Results in 3-Majority Tournaments

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In this project we prove a number of results related to 3-majority tournaments and extend the famous Erdős-Szekeres theorem (1935) which says that for any sequence of at least $(r-1)(s-1)+1$ distinct numbers, there exists an increasing subsequence of length at least r or a decreasing subsequence of length at least s. We show that this bound is tight for pairs of sequences among three. We prove an upper bound using a construction of points in two dimensions and a recursive inequality related to the guaranteed minimum size of neutral of consistent subsets in pairs among three linear orders, and are able to obtain a family of values of n such that the lower bound is tight. We also create a new function related to bipartite subgraphs, in a notation similar to that of the Ramsey Numbers. We compute a set of specific values, and establish a general upper bound. These results, among others, may be useful to improving the bounds on acyclic sets in k-majority tournaments given by Milans, Schreiber, and West [1]. These are useful in modeling voting scenarios and paradoxes.

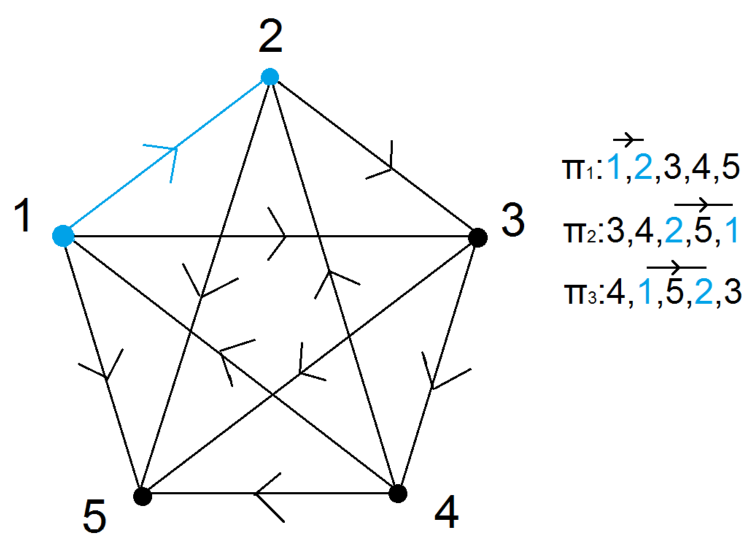
1 Introduction

The basic definition of a graph is a collection of a set of vertices V with a set edges E connecting pairs of V. A digraph is a graph with directed edges. In other words, all edges connecting two vertices in a graph G have an arrow pointing towards one of the vertices. If an edge points from v_1 to v_2, we say that $v_1v_2 \in G$, and if an edge points from v_2 to v_1, we say that $v_2v_1 \in G$. A tournament is a graph that has an edge connecting every pair of vertices.

A subgraph S of a graph G includes a subset of vertices $v \in G$ and edges v_1v_2 if $v_1, v_2 \in S$.

For a finite set L of elements, a linear order π is a permutation of the elements in L. In other words, the elements of L are given a definite rank, such that elements $e_1 > e_2$ if e_1 precedes e_2 according to π.

A k-majority tournament T, where k is odd, is a resulting tournament given by k linear orders, where $v_{e_1}v_{e_2} \in T$ if $e_1 > e_2$ in at least $\frac{k+1}{2}$ of the k linear orders. In the example below, the graph is the result of a 3-majority tournament. For example 1 points to 2, because both π_1 and π_3 rank 1 before 2, which is a majority of the linear orders.



A complete bipartite graph consists of two groups of vertices V_1 and V_2 such that for any two vertices $v_1 \in V_1$ and $v_2 \in V_2$, the directed edge $v_1 v_2$ exists.

In this paper, we use the Erdős-Szekeres Theorem extensively. The theorem states that, for any sequence of at least $(r-1)(s-1)+1$ distinct numbers, there exists an increasing subsequence of length at least r or a decreasing one of length at least s.

1.1 Background

A $\{\pi_1, \pi_2\}$-consistent subsequence of 2 linear orders π_1, π_2 is a set of values in π_1 and π_2, ordered so that it matches the rankings of both linear orders. On the other hand, a $\{\pi_1, \pi_2\}$-neutral set of 2 linear orders π_1, π_2 is a set of values, ordered so that π_2 ranks the items in reverse order as π_1, or vice versa. For example, if $\pi_1 = [2, 3, 1, 5, 4]$ and $\pi_2 = [1, 5, 2, 4, 3]$, then the longest $\{\pi_1, \pi_2\}$-consistent set is $\{1, 5, 4\}$, while the longest $\{\pi_1, \pi_2\}$-neutral set is of size 2, one of which is $\{1, 2\}$.

