2022 Alibaba Global Mathematics Competition Qualifying Round

Name Age Occupation

Question	1	2	3	4	5	6	7	8	Total
Score									

1. Single-Choice Problem: Magic Magnetic Cube

Divide a solid cube $ABCD - A_1B_1C_1D_1$ (with $AB = 1$) into 12 pieces (Figure 1) as follows:

1) Take 6 diagonals of its surfaces AC,AB_1, AD_1, C_1B, C_1D, C_1A_1;
2) Consider all triangles with the center of the cube as a vertex, and one of the above 6 diagonals and 12 edges as the opposite side;
3) These 18 triangles cut the cube into 12 tetrahedra, and each tetrahedron has two edges that are *cube edges*;
4) Each tetrahedron is connected to other tetrahedra **only** by its two cube edges.

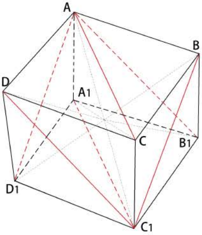

Figure 1: Magic magnetic cube

Such a toy can take on a variety of shapes (Figure 2).

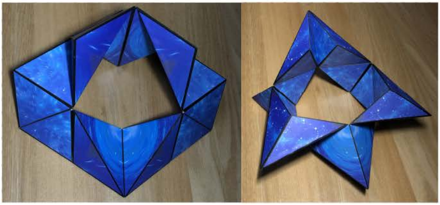

Figure 2: Examples of various shapes

Question: Of all the possible shapes of this toy, what is the maximum distance (in space) between two points on it?

A. $\sqrt{11}$ B. $\sqrt{7+4\sqrt{2}}$ C. $\sqrt{13}$ D. $1+2\sqrt{2}$ E. None of the above

2. Single-Choice Problem: Onlook With Distance

One day, there is a Street Art Show at somewhere, and there are some spectators around. We consider this place as an Euclidean plane. Let K be the center of the show. And name the spectators by $A_1, A_2, \ldots, A_n, \ldots$. They pick their positions

$$P_1, P_2, \ldots, P_n, \ldots,$$

one by one. The positions need to satisfy the following three conditions simultaneously.

(i) The distance between K and A_n is no less than 10 meters, that is, $KP_n \geq 10\text{m}$ holds for any positive integer n.

(ii) The distance between A_n and any previous spectator is no less than 1 meter, that is, $P_m P_n \geq 1\text{m}$ holds for any $n \geq 2$ and any $1 \leq m \leq n-1$.

(iii) A_n always choose the position closest to K that satisfies (i) and (ii), that is, KP_n reaches its minimum possible value. If there are more than one point that satisfy (i) and (ii) and have the minimum distance to K, A_n may choose any one of them.

For example, A_1 is not restricted by (ii), so he may choose any point on the circle C which is centered at K with radius 10 meters. For A_2, since there are lots of points on C which are at least 1 meter apart from P_1, he may choose anyone of them.

(1) Which of the following statement is true?

A. There exist positive real numbers c_1, c_2 such that for any positive integer n, no matter how $A_1, A_2, \ldots, A_n$ choose their positions, $c_1 \leq KP_n \leq c_2$ always hold (unit: meter);

B. There exist positive real numbers c_1, c_2 such that for any positive integer n, no matter how $A_1, A_2, \ldots, A_n$ choose their positions, $c_1\sqrt{n} \leq KP_n \leq c_2\sqrt{n}$ always hold (unit: meter);

C. There exist positive real numbers c_1, c_2 such that for any positive integer n, no matter how $A_1, A_2, \ldots, A_n$ choose their positions, $c_1 n \leq KP_n \leq c_2 n$ always hold (unit: meter);

D. There exist positive real numbers c_1, c_2 such that for any positive integer n, no matter how $A_1, A_2, \ldots, A_n$ choose their positions, $c_1 n^2 \leq KP_n \leq c_2 n^2$ always hold (unit: meter).

(2) Since human bodies are 3-dimensional, if one spectator's position is near another spectator's path of view, then the second one's sight will be blocked by the first one. Suppose that for different i, j, if the circle centered at P_i with radius $\frac{1}{6}$ meter intersects with segment KP_j, then A_j's sight will be blocked by A_i, and A_j could not see the entire show.

Which of the following statement is true?

A. If there were 60 spectators, then some of them could not see the entire show;

B. If there were 60 spectators, then it is possible that all spectators could see the entire show, but if there were 800 spectators, then some of them could not see the entire show;

C. If there were 800 spectators, then it is possible that all spectators could see the entire show, but if there were 10000 spectators, then some of them could not see the entire show;

D. If there were 10000 spectators, then it is possible that all spectators could see the entire show.

3. Single-Choice Problem: Tiger Mystery Box

A milk drink company organizes a promotion during the Chinese New Year: one gets a red packet when buying a carton of milk of their brand, and there is one of the following characters in the red packet " 虎"(Tiger), "生"(Gain), "威"(Strength).

