\alpha^2 e^{\beta+\frac{1}{2}\gamma^2} + 3S_0^3\alpha e^{2\beta+2\gamma^2}
\end{aligned}
$$

For every x and y, solve the following equation system,

$$
\begin{aligned}
S_0\alpha_y + S_0 e^{\beta_y+\frac{1}{2}\gamma_y^2} &= E(\Lambda_T|G = x)\\
S_0^2\alpha_y^2 + 2S_0^2\alpha e^{\beta_y+\frac{1}{2}\gamma_y^2} + S_0^2 e^{2\beta_y+2\gamma_y^2} &= E({\Lambda_T}^2|G = x)\\
S_0^3\alpha_y^3 + S_0^3 e^{3\beta_y+\frac{9}{2}\gamma_y^2} + 3S_0^3\alpha_y^2 e^{\beta_y+\frac{1}{2}\gamma_y^2} + 3S_0^3\alpha_y e^{2\beta_y+2\gamma_y^2} &= E({\Lambda_T}^3|G = x)
\end{aligned}
$$

Finally, I can calculate C_1:

$$
\begin{aligned}
C_1 &= e^{-rT}\int_0^K E\left[\left(\frac{1}{T}\int_0^T S_u du - K\right)^+ |G = x\right] dP(G\leq x)\\
&= e^{-rT}\int_0^K \int_{\frac{1}{\gamma_y}\left(\log\left(\frac{KT}{S_0}-\alpha_y\right)-\beta_y\right)}^{\infty} \left(\frac{S_0}{T}\left(\alpha_y + e^{\beta_y+\gamma_y z}\right) - K\right) dP(Z\leq z)dP(G\leq x)\\
&= e^{-rT}\int_{-\infty}^{\log(\frac{K}{S_0})-\frac{T}{2}(r-\frac{\sigma^2}{2})} \int_{\frac{1}{\gamma_y}\left(\log\left(\frac{KT}{S_0}-\alpha_y\right)-\beta_y\right)}^{\infty} \left(\frac{S_0}{T}\left(\alpha_y + e^{\beta_y+\gamma_y z}\right) - K\right) dP(Z\leq z)dP(Y\leq y) \qquad (6)
\end{aligned}
$$

5 Lower Bound

Since $C2$ can be exactly calculated, I can approximate the option value more tightly by calculating the lower bound for C_1.

5.1 Conditioning on geometric stock price

Use the Rogers and Shi (1995) lower bound for C_1,

$$
\begin{aligned}
C_1 &= e^{-rT}\int_0^K E\left[\left(\frac{1}{T}\int_0^T S_u du - K\right)^+ \middle| G = x\right] dP(G\leq x)\\
&\geq e^{-rT}\int_0^K \left(E\left[\frac{1}{T}\int_0^T S_u du - K\middle| G = x\right]\right)^+ dP(G\leq x)
\end{aligned}
$$

Thus

$$
\begin{aligned}
& e^{-rT}\int_0^K \left(E\left[\frac{1}{T}\int_0^T S_u du - K\middle| G = x\right]\right)^+ dP(G = x)\\
= \quad & e^{-rT}\int_{-\infty}^{\log\frac{K}{S_0}-\frac{T}{2}(r-\frac{\sigma^2}{2})} \left(\int_0^T \frac{1}{T} S_0 e^{(r-\frac{\sigma^2}{2})u}\exp\left(\sigma\mu_{B_u|Y} + \frac{1}{2}\sigma^2\sigma_{B_u|Y}^2\right) du - K\right)^+ dP(Y\leq y)
\end{aligned}
$$

I get

$$
LB_{GA} = e^{-rT}\int_0^K \left(E\left[\frac{1}{T}\int_0^T S_u du - K\middle| G = x\right]\right)^+ dP(G\leq x) + C_2 \qquad (7)
$$

5.2 Conditioning on FA

Consider,

$$\begin{aligned}\text{FA} &= \int_0^T \frac{1}{T} S_0 e^{(r-\frac{1}{2}\sigma^2)u}(1+\sigma B_u)du \\ &= \frac{S_0}{T(r-\frac{1}{2}\sigma^2)}\left(e^{(r-\frac{1}{2}\sigma^2)T}-1\right)+\frac{S_0\sigma}{T}\int_0^T B_u du\end{aligned}$$

