Introduction

1
Wow I don't recall ever proving that e is irrational but I think that one can obtain a contradiction sandwiching on the denominator between 2 rationals.

2
Sylvester 2^n from $\begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}$ and $\begin{bmatrix} A & A \\ A & -A \end{bmatrix}$

4
The statement is equivalent with having disjoint sets of points and their distance 1 neighbours whence we obtain the upper bound of $\frac{2^7}{1+\binom{7}{1}} = \frac{2^7}{8} = 16$. This bound is obtained Hamming $(7,4)$ for the 16 binary strings of length 4 augmented with 3 parity bits covering $\{d_1, d_2, d_4\}, \{d_1, d_3, d_4\}, \{d_2, d_3, d_4\}$

1
Induction like $3^2 + 4^2 = 5^2$ and multiply all representations by 5^2 and then to generate a new $n+1$ break down 1 dude in to 2 via that Pythagorean triple.

2
An inductive construction is take 2 points at a unit distance from each other now copy and paste that set at some non intersecting unit offset so that each point has precisely 1 new unit distance point in the set, namely its corresponding point in the copy. Iterate. This translates in to taking n sufficiently close unit vectors and for each binary string, each subset, add those unit vectors in the subset so generate a quasi embedding of the hypercube of n-dimensions in to $\mathbb{R}^2$ essentially on the underlying unit-distance graph.

3
Contradiction consider the face with the maximal number of edges and contradiction if all distinct on its adjacent faces.

4
Contradiction. Indeed the condition says that for each of the positive integers it must be the case that it contains the strict majority of at least 1 prime.

5
Putnam And Beyond. Without loss of generality $r \geq g \geq b$ not obtained then sphere, circle, segment contradiction.

6

Induction.

7

Impossible for odd n as $(2n-1)\binom{\frac{n+1}{2}}{} > n^2$ counting incidences as each diagonal element contributes to 1 of these union sets, and each off-diagonal element contributes to 2 of these union sets, and we must hit each set thus requiring at least $\frac{n+1}{2}$ of each of the $2n-1$ values to appear in the matrix. For the even case, and the particular task of demonstrating existence for infinitely many values of n, one may do an inductive/recursive construction for powers of 2. Namely, create 4 copies, and shift the upper right and lower right copies by 2^n and then shift the upper right diagonal by -1, so for example at step 4 we have:

1	2	4	6	8	10	12	14
3	1	7	4	11	8	15	12
5	6	1	2	13	14	8	10
7	5	3	1	15	13	11	8
9	10	12	14	1	2	4	6
11	9	15	12	3	1	7	4
13	14	9	10	5	6	1	2
15	13	11	9	7	5	3	1

8

Eventual cyclicity argumentation.

9

USAMO. $\boxed{1999}$ contradiction take a row and column excluding their intersection and in the other direction argue on rows with precisely 1 coloured square.

10

USAMO. $\boxed{10^5} = 10^{n-1}$ bound and construction final bit error correcting sum of first $n-1$ digits modulo 10 e.g.

11

Induction.

12

Contradiction otherwise we know precisely the full set of values taken on whence a contradiction arises by summing modulo n!

13

14

IMO. Splitting lines left right argumentation.

Polynomials

1
$\boxed{x^6 - 6x^4 - 6x^3 + 12x^2 - 36x + 1}$ is the minimal polynomial.

2
Contradiction degree blowup magnitude argumentation.

3
Smoothing, first derivative argumentation should work.

4
$1, i, -i$ work and e^{ia} substitution or Vieta in reverse cubic formula should work for uniqueness if not some more obvious argumentation.

