Mathematical Statistics Solutions

1.1
1. $\left(\frac{1}{2}\right)^5 = \frac{1}{32}$
2. $1 - \frac{1}{32} = \frac{31}{32}$
3. $\frac{5}{6} \cdot \frac{4}{6} \cdot \frac{3}{6} \cdot \frac{2}{6} = \frac{5}{54}$

1.2
1. $[1, 2]$
2. $[0, 1, 4, \ldots, n^2]$
3. $[-5, -4, \ldots]$
4. $[0, 2, 4, \ldots]$

1.3
(a) - definition

1.4
$\sum_{n=16}^{23} \binom{23}{n} (0.6)^n (0.4)^{23-n}$

1.5
(e) - (a) if independent

1.6
1. Sum of even indexed Ps is $e^{-l} \cosh(l)$
2. Bob as $> \frac{1}{2}$

1.7
$\mathrm{Var}[Y] = \mathrm{E}[Y^2] - \mathrm{E}[Y]^2 = \mathrm{E}[Y^2] = \mathrm{E}[Y_1^2]^2 = 0 \cdot \frac{1}{4} + 1 \cdot \frac{1}{2} + 4 \cdot \frac{1}{4} = \frac{3}{2}$

1.8
1. $[1, 2, 3, 4, 5, 6, 8, 10, 12]$ with $[1, 2, 1, 2, 1, 2, 1, 1, 1]/12$
2. $\mathrm{E}[Y] = \frac{7}{2} \cdot \frac{1}{2} + 7 \cdot \frac{1}{2} = \frac{21}{4}$ and

$\mathrm{Var}[Y] = \frac{1}{12} \cdot 1^2 + \frac{2}{12} \cdot 2^2 + \frac{1}{12} \cdot 3^2 + \frac{2}{12} \cdot 4^2 + \frac{1}{12} \cdot 5^2 + \frac{2}{12} \cdot 6^2 + \frac{1}{12} \cdot 8^2 + \frac{1}{12} \cdot 10^2 + \frac{1}{12} \cdot 12^2 - \left(\frac{21}{4}\right)^2 = \frac{497}{48}$

1.9
(d) - definition

1.10
$\mathrm{E} = \frac{n}{2}, \mathrm{Var} = \frac{(n+1)^2 - 1}{12}$

1.11
Interesting with respect to Rolling A Die Until The First 1, I mean for example if

we have this as a staircase over the unit interval $[0,1]$ then the integral area under the curve is by definition the left hand side summing over columns and is the right hand side summing over rows gives a nice visual.

1.12
$\left[1, \frac{1}{2}, \left(\frac{1}{2}\right)^2, \ldots\right]$ with $\left[p, p(1-p), p(1-p)^2, \ldots\right]$ so $\mathrm{E}[Y] = \frac{p}{1-\frac{1-p}{2}} = \frac{2p}{1+p}$

1.13
For X, Y uncorrelated $\mathrm{Var}[aX+bY] = \mathrm{Var}[aX] + \mathrm{Var}[bY] = a^2\mathrm{Var}[X] + b^2\mathrm{Var}[Y]$ so in this case we have the system of linear equations $4\mathrm{Var}[Y_1] + \mathrm{Var}[Y_2] = 17$ and $\mathrm{Var}[Y_1] + 4\mathrm{Var}[Y_2] = 5$ implies $\mathrm{Var}[Y_1] = \frac{21}{5}$, $\mathrm{Var}[Y_2] = \frac{1}{5}$, and $\mathrm{Var}[Y_1 - Y_2] = \frac{22}{5}$

1.14
(d) - definition

1.15
$n\left(1 - \left(\frac{n-1}{n}\right)^l\right)$

2.1
1. $f(y) = cy^2 1_{[-1,1]}$
2. $c = \frac{3}{2}$
3. $\mathrm{E}[Y] = \int_{-1}^{1} y \frac{3}{2} y^2 dy = 0$ symmetry and $\mathrm{Var}[Y] = \int_{-1}^{1} y^2 \frac{3}{2} y^2 dy = \frac{3}{5}$

2.2
$Y^2 \leq \frac{1}{4}$ if and only if $-\frac{1}{2} \leq Y \leq \frac{1}{2}$ i.e. $P[Y^2 \leq \frac{1}{4}] = \int_{-\frac{1}{2}}^{\frac{1}{2}} f(y) dy = \frac{17}{64}$

2.3
i.e. $1 - \int_{-\frac{1}{\sqrt{2}}}^{\frac{1}{\sqrt{2}}} f(y) dy = 1 - \frac{1}{2\sqrt{2}}$

