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Homework 1

CAP 5416
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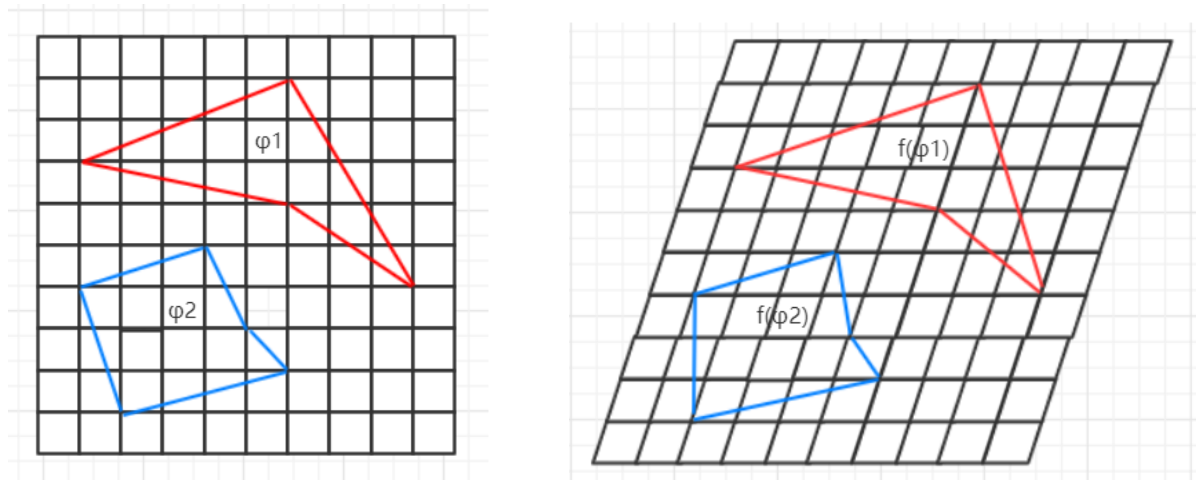
1 Solutions for Homework1

1. Prove that ratio of areas is invariant to planar affine transforms. Is this ratio invariant for orientation reversing affinities (affine transforms) as well?



(a) original coordination

(b) affined coordination

Figure 1: An affine transformation example.

Proof. First, impose upon the plane a grid of congruent squares. (See Figure.1(a).) Let $f(\vec{x}) = \mathbf{A}\vec{x} + \vec{b}$ be an affine transformation. Then f:

1. maps a line to a line,
2. maps a line segment to a line segment,
3. preserves the property of parallelism among lines and line segments,
4. maps an n-gon to an n-gon,
5. preserves the ratio of lengths of two parallel segments.

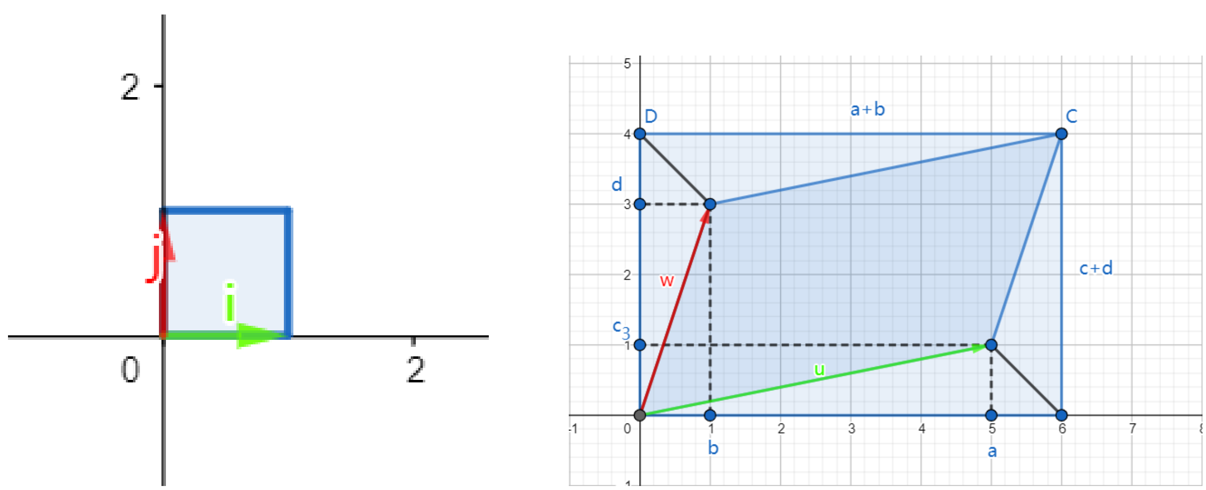
The first four properties imply that an affine transformation f will map this grid of squares into a grid of parallelograms, the last one implies that these parallelograms are all congruent to each other. (See Figure 1(b).)

