Fourier Transforms

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1 Image Processing (Chapter 3 in Szeliski and 6 in Horn)

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Fourier Transforms

- Here's a fun and very intutive (Smoothies and Recipes!) explanation of Fourier Transforms
  http://betterexplained.com/articles/an-interactive-guide-to-the-fourier-transform/
- One way to think: FT is an expansion of a function (image) in terms of a sum of complex exponentials. Where, the complex exponentials serve the purpose of basis functions from a linear algebraic view point.

$$F(u,v) = \iint_{-\infty}^{\infty} f(x,y)\exp\{-j2\pi(ux+vy)\}dxdy \qquad (1)$$

$$f(x,y) = \iint_{-\infty}^{\infty} F(u,v)\exp\{j2\pi(ux+vy)\}dudv \qquad (2)$$

Explanation

- Existence: (i) f must be absolutely integrable over the entire (x, y) (picture) plane.(ii) f must have only a finite number of discontinuities and a finite number of maxima and minima in any finite rectangle. (iii) f must have no infinite discontinuities.
- View equation 2 as an expansion of the picture function in terms of a sum of complex exponentials.
- For each pair of “spatial frequencies”, (u, v), we have ONE exponential in the sum with a weighting coefficient $F(u, v)$ from 1.
- Hence, the Fourier Transform of f is merely the weighting coefficients in the expansion of f in a sum of complex exponentials.

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More Explanation

- How does $\exp\{2\pi j(ux + vy)\}$ look?

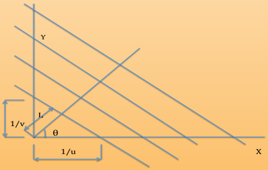

Figure: Zero Phase Plot

- It's complex valued. It's the locus of points in (x, y) plane for which its real and positive value is obtained by seting,

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$$2\pi j(ux + vy) = 2\pi jn \;\; \forall \;\; \text{integer values of } n$$
$$\Rightarrow y = -\frac{u}{v}x + \frac{n}{v}$$

- This is a set of parallel lines whose spatial period (distance between them) $L = \frac{1}{\sqrt{(u^2+v^2)}}$ and the perpendicular to them is oriented at $\theta = \tan^{-1}(\frac{v}{u})$.
- Higher the spatial frequencies $\Rightarrow$ closer together are the lines in the zero phase plot.

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- Two geometrical aspects associated with a point (u, v) inthe spatial frequency plane.
  - An orientation θ and
  - Spacing L.
- If the FT $\mathcal{F}(g(x, y)) = G(u, v)$ has a large magnitude at some particular spatial frequency (u, v), then, $G(u, v)\exp\{2\pi j(ux + vy)\}$ will be a large contribution in the IFT.
- Since $g(x, y)$ is real valued, we have $G(u, v) = G^*(-u, -v)$, where G^* is the complex conjugate of G. Fact: Complex conjugates have same magnitude. Hence, $G(-u, -v)$ has same magnitude as $G(u, v)$.

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- Hence, $G(-u,-v)\exp\{2\pi j(ux+vy)\}$ will also have an important contribution to the IFT.
- If $G(u,v)$ is small except at (u,v) and $(-u,-v)$, then the picture looks like,

  $$G(u,v)\exp\{2\pi j(ux+vy)\}+G(-u,-v)\exp\{2\pi j(-ux-vy)\}$$

  since this dominates the IFT equation. Above equation is purely real.
- If plotted, its a sinusoidally undulating surface whose *crests* are a set of parallel lines. Hence, each symmetric pair of spatial frequencies (u,v) and $(-u,-v)$ contributes a picture consisting of sinusoidally varying intensity.
- Greater the magnitude of the transform, greater is this contribution.

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Examples

Prove the following:

$$\mathcal{F}\left(\frac{1}{2}\delta(x+a,y)+\frac{1}{2}\delta(x-a,y)\right)=\cos(2\pi au) \qquad (3)$$

$$\text{LHS of (3)} = \iint_{-\infty}^{\infty}\frac{1}{2}\delta(x+a,y)\exp\{-j2\pi(ux+vy)\}dxdy =$$

$$\frac{1}{2}\int_{-\infty}^{\infty}\left(\int_{-\infty}^{\infty}\delta(x+a)\exp\{-j2\pi ux\}dx\right)\delta(y)\exp\{-j2\pi vy\}dy$$

$$= (1/2)\exp\{j2\pi au\}\cdot 1 \qquad (4)$$

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Example(Contd.)

- Note that the FT of $\delta(y)$ is 1 (can show using limiting definition of the delta function).
- Similarly, we can show that the FT of the second term $(1/2)\delta(x-a,y)$ is $(1/2)\exp\{-j2\pi au\}$ i.e.,

$$\mathcal{F}\left(\frac{1}{2}\delta(x-a,y)\right)=\frac{1}{2}\exp\{-j2\pi au\} \tag{5}$$

- Summing equations (4) and (5), we get the desired result of $\cos(2\pi au)$.

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Example FTs

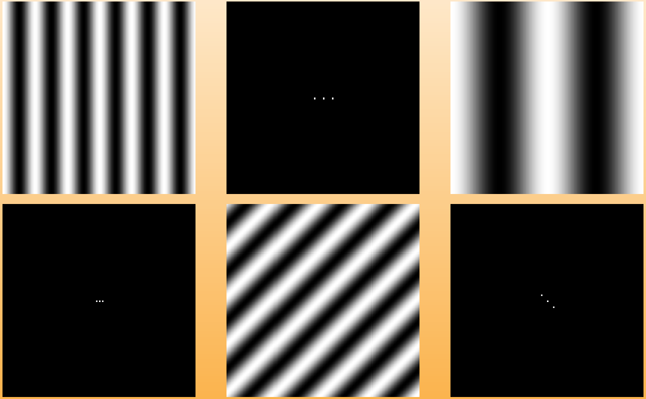

Figure: Images and their FTs

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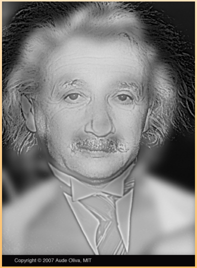
Hybrid Images: Blending Frequencies



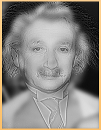


Figure: (L) Hybrid Image of Monroe & Einstien, (R) Down sampled version. courtesy: Aude Oliva -MIT

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