2.4
i.e. by symmetry $2 \int_{\frac{1}{2}}^{2} f(y) dy = \frac{\tan^{-1}(\frac{24}{7})}{\pi}$

2.5
1. 0 definition
2. 0 definition
3. $\int_0^y f(y) dy = 1 - e^{-\frac{y}{t}}$ for $y > 0$ and 0 for $y \leq 0$
4. $e^{-\frac{1}{t}}$
5. $1 - e^{-\frac{3}{t}}(e^{\frac{2}{t}} - 1)$
6. t
7. $2t^2$
8. t^2
9. 0
10. $t \ln 2$
11. even dominating i.e. 0 to 1 greater than 1 to 2 etc. integral rectangle trapezoidal sums style argumentation

2.6
1. $\frac{4}{(r-l)^2}$
2. $\mathrm{E}[Y] = \frac{r+l}{2}$ symmetry and $\mathrm{Var}[Y] = \frac{(r-l)^2}{24}$
3. $\frac{7-2\sqrt{6}}{6}$

2.7
$u_n = \frac{b^{n+1} - a^{n+1}}{(n+1)(b-a)}$ and setting a, b to $-\frac{b-a}{2}, \frac{b-a}{2}$ one obtains $u_n^c = \frac{(b-a)^n}{(n+1)2^n}$ for even n and 0 for odd n

3.1
$[0, 1, 2]$ stair steps up from 0 to $\frac{1}{4}$ then $\frac{3}{4}$ then 1.

3.2
(c) - definition

3.3
Inverse, reflect over $x = y$

3.4
1. $c = 12$
2. $F(y) = (4 - 3y)y^3, S(y) = 1 - (4 - 3y)y^3$ on $[0, 1]$
3. $[0, 1], h(y) = \frac{f(y)}{S(y)} = \frac{12y^2}{-3y^3+y^2+y+1}$
4. $\frac{2}{3}$
5. $\frac{1}{2}$

3.5
$F(y) = y^2, S(y) = 1 - y^2, h(y) = \frac{f(y)}{S(y)} = \frac{2y}{1-y^2}$

3.6
(b) - definition

3.7
1. $\frac{1}{e}$ integrating past the mean
2. $t \ln 2$

3.8
7

4.1

$g(y) = y^3$, $g^{-1}(w) = w^{\frac{1}{3}}$, $(g^{-1})'(w) = \frac{1}{3}w^{-\frac{2}{3}}$, $F_W(w) = 1 - e^{-\frac{w^{\frac{1}{3}}}{t}}$,

$f_W(w) = \frac{1}{t}e^{-\frac{w^{\frac{1}{3}}}{t}}\frac{1}{3}w^{-\frac{2}{3}}$

4.2

$g(y) = e^y$, $g^{-1}(w) = \ln w$, $(g^{-1})'(w) = \frac{1}{w}$, $\mathrm{E}[W] = \int_0^1 e^y dy = e - 1$, $F_W(w) = \ln w$,

$f_W(w) = \frac{1}{w}$

4.3

1. $g(y) = y^3$, $g^{-1}(w) = w^{\frac{1}{3}}$, $(g^{-1})'(w) = \frac{1}{3}w^{-\frac{2}{3}}$, $f_W(w) = w^{\frac{1}{3}}\frac{1}{\sqrt{2\pi 0.1^2}}e^{-\frac{(w^{\frac{1}{3}}-1)^2}{2\cdot 0.1^2}}\frac{1}{3}w^{-\frac{2}{3}}$
2. $1.1^{\frac{1}{3}} \approx 1.03228$, z-score ≈ 0.3228, P ≈ 0.3734

4.4

(b) - $g(y) = y^2$, $g^{-1}(w) = w^{\frac{1}{2}}$, $(g^{-1})'(w) = \frac{1}{2}w^{-\frac{1}{2}}$, $f_W(w) = \frac{2+3w^{\frac{1}{2}}}{16}\frac{1}{2}w^{-frac12}$

4.5

$g(y) = my^2$, $g^{-1}(w) = (\frac{w}{m})^{\frac{1}{2}}$, $(g^{-1})'(w) = \frac{1}{2}(\frac{w}{m})^{-\frac{1}{2}}\frac{1}{m}$, $f_W(w) =$ blahblahblah

4.6

(b) - $g(y) = -\frac{1}{2}\ln y$, $g^{-1}(w) = e^{-2w}$, $(g^{-1})'(w) = -2e^{-2w}$, $f_W(w) = 2e^{-2w}$