Due to the convexity of the exponential,

$$\int_0^T \frac{1}{T} S_u du \geq \text{FA}$$

Use conditional expectation to calculate C_2 in the Curran approximation:

$$\begin{aligned}E\left[\int_0^T S_u du|\text{FA}=x\right] &= \int_0^T E(S_u|\text{FA}=x)du \\ &= \int_0^T E\left[S_u\middle|\frac{\sigma}{T}\int_0^T B_u du = \left(x-\frac{S_0}{T(r-\frac{1}{2}\sigma^2)}\left(e^{(r-\frac{1}{2}\sigma^2)T}-1\right)\right)\Big/ S_0\right]du\end{aligned}$$

Let $y = \left(x-\frac{S_0}{T(r-\frac{1}{2}\sigma^2)}\left(e^{(r-\frac{1}{2}\sigma^2)T}-1\right)\right)\Big/ S_0$, so y is a zero-mean Gaussian variable. Calculate the first two moments,

$$\begin{aligned}E\left[B_u|Y\right] &= \mu_{B_u|Y} \\ &= \frac{3u(T-\frac{1}{2}u)}{\sigma T^2}Y \\ \text{Var}\left(B_u|Y\right) &= \sigma^2_{B_u|Y} \\ &= u-\frac{3u^2(T-\frac{1}{2}u)^2}{T^3}\end{aligned}$$

Also, I consider

$$\begin{aligned}P(\text{FA}\geq K) &= P\left(\int_0^T B_u du \geq \frac{T}{S_0\sigma}\left(K-\frac{S_0}{T(r-\frac{1}{2}\sigma^2)}\left(e^{(r-\frac{1}{2}\sigma^2)T}-1\right)\right)\right) \\ &= \Phi\left(\frac{\sqrt{3}}{S_0\sigma\sqrt{T}}\left(-K+\frac{S_0}{T(r-\frac{1}{2}\sigma^2)}\left(e^{(r-\frac{1}{2}\sigma^2)T}-1\right)\right)\right)\end{aligned}$$

Finally C_2 can be calculated by conditioning to FA:

$$\begin{aligned}C_2 &= \frac{e^{-rT}}{T}\int_K^\infty E\left[\int_0^T S_u du|G=x\right]dP(\text{FA}\leq x) - Ke^{-rT}\int_K^\infty dP(\text{FA}\leq x) \\ &= \frac{e^{-rT}}{T}\int_{\left(K-\frac{S_0}{T(r-\frac{1}{2}\sigma^2)}\left(e^{(r-\frac{1}{2}\sigma^2)T}-1\right)\right)/S_0}^\infty \int_0^T S_0 e^{(r-\frac{\sigma^2}{2})u}\exp\left(\sigma\mu_{B_u|Y}+\frac{1}{2}\sigma^2\sigma^2_{B_u|Y}\right)dudP(Y\leq y) \\ &- Ke^{-rT}\Phi\left(\frac{\sqrt{3}}{S_0\sigma\sqrt{T}}\left(-K+\frac{S_0}{T(r-\frac{1}{2}\sigma^2)}\left(e^{(r-\frac{1}{2}\sigma^2)T}-1\right)\right)\right)\end{aligned}$$

Use the Rogers and Shi (1995) lower bound plus conditioning on FA for C_1,

$$\begin{aligned}C_1 &= e^{-rT}\int_0^K E\left[\left(\frac{1}{T}\int_0^T S_u du - K\right)^+\middle|\text{FA}=x\right]dP(\text{FA}\leq x) \\ &\geq e^{-rT}\int_0^K \left(E\left[\frac{1}{T}\int_0^T S_u du - K\middle|\text{FA}=x\right]\right)^+ dP(\text{FA}\leq x)\end{aligned}$$

Thus,

$$
\begin{aligned}
& e^{-rT}\int_0^K \left(E\left[\frac{1}{T}\int_0^T S_u du - K \middle| \text{FA}=x\right]\right)^+ dP(\text{FA}\le x) \\
= \quad & e^{-rT}\int_{-\infty}^{\left(K-\frac{S_0}{T(r-\frac{1}{2}\sigma^2)}\left(e^{(r-\frac{1}{2}\sigma^2)T}-1\right)\right)/S_0} \left(\int_0^T \frac{1}{T}S_0 e^{(r-\frac{\sigma^2}{2})u}\exp\left(\sigma\mu_{B_u|Y}+\frac{1}{2}\sigma^2\sigma^2_{B_u|Y}\right)du - K\right)^+ dP(Y\le y)
\end{aligned}
$$

Finally, I get

$$
LB_{FA} = e^{-rT}\int_0^K \left(E\left[\frac{1}{T}\int_0^T S_u du - K \middle| \text{FA}=x\right]\right)^+ dP(\text{FA}\le x) + C_2 \tag{8}
$$

