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Homework 3

CAP 5416
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1. Given n data points $\{x_i, y_i\}_{i=1,..,n}$ such that $f(x_i) = y_i$, how many unknowns are there if a quadratic spline is to be fitted to this data? Deduce the conditions required to solve the unknowns. Do you have enough conditions? If not, what conditions can you impose?

Answer:

To derive a mathematical model of a quadratic spline, suppose the data are$\{(x_i, f_i)\}_{i=0}^{n}$, where, as for linear splines,

$$a = x_0 < x_1 < \cdots < x_n = b, h \equiv \max_i |x_i - x_{i-1}| \tag{1}$$

A quadratic spline $S_{2,n}(x)$ is a C^1 piecewise quadratic polynomial. This means that:

1. $S_{2,n}(x)$ is piecewise quadratic; that is, between consecutive knots x_i,

$$S_{2,n}(x) = \begin{cases} p_1(x) = a_1 + b_1 x + c_1 x^2, & x \in [x_0, x_1], \\ p_2(x) = a_2 + b_2 x + c_2 x^2, & x \in [x_1, x_2], \\ \vdots & \\ p_n(x) = a_n + b_n x + c_n x^2, & x \in [x_{n-1}, x_n]; \end{cases} \tag{2}$$

2. $S_{2,n}(x)$ is C^1; that is, $S_{2,n}(x)$ is continuous and has continuous first derivative everywhere in the interval $[a, b]$, in particular, at the knots.

For $S_{2,n}(x)$ to be an interpolatory quadratic spline, we must also have:

$S_{2,n}(x)$ interpolates the data, that is,

$$S_{2,n}(x_i) = f_i, \quad i = 0, 1, \cdots, n \tag{3}$$

Within each interval (x_{i-1}, x_i), the corresponding quadratic polynomial is continuous and has continuous derivatives of all orders. Therefore, $S_{2,n}(x)$ or one of its derivatives can be discontinuous only at a knot. Observe that the function $S_{2,n}(x)$ has two quadratic pieces incident at the interior knot x_i; to the left of x_i, it is a quadratic $p_i(x)$ while to the right it is a quadratic $p_{i+1}(x)$. Thus, a necessary and sufficient condition for $S_{2,n}(x)$ to have continuous first derivative is for these two quadratic polynomials incident at the interior knot to match in first derivative value. So we have a set of smoothness conditions: that is, at each interior knot,

$$p'_i(x_i) = p'_{i+1}(x_i), i = 1, 2, \cdots, n-1 \tag{4}$$

In addition, to interpolate the data, we have a set on interpolation conditions: that is, on the i-th interval,

$$p_i(x_{i-1}) = f_{i-1}, \quad p_i(x_i) = f_i \quad i = 1, 2, \cdots, n \tag{5}$$

This way of writing the interpolation conditions also forces $S_{2,n}(x)$ to be continuous at the knots.

Since each of the n quadratic pieces has three unknown coefficients, our description of the function $S_{2,n}(x)$ involves $3n$ unknown coefficients. Assuring continuity of the first derivative imposes $(n-1)$ linear constraints on its coefficients, and interpolation imposes an additional $2n$ linear constraints. Therefore, there are a total of $3n-1$ linear constraints on the $3n$ unknown coefficients. In order that we have the same number of equations as unknowns, we need 1 more (linear) constraints, such as $p_1'(x_0) = 0$.

To construct an interpolatory quadratic spline, we first define the numbers

$$z_i = S_{2,n}'(x_i), 0 \leq i \leq n. \tag{6}$$

Since $S_{2,n}(x)$ is a quadratic spline, $S_{2,n}'(x)$ is a linear spline (by checking the definitions). Therefore $S_{2,n}'(x)$ is given by the straight line joining the points (x_{i-1}, z_{i-1}) and (x_i, z_i):

$$p_i'(x) = z_{i-1} + \frac{z_i - z_{i-1}}{h_i}(x - x_{i-1}) \tag{7}$$

where $h_i \equiv x_i - x_{i-1}$. If this expression is integrated once, we obtain

$$p_i(x) = z_{i-1}(x - x_{i-1}) + \frac{z_i - z_{i-1}}{2h_i}(x - x_{i-1})^2 + C_i \tag{8}$$

