RIT Student-Run Potluck Mathematics Competition 2022

Compiled by Quinn Kolt and Andrew Searns

June 16th-June 23rd, 2022

- Included in this PDF is a set of **nine problems and one bonus puzzle**. The problems and puzzle are labeled with their creators' names or "Anonymous". The problems are ordered alphabetically by their creators' last names. Anonymous problems are placed at the end in randomized order. Starred problems are original.
- 10 points are assigned to each problem and the bonus puzzle. To receive full credit for these problems, please write in full English sentences and **justify your answers rigorously**.
- While this is a virtual competition, we strongly **discourage the use of online or physical resources, software, calculators, etc.**, unless the problem indicates otherwise. If you decide to use such resources, please explicitly state how (and for which problems) when you send in your solutions.
- We allow competitors to solve problems in teams of two. As such, you are welcome to use any form of virtual communication with your teammate. Please do not work in groups with more than two members. Only one set of solutions needs to be submitted, but please include both members' names if you are participating in a team.
- **Score statistics will be publicly released** to all competitors without names, except for the highest scoring individual/team. Individual problem scores will also be emailed privately to each competitor.
- All solutions must be sent to kolt.math.problems@gmail.com by **11:59pm EST, June 23rd, 2022**. The solutions may be sent in any common format (e.g. PNG of handwritten work, PDF made in LaTeX, DOCX made in Google Docs) as long as all work is clear and without ambiguity. Feel free write your work on a printed copy of the exam. However, keep in mind that some diagrams may be ink-intensive.
- If solutions are not sent in by 11:59pm EST, June 23rd, 2022, they will **not be included when scores are publicly released**. However, your individual/team score will still be sent privately.
- **NEW THIS YEAR:** Many problems were posed by fellow solvers! The non-anonymous contributors were: Anurag Agarwal, Josh Faber, Quinn Kolt, and Andrew Searns from RIT, Gerry Wildenberg from St. John Fisher College.
- You may begin this exam **immediately**!

1⋆ (Anurag Agarwal). Let $p(x) = (x^{18} - 1)\ (x^2 - 1)(x^3 - 1)(x^4 - 1) \ldots (x^{15} - 1)(x^{16} - 1)$ and $s(x) = 1 + x + x^2 + \cdots + x^{16}$. Find the remainder when $p(x)$ is divided by $s(x)$.

2 (Josh Faber). Inscribe the rectangle with the largest possible area inside a right triangle along the hypotenuse. Show that the radii r_1, r_2, r_3 of the circles inscribed in the remaining three regions outside the rectangle and within the triangle satisfy $r_1^2 + r_2^2 = r_3^2$. A visual depiction of this problem is included in Figure 1.

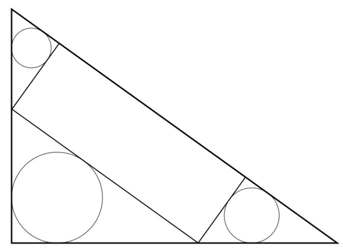

Figure 1: An example right triangle with inscribed rectangle and further inscribed circles. This figure is not to scale.

3⋆ (Quinn Kolt). Given a real number y, let $\lceil y \rceil$ denote the smallest integer greater than or equal to y and $\lfloor y \rfloor$ denote the largest integer less than or equal to y (e.g., $\lfloor \pi \rfloor = \lfloor 3 \rfloor = \lceil 3 \rceil = 3$ but $\lceil \pi \rceil = 4$).
Prove that there are no positive integers a, b, c such that

$$\left\lceil (1+\sqrt{3})^a \right\rceil + \left\lfloor (1+\sqrt{3})^b \right\rfloor - c^2 + c = 2022.$$

4⋆ (Andrew Searns). Conway's Game of Life depicts that chaotic behavior can emerge from a simple set of rules. In this spirit, we introduce the following three-step procedure for graphs with black and white vertices:

1. For each black vertex with no black neighbors, color its neighbors red and it white.
2. If u is colored black and has a black neighbor v, contract u and v into a single black vertex w. The neighbors of w are the union of the neighbors of u and the neighbors of v. For each white neighbor of w, set its color to red. Repeat this this until no connected black vertices remain.
3. Change the color of all red vertices to black.