6 PDE

6.1 Rogers and Shi's PDE

Rogers and Shi (1995) PDE comes from a self-financing portfolio strategy. Consider (η_t, ϵ_t) be a self-financing protfolio $V_t := \eta_t A_t + \epsilon_t S_t$, where A_t is a unit of risk-free asset. V_t could also be given by

$$
V_t = S_t f(t, Z_t)
$$

where $Z_t = \frac{1}{S_t}\left(\frac{\Lambda_t}{T} - K\right)$. Imposing the self-financing condition, the function $f(t,z)$ could satisfy the PDE

$$
\frac{\partial f}{\partial t}(t,z) + \left(\frac{1}{T} - rz\right)\frac{\partial f}{\partial z}(t,z) + \frac{1}{2}\sigma^2 z^2 \frac{\partial^2 f}{\partial z^2}(t,z) = 0
$$

under the terminal condition $f(T,z) = z^+$. Thus, consider the termainal and boundary conditions for Rogers and Shi's PDE:

- TC: $f_{N,j} = z_j^+$, for $j = 0, 1, \ldots M$
- BC: $f_{i,0} = 0$, for $i = 0, 1, \ldots N$
- BC: $f_{i,M} = z_M + \frac{1-e^{-r(T-t)}}{rT}$, for $i = 0, 1, \ldots N$

Let

$$
z = z_0 + j\Delta z
$$

6.1.1 Implicit Scheme

Symmetric in z:

$$
\frac{\partial z}{\partial S} = \frac{f(i,j+1) - f(i,j-1)}{2\Delta z}
$$

Forward in Time

$$
\frac{\partial f}{\partial t} = \frac{f(i+1,j) - f(i,j)}{\Delta t}
$$

Centeral in z

$$
\frac{\partial^2 f}{\partial z^2} = \frac{f(i,j+1) - 2f(i,j) + f(i,j-1)}{\Delta z^2}
$$

Consider

$$
a_j f_{i,j-1} + b_j f_{i,j} + c_j f_{i,j+1} = f_{i+1,j}, \text{for}\, i = 0, \ldots, N-1, \text{and}\, j = 1, 2, \ldots, M-1
$$

where

$$
\begin{aligned}
a_j &= \left(\frac{1-rzT}{2\Delta z T} - \frac{\sigma^2 z^2}{2\Delta z^2}\right)\Delta t \\
b_j &= 1 + \frac{\sigma^2 z^2 \Delta t}{\Delta z^2} \\
c_j &= -\left(\frac{1-rzT}{2\Delta z T} + \frac{\sigma^2 z^2}{2\Delta z^2}\right)\Delta t
\end{aligned}
$$

I get the implicit scheme

$$\begin{bmatrix} b_1 & c_1 & & & & \\ a_2 & b_2 & c_2 & & & \\ & a_3 & b_3 & c_3 & & \\ & & \cdots & \cdots & \cdots & \\ & & & \cdots & \cdots & \cdots \\ & & & & a_{M-1} & b_{M-1} \end{bmatrix} \begin{bmatrix} f(i,1) \\ f(i,2) \\ f(i,3) \\ \cdots \\ \cdots \\ f(i,M-1) \end{bmatrix} = \begin{bmatrix} f(i+1,1) - a_1 f(i,0) = b_1 f(i,1) + c_1 f(i,2) \\ f(i+1,2) = a_2 f(i,1) + b_2 f(i,2) + c_2 f(i,3) \\ f(i+1,3) = a_3 f(i,2) + b_3 f(i,3) + c_3 f(i,4) \\ \cdots \\ \cdots \\ f(i+1,M-1) - c_{M-1} f(i,M) \end{bmatrix}$$

6.1.2 Explicit scheme

Symmetric in z:

$$\frac{\partial f}{\partial z} = \frac{f(i+1,j+1) - f(i+1,j-1)}{2\Delta z}$$

Forward in Time

$$\frac{\partial f}{\partial t} = \frac{f(i+1,j) - f(i,j)}{\Delta t}$$

Centeral in z

$$\frac{\partial^2 f}{\partial z^2} = \frac{f(i+1,j+1) - 2f(i+1,j) + f(i+1,j-1)}{\Delta z^2}$$

