1

1

Note that $f_1(11) = 1^2 + 1^2 = 1 + 1 = 2$, $f_2(11) = f(2) = 2^2 = 4$, $f_3(11) = f(4) = 4^2 = 16$, $f_4(11) = 37$ similarly, $f_5(11) = 58$, $f_6(11) = 89$, $f_7(11) = 145$, $f_8(11) = 42$, $f_9(11) = 20$, $f_{10}(11) = 4$, and thus the involution becomes cyclic from hereon out and thus $f_{2021}(11) = f_5(11) = \boxed{58}$.

2

$\boxed{c(n) = n}$. Indeed, for each $i = 1, 2, \ldots, n$ there exists precisely 1 such representation using i summands. This can be explicitly derived in terms of multiplicities of the floor and ceiling function of $\frac{n}{i}$ in conjunction with the 1 gap bound.

3

Indeed, the right hand side naturally factors as $(1+1)(1+1)\ldots(1+1)(1+x_1x_2\ldots x_n)$ but another idea is smoothing, and yet another idea is induction. It suffices to note that $(1+x_1)(1+x_2) = 1 + x_1 + x_2 + x_1x_2 \leq 2 + 2x_1x_2$ as indeed $x_1 + x_2 \leq 1 + x_1x_2$ as indeed $(x_1 - 1)(x_2 - 1) \geq 0$. Now the induction step follows as it suffices to note that $2^{n-1}(1 + x_1x_2\ldots x_n)(1 + x_{n-1}) \leq 2^n(1 + x_1x_2\ldots x_{n-1})$ as indeed this is the $n = 2$ case in disguise with the first term replaced by the previous product!

4

The determinant is $\boxed{0}$. Indeed the first row plus the third row is, by $\cos(a) + \cos(b) = 2\cos\left(\frac{a+b}{2}\right)\cos\left(\frac{a-b}{2}\right)$, $2\cos(3)$ times the second row, and thus the rows are linearly dependent.

5

There exists a fundamental theorem which leads to a one liner of this assertion. However, one can note that $a^2 = (b\sqrt{2} + c\sqrt{3})^2 = 2b^2 + 3c^2 + 2bc\sqrt{6}$ which re arranges to $a^2 - 2b^2 - 3c^2 = 2bc\sqrt{6}$ and if both $b = c = 0$ we deduce $a = 0$, if one of them is 0 then we deduce a contradiction on irrationality a la canon, or neither are 0 whence $\frac{a^2 - 2b^2 - 3c^2}{2bc} = \sqrt{6}$ and again a contradiction on irrationality.

6

Intuitively one notes ideas about translations to the origin, affine transformations to algebraic manipulations, case work on the pair of complex conjugates root case, a derivative producing a tangent line which can be slightly shifted or wiggled, but an asymptotic growth analysis gives rise to the solution $y = cx$ for sufficiently large c in the without loss of generality $p(x) = x^3 + ax^2 + bx$ case as indeed the intersection will be precisely when the difference is 0 i.e. $x^3 + ax^2 + (b - c)x = 0$ and for sufficiently large c this has two nonzero real roots as the discriminant of the resultant quadratic ignoring the origin $x^2 + ax + b - c = 0$ is $a^2 - 4(b - c) = a^2 - 4b + 4c$ grows arbitrarily large and e.g. clears the 0 threshold.

2

1
Note that this is $\frac{1-2(3x+b)}{x(3x+b)} = \frac{(1-b)-6x}{bx+3x^2}$ whence for $\boxed{b=1}$ we obtain $\boxed{-6}$

2
$\int_0^1 \frac{x^4(1-x)^4}{1+x^2} = \int_0^1 x^6 - 4x^5 + 5x^4 - 4x^2 + 4 - \frac{4}{1+x^2} = \boxed{\dfrac{22}{7} - \pi}$

3
Putnam.

4
Recall that $\ln(ab) = \ln(a) + \ln(b)$ and that $a^3 + b^3 = (a+b)(a^2 - ab + b^2)$ whence one obtains that the desired is $\sum_{n=2}^{\infty} \ln\left(\frac{(n-1)(n^2+n+1)}{(n+1)(n^2-n+1)}\right) =$