where C_i is constant of integration. The interpolation conditions $p_i(x_{i-1}) = f(x_{i-1})$ and $p_i(x_i) = f(x_i)$ give

$$C_i = f(x_{i-1}), i = 1, 2, \cdots n \tag{9}$$

and

$$p_i(x) = z_{i-1}h_i + \frac{z_i - z_{i-1}}{2}h_i + C_i = f(x_i), \quad i = 1, 2, \cdots, n \tag{10}$$

or equivalently

$$\frac{z_i + z_{i-1}}{2}h_i = f(x_i) - f(x_{i-1}), \quad i = 1, 2, \cdots, n \tag{11}$$

This equation is a system of n linear equations for the $n+1$ unknowns $z_0, z_1, \cdots, z_n$. To obtain one additional equation, we apply the endpoint derivative conditions, e.g. $p_1'(x_0) = 0$ which gives $z_0 = 0$.

Now we obtain the system of linear equations, $A\mathbf{z} = \mathbf{d}$, where

$$\mathbf{z} = [z_0, z_1, \cdots, z_n]^T, \quad \mathbf{d} = [d_0, d_1, \cdots, d_n]^T \tag{12}$$

with

$$d_i = \begin{cases} 0, i = 0, \\ \frac{2(f(x_i) - f(x_{i-1}))}{x_i - xi - 1}, 1 \leq i \leq n \end{cases} \tag{13}$$

and A is the tridiagonal matrix

$$A = \begin{bmatrix} 1 & 0 & 0 & \cdots & 0 \\ 1 & 1 & 0 & \cdots & 0 \\ 0 & 1 & 1 & \cdots & 0 \\ \vdots & \ddots & \ddots & \ddots & \vdots \\ 0 & \cdots & 0 & 1 & 1 \end{bmatrix} \tag{14}$$

The system can be solved, therefore the quadratic spline function (8).

2. Given a set of sample measurements of a one dimensional curve, $f(x)$, in the image plane, what is the purpose of the following minimization:

$$\min_S E(S) = \min_S \int \{\lambda(S''(x))^2 + (f(x) - S(x))^2 \sum_k \delta(x - x_k)\} dx \tag{15}$$

Describe the significance of each term on the right hand side of the above equation. Assume that λ is a constant regularization parameter and write down the Euler-Lagrange equation for this minimization problem.

Answer:

The first term is the regularization term, it measures the amount of noise in the image and minimizing this term encourages the removal of noise in the restored image.

The second term is data fidelity term, it encourages the restored image to be close to the observed noisy image f.

The purpose is to find $S(x)$ which is smooth and best approximate $f(x)$.

Solution for Euler-Lagrange equation:

$$L(x, S(x), S'(x), S''(x)) = \lambda(S''(x))^2 + (f(x) - S(x))^2 \sum_k \delta(x - x_k) \tag{16}$$

In dimension n = 1, the Euler-Lagrange equation is

$$\begin{aligned} &L_{S(x)}(x, S(x), S'(x), S''(x), \cdots) - \frac{d}{dx} L_{S'(x)}(x, S(x), S'(x), S''(x), \cdots) \\ &+ \frac{d^2}{dx^2} L_{S''(x)}(x, S(x), S'(x), S''(x), \cdots) - \cdots = 0 \end{aligned} \tag{17}$$

Since the formula only contain $S(x), S'(x), S''(x)$, It should be written as

$$\begin{aligned} &L_{S(x)}(x, S(x), S'(x), S''(x)) - \frac{d}{dx} L_{S'(x)}(x, S(x), S'(x), S''(x)) \\ &+ \frac{d^2}{dx^2} L_{S''(x)}(x, S(x), S'(x), S''(x)) = 0 \end{aligned} \tag{18}$$

We calculate it one by one.