Consider

$$f_{i,j} = a_j^* f_{i+1,j-1} + b_j^* f_{i+1,j} + c_j^* f_{i+1,j+1}$$

where

$$\begin{aligned} a_j^* &= \frac{1}{1+r\Delta t}(-\frac{1}{2}rj\Delta t + \frac{1}{2}\sigma^2 j^2 \Delta t) \\ b_j^* &= \frac{1}{1+r\Delta t}(1 - \sigma^2 j^2 \Delta t) \\ c_j^* &= \frac{1}{1+r\Delta t}(\frac{1}{2}rj\Delta t + \frac{1}{2}\sigma^2 j^2 \Delta t) \end{aligned}$$

I get the explicit scheme

$$\begin{bmatrix} b_1 & c_1 & & & & \\ a_2 & b_2 & c_2 & & & \\ & a_3 & b_3 & c_3 & & \\ & & \cdots & \cdots & \cdots & \\ & & & \cdots & \cdots & \cdots \\ & & & & a_{M-1} & b_{M-1} \end{bmatrix} \begin{bmatrix} f(i+1,1) \\ f(i+1,2) \\ f(i+1,3) \\ \cdots \\ \cdots \\ f(i+1,M-1) \end{bmatrix} = \begin{bmatrix} f(i,1) - a_1 f(i+1,0) = b_1 f(i+1,1) + c_1 f(i+1,2) \\ f(i,2) = a_2 f(i+1,1) + b_2 f(i+1,2) + c_2 f(i+1,3) \\ f(i,3) = a_3 f(i+1,2) + b_3 f(i+1,3) + c_3 f(i+1,4) \\ \cdots \\ \cdots \\ f(i,M-1) - c_{M-1} f(i+1,M) \end{bmatrix}$$

6.1.3 Crank-Nicolson scheme

Crank-Nicolson scheme is the average of implicit scheme and explicit scheme:

$$\frac{\partial f}{\partial z} = \frac{1}{2}\left(\frac{f(i,j+1) - f(i,j-1)}{2\Delta z} + \frac{f(i-1,j+1) - f(i-1,j-1)}{2\Delta z}\right)$$

$$\frac{\partial f}{\partial t} = \frac{f(i,j) - f(i-1,j)}{\Delta t}$$

$$\frac{\partial^2 f}{\partial z^2} = \frac{1}{2}\left(\frac{f(i,j+1) - 2f(i,j) + f(i,j-1)}{\Delta z^2} + \frac{f(i-1,j+1) - 2f(i-1,j) + f(i-1,j-1)}{\Delta z^2}\right)$$

Consider

$$\widetilde{a}_j f_{i,j-1} + (1 - \widetilde{b}_j) f_{i,j} + \widetilde{c}_j f_{i,j+1} = -\widetilde{a}_j f_{i-1,j-1} + (1 + \widetilde{b}_j) f_{i-1,j} - \widetilde{c}_j f_{i-1,j+1}$$

where

$$\widetilde{a}_j = \left(\frac{\sigma^2 z^2}{4\Delta z^2} - \frac{1 - rzT}{4\Delta zT}\right)\Delta t$$
$$\widetilde{b}_j = \frac{\sigma^2 z^2 \Delta t}{2\Delta z^2}$$
$$\widetilde{c}_j = \left(\frac{\sigma^2 z^2}{4\Delta z^2} + \frac{1 - rzT}{4\Delta zT}\right)\Delta t$$

I get the Crank-Nicolson scheme

$$\begin{bmatrix} 1-\widetilde{b}_1 & \widetilde{c}_1 & & & & & \\ \widetilde{a}_2 & 1-\widetilde{b}_2 & \widetilde{c}_2 & & & & \\ & & \cdots & \cdots & \cdots & & \\ & & & \cdots & \cdots & \cdots & \\ & & & & & \widetilde{a}_{M-1} & 1-\widetilde{b}_{M-1} \end{bmatrix} \begin{bmatrix} f(i,1) \\ f(i,2) \\ f(i,3) \\ \cdots \\ \cdots \\ f(i,M-1) \end{bmatrix} + \begin{bmatrix} \widetilde{a}_1 f_{i,0} \\ \cdots \\ \cdots \\ \cdots \\ \cdots \\ \widetilde{a}_{M-1} f_{i,M} \end{bmatrix} + \begin{bmatrix} \widetilde{a}_1 f_{i-1,0} \\ \cdots \\ \cdots \\ \cdots \\ \cdots \\ \widetilde{a}_{M-1} f_{i-1,M} \end{bmatrix}$$
$$= \begin{bmatrix} 1+\widetilde{b}_1 & \widetilde{c}_1 & & & & & \\ \widetilde{a}_2 & 1+\widetilde{b}_2 & \widetilde{c}_2 & & & & \\ & & \cdots & \cdots & \cdots & & \\ & & & \cdots & \cdots & \cdots & \\ & & & & & \widetilde{a}_{M-1} & 1+\widetilde{b}_{M-1} \end{bmatrix} \begin{bmatrix} f(i-1,1) \\ f(i-1,2) \\ f(i-1,3) \\ \cdots \\ \cdots \\ f(i-1,M-1) \end{bmatrix}$$