Milans, Schreiber, and West [1] showed that there always exists either a $\{\pi_1, \pi_2\}$-consistent or $\{\pi_1, \pi_2\}$-neutral subsequence of size at least $\sqrt{n}$, derived from a version of the Erdős-Szekeres Theorem.

Lemma 1.1: Given linear orderings π_1 and π_2 of a set X of n elements with $n > (r-1)(s-1)$, there exists either a $\{\pi_1, \pi_2\}$-consistent set of size r or a $\{\pi_1, \pi_2\}$-neutral set of size s.

Proof: Relabel the elements of π_1 as $[1, 2, 3, \ldots, n]$ in that particular order, and perform the same substitutions on the elements of π_2. By the Erdős-Szekeres Theorem, there must exist either an increasing subsequence of size r or a decreasing subsequence of size s in π_2.

The former subsequence is equivalent to a $\{\pi_1, \pi_2\}$-consistent set, and the latter is equivalent to a $\{\pi_1, \pi_2\}$-neutral set, hence we are done.

Note that we can set $r = s \approx \sqrt{n}$ in order to determine the guaranteed minimum size of a consistent or neutral subset.

2 New Results

We say that one vertex precedes another if it is ranked higher than the other in at least 2 of the 3 linear orders. Define an (x, y)-directed bipartite subgraph to consist of two non-intersecting subsets of vertices, P and Q of size x and y respectively, such that for any two vertices, one in P and one in Q, the vertex in P precedes the vertex in Q. Define the degree of a vertex to be the number of vertices it precedes.

Let the function $r(m, n) = k$ denote the minimum value of k such that for all k-vertex 3-majority tournaments there exists an (m, n)-directed bipartite subgraph. Firstly, note that the function is symmetric; it is easy to see that $r(m, n) = r(n, m)$ by simply reversing the elements of all three linear orders. We will prove that $r(m, n) \leq n \cdot 2^m$.

Theorem 2.1: $r(1, n) \leq 2n$

Proof: Suppose that we have a $2n$-vertex tournament. Because each edge adds 1 to the total degree of the vertices, the sum of the degrees of all of the vertices is the total number of edges, $\binom{2n}{2} = \frac{(2n)(2n-1)}{2} = n(2n-1)$. Now, because there are $2n$ vertices, by the Pigeonhole Principle, there must exist a vertex with degree at least $\frac{n(2n-1)}{2n} = n - \frac{1}{2}$, which, rounded up to the nearest integer, is n. However, we can now choose the vertex in one subgraph and choose n of the vertices it precedes in the other subgraph to obtain a $(1, n)$-directed bipartite subgraph. Thus we have established the desired result.

Theorem 2.2: $r(m, n) \leq 2 \cdot r(m-1, n)$.

Proof: By lemma 1, in a $2 \cdot r(m-1, n)$-vertex tournament, there must exist a vertex that precedes at least $r(m-1, n)$ vertices. However, note that, since there necessarily exists an $(m-1, n)$-directed bipartite subgraph in the subgraph induced by the vertices the original vertex precedes, we can choose the vertices of the $(m-1,n)$-directed bipartite subgraph and add the original vertex to the preceding set, to obtain an (m,n)-directed bipartite subgraph. This proves the lemma.

Now, we can use induction on m to prove the desired result. We already have our base case, that is, $r(1,n) \leq 2n$. Suppose that $r(k,n) \leq n \cdot 2^k$, for all n, and a given value k. Then, by Lemma 2, $r(k+1,n) \leq 2 \cdot n \cdot 2^k = n \cdot 2^{k+1}$. Thus, we have established that $r(m,n) \leq n \cdot 2^m$.

Note that m and n are interchangeable, because the r function is symmetric, so we can assume $m \leq n$, and reverse the variables if $n \leq m$, to obtain a tighter bound.