$\sum_{n=2}^{\infty} \ln(n-1) + \ln(n^2+n+1) - \ln(n+1) - \ln(n^2-n+1) = \boxed{\ln\left(\dfrac{2}{3}\right)}$

5
Let $I(a) = \int_0^{\infty} \frac{\arctan(ax) - \arctan(x)}{x} dx$. Then one obtains that
$I'(a) = \int_0^{\infty} \frac{1}{1+a^2x^2} dx = \frac{1}{a} \int_0^{\infty} \frac{a}{1+a^2x^2} dx = \frac{1}{a} \cdot \arctan(ax)|_0^{\infty} = \frac{\pi}{2a}$. $I(1) = 0$ and the desired is $I(\pi)$ by construction whence we obtain by integrating
$\int_1^{\pi} \frac{\pi}{2x} dx = \frac{\pi}{2} \ln(x)|_1^{\pi} = \boxed{\dfrac{\pi}{2} \ln(\pi)}$

6
The derivative $\frac{f'(x)x - f(x)}{x^2} > 0 \iff f'(x)x > f(x)$ graphically as e.g. the tangent line to the curve intersects the y-axis below the origin, and thus shifting this line up to the origin produces precisely this value above that value. But mathematically one would write
$f(x) = f(x) - f(0) = f(x) - 0 = \int_0^x f'(t)dt < \int_0^x f'(x)dt = f'(x)x$

7
Noting $(1+2)^3 = 1 + 6 + 12 + 8$ one obtains that $(e^{\frac{x}{2}} f(x))' = \frac{1}{2} e^{\frac{x}{2}} f(x) + e^{\frac{x}{2}} f'(x)$ and that $(e^{\frac{x}{2}} f(x))'''$ implies the desired in conjunction with the fact that $e^{\frac{x}{2}}$ has no zeros.

8
By Cauchy-Schwarz one obtains $\left(\frac{1}{x} + \frac{1}{y} + \frac{1}{z}\right)(x+y+z) \geq (1+1+1)^2 = 3^2 = \boxed{9}$ with equality for $x = y = z = \frac{1}{3}$

9

Indeed $n! = 1 \cdot n \cdot 2 \cdot (n-1) \dots$ and for each pair one obtains that $a \cdot (n+1-a) < \left(\frac{n+1}{2}\right)^2$

10
AM-GM directly to 9 symmetric terms on the LHS.

3

1

For each i it can not be in 0 or 3 of the sets, thus $(2^3-2)^{10} = \boxed{2^{10}3^{10}}$

2

Putnam. Contradiction otherwise one obtains that $12(0+1+2+3+4) \leq 3 \cdot 39$

3

4 squares partition, $\boxed{\text{no}}$ consider the set of the center and arbitrarily close to the 4 corners.

4

Probably Putnam. $\boxed{F_n}$ indeed case work on whether or not n is in the subset, if it is not we recurse, otherwise we deduce that 1 is not in the subset and that by definition the subset is selfish if and only if the subset from reducing each other element by 1 was selfish whence we obtain the Fibonacci recursion.

5

Add $\boxed{7}$

6

Putnam. Note this expression appears in Stirling's Approximation that $\lim_{n\to\infty} \frac{n}{(n!)^{\frac{1}{n}}} = e$ and that log transforming this statement becomes $g(m+n) < g(m) + g(n)$. In any case $(m+n)^{m+n} > \binom{m+n}{m} m^m n^n$ because the binomial expansion includes the term on the right as well as some others. Or combinatorial interpretation.

7

Isomorphs in to a statement about binary matrices and appears in combinatorics handouts. One solution is to note that counting in 2 ways incidences linearity of expectation there exists a student who solved $\geq \frac{120}{200} \cdot 6 = \frac{720}{200}$ problems e.g. they solved 4 problems. Now consider that there were 240 solves on the other 2 problems, thus there exists some student who solved those 2. Now these together solved all 6 problems.

8

$\boxed{2n-3}$ indeed $\frac{a_1+a_2}{2} < \frac{a_1+a_3}{2} < \cdots < \frac{a_1+a_n}{2} < \frac{a_2+a_n}{2} < \cdots < \frac{a_{n-1}+a_n}{2}$ equality obtained for arithmetic progressions.