4.7

(b) - integral of $pdf \cdot f$

4.8

(b) - $g(y) = \frac{1}{y^2}$, $g^{-1}(w) = (\frac{1}{w})^{\frac{1}{2}}$, $(g^{-1})'(w) = -\frac{1}{2}w^{-frac32}$, $f_W(w) =$(b)

4.9

(d) - $g(y) = \frac{1}{y^2}$, $g^{-1}(w) = (\frac{1}{w})^{\frac{1}{2}}$, $(g^{-1})'(w) = -\frac{1}{2}w^{-\frac{3}{2}}$, $f_W(w) =$(d)

4.10

(c) - same as 11 by symmetry

4.11

(c) - $g(y) = y^2$, $g^{-1}(w) = w^{\frac{1}{2}}$, $(g^{-1})'(w) = \frac{1}{2}w^{-frac12}$, $f_W(w) =$(c)

4.12

$g(y) = \frac{235.24}{y}$, $g^{-1}(w) = \frac{235.24}{w}$, $(g^{-1})'(w) = -\frac{235.24}{w^2}$, $f_W(w) = \frac{1}{20}\frac{235.24}{w^2}$,

$E = \int_{10}^{30}\frac{1}{20}\frac{235.24}{y}dy \approx 12.922$

4.13

$g(y) = -\ln y,\ g^{-1}(w) = e^{-w},\ (g^{-1})'(w) = -e^{-w},\ f_W(w) = e^{-w}$

5.1
1. $[0, 1, 2, 3]$ with $[\frac{1}{4}, \frac{1}{4}, \frac{1}{4}, \frac{1}{4}]$
2.
$$\begin{array}{cc} & \begin{array}{cccc} 0 & 1 & 2 & 3 \end{array} \\ \begin{array}{c} 0 \\ 1 \\ 2 \\ 3 \end{array} & \begin{bmatrix} \frac{1}{8} & \frac{1}{8} & 0 & 0 \\ \frac{1}{8} & \frac{1}{8} & 0 & 0 \\ 0 & 0 & \frac{1}{8} & \frac{1}{8} \\ 0 & 0 & \frac{1}{8} & \frac{1}{8} \end{bmatrix} \end{array}$$
3. No - definition

5.2
(b) - definition

5.3
(c) - definition in particular the mean of W is $1 \cdot 1 + \frac{1}{2} \cdot 2 + \frac{1}{3} \cdot 3 = 1$ and the variance is $1 \cdot 1 + \frac{1}{4} \cdot 4 + \frac{1}{9} \cdot 9 = 3$

5.4
Geometric probability the distribution is uniform on the unit square and the success region gives $\frac{3}{4}$

5.5
(b) - definition

5.6
1. $f(y_1, y_2) = \frac{1}{t_1} e^{-\frac{y_1}{t_1}} \frac{1}{t_2} e^{-\frac{y_2}{t_2}}$
2. $\frac{t_1}{t_1+t_2}$ by integrating or simply considering the ratio of annihilation at each point in time

5.7
1. No, if we know that $y_1 \approx 1$ for example our posterior on y_2 is very precise around 0 i.e. this is clearly not a product of 2 distributions
2. No, $f(y_1) = \frac{1}{\pi} \cdot 2(1 - y_1^2)^{\frac{1}{2}}$ and similar by definition
3. Geometric probability the success region is symmetry on 1 side of the line $x = y$ so $\frac{1}{2}$
4. $\frac{1}{2}$

5.8
(d) - definition $\frac{2}{3} \cdot \frac{2}{3} + \frac{1}{3} \cdot \frac{1}{3} = \frac{5}{9}$

5.9
1. $c = \frac{8}{3}$

2. $\frac{4}{5}$ and $\frac{56}{45}$
3. $\frac{28}{27}$
4. $\frac{28}{27} - \frac{4}{5} \cdot \frac{56}{45} = \frac{28}{675} \neq 0$, No

5.10
1. $f(y) = ye^{-y}$ - definition integral
2. one could certainly log transform and then find the pdf of $\ln y_2 - \ln y_1$ and then exponentially transform back

6.1
(e) - definition the underlying distribution of Y is $[1, 2, 3, 4]$ with $[0.1, 0.2, 0.3, 0.4]$

6.2
1. $\frac{1}{1-\frac{1}{2}t}$ exponential $\tau = \frac{1}{2}$
2. Poisson $\lambda = 2$
3. $N(-2, 2^{\frac{1}{2}})$
4. $\frac{\frac{1}{3}}{1-\frac{2}{3}e^t}$ Geometric $p = \frac{1}{3}$
5. Binomial $(2, \frac{2}{3})$
6. Uniform $[-1, 0]$
7. $[-1, 4]$ with $[\frac{3}{4}, \frac{1}{4}]$

