Distribution of functions of Random Variables

$X \to$ random variable $\quad F_X(x) \to$ CDF

then any function of X, $g(x)$ is also a random variable

$$Y = g(X)$$

$$P(Y \in A) = P(g(X) \in A)$$

$y = g(x) \leftarrow$ mapping from original sample space $\mathcal{X}$ to a new sample space $\mathcal{Y}$

$$g(x) : \mathcal{X} \rightarrow \mathcal{Y}$$

$$g^{-1}(A) = \{ x \in \mathcal{X} : g(x) \in A \}$$

For any set $A \subset \mathcal{Y}$

$$P(Y \in A) = P(g(x) \in A)$$
$$= P(\{x \in \mathcal{X} : g(x) \in A\})$$
$$= P(X \in g^{-1}(A))$$

$$f_Y(y) = P(Y=y) = \sum_{x \in g^{-1}(y)} P(X=x)$$

$$= \sum_{x \in g^{-1}(y)} f_X(x) \quad \text{for } y \in \mathcal{Y}$$

and 0 otherwise

Binomial Distribution

$$f_X(x) = P(X=x) = \binom{n}{x} p^x (1-p)^{n-x} \quad , x = 0, 1, 2 \ldots n.$$

$n \rightarrow$ positive integer and

$$0 \leq p \leq 1$$

$Y = g(X)$, where $g(x) = n - x$

what is $f_Y(y)$?

Homework

Note on the Continuous Random Variable

$$F_Y(y) = P(Y \le y)$$
$$= P(g(X) \le y)$$
$$= P(\{x \in \mathcal{X} : g(x) \le y\})$$

$$= \int_{\{x \in \mathcal{X} : g(x) \le y\}} f_X(x)\, dx$$

* Monotone (increasing or decreasing) function $g(x)$

$$u > v \Rightarrow g(u) > g(v) \quad (\text{inc.})$$

$$u < v \Rightarrow g(u) > g(v) \quad (\text{dec.})$$

Q. $Y = X^2$

is this transformation monotone?

find $f_Y(y)$

Expected Value (of a random variable $g(X)$)
(mean)

$$E\,g(X) = \begin{cases} \displaystyle\int_{-\infty}^{\infty} g(x)\, f_X(x)\, dx, & X \to \text{continuous} \\ \displaystyle\sum_{x \in \mathcal{X}} g(x)\, f_X(x) = \sum_{x \in \mathcal{X}} g(x)\, P(X = x) & \end{cases}$$

$x \to$ discrete

Binomial mean.

$$P(X=x) = \binom{n}{x} p^x (1-p)^{n-x} \quad , \quad x = 0, 1 \ldots n.$$

$$Ex = \sum_{x=0}^{n} x \binom{n}{x} p^x (1-p)^{n-x}$$

$$= \sum_{x=1}^{n} x \binom{n}{x} p^x (1-p)^{n-x}$$

$$\vdots$$

$$= np$$

Things to remember about Expectations

* $E\left(a g_1(x) + b g_2(x) + c\right) = a E\left(g_1(x)\right) + b E\left(g_2(x)\right) + c$

* If $g_1(x) \geq 0$ for all x then $E g_1(x) \geq 0$

* If $g_1(x) \geq g_2(x)$ . . . . . " . . . $E g_1(x) \geq E g_2(x)$

* If $a \leq g_1(x) \leq b$ " $a \leq E g_1(x) \leq b$

Homework : Prove at least the first one

Suppose we measure the distance between a random variable X and a constant b

as $(X-b)^2$

what 'b' minimizes $E(X-b)^2$

$$E(X-b)^2 = E(X-EX)^2 + (EX-b)^2$$

[How do you get this?
Homework]

Variance of Random Variable

$$\text{Var } X = E\,(X - EX)^2$$

$$+\sqrt{\text{Var } X} \longrightarrow \text{Standard deviation.}$$

* Measure of the degree of spread of a distribution around its mean

H.W. $\rightarrow$ Binomial variance

$$X \sim \text{binomial}\,(n, p)$$

$\text{Var } X$?