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Homework 2

1. Let $\Pi(x, y)$ denote the rectangle function defined as,

$$\Pi(x,y) = \begin{cases} 1, \text{for } |x| < 1/2, |y| < 1/2 \\ 0, Otherwise \end{cases} \tag{1}$$

Derive the Fourier transform of $\Pi(\frac{x-b}{c}, y)$. Note that the FT of $\Pi(x, y) = \Pi(x)\Pi(y)$ is $sinc(x, y) = sinc(x)sinc(y)$, and $sinc(x) = \frac{sin(\pi x)}{\pi x}$, for $x \neq 0$.

Solution, Method 1:

$$\Pi(\frac{x-b}{c}, y) = \begin{cases} 1, \text{for } \left|\frac{x-b}{c}\right| < \frac{1}{2}, |y| < 1/2 \\ 0, Otherwise \end{cases} \tag{2}$$

we assume that $c > 0$, then we have:

$$\begin{aligned} \mathscr{F}\{\Pi(u,v)\} &= \mathscr{F}\{\Pi(\frac{x-b}{c}, y)\} \\ &= \int_{-\frac{1}{2}}^{\frac{1}{2}} \int_{b-\frac{c}{2}}^{b+\frac{c}{2}} e^{-j2\pi(ux+vy)} dxdy \quad = \int_{-\frac{1}{2}}^{\frac{1}{2}} e^{-j2\pi vy} dy \int_{b-\frac{c}{2}}^{b+\frac{c}{2}} e^{-j2\pi ux} dx \end{aligned} \tag{3}$$

Here, we first calculate the first part of the eq.3, based on Euler's formula, we have:

$$\begin{aligned} \int_{-\frac{1}{2}}^{\frac{1}{2}} e^{-j2\pi vy} dy &= \int_{-\frac{1}{2}}^{\frac{1}{2}} [cos(2\pi vy) - jsin(2\pi vy)] dy \\ &= [\frac{sin(2\pi vy)}{2\pi v} + j\frac{cos(2\pi vy)}{2\pi v}]\Big|_{-\frac{1}{2}}^{\frac{1}{2}} \\ &= \frac{sin(\pi v)}{2\pi v} - \frac{sin(-\pi v)}{2\pi v} + j(\frac{cos(\pi v)}{2\pi v} - \frac{cos(\pi(-v))}{2\pi v}) \\ &= \frac{sin(\pi v)}{\pi v} = sinc(v) \end{aligned} \tag{4}$$

Then we calculate the second part of eq.3.

$$\begin{aligned} \int_{b-\frac{c}{2}}^{b+\frac{c}{2}} e^{-j2\pi ux} dx &= \int_{b-\frac{c}{2}}^{b+\frac{c}{2}} [cos(2\pi ux) - jsin(2\pi ux)] dx \\ &= [\frac{sin(2\pi ux)}{2\pi u} + j\frac{cos(2\pi ux)}{2\pi u}]\Big|_{b-\frac{c}{2}}^{b+\frac{c}{2}} \\ &= \frac{sin(2\pi u(b+\frac{c}{2})) - sin(2\pi u(b-\frac{c}{2}))}{2\pi u} + j\frac{cos(2\pi u(b+\frac{c}{2})) - cos(2\pi u(b-\frac{c}{2}))}{2\pi u} \end{aligned} \tag{5}$$

Based on sum and product formulae:

$$\begin{aligned} sinA - sinB &= 2cos\frac{A+B}{2}sin\frac{A-B}{2} \\ cosA - cosB &= -2sin\frac{A+B}{2}sin\frac{A-B}{2} \end{aligned} \tag{6}$$

we can rewrite equation.5 into:

$$\begin{aligned}\int_{b-\frac{c}{2}}^{b+\frac{c}{2}} e^{-j2\pi ux}dx &= \frac{cos(2\pi ub)sin(\pi uc)}{\pi u} - j\frac{sin(2\pi ub)sin(\pi uc)}{\pi u} \\ &= c \cdot sinc(cu)(cos(2\pi ub) - jsin(2\pi ub)) \\ &= c \cdot sinc(cu)e^{-j2\pi ub}\end{aligned} \tag{7}$$