6.3
(b) - definition

6.4
Sum of 3 independent geometric with $p = \frac{1}{2}$ so $\text{Var}[Y] = 3\frac{1-p}{p^2} = 6$

6.5
(c) - definition the n variances each scaled $\frac{1}{n^2}$

6.6
1. Binomial (k, p)
2. Binomial (n, p)
3. Poisson $P(\lambda)$

6.7
1. definition
2. $(1 - 2t)^{-\frac{1}{2}}$
3. $(1 - 2t)^{-1}$ Exponential $\tau = 2$

6.8
1. $\alpha = (\frac{1-p}{p})^{\frac{1}{2}}$
2. Normal
3. Normal approximation to Binomial

7.1
1. direct sum or Normal approximation
2. direct P of 0 or Poisson approximation

7.2
(d) - definition

7.3
(c) - definition

7.4
(d) - definition

7.5
(d) - definition

7.6
(d) - definition

7.7
(a) - definition

7.9
(e) - definition

7.10
1. factor in the $\frac{1}{2}m$ and σ^2 so $\Gamma(\frac{3}{2}, \sigma^2 m)$
2. $\Gamma(\frac{3n}{2}, \sigma^2 m)$

7.11
1. $(n-1)!$
2. $\frac{1}{\Gamma(k)\tau^k}$
3. Yes

9.1
(e) - definition

9.2
(e) - definition

9.3
(c) - definition

9.4
1. Yes. $\mathrm{Var}[\hat{\mu}] = \frac{\sigma^2}{2}$

9.5
1. 3 and No
2. $\mathrm{Var}[\hat{\theta}] = (\frac{3}{n})^2 \cdot n \cdot \frac{\theta^2}{12} = \frac{3\theta^2}{4n}$

9.6
(c) - $\mathrm{bias}(\hat{\theta}) = \mathrm{E}[\hat{\theta}] - \theta = c\frac{\theta}{2} - \theta = \theta(\frac{c}{2} - 1)$ and $\mathrm{Var}[\hat{\theta}] = \frac{c^2}{n}\frac{\theta^2}{12}$ and (c) from minimizing the relevant sum

9.7
(b) - definition the Variance is constant and minimizes to unbiased

9.8
(b) - definition all unbiased Variance order

9.9
(c) - This is the unbiased estimator for Variance in the known mean setting whence it suffices to compute $\mathrm{Var}[\hat{\theta}]$ sum or Γ i.e.

9.10
1. 2 and $\frac{3}{2}$
2. $\frac{\theta^2}{3}$ and $\frac{\theta^2}{8}$
3. $\hat{\theta}_3$ of course strictly dominating

9.11
1. 0 and τ definition symmetry
2. Yes
3. sum or Γ i.e. $\frac{\tau^2}{n}$

10.1
1. $\Gamma(n, 2)$ Yes

10.2
just do it

10.3
just do it

10.4
just do it

10.5
(c) - definition

10.6
(a) most likely

10.7
just do it

10.8
just do it

10.9
1. by definition the cdf is $(\frac{y}{\theta})^n$ whence pdf is $\frac{ny^{n-1}}{\theta^n}$
2. multiply by θ^n so y^n and ny^{n-1} on the interval $[0, 1]$ of course
3. just do it

10.10
1. just do it
2. just do it
3. just do it

11.1
1. i.e. maximize $e^{-n\theta}\theta^{\sum y}$ at $\theta = \frac{\sum y}{n}$
2. maximize $\sum(\ln y - \theta)^2 = \sum \theta^2 - 2\ln y\theta = n\theta^2 - 2\theta(\sum \ln y)$ at $\theta = \frac{\sum \ln y}{n}$
3. just do it
4. just do it probably biased the MLE being some value below the sample whence its expected value is less than the true mean

11.2
(e) - 1-(d)

11.3
(d) - definition

11.4
(e) - mean

11.5
1. indeed can be converted into sum into product of f terms exponent
2. indeed the overall P is based only on total number of 0s and 1s which this counts implicitly
3. well the product of these f terms decomposes precisely into these dudes by log sum product and the other dudes can go in h for example without reference to β

11.6
(c) - can't be mapped into the requisite sum in the exponent in the product

11.7
(d) - definition if cleared then uniform random point in that space $\frac{1}{\theta^n}$

11.8
2. 11.3 (d)
3. Yes, definition

11.9
1. Mean and as in 11.3 (d) for Variance
2. Mean is unbiased, Variance is biased
3. immediate via hint i.e.