$$L_{S(x)}(x, S(x), S'(x), S''(x)) = -2(f(x) - S(x))\sum_k \delta(x - x_k) \tag{19}$$

$$L_{S'(x)}(x, S(x), S'(x), S''(x)) = 0 \tag{20}$$

$$L_{S''(x)}(x, S(x), S'(x), S''(x)) = 2\lambda S''(x) \tag{21}$$

Hence, the Euler-Lagrange equation is

$$-2(f(x) - S(x))\sum_k \delta(x - x_k) + \frac{d^2}{d^2x}(2\lambda S''(x)) = 0 \tag{22}$$

i.e.

$$\lambda S''''(x) - (f(x) - S(x))\sum_k \delta(x - x_k) = 0 \tag{23}$$

3. Let $C(s) = (x(s), y(s))$, where s is the arc length parameter. Let n be the curve normal and $V = (u(x, y), v(x, y))$ a given vector field, for example, the image gradient. Maximize the following edge alignment functional

$$\int_C | < \mathbf{v}, \mathbf{n} > | ds \tag{24}$$

Answer: Let $x = x(p), y = y(p)$ by an arbitrary parameterization, so

$$\begin{aligned} ds &= \sqrt{(dx)^2 + (dy)^2} \\ &= \sqrt{(\frac{dx(p)}{dp})^2 + (\frac{dy(p)}{dp})^2} dp \\ &= |C_p| dp \end{aligned} \tag{25}$$

$$\begin{aligned} \varepsilon(c) &= \int_C | < \mathbf{v}, \mathbf{n} > | ds \\ &= \int_C | < (u, v), (-y_s, x_s) > | ds \\ &= \int_C | < (u, v), \frac{(-y_p, x_p)}{|C_p|} > || C_p | dp \\ &= \int_C | < (u, v), (-y_p, x_p) > | dp \\ &= \int_C \sqrt{(vx_p - uy_p)^2} dp \end{aligned} \tag{26}$$

the E-L eqution of $\varepsilon(c)$ is

$$\frac{\delta\varepsilon(c)}{\delta c} = \begin{pmatrix} \frac{\partial}{\partial x} - \frac{d}{dp}\frac{\partial}{\partial x_p} \\ \frac{\partial}{\partial y} - \frac{d}{dp}\frac{\partial}{\partial y_p} \end{pmatrix} \cdot \sqrt{(vx_p - uy_p)^2} \tag{27}$$

consider just the x-component of E-L equation:

$$\begin{aligned}
&(\frac{\partial}{\partial x} - \frac{d}{dp}\frac{\partial}{\partial x_p})\sqrt{(vx_p - uy_p)^2} \\
&= \frac{2(vx_p - uy_p)}{2\sqrt{(vx_p - uy_p)^2}}(v_x x_p - u_x y_p) - \frac{2(vx_p - uy_p)}{2\sqrt{(vx_p - uy_p)^2}}(\frac{d}{dp}v) \\
&= \frac{(vx_p - uy_p)}{\sqrt{(vx_p - uy_p)^2}}[(v_x x_p - u_x y_p) - (v_x x_p + v_y y_p)] \\
&= -\frac{(vx_p - uy_p)}{\sqrt{(vx_p - uy_p)^2}} y_p(u_x + v_y) \\
&= -\frac{(vx_p - uy_p)}{\sqrt{(vx_p - uy_p)^2}} y_p div(V) \\
&= \frac{(vx_p - uy_p)}{|vx_p - uy_p|}(-y_p) div(V)
\end{aligned} \tag{28}$$

where $div(V)$ denotes the divergence of V. Similarly, we can derive the y-compoonent of the EL-equation.

$$\begin{aligned}
&(\frac{\partial}{\partial y} - \frac{d}{dp}\frac{\partial}{\partial y_p})\sqrt{(vx_p - uy_p)^2} \\
&= \frac{2(vx_p - uy_p)}{2\sqrt{(vx_p - uy_p)^2}}(v_y x_p - u_y y_p) + \frac{2(vx_p - uy_p)}{2\sqrt{(vx_p - uy_p)^2}}(\frac{d}{dp}u) \\
&= \frac{vx_p - uy_p}{\sqrt{(vx_p - uy_p)^2}}[(v_y x_p - u_y y_p) + (u_x x_p + u_y y_p)] \\
&= \frac{vx_p - uy_p}{\sqrt{(vx_p - uy_p)^2}} x_p(u_x + v_y) \\
&= \frac{vx_p - uy_p}{\sqrt{(vx_p - uy_p)^2}} x_p div(V) \\
&= \frac{vx_p - uy_p}{|vx_p - uy_p|} x_p div(V)
\end{aligned} \tag{29}$$