6.2 Vecer's PDE

Vecer (2001) PDE comes from an idea of traded account. Vecer (2001) finds that

$$X_t := \int_0^t (1 - \frac{u}{T}) dS_u + S0 - K$$

Then Vecer (2001) denotes $\tilde{Z}_t := \frac{X_t}{S_t}$ and $\tilde{X}_t$ is the hedging portfolio that the seller of the option uses to hedge against the holder of the option *(Note here $\tilde{X}_t$ is not a self-financing portfolio)*. The Asian fixed strike call can thus be achieved by $q_t = 1 - \frac{t}{T}$ and $X_0 = S_0 - K$. Further denote $u(T, \tilde{z}) := \tilde{Z_T}^+ = \frac{1}{S_T}\left(\frac{\Lambda_T}{T} - K\right)^+$. By using the HJB equation, I can get

$$\frac{\partial u}{\partial t}(t, \tilde{z}) + r(q_t - \tilde{z})\frac{\partial u}{\partial \tilde{z}}(t, \tilde{z}) + \frac{1}{2}\sigma^2(q_t - \tilde{z})^2 \frac{\partial^2 u}{\partial \tilde{z}^2}(t, \tilde{z}) = 0$$

where the terminal condition is $u(T, \tilde{z}) = \tilde{z}_T^+$.

6.2.1 Crank-Nicolson scheme

Use Vecer (2001) settings:

$$u_{i,j} = u(t_j, z_i)$$
$$z_i = z_0 + i\Delta z$$
$$t_j = j\Delta t$$
$$q_t = 1 - \frac{t}{T}$$

Consider

$$\widetilde{a}_j u_{i-1,j} + (1 - \widetilde{b}_j)u_{i,j} + \widetilde{c}_j u_{i+1,j} = -\widetilde{a}_j u_{i-1,j-1} + (1 + \widetilde{b}_j)u_{i,j-1} - \widetilde{c}_j u_{i+1,j-1}$$

where

$$\widetilde{a}_j = \left(\frac{\sigma^2 (q_t - z)^2}{4\Delta z^2} - \frac{r(q_t - z)}{4\Delta z} \right) \Delta t$$
$$\widetilde{b}_j = \frac{\sigma^2 (q_t - z)^2 \Delta t}{2\Delta z^2}$$
$$\widetilde{c}_j = \left(\frac{\sigma^2 (q_t - z)^2}{4\Delta z^2} + \frac{r(q_t - z)}{4\Delta z} \right) \Delta t$$

7 Numerical examples

Since there is very little hope that the approximations admits a closed form solution, one must compute the price of the Asian option numerically. I further use the Monte Carlo method with antithetic variables to calibrate the exact price. The paths are 5 million and 2 million respectively for T=5y and T=30y. The moneyness defined in the table is:

$$\text{moneyness} = \frac{K}{\frac{1}{T}E[\Lambda_T]} - 1 = \frac{KTr}{S_0(e^{rT} - 1)} - 1$$

7.1 5-year maturity

Table 1a: Price

Strike	Moneyness	PDE	Curran 2M+	2M+Uniform	Curran 3M+	LB_FA
56.81	-0.5	46.730	46.697	46.698	46.438	46.013
113.61	0	22.822	22.880	22.882	22.650	22.688
170.42	0.5	12.185	12.295	12.299	11.934	10.912

Strike	Moneyness	LB_GA	Levy/2M	Con Gamma	Con Lognormal	Inv Gamma
56.81	-0.5	46.570	61.403	46.861	46.705	45.835
113.61	0	22.761	31.103	22.995	22.878	21.281
170.42	0.5	12.116	16.460	12.229	12.203	11.174

* The parameters are: r=0.05, σ=0.5, T=5

Table 1b: Pricing Error

Strike	Moneyness	PDE	Curran 2M+	2M+Uniform	Curran 3M+	LB_FA
56.81	-0.5	0.051	0.017	0.018	-0.242	-0.667
113.61	0	-0.058	0.000	0.002	-0.230	-0.192
170.42	0.5	-0.062	0.048	0.052	-0.313	-1.335

Strike	Moneyness	LB_GA	Levy/2M	Con Gamma	Con Lognormal	Inv Gamma
56.81	-0.5	-0.110	14.723	0.181	0.026	-0.845
113.61	0	-0.119	8.223	0.115	-0.002	-1.599
170.42	0.5	-0.132	4.213	-0.018	-0.044	-1.073

* The benchmark is a 5-million path Monte Carlo simulation with antithetic variables.