Let φ_1 and φ_2 be two figures in the plane, with images $f(\varphi_1)$ and $f(\varphi_2)$, respectively, under the map. If the grid of squares is sufficiently fine, then the ratio of the number of squares in the interior of φ_1 to the number of squares in the interior of φ_2 will infinitely close to the ratio $Area(\varphi_1)/Area(\varphi_2)$. Similarly, the ratio of the number of parallelograms in the interior of $f(\varphi_1)$ to the number of parallelograms in the interior of $f(\varphi_2)$ will infinitely close to the ratio $Areaf(\varphi_1)/Areaf(\varphi_2)$.

Now, restate this question in an equivalent way: for every affine transformation f , does there exists a positive real number k such that the area of every figure is altered by a factor of k? i.e., $Area(f(\varphi)) = k \cdot Area(\varphi)$.

In order to find k, we may concentrate on the change of area of the unit square defined by vectors $\vec{i}$ and $\vec{j}$, shown in Figure.2(a). As previously noted, if $\mathbf{A} = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ is the 2×2 matrix corresponding to an affine transformation f , the first column of $\mathbf{A}$ is $\vec{v} = f(\vec{i})$ and the second column is $\vec{w} = f(\vec{j})$. Under f, the unit square with sides given by $\vec{i}$ and $\vec{j}$ is mapped to a parallelogram with sides defined by $\vec{v} = \langle a, c \rangle$ and $\vec{w} = \langle b, d \rangle$. The area of the parallelogram can be found by subtracting the areas of two pairs of congruent triangles from the area of a rectangle. This is pictured in Figure.2(b) for the case when $a > b > 0$, and $d > c > 0$. Therefore, the area of the parallelogram is:

$$(a+b)(c+d) - 2\left(\frac{1}{2}(a+b)c\right) - 2\left(\frac{1}{2}(c+d)b\right) = ad - bc = det\mathbf{A} \tag{1}$$



(a) (b)

Figure 2: Unit square is mapped to a parallelogram in an affine transformation.

By similar arguments one can show that essentially the same result holds if we remove the conditions imposed on a, b, c, and d. More precisely, the unit square defined by $\vec{i}$ and $\vec{j}$ is always mapped to a parallelogram having area equal to $|det(\mathbf{A})|$. From this we conclude that the area of any figure is altered by a factor equalling the absolute value of the determinant of $\mathbf{A}$ under the transformation f. If the transformation is orientation-reversing, $det(\mathbf{A}) < 0$, but the ratio is still invariant. $\square$

2. Given a projective transformation that takes points $\mathbf{x} \to \mathbf{x'}$ via $\mathbf{x'} = H\mathbf{x}$, prove that $Cross(x'_1, x'_2, x'_3, x'_4) = Cross(x_1, x_2, x_3, x_4)$ i.e., the cross ratio is a projective invariant.

Proof. we know:

$$Cross(x_1, x_2, x_3, x_4) = \frac{|x_1 x_2|\,|x_3 x_4|}{|x_1 x_3|\,|x_2 x_4|} \tag{2}$$

where$|x_i x_j|$ represents a determinant with columns x_i and x_j. Then we have

$$\left|x'_i x'_j\right| = |Hx_i Hx_j| = det(H)\,|x_i x_j| \tag{3}$$

this gives:

$$\begin{aligned} Cross(x'_1, x'_2, x'_3, x'_4) &= \frac{|x'_1 x'_2|\,|x'_3 x'_4|}{|x'_1 x'_3|\,|x'_2 x'_4|} \\ &= \frac{det(H)^2\,|x_1 x_2|\,|x_3 x_4|}{det(H)^2\,|x_1 x_3|\,|x_2 x_4|} \\ &= \frac{|x_1 x_2|\,|x_3 x_4|}{|x_1 x_3|\,|x_2 x_4|} = Cross(x_1, x_2, x_3, x_4) \end{aligned} \tag{4}$$

Canceling all determinants is feasible, since H was assumed to be invertible. The equality of the leftmost and the rightmost term is exactly the claim. □

3. Show that a line at infinity l_∞ remains a line at infinity under the projectivity H if H is an affinity.**Bonus**: Is the converse true, if no (yes) then, prove it.