Hence,

$$\frac{\delta\varepsilon(c)}{\delta c} = \frac{vx_p - uy_p}{|vx_p - uy_p|} div(V)\mathbf{n} \tag{30}$$

Where $\mathbf{n} = (-y_p, x_p)$. Since V stands for the mage gradient, $V(u(x,y), v(x,y)) = \nabla I(x,y)$, so $div(V) = \nabla \cdot V = \Delta I$, it's laplacian of I.

$$\frac{\delta\varepsilon(c)}{\delta c} = \frac{vx_p - uy_p}{|vx_p - uy_p|}\Delta I \cdot \mathbf{n} \tag{31}$$

to maximize, $\frac{\partial C}{\partial t} = \frac{vx_p - uy_p}{|vx_p - uy_p|}\Delta I \cdot n = 0$ should be promised with any $C(p,t) = (x(p,t), y(p,t))$.

4. We know that the length element of a function y(x) is given by $ds = \sqrt{1+y_x^2}dx$. Show that the area element of the surface $z(x,y)$ is given by $dA = \sqrt{1+z_x^2+z_y^2}dxdy$. Next, find the Euler-Lagrange equations for the minimization of the functional $\iint_\Omega dA$.

Answer:

Construct a tiny area so that it can be considered as a tiny plane. Then the absolute value of the cross product of its two side vector denotes the area, that is,

$$\begin{aligned} dA &= \parallel (dx, 0, \frac{\partial z}{\partial x}dx)^t \times (dy, 0, \frac{\partial z}{\partial y}dy)^t \parallel \\ &= \parallel (-\frac{\partial z}{\partial x}, -\frac{\partial z}{\partial y}, 1)^t \parallel \\ &= \sqrt{1+(\frac{\partial z}{\partial x})^2+(\frac{\partial z}{\partial y})^2}dxdy \\ &= \sqrt{1+z_x^2+z_y^2}dxdy \end{aligned} \tag{32}$$

To find a $z(x,y)$ that yields an extremum of the integral

$$\iint_\Omega dA = \iint_\Omega \sqrt{1+z_x^2+z_y^2}dxdy \tag{33}$$

Let $F(x,y,z,z_x,z_y) = \sqrt{1+z_x^2+z_y^2}$. the integral is over some simply-connected closed region Ω. We introduce a test function $\eta(x,y)$ and add $\epsilon\eta(x,y)$ to $z(x,y)$. We are given the values of f(x,y) on the boundary ∂D of the region, so the test function must be zero on the boundary. Use Taylor series expansion:

$$\begin{aligned} &F(x,y,z+\epsilon\eta, z_x+\epsilon\eta_x, z_y+\epsilon\eta_y) \\ &\quad = F(x,y,z,z_x,z_y) + \epsilon\frac{\partial}{\partial z}F(x,y,z,z_x,z_y)\eta(x,y) \\ &\quad + \epsilon\frac{\partial}{\partial z_x}F(x,y,z,z_x,z_y)\eta_x(x,y) + \epsilon\frac{\partial}{\partial z_y}F(x,y,z,z_x,z_y)\eta_y(x,y) + e \end{aligned} \tag{34}$$

where e consists of terms in higher powers of ϵ. Thus

$$I = \iint_\Omega dA = \iint_\Omega (F + \epsilon\eta F_z + \epsilon\eta_x F_{z_x} + \epsilon\eta_y F_{z_y})dxdy \tag{35}$$

since we want

$$\frac{dI}{d\epsilon}|_{\epsilon=0} = \iint_\Omega (\eta F_z + \eta_x F_{z_x} + \eta_y F_{z_y})dxdy = 0 \tag{36}$$

From Green theorem

$$\iint_D (\frac{\partial Q}{\partial x} + \frac{\partial Q}{\partial y})dxdy = \int_{\partial D} (Qdy - Pdx) \tag{37}$$

so that

$$\iint_{\Omega}(\frac{\partial}{\partial x}(\eta F_{z_x})+\frac{\partial}{\partial y}(\eta F_{z_y}))dxdy=\int_{\partial\Omega}(\eta F_{z_x}dy-\eta F_{z_y}dx) \tag{38}$$