We now compute the exact value of $r(m,n)$, for small m and n. By Lemma 1, $r(1,n) \leq 2n$. We now claim that $r(1,n) = 2n$ for all n: the sequences for $2n-1$ vertices are:

$1, 2, 3, 4, \ldots, 2n-1$

$n, n+1, n+2, \ldots, 2n-1, 1, 2, \ldots, n-1$

$2n-1, n-1, 2n-2, n-2, 2n-3, \ldots, 1, n.$

For the purposes of this proof, let $x \mod y$ denote the value n such that $x \equiv n \pmod{y}$ and $0 \leq n \leq y-1$. We claim that x precedes y if and only if $1 \leq y-x \mod (2n-1) \leq n-1$. Partition the integers $1, 2, 3, \ldots, 2n-1$ into two groups: $1, 2, ..., n-1$ and $n, n+1, \ldots, 2n-1$. If both x and y are in the same group, by the first two sequences, the lower value precedes, and the difference is always at most $n-1$. If they are in different groups, they are neutral in the first two. However, in the third sequence, if $1 \leq y-x \mod (2n-1) \leq n-1$, then x precedes y, because k is directly before $k+(n-1) \mod (2n-1)$, and thus precedes all terms that are lower and in the other group, based in the construction of the sequence. The argument works in reverse because, assuming x and y are different, $y-x \mod (2n-1) \leq n-1$ if and only if $x-y \mod (2n-1) \geq (2n-1)-(n-1) = n$.

Thus, since every vertex precedes precisely the following $n-1$ vertices, "wrapping around" at the end, there is no $(1, n)$-bipartite subgraph, so this is a counterexample for $2n-1$ vertices and we have established that $r(1, n) = 2n$ for all n.

We now compute $r(2, 2)$. By the result we proved, $r(2, 2) \leq 2 \cdot 2^2 = 8$. By inspection, t is easy to verify that the 7-vertex tournament induced by the following three linear orderings does not contain a $(2, 2)$-directed bipartite subgraph:

$[1, 2, 3, 4, 5, 6, 7]$ $[5, 7, 2, 4, 6, 1, 3]$ $[6, 3, 7, 4, 1, 5, 2]$

Thus, $r(2, 2) > 7$, and we have established that $r(2, 2) = 8$.

In summary, we have so far shown that $r(m, n) = r(n, m) \leq \min(n \cdot 2^m, m \cdot 2^n)$, $r(1, n) = 2n$ for all n, and $r(2, 2) = 8$. Using our results, we can generate a table of maximums for $r(m, n)$, where m is the row value and n is the column value:

	1	2	3	4	5	6
1	2	4	6	8	10	12
2	4	8	12	16	20	24
3	6	12	24	32	40	48
4	8	16	32	64	80	96
5	10	20	40	80	160	192
6	12	24	48	96	192	384

2.1 Largest Common Subsequence for n Linear Orders

Beame and Huynh-Ngoc determined that there always exists a consistent set among some pair of three linear orders of π_1, π_2, π_3, or a subset of elements that appears in the same ranking in two of the three orders, with size of at least $m^{\frac{1}{3}}$, where m is the number of elements in each linear order [2]. We extend this to a general n linear orders, where $n \geq 3$.

Theorem 2.3: Let $\pi_1, \pi_2, \pi_3, \ldots, \pi_n$ be n linear orders of S, which contains m elements. Then the size of the largest common subsequence in $\dfrac{n+1}{2}$ out of n orderings is at least $m^{1/\binom{n}{\frac{n+1}{2}}}$.