If one collects two "虎", one "生" and one "威", then they form a Chinese phrases "虎虎生威"(Pronunciation: hu hu sheng wei), which means "Have the courage and strength of the tiger". This is a nice blessing because the Chinese zodiac sign for the year 2022 is tiger. Soon, the product of Brave NiuNiu becomes quite popular and people hope to get a collection of "虎虎生威". Suppose that the characters in every packet are independently random, and each character has probability $\frac{1}{3}$.

(1) What is the expectation of cartons of milk to collect "虎虎生威"(i.e. one collects at least 2 copies of "虎", 1 copy of "生", 1 copy of "威")?

A. $6\frac{1}{3}$ B. $7\frac{1}{3}$ C. $8\frac{1}{3}$ D. $9\frac{1}{3}$ E. None of the above

(2) In a weekly meeting of Brave NiuNiu, its market team notices that one often has to collect too many "生" and "威", before getting a collection of "虎虎生威". Thus an improved plan is needed for the proportion of characters. Suppose that the probability distribution of "虎", "生" and "威" is (p, q, r), then which of the following plans has the smallest expectation (among the 4) for a collection of "虎虎生威"?

A. $(p, q, r) = (\frac{1}{3}, \frac{1}{3}, \frac{1}{3})$

B. $(p, q, r) = (\frac{1}{2}, \frac{1}{4}, \frac{1}{4})$

C. $(p, q, r) = (\frac{2}{5}, \frac{3}{10}, \frac{3}{10})$

D. $(p, q, r) = (\frac{3}{4}, \frac{1}{8}, \frac{1}{8})$

4. Proof Question

Given a set X and a function $f : X \times X \to [0, 1]$, we say f is right uniform if for any $\epsilon > 0$, there exist finitely many elements $b_1, b_2, ..., b_m$ in X such that for any $t \in X$, the following holds for some $b_{i(t)}$:

$$|f(x, t) - f(x, b_{i(t)})| < \epsilon, \forall x \in X.$$

Similarly, we say f is left uniform if for any $\epsilon > 0$, there exist finitely many elements $a_1, a_2, ..., a_n$ in X such that for any $t \in X$, the following holds for some $a_{i(t)}$:

$$|f(t, x) - f(a_{i(t)}, x)| < \epsilon, \forall x \in X.$$

Prove that f is right uniform if and only if it's left uniform.

5. Proof Question

Let n be a positive integer and $V = \mathbb{R}^n$ be an n-dimensional Euclidean space with a basis $e_i = (\underbrace{0,\ldots,0}_{i-1}, 1, \underbrace{0,\ldots,0}_{n-i})$ $(1 \le i \le n)$ and with an inner product $(\cdot,\cdot)$ defined by

$$(e_i, e_j) = \delta_{i,j}$$

where

$$\delta_{i,j} = \begin{cases} 1 \text{ if } i = j \\ 0 \text{ if } i \neq j \end{cases}$$

is Kronecker's symbol. For a nonzero vector $v \in V$, define $s_v : V \to V$ by

$$s_v(u) = u - \frac{2(u,v)}{(v,v)}v, \ \forall u \in V.$$

For an integer k between 0 and n, write $\mathrm{Gr}_k(V)$ for the set of k-dimensional subspaces of V. For a k-dimensional subspace W of V, write $[W]$ for the corresponding element of $\mathrm{Gr}_k(V)$. Choose an orthonomal basis $\{v_1, \ldots, v_k\}$ of W, define $s_{[W]} : V \to V$ by

$$s_{[W]} = s_{v_1} \cdots s_{v_k}.$$

(1) Prove that $s_{[W]}$ is independent of the choice of an orthonomal basis $\{v_1, \ldots, v_k\}$.

(2) Prove that $s_{[W]}^2 = \mathrm{id}$.

(3) For another element $[W'] \in \mathrm{Gr}_k(V)$, define

$$t_{[W]}([W']) = [s_{[W]}(W')],$$

where $s_{[W]}(W')$ is the image of W' under $s_{[W]}$. We call a subset X of $\mathrm{Gr}_k(V)$ a "nice set" if

$$t_{[W]}([W']) = [W'], \ \forall [W], [W'] \in X.$$

Find the maximal cardinality of a "nice set" in $\mathrm{Gr}_k(V)$ and prove it.