Given the boundary conditions, the term on the right must be zero, so that

$$\int_{\partial\Omega}(\eta_x F_{z_x}dy+\eta_y F_{z_y}dx)=-\iint_{\Omega}(\frac{\partial}{\partial x}(\eta F_{z_x})+\frac{\partial}{\partial y}(\eta F_{z_y}))dxdy \tag{39}$$

Consequently,

$$0=\iint_{\Omega}\eta(F_z-\frac{\partial}{\partial x}F_{z_x}-\frac{\partial}{\partial y}F_{z_y}) \tag{40}$$

This must be true for all test function $\eta(x)$. then we have

$$F_z-\frac{\partial}{\partial x}F_{z_x}-\frac{\partial}{\partial y}F_{z_y}=0 \tag{41}$$

for z(x,y)

$$F_z=0 \tag{42}$$

$$F_{z_x}=\frac{z_x}{\sqrt{1+z_x^2+z_y^2}} \tag{43}$$

$$F_{z_y}=\frac{z_y}{\sqrt{1+z_x^2+z_y^2}} \tag{44}$$

$$\begin{aligned}\frac{\partial}{\partial x}F_{z_x}&=\frac{\partial}{\partial x}\frac{z_y}{\sqrt{1+z_x^2+z_y^2}}\\&=\frac{z_{xx}\sqrt{1+z_x^2+z_y^2}-z_x(\frac{2z_{xx}z_x+2z_yz_{yx}}{2\sqrt{1+z_x^2+z_y^2}})}{1+z_x^2+z_y^2}\\&=\frac{z_{xx}(1+z_x^2+z_y^2)-[z_x^2z_{xx}+z_xz_yz_{yx}]}{(1+z_x^2+z_y^2)^{\frac{3}{2}}}\\&=\frac{z_{xx}+z_y^2z_{xx}-z_xz_yz_{yx}}{(1+z_x^2+z_y^2)^{\frac{3}{2}}}\end{aligned} \tag{45}$$

Symmetrically, we have

$$\frac{\partial}{\partial y}F_{z_y}=\frac{z_{yy}+z_x^2z_{yy}-z_xz_yz_{xy}}{(1+z_x^2+z_y^2)^{\frac{3}{2}}} \tag{46}$$

therefore the E-L eqution should be the formula(eliminating the denominator is also right)

$$\frac{z_{xx}+z_y^2z_{xx}+z_{yy}+z_x^2z_{yy}-z_xz_yz_{xy}-z_xz_yz_{yx}}{(1+z_x^2+z_y^2)^{\frac{3}{2}}}=0 \tag{47}$$

5. Image restoration is a very important problem in computer vision and image processing. Let $\Omega \subset R^2$ denote the image domain and $f(x,y)$ represent the given observed image. The restored image $u(x,y)$ can be computed via the following minimization.

$$\min_u E(u) = \int_\Omega |\nabla u| dx + \frac{\lambda}{2}\|u - f\|_2^2 \tag{48}$$

Write down the Euler-Lagrange equation for this minimization.

Answer: Let z represents the variable where we substitute $u(x)$, and p is the variable where we substitute $\nabla u(x)$. the Lagrangian

$$L(x,z,p) = \frac{\lambda}{2}(f - z)^2 + |p| \tag{49}$$

is not differentiable at $p = 0$. This causes some minor issues with numerical simulations, so it is common to take an approximation of the functional that is differentiable. One popular choice is

$$E_\varepsilon(u) = \int_\Omega \frac{\lambda}{2}(f - u)^2 + \sqrt{|\nabla u|^2 + \varepsilon^2} dx \tag{50}$$

where $\varepsilon > 0$ is a small parameter. When $\varepsilon = 0$ we get the functional. In this case the Lagrangian is

$$L_\varepsilon(x,z,p) = \frac{\lambda}{2}(f - z)^2 + \sqrt{|p|^2 + \varepsilon^2} \tag{51}$$

which is differentiable in both z and p. The minimizers of I_ε converge to minimizers of E as $\varepsilon \to 0$ So the idea is to fix some small value of $\varepsilon > 0$ and minimize E_ε. To compute the Euler-Lagrange equation note that

$$L_{\varepsilon,z}(x,z,p) = \lambda(f - z) \tag{52}$$

and

$$\nabla_p L_\varepsilon(x,z,p) = \frac{p}{\sqrt{|p|^2 + \varepsilon^2}} \tag{53}$$