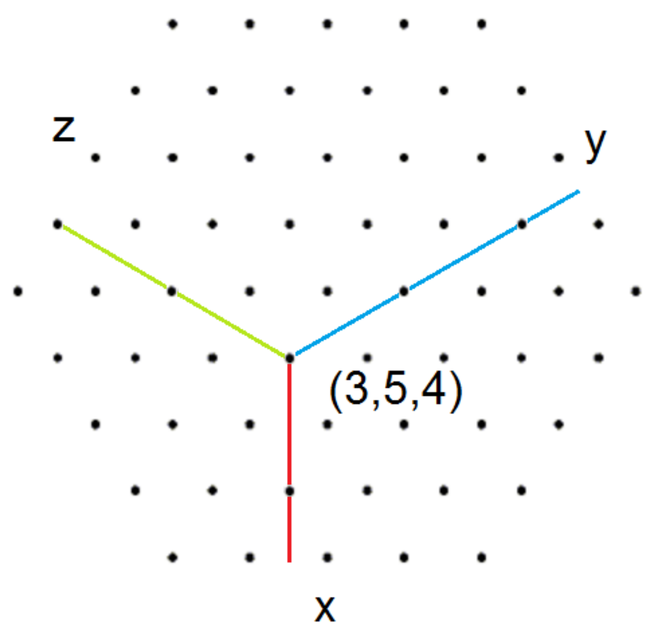
Proof: Suppose, by contradiction, that there is no such common subsequence of this size. For every element $i \in S$, define a tuple t_i with $\binom{n}{\frac{n+1}{2}}$ values, each with a value v such that $v < \lceil m^{1/\binom{n}{\frac{n+1}{2}}} + 1\rceil$ (bounded like so due to our assumption). Additionally, each value within each tuple maps to $\frac{n+1}{2}$ specific linear orderings, without loss of generality, π_1, π_2,..., $\pi_t frac{n+1}2$ and the value represents the longest common subsequence of the elements from $\pi_1[1]$ to $\pi_1[a]$ and the elements from $\pi_2[1]$ to $\pi_2[b]$, ...where $\pi_1[a] = \pi_2[b] = ... = i$. For any elements i, j we must have $t_i \neq t_j$, because there must be some set of $\frac{n+1}{2}$ linear orders, say π_1, π_2,..., $\pi_{\frac{n+1}{2}}$ so that either i precedes j in these linear orderings or j precedes i in these since a majority must have one way. Either way, that means that $t_i[p] \neq t_j[p]$, where p is the number that corresponds to the set π_1, π_2, $\pi_{\frac{n+1}{2}}$. But since there are m different tuples t, which can only take $(\lceil m^{1/\binom{n}{\frac{n+1}{2}}} + 1\rceil)^{\binom{n}{\frac{n+1}{2}}} \leq m$ distinct values, there must exist a tuple with a value v such that $v \geq \lceil m^{1/\binom{n}{\frac{n+1}{2}}} + 1\rceil$, a contradiction. Therefore, the size of the largest common subsequence in the n orderings is at least $m^{1/\binom{n}{\frac{n+1}{2}}}$, as desired.

2.2 Bounding the Maximum

Define the function $h(n)$ to represent the guaranteed length of the longest $\{\pi_i\pi_j\}$-consistent or $\{\pi_i\pi_j\}$-neutral subsequence of three linear orders π_1, π_2, π_3 for all $1 \leq i < j \leq 3$.

Theorem 2.4: $h(n) \leq \sqrt{\frac{4n}{3}}$

Proof: Consider a set of lattice points arranged in a hexagonal formation. Let coordinates x, y, and z denote the distances from three of the six sides, where no three sides are adjacent. In the figure below, the example point is 3 "rows" away from the bottom row of the hexagon, which has an x-coordinate of 0, so the x-coordinate of the example point is 3. Similarly, the y-coordinate and z-coordinate are 5 and 4, respectively.



Lemma 2.5: A hexagonal grid with $2s-1$ points in the diameter contains $1+3(s)(s-1)$ points overall. Proof: Notice that $2s-1$ points in the diameter implies that there are s points in a single edge. We prove this by induction. When $s=1$, the grid is a single point, and $1+3(1)(1-1) = 1$. Next, consider a grid with $s+1$ points in one edge, which is essentially a hexagonal grid of s with a thin shell of points attached around the grid. This shell contains $6s$ points, because there are $6(s+1)$ points in all the edges, minus the 6 corners due to overcounting. Additionally, $1+3(s)(s-1)+6s = 1+3(s+1)(s)$, and the induction

is complete.

We now define three linear orderings of the points in the hexagonal array with s points per edge: π_1, which ranks $(x_1, y_1, z_1) > (x_2, y_2, z_2)$ if $x_1 > x_2$ or $x_1 = x_2$ and $y_1 > y_2$, π_2, which ranks $(x_1, y_1, z_1) > (x_2, y_2, z_2)$ if $y_1 > y_2$ or $y_1 = y_2$ and $z_1 > z_2$, and π_3, which ranks $(x_1, y_1, z_1) > (x_2, y_2, z_2)$ if $z_1 > z_2$ or $z_1 = z_2$ and $x_1 > x_2$. Without loss of generality, we may consider only π_1 and π_2, and attempt to find the longest $\{\pi_1, \pi_2\}$-consistent subset of points in the grid. Note that two points $(x_1, y_1, z_1) > (x_2, y_2, z_2)$ for both π_1 and π_2 if $x_1 = x_2, y_1 > y_2$ or $x_1 > x_2, y_1 > y_2$ or $x_1 > x_2, y_1 = y_2, z_1 > z_2$, although the latter case is nonexistent, as there are no points that satisfy the bounds since the sum of the coordinates is constant for all points in our construction (it is known that the sum of the distances from a point inside an equilateral triangle to the sides is constant from sum of areas and this generalizes to a simplex). Therefore, we see that a different y-coordinate is needed to determine consistency. As the points occupy a maximum of $2s - 1$ different y-coordinates, this is the maximum possible size of A $\{\pi_1, \pi_2\}$-consistent subset.

