1

The rectangularity for a positive integer results from choosing a to be closest factor of n to its square root which is less than or equal to that quantity. Hence it is 1 for squares and p for primes.

2

Whether or not integer factorization is in P is an open problem in theoretical computer science. If an oracle outputs the full factorization then computing the requisite setting is fast for example by sorting and viewing the middle two or one. However, if the number is large and the oracle gives a prime factorization then it could be rather hard to compute all of the factors and sort them.

3

A natural precise interpretation for smallest is truly minimal over the domain and by definition this is the value 1 obtained for all squares. As for largest certainly in ratio to n it is upper bounded by n itself and this equality case occurs if and only if n is prime. The local structure and variance of this function is of interest as well as asymptotics.

4

There is, indeed $n = 3$ has rectangularity 3. There is not, as any such positive integer would have factors 5 and 12 and hence factors 6 and 10 which immediately gives rise to a lesser potential rectangularity. Fully characterizing by analagous analysis and say if it's just powers of 2 and 3 generates an infinite relevant set related to ratio but again go in depth. This gives rise to the fact that $r(ab) \leq r(a)r(b)$ by transfer of construction i.e. r is submultiplicative.

5

Another natural measure would be more closely related to the delta between it and its nearest squares or the remainder of its square root modulo 1 normed appropriately to satisfy some desiderata. In terms of this task we define the analagous term prismicism or perhaps cubicism to squarism. That is, the minimum value of $\frac{c}{a}$ where a, b, c are positive integers such that $a \leq b \leq c$ and $abc = n$. So then primes and squares of primes both have prismicism p and cubes 1. However, casting proximity in another way can lead to $\frac{c^2}{ab}$ and $\frac{cb}{a^2}$ which go to higher dimensions. These follow from volume ratios of inscribed and circumscribed general cubes. These may be seen to satisfy potentially desired desiderata.

6

Yes for infinitely many. No for nonzero finitely many we can always introduce two copies of a large enough prime into the prime factorization. This results in the sorted factors being precisely the original factors followed strictly by those multiplied by the prime followed strictly by those multiplied by the prime squared which yields the same ratio of middle two elements.

7

The relationship is the entire point of this task! $1 \leq r(n) \leq n$ but the expectation of $r(n)$ when n is chosen uniformly and at random from $[1, n]$ and under other distributions is of interest and can be computed for some values.

8

No, almost by definition if it maps into an integer which results in a different value. But in particular as the number of prime factors goes up what happens? Among the first n with a certain number of prime factors.

9

It is $\frac{q}{p}$ for $k = 1$ and 1 for even k and does not eventually have only repeated values. Indeed taking the natural logarithm and considering the density of linear combinations of integer multiples of irrational numbers modulo 1 one can deduce that any nontrivial interval from 1 will contain a real which can be realized as one of the rectangularities in this sequence.

10

Yes for example $(pq)^{2k-1}$ or really and such sequence where the ratio between consecutive terms is a not too large integer and the terms are never squares.

