Scientific Computation Notes 2

The condition number of a square matrix A, cond(A), is the maximum possible error magnification factor for solving $Ax = b$, over all right hand sides b. cond$(A) = ||A||_\infty \cdot ||A^{-1}||_\infty$.

The matrix norm of an $n \times n$ matrix A is
$||A||_\infty =$ maximum absolute row sum, that is total the absolute values of each row, and assign the maximum of these n numbers to be the norm of A.

Let x_a be an approximate solution of the linear system $Ax = b$. The residual is the vector $r = b - Ax_a$. The backward error is the norm of the residual $||b - Ax_a||_\infty$, and the forward error is $||x - x_a||_\infty$. The relative backward error is $\frac{||r||_\infty}{||b||_\infty}$ and the relative forward error is $\frac{||x-x_a||_\infty}{||x||_\infty}$. The error magnification error is the ratio of those two, or error magnification error = $\frac{\text{relative forward error}}{\text{relative backward error}} = \frac{\frac{||x-x_a||_\infty}{||x||_\infty}}{\frac{||r||_\infty}{||b||_\infty}}$.

$PA = LU$ Factorisation [Probably For $n = 2, 3$]: Row Pivoting Swapping To Maximum Magnitude Entry In Column On Diagonal Followed With Usual Tracking Zeroing Of LU

General Reasons For $PA = LU$ Factorisation Over $A = LU$ Factorisation:
Ensures that all multipliers, entries of L, will be no greater than 1 in absolute value. Also solves the problem of 0 pivots. Which are immediately exchanged.

Lagrange Interpolation:
$P(x) = \sum y_j \prod_{k \neq j} \frac{x - x_k}{x_j - x_k}$

Theorem 3.3: Assume that $P(x)$ is the (degree $n-1$ or less) interpolating polynomial fitting the n points $(x_1, y_1), \dots, (x_n, y_n)$. The interpolation error is
$f(x) - P(x) = \frac{(x-x_1)(x-x_2)\dots(x-x_n)}{n!} f^{(n)}(c)$, where c lies in the range i.e. between the smallest and largest of the numbers $x, x_1, \dots, x_n$.

Chebyshev Interpolation Nodes: On the interval $[a, b]$,
$x_i = \frac{b+a}{2} + \frac{b-a}{2} \cos\left(\frac{(2i-1)\pi}{2n}\right)$ for $i = 1, 2, \dots, n$. The inequality $|(x - x_1)(x - x_2)\dots(x - x_n)| \leq \frac{\left(\frac{b-a}{2}\right)^n}{2^{n-1}}$ holds on $[a, b]$.

Interpolation Error For Approximating $f(x)$: for nth degree approximation I think it is
$|f(x) - Q_n(x)| \leq \frac{|(x-x_1)(x-x_2)\dots(x-x_{n+1})|}{(n+1)!} \cdot |f^{(n+1)}(c)| \leq \frac{\left(\frac{b-a}{2}\right)^{n+1}}{(n+1)!2^n} \cdot |f^{(n+1)}(c)|$ of course the 6th derivative of $f(x) = e^x$ is simply e^x which is maximised at $x = 1$ for the value of e. And thus one obtains $\frac{1}{6! \cdot 2^5} \cdot e \approx 0.00011798$ so 3 expected correct decimal places after the decimal.

It would seem that for a degree n spline the condition at the joints is that they agree on the 0th, 1st,... ,$n-1$th derivatives. In generality the smoothness vector of desired agreements is defined.

But for cubic splines the properties are also of the form for n given data points $(x_1, y_1), (x_2, y_2), \ldots, (x_n, y_n)$:

$S_1(x) =$
$y_1 + b_1(x - x_1) + c_1(x - x_1)^2 + d_1(x - x_1)^3$
on $[x_1, x_2]$
$S_2(x) =$
$y_2 + b_2(x - x_2) + c_2(x - x_2)^2 + d_2(x - x_2)^3$
on $[x_2, x_3]$
$\ldots$
$S_{n-1}(x) = y_{n-1} + b_{n-1}(x - x_{n-1}) +$
$c_{n-1}(x - x_{n-1})^2 + d_{n-1}(x - x_{n-1})^3$ on
$[x_{n-1}, x_n]$