5
Quora. Assume not. Say $n = \deg(P(x)) \geq \deg(Q(x))$ and $P(x)$ has A distinct roots: A_1 distinct single roots and A_2 distinct multiple roots. Then $P'(x)$ has $n - A$ roots in common with $P(x)$ counting multiplicity. Similarly $P(x) - 1$ has $n - B$ roots in common with $P'(x) - 1 = P'(x)$. But these are distinct thus $P'(x)$ has at least $2n - A - B$ roots and degree $n - 1$ whence $A + B \geq n + 1$. But then as these A roots of $P(x)$ and B roots of $P(x) - 1$ are also roots of $Q(x)$ and $Q(x) - 1$ we obtain that they are $A + B$ distinct roots of $P(x) - Q(x)$ which is a polynomial of degree at most n, and thus is the 0 polynomial contradiction.

6

Number Theory

1
$(x-y)^{p-1} \equiv x^{p-1} + x^{p-2}y + \cdots + y^{p-1} \pmod{p}$ or note by Fermat that this expression must be 1 or 0 if and only if $x \equiv y \pmod{p}$ if and only if $x^p \equiv y^p \pmod{p}$ and norming.

2
$\boxed{7}$ indeed $3^{\text{odd}} \equiv (-1)^{\text{odd}} \equiv -1 \pmod{4}$ thus $3, 9, 7, 1, 3, 9, 7, 1, \ldots$ implies this.

3
$\boxed{87}$ indeed find the fixed point is easier than modulo 25 and 4 Chinese Remainder Theorem ϕ reductions.

4
StackExchange. For example letting C be the number of consecutive pairs in the partition and S be the number of pairs that differ by 6 one obtains that $C + S = 999$ and the sum of the differences is $C + 6S = 999 + 5S$ whence it suffices to note that $a + b \equiv a - b \pmod{2}$ means the sum of differences is equivalent with the overall sum $1 + 2 + \cdots + 1998 = \frac{1998 \cdot 1999}{2} = 999 \cdot 1999 \equiv 1 \pmod{2}$ and thus one obtains as desired $9 \pmod{10}$.

5
$\boxed{\text{Yes}}$, for example converting this in to a statement about a sequence of elements in Z_2^∞ e.g. binary strings corresponding with prime factors so $2^1 \cdot 3^0 \cdot 5^1 \cdot 7^1 = 70$ corresponds with $101100\ldots$ one obtains for example:
$1100000000000000\ldots$
$0011000000000000\ldots$
$1000110000000000\ldots$
$0110001100000000\ldots$
$1001100011000000\ldots$
$0110011000110000\ldots$
$1001100110001100\ldots$
$0110011001100011\ldots$

6
USAMO. $\boxed{\text{GCD}(r, s)}$

7
USAMO. Let b be the least n for which the problem statement is false. Note that for all integers b, the sequence $2^0, 2^1, 2^2, \ldots$ eventually becomes cyclic modulo b.

Let k be the period of this cycle. Since there are $b-1$ nonzero residues modulo b one obtains that $1 \leq k \leq b-1 < b$ and then it follows that as the sequence does not become constant modulo b then the sequence of exponents must not become constant modulo k contradicting the constructed minimality of b.

8

RMM. $\boxed{2}$

9

IMO. Let $p_1 < p_2 < p_3 < \dots$ be the prime numbers. Let A consist of all products of n distinct primes such that the smallest is greater than p_n. Now let $S = \{p_{a_1}, p_{a_2}, \dots\}$ be any infinite set of primes. Then $p_{a_1} p_{a_2} \dots p_{a_{a_1}}$ is not in A but $p_{a_2} p_{a_3} \dots p_{a_{a_1}+1}$ is.

Calculus

1
$\boxed{(2x+1)\cos(x^2)}$ and for example to deduce that the 10th derivative of this is 0 at 0 note the Taylor series implies vanishing for $n \equiv 2, 3 \pmod 4$

2
$\boxed{\dfrac{\pi}{4}}$ from $\frac{x}{2} + \frac{\ln(\sin(x)+\cos(x))}{2}$

3
$\boxed{0}$ as $u = \sqrt{x} = x^{\frac{1}{2}}$ has $du = \frac{1}{2}x^{-\frac{1}{2}}dx$ and thus $\int_1^2 \frac{2(u^2-2)}{u^4+4}$ by partial fractional decomposition in conjunction with Sophie Germain per usual one obtains $\frac{1}{2}(\ln(u^2-2u+2) - \ln(u^2+2u+2))|_1^2 = 0$