Where $sinc(cu) = \frac{sin(\pi uc)}{\pi uc}$. For $c < 0$, we have similar derivation process just by reversing the upper and lower marks of the integral of the second part of eq.3. Finally, we get:

$$\int_{b-\frac{c}{2}}^{b+\frac{c}{2}} e^{-j2\pi ux}dx = -c \cdot sinc(cu)e^{-j2\pi ub} \tag{8}$$

Hence, for $|c| > 0$, we have:

$$\begin{aligned}\mathscr{F}\{\Pi(u,v)\} &= \mathscr{F}\{\Pi(\frac{x-b}{c}, y)\} \\ &= |c| \cdot sinc(cu)sinc(v)e^{-j2\pi ub}\end{aligned} \tag{9}$$

where $sinc(cu) = \frac{sin(\pi uc)}{\pi uc}, sinc(v) = \frac{sin(\pi v)}{\pi v}$

Method 2: In fact, there exists a shorter derivation and it needs two theorem:

Similarity Theorem. If $\mathscr{F}\{g(x,y)\} = G(u,v)$, then

$$\mathscr{F}\{g(ax,by)\} = \frac{1}{|ab|}G(\frac{u}{a}, \frac{v}{b}) \tag{10}$$

Shift Theorem. If $\mathscr{F}\{g(x,y)\} = G(u,v)$, then

$$\mathscr{F}\{g(x-a, y-b)\} = G(u,v)e^{-j2\pi(ua+vb)} \tag{11}$$

Hence, we have:

$$\begin{aligned}\Pi(x,y) &\leftrightarrow sinc(x)sinc(y) \\ \Pi(x-b,y) &\leftrightarrow sinc(x)sinc(y)e^{-j2\pi xb} \\ \Pi(\frac{x-b}{c}, y) &\leftrightarrow |c|sinc(cx)sinc(y)e^{-j2\pi xb}\end{aligned} \tag{12}$$

It gets the same result as method 1, just replace u, v with x, y and method 1 itself can be regarded as a proof of these two theorem.

2. Prove the rotation theorem for Fourier transforms i.e., prove that

$$f(xcos\theta - ysin\theta, xsin\theta + ycos\theta) \leftrightarrow F(ucos\theta - vsin\theta, usin\theta + vcos\theta) \tag{13}$$

The double arrow denotes the Fourier transform pair.

Solution, Method 1:

The first proof method uses the rotation formula.

Proof. We can define a new coordinate system $(\tilde{x}, \tilde{y})$, where

$$\begin{bmatrix}\tilde{x}\\ \tilde{y}\end{bmatrix} = \begin{bmatrix}cos\theta & -sin\theta\\ sin\theta & cos\theta\end{bmatrix}\begin{bmatrix}x\\ y\end{bmatrix} \tag{14}$$

We can also express (x, y) in terms of the new coordinate system, where

$$\begin{bmatrix}x\\ y\end{bmatrix} = \begin{bmatrix}cos\theta & sin\theta\\ -sin\theta & cos\theta\end{bmatrix}\begin{bmatrix}\tilde{x}\\ \tilde{y}\end{bmatrix} = \begin{bmatrix}\tilde{x}cos\theta + \tilde{y}sin\theta\\ -\tilde{x}sin\theta + \tilde{y}cos\theta\end{bmatrix} \tag{15}$$

Since the formula of Two-dimensional continuous Fourier Transform is

$$\mathscr{F}\{u, v\} = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(x, y)e^{-j2\pi(ux+vy)}dxdy \tag{16}$$

