Linear Shift Invariant Systems

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Linear Shift Invariant Systems

1. Image Processing (Chapter 3 from Szeliski and Chapter 6 from Horn)

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Continuous Domain

- Key task in Image Processing: Transform an image to a form more amenable to further manipulation.
- Imaging systems or image formation systems can be approximated by Linear Shift Invariant Systems (LSI), a powerful analytic tool.
- Consider a 2D system with inputs $f_1(x, y)$ and $f_2(x, y)$ and outputs $g_1(x, y)$ and $g_2(x, y)$.

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- System is linear if $Input = \alpha f_1 + \beta f_2$ and $Output = \alpha g_1 + \beta g_2$.
- Which of the following systems are linear? (a) $g(x) = e^{\pi} f(x)$, (b) $g(x) = f(x) + 1$, (c) $g(x) = xf(x)$, (d) $g(x) = (f(x))^2$
- **Shift Invariant**: $Input = f(x-a, y-b)$, $Output = g(x-a, y-b)$, for arbitrary a, b. In practice, images are limited in area and so, shift invariance holds only for limited areas.

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Why are LSIs important?

- Can model image formation systems using LSIs
- System shortcomings can be discussed in terms of the LSI model.
- Study of LSI leads to useful algorithms for processing images digitally or even optically.

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Convolution and Point Spread Functions

- Consider a system with $f(x, y)$ as input that produces,

$$g(x, y) = \iint_{-\infty}^{\infty} f(x - \xi, y - \eta) h(\xi, \eta) d\xi d\eta \qquad (1)$$

$$i.e.\ \ g_{out} = f_{in} \otimes h_{in} \quad \text{(convolution integral)} \qquad (2)$$

- This system is an LSI system:

$$Input : \alpha f_1 + \beta f_2;\ Output : \alpha g_1 + \beta g_2$$

$$Input :\ f(x - a, y - b);\ \ Output : g(x - a, y - b)$$

- Thus, *a system whose response is described by a convolution,* $\otimes$ *is an LSI and conversely, any LSI system performs a convolution.*

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Convolution Example

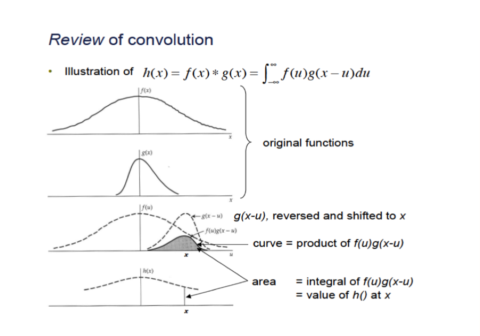

Figure: Convolution Example

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Impulse Functions

- **Question**: Given an arbitrary function $h(x, y)$, can we find a function $f(x, y)$ that causes the output to be $h(x, y)$ i.e.,

$$h(x, y) = \iint_{-\infty}^{\infty} f(x - \xi, y - \eta) h(\xi, \eta) d\xi d\eta \ ? \qquad (3)$$

- **Ans**: Yes; called an *impulse function*.

$$\delta(x, y) = 0 \ \text{ if } (x, y) \neq 0$$
$$\iint_{\infty}^{\infty} \delta(x, y) = 1$$

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Impulse Function Definition & Properties

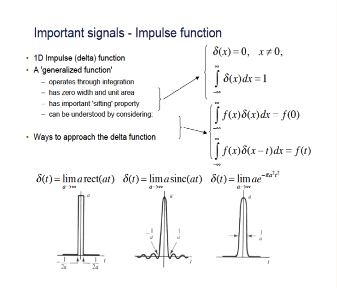

Figure: Impulse Function

- Further, $\delta(x, y) \otimes f(x, y) = f(x, y) \otimes \delta(x, y) = f(x, y)$.
- $\delta(ax, by) = (1/|ab|)\delta(x, y)$

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Sampling and Replicating Property

- Consider an infinite sequence of impulses denoted by the "comb" or "shah" symbol,

$$\text{Ш}(x, y) = \sum_{-\infty}^{\infty}\sum \delta(x - n, y - m) \tag{4}$$

- Properties:

$$\text{Ш}(ax, by) = \frac{1}{|ab|}\sum_{-\infty}^{\infty}\sum \delta(x - n/a, y - m/b) \tag{5}$$

$$\text{Ш}(-x, -y) = \text{Ш}(x, y) \tag{6}$$

$$\iint_{-1/2}^{1/2} \text{Ш}(x, y)dxdy = 1 \text{ because its periodic with unit period} \tag{7}$$

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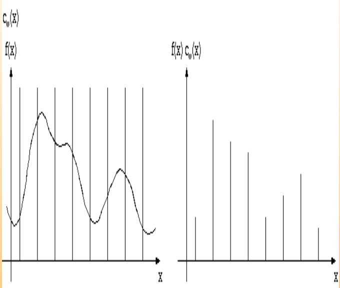

Figure: Sampling a function

- Multiplication of a function by $Ш(x, y)$ effectively samples it at unit intervals.

$$Ш(x, y)f(x, y) = \sum_{-\infty}^{\infty}\sum f(n, m)\delta(x - n, y - m) \quad (8)$$

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- $$Ш(x, y) \otimes f(x, y) = \sum_{-\infty}^{\infty}\sum f(x - n, y - m)$$
- **Seaparability**: $\delta(x, y) = \delta(x)\delta(y)$, similarly, $Ш(x, y) = Ш(x)Ш(y)$.
- **Derivative of the Impulse function**: We use the limit definition of the impulse function and take the derivative of the function whose limit defines the impulse function. Then take the limit.
- **Derivative Sifting Property**:

$$\delta'(x, y) \otimes f(x, y) = \iint_{-\infty}^{\infty} \delta'(x - \xi, y - \eta) f(\xi, \eta) d\xi d\eta = f'(x, y)$$

- Similarly, $\delta''(x, y) \otimes f(x, y) = f''(x, y)$

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LSI Performs Convolution

- **Proof**: Using the sifting property of the impulse function,

$$f(x,y) = \iint_{-\infty}^{\infty} f(\xi,\eta)\delta(x-\xi, y-\eta)d\xi d\eta \qquad (9)$$

- Decompose $f(x,y)$ into elementary functions and then we can determine the overall output $g(x,y)$ by summing the shifted scaled impulses. Why? (because we can use the fact that the system is linear)
- Response of the LSI system to $\alpha\delta(x-\xi, y-\eta)$ is $\alpha h(x-\xi, y-\eta)$ as the system is shift invariant. But $\alpha = f(\xi,\eta)$.
- Hence,

$$g(x,y) = \iint_{-\infty}^{\infty} f(\xi,\eta)h(x-\xi, y-\eta)d\xi d\eta \qquad (10)$$

Which is the same as $g(x,y) = f(x,y) \otimes h(x,y)$. QED.

Properties of Convolution

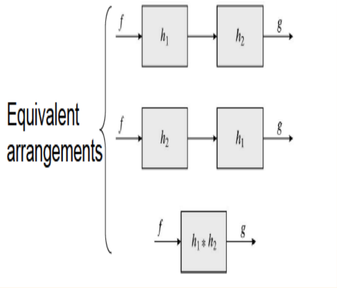

Figure: Convolution Properties

- Convolution is commutative: $f \otimes g = g \otimes f$.
- Convolution is associative: $(f \otimes g) \otimes h = f \otimes (g \otimes h)$.
- Can cascade systems etc.

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