4
Steinmetz solid. $\boxed{8(2-\sqrt{2})}$

5
$\boxed{\ln(2)}$

6
$\boxed{\dfrac{\pi\ln(2)}{8}}$

7
Let $I(a) = \int_0^\infty \frac{\arctan(ax)-\arctan(x)}{x}dx$. Then one obtains that $I'(a) = \int_0^\infty \frac{1}{1+a^2x^2}dx = \frac{1}{a}\int_0^\infty \frac{a}{1+a^2x^2}dx = \frac{1}{a}\cdot\arctan(ax)|_0^\infty = \frac{\pi}{2a}$. $I(1) = 0$ and the desired is $I(\pi)$ by construction whence we obtain by integrating $\int_1^\pi \frac{\pi}{2x}dx = \frac{\pi}{2}\ln(x)|_1^\pi = \boxed{\dfrac{\pi}{2}\ln(\pi)}$

8

Functional Equations

1
$f(x^2) = f(x) = f(x^{\frac{1}{2}}) = \dots$ in the limiting case as $n \to \infty$ one obtains by continuity that $f(x) = f(1)$ and it follows too that $f(0) = f(1)$ as desired.

2
Note that $\boxed{f(x) = cx + 1}$ works. So for example $g(x) = f(x) - 1$ whence one obtains $g(x + y) = g(x) + g(y)$ and the desired by Cauchy.

3
$\frac{1}{2} = f(0 + 0) = f(0) = f(a)$ whence $f(0 + y) = \frac{1}{2}f(a - y) + \frac{1}{2}f(y)$ so $f(y) = f(a - y)$ and $f(x + y) = 2f(x)f(y)$ so $2f(x + y) = 2f(x)2f(y)$ so $\ln(2f(x + y)) = \ln(2f(x)2f(y)) = \ln(2f(x)) + \ln(2f(y))$ and $g(x) = \ln(2f(x))$ satisfies $g(x + y) = g(x) + g(y)$ I don't know use the flip flop somehow. Balkan. $f(x + a - x) = f(x) = (f(x))^2 + (f(a - x))^2 = 2(f(x))^2$ so $f(x) = \pm\frac{1}{2}$ but then $f(x) = f\left(\frac{x}{2} + \frac{x}{2}\right) = 2f(x)f\left(a - \frac{x}{2}\right) = \frac{1}{2}$

4
StackExchange. $f(x + y) + 1 = (f(x) + 1)(f(y) + 1)$ so $g(x + y) = g(x)g(y)$ is Cauchy $g(x) = x^c$ whence $\boxed{f(x) = x^c - 1}$

5
StackExchange. A substitution takes this sort of inequality to $f(a) + f(b) + f(c) \geq f(2b + c)$ and then e.g. the summation of cyclic permutations of this inequality gives something but in this task plugging in $(a, 0, 0)$ and $\left(\frac{a}{2}, \frac{a}{2}, -\frac{a}{2}\right)$ gives $f(0) \geq f(a)$ and $f(a) \geq f(0)$ thus $f(a) = f(0)$ as desired.

6
StackExchange. this means $f(f(f(x))) = x$ has solutions x, y, z but as a composition one obtains $f(f(x))$ is increasing and $f(f(f(x)))$ is decreasing which means it has a unique zero, the unique zero of $f(x) = x$.

7
IMO.

8
StackExchange. Continuous involutions 2 fixed points surjective injective invertible symmetric around the line $y = x$ contradiction if there exists interval without fixed point on graph lying above or below this diagonal on that interval hence identity.

