Draft Solutions Homework

Stats 1
Lect 1

1. Proof of DeMorgan's Law

$x \in (A \cup B)^c \iff x \notin A$ or $x \notin B \iff$
$x \in A^c$ and $x \in B^c \iff x \in A^c \cap B^c$

$x \in (A \cap B)^c \iff x \notin A \cap B \iff x \notin A$ and
$x \notin B \iff x \notin A^c$ or $x \in B^c$

2. legitimate probability function

Proof for finite S

For any $A \in \mathcal{B}$, $P(A) = \sum_{\{i : s_i \in A\}} p_i \geq 0$,

because every $p_i \geq 0$. Thus axiom 1 is true.

Now,

$P(S) = \sum_{(i : s_i \in S)} p_i = \sum_{i=1}^{n} p_i = 1$ ∴ Thus axiom 2 is true.

Let $A_1, \cdots A_k$ denote pairwise disjoint events.

B contains finite sets so we need to consider only finite disjoint unions.

Then

$$P\left(\bigcup_{i=1}^{k} A_i\right) = \sum_{\{j: s_j \in \bigcup_{i=1}^{k} A_i\}} p_j = \sum_{i=1}^{k} \sum_{\{j: s_j \in A_i\}} p_j$$

$$= \sum_{i=1}^{k} P(A_i) .$$

First and third equality are true by definition. Second comes from disjointness.

Thus Axiom 3 is also true.

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$$P(B_1) = P(B_1 \cap A_1) + P(B_1 \cap A_2)$$

$$= \cancel{P(dot} = P(B_1|A_1)\, P(A_1) + P(B_1|A_2) P(A_2)$$

$$= 7/8 \times \frac{3}{7} + \frac{1}{8} \times \frac{4}{7} = \frac{25}{56}.$$

4. Sample space for this experiment has 8 points

$\{HHH, HHT, HTH, THH, TTH, THT, HTT, TTT\}$

let H_i, $i = 1,2,3$ denote the event that the ith toss is head.

e.g. $H_1 = \{HHH, HHT, HTH, HTT\}$

Assuming fairness each sample point's probability is $1/8$, then by enumeration

$$P(H_1) = P(H_2) = P(H_3) = \frac{1}{2}$$

now,

$$P(H_1 \cap H_2 \cap H_3) = P(\{HHH\}) = \frac{1}{8} = \frac{1}{2} \cdot \frac{1}{2} \cdot \frac{1}{2} = P(H_1)\,P(H_2)\,P(H_3)$$

but, we should check all pairs

$$P(H_1 \cap H_2) = P(\{HHH, HHT\}) = \frac{2}{8} = \frac{1}{2} \cdot \frac{1}{2} = P(H_1)\,P(H_2)$$

similarly for other pairs

Thus, H_1, H_2, H_3 are mutually independent.

5. Tossing for a head (Page 20)

$$P(X=x) = (1-p)^{x-1} p$$

since we must get $x-1$ tails followed by a head for the event to occur and all trials are independent.

Now

$$P(X \leq x) = \sum_{i=1}^{x} P(X=i) = \sum_{i=1}^{x} (1-p)^{i-1} p.$$

Partial sum of Geometric series

$$\sum_{k=1}^{n} t^{k-1} = \frac{1-t^n}{1-t}, \quad t \neq 1.$$

Applying ~~ato~~ above to our case.

$$F_X(x) = P(X \leq x)$$

$$= \frac{1-(1-p)^x}{1-(1-p)} p$$

$$= 1-(1-p)^x, \quad x = 1, 2, \ldots$$

and flat between non-negative integers.

6.

Since $\{X = x\} \subset \{x - \varepsilon < X \leq x\}$ for any $\varepsilon > 0$,

we have from theorem :

"If $A \subset B$, then $P(A) \leq P(B)$"

$$P(X = x) \leq P(x - \varepsilon < X \leq x)$$

$$= F_X(x) - F_X(x - \varepsilon) \quad \text{for any } \varepsilon > 0$$

Therefore,

$$0 \leq P(X = x) \leq \lim_{\varepsilon \downarrow 0} [F_X(x) - F_X(x - \varepsilon)] = 0$$

by the right continuity of F_X.