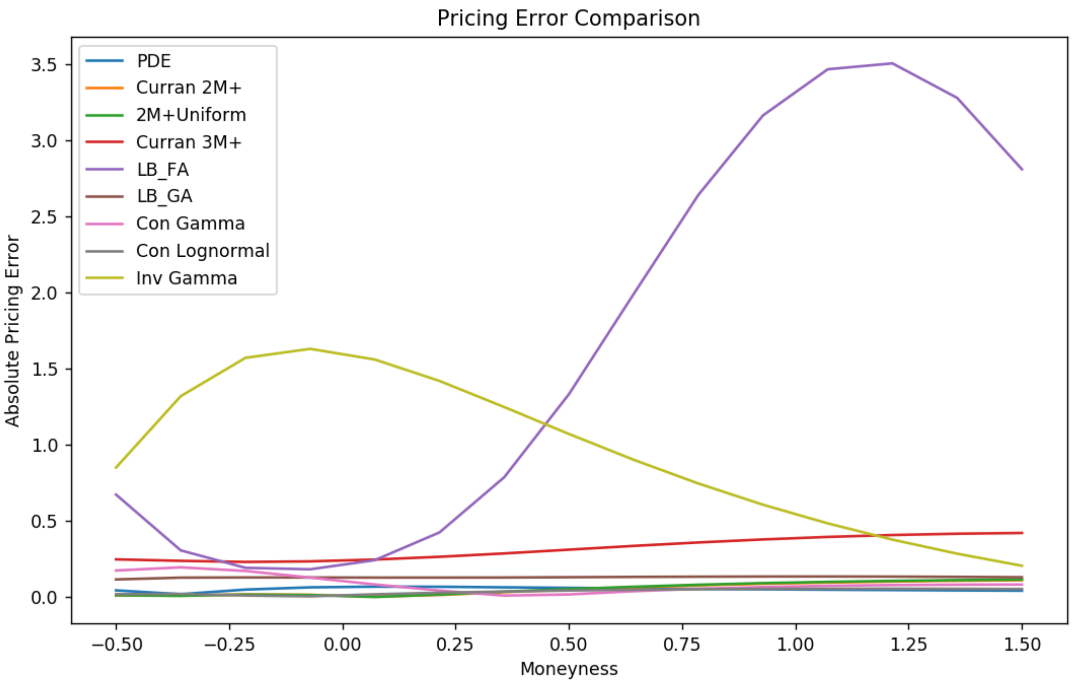


Table 1c: Implied Volatility

Strike	Moneyness	Curran 2M+	2M+Uniform	Curran 3M+	LB_FA
56.81	-0.5	-0.0002	-0.0003	0.0181	0.0436
113.61	0	0.0004	0.0003	0.0059	0.0049
170.42	0.5	0.0000	-0.0001	0.0081	0.0292

Strike	Moneyness	Levy/2M	LB_GA	Con Gamma	Con Lognormal
56.81	-0.5	-0.5014	0.0084	-0.0103	-0.0007
113.61	0	-0.1272	0.0032	-0.0022	0.0004
170.42	0.5	-0.0634	0.0040	0.0015	0.0020

* The figures in the table indicate the shift of implied volatility to calibrate the exact price.