9
IMO.

10
IMO.

11
$\boxed{\text{Yes } f(x) = \dfrac{1+\sqrt{5}}{2} \cdot x, \text{ rounded}}$ works. Indeed suppose without loss of generality that $\frac{1+\sqrt{5}}{2} \cdot x = y + r$ with $r < \frac{1}{2}$ so that $f(f(x)) = f(y) = \frac{1+\sqrt{5}}{2} \cdot \left(\frac{1+\sqrt{5}}{2} \cdot x - r\right)$ rounded and it suffices to note that the literal integer value of $f(x) + x = x + \frac{1+\sqrt{5}}{2} \cdot x - r$ and the difference in the unrounded is $\frac{1-\sqrt{5}}{2} \cdot r$ which is even lower in magnitude as desired hence the remainder rounding term is correct and the desired functional equation is satisfied.

Inequalities

1
Yes this is a very classical result of partial/prefix/interval sums and the solution is traditionally presented as going around letting the gas metric go negative and starting at the global minimum.

2
We can always smooth regardless of sign keeping the sum fixed or increasing it with a flip flop while strictly increasing the sum of the squares in to the $2, 2, \dots, 2, n - 2(n-1) = -(n-2)$ case and observe the inequality equality case threshold.

3
i.e. the sum of 2 non negative real numbers is 4 what is the smallest possible value of the sum of their squares i.e. $\boxed{8}$

4
ISL. Monovariant with bound the product of all the numbers becomes at least 4 times larger after each step and then $AM - GM$ gives the desired equality at the symmetric pair off all the 1s in to 2s then 2s in to 4s etc. etc.

5
IMO. Inded pairing off symmetric elements it suffices to show that $a_k + a_{m+1-k} \geq n+1$ but if not then the k distinct numbers $a_1 + a_{m+1-k}, a_2 + a_{m+1-k}, \dots, a_k + a_{m+1-k}$ are all $\leq n$ and hence equal to some a_i. But then they'd all be greater than a_{m+1-k} but there are only $k-1$ such a_i in that range contradiction.

6
TSTST.

7
Putnam. Recall the curve length and Taylor series.

8
ISL.

Convergence

1

$\boxed{\text{Divergent}}$ as $\int \frac{1}{x\ln(x)} = \ln(\ln(x))$

2

$\boxed{\text{Convergent}}$ as $\sum_{n=2}^{\infty} \frac{1}{n(\ln(n))^2} \leq \sum_{n=1}^{\infty} \frac{2^n}{2^n(\ln(2^n))^2} = \sum_{n=1}^{\infty} \frac{1}{n^2(\ln(2))^2} = \frac{\pi^2}{6(\ln(2))^2}$ by Cauchy Condensation Test.

3

$\boxed{\text{Divergent}}$ as $\int \frac{1}{x\ln(x)} = \ln(\ln(\ln(x)))$

4

Riemann Series/Rearrangement Theorem.

5

VTRMC. $\boxed{\text{Convergent}}$. Direct comparison with the maximum $a_1 + a_2 + a_4 - a_3 - a_5 - a_6 + \ldots$

6

$\boxed{\text{Convergent}}$

7

8

$\boxed{\ln(2)}$

9

VTRMC. Comparison, AM-GM, and the convergence of $\sum \frac{1}{n^2}$

10

Putnam.

11

12

Recursions

1
Induction.

2
No $91 \neq 89$

3
$\boxed{\dfrac{1+\sqrt{5}}{2}}$ by Binet e.g.

4
Sequence A000930 on the OEIS. $\boxed{277}$

5
$n^2-1=(n+1)(n-1)=101\cdot 99=\boxed{9999}$

6
USAMO. Bijection.

7
Zeckendorf binary strings algorithms and so on and so on.

8
Note $\frac{1}{F_nF_{n+2}}=\frac{F_{n+1}}{F_nF_{n+1}F_{n+2}}=\frac{F_{n+2}-F_n}{F_nF_{n+1}F_{n+2}}=\frac{1}{F_nF_{n+1}}-\frac{1}{F_{n+1}F_{n+2}}$ whence the summation telescopes to $\frac{1}{F_1F_2}=\boxed{1}$

9
$\boxed{14}$ cyclic modulo $16, 17$ probably works e.g.