Now we attempt to find the largest $\{\pi_1, \pi_2\}$-neutral subset of points. (x_1, y_1, z_1) and (x_2, y_2, z_2) are neutral with respect to these two linear orderings if $x_1 > x_2, y_1 < y_2$ or $x_1 < x_2, y_1 > y_2$ or $x_1 > x_2, y_1 = y_2, z_1 = z_2$ or $x_1 > x_2, y_1 = y_2, z_1 \leq z_2$, where no point on the hexagonal grid can fill the last two inequality sets, as $x+y+z$ is constant. In other words, two points are neutral to each other if both their x and y coordinates differ. Again, there are only $2s - 1$ different x and the same number of y coordinates, the maximum possible size of A $\{\pi_1, \pi_2\}$-neutral subset. By symmetry, the same can be said for the sizes of the $\{\pi_2, \pi_3\}$-consistent,$\{\pi_2, \pi_3\}$-neutral,$\{\pi_3, \pi_1\}$-consistent, and $\{\pi_3, \pi_1\}$-neutral subsets.

This implies that $h(1+3(s)(s-1)) \leq 2s-1$, which simplifies to $h(3s^2) \leq 2s = \sqrt{\frac{4(3n^2)}{3}}$. Making the substitution $3s^2 = n$, we arrive at $h(n) \leq \sqrt{\frac{4n}{3}}$, as desired.

2.3 Erdős-Szekeres Pairwise in 3 Linear Orderings, Lower Bound

Inspired by the previous works relating to three linear orderings, we became interested in the tightness of a bound related to a different problem concerning three linear orderings on $[n]$. We investigated bounds on the size of the largest common subsequence or common reversed subsequence (where the terms are in the opposite order) that is guaranteed among two out of three linear orderings. In particular we wanted to determine if the Erdős-Szekeres bound (which says that among two linear orderings on $[n]$ there is a common subsequence or common reversed subsequence of length $\sqrt{n}$) can be tight for each pair among three linear orderings. This would have potentially been able to improve the results in the paper Acyclic sets in k-majority tournaments by Milans, Schreiber, and West.

Using the hexagonal construction we found an upper bound of $h(n)$ to be $\sqrt{\frac{4n}{3}}$ and we also found a construction for the tightness of the bound, in particular

Theorem 2.5: $h(n) \geq \sqrt{n}$ with equality occurring when $n = 9^m$ for any positive integer m

Proof: The following are three permutations we found for which each pair has the largest common subsequence and reverse subsequence having length $\sqrt{9} = 3$:

$[1, 2, 3, 4, 5, 6, 7, 8, 9]$

$[3, 2, 6, 9, 5, 1, 4, 8, 7]$

$[6, 2, 1, 7, 5, 3, 9, 8, 4]$

We will prove that $h(a)h(b) \geq h(ab)$ by applying a constructive process that merges linear orderings.

Theorem 2.6: $h(a)h(b) \geq h(ab)$.

Proof: Let $x \mod y$ denote the value n such that $x \equiv n \pmod y$ and $0 \leq n \leq y-1$. We aim to merge orderings in a way to preserve the desired property. Indeed say three linear orders of $[a]$ that give the optimal maximum guaranteed length of common subsequence or reversed subsequence among pairs are L, M, and N, where $L = [L_1, L_2, \ldots, L_a]$ and similarly for the others. Say the three on $[b]$ are X, Y, and Z which are defined similarly as above. Then three linears of $[ab]$ that give the equality in the inequality are A, B, and C where we define

$A_n = (L_{\lceil \frac{n}{b} \rceil} - 1)b + X_{1+((n-1) \mod b)}$ and similarly