Thus,

$$\begin{aligned}\mathscr{F}\{f(\tilde{x}, \tilde{y})\} &= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(\tilde{x}, \tilde{y})e^{-j2\pi(ux+vy)}dxdy\\ &= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(\tilde{x}, \tilde{y})e^{-j2\pi[u(\tilde{x}cos\theta+\tilde{y}sin\theta)+v(-\tilde{x}sin\theta+\tilde{y}cos\theta)]}dxdy\\ &= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(\tilde{x}, \tilde{y})e^{-j2\pi[\tilde{x}(ucos\theta-vsin\theta)+\tilde{y}(usin\theta+vcos\theta)]}d\tilde{x}d\tilde{y}\end{aligned} \tag{17}$$

In the derivation above, we have made use of the fact that

$$d\tilde{x}d\tilde{y} = \begin{vmatrix}\frac{\partial\tilde{x}}{\partial x} & \frac{\partial\tilde{x}}{\partial y}\\ \frac{\partial\tilde{y}}{\partial x} & \frac{\partial\tilde{y}}{\partial y}\end{vmatrix}dxdy = \begin{vmatrix}cos\theta & sin\theta\\ -sin\theta & cos\theta\end{vmatrix}dxdy = dxdy \tag{18}$$

Where the matrix is the Jacobi matrix $|J|$. Hence,

$$\begin{aligned}\mathscr{F}\{f(\tilde{x}, \tilde{y})\}\mathscr{F}\{f(\tilde{x}, \tilde{y})\} &= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(\tilde{x}, \tilde{y})e^{-j2\pi[\tilde{x}(ucos\theta-vsin\theta)+\tilde{y}(usin\theta+vcos\theta)]}d\tilde{x}d\tilde{y}\\ &= F(ucos\theta - vsin\theta, usin\theta + vcos\theta)\end{aligned} \tag{19}$$

□

Method 2: The second proof method uses the rotation matrix.

Proof. Let $\mathbf{x}$=$\begin{pmatrix}x\\ y\end{pmatrix}$,$\mathbf{u}$=$\begin{pmatrix}u\\ v\end{pmatrix}$, then $f(\mathbf{x}) \leftrightarrow F(\mathbf{u})$,

$$F(\mathbf{u}) = \int_{-\infty}^{\infty} f(\mathbf{x})e^{-j2\pi\mathbf{x}^T\mathbf{u}}dx \tag{20}$$

when rotate counterclockwise θ,

$$\mathbf{x}' = \begin{pmatrix}cos\theta & -sin\theta\\ sin\theta & cos\theta\end{pmatrix}\mathbf{x} = \mathbf{R}\mathbf{x} \tag{21}$$

where R is an orthogonal matrix, $\mathbf{R}\mathbf{R}^T = \mathbf{I}$. So $x = R^{-1}x' = R^T x'$ we have

$$f(\mathbf{Rx}) = f(\mathbf{x}') \tag{22}$$

$$\begin{aligned}
\mathscr{F}\{f(xcos\theta - ysin\theta, xsin\theta + ycos\theta)\} &= \mathscr{F}\{f(\mathbf{Rx})\} \\
&= \int_{-\infty}^{\infty} f(\mathbf{x}')e^{-j2\pi\mathbf{x}^T\mathbf{u}}d\mathbf{x} \\
&= \int_{-\infty}^{\infty} f(\mathbf{x}')e^{-j2\pi(\mathbf{R}^T\mathbf{x}')^T\mathbf{u}}d\mathbf{x} \\
&= \int_{-\infty}^{\infty} f(\mathbf{x}')e^{-j2\pi(\mathbf{x}')^T\mathbf{R}\mathbf{u}}d\mathbf{x} \\
&= F(\mathbf{Ru}) = F(ucos\theta - vsin\theta, usin\theta + vcos\theta)
\end{aligned} \tag{23}$$