9

The extended binary representation framing is more natural. $\boxed{\left\lfloor \frac{n+2}{2} \right\rfloor}$

4

1
$\boxed{[332, 350]}$ by examining delta differences between squares, resultant inequalities cases.

2
Putnam. $\boxed{3987} = 1993 + 1994$. Indeed note $\frac{a}{b} < \frac{a+c}{b+d} < \frac{c}{d}$ therefore $\frac{m}{1993} < \frac{2m+1}{3987} < \frac{m+1}{1994}$ and the bound results from manipulations of $\frac{1992}{1993} < \frac{k}{n} < \frac{1993}{1994}$

3
$\boxed{1 : 101}$ as one can otherwise factor out the 9 and the 11 terms in the representation $\frac{10^{2n}-1}{99} = \frac{(10^n+1)(10^n-1)}{9 \cdot 11}$

4
A trollish task as $z = 1$ gives
$xy + x + y + 1 = (x + 1)(y + 1) = ((a - 1) + 1)((b - 1) + 1) = ab$

5
Rusin. $\boxed{(2^r, 2^r, 2r, 2r + 1) | r \in \mathbb{Z}_{\geq 0}}$

6
Rusin. Analysis for $n = 1, 2, 3, 4$ modulo 4 works.

7
Thousandth might lead some readers to see the 1000 precision in the task statement as less obvious (numerics in task statements are often very important for they are in fact precisely the task) but in any case $N = \frac{10^{2016}-1}{9} \approx \frac{10^{2016}}{9}$ whence, with a Taylor series type first order second order term break down decomposition e.g. one obtains that $\sqrt{N} \approx \frac{10^{1008}}{3} = 3 \ldots 3.3 \ldots \boxed{3} \ldots$. This sort of approach can be useful as one can engineer a solution backwards by writing out an inequality with that value and an error term.

8
Putnam. And I quote, let n be the smallest positive integer such that $\mathrm{GCD}(s, n) > 1$ for all s in n; note that n has no repeated prime factors. By the condition on S, there exists $s \in S$ which divides n. On the other hand, if p is a prime divisor of s, then by the minimality of n, $\frac{n}{p}$ is relatively prime to some element t of S. Since n can not be relatively prime to t, t is divisible by p, but not by any other prime divisor of s (any such prime divides $\frac{n}{p}$). Thus $\mathrm{GCD}(s, t) = p$, as desired.

9

Rusin. One can do this algebraically after clearing out denominators but I want to note a key idea here prior to the break down which is of the ratio that to move from $\binom{n}{r}$ to $\binom{n}{r+1}$ one multiplies by $\frac{n-r}{r+1}$ whence this task transforms in to an arithmetic progression of $1, \frac{n-r}{r+1}, \frac{(n-r)(n-r-1)}{(r+1)(r+2)}, \frac{(n-r)(n-r-1)(n-r-2)}{(r+1)(r+2)(r+3)}$

5

1
Well I think it's very important that they commute and also that there is a subscript of 4 in M_4 denoting that these are 4×4 matrices which means there exists I think one would admit there exists a computery computational proof at the very least perhaps Wolfram Alpha could output a proof.

2
Putnam.

3
Putnam.

4
Putnam.

5
There exists a linearly independent basis set of r columns from which the column span can be generated i.e. all of the other columns can be written in terms of them as a linear combination and one can thus take precisely that set of columns for B and then a bunch of one hots and other coefficient linear combination representations for C like if the first column of C is $[1, 0, 0, \ldots, 0]$ then the first column of the product will be the first column of B e.g.

6
Putnam.

7
Putnam.

8
Putnam.

9
StackExchange. Commutativity Of Addition VS2. Perhaps Rusin intends something different and perhaps there exists a computery proof for this task as well.

6

1
$\boxed{28}$ I don't know locus Euler line coordinate bash something.

2
Putnam. USAMO.

3
$\boxed{2}$ otherwise the circumcenter is a rational point via perpendicular bisector lines intersection formula for example.

4
Putnam.

5
Putnam.

6
$\boxed{\dfrac{3\sqrt{3}}{2}}$ unit regular hexagon.