11

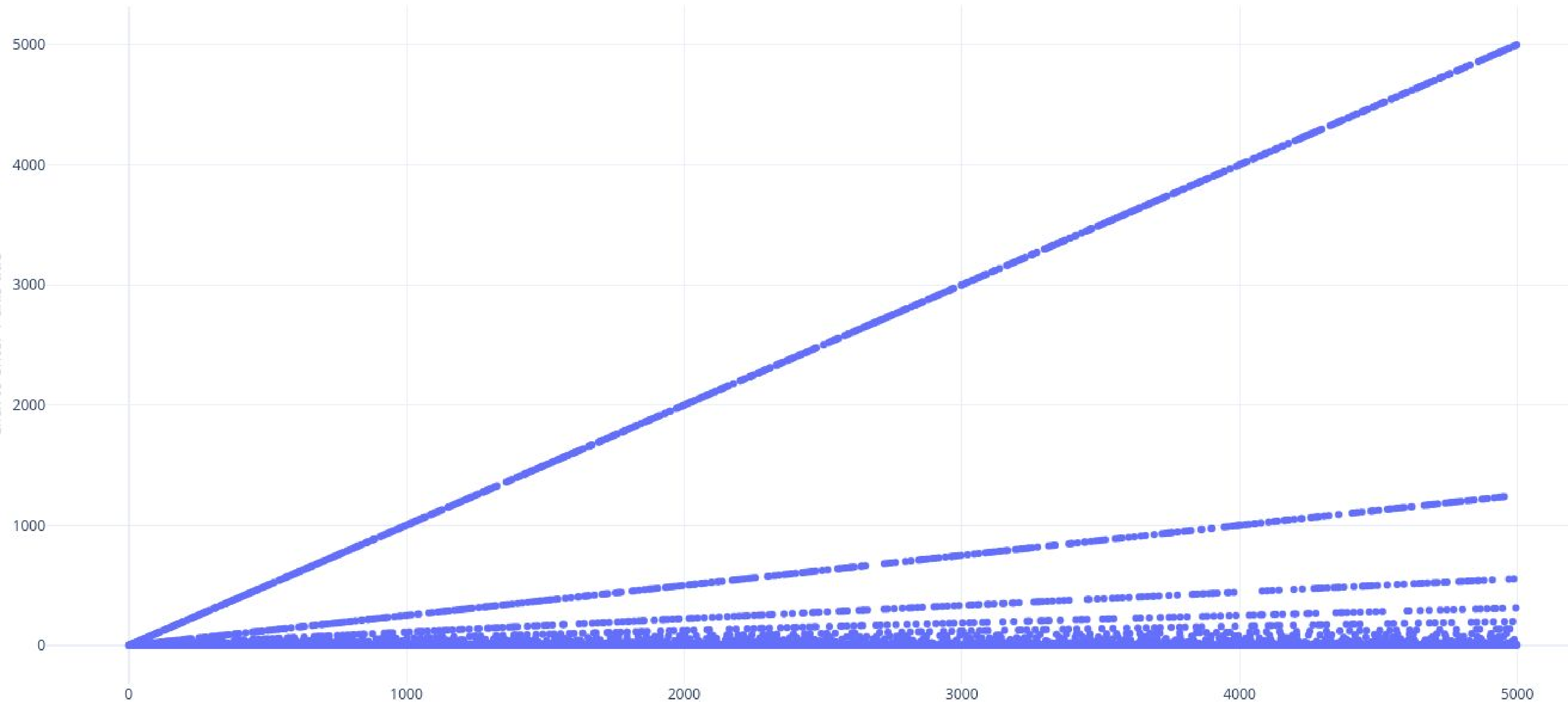
Estimate? Diverges. Indeed by the rough analysis below for any fixed real greater than 1 the density of corresponding numbers with less rectangularity is nontrivial implies the bound by the canonical divergence of the harmonic series.

12

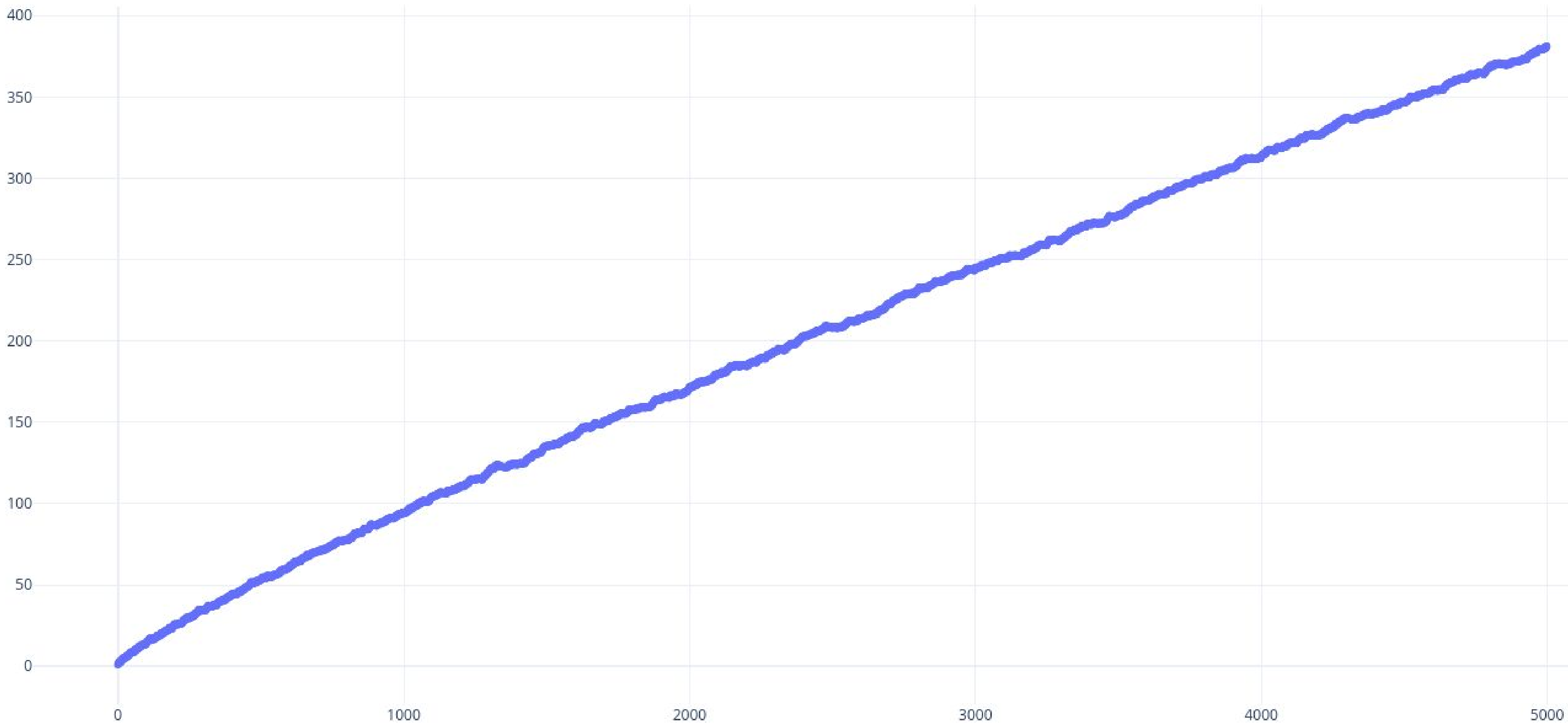
More than 2. Consider the first n numbers. The average number of divisors. The roughly expected distribution of those and how central they will be. A rough estimate assumes independence and says the probability the recangularity is greater than $\frac{1}{2}$ is the probability none of the numbers between $(\frac{n}{2})^{\frac{1}{2}}$ and $n^{\frac{1}{2}}$ divide n which is $2^{\frac{-1}{2}}$ 0.7071.