□

3.Consider the discrete approximation of the Laplacian given by the following convolution mask: Where, ϵ is the spacing between pixels. Write down this weighting scheme (mask) as a sum of nine impulse functions. Find the Fourier transform of this mask and evaluate it as $\epsilon \to 0$.
Answer:

$$h(x, y) = \delta(x, y) \tag{24}$$

thus, based on the Theorem in Problem 1, we get

$$h(x, y) \leftrightarrow 1 \tag{25}$$

$$\begin{aligned}
h(x-\epsilon, y-\epsilon) &\leftrightarrow e^{-j2\pi(u+v)\epsilon} \\
h(x-\epsilon, y) &\leftrightarrow e^{-j2\pi u\epsilon} \\
h(x-\epsilon, y+\epsilon) &\leftrightarrow e^{-j2\pi(u-v)\epsilon} \\
h(x, y-\epsilon) &\leftrightarrow e^{-j2\pi v\epsilon} \\
h(x, y+\epsilon) &\leftrightarrow e^{-j2\pi v\epsilon} \\
h(x+\epsilon, y-\epsilon) &\leftrightarrow e^{j2\pi(u-v)\epsilon} \\
h(x+\epsilon, y) &\leftrightarrow e^{j2\pi u\epsilon} \\
h(x+\epsilon, y+\epsilon) &\leftrightarrow e^{j2\pi(u+v)\epsilon}
\end{aligned} \tag{26}$$

so, the weighted scheme should be written as

$$\begin{aligned}
\frac{\partial^2 I}{\partial x^2} + \frac{\partial^2 I}{\partial y^2} \approx \frac{1}{6\epsilon^2}[&h(x-\epsilon, y-\epsilon) + 4*h(x-\epsilon, y) + h(x-\epsilon, y+\epsilon)+ \\
&4*h(x, y-\epsilon) - 20*h(x, y) + 4*h(x, y+\epsilon)+ \\
&h(x+\epsilon, y-\epsilon) + 4*h(x+\epsilon, y) + h(x+\epsilon, y+\epsilon)]
\end{aligned} \tag{27}$$

The corresponding Fourier Transform is also sum of each corresponded Fourier Transform:

$$\frac{1}{6\epsilon^2}[e^{-j2\pi(u+v)\epsilon} + 4e^{-j2\pi u\epsilon} + e^{-j2\pi(u-v)\epsilon} + \quad 4e^{-j2\pi v\epsilon} - 20 + 4e^{-j2\pi v\epsilon} + \\ e^{j2\pi(u-v)\epsilon} + 4e^{j2\pi u\epsilon} + e^{j2\pi(u+v)\epsilon}] \tag{28}$$

when $\epsilon \to 0$, we calculate the limit. By using L'Hôpital's rule two times we get

$$\begin{aligned} &\lim_{\epsilon \to 0} \frac{1}{6\epsilon^2}[e^{-j2\pi(u+v)\epsilon} + 4e^{-j2\pi u\epsilon} + e^{-j2\pi(u-v)\epsilon} + 4e^{-j2\pi v\epsilon} - 20 + 4e^{-j2\pi v\epsilon} + \\ &e^{j2\pi(u-v)\epsilon} + 4e^{j2\pi u\epsilon} + e^{j2\pi(u+v)\epsilon}] \\ &= \frac{-4\pi^2}{12}[(u+v)^2 + 4u^2 + (u-v)^2 + 4v^2 + (u-v)^2 + 4u^2 + (u+v)^2] \\ &= -\frac{16\pi^2}{3}(u^2+v^2) \end{aligned} \tag{29}$$

note that since $\epsilon \to 0$, all items involve exponent will equal to 1. PS: this is the fixed version, I saw the result should be $4\pi^2(u^2+v^2)$, I asked TA but he didn't show me the corrected procedure. The coefficient is not important, it may be that he miscalculated