Table 1d: Greeks

Moneyness	Strike	Method	Delta	Gamma	Vega
-0.5	56.81	PDE	0.8143	N/A	0.1600
-0.5	56.81	LB_GA	0.8091	0.0023	0.1458
-0.5	56.81	LB_FA	0.7833	0.0018	0.1474
-0.5	56.81	Inv Gamma	0.8203	0.0025	0.0964
-0.5	56.81	Curran 3M+	0.8069	0.0024	0.1385
-0.5	56.81	Curran 2M+	0.8104	0.0024	0.1538
-0.5	56.81	Con Lognormal	0.8093	0.0023	0.1547
-0.5	56.81	Con Gamma	0.8094	0.0023	0.1644
-0.5	56.81	2M+Uniform	0.8104	0.0024	0.1538
0	113.61	PDE	0.5369	N/A	0.4319
0	113.61	LB_GA	0.5355	0.0050	0.4266
0	113.61	LB_FA	0.5388	0.0046	0.4285
0	113.61	Inv Gamma	0.5157	0.0055	0.3395
0	113.61	Curran 3M+	0.5353	0.0050	0.4208
0	113.61	Curran 2M+	0.5359	0.0051	0.4325
0	113.61	Con Lognormal	0.5377	0.0050	0.4338
0	113.61	Con Gamma	0.5411	0.0050	0.4413
0	113.61	2M+Uniform	0.5359	0.0051	0.4326
0.5	170.42	PDE	0.3344	N/A	0.4556
0.5	170.42	LB_GA	0.3346	0.0046	0.4520
0.5	170.42	LB_FA	0.3870	0.0034	0.4689
0.5	170.42	Inv Gamma	0.3066	0.0046	0.3608
0.5	170.42	Curran 3M+	0.3351	0.0046	0.4456
0.5	170.42	Curran 2M+	0.3344	0.0046	0.4587
0.5	170.42	Con Lognormal	0.3358	0.0047	0.4582
0.5	170.42	Con Gamma	0.3380	0.0047	0.4631
0.5	170.42	2M+Uniform	0.3344	0.0046	0.4589

* I use linear interpolation to calibrate the PDE price on the grid. So the gamma value is not applicable.

7.2 30-year maturity

Table 2a: Price

Strike	Moneyness	PDE	Curran 2M+	2M+Uniform	Curran 3M+	LB_FA
116.06	-0.5	38.025	40.108	40.503	36.870	37.262
232.11	0	29.931	35.847	36.557	31.157	31.301
348.17	0.5	22.868	33.838	31.144	27.507	23.143

Strike	Moneyness	LB_GA	Levy/2M	Con Gamma	Con Lognormal	Inv Gamma
116.06	-0.5	37.305	189.703	38.709	37.911	85.940
232.11	0	31.815	170.491	32.840	32.496	199.988
348.17	0.5	28.410	157.338	29.194	29.151	314.594

* The parameters are: r=0.05, σ=0.5, T=30

Table 2b: Pricing Error

Strike	Moneyness	PDE	Curran 2M+	2M+Uniform	Curran 3M+	LB_FA
116.06	-0.5	0.617	2.700	3.095	-0.538	-0.146
232.11	0	-2.030	3.886	4.596	-0.804	-0.660
348.17	0.5	-5.752	5.218	2.524	-1.113	-5.476

Strike	Moneyness	LB_GA	Levy/2M	Con Gamma	Con Lognormal	Inv Gamma
116.06	-0.5	-0.103	152.295	1.301	0.503	-8.035
232.11	0	-0.146	138.530	0.879	0.535	-12.965
348.17	0.5	-0.209	128.719	0.574	0.531	-14.696

* The benchmark is a 2-million path Monte Carlo simulation with antithetic variables.