6. A Busy Courier

A courier picks up a package at coordinate $(n, 0)$ in the two-dimensional lattice and then does a discrete time simple random walk on $\mathbb{Z}^2$. His station locates at the origin $(0, 0)$. In the rest of this question, you may without loss of generality assume n is sufficiently large.

(1) Let $P_{1,n}$ be the probability that at his $\lfloor n^{1.5} \rfloor$th step, the distance between this courier and his station is greater than $\frac{n}{2}$. Prove that

$$\lim_{n \to +\infty} P_{1,n} = 1$$

(2) Let $P_{2,n}$ be the probability that the courier has ever reached the station within his first $\lfloor n^{1.5} \rfloor$ steps. Prove that

$$\lim_{n \to +\infty} P_{2,n} = 0$$

(3) Let $P_{3,n}$ be the probability that the courier has ever reached the station within his first 2^n steps. Prove that

$$\lim_{n \to +\infty} P_{3,n} = 1$$

7. Problem-Solving Question

(1) Show that there is no *periodic* sequence $a_1, a_2, a_3, \ldots$ of signs $a_n \in \{\pm 1\}$ such that, for every *rational* number θ,

$$\sup_{N\in\mathbb{N}} \left| \sum_{n=1}^{N} a_n e^{2\pi i n\theta} \right| < +\infty.$$

(2) Show that there is no sequence $a_1, a_2, a_3, \ldots$ of signs $a_n \in \{\pm 1\}$ such that, for every *rational* number θ

$$\sup_{N\in\mathbb{N}} \left| \sum_{n=1}^{N} a_n e^{2\pi i n\theta} \right| \leq 2022\,?$$

(3) Give an example of a sequence of signs $a_n \in \{\pm 1\}$ such that, for every $\theta \in \mathbb{Q} - \mathbb{Z}$,

$$\sup_{N\in\mathbb{N}} \left| \sum_{n=1}^{N} a_n e^{2\pi i n\theta} \right| < +\infty,$$

and which takes each of the values $+1$ and -1 an infinite number of times.

8. Program Design For The Opening Ceremony

Suppose you are chosen as a technology assistant by the director of the opening ceremony for the 2022 Winter Olympics, and your job is to evaluate the program proposals. One of the backup programs is a skating show of a ensemble of drones dressed as mascots, which are moving along a circle. Since the number of the drones is sufficiently large, we can use a probability density function $\rho(t,v)(\geq 0)$ to represent the distribution of the drones. Here, $v \in \mathbb{R}$ is the line speed, and for a given time t, and two speeds $v_1 < v_2$,

$$\int_{v_1}^{v_2} \rho(t,v)\, dv$$

is the probability of find a drone with its speed between v_1 and v_2.

Suppose that the dynamics of the density function is governed by

$$\rho_t + \big((u(t)-v)\rho\big)_v = \rho_{vv}, \quad v \in \mathbb{R}, \quad t > 0,$$

where $u(t)$ is the command speed.

(1) To investigate proper choices of the command speed, D.B. propose that we shall choose

$$u(t) = u_0 + u_1 N(t)$$

where $u_0 > 0$, $u_1 > 0$ and $N(t)$ is the average of the positive part of the speed $v_+ = \max\{0, v\}$, i.e.,

$$N(t) = \int_{-\infty}^{+\infty} v_+ \rho(t,v)\, dv = \int_0^{+\infty} v\rho(t,v)\, dv.$$

But you claim that if $u_1 > 1$, $N(t)$ may become unbounded in evolution, such that the drones may be out of order. Can you prove it? (For simplicity, the contributions of ρ and its derivatives at $|v| \to +\infty$ are neglected.)

(2) After taking those advices, the directly is wondering whether the drones will be evenly distributed along the circle. Thus, we shall consider the joint density function $p(t,x,v)(\geq 0)$ of position and speed, where $x \in [0, 2\pi]$ is the position coordinate on the circle. Clearly, $\int_0^{2\pi} p(t,x,v)dx = \rho(t,v)$. Suppose that the governing equation for $p(t,x,v)$ is

$$p_t + vp_x + \big((u(t)-v)p\big)_v = p_{vv}, \quad x \in [0,2\pi], \quad v \in \mathbb{R}, \quad t > 0.$$

Because, the drones are circulating around, the following boundary condition is satisfied

$$p(t,0,v) = p(t,2\pi,v), \quad v \in \mathbb{R}, \quad t > 0.$$

You have a feeling that, no matter how the drones are distributed initially, they will become almost evenly distributed very quickly. Can you prove or disprove this statement? (For simplicity, the contributions of p and its derivatives at $|v| \to +\infty$ are neglected.)