Projective Geometry

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Uses of Projective Geometry

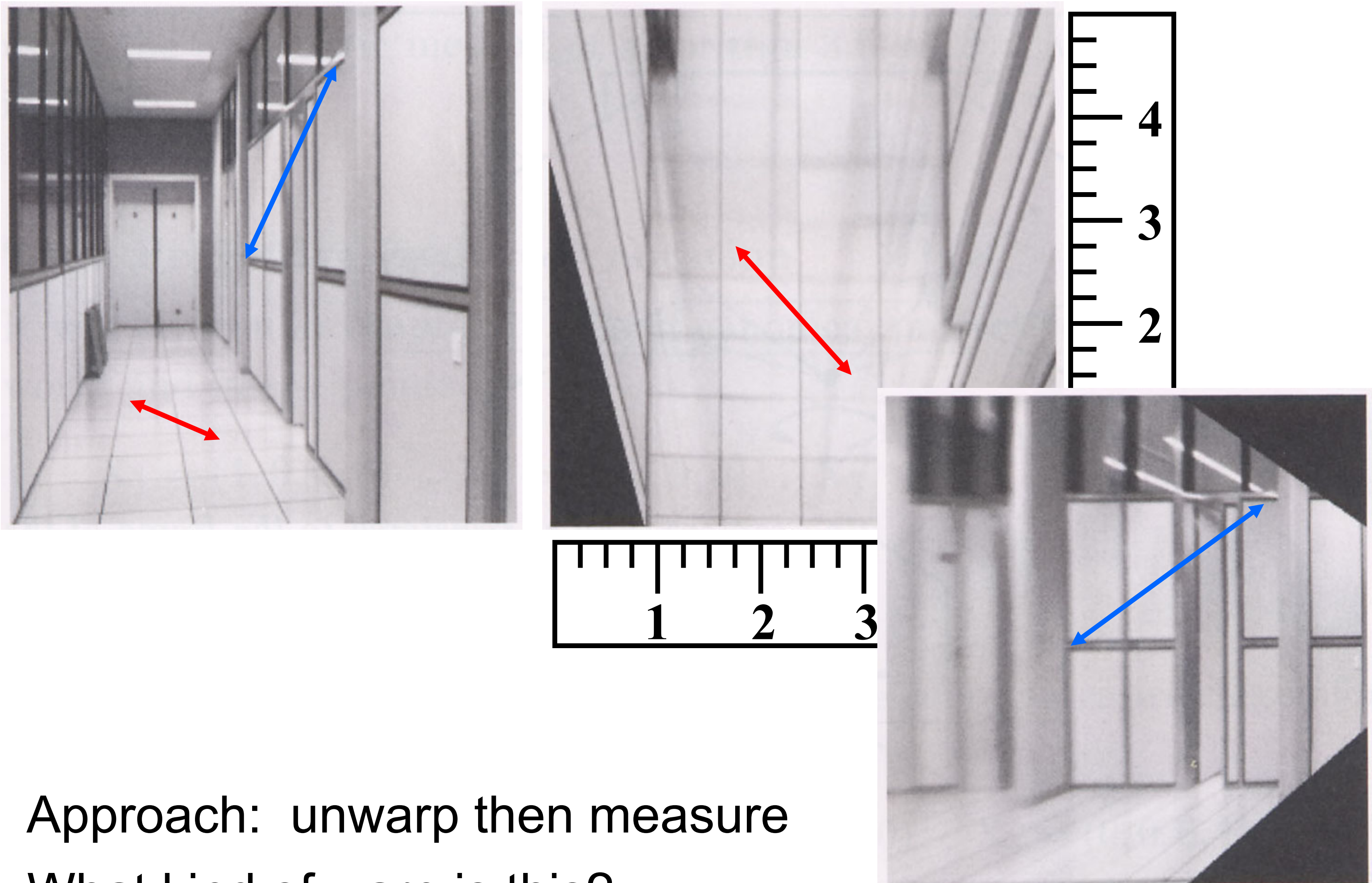
- Drawing and mesurements
- Mathematics for projection
- Undistorting images, Focus of expansion
- Camera pose estimation, matching
- Object recognition

Applications of projective geometry

Vermeer's *Music Lesson*

Reconstructions by Criminisi et al.

Measurements on planes



Approach: unwarp then measure

What kind of warp is this?

Homogeneous Rep. of Lines: Lecture -1 (Contd.)

- A line $ax + by + c = 0$ can be represented by $(a, b, c)^t$.
- Correspondence between lines and vectors is not one to one e.g. $ax + by + c = 0$ and $k\ ax + k\ by + k\ c = 0$ are the same $\forall k \neq 0$. Thus, $(a, b, c)^t$ and $k(a, b, c)^t$ represent the same line.
- The equivalence class of vectors under this equivalence relation is known as a homogeneous vector.
- The set of equivalence class of vectors in $\Re^3 - (0, 0, 0)^t$ forms the projective space $\mathbb{P}^2$.

