Projective Geometry

Baba C. Vemuri

Professor, Department of Computer & Information Science and Engineering
University of Florida

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Homogeneous Linear Systems

- Let $A \in \Re^{(m \times n)}$ and $\mathbf{x} \in \Re^n$, $Rank(A) = (n-1)$, find the non-trivial solution for $A\mathbf{x} = 0$. Note, the trivial solution is $\mathbf{x} = \mathbf{0}$.
- A solution unique upto a scale factor is easily found by using SVD decomposition.
- **Theorem:** This solution is proportional to the eigen-vector corresponding to the only zero eigen value of $A^t A$ (all other eigen values are strictly positive, why?).

Proof of the Theorem

- Look for a unit-norm solution in the least-squares sense, since norm of the solution otherwise can be arbitrary.
- Therefore, minimize $||A\mathbf{x}||^2 = (A\mathbf{x})^t A\mathbf{x} = \mathbf{x}^t A^t A\mathbf{x}$, subject to $\mathbf{x}^t\mathbf{x} = 1$. Using Lagrange multiplier λ, this is equivalent to minimizing,

$$\mathcal{L}(\mathbf{x}) = \mathbf{x}^t A^t A\mathbf{x} - \lambda(\mathbf{x}^t\mathbf{x} - 1) \tag{1}$$

- Taking the gradient of the above eqn. and equating to zero, we get $A^t A\mathbf{x} - \lambda\mathbf{x} = 0$.
  $\Rightarrow \lambda$ is the eigen value of $A^t A$ and $\mathbf{x} = \mathbf{e}_\lambda$ the corresponding eigen vector.

Proof (Contd.)

- Replacing $\mathbf{x}$ with $\mathbf{e}_\lambda$ and $A^t A\mathbf{e}_\lambda$ with $\lambda\mathbf{e}_\lambda$ in 1, we get
- $\mathcal{L}(\mathbf{e}_\lambda) = \lambda \Rightarrow$ minimum is reached at $\lambda = 0$, the least eigen value of $A^t A$.
- Thus the solution is the column of V in the SVD of A that corresponds to the only singular value of A. Why columns of V? This complets the proof.

Singular Value Decomposition (SVD)

- Let $A \in \Re^{(m \times n)}$, $U \in \Re^{(m \times m)}$ of Orthogonal matrices, $\Sigma \in \Re^{(m,n)}$ be diagonal containing the singular values, and $V \in \Re^{(n \times n)}$ of Orthogonal matrices, then, $A = U \Sigma V^t$, with $\sigma_1 \geq \sigma_2 ... \geq 0$. U and V have unit norm columns.
- **Proposition:** Columns of U corresponding to the nonzero singular values span the range of A, columns of V corresponding to the zero singular values span the null space of A.

Class-1: Isometries

- Transforms that preserve distances in the plane $\Re^2$. Example: Rotation, rigid (rotation+tranlation) transformation.

$$\begin{pmatrix} x' \\ y' \\ 1 \end{pmatrix} = \begin{pmatrix} \epsilon\cos\theta & -\sin\theta & t_x \\ \epsilon\sin\theta & \cos\theta & t_y \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} x \\ y \\ 1 \end{pmatrix} \qquad (2)$$

- $\epsilon = 1 \Rightarrow$ orientation preserving, $\epsilon = -1 \Rightarrow$ orientation reversing (reflection).

Rigid Motion

- Can write 2 as

$$\mathbf{x}' = H_e \mathbf{x} = \begin{pmatrix} R & \mathbf{t} \\ \mathbf{0}^t & 1 \end{pmatrix} \mathbf{x}$$

  $R : (2,2)$ matrix; $R^t R = I = RR^t$ $\mathbf{t} = (2,1)$-vector; $\mathbf{0} : (2,1)$ null-vector.
- DOF: 3; can compute H_e from 2-point correspondences.
- **Invariants:** lengths, angle and area.

Class-II: Planar Similarity

- Isotropic scaling:

$$\begin{pmatrix} x' \\ y' \\ 1 \end{pmatrix} = \begin{pmatrix} s\cos\theta & -s\sin\theta & t_x \\ s\sin\theta & s\cos\theta & t_y \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} x \\ y \\ 1 \end{pmatrix} \tag{3}$$

- Consisely,

$$\mathbf{x}' = H_s\mathbf{x} = \begin{pmatrix} sR & \mathbf{t} \\ \mathbf{0}^t & 1 \end{pmatrix} \mathbf{x} \tag{4}$$

  Where, s is an isotropic scaling factor.
- DOF: 4, can be computed from 2 point correspondences.
- **Invariants:** (i) Angles between lines, (ii) ratio of two lengths (iii) ratio of areas.