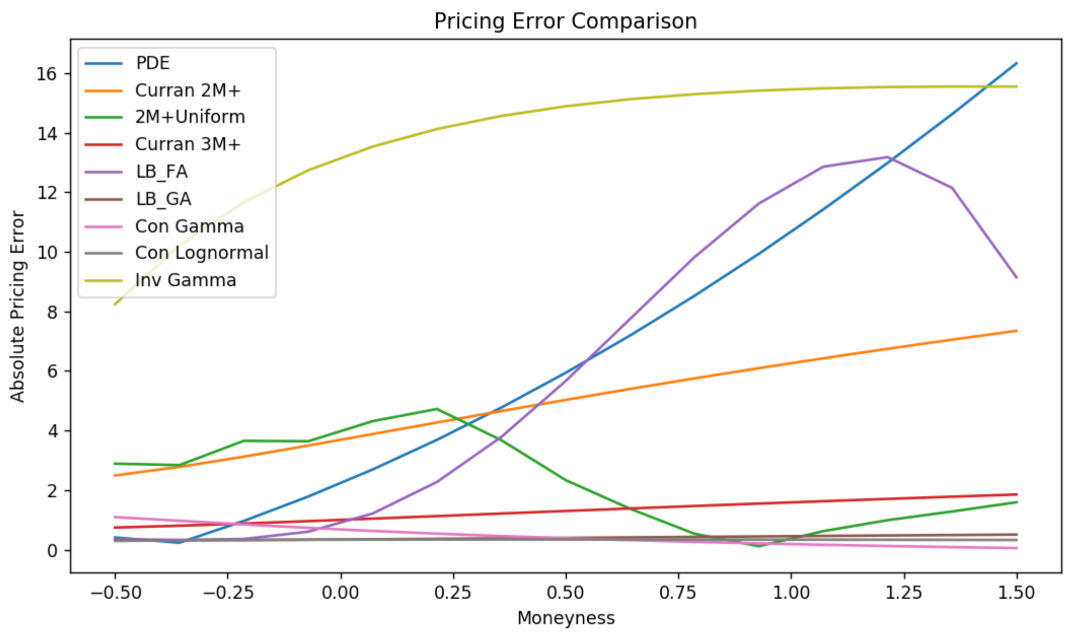


Table 2c: Implied Volatility

Strike	Moneyness	Curran 2M+	2M+Uniform	Curran 3M+	LB_FA
116.06	-0.5000	-0.0499	-0.0502	0.0520	0.0354
232.11	0.0000	-0.0705	-0.0858	0.0175	0.0100
348.17	0.5000	-0.0766	-0.0501	0.0281	0.0606

Strike	Moneyness	Levy/2M	LB_GA	Con Gamma	Con Lognormal
116.06	-0.5000	-0.4959	0.0289	-0.0191	0.0045
232.11	0.0000	-0.4083	0.0008	-0.0223	-0.0141
348.17	0.5000	-0.4973	0.0089	-0.0062	-0.0053

* The figures in the table indicate the shift of implied volatility to calibrate the exact price.

Table 2d: Greeks

Moneyness	Strike	Method	Delta	Gamma	Vega
-0.5	116.06	PDE	0.4760	N/A	0.2954
-0.5	116.06	LB_GA	0.4478	0.0006	0.2619
-0.5	116.06	LB_FA	0.4472	0.0007	0.2408
-0.5	116.06	Inv Gamma	0.4480	0.0014	0.0064
-0.5	116.06	Curran 3M+	0.4452	0.0006	0.2404
-0.5	116.06	Curran 2M+	0.4672	0.0007	0.4147
-0.5	116.06	Con Lognormal	0.4529	0.0006	0.2914
-0.5	116.06	Con Gamma	0.4657	0.0006	0.3762
-0.5	116.06	2M+Uniform	0.4642	0.0073	0.2462
0	232.11	PDE	0.4446	N/A	0.3031
0	232.11	LB_GA	0.4009	0.0008	0.4223
0	232.11	LB_FA	0.4371	-0.0013	0.5607
0	232.11	Inv Gamma	0.3272	0.0019	0.0136
0	232.11	Curran 3M+	0.3992	0.0007	0.4037
0	232.11	Curran 2M+	0.4132	0.0008	0.5453
0	232.11	Con Lognormal	0.4064	0.0007	0.4506
0	232.11	Con Gamma	0.4174	0.0008	0.4973
0	232.11	2M+Uniform	0.4019	0.0173	0.5775
0.5	348.17	PDE	0.4389	N/A	0.2395
0.5	348.17	LB_GA	0.3689	0.0008	0.5138
0.5	348.17	LB_FA	0.5235	-0.0012	1.0293
0.5	348.17	Inv Gamma	0.2515	0.0018	0.0159
0.5	348.17	Curran 3M+	0.3671	0.0008	0.4986
0.5	348.17	Curran 2M+	0.3825	0.0007	0.6152
0.5	348.17	Con Lognormal	0.3747	0.0008	0.5408
0.5	348.17	Con Gamma	0.3822	0.0009	0.5490
0.5	348.17	2M+Uniform	0.3713	0.0407	0.4762

* I use linear interpolation to calibrate the PDE price on the grid. So the gamma value is not applicable.

References

Curran, M. (1994). Valuing asian and portfolio options by conditioning on the geometric mean price. *Management science*, *40*(12), 1705–1711.

Dufresne, D. (1989). Weak convergence of random growth processes with applications to insurance. *Insurance: Mathematics and Economics*, *8*(3), 187–201.

Geman, H., & Yor, M. (1993). Bessel processes, asian options, and perpetuities. *Mathematical finance*, *3*(4), 349–375.

Privault, N., & Yu, J. (2016). Stratified approximations for the pricing of options on average. *Journal of Computational Finance*, *19*(4).

Rogers, L. C. G., & Shi, Z. (1995). The value of an asian option. *Journal of Applied Probability*, 1077–1088.

Thompson, G. (2002). *Fast narrow bounds on the value of asian options.* Citeseer.

Vecer, J. (2001). A new pde approach for pricing arithmetic average asian options. *Journal of computational finance*, *4*(4), 105–113.