Homogeneous Rep. of Points

- We have already seen how to represent points in homogeneous coordinates. When points lie on lines, in homogeneous coordinates we have,
- A point $(x, y)^t$ lies on the line $L = (a, b, c)^t \iff (a, b, c) \bullet (x, y, 1) = 0$, i.e., $X^t L = 0$, where $X = (x, y, 1)^t$.

Intersection of Lines

- Let $L = (a, b, c)^t$ and $L' = (a', b'c')^t$, and let $X = L \times L'$.
- Form the triple product, $L \bullet (L \times L') = L' \bullet (L \times L') = 0$. Note that scalar triple products $\mathbf{a} \bullet (\mathbf{b} \times \mathbf{c}) = 0$ if any two of the vectors are the same.
- We see that $L^t X = L'^t X = 0, \Rightarrow X$ lies on $L \& L'$ i.e., $X = L \cap L'$.

Lecture -2

- Example: Intersection of two lines $x = 1$ and $y = 1$, line $x = 1$ is equivalent to $(-1)\,x + 1 = 0$ and has a homogeneous representation, $L = (-1, 0, 1)^t$ and line $y = 1$ is $(-1)\,y + 1 = 0$ which in homogeneous coordinates is $L' = (0, -1, 1)^t$ and intersection point, $X = L \times L' = \begin{pmatrix} i & j & k \\ -1 & 0 & 1 \\ 0 & -1 & 1 \end{pmatrix} = (1, 1, 1)^t$ and in inhomogeneous coordinates $(1, 1)^t$.
- **Line Joining Points X, X':** Similar to the above, we can get $L = X \times X'$.

Intersection of Parallel Lines!

- $ax + by + c = 0$ and $ax + by + c' = 0$; $L = (a, b, c)^t$ and $L' = (a, b, c')^t$
- $L \times L' = (c - c')(b, -a, 0)^t$, ignoring the scale $(c - c')$, the point is $(b, -a, 0)^t$, which in inhomogenous representation is, $(b/0, -a/0)^t$, a point at infinity!

Ideal Points and Line at Infinity

- Homogeneous vectors $\mathbf{x} = (x_1, x_2, x_3)^t$ with $x_3 \neq 0$ are finite points in $\Re^2$.
- Can augment $\Re^2$ by adding points with last coordinate $x_3 = 0$, called *ideal points*. Set of all *ideal points* is $(x_1, x_2, 0)^t$.
- Resulting space is the set of all homogeneous vectors, called the *Projective Space* $\mathbb{P}^2$.

- Set of ideal points lies on a single line, the *line at infinity*, $L_\infty = (0,0,1)^t$. Obviously $(x_1, x_2, 0)^t$ lies on L_∞ since $L_\infty^t (x_1, x_2, 0)^t = 0$.
- $L = (a,b,c)^t$ intersects L_∞ in the *ideal point* $(b, -a, 0)^t$ since $(b, -a, 0)^t L = 0$. All lines $L' = (a, b, c')^t$ parallel to L regardless of the value of c' intersect L_∞ at the same location.
- As the line direction varies, the *ideal point* varies over L_∞.
- L_∞ can be thought of as a set of directions of lines in the plane!

Removing Planar Projective Distortion

- A planar projective transformation is a linear transformation on homogeneous 3-vectors represented by a nonsingular $(3,3)$ matrix H.

$$\begin{pmatrix} x'_1 \\ x'_2 \\ x'_3 \end{pmatrix} = \begin{pmatrix} h_{11} & h_{12} & h_{13} \\ h_{21} & h_{22} & h_{23} \\ h_{31} & h_{32} & h_{33} \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} \; or \;\; X' = HX \qquad (1)$$

- Let $X = (x, y)$ and $X' = (x', y')$ be the inhomogeneous coordinates of corresponding points in the world and image plane respectively.
- In inhomogeneous coordiantes, can write eqn. 1 as,

$$x' = \frac{x'_1}{x'_3} = \frac{h_{11}x + h_{12}y + h_{13}}{h_{31}x + h_{32}y + h_{33}}, y' = \frac{x'_2}{x'_3} = \frac{h_{21}x + h_{22}y + h_{23}}{h_{31}x + h_{32}y + h_{33}};$$

- Each point correspondence generates two equations for elements of the unknown H which can be written as,

$$\begin{aligned} x'(h_{31}x + h_{32}y + h_{33}) &= h_{11}x + h_{12}y + h_{13} \\ y'(h_{31}x + h_{32}y + h_{33}) &= h_{21}x + h_{22}y + h_{23} \end{aligned}$$

- Note that the H matrix has only 8 unknowns because we can divide all elements by the last element h_{33} and the matrix H is then specified upto a scale.

- Thus, we need 4 point correspondences to get 8 equations in 8 unknowns, to solve for the elements of the H matrix.
- Once H is computed, apply H^{-1} to the whole image to undo the effects of perspective distortion on the selected plane.