Therefore the Euler-Lagrange equation is

$$\lambda u - div(\frac{\nabla u}{\sqrt{|\nabla u|^2 + \varepsilon^2}}) = f, \quad in\ \Omega \tag{54}$$

with homogeneous Neumann boundary conditions $\frac{\partial u}{\partial v} = 0$ on $\partial\Omega$. It is almost always impossible to find a solution of eq.54 analytically, so we are to use numerical approximations.

A standard numerical method for computing solutions of eq.54 is like gradient descent, The gradient descent partial differential equation is

$$\lambda(u_t + u) - div\left(\frac{\partial u}{\sqrt{|\partial u|^2 + \varepsilon^2}}\right) = f, \quad for\ x \in \Omega, t > 0, \tag{55}$$

with initial condition $u(x,0) = f(x)$ and boundary conditions $\frac{\partial u}{\partial v} = 0$ on $\partial\Omega$.

CORRECTED

$$\begin{aligned} E(u) &= \int_\Omega |\nabla u| + \frac{\lambda}{2}\|u-f\|_2^2 \\ &= \int_\Omega \sqrt{u_x^2+u_y^2} + \frac{\lambda}{2}(u(x,y)-f(x,y))^2 dxdy \end{aligned} \tag{56}$$

Let $F(x,y,u,u_x,u_y)$ denote it.

E(u) get its minimum when

$$\frac{\partial F}{\partial u} - \left(\frac{d}{dx}\frac{\partial F}{\partial u_x} + \frac{d}{dy}\frac{\partial F}{\partial u_y}\right) = 0 \tag{57}$$

$$\frac{\partial F}{\partial u} = \lambda(u-f) \tag{58}$$

$$\begin{aligned} \frac{d}{dx}\frac{\partial F}{\partial u_x} &= \frac{d}{dx}\frac{2u_x}{2(\sqrt{u_x^2+u_y^2})} \\ &= u_{xx}(u_x^2+u_y^2)^{-\frac{1}{2}} + u_x[-\frac{1}{2}(u_x^2+u_y^2)^{-\frac{3}{2}}(2u_xu_{xx}+2u_yu_{yx})] \\ &= \frac{u_{xx}(u_x^2+u_y^2) - u_x(u_xu_{xx}+u_yu_{yx})}{(u_x^2+u_y^2)^{\frac{3}{2}}} \\ &= \frac{u_{xx}u_y^2 - u_xu_yu_{yx}}{(u_x^2+u_y^2)^{\frac{3}{2}}} \end{aligned} \tag{59}$$

$$\begin{aligned} \frac{d}{dy}\frac{\partial F}{\partial u_y} &= \frac{d}{dy}\frac{2u_y}{2(\sqrt{u_x^2+u_y^2})} \\ &= u_{yy}(u_x^2+u_y^2)^{-\frac{1}{2}} + u_y[-\frac{1}{2}(u_x^2+u_y^2)^{-\frac{3}{2}}(2u_xu_{xy}+2u_yu_{yy})] \\ &= \frac{u_{yy}(u_x^2+u_y^2) - u_y(u_xu_{xy}+u_yu_{yy})}{(u_x^2+u_y^2)^{\frac{3}{2}}} \\ &= \frac{u_{yy}u_x^2 - u_xu_yu_{xy}}{(u_x^2+u_y^2)^{\frac{3}{2}}} \end{aligned} \tag{60}$$

$$\lambda(u-f) - \frac{u_{xx}u_y^2 + u_{yy}u_x^2 - 2u_xu_yu_{xy}}{(u_x^2+u_y^2)^{\frac{3}{2}}} = 0 \tag{61}$$