An example of repeatedly applying this procedure can be seen in Figure 2.

If we start this sequence by coloring a single vertex black, for which graphs will repeating this procedure eventually lead to a graph with only a single black vertex (and no other vertices)?

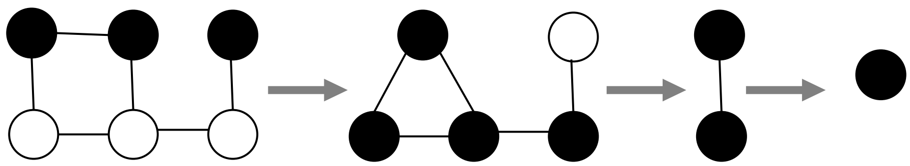

Figure 2: An example of repeatedly applying the update rules.

5⋆ (Gerry Wildenberg). Find all pairs of nonzero integers (x, y) such that

$$\frac{17}{x} + \frac{19}{y} = 3.$$

6⋆ (Anonymous). Is there a sequence of positive real numbers $\{a_n\}_{n=1}^{\infty}$ such that $\lim_{n\to\infty} a_n = 0$ for which the series

$$\sum_{n=1}^{\infty} \frac{1}{n^{1+a_n}}$$

converges? Justify your answer.

7 (Anonymous). The function f is continuously differentiable and nonzero for all $x \in [4, 8]$. Suppose

$$f(4) = \frac{1}{4},$$
$$f(8) = \frac{1}{2},$$
$$\int_4^8 \frac{[f'(x)]^2}{[f(x)]^4} dx = 1.$$

Find $f(6)$.

8 (Anonymous). Is it possible to cover a 6×5 tiled board with dominoes such that the tiling doesn't decompose into a tiling of two smaller rectangles? What about 6×6?

9⋆ (Anonymous). Evaluate

$$\lim_{t\to 1^-} \sqrt{\ln(\tfrac{1}{t})}(1 + t + t^4 + t^9 + \dots) = \lim_{t\to 1^-} \sqrt{\ln(\tfrac{1}{t})} \sum_{k=0}^{\infty} t^{k^2}.$$

Bonus⋆ (Anonymous). You may use external resources/calculators/computer programming/Excel sheets to solve this puzzle. No rigorous proof is needed, but your process for solving the puzzle is required.

A One-Time Pad is a cryptographic method which can guarantee an encrypted message cannot be deciphered by any adversary, as long as the key is not found. Given the plaintext (the message to encrypt) and the key (used for encryption and decryption), we can encrypt a message using the One-Time Pad as follows:

```
plaintext:  ilovebunnies  <-> 08 11 14 21 04 01 20 13 13 08 04 18
key:        CRYPTOGRAPHY  <-> 02 17 24 15 19 14 06 17 00 15 07 24
-----------       +           -----------------------------------
ciphertext: KCMKXPAENXLQ  <-> 10 02 12 10 23 15 00 04 13 23 11 16
```

We note that each letter of the alphabet corresponds to the number for its position in the alphabet (i.e. $\texttt{A} = 0$, $\texttt{B} = 1$, ..., $\texttt{Z} = 25$). Addition in this system is done mod 26, so in the second column $11+17 = 28 \equiv 28-26 \equiv 2 \pmod{26}$. Adding or subtracting 26 from a number doesn't change which letter it represents.

The following ciphertext has been encrypted with a One-Time Pad using a key made of English words (with some padding at the end of the plaintext). The word "`you`" appears exactly twice in the plaintext, and the start of these "`you`"s are also the start of words in the key:

```
OPVGBMFNLWNQOIPSXAGRIIFCASLRLWSPAMUQAWTBKQX
```

Find the plaintext.