Class-III: Affine Transforms

- These are non-singular linear transforms followed by translation.

$$\begin{pmatrix} x' \\ y' \\ 1 \end{pmatrix} = \begin{pmatrix} a_{11} & a_{12} & t_x \\ a_{21} & a_{22} & t_y \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} x \\ y \\ 1 \end{pmatrix} \tag{5}$$

- Consisely,

$$\mathbf{x}' = H_a \mathbf{x} = \begin{pmatrix} A & \mathbf{t} \\ \mathbf{0}^t & 1 \end{pmatrix} \mathbf{x} \tag{6}$$

  A is a non-singular linear transformation.

- Can represent $A = R(\theta)R(-\phi)DR(\phi)$, why? Hint:.....

Affine Transform (Contd.)

- **Invariants:** (i) Parallel lines map to parallel lines.
  - Two parallel lines intersect at an ideal point $(x_1, x_2, 0)^t$.
  - Under an Affine transform, it is mapped to another ideal point $\Rightarrow$ parallel lines remain parallel.
- Ratio of lengths of parallel line segments is unchanged (prove it).
- Ratio of areas is unchaged (prove it).

Projective Transformations

- It is a general non-singular linear transform of homogeneous coordinates. $\mathbf{x}' = H_p\mathbf{x}$.

$$\mathbf{x}' = \begin{pmatrix} A & \mathbf{t} \\ \mathbf{v}^t & v \end{pmatrix} \mathbf{x} \tag{7}$$

- Where, $\mathbf{v} = (v_1, v_2)^t$. It is not in general possible to scale the matrix to make $v = 1$ as v could be 0.
- Note that Affine transforms map ideal points to ideal points but projective transforms DON'T.

Decomposition of Projective Transforms

- Projective transforms can be decomposed as:

$$H = H_s H_a H_p = \begin{pmatrix} sR & \mathbf{t} \\ \mathbf{0}^t & 1 \end{pmatrix} \begin{pmatrix} K & \mathbf{0} \\ \mathbf{0}^t & 1 \end{pmatrix} \begin{pmatrix} I & \mathbf{0} \\ \mathbf{v}^t & v \end{pmatrix} = \begin{pmatrix} A & \mathbf{t} \\ \mathbf{v}^t & v \end{pmatrix} \tag{8}$$

  Where, $A = sRK + \mathbf{t}\mathbf{v}^t$, K is an upper triangular matrix with $det(K) = 1$. Decomposition is valid if $v \neq 0$ and unique if $s > 0$.
- DOF= 8.

Invariants of Projective Transform

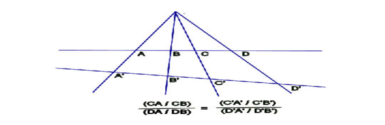

Figure: Cross Ratio as an Invariant

- Cross Ratio is the ratio that is preserved between two sets of points that differ by a projectivity (projective transform).

$$Cross(x_1', x_2', x_3', x_4') = Cross(x_1, x_2, x_3, x_4) \tag{9}$$

$$Cross(x_1, x_2, x_3, x_4) = \frac{|x_1 x_2||x_3 x_4|}{|x_1 x_3||x_2 x_4|} \tag{10}$$

$$|x_i x_j| = det \begin{pmatrix} x_{i1} & x_{j1} \\ x_{i2} & x_{j2} \end{pmatrix} \tag{11}$$

Back to DLT and Variants

- In DLT, we minimize $||A\mathbf{h}||^2$ and SVD to solve the ensuing homogenous linear system. This is the "Algebraic distance" as a cost function, $d_{alg}(\mathbf{x}_i', H\mathbf{x}_i) = ||\epsilon_i||^2 = ||A_i\mathbf{h}||^2$.
- Geometric distance: also called transfer error (we fix one of the images as a calibration pattern wehre measurements are highly accurate) – $\underset{H}{argmin} \sum_i \left(d(\mathbf{x}_i, H^{-1}\mathbf{x}_i')\right)^2$
- Symmetric transfer error: $\underset{H}{argmin} \sum_i \left(d(\mathbf{x}_i, H^{-1}\mathbf{x}_i')\right)^2 + \sum_i \left(d(\mathbf{x}_i', H\mathbf{x}_i)\right)^2$.

Reprojection Error

- Points $\mathbf{x}$ and $\mathbf{x}'$ are measured noisy points.
- Under estimated homography, $\mathbf{x}'$ and $H\mathbf{x}$ do not correspond perfectly and neither do $\mathbf{x}$ and $H^{-1}\mathbf{x}'$.
- However, $\hat{\mathbf{x}}$ and $\hat{\mathbf{x}}'$ match perfectly via $\hat{\mathbf{x}}' = \hat{H}\hat{\mathbf{x}}$.
- Hence we want to minimize:

$$\underset{H,\hat{\mathbf{x}}_i',\hat{\mathbf{x}}_i'}{argmin} \Sigma_i \left(d(\mathbf{x}_i, \hat{\mathbf{x}}_i)\right)^2 + \left(d(\mathbf{x}_i', \hat{\mathbf{x}}_i')\right)^2$$

Such that, $\hat{\mathbf{x}}_i' = \hat{H}\hat{\mathbf{x}}_i \cdots \forall i$.