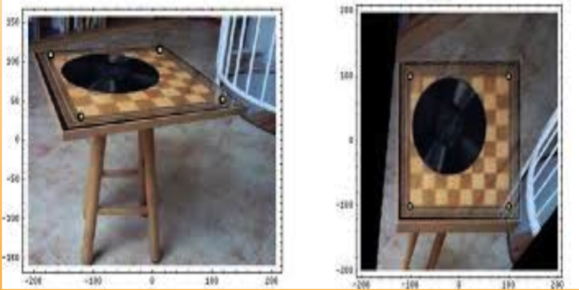


Figure: Rectification of Planar Projective Distortion

Computing H

- The planar homography equation can be written as,

$$X_i' = HX_i \tag{2}$$

- H is a $(3, 3)$ matrix defining a homography (correspondence). Given 4 point correspondences, can rewrite equation 2 as, $X_i' \times HX_i = 0$, (cross product of a vector with itself is zero).
- This form allows a simple linear solution to be derived.
- Let the j^{th} row of the H matrix be $(h^j)^t$, then,

$$HX_i = \begin{pmatrix} (h^1)^t X_i \\ (h^2)^t X_i \\ (h^3)^t X_i \end{pmatrix}$$

- Let $X_i' = (x_i', y_i', w_i')^t$ then,

$$X_i' \times HX_i = \begin{pmatrix} y_i'(h^3)^t X_i - w_i(h^2)^t X_i \\ w_i'(h^1)^t X_i - x_i'(h^3)^t X_i \\ x_i'(h^2)^t X_i - y_i'(h^1)^t X_i \end{pmatrix}$$

- Since $(h^j)^t X_i = X_i^t h^j$ for $j = 1, 2, 3$, it gives 3 equations in entries of H that can be written as,

$$\begin{pmatrix} \mathbf{0}^t & -w_i' X_i^t & y_i X_i^t \\ w_i' X_i^t & \mathbf{0}^t & -x_i' X_i^t \\ -y_i' X_i^t & x_i' X_i^t & \mathbf{0}^t \end{pmatrix} \begin{pmatrix} h^1 \\ h^2 \\ h^3 \end{pmatrix} = 0 \qquad (3)$$

- These equations are of the form $A_i\mathbf{h} = 0$, A_i is $(3, 9)$ matrix and $\mathbf{h}$ is a 9-vector made of the entries of H-matrix.

- Although there are 3 equations in 3, only 2 are linearly independent. 3rd row is $x_i' * Row - 1 + y_i' * Row - 2$. Thus giving just 2 equations in entries of H.
- Hence, Delete Row-3 from 3 and write it as $A_i\mathbf{h} = 0$, where A_i is a $(2, 9)$ matrix.
- Solve using SVD (singular value decomposition). Rewrite the linear system as $A\mathbf{h} = 0$, where A is $(8, 9)$ matrix and Rank(A)=8.
- $\mathbf{h}$ lies in the null space of A, can find the solution only upto a scale factor. Fix the scale of $\mathbf{h}$ by imposing $||\mathbf{h}|| = 1$.

Conics and Duals

- Conic: A second degree curve in a plane, e.g. hyperbola, ellipse, parabola (obtained by intersecting a plane of varying orientation with a cone).
- In projective geometry all these conics are equivalent under projective transformations.
- Equation of a conic (poly. of degree 2):

$$ax^2 + bxy + xy^2 + dx + ey + f = 0$$

- Homogenize by replacing $x = \frac{x_i}{x_3}, y = \frac{x_2}{x_3}$ giving,

- $ax_1^2 + bx_1x_2 + cx_2^2 + dx_1x_3 + ex_2x_3 + fx_3^2 = 0$
- In matrix form: $\mathbf{x}^t C\mathbf{x} = 0$, where,

$$C = \begin{pmatrix} a & b/2 & d/2 \\ b/2 & c & e/2 \\ d/2 & e/2 & f \end{pmatrix}$$

- As in homogeneous rep. of points & lines, only ratios of matrix elements are important, since multiplying C in $\mathbf{x}^t C\mathbf{x} = 0$ by a scalar doesn't effect anything.
- Thus C is a homogeneous rep. of a conic and has 5 degrees of freedom.

- How many points do we need to specify the conic uniquely?
- Each point (x_i, y_i) gives one constraint and if we stack all the 5 constraints into a matrix form, we get a $(5, 6)$ linear system:

$$\begin{pmatrix} x_1^2 & x_1 y_1 & y_1^2 & x_1 & y_1 & 1 \\ x_2^2 & x_2 y_2 & y_2^2 & x_2 & y_2 & 1 \\ x_3^2 & x_3 y_3 & y_3^2 & x_3 & y_3 & 1 \\ x_4^2 & x_4 y_4 & y_4^2 & x_4 & y_4 & 1 \\ x_5^2 & x_5 y_5 & y_5^2 & x_5 & y_5 & 1 \end{pmatrix} \begin{pmatrix} a \\ b \\ c \\ d \\ e \\ f \end{pmatrix} = 0$$

- The conic is the solution to the above equation and lies in the null space of the $(5, 6)$ matrix.
- READING Assignment: Read the dual conics from the tutorial handout on multiview geometry.