4. Compute the following convolutions and write the solution in matrix form.

$$\begin{bmatrix} 1 & 4 & 6 & 4 & 1 \\ 4 & 16 & 24 & 16 & 4 \\ 6 & 24 & 36 & 24 & 6 \\ 4 & 16 & 24 & 16 & 4 \\ 1 & 4 & 6 & 4 & 1 \end{bmatrix} * \begin{bmatrix} 1 & -1 \end{bmatrix} \tag{30}$$

Answer:

First, by flipping the second matrix both vertically and horizontally, we get $[1 \quad -1]$. For the first raw of matrix, we align -1 of the flipped matrix and 1, the first element of the first matrix, we get the first element of the result: $-1*0+1*1=1$, note that for the first multiplication formula, we fill it with 0. We take the same operation to the rest, we get

$$\begin{bmatrix} -1*0+1*1 & -1*1+1*4 & -1*4+1*6 & -1*6+1*4 & -1*4+1*1 \\ -1*0+1*4 & -1*4+1*16 & -1*16+1*24 & -1*24+1*16 & -1*16+1*4 \\ -1*0+1*6 & -1*6+1*24 & -1*24+1*36 & -1*36+1*24 & -1*24+1*6 \\ -1*0+1*4 & -1*4+1*16 & -1*16+1*24 & -1*24+1*16 & -1*16+1*4 \\ -1*0+1*1 & -1*1+1*4 & -1*4+1*6 & -1*6+1*4 & -1*4+1*1 \end{bmatrix}$$

$\begin{bmatrix} -1*1+1*0 \\ -1*4+1*0 \\ -1*6+1*0 \\ -1*4+1*0 \\ -1*1+1*0 \end{bmatrix}$ The output is:

$$\begin{bmatrix} 1 & 3 & 2 & -2 & -3 & -1 \\ 4 & 12 & 8 & -8 & -12 & -4 \\ 6 & 18 & 12 & -12 & -18 & -6 \\ 4 & 12 & 8 & -8 & -12 & -4 \\ 1 & 3 & 2 & -2 & -3 & -1 \end{bmatrix} \tag{31}$$

$$\begin{bmatrix} 3 & 2 & 3 \\ 2 & 3 & 2 \\ 1 & 2 & 3 \end{bmatrix} * \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \tag{32}$$

flip the second matrix, we get $\begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix}$, then do the same operation to the rest

$$\begin{bmatrix} -1*0+0*0+0*0+1*3 & -1*0+0*0+0*3+1*2 & -1*0+0*0+0*2+1*3 & -1*0+0*0+0*3+1*0 \\ -1*0+0*3+0*0+1*2 & -1*3+0*2+0*2+1*3 & -1*2+0*3+0*3+1*2 & -1*3+0*0+0*2+1*0 \\ -1*0+0*2+0*0+1*1 & -1*2+0*3+0*1+1*2 & -1*3+0*2+0*2+1*3 & -1*2+0*0+0*3+1*0 \\ -1*0+0*1+0*0+1*0 & -1*1+0*2+0*0+1*0 & -1*2+0*3+0*0+1*0 & -1*3+0*0+0*0+1*0 \end{bmatrix}$$

the output is

$$\begin{bmatrix} 3 & 2 & 3 & 0 \\ 2 & 0 & 0 & -3 \\ 1 & 0 & 0 & -2 \\ 0 & -1 & -2 & -3 \end{bmatrix} \tag{33}$$

5.One way to find edges in an image, $I(x, y)$, is to find the zero-crossings in the Laplacian of the image function. It however suffers from a poor signal to noise performance. One might instead use the second derivative of brightness in the direction of the brightness gradient.
(a) Given a unit vector in the direction of the brightness gradient, find the sine and cosine of the angle θ between this vector and the x-axis. Hint: depending on your approach to this problem, you might need the trigonometric identities $(1+tan^2\theta) = sec^2\theta$ and $cos^2\theta+sin^2\theta = 1$.
(b) What is the first directional derivative of brightness in the direction of the brightness gradient? How is this related to the magnitude of the brightness gradient?
(c) Express the second directional derivative, $I^{''}(x, y)$ (in any direction v) in terms of the first and second partial derivatives with respect of x and y.