10
Advanced Problems And Solutions. Wow. With Fibonacci $F_1=1, F_2=1, F_{r+1}=F_r+F_{r-1}$ and Lucas $L_1=1, L_2=3, L_{r+1}=L_r+L_{r-1}$ one obtains that $(-1)^r+F_r^2=F_{r+1}F_{r-1}, L_r=F_{r+1}+F_{r-1}$ and subtracting 4 times the first from the square of the second one obtains that $5F_r^2+4(-1)^r=L_r^2$. The converse is a little tricky.

11
StackExchange.

Linear Algebra

1
$(I+aP)(I-cP) = I+(a-c)P - acP^2 = I+(a-c-ac)P$ so if $a-c-ac=0$ then $I+aP$ is the desired inverse and one obtains $a(1-c)=c \to a = \frac{c}{1-c}$ and
$\boxed{I+\dfrac{c}{1-c}P}$

2
$$\begin{vmatrix} 0 & 1 & 1 & 1 & 1 \\ 1 & 0 & 1 & 1 & 1 \\ 1 & 1 & 0 & 1 & 1 \\ 1 & 1 & 1 & 0 & 1 \\ 1 & 1 & 1 & 1 & 0 \end{vmatrix} = \begin{vmatrix} -1 & 0 & 0 & 0 & 1 \\ 0 & -1 & 0 & 0 & 1 \\ 0 & 0 & -1 & 0 & 1 \\ 0 & 0 & 0 & -1 & 1 \\ 1 & 1 & 1 & 1 & 0 \end{vmatrix} = \begin{vmatrix} -1 & 0 & 0 & 0 & 1 \\ 0 & -1 & 0 & 0 & 1 \\ 0 & 0 & -1 & 0 & 1 \\ 0 & 0 & 0 & -1 & 1 \\ 0 & 0 & 0 & 0 & n-1 \end{vmatrix} =$$
$\boxed{(-1)^{n-1}(n-1)}$

3
$abc = \boxed{3}$ given that by Spectral Mapping one obtains
$a+b+c=1, a^2+b^2+c^2=-3, a^3+b^3+c^3=4$ by for example the usual Vieta symmetric expressions manipulations.

4
Vandermonde argumentation gives $(x_i - x_j)|\det$ whence one can deduce the desired on residues and primes and number theoretic argumentation.

5
No as $(A^2+B^2)(A-B) = A^3 - A^2B + B^2A - B^3 = 0$ and $A \neq B$ one would have $(A-B) = (A^2+B^2)^{-1}(A^2+B^2)(A-B) = (A^2+B^2)^{-1}0 = 0$ contradiction.

6
7
Probably some algebraic manipulations related to traces and nilpotent matrices again perhaps utilizing the 0 exponential traces fact and 2 eigenvalues algebra.

8
StackExchange. Suppose that $A=UV$ is a rank decomposition. We want $UVBUV = ABA = A = UV$ hence it suffices to find a B such that $VBU = I$

9
Probably follows from Cayley-Hamilton.

Combinatorics

IMC $\boxed{6}$ logic.

1
The 1 and 0 multiplicities line up system of equations perhaps kind of inductively.

2
Same.

3
Some quasi greedy pairing symmetry like try and distribute as evenly as possible with respect to opponent possibly a greedy ranking based with opponent edge degree works if not one based on explicit decomposition in to matchings.

4
IMC. Yoav Krauz pairing argumentation.

5
StackExchange. A for $2, 3$ and B for > 3 due to parity of maximal minimal generating sets.

6
ISL. Constructed line set, incidences, monovariant, yadda yadda.

Integer Polynomials

1

Shift to $P(-6)-2P(0)+P(6)$ whence on degree one obtains it suffices to examine terms of degree ≥ 2 and thus $\boxed{72}$

2

$(2x-y)(2x+1-y)$

3

Cyclotomic polynomials, roots of unity, Eisenstein
$\frac{x^p-1}{x-1}=\frac{(y+1)^p-1}{y}=y^{p-1}+\binom{p}{1}y^{p-2}+\binom{p}{2}y^{p-3}+\cdots+\binom{p}{1}$

4

$(P(x)-1)(P(x)+1)=(x-a)(x-b)(x-c)Q(x)=-1$ magnitude contradiction e.g.