$B_n = (M_{\lceil \frac{n}{b} \rceil} - 1)b + Y_{1+((n-1) \mod b)}$

$C_n = (N_{\lceil \frac{n}{b} \rceil} - 1)b + Z_{1+((n-1) \mod b)}$

For example, if

$L = [1, 2, 3, 4, 5], M = [3, 2, 5, 1, 4], N = [2, 1, 5, 3, 4]$

$X = [1, 2, 3], Y = [2, 3, 1], Z = [3, 1, 2]$

then the three resulting linear orderings would be

$[1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12, 13, 14, 15]$

$[8, 9, 7, 5, 6, 4, 14, 15, 13, 2, 3, 1, 11, 12, 10]$

$[6, 4, 5, 3, 1, 2, 15, 13, 14, 9, 7, 8, 12, 10, 11]$

We have created a blocks(ranges) of 1 to b, from $b+1$ to $2b$, $\ldots$ and then we arrange the blocks according to the ordering in L, M, and N and then inside each block we arrange the elements according to X, Y, and Z. In particular we have that the elements in the first b positions will be all from one and exactly one of the above defined ranges and similarly in the next b positions etc.

Indeed assume that there was among a pair of A, B, and C a common subsequence or reversed subsequence of length greater than $h(a)h(b)$. Then either we have in this common subsequence or reversed subsequence elements from more than $h(a)$ different ranges or we have more than $h(b)$ elements in the subsequence in one range. But this would be a contradiction since the former would contradict the fact that L, M, and N had the maximum length being $h(a)$ and X, Y, and Z had the maximum length being $h(b)$.

Then by induction we can prove that $h(n) \geq \sqrt{n}$ with equality occurring when $n = 9^m$ as claimed, since we have by construction above that $h(9) = 3$ and thus the base case of $m = 1$ is done. Then the inductive step is that $h(9^m)h(9) \geq h(9^{m+1})$ which means $3^{m+1} \geq h(9^{m+1})$ but we also know from the Erdős-Szekeres theorem that $3^{m+1} \leq h(9^{m+1})$ hence equality holds.

3 Future Work and Applications

We will work to strengthen the upper bound of the problem of finding the minimum number of vertices n to ensure a complete bipartite graph.

We will also attempt to extend the 3 linear orders problem to a general $2k-1$ linear orders, where we find the largest common consistent or neutral subsequence found in k of the linear orders. We expect the upper bound to be about $n^{\frac{1}{k}}$, where n is the number of elements in each linear order.

Also we hope to improve on the bound presented by Milans, Schreiber, and West for $f_5(n)$ in a perhaps somewhat analogous fashion to the improvement on the bounds of $f_3(n)$ by Shen, Shen, and Ilic [4]. In particular we tried to continue the idea for a construction that they presented but first had difficulties visualizing in 4 dimensions and then realized that a construction is not possible which inductively works in a similar fashion immediately since the premise for $n = 3$ relies on the fact that any two points have one dominate the other one by having a higher coordinate in 2 out of 3 coordinates (with edge cases accounted for) but for $n = 5$ we see that the locus of points which have a higher coordinate for 3 out of 5 is not 5 subgraphs which are similar to the whole graph, which would be what we would need 4 out of 5 coordinates so essentially two points have an indeterminate domination when this occurs if we try to construct a similar set of linear orderings. We have considered ways to construct sets of points to account for this or ways to fudge locally to improve bounds.

Also we would like to tighten the bounds on $f_3(n)$ using our results from the bounds of consistent and neutral sets of three linear orderings by examining the relationship between these types of sets and acyclic subgraphs. In particular we would ideally like to completely solve this by either considering some new construction or creating a more powerful way to analyze the problem since there is some power when dealing with just 3 linear orderings.

4 Conclusion

In graph theory, k-majority tournaments represent an essential way to visualize voting or ranking preferences between judges and objects, by utilizing linear orders, one for each judge. Our paper proves a recursive inequality $h(mn) \leq h(m)h(n)$ for how similar or dissimilar two votes in a group of 3 can be, by analyzing the minimax of consistent and neutral subsequences, which tends to $\sqrt{n}$. We also tighten the upper bound to $h(n) \leq \sqrt{\frac{4n}{3}}$ by considering a hexagonal lattice representation of n objects with a triangular coordinate system. This provides additional insight into the nature of the Erdős-Szekeres theorem in the context of applying pairwise among three orderings. If we could find analogous results for more than 3 this could directly improve bounds in the subject of k-majority tournaments

The numbers $r(m, n)$ helps us to find bipartite subgraphs, which are very useful for finding groups that win over other groups in voting sfcenarios, and also helps us to find acyclic subsets because, after we have found a bipartite subgraph, if we verify that the individual groups of vertices are acyclic, we can conclude that the union of the individual groups forms an acyclic subset.

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