FT Properties

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Fourier Transform Properties

1. Image Processing (Chapter 6 from BKP Horn)

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Some Useful Fourier Transforms

- If $f(x, y)$ and $F(u, v)$ are a transform pair i.e., $F(u, v)$ is the FT of $f(x, y)$ and $f(x, y)$ is the IFT of $F(u, v)$, denoted by, $f(x, y) \longleftrightarrow F(u, v)$, what are the FTs of the $\frac{\partial f}{\partial x}$ and $\frac{\partial f}{\partial y}$.

$$\mathcal{F}\left(\frac{\partial f}{\partial x}\right) = \int\int_{\infty}^{\infty} \frac{\partial f}{\partial x} \exp\{-j2\pi(ux + vy)\}dxdy \tag{1}$$

$$= \int_{-\infty}^{\infty} \left(\int_{-\infty}^{\infty} \frac{\partial f}{\partial x} \exp\{-j2\pi ux\}dx\right) \exp\{-j2\pi vy\}dy \tag{2}$$

Applying integration-by-parts for the inner integral, we get,

$$\{f(x, y) \exp\{-j2\pi ux\}\}_{-\infty}^{\infty} + (j2\pi u) \int_{-\infty}^{\infty} f(x, y) \exp\{-j2\pi ux\}dx \tag{3}$$

- Unless $f(x, y) \to 0$ as $x \to +(-)\infty$, can not proceed further. If its is true (otherwise use convergence factors as in the book), we get,

$$\int_{-\infty}^{\infty}(j2\pi u)\int_{-\infty}^{\infty}\exp\{-j2\pi(ux+vy)\}dxdy = (j2\pi u)F(u,v) \tag{4}$$

  Applying similar steps to $\frac{\partial f}{\partial y}$ we get,
  $\mathcal{F}\{\frac{\partial f}{\partial y}\} = (j2\pi v)F(u, v)$.

- **Laplacian**: $\nabla^2 f = \frac{\partial^2 f}{\partial^2 x} + \frac{\partial^2 f}{\partial^2 y}$. Its Fourier transform is,

$$\mathcal{F}\{\nabla^2 f\} = -(u^2 + v^2)F(u, v). \tag{5}$$

- Note: Taking derivatives in the spatial domain corresponds to multiplication in the spatial-frequency domain. Why?

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Useful Properties of FT

- **Linearity**: $\mathcal{F}\{af(x,y) + bg(x,y)\} = aF(u,v) + bG(u,v)$.
- **Similarity Theorem**: $\mathcal{F}\{f(ax,by)\} = \frac{1}{|ab|}F(u/a, v/b)$.
- **Shift Theorem**:
  $\mathcal{F}\{f(x-a, y-b)\} = \exp\{-j2\pi(au+bv)\}F(u,v)$.
- **Convolution Theorem**:
  $\mathcal{F}\{f(x,y) \otimes g(x,y)\} = F(u,v)G(u,v)$.

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Useful Properties (Contd.)

- **Rotation Theorem:** $\mathcal{F}\{f(x\cos\theta - y\sin\theta, x\sin\theta + y\cos\theta)\} = F(u\cos\theta - v\sin\theta, u\sin\theta + v\cos\theta)$
- **Shear Theoreom**: $\mathcal{F}\{f(x + by, y)\} = F(u, v - bu)$
- **Affine Theorem**: Let $g(x, y) = f(ax + by + c, dx + ey + f)$ then,

$$G(u, v) = \frac{1}{|\Delta|}\exp\{\frac{j2\pi}{\Delta}((ec - bf)u + (af - cd)v)\} \cdot F\left(\frac{eu - dv}{\Delta}, \frac{av - bu}{\Delta}\right)$$

Where, $\Delta = det\begin{pmatrix} a & b \\ d & e \end{pmatrix} = (ae - bd)$.

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Important Functions & their FTs

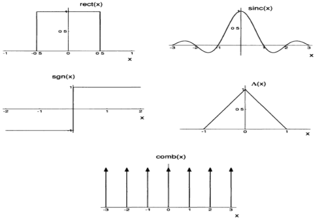

FIGURE 2.2
Special functions.

Rectangle function	$\mathrm{rect}(x) = \begin{cases} 1 & \lvert x\rvert < \frac{1}{2} \\ \frac{1}{2} & \lvert x\rvert = \frac{1}{2} \\ 0 & \text{otherwise} \end{cases}$
Sinc function	$\mathrm{sinc}(x) = \dfrac{\sin(\pi x)}{\pi x}$
Signum function	$\mathrm{sgn}(x) = \begin{cases} 1 & x > 0 \\ 0 & x = 0 \\ -1 & x < 0 \end{cases}$
Triangle function	$\Lambda(x) = \begin{cases} 1 - \lvert x\rvert & \lvert x\rvert \leq 1 \\ 0 & \text{otherwise} \end{cases}$
Comb function	$\mathrm{comb}(x) = \sum_{n=-\infty}^{\infty} \delta(x - n)$

Figure: Some important functions (cf. Goodman)

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Table of FTs

TABLE 2.1
Transform pairs for some functions separable in rectangular coordinates.

Function	Transform
$\exp[-\pi(a^2x^2 + b^2y^2)]$	$\frac{1}{\lvert ab\rvert} \exp\left[-\pi\left(\frac{f_X^2}{a^2} + \frac{f_Y^2}{b^2}\right)\right]$
$\mathrm{rect}(ax)\ \mathrm{rect}(by)$	$\frac{1}{\lvert ab\rvert} \mathrm{sinc}(f_X/a)\ \mathrm{sinc}(f_Y/b)$
$\Lambda(ax)\ \Lambda(by)$	$\frac{1}{\lvert ab\rvert} \mathrm{sinc}^2(f_X/a)\ \mathrm{sinc}^2(f_Y/b)$
$\delta(ax, by)$	$\frac{1}{\lvert ab\rvert}$
$\exp[j\pi(ax + by)]$	$\delta(f_X - a/2, f_Y - b/2)$
$\mathrm{sgn}(ax)\ \mathrm{sgn}(by)$	$\frac{ab}{\lvert ab\rvert} \frac{1}{j\pi f_X} \frac{1}{j\pi f_Y}$
$\mathrm{comb}(ax)\ \mathrm{comb}(by)$	$\frac{1}{\lvert ab\rvert} \mathrm{comb}(f_X/a)\ \mathrm{comb}(f_Y/b)$
$\exp[j\pi(a^2x^2 + b^2y^2)]$	$\frac{j}{\lvert ab\rvert} \exp\left[-j\pi\left(\frac{f_X^2}{a^2} + \frac{f_Y^2}{b^2}\right)\right]$
$\exp[-(a\lvert x\rvert + b\lvert y\rvert)]$	$\frac{1}{\lvert ab\rvert} \frac{2}{1 + (2\pi f_X/a)^2} \frac{2}{1 + (2\pi f_Y/b)^2}$

Figure: Table of Fourier Transforms (cf. Goodman)

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