Answer:
(a) The unit vector in the direction of the brightness gradient

$$\vec{u} = \frac{(\frac{\partial I}{\partial x}, \frac{\partial I}{\partial y})^T}{\sqrt{(\frac{\partial I}{\partial x})^2 + (\frac{\partial I}{\partial y})^2}} \tag{34}$$

$$\begin{aligned} cos\theta &= \frac{\left|\frac{\partial I}{\partial x}\right|}{\sqrt{(\frac{\partial I}{\partial x})^2 + (\frac{\partial I}{\partial y})^2}} \\ sin\theta &= \frac{\left|\frac{\partial I}{\partial y}\right|}{\sqrt{(\frac{\partial I}{\partial x})^2 + (\frac{\partial I}{\partial y})^2}} \end{aligned} \tag{35}$$

(b) The first dictional derivative of brightness in the direction $\vec{u}$ is

$$\begin{aligned} I' = \frac{dI}{ds} &= I_x cos\theta + I_y sin\theta \\ &= \frac{(\frac{\partial I}{\partial x})^2 + (\frac{\partial I}{\partial y})^2}{\sqrt{(\frac{\partial I}{\partial x})^2 + (\frac{\partial I}{\partial y})^2}} \\ &= \sqrt{(\frac{\partial I}{\partial x})^2 + (\frac{\partial I}{\partial y})^2} \end{aligned} \tag{36}$$

which is the magnitude of the brightness gradient.
(c)

$$\begin{aligned} I'' &= \frac{d^2 I}{ds^2} = \frac{d\frac{dI}{ds}}{ds} \\ &= \frac{\partial\frac{dI}{ds}}{\partial x} cos\theta + \frac{\partial\frac{dI}{ds}}{\partial y} sin\theta \\ &= (I_{xx} cos\theta + I_{yx} sin\theta) cos\theta + (I_{xy} cos\theta + I_{yy} sin\theta) sin\theta \\ &= I_{xx} cos^2\theta + 2 I_{yx} sin\theta cos\theta + I_{yy} sin^2\theta \end{aligned} \tag{37}$$

Substitute it with the first and second partial derivatives with respect of x and y, we get:

$$\begin{aligned} I'' &= I_{xx} cos^2\theta + 2 I_{yx} sin\theta cos\theta + I_{yy} sin^2\theta \\ &= \frac{\partial^2 I}{\partial x^2} \frac{(\frac{\partial I}{\partial x})^2}{(\frac{\partial I}{\partial x})^2 + (\frac{\partial I}{\partial y})^2} + 2\frac{\partial^2 I}{\partial x \partial y} \frac{(\frac{\partial I}{\partial x})(\frac{\partial I}{\partial y})}{(\frac{\partial I}{\partial x})^2 + (\frac{\partial I}{\partial y})^2} + \frac{\partial^2 I}{\partial y^2} \frac{(\frac{\partial I}{\partial y})^2}{(\frac{\partial I}{\partial x})^2 + (\frac{\partial I}{\partial y})^2} \\ &= \frac{1}{(\frac{\partial I}{\partial x})^2 + (\frac{\partial I}{\partial y})^2} \left[\frac{\partial^2 I}{\partial x^2} \cdot (\frac{\partial I}{\partial x})^2 + 2\frac{\partial^2 I}{\partial x \partial y} \cdot \frac{\partial I}{\partial x}\frac{\partial I}{\partial y} + \frac{\partial^2 I}{\partial y^2} \cdot (\frac{\partial I}{\partial y})^2 \right] \end{aligned} \tag{38}$$

For solving the problem 3 and 5, I referenced the chapter 8 of this book[1].

References

[1] B. Horn, B. Klaus, and P. Horn, *Robot vision.* MIT press, 1986.