5

This is more commonly stated in the form that it is impossible to have $P(a)=b, P(b)=c, P(c)=a$ but this is the same it follows from distinct and divisibility that $(b-a)|(c-b),(c-b)|(a-c),(a-c)|(b-a)$ whence $|b-a|\leq|c-b|\leq|a-c|\leq|b-a|$ whence $|b-a|=|c-b|=|a-c|$ contradiction.

6

Otherwise a contradiction from the polynomial GCD of $p(x)$ and $p'(x)$

7

Perron.

8

9

Buffet Contest.

10

Probably Chebyshev.

Probability

1

In the Nash Equilibrium one obtains that $-2ab+3(1-a)b = 3a(1-b)-4(1-a)(1-b), -2ab+3a(1-b) = 3(1-a)b-4(1-a)(1-b)$ thus $a=b=2-\sqrt{2}$ thus the game has expected value $34\sqrt{2}-48 \neq 0$ and thus is not fair.

2

$98:1:1$ prior ratio updates to $98:16:0$ e.g. $\frac{16}{16+98} = \boxed{\frac{8}{57}}$

3

$\boxed{\frac{1}{2}}$

4

There is a 14×14 of equally likely potential first/second child sex/day pairs and in $\boxed{\frac{13}{27}}$ both children are girls.

5

Higher certainly strictly dominating and one could also write out the delta distributions or note the distribution over the final 7 game delta as a sum of 4 and 3 Bernoulli point masses distributions where a positive delta corresponds with win probability as winning a best-of series is isomorphic with winning a full series.

6

For example let $P(0), P(1), \dots$ be the probabilities we are ever behind i.e. we hit -1 delta from starting state i. So $P(0) = .49 + (.51)P(1), P(1) = (.49)P(0) + (.51)P(2), P(2) = (.49)P(1) + (.51)P(3), \dots$ and obtain $P(0) = \frac{1-.51}{.51} = \frac{.49}{.51}$. Now the expected maximum is isomorphic with repeatedly asking each time we obtain a new maximum: do we obtain a new maximum from this level or not? And thus, it's a geometric series $\frac{\frac{.49}{.51}}{1-\frac{.49}{.51}} = \frac{49}{2}$

7

Probably follows from some algebraic manipulations.

8

$\sum_{k=0}^{\infty} \left(1-\frac{1}{2^k}\right)^n$ so for example by $x = 1-\frac{1}{2^k}$ one obtains $\frac{1}{\ln(2)}\int_0^1 (1-x^n)\frac{dx}{1-x} = \frac{1}{\ln(2)}\left(1+\frac{1}{2}+\cdots+\frac{1}{n}\right)$ and so the asymptotic $\frac{\ln(n)}{\ln(2)} = \boxed{\log_2(n)}$

Geometry

1
VTRMC. $\boxed{3\sqrt{7}}$

2
Induce symmetry in to a square of side length 7 and obtain $\boxed{7\sqrt{2}}$ or on sight the fact that the powerful hammer of coordinate bashing works and execute in a minute.

3
$(a+b+c)\times b = 0 \iff a\times b+b\times b+c\times b = 0 \iff a\times b+c\times b = 0 \iff a\times b = b\times c$

4
Consider expanding/projecting the original polygon outward each edge perpendicularly to itself.

5
Classical result.

6
Classical result extend to sphere argumentation Putnam Notes.

7
IMO. Convex hull $\boxed{\text{regular n-gons}}$ interior contradiction.

8
USAMO. The convex hull hexagon formed by the diagonals is always preserved works for the bound.

9
VTRMC. $\boxed{42}$ by distance formula for example using the usual reflection isomorphic with extension through $\mathbb{Z}^3$ mapping.

10
Putnam. Note, by e.g Pick, that $S \geq \frac{1}{2}$ whence the desired follows from $S = \frac{abc}{4R}$