Proof. Under the point transformation $x' = Hx$, a line transforms as $l' = H^{-T}l$, thus we have

$$l'_\infty = H_A^{-T} l_\infty = \begin{bmatrix} A^{-T} & 0 \\ -t^T A^{-T} & 1 \end{bmatrix} \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} = l_\infty \tag{5}$$

□

The converse is also true, i.e. an affine transformation is the most general linear transformation that fixes l_∞.

Proof. We require that a point at infinity, say $x = (1, 0, 0)^T$, be mapped to a point at infinity. This requires that $h_{31} = 0$. Similarly, $h_{32} = 0$, so the transformation is an affinity. □

4. In DLT, if $\|A\mathbf{h}\|$ is minimized subject to the constraint $h_9 = H_{33} = 1$, then the result is invariant to scaling but not translation of coordinates.

Proof. Since $\|\mathbf{A}h\|$ is minimized subject to the constraint $h_9 = H_{33} = 1$, $\mathbf{A}h$=$\mathbf{0}$ can be written as:

$$\begin{bmatrix} 0 & 0 & 0 & -x_i w_i' & -y_i w_i' & -w_i w_i' & x_i y_i' & y_i y_i' \\ x_i' w_i & -y_i' w_i & -w_i' w_i & 0 & 0 & 0 & -x_i x_i' & -y_i x_i' \end{bmatrix} \tilde{h} = \begin{bmatrix} w_i y_i' \\ -w_i x_i' \end{bmatrix} \tag{6}$$

where $\tilde{h}$ is a 8-vector made of the first eight entries of H matrix and $X_i = (x_i, y_i, w_i)^t$. With more that 4 point pair we may solve $\min_{\tilde{h}} \left\| M\tilde{h} - b \right\|$ with a least squares method. For scaling, we have scaling factor s for A, we should find the eigenvalue of the equation $det(sA^T sA) = det(sA^T)det(sA) = s^4 det(A) = 0$. The scaling factor s occurs no influence on the eigenvalue. For translation, $det((A+T)^T(A+T)) \neq det(A^T A)$, so the eigenvalue is different, thus,the result is invariant to scaling but not translation of coordinates.

Plus, if the coordinate origin is mapped to a point at infinity by H.Since $(0,0,1)^T$ represents the coordinate origin x_0, and also $(0,0,1)^T$ represents the line at infinity l, this condition may be written as $l^T H x_0 = (0,0,1)H(0,0,1)^T = 0$, thus $H_{33} = 0$, which means t this method works poorly if the correct solution has $H_{33} = 0$. □

5. Prove that an image line l defines a plane through the camera center with normal direction $\mathbf{n} = K^t l$ measured in the cameras Euclidean coordinate frame. The matrix K is the camera matrix.

Proof. Let $\tilde{\mathbf{X}} = \lambda \mathbf{d}$ be the back projected points of x, then

$$\mathbf{x} = \mathbf{K}[\text{-}\mathbf{I}|\mathbf{0}](\lambda \mathbf{d}^T, 1)^T = \mathbf{K}\mathbf{d} \tag{7}$$

so

$$\mathbf{d} = \mathbf{K}^{-1} x \tag{8}$$

Since $\mathbf{d}$ is on the plane, we have

$$\begin{aligned} \mathbf{d}^T \mathbf{n} = 0 &\Rightarrow (\mathbf{K}^{-1} x)^T \mathbf{n} = 0 \\ &\Rightarrow x^T (\mathbf{K}^{-t} \mathbf{n}) = 0 \\ &\Rightarrow l = \mathbf{K}^{-t} \mathbf{n} \end{aligned} \tag{9}$$

where l contains the image point x. □

Reference are listed as follows[1].

References

[1] R. Hartley and A. Zisserman, *Multiple view geometry in computer vision.* Cambridge university press, 2003.