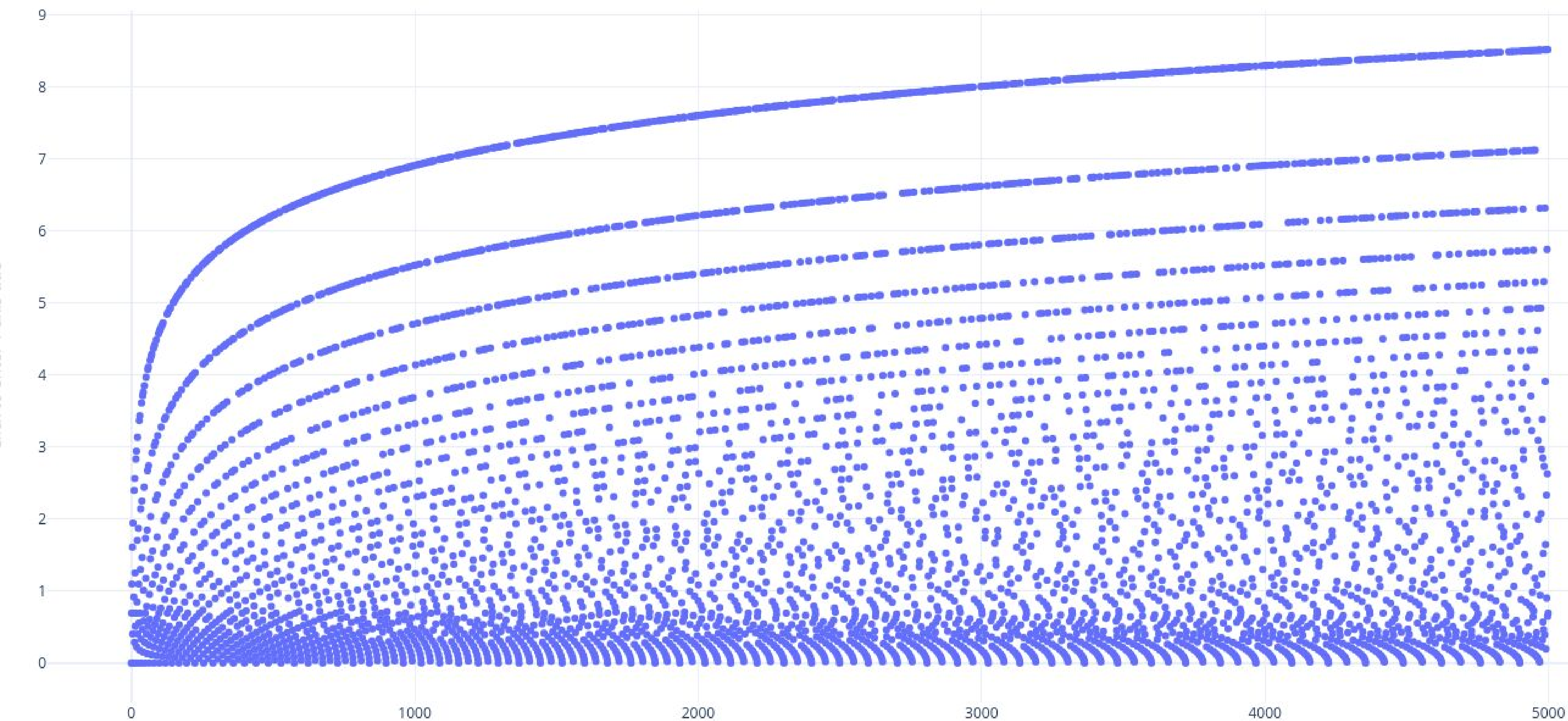
13

Average number of divisors is theta log n by Dirichlet. One could curve fit. The graphs of the first 5000 values of $r(n)$, average over first n, $\log r(n)$, and average of that over first n follow. The averages look linear and shifted logarithmic in n roughly (these are distinct observations about the underlying set) and expected distributions can emerge from Dirichlet and beyond number theoretic analysis. One can logarithmically transform this into a question about sumsets of arithmetic progressions. Distributions emerge from somewhat standard number theoretic analysis.

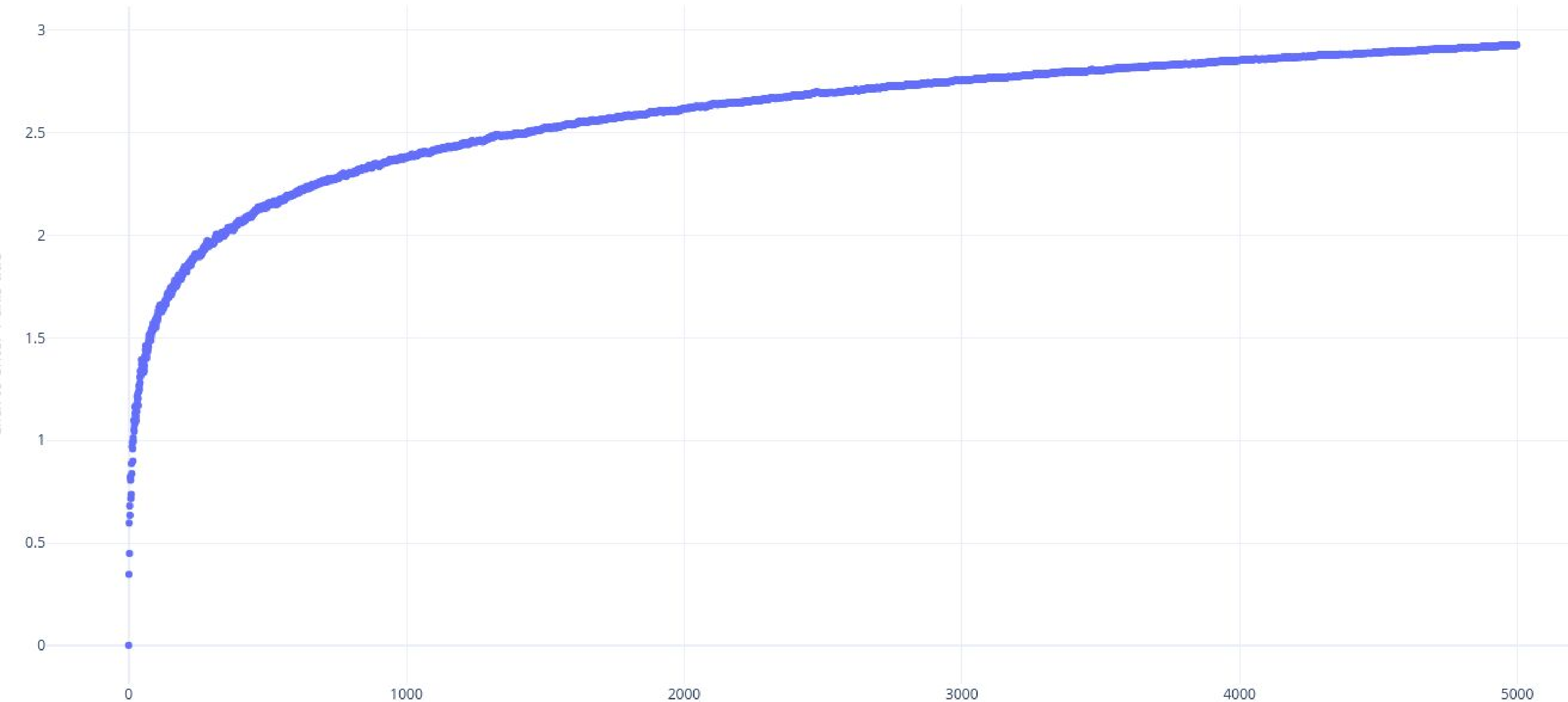


5000
4000
3000
2000
1000
0
0
1000
2000
3000
4000
5000


400
350
300
250
200
150
100
50
0
0
1000
2000
3000
4000
5000



9
8
7
6
5
4
3
2
1
0
0
1000
2000
3000
4000
5000



3
2.5
2
1.5
1
0.5
0
0